

CO-HOPFIAN AND BOUNDEDLY ENDO-RIGID MIXED GROUPS

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ABSTRACT. For a given cardinal λ and a torsion abelian group K of cardinality less than λ , we present, under some mild conditions (for example $\lambda = \lambda^{\aleph_0}$), boundedly endo-rigid abelian group G of cardinality λ with $\text{Tor}(G) = K$. Essentially, we give a complete characterization of such pairs (K, λ) . Among other things, we use a twofold version of the black box. We present an application of the construction of boundedly endo-rigid abelian groups. Namely, we turn to the existing problem of co-Hopfian abelian groups of a given size, and present some new classes of them, mainly in the case of mixed abelian groups. In particular, we give useful criteria to detect when a boundedly endo-rigid abelian group is co-Hopfian and completely determine cardinals $\lambda > 2^{\aleph_0}$ for which there is a co-Hopfian abelian group of size λ .

CONTENTS

§ 1. Introduction	2
§ 2. Preliminaries	6
§ 3. The ZFC construction of boundedly rigid mixed groups	10
§ 4. Co-Hopfian and boundedly endo-rigid abelian groups	41

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§ 1. INTRODUCTION

By a torsion (resp. torsion-free) group we mean an abelian group such that all its non-zero elements are of finite (resp. infinite) order. A mixed group G contains both non-zero elements of finite order and elements of infinite order, and these are connected via the celebrated short exact sequence

$$(*) \quad 0 \longrightarrow \text{Tor}(G) \longrightarrow G \longrightarrow \frac{G}{\text{Tor}(G)} \longrightarrow 0.$$

Despite the importances of $(*)$, there are series of questions concerning how to glue the issues from torsion and torsion-free parts and put them together to check the desired properties for mixed groups.

Reinhold Baer was interested to find an interplay between abelian groups and rings, see [1] and [2]. In this regard, he raised the following general problem:

Problem 1.1. Which rings can be the endomorphism ring of a given abelian group G ?

There are a lot of interesting research papers and books that study this problem, see for example the books [11] and [17]. According to the recent book of Fuchs [15], for mixed groups, only very little can be said. As an achievement, we cite the works of Corner-Göbel [7] and Franzen-Goldsmith [12].

For any group G , by $E_f(G)$ we mean the ideal of $\text{End}(G)$ consisting of all elements of $\text{End}(G)$ whose image is finitely-generated. In [8], Corner has constructed an abelian group $G := (M, +)$, for some ring R and R -module M , such that any of its endomorphisms is of the form multiplication by some $r \in R$ plus a distinguished function from $E_f(G)$. One can allow such a distinguished function ranges over other classes such as finite-range, countable-range, inessential range or even small homomorphism, and there are a lot of work trying to clarify such situations. As a short list, we may mention the papers Corner-Göbel [7], Dugas-Göbel [10], Corner [8], Thome [33] and Pirece [18].

Here, by a bounded group, we mean a group G such that $nG = 0$ for some fixed $0 < n \in \mathbb{N}$. By a theorem of Baer and Prüfer a bounded group is a direct sum of cyclic groups. The converse is not true. However, there is a partial converse for countable p -groups. For more details see the book of Fuchs [15]. A homomorphism $h \in G_1 \rightarrow G_2$ of abelian groups is called bounded if $\text{Rang}(h)$ is bounded.

Definition 1.2. An abelian group G is *boundedly rigid* when every endomorphism of it has the form $\mu_n + h$, where μ_n is multiplication by $n \in \mathbb{Z}$ and h has bounded range. By $E_b(G)$ we mean the ideal of $\text{End}(G)$ consisting of all elements of $\text{End}(G)$ whose image is bounded.

Let us explain some motivation. The concept of a rigid system of torsion-free groups has a natural analogue for the class of separable p -primary groups: a family $\{G_i : i \in I\}$ of separable p -primary groups is called rigid-like if for all $i \neq j \in I$ every homomorphism $G_i \rightarrow G_j$ is small, and also for all $i \in I$, every endomorphism of G_i is the sum of a small endomorphism and multiplication by a p -adic integer. In his paper [25], Shelah confirmed a conjecture of Pierce [18] by showing that if μ is an uncountable strong limit cardinal, then there is a rigid-like system $\{G_i : i \in I\}$ of separable p -primary groups such that $|G_i| = \mu$ and $|I| = 2^\mu$, see also [23] for more results in this direction.

Let us now turn to the paper and state our main results. Section 2 contains the preliminaries and basic definitions and notations that we need. The reader may skip it, and come back to it when needed later. In Section 3, and as one of the main results, we prove the following.

Theorem 1.3. *Given a cardinal λ such that $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$ and a torsion group K of cardinality less than λ , there is a boundedly rigid abelian group G of cardinality λ with $\text{Tor}(G) = K$.*

To prove this, we introduce a series of definitions and claims. The first one is the rigidity context, denoted by \mathbf{k} , see Definition 3.1. Also, the main technical tool is a variation of “Shelah’s black box”, and we refer to it as *twofold λ -Black Box*. For more details, see Definition 3.13. It may be worth to note that the

black boxes were introduced by Shelah in [28] and [29], where he showed that they follow from ZFC. We can consider them as a general method to generate a class of diamond-like principles provable in ZFC. Then, we continue by introducing the approximation blocks, denoted by $\text{AP}_{\mathbf{k},\lambda}$, more precisely, see Definition 3.18. There is a distinguished object \mathbf{c} in $\text{AP}_{\mathbf{k},\lambda}$ that we call them full. The twofold λ -Black Box, helps us to find such distinguished objects, see Lemma 3.28. Here, one may define the group $G := G_{\mathbf{c}}$. Let $h \in \text{End}(G)$. In order to show h is boundedly rigid, we apply a couple of reductions (see Lemmas 3.33–3.41), to reduce to the case that h factors throughout $G \rightarrow \text{Tor}(G)$. Finally, in Lemma 3.29 we handle this case, by showing that any map $G \rightarrow \text{Tor}(G)$ is boundedly rigid.

In the course of the proof of Theorem 1.3, we develop a general method that allows us to prove $0 \rightarrow \mathbb{Z} \rightarrow \text{End}(G) \rightarrow \frac{\text{End}(G)}{E_b(G)} \rightarrow 0$ is exact, and also enables us to present a connection to Problem 1.1. In order to display the connection, let R be a ring coming from the rigidity context. For the propose of the introduction, we may assume that $(R, +)$ is cotorsion-free, see Definition 2.8 (with the convenience that the argument becomes easier if we work with $R := \mathbb{Z}$, or even $(R, +)$ is \aleph_1 -free). Following our construction, every endomorphism of G has the form $\mu_r + h$, where μ_r is multiplication by $r \in R$ and h has bounded range, i.e., the sequence

$$0 \longrightarrow R \longrightarrow \text{End}(G) \longrightarrow \frac{\text{End}(G)}{E_b(G)} \longrightarrow 0$$

is exact.

Essentially, we give complete characterization of the pairs (K, λ) by relating our work with the recent works of Paolini and Shelah, see [20], [21] and [22]. To this end, first we recall the following folklore problem:

Problem 1.4. Construct co-Hopfian groups of a given size.

Baer [3] was the first to investigate Problem 1.4 for abelian groups. A torsion-free abelian group is co-Hopfian if and only if it is divisible of finite rank, hence the problem naturally reduces to the torsion and mixed cases. In their important paper [4], Beaumont and Pierce proved that if G is co-Hopfian, then $\text{Tor}(G)$ is of size at most continuum, and further that G cannot be a p -groups of size \aleph_0 . This naturally

left open the problem of the existence of co-Hopfian p -groups of uncountable size $\leq 2^{\aleph_0}$, which was later solved by Crawley [6] who proved that there exist p -groups of size 2^{\aleph_0} . Braun and Strüngmann [5] showed that the existence of three types of infinite abelian p -groups of size $\aleph_0 < |G| < 2^{\aleph_0}$ are independent of ZFC:

- (a) both Hopfian and co-Hopfian,
- (b) Hopfian but not co-Hopfian,
- (c) co-Hopfian but not Hopfian.

Also, they proved that the above three types of groups of size 2^{\aleph_0} exist in ZFC. So, in the light of Theorem 1.3, the remaining part is $2^{\aleph_0} < \lambda < \lambda^{\aleph_0}$. Very recently, and among other things, Paolini and Shelah [21] proved that there is no co-Hopfian group of size λ for such a λ . As an application, in Section 4, we completely determine cardinals $\lambda > 2^{\aleph_0}$ for which there is a co-Hopfian group of size λ . For the precise statement, see Corollary 4.12.

Let us recall a connection between the concepts boundedly endo-rigid groups and Hopfian and co-Hopfian groups. First, recall from the seminal paper [24], for any λ less than the first beautiful cardinal, Shelah proved that there is an endo-rigid torsion-free group of cardinality λ . By definition, for any $f \in \text{End}(G)$ there is $m_f \in \mathbb{Z}$ such that $f(x) = m_f x$. So, f is onto iff $m_f = \pm 1$. In other words, G is Hopfian. This naturally motives us to detect co-Hopfian property by the help of some boundedly endo-rigid groups. This is what we want to do in §4. Namely, our first result on co-Hopfian groups is stated as follows:

Theorem 1.5. *Let $K = \bigoplus \{ \frac{\mathbb{Z}}{p^n \mathbb{Z}} : p \in \mathbb{P} \text{ and } 1 \leq n < \omega \}$, where \mathbb{P} is the set of prime numbers. If G is a boundedly endo-rigid abelian group and $K = \text{Tor}(G)$, then G is co-Hopfian.*

In fact, we prove a little more, see Observation 4.9. Let h be a natural number. One of the tools that we use is the h -power torsion subgroup of G , denoted by $\text{Tor}_h(G)$, which is defined as

$$\text{Tor}_h(G) = \{g \in G : \exists n \in \mathbb{N} \text{ such that } h^n g = 0\}.$$

It may be worth to mention that, in the style of Grothendieck, this is called section functor and he denoted this by $\Gamma_h(G)$, and also by $H_h^0(G)$. In our study of the co-Hopfian property of G , the following subset of prime numbers appears:

$$S_G := \{p \in \mathbb{P} : G/\mathrm{Tor}_p(G) \text{ is not } p\text{-divisible}\}.$$

The set S_G helps us to present a useful criteria to detect when a boundedly endo-rigid abelian group is co-Hopfian:

Theorem 1.6. *Assume $\lambda > 2^{\aleph_0}$ and G is a boundedly endo-rigid abelian group of size λ . Then G is co-Hopfian if and only if:*

- (a): S_G is a non-empty set of primes,
- (b): (b₁) $\mathrm{Tor}(G) \neq G$,
 (b₂) if $p \in S_G$, then $\mathrm{Tor}_p(G)$ is not bounded,
 (b₃) if $\mathrm{Tor}_p(G)$ is bounded, then it is finite.

Let G be an abelian group. In order to show that G is (not) co-Hopfian, and also to see a connection to bounded morphisms, we introduce a useful set $\mathrm{NQR}_{(m,n)}(G)$ consisting of those bounded $h \in \mathrm{End}(\mathrm{Tor}_n(G))$ such that:

- (1) $h' := m \cdot \mathrm{id}_{\mathrm{Tor}_n(G)} + h \in \mathrm{End}(\mathrm{Tor}_n(G))$ is 1-to-1,
- (2) h' is not onto or $m > 1$ and $G/\mathrm{Tor}_n(G)$ is not m -divisible.

In a series of nontrivial cases we check $\mathrm{NQR}_{(m,n)}(G)$ and its negation. This enables us to present some new classes of co-Hopfian and non co-Hopfian groups (see items 4.3–4.7).

For all unexplained definitions from set theoretic algebra see the books by Eklof-Mekler [11] and Göbel-Trlifaj [17]. Also, for unexplained definitions from the group theory see the books of Fuchs [15], [14] and [13].

§ 2. PRELIMINARIES

In this section we recall some basic definitions and facts that will be used for later sections of the paper.

Definition 2.1. An abelian group G is called \aleph_1 -free if every countable subgroup of G is free. More generally, an abelian group G is called λ -free if every subgroup of G of cardinality $< \lambda$ is free.

Definition 2.2. Let κ be a regular cardinal. An abelian group G is said to be strongly κ -free if there is a set \mathcal{S} of $< \kappa$ -generated free subgroups of G containing 0 such that for any subset S of G of cardinality $< \kappa$ and any $N \in \mathcal{S}$, there is $L \in \mathcal{S}$ such that $S \cup N \subset L$ and L/N is free.

The abelian group G is pure in H if $G \subseteq H$ and $nG = nH \cap G$ for every $n \in \mathbb{Z}$. The common notation for this notion is $G \subseteq_* H$.

Fact 2.3. *Suppose G is a torsion-free group. Then the intersection of pure subgroups of G is again pure. In particular, for every $S \subset G$, there exists a minimal pure subgroup of G containing S . The common notation for this subgroup is $\langle S \rangle_G^*$.*

Fact 2.4 ([16, Theorem 7]). *Let G be an abelian group and H a pure subgroup of G of bounded exponent. Then H is a direct summand of G .*

Fact 2.5 ([16, Theorem 8]). *Let G be an abelian group and $T \subseteq_* \text{Tor}(G)$. If T is the direct sum of a divisible group and a group of bounded exponent, then T is a direct summand of G . The same result holds if $T \subseteq_* G$.*

Fact 2.6 ([4]). (i) *Let G be a countable p -group. Then G is co-Hopfian if and only if G is finite.*

(ii) *If a group G is co-Hopfian, then $\text{Tor}(G)$ is of size at most continuum, and further that G cannot be a p -groups of size \aleph_0 .*

Fact 2.7 ([14, Theorem 17.2]). *If G is a p -group of bounded exponent, then G is a direct sum of (finitely many, up to isomorphism) finite cyclic groups.*

Definition 2.8. i) An abelian group G is called *cotorsion* if $\text{Ext}(J, G) = 0$ for all torsion-free abelian groups J . In other words, G is cotorsion provided that it is a direct summand of every abelian group H containing G with the property that H/G is torsion-free.

ii) An abelian group G is called *cotorsion-free* if it has no nonzero co-torsion subgroup.

Fact 2.9. ([11, Corollary 2.10(ii)]) *Any \aleph_1 -free group is cotorsion-free.*

The p -torsion parts of a group G are important sources to produce pure subgroups.

Notation 2.10. Let \mathbb{P} denote the set of all prime numbers.

(i) Let $p \in \mathbb{P}$. The p -power torsion subgroup of G is

$$\mathrm{Tor}_p(G) = \{g \in G : \exists n \in \mathbb{N} \text{ such that } p^n g = 0\}.$$

(ii) For $1 \leq m < \omega$ we let $\mathrm{Tor}_m(G) = \bigoplus \{\mathrm{Tor}_p(G) : p \mid m\}$.

(iii) The notation $\mathrm{Tor}(G)$ stands for the full torsion subgroup of G .

Suppose G is torsion. Then

$$G = \bigoplus_{p \in \mathbb{P}} \mathrm{Tor}_p(G).$$

Notation 2.11. In this paper, by $\mathrm{End}(-)$ we mean $\mathrm{End}_{\mathbb{Z}}(-)$ where $(-)$ is at least an abelian group, otherwise we specify it.

The following notion of boundness plays an important role in establishing the main theorems:

Definition 2.12. Let G be an abelian group of size λ . We say G is *boundedly endo-rigid* when for every $f \in \mathrm{End}(G)$ there is $m \in \mathbb{Z}$ such that the map $x \mapsto f(x) - mx$ has bounded range.

The next fact follows from the definition.

Fact 2.13. *G is boundedly endo-rigid if and only if for every $f \in \mathrm{End}(G)$ there is $m \in \mathbb{Z}$ and bounded $h \in \mathrm{End}(G)$ such that $f(x) = mx + h(x)$.*

Fact 2.14. *Let K be a bounded torsion abelian group and let $G \subseteq_* H$. If $g \in \text{Hom}(G, K)$, then there is $h \in \text{Hom}(H, K)$ extending g . This property is conveniently summarized by the subjoined diagram:*

$$\begin{array}{ccc} 0 & \longrightarrow & G & \xrightarrow{\subseteq_*} & H \\ & & \downarrow g & \swarrow \exists h & \\ & & K & & \end{array}$$

Fact 2.15. *Let G be abelian group and suppose that G is not bounded, then the bounded endomorphisms of G (i.e., those $f \in \text{End}(G)$ with bounded range) form an ideal of the ring $\text{End}(G)$, we denote this ideal by $E_b(G)$. With respect to this terminology, G is boundedly rigid if and only if the quotient ring $\text{End}(G)/E_b(G) \cong \mathbb{Z}$.*

Remark 2.16. Recall that torsion subgroups are pure. Let f be a bounded endomorphism of $\text{Tor}(G)$. By Fact 2.14, we have

$$\begin{array}{ccc} 0 & \longrightarrow & \text{Tor}(G) & \xrightarrow{\subseteq_*} & G \\ & & \downarrow f & \swarrow \exists h & \\ & & \text{Tor}(G) & & \end{array}$$

Let $\hat{f} : G \xrightarrow{h} \text{Tor}(G) \xrightarrow{\subseteq} G$. In sum, f extends to an endomorphisms \hat{f} of G with the same range:

$$\begin{array}{ccc} \text{Tor}(G) & \xrightarrow{f} & \text{Tor}(G) \\ \subseteq \downarrow & & \downarrow \subseteq \\ G & \xrightarrow{\hat{f}} & G \end{array}$$

Hence, the notion of boundedly rigid is really the right notion of endo-rigidity for mixed groups (for G torsion-free abelian group, we say that G is endo-rigid when $\text{End}(G) \cong \mathbb{Z}$). For instance, we look at $K = \bigoplus \{ \frac{\mathbb{Z}}{p^{\ell+1}\mathbb{Z}} : \ell < \omega \}$, and recall that this has many bounded endomorphisms. The same will happen for any G extending it.

In what follows we will use the concept of reduced group several times. Let us recall its definition.

Definition 2.17. Let G be an abelian group.

- (a) G is called reduced if it contains no divisible subgroup other than 0.
- (b) G is called injective if for any inclusion $G_1 \subseteq G_2$ of abelian groups, any morphism $f : G_1 \rightarrow G$ can be extended into G_2 :

$$\begin{array}{ccc} 0 & \longrightarrow & G_1 & \xrightarrow{\subseteq} & G_2 \\ & & \downarrow f & \swarrow \exists h & \\ & & G & & \end{array}$$

One can show that an abelian group G is reduced if and only if it is injective, see [15].

§ 3. THE ZFC CONSTRUCTION OF BOUNDEDLY RIGID MIXED GROUPS

In this section we show that for any cardinal $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$ and any torsion abelian group K of size less than λ , there exists a boundedly rigid abelian group G with $\text{Tor}(G) = K$, see Theorem 3.11.

To this end, we define the notion of rigidity context \mathbf{k} which in particular codes a torsion group K , and assign to it a collection of objects \mathbf{m} , which among other things have a group G with $\text{Tor}(G) = K$. We show that under the above assumptions on λ and K , we can always find such an \mathbf{m} such that the associated group G is boundedly rigid.

Definition 3.1. (1) We say a tuple \mathbf{k} is a *rigidity context* when

$$\mathbf{k} = (K_{\mathbf{k}}, R_{\mathbf{k}}, \phi_r^{\mathbf{k}}, \Psi_{r,s}^{\mathbf{k}}, \Psi_{(r,s)}^{\mathbf{k}}, S_{\mathbf{k}})_{r,s \in R_{\mathbf{k}}} = (K, R, \phi_r, \Psi_{r,s}, \Psi_{(r,s)}, S)_{r,s \in R}$$

where

- (a) K is a reduced torsion abelian group,
- (b) R is a ring,
- (c) S is a set of prime numbers, $S_{\mathbf{k}}^{\perp} = \mathbb{P} \setminus S$ is its complement, and R is $S_{\mathbf{k}}^{\perp}$ -divisible. This means that R is divisible for any $p \in S_{\mathbf{k}}^{\perp}$.

- (d) for $r \in R$, $\phi_r \in \text{End}(K)$ has bounded range,
- (e) if $r, s \in R$, then $\Psi_{r,s} = \phi_r + \phi_s - \phi_{r+s} \in \text{End}(K)$,
- (f) if $r, s \in R$, then $\Psi_{(r,s)} \in \text{End}(K)$ has bounded range and, letting $t = rs$, for $x \in K$ we have

$$\Psi_{(r,s)}(x) = \phi_r(\phi_s(x)) - \phi_t(x).$$

- (2) We say \mathbf{k} is nontrivial when for some prime $p \in S_{\mathbf{k}}$ the p -torsion $\text{Tor}_p(K)$ is infinite, or the set

$$\{p \in S_{\mathbf{k}} : \text{Tor}_p(K) \neq 0\}$$

is infinite.

- (3) By $\mathbb{Z}_{\mathbf{k}}$ we mean the subring of \mathbb{Q} generated by $\{1\} \cup \{\frac{1}{p} : p \in S_{\mathbf{k}}^{\perp}\}$.

Observation 3.2. *Suppose $(R_{\mathbf{k}}, +)$ is cotorsion-free as an abelian group. Then $S_{\mathbf{k}} \neq \emptyset$.*

Proof. Suppose $S_{\mathbf{k}} = \emptyset$. In other words, $S_{\mathbf{k}}^{\perp}$ is the set of prime numbers. By definition, R is $S_{\mathbf{k}}^{\perp}$ -divisible. This means that $\mathbb{Q} \subseteq R_{\mathbf{k}}$. It turns out that $(R_{\mathbf{k}}, +)$ is not cotorsion-free, a contradiction. \square

Definition 3.3. Let \mathbf{k} be a rigidity context. By $\mathbf{M}_{\mathbf{k}}$ we mean the family of all tuples

$$\mathbf{m} = (\mathbf{k}_{\mathbf{m}}, G_{\mathbf{m}}, F_r^{\mathbf{m}}, F_{r,s}^{\mathbf{m}}, F_{(r,s)}^{\mathbf{m}})_{r,s \in R_{\mathbf{k}_{\mathbf{m}}}} = (\mathbf{k}, G, F_r, F_{r,s}, F_{(r,s)})_{r,s \in R_{\mathbf{k}}}$$

where

- (a) G is an abelian group,
- (b) $\text{Tor}(G) = K_{\mathbf{k}}$,
- (c) for $r \in R_{\mathbf{k}}$, F_r is an endomorphism of G extending $\phi_r^{\mathbf{k}}$:

$$\begin{array}{ccc} K & \xrightarrow{\phi_r} & K \\ \subseteq \downarrow & & \downarrow \subseteq \\ G & \xrightarrow{F_r} & G \end{array}$$

(d) for $r, s \in R_{\mathbf{k}}$, $F_{r,s} \in \text{End}(G)$ extends $\Psi_{r,s}$

$$\begin{array}{ccc} K & \xrightarrow{\Psi_{r,s}} & K \\ \subseteq \downarrow & & \downarrow \subseteq \\ G & \xrightarrow{F_{r,s}} & G \end{array}$$

and they have the same range $F_{r,s}[G] = \Psi_{r,s}[K]$.

(e) for $r, s \in R_{\mathbf{k}}$, $F_{(r,s)} \in \text{End}(G)$ extends $\Psi_{(r,s)}^{\mathbf{k}}$:

$$\begin{array}{ccc} K & \xrightarrow{\Psi_{(r,s)}} & K \\ \subseteq \downarrow & & \downarrow \subseteq \\ G & \xrightarrow{F_{(r,s)}} & G \end{array}$$

and thereby they have the same range $F_{(r,s)}[G] = \Psi_{(r,s)}[K]$.

(f) if $r, s, t \in R$ and $t = r + s$, then for $x \in G$,

$$F_{r,s}(x) = F_r(x) + F_s(x) - F_t(x),$$

(g) if $r, s, t \in R$ and $t = rs$, then for $x \in G$,

$$F_{(r,s)}(x) = F_r(F_s(x)) - F_t(x).$$

Definition 3.4. Adopt the previous notation, and let

$$\mathbf{M} = \bigcup \{ \mathbf{M}_{\mathbf{k}} : \mathbf{k} \text{ is a rigidity context} \}.$$

(1) We define $\leq_{\mathbf{M}}$ as the following partial order on \mathbf{M} . Namely, $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ iff

- (a) $\mathbf{m}, \mathbf{n} \in \mathbf{M}$,
- (b) $\mathbf{k}_{\mathbf{m}} = \mathbf{k}_{\mathbf{n}}$,
- (c) $G_{\mathbf{m}} \subseteq G_{\mathbf{n}}$,
- (d) $F_r^{\mathbf{m}} \subseteq F_r^{\mathbf{n}}$.

(2) By $\leq_{\mathbf{M}_{\mathbf{k}}}$ we mean $\leq_{\mathbf{M}} \upharpoonright \mathbf{M}_{\mathbf{k}}$.

Notation 3.5. Let $r \in R$ and $x \in G_{\mathbf{m}}$. By rx we mean $rx := F_r^{\mathbf{m}}(x) \in G_{\mathbf{m}}$.

Definition 3.6. Suppose \mathbf{k} is a rigidity context and $\mathbf{m} \in \mathbf{M}_{\mathbf{k}}$.

- (1) We say \mathbf{m} is *boundedly rigid* when for every $f \in \text{End}(G_{\mathbf{m}})$ there are $r \in R$ and $h \in \text{End}_b(G_{\mathbf{m}})$ ¹ and

$$x \in G_{\mathbf{m}} \implies f(x) = rx + h(x).$$

- (2) We say \mathbf{m} is *free* when it has a base B which means that the set $\{x + K_{\mathbf{k}} : x \in B\}$ is a free base of the abelian group $G_{\mathbf{m}}/K$.

- (3) We say \mathbf{m} is λ -*free* when $G_{\mathbf{m}}/K$ is.

- (4) We say \mathbf{m} is *strongly λ -free* when $G_{\mathbf{m}}/K$ is.

- (5) Let $M_{\mathbf{m}}$ be the R -module obtained by expanding $G_{\mathbf{m}}/K$ such that for $x, y \in G_{\mathbf{m}}$ and $r \in R$

$$rx + K = y + K \iff F_r^{\mathbf{m}}(x) = y.$$

The next easy lemma shows that $M_{\mathbf{m}}$ as defined above is well-defined.

Lemma 3.7. $M_{\mathbf{m}}$ is an R -module structure.

Proof. Since $M_{\mathbf{m}}$ is an expansion for $G_{\mathbf{m}}/K$, it is an abelian group. Let $r \in R$ and $m := g + K \in M_{\mathbf{m}}$ where $g \in G$. The assignment

$$(r, m) \mapsto rm := F_r^{\mathbf{m}}(g) + K \in G_{\mathbf{m}}/K = M_{\mathbf{m}},$$

defines the desired module structure on $M_{\mathbf{m}}$. □

Lemma 3.8. Suppose \mathbf{k} is a rigidity context and $\mathbf{m} \in \mathbf{M}_{\mathbf{k}}$. The following assertions hold.

- (1) Suppose $R_{\mathbf{k}} = \mathbb{Z}$ (so, $S_{\mathbf{k}}^{\perp} = \emptyset$). Then \mathbf{m} is boundedly rigid iff $G_{\mathbf{m}}$ is boundedly rigid.
- (2) Let $R_{\mathbf{k}} = \mathbb{Z}_{\mathbf{k}}$ (see Definition 3.1(3)). Then \mathbf{m} is boundedly rigid iff $G_{\mathbf{m}}$ is boundedly rigid.
- (3) if $\phi_r^{\mathbf{k}}$ is zero for every $r \in R$, then $G_{\mathbf{m}}$ is an R -module.

¹so, h has a bounded range.

Proof. (1) and (2) are trivial and follow from the definitions.

(3): For each $x \in G_{\mathbf{m}}$ and $r \in R$, we set $rx := F_r^{\mathbf{m}}(x)$. It is straightforward to furnish the following three properties:

- the identity $r(x + y) = rx + ry$ follows from Definition 3.1(2)(c),
- the equality $(r + s)x = rx + sx$ follows from Definition 3.1(2)(d),
- the equality $r(sm) = (rs)m$ follows from items (e) and (f) from Definition 3.1(2).

From these, $G_{\mathbf{m}}$ is equipped with an R -module structure. □

In what follows, the notation $\text{lg}(-)$ stands for the length function.

Definition 3.9. Let $\alpha \in \text{Ord}$.

(1) By $\Lambda_{\omega}[\alpha]$ we mean

$\{\eta : \text{lg}(\eta) = \omega \text{ and } \eta(n) = (\eta(n, 1), \eta(n, 2)) \text{ where } \eta(n, 1) \leq \eta(n, 2) < \eta(n+1, 1) < \alpha\}$.

(2) For each $\eta \in \Lambda_{\omega}[\alpha]$, we let $\mathbf{j}(\eta) = \bigcup\{\eta(n, 1) : n < \omega\}$.

(3) $\Lambda_{<\omega}[\alpha] := \{\langle \rangle\} \cup \bigcup_{k < \omega} \Lambda_k[\alpha]$, where $\Lambda_k[\alpha]$ is the set of all η furnished with the following four properties:

- (a) $\text{lg}(\eta) = k + 1$,
- (b) $\eta(k) < \alpha$,
- (c) Suppose $\ell < k$. Then
 - (c1) $\eta(\ell) = (\eta(\ell, 1), \eta(\ell, 2))$, where $\eta(\ell, 1) \leq \eta(\ell, 2) < \alpha$, and
 - (c2) If in addition $\ell + 1 < k$, then $\eta(\ell, 2) < \eta(\ell + 1, 1)$,
- (d) if $\ell < k$, then $\eta(\ell, 1) = \eta(\ell, 2)$ iff $\ell = 0$.

(4) $\Lambda[\alpha] := \Lambda_{\omega}[\alpha] \cup \Lambda_{<\omega}[\alpha]$.

(5) If $\eta \in \Lambda[\alpha]$ and $k + 1 < \text{lg}(\eta)$, then we set

(5.1) $\eta \upharpoonright_L k := \langle (\eta(\ell, 1), \eta(\ell, 2)) : \ell < k \rangle \widehat{\langle \eta(k, 1) \rangle}$, and

(5.2) $\eta \upharpoonright_R k := \langle (\eta(\ell, 1), \eta(\ell, 2)) : \ell < k \rangle \widehat{\langle \eta(k, 2) \rangle}$.

Note that $\eta \upharpoonright_L k$ and $\eta \upharpoonright_R k$ belong to $\Lambda_{k+1}[\alpha]$.

(6) We say $\Lambda \subseteq \Lambda[\alpha]$ is *downward closed* while for each $\eta \in \Lambda$ and $k + 1 < \text{lg}(\eta)$ we have $\eta \upharpoonright_L k, \eta \upharpoonright_R k \in \Lambda$.

We next define when a subset of $\Lambda_\omega[\alpha]$ is free.

Definition 3.10. Suppose $\alpha \in \text{Ord}$ and $\Lambda \subseteq \Lambda_\omega[\alpha]$.

- (1) We say Λ is free whenever there is a function $h : \lambda \rightarrow \omega$ such that the sequence

$$\langle \{\eta \upharpoonright_L n, \eta \upharpoonright_R n : h(\eta) \leq n < \omega\} : \eta \in \Lambda \rangle$$

is a sequence of pairwise disjoint sets.

- (2) We say Λ is μ -free when every $\Lambda' \subseteq \Lambda$ of cardinality $< \mu$ is free.

We can now state the main result of this section as follows.

Theorem 3.11. *Let $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$. Let \mathbf{k} be a nontrivial rigidity context such that $K_{\mathbf{k}}$ and $R_{\mathbf{k}}$ are of cardinality $\leq \lambda$. Then there exists an abelian group G such that $\text{Tor}(G) = K_{\mathbf{k}}$ and G is boundedly rigid. In particular, the sequence*

$$0 \longrightarrow R_{\mathbf{c}} \longrightarrow \text{End}(G) \longrightarrow \frac{\text{End}(G)}{E_b(G)} \longrightarrow 0$$

is exact.

The rest of this section is devoted to the proof of above theorem.

Definition 3.12. Suppose γ is an ordinal, $\eta \in \Lambda[\lambda]$ and $\Lambda \subseteq \Lambda[\lambda]$. Then

- (1) S_γ is the closure of $\omega \cup \gamma$ under taking finite subsets, so including finite sequences.
- (2) $\gamma(\eta) = \eta(0, 1)$.
- (3) $\Lambda_\gamma = \{\eta \in \Lambda : \gamma(\eta) < \gamma\}$.
- (4) We set
- (4.1) $\Lambda_{<\omega} = \Lambda \cap \Lambda_{<\omega}[\alpha]$, and
- (4.2) $\Lambda_\omega = \Lambda \cap \Lambda_\omega[\alpha]$.

In order to prove Theorem 3.11, we need a twofold version of Black Box, that we now introduce. On simple Black Boxes see [27], [30] and [31]. The presentation here is a special case of the n -fold λ -Black Box from [32], when $n = 2$.

Definition 3.13. We say \mathbf{b} is a *twofold λ -Black Box* when it consists of:

- (1) $\bar{g} = \langle g_\eta : \eta \in \Lambda_{\mathbf{b}} \rangle$, where $\Lambda_{\mathbf{b}} \subseteq \Lambda_\omega[\lambda]$,
- (2) g_η is a function from ω into $S_{\gamma(\eta)}$,
- (3) Suppose $\langle \Lambda_{\mathbf{b},\epsilon} : \epsilon < \lambda \rangle$ is a partition of $\Lambda_{\mathbf{b}}$, $g : \Lambda_{<\omega}[\lambda] \rightarrow S_\lambda$ is a function, $\epsilon < \lambda$ and $f : \Lambda_{<\omega}[\lambda] \rightarrow \gamma$ where $\gamma < \lambda$. Then for some $\eta \in \Lambda_{\mathbf{b},\epsilon}$ the following holds:
 - (a) $\gamma(\eta) > \gamma$,
 - (b) $g_\eta(0) = g(\langle \rangle)$,
 - (c) $g_\eta(n+1) = (g(\eta \upharpoonright_L n), g(\eta \upharpoonright_R n))$,
 - (d) $\eta(n, 1) < \eta(n, 2)$ and $f(\eta \upharpoonright_L n) = f(\eta \upharpoonright_R n)$ for all $1 \leq n < \omega$.

The following theorem is proved in [32].

Theorem 3.14. *Assume $\lambda = \lambda^{\aleph_0}$. Then there exists an \aleph_1 -free twofold λ -Black Box.*

Assuming hypotheses beyond ZFC, we can get stronger versions of twofold λ -Black Box (see again [32]).

Observation 3.15. *Assume $\lambda = \text{cf}(\lambda) \geq \aleph_1$. Let*

$$S \subseteq \{\alpha < \lambda : \text{cf}(\alpha) = \aleph_0\}$$

be a stationary and non-reflecting subset of λ such that the principle \diamond_S holds. Then there is a λ -free twofold λ -Black Box \mathbf{b} such that $\Lambda_{\mathbf{b}} = \{\eta_\delta : \delta \in S\}$ and $\mathbf{j}(\eta_\delta) = \delta$ for every $\delta \in S$.

Recall that Jensen's diamond principle \diamond_S is a kind of prediction principle whose truth is independent of ZFC. The point in the above proof is that if $\Lambda_{\mathbf{b}} = \{\eta_\delta : \delta \in S\}$ and $\mathbf{j}(\eta_\delta) = \delta$ for every $\delta \in S$, then as S does not reflect, the set $\Lambda_{\mathbf{b}}$ is λ -free.

Hypothesis 3.16. For the rest of this section we adopt the following hypotheses, otherwise specializes:

- $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$.
- \mathbf{k} is a rigidity context as in Definition 3.1.

- $K = K_{\mathbf{k}}, R = R_{\mathbf{k}}$ are of cardinality $< \lambda$. Without loss of generality the set of elements of K and R are subsets of λ .
- $(R, +)$ is cotorsion-free.
- \mathbf{b} is a twofold λ -Black Box.

Remark 3.17. Recall from [5] that co-Hopfian (resp. Hopfian) abelian group of size $\lambda = 2^{\aleph_0}$ exist in ZFC. We can also deal with the case of $\lambda = 2^{\aleph_0}$, but all is known in this case, so we just concentrate on the case $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$.

Definition 3.18. Let $\text{AP} = \text{AP}_{\mathbf{k},\lambda}$ be the set of all quintuples

$$\mathbf{c} = (\Lambda_{\mathbf{c}}, \mathbf{m}_{\mathbf{c}}, \Gamma_{\mathbf{c}}, X_{\mathbf{c}}, \langle a_{\eta,n}^{\mathbf{c}} : \eta \in \Lambda_{\mathbf{c}}, n < \omega \rangle)$$

such that:

- $\Lambda_{\mathbf{c}} \subseteq \Lambda[\lambda]$ is downward closed.
- $\mathbf{m}_{\mathbf{c}} \in \mathbf{M}_{\mathbf{k}}$. We may write $G_{\mathbf{c}}, M_{\mathbf{c}}$ instead of $G_{\mathbf{m}_{\mathbf{c}}}, M_{\mathbf{m}_{\mathbf{c}}}$ respectively, etc.
- $X_{\mathbf{c}}$ is the following set:

$$\{rx_{\nu} : r \in R, \nu \in \Lambda_{\mathbf{c}, < \omega}\} \cup \{ry_{\eta,n} : r \in R, \eta \in \Lambda_{\mathbf{c}, \omega}, n < \omega\}.$$

- $G_{\mathbf{c}}$ is generated, as an abelian group, by the sets K and $X_{\mathbf{c}}$. The relations presented in item (f), see below.
- for any ordinal α , let $G_{\mathbf{c},\alpha}$ be the subgroup of $G_{\mathbf{c}}$ generated by the set K and

$$\{rx_{\nu} : r \in R, \nu \in \Lambda_{\mathbf{c}, < \omega} \cap \Lambda[\alpha]\} \cup \{ry_{\rho,n} : r \in R, \rho \in \Lambda_{\mathbf{c}, \omega} \cap \Lambda[\alpha], n < \omega\}.$$

- $M_{\mathbf{c}}$, as an R -module, is generated by $X_{\mathbf{c}} \cup K$, freely except the following set $\Gamma_{\mathbf{c}}$ of equations:

$$\bullet y_{\eta,n} = a_{\eta,n}^{\mathbf{c}} + (n!)y_{\eta,n+1} + (x_{\eta \upharpoonright_L n} - x_{\eta \upharpoonright_R n}),$$

where $a_{\eta,n}^{\mathbf{c}} \in G_{\mathbf{c},\eta(0,1)}$.

Definition 3.19. Suppose $\mathbf{c} \in \text{AP}_{\mathbf{k},\lambda}$.

- $\gamma_{\mathbf{c}} = \min\{\gamma \leq \lambda : \Lambda_{\mathbf{c}} \subseteq \Lambda[\gamma]\}$.
- Let $\Omega_{\mathbf{c}} := \Lambda_{\mathbf{c}, < \omega} \cup (\Lambda_{\mathbf{c}, \omega} \times \omega)$ and define $\langle x_{\rho} : \rho \in \Omega_{\mathbf{c}} \rangle$ by the following rule
 - (2.1) If $\rho \in \Lambda_{\mathbf{c}, < \omega}$, then x_{ρ} is defined as in Definition 3.18(c).

(2.2) If $\rho = (\eta, n) \in \Lambda_{\mathbf{c}, \omega} \times \omega$, we define $x_\rho := y_{\eta, n}$.

(3) For $b \in G_{\mathbf{c}}$ choose the sequence

$$\langle r_{b, \ell}, \eta_{b, \ell}, m_{b, \ell} : \ell < n_b \rangle$$

such that

$$b - \sum_{\ell < n_b} r_{b, \ell} y_{\eta_{b, \ell}, m_{b, \ell}} \in \sum_{\rho \in \Lambda_{\mathbf{c}, < \omega}} Rx_\rho + K,$$

where $r_{b, \ell} \in R \setminus \{0\}$ and $(\eta_{b, \ell}, m_{b, \ell}) \in \Lambda_{\mathbf{c}, \omega} \times \omega$.

(4) By $\text{supp}_o(b)$ we mean $\{\eta_{b, \ell} : \ell < n_b\}$.

Definition 3.20. Suppose $\mathbf{c} \in \text{AP}_{\mathbf{k}, \lambda}$ and let $a \in G_{\mathbf{c}}$.

(a) There is a finite set $\Lambda_a \subseteq \Lambda_{\mathbf{c}}$, a sequence $S := \langle r_\rho : \rho \in \Lambda_a \rangle$ of non-zero elements of R , an $n(a) < \omega$ and $d_a \in K$ such that

$$a = \sum_{\eta \in \Lambda_{a, < \omega}} r_\eta x_\eta + \sum_{\nu \in \Lambda_{a, \omega}} r_\nu y_{\nu, n(a)} + d_a,$$

where $\Lambda_{a, < \omega} = \Lambda_a \cap \Lambda_{\mathbf{c}, < \omega}$ and $\Lambda_{a, \omega} = \Lambda_a \cap \Lambda_{\mathbf{c}, \omega}$.

(b) Let $\text{supp}_{\mathbf{c}}(a) = \text{supp}(a)$ be the minimal set $\Lambda \subseteq \Lambda_{\mathbf{c}}$ minimal with respect to the following two properties:

(b.1) $\Lambda_a \subseteq \Lambda$.

(b.2) If $\nu \in \Lambda_a \cap \Lambda_{\mathbf{c}, \omega}$ and $n < \omega$ then $\Lambda_{a_{\nu, n}} \subset \Lambda$ and $\eta \upharpoonright_L n, \eta \upharpoonright_R n \in \Lambda$.

for $a \in G_{\mathbf{c}}$ let $\text{supp}_{\mathbf{c}}(a) = \text{supp}(a)$ be the minimal set $\Lambda \subseteq \Lambda_{\mathbf{c}}$ such that

$$a \in \langle \{x_\eta, y_{\nu, n} : \eta \in \Lambda(L, R), \nu \in \Lambda, n < \omega\} \cup K \rangle_{G_{\mathbf{c}}}^*.$$

Remark 3.21. Adopt the previous notation. The following holds.

(1) The set $\text{supp}_{\mathbf{c}}(a)$ is countable.

(2) If $a = x_\nu$ for some $\nu \in \Lambda_{\mathbf{c}}$, then

$$\text{supp}(a) \setminus S_{\eta(\nu, 1)} = \{\nu\} \cup \{\nu \upharpoonright_L n, \nu \upharpoonright_R n : n < \omega\}.$$

Definition 3.22. Let \leq_{AP} be the following partial order on $\text{AP} = \text{AP}_{\mathbf{k}, \lambda}$. For any $\mathbf{c}, \mathbf{d} \in \text{AP}$ we say $\mathbf{c} \leq_{\text{AP}} \mathbf{d}$ when the following holds:

(a) $\Lambda_{\mathbf{c}} \subseteq \Lambda_{\mathbf{d}}$,

- (b) $\mathbf{m}_c \leq_M \mathbf{m}_d$, hence $G_c \subseteq G_d$, etc.
- (c) $a_{\eta,\ell}^c = a_{\eta,\ell}^d$ for $\eta \in \Lambda_c, \ell < \omega$,
- (d) $x_\eta^c = x_\eta^d$ for $\eta \in \Lambda_{c,<\omega}$,
- (e) $y_{\eta,\ell}^c = y_{\eta,\ell}^d$ for $\eta \in \Lambda_{c,\omega}$ and $\ell < \omega$.

Lemma 3.23. *The following two assertions are valid:*

- (1) \leq_{AP} is indeed a partial order,
- (2) If $\bar{\mathbf{c}} = \langle \mathbf{c}_\alpha : \alpha < \delta \rangle$ is \leq_{AP} -increasing, then there exists $\mathbf{c}_\delta = \bigcup_{\alpha < \delta} \mathbf{c}_\alpha$ in AP which is the \leq_{AP} -least upper bound of the sequence $\bar{\mathbf{c}}$.

Proof. Clause (1) is clear, for clause (2), let

$$\mathbf{c}_\delta = (\Lambda, \mathbf{m}, \Gamma, X, \langle a_{\eta,n} : \eta \in \Lambda, n < \omega \rangle),$$

where:

- $\Lambda = \bigcup_{\alpha < \delta} \Lambda_{\mathbf{c}_\alpha}$,
- $\mathbf{m} = (G, F_r, F_{r,s}, F_{(r,s)})$, where
 - $G = \bigcup_{\alpha < \delta} G_{\mathbf{c}_\alpha}$,
 - $F_r = \bigcup_{\alpha < \delta} F_r^{\mathbf{c}_\alpha}$, and similarly for $F_{r,s}$ and $F_{(r,s)}$.
- $\Gamma = \bigcup_{\alpha < \delta} \Gamma_{\mathbf{c}_\alpha}$,
- $X = \bigcup_{\alpha < \delta} X_{\mathbf{c}_\alpha}$,
- for $\eta \in \Lambda_\omega$ and $n < \omega$, we have $a_{\eta,n} = a_{\eta,n}^{\mathbf{c}_\alpha}$, for some and hence any $\alpha < \delta$ such that $\eta \in \Lambda_{\mathbf{c}_\alpha,\omega}$.

It is easily seen that \mathbf{c}_δ is as required. □

An R -module M is called \aleph_1 -free, if every countably generated submodule of M is contained in a free submodule of M . Similarly, μ -free can be defined. For more details, see [11, IV. Definition 1.1].

Lemma 3.24. *Let $\mathbf{c} \in AP$. The following claims hold:*

- (1) $\text{Tor}[G_{\mathbf{c}}] = K$,
- (2) The group

$$G_{\mathbf{c}} / \langle K \cup \{rx_\nu : r \in R, \nu \in \Lambda_{\mathbf{c},<\omega}\} \rangle$$

is divisible and torsion-free. Also, the parallel result holds for the R -module:

$$M_{\mathbf{c}} / \langle K \cup \{rx_\nu : r \in R, \nu \in \Lambda_{\mathbf{c}, < \omega}\} \rangle$$

(3) the following hold:

- (a) $\Lambda_{\mathbf{c}}$ is \aleph_1 -free.
- (b) If $\Lambda_{\mathbf{c}}$ is μ -free, then $M_{\mathbf{c}}$ is μ -free.
- (c) If $\Lambda_{\mathbf{c}}$ is μ -free and $(R, +)$ is μ -free, then $G_{\mathbf{c}}/K$ is a μ -free abelian group.

(4) If $\gamma \leq \gamma_{\mathbf{c}}$ and $\Lambda \subseteq \Lambda_{\mathbf{c}}$, then there exists a unique $\mathbf{d} \in \text{AP}$ such that

- (a) $\Lambda_{\mathbf{d}} = \Lambda \cap \Lambda[\gamma]$,
- (b) $G_{\mathbf{d}} \subseteq G_{\mathbf{c}}$.

Such a unique object is denoted by $\mathbf{d} := \mathbf{c} \upharpoonright (\gamma, \Lambda)$.

(5) Assume $\eta \in \Lambda_{\omega}[\lambda] \setminus \Lambda_{\mathbf{c}}$, $\ell < \omega$ and $a_\ell \in G_{\mathbf{c}}$ are such that $a_\ell \in G_{\mathbf{c}, \eta(0,1)}$ for each ℓ . Then there is $\mathbf{d} \in \text{AP}$ equipped with the following three properties:

- (a) $\Lambda_{\mathbf{d}} = \Lambda_{\mathbf{c}} \cup \{\eta\} \cup \{\eta \upharpoonright_L n, \eta \upharpoonright_R n : n < \omega\}$,
- (b) $\mathbf{c} \leq_{\text{AP}} \mathbf{d}$ and so $G_{\mathbf{c}} \subseteq G_{\mathbf{d}}$,
- (c) $a_{\eta, \ell}^{\mathbf{d}} = a_\ell$ for $\ell < \omega$.

Proof. (1)-(2): These are easy.

(3): (a) : Let $\Lambda \subseteq \Lambda_{\mathbf{c}, \omega}$ be countable, and let $\{\eta_n : n < \omega\}$ be an enumeration of it. Define the maps h_1 and h_2 from Λ to ω as follows:

$$h_1(\eta_n) := \min \left\{ k : \forall j < n, \forall \ell, r \in \{L, R\} \text{ we have } \eta_j \upharpoonright_\ell k \neq \eta_n \upharpoonright_r k \right\},$$

and

$$h_2(\eta_n) := \min \left\{ k : \eta_n \upharpoonright_L k \neq \eta_n \upharpoonright_R k \right\}.$$

Finally, we set

$$h(\eta_m) := \max\{h_1(\eta_m), h_2(\eta_m)\} + 1.$$

Having Definition 3.10 in mind, we are going to show h is as required. Let $j < i < \omega$ and let

- $h(\eta_j) \leq n_j < \omega$
- $h(\eta_i) \leq n_i < \omega$.

We will show that $\eta_j \upharpoonright_\ell n_i \neq \eta_i \upharpoonright_r n_j$, where $\ell, r \in \{L, R\}$. To see this, we note that there is nothing to prove if $n_i \neq n_j$. So, we may and do assume that $n := n_i = n_j$. Thus, $h(\eta_j), h(\eta_i) \leq n$. We look at $m := h_1(\eta_i)$. According to the definition of h_1 , we know that $\eta_j \upharpoonright_\ell m \neq \eta_i \upharpoonright_r m$. As $m \leq n$ one has

$$\eta_i \upharpoonright_\ell n \neq \eta_j \upharpoonright_r n.$$

Also given any $i < \omega$, if $n \geq h(\eta_i)$, then by the definition of h_2 and as $n \geq h_2(\eta_i)$, we have

$$\eta_i \upharpoonright_L n \neq \eta_i \upharpoonright_R n.$$

It follows that the sequence

$$\langle \{\eta \upharpoonright_L n, \eta \upharpoonright_R n : h(\eta) \leq n < \omega\} : \eta \in \Lambda \rangle$$

is a sequence of pairwise disjoint sets. By definition, $\Lambda_{\mathbf{c}}$ is \aleph_1 -free.

(b) : For simplicity, we present the proof when $\mu := \aleph_1$. Let $X \subseteq M_{\mathbf{c}}$ be countable. We are going to show that it is included into a countably generated free R -submodule of $M_{\mathbf{c}}$. As X countable,

- $\exists \Lambda \subseteq \Lambda_{\mathbf{c}, \omega}$ countable,
- $\exists \Lambda_* \subseteq \Lambda_{\mathbf{c}, < \omega}$ countable

such that

$$X \subseteq \sum \{Ry_{\eta, n} : \eta \in \Lambda \text{ and } n < \omega\} + \sum \{Rx_\rho : \rho \in \Lambda_*\}.$$

As $\Lambda_{\mathbf{c}}$ is \aleph_1 -free and Λ is countable, there is a function $h : \Lambda \rightarrow \omega$ such that the sequence

$$\langle \{\eta \upharpoonright_L n, \eta \upharpoonright_R n : h(\eta) \leq n < \omega\} : \eta \in \Lambda \rangle$$

is a sequence of pairwise disjoint sets. Now, we note the following two properties:

(b)₁: the R -module

$$M_\Lambda := \langle x_{\eta \upharpoonright_L n}, x_{\eta \upharpoonright_R n}, y_{\eta, n} : \eta \in \Lambda : h(\eta) \leq n < \omega \rangle$$

is free;

(b)₂: Set $M_{\Lambda \cup \Lambda_*} := \langle M_\Lambda \cup \{x_\nu : \nu \in \Lambda_*\} \rangle$. Then the R -module $\frac{M_{\Lambda \cup \Lambda_*}}{M_{\Lambda_*}}$ is free.

In view of $(b)_2$ the short exact sequence

$$0 \longrightarrow M_\Lambda \longrightarrow M_{\Lambda \cup \Lambda_*} \longrightarrow M_{\Lambda \cup \Lambda_*} / M_\Lambda \longrightarrow 0,$$

splits. Combining this along with $(b)_1$, we observe that $M_{\Lambda \cup \Lambda_*}$ is free. Since it includes X , we get the desired claim.

(c) : Now, suppose $(R, +)$ is μ -free. Let H be a subset of $(G_{\mathbf{c}}/K, +)$ of size $< \mu$. There is a free R -module F such that $H \subset F$. There is a subset S of R of size $< \mu$ such that any element of H can be written from a linear combination from F with coefficients taken from S . As $(R, +)$ is μ -free, there is a free subgroup $(T, +)$ of it containing S . Then,

$$H \subseteq T * F := \left\langle \sum \{t_i f_i : t_i \in T, f_i \in F\} \right\rangle.$$

Since $(T * F, +)$ is free as an abelian group, we get the desired claim.

(4): Let \mathbf{d} be such that:

$$4.1) \Lambda_{\mathbf{d}} = \Lambda \cap \Lambda[\gamma],$$

4.2) $X_{\mathbf{d}}$ is defined using $\Lambda_{\mathbf{d}}$ naturally,

$$4.3) \text{ for } \nu \in \Lambda_{\mathbf{d}, \omega} \text{ and } n < \omega, a_{\nu, n}^{\mathbf{d}} = a_{\nu, n}^{\mathbf{c}},$$

4.4) $\Gamma_{\mathbf{d}}$ is defined naturally as the set of equations in (1), but only for $\eta \in \Lambda_{\mathbf{d}, \omega}$.

This is straightforward to check that \mathbf{d} is as required.

(5): Let \mathbf{d} be defined in the natural way, so that:

$$5.1) \Lambda_{\mathbf{d}} = \Lambda_{\mathbf{c}} \cup \{\eta\} \cup \{\eta \upharpoonright_L n, \eta \upharpoonright_R n : n < \omega\},$$

$$5.2) X_{\mathbf{d}} = X_{\mathbf{c}} \cup \{x_{\eta \upharpoonright_L n}, x_{\eta \upharpoonright_R n} : n < \omega\} \cup \{y_{\eta, n} : n < \omega\},$$

$$5.3) \text{ for } \nu \in \Lambda_{\mathbf{c}, \omega} \text{ and } n < \omega, a_{\nu, n}^{\mathbf{d}} = a_{\nu, n}^{\mathbf{c}},$$

$$5.4) a_{\eta, n}^{\mathbf{d}} = a_n \text{ for } n < \omega,$$

5.5) in addition to the equations displayed in $\Gamma_{\mathbf{c}}$, $\Gamma_{\mathbf{d}}$ contains equations of the following forms

$$y_{\eta, n} = a_n + (n!)y_{\eta, n+1} + (x_{\eta \upharpoonright_L n} - x_{\eta \upharpoonright_R n}),$$

where $n < \omega$.

The assertion is now obvious by the above definition of \mathbf{d} . □

Lemma 3.25. *Let $\mathbf{c} \in \text{AP}$. Then the abelian group $G_{\mathbf{c}}/K$ is reduced.*

Proof. Suppose on the way of contradiction that $G_{\mathbf{c}}/K$ is not reduced. Then it has divisible direct summand I . Since $G_{\mathbf{c}}/K$ is torsion-free, I is both injective (see Discussion 2.17) and torsion-free. This yields that $(\mathbb{Q}, +)$ is a directed summand of $G_{\mathbf{c}}/K$. Recall from Lemma 3.24 that $M_{\mathbf{c}}$ is \aleph_1 -free as an R -module. We have two possibilities: 1) \mathbf{k} is trivial, and 2) \mathbf{k} is nontrivial.

1) \mathbf{k} is trivial: Then $R := \mathbb{Z}$. Recall that $M_{\mathbf{c}} = G_{\mathbf{c}}/K$ is \aleph_1 -free. Since $(\mathbb{Q}, +)$ is countable, it should be free, a contradiction.

2) \mathbf{k} is nontrivial: Recall that R is $S_{\mathbf{k}}^{\perp}$ -divisible. Since the context is nontrivial, there is $p \in S_{\mathbf{k}}^{\perp}$ such that $\{1/p^n : n \gg 0\} \subseteq R$. For simplicity, we assume that $\{1/p^n : n > 0\} \subseteq R$. Since $M_{\mathbf{c}}$ is \aleph_1 -free and that $\{1/p^n : n > 0\} \subseteq \mathbb{Q} \subseteq M_{\mathbf{c}}$, there is a free R -module $F \subseteq M_{\mathbf{c}}$ such that $\{1/p^n : n > 0\} \subseteq F$. Let $F = \bigoplus R$. So,

$$\begin{aligned} \{r/p^n : n > 0, r \in R\} &= \bigcap_{\ell > 0} p^{\ell} \{r/p^n : n > 0, r \in R\} \\ &\subseteq \bigcap_{\ell > 0} p^{\ell} F \\ &= \bigoplus (\bigcap_{\ell > 0} p^{\ell} R) \\ &\subseteq \bigoplus (\bigcap_{\ell > 0} \ell R) \\ &= 0, \end{aligned}$$

where we the last equality comes from the fact that $(R, +)$ is cotorsion-free. This is the contradiction that we searched for it. \square

The following easy lemma will be used later at several places.

Lemma 3.26. *For any $n < \omega$,*

$$y_{\eta,0}^{\mathbf{c}} = \sum_{i=0}^n (\prod_{j<i} j!) a_{\eta,i}^{\mathbf{c}} + (\prod_{i=1}^n i!) y_{\eta,n+1}^{\mathbf{c}} + \sum_{i=0}^n (\prod_{j<i} j!) (x_{\eta \upharpoonright_L i}^{\mathbf{c}} - x_{\eta \upharpoonright_R i}^{\mathbf{c}}).$$

Proof. It clearly holds for $n = 0$. Suppose it holds for n . Apply the induction assumption along with the relation

$$y_{\eta,n+1}^{\mathbf{c}} = a_{\eta,n+1}^{\mathbf{c}} + (n+1)! y_{\eta,n+2}^{\mathbf{c}} + (x_{\eta \upharpoonright_{L^{n+1}}}^{\mathbf{c}} - x_{\eta \upharpoonright_{R^{n+1}}}^{\mathbf{c}})$$

to deduce

$$\begin{aligned}
 y_{\eta,0}^{\mathbf{c}} &= \sum_{i=0}^n (\prod_{j<i} j!) a_{\eta,i}^{\mathbf{c}} + (\prod_{i=1}^n i!) y_{\eta,n+1}^{\mathbf{c}} + \sum_{i=0}^{n+1} (x_{\eta \upharpoonright_L i}^{\mathbf{c}} - x_{\eta \upharpoonright_R i}^{\mathbf{c}}) \\
 &= \sum_{i=0}^n (\prod_{j<i} j!) a_{\eta,i}^{\mathbf{c}} + (\prod_{i=0}^n i!) a_{\eta,n+1}^{\mathbf{c}} + (\prod_{i=1}^n i!) (n+1)! y_{\eta,n+2}^{\mathbf{c}} \\
 &\quad + (\prod_{i=0}^n i!) (x_{\eta \upharpoonright_L n+1}^{\mathbf{c}} - x_{\eta \upharpoonright_R n+1}^{\mathbf{c}}) + \sum_{i=0}^n (\prod_{j<i} j!) (x_{\eta \upharpoonright_L i}^{\mathbf{c}} - x_{\eta \upharpoonright_R i}^{\mathbf{c}}) \\
 &= \sum_{i=0}^{n+1} (\prod_{j<i} j!) a_{\eta,i}^{\mathbf{c}} + (\prod_{i=1}^{n+1} i!) y_{\eta,n+2}^{\mathbf{c}} + \sum_{i=0}^{n+1} (\prod_{j<i} j!) (x_{\eta \upharpoonright_L i}^{\mathbf{c}} - x_{\eta \upharpoonright_R i}^{\mathbf{c}}).
 \end{aligned}$$

Thus the claim holds for $n+1$ as well. \square

There are some distinguished and useful objects in $\text{AP}_{\mathbf{k},\lambda}$:

Definition 3.27. We say $\mathbf{c} \in \text{AP}_{\mathbf{k},\lambda}$ is full when:

- (a) $\Lambda_{\mathbf{c}} \supseteq \Lambda_{<\omega}[\lambda]$,
- (b) if $a_n \in G_{\mathbf{c}}$ for $n < \omega$ and $f : \Lambda_{<\omega}[\lambda] \rightarrow \gamma$, where $\gamma < \lambda$, then for some $\eta \in \Lambda_{\mathbf{c}}$ and all $n < \omega$ we have $a_{\eta,n}^{\mathbf{c}} = a_n$ and $f(\eta \upharpoonright_L n) = f(\eta \upharpoonright_R n)$.

Now, we study the existence problem for fullness in AP :

Lemma 3.28. *There are some full $\mathbf{c} \in \text{AP}_{\mathbf{k},\lambda}$.*

Proof. Let \mathbf{b} be a twofold λ -Black Box, which exists by Theorem 3.14. Let $\Omega := \Lambda_{<\omega}[\lambda] \cup (\Lambda_{\omega}[\lambda] \times \omega)$, and for each ordinal $\alpha < \lambda$ set $\Omega_{\alpha} = \Lambda_{<\omega}[\alpha] \cup (\Lambda_{\omega}[\alpha] \times \omega)$. Fix a bijection map

$$h : S_{\lambda} \xrightarrow{\cong} \left(\bigoplus_{\rho \in \Omega} Rx_{\rho} \right) \oplus K$$

such that for each ordinal $\alpha < \lambda$ one has

$$h''[S_{\alpha}] \subseteq \left(\bigoplus_{\rho \in \Omega_{\alpha}} Rx_{\rho} \right) \oplus K \quad (*),$$

This is possible, as for each α ,

$$|S_{\alpha}| \leq \aleph_0 + |\alpha| \leq \left| \left(\bigoplus_{\rho \in \Omega_{\alpha}} Rx_{\rho} \right) \oplus K \right| < \lambda.$$

Let \mathbf{c} be defined as

- (1) $\Lambda_{\mathbf{c}} = \Lambda_{\mathbf{b}} \cup \Lambda_{<\omega}[\lambda]$.
- (2) $X_{\mathbf{c}}$ is the following set:

$$\{rx_{\nu} : r \in R, \nu \in \Lambda_{\mathbf{c},<\omega}\} \cup \{ry_{\eta,n} : r \in R, \eta \in \Lambda_{\mathbf{c},\omega}, n < \omega\}.$$

- (3) $a_{\eta,n}^c = h(g_\eta^b(n+1))$, where g_η^b is given by the twofold λ -Black Box.
- (4) $G_{\mathbf{c}}$ is generated, as an abelian group, freely by the sets K and $X_{\mathbf{c}}$ except the following set of relations:

$$y_{\eta,n} = a_{\eta,n}^c + (n!)y_{\eta,n+1} + (x_{\eta \upharpoonright_L n} - x_{\eta \upharpoonright_R n}),$$

with the convenience that $a_{\eta,n}^c$ is regarded as an element of $G_{\mathbf{c}}$ via the quotient map

$$\left(\bigoplus_{\rho \in \Omega} Rx_\rho \right) \oplus K \twoheadrightarrow G_{\mathbf{c}}.$$

From this identification and (*), $a_{\eta,n}^c \in G_{\mathbf{c},\eta(0,1)}$.

- (5) $\Gamma_{\mathbf{c}}$ is defined naturally as in Definition 3.18.

Let us show that \mathbf{c} is as required. It clearly satisfies clause (a) of Definition 3.27. To show that clause (b) of Definition 3.27 is satisfied, let $\langle a_n : n < \omega \rangle \in {}^\omega G_{\mathbf{c}}$ and $f : \Lambda_{<\omega}[\lambda] \rightarrow \gamma$, where $\gamma < \lambda$. Let $g : \Lambda_{<\omega}[\lambda] \rightarrow S_\lambda$ be defined such that for all $\nu \in \Lambda_{<\omega}[\lambda] \setminus \{\langle \rangle\}$,

$$h(g(\nu)) = a_{\text{lg}(\nu)-1} \quad (+).$$

We are going to apply the twofold λ -Black Box \mathbf{b} . According to its properties, there is an $\eta \in \Lambda_{\mathbf{b}}$ such that:

- (6) $\gamma(\eta) > \gamma$,
- (7) $g_\eta^b(0) = g(\langle \rangle)$,
- (8) $g_\eta^b(n+1) = g(\eta \upharpoonright_L n)^2$,
- (9) $\eta(n,1) < \eta(n,2)$ and $f(\eta \upharpoonright_L n) = f(\eta \upharpoonright_R n)$ for all $1 \leq n < \omega$.

Applying h to the both sides of (8), one has

$$a_{\eta,n}^c \stackrel{(**)}{=} h(g_\eta^b(n+1)) = h(g(\eta \upharpoonright_L n)) \stackrel{(+)}{=} a_n,$$

thereby completing the proof. \square

²Here we are using a modified version of the twofold λ -Black Box \mathbf{b} , which can be easily obtained from the original one.

Lemma 3.29. *Assume $\mathbf{c} \in \text{AP}$ is full and let $h \in \text{Hom}(G_{\mathbf{c}}, K)$ be unbounded.*

Then there is a sequence

$$\langle a_n : n < \omega \rangle \in {}^\omega \text{Rang}(h)$$

such that the following set of equations Γ has no solution, not only in $G_{\mathbf{c}}$, but in any $G_{\mathbf{d}}$ with $\mathbf{c} \leq \mathbf{d} \in \text{AP}$, where

$$\Gamma := \{z_n = a_n + n!z_{n+1} : n < \omega\}.$$

Proof. If for some prime number p , $\text{Tor}_p[\text{Rang}(h)]$ is infinite, let p be the first such prime number and let $p_n = p$ for all $n < \omega$. Otherwise let

$$p_n \in \{p : \text{Tor}_p[\text{Rang}(h)] \neq 0\}$$

be a strictly increasing sequence of prime numbers. As h is not bounded, we can find by induction on n , the pair (H_n, a_n) such that:

- (+) (a) $H_0 = \text{Rang}(h)$,
- (b) $H_n = a_n\mathbb{Z} \oplus H_{n+1}$,
- (c) a_n has order $p_n^{l_n}$, where for $n = m + 1$ we assume

$$l_n > l_m + \left(\prod_{i=0}^{n+1} i! \right).$$

To see this, let $H_0 = \text{Rang}(h)$ and let $a_0 \in \text{Tor}_{p_0}[\text{Rang}(h)]$. Now suppose that $n > 0$ and we have defined $\langle H_i : i \leq n \rangle$ and $\langle a_i : i < n \rangle$. We shall now define a_n and H_{n+1} . By our induction assumption,

$$\text{Rang}(h) = \left(\bigoplus_{i < n} a_i\mathbb{Z} \right) \oplus H_n.$$

In particular, H_n is torsion. Using facts 2.5 and 2.7, we can find for some ℓ_n and an element a_n such that a_n has order $p_n^{l_n}$ and $a_n\mathbb{Z}$ is a direct summand of H_n . We may further suppose that $l_n > l_m + \left(\prod_{i=0}^{n+1} i! \right)$. Let H_{n+1} be such that $H_n = a_n\mathbb{Z} \oplus H_{n+1}$.

To prove that the sequence $\langle a_n : n < \omega \rangle$ is as required, assume towards a contradiction that there is $\mathbf{c} \leq \mathbf{d} \in \text{AP}$ such that $\langle c_n : n < \omega \rangle$ is a solution of Γ in

$G_{\mathbf{d}}$. So

$$G_{\mathbf{d}} \models \bigwedge_{n < \omega} (c_n = a_n + n!c_{n+1}) \quad (*)$$

Since for each $n, a_n \in K$, it follows that

$$G_{\mathbf{d}}/K \models \bigwedge_{n < \omega} (c_n + K = n!c_{n+1} + K).$$

By Lemma 3.25, $G_{\mathbf{c}}/K$ is reduced, hence necessarily,

$$\bigwedge_{n < \omega} (c_n + K = 0 + K).$$

In other words, $c_n \in K$ for all $n < \omega$.

We now show that for each n ,

$$\left(\prod_{i < n} i! \right) c_n \in H_n \quad (**)$$

This is true for $n = 0$, because $c_0 \in K = H_0$. Suppose it holds for n . Then multiplying both sides of $(*)$ into $\prod_{i < n} i!$ we get

$$\left(\prod_{i < n} i! \right) c_n = \left(\prod_{i < n} i! \right) a_n + \left(\prod_{i < n+1} i! \right) c_{n+1}.$$

Using the induction hypothesis and $(\star)(b)$ we get

$$\left(\prod_{i < n+1} i! \right) c_{n+1} \in H_{n+1},$$

as requested.

By an easy induction, for each n we have

$$c_0 = a_0 + \sum_{\ell \leq n} \left(\prod_{i=1}^{\ell} i! \right) a_{\ell} + \left(\prod_{i=1}^n i! \right) c_{n+1} \quad (***)_n$$

Indeed this is true for $n = 0$, as $c_0 = a_0 + c_1$. Suppose it holds for n , then using $(*)$ and the induction hypothesis

$$\begin{aligned} c_0 &= a_0 + \sum_{\ell \leq n} \left(\prod_{i=1}^{\ell} i! \right) a_{\ell} + \left(\prod_{i=1}^n i! \right) c_{n+1} \\ &= a_0 + \sum_{\ell \leq n} \left(\prod_{i=1}^{\ell} i! \right) a_{\ell} + \left(\prod_{i=1}^n i! \right) (a_{n+1} + (n+1)!c_{n+2}) \\ &= a_0 + \sum_{\ell \leq n+1} \left(\prod_{i=1}^{\ell} i! \right) a_{\ell} + \left(\prod_{i=1}^{n+1} i! \right) c_{n+2}. \end{aligned}$$

We are now ready to complete the proof. Let $m(*)$ be the order of c_0 . We consider two cases.

Case 1. $p_n = p$ for all n : Let t be an integer such that

$$m(*) = tp^{\ell(*)} > 1$$

where $\ell(*) \geq 0$ and $(p, t) = 1$, i.e., p does not divide t . Let k be the least natural number such that $\mathbf{l}_k > \ell(*)$. By multiplying both sides of $(***)_{k+1}$ into $tp^{\mathbf{l}_k}$ we get

$$tp^{\mathbf{l}_k} c_0 = tp^{\mathbf{l}_k} a_0 + tp^{\mathbf{l}_k} \sum_{\ell \leq k+1} \left(\prod_{i=1}^{\ell} i! \right) a_{\ell} + tp^{\mathbf{l}_k} \left(\prod_{i=1}^{k+1} i! \right) c_{k+2}.$$

As the sequence $\langle \mathbf{l}_{\ell} : \ell \leq k \rangle$ is increasing, $p^{\mathbf{l}_k} a_{\ell} = 0$ for all $\ell \leq k$, so

$$0 = tp^{\mathbf{l}_k} \left(\prod_{i=1}^{k+1} i! \right) a_{k+1} + tp^{\mathbf{l}_k} \left(\prod_{i=1}^{k+1} i! \right) c_{k+2} \quad (\dagger)$$

According to $(+)_b$, we know $a_{k+1} \mathbb{Z} \cap H_{k+2} = 0$, and by using $(**)$ along with (\dagger) we get that

$$tp^{\mathbf{l}_k} \left(\prod_{i=1}^{k+1} i! \right) a_{k+1} = 0.$$

As the order of a_{k+1} is a power of p and $(p, t) = 1$, we get that

$$p^{\mathbf{l}_k} \left(\prod_{i=1}^{k+1} i! \right) a_{k+1} = 0.$$

So,

$$p^{\mathbf{l}_{k+1}} = \text{ord}(a_{k+1}) \leq p^{\mathbf{l}_k} \left(\prod_{i=1}^{k+1} i! \right) \leq p^{\mathbf{l}_k + (\prod_{i=1}^{k+1} i!)}.$$

But, this contradicts the choice of \mathbf{l}_{k+1} (see $(+)_c$). The result follows.

Thereby, without loss of generality we deal with:

Case 2. Otherwise: Then the sequence $\langle p_n : n < \omega \rangle$ is strictly increasing. Let k be the least integer such that

$$p_{k+1} > m(*) \times \left(\prod_{i=1}^{k+1} i! \right) \quad (\dagger\dagger)$$

By multiplying both sides of $(***)_{k+1}$ into $m(*) \times \left(\prod_{i=1}^k p_i^{\mathbf{l}_i} \right)$ we get

$$\begin{aligned} 0 &= m(*) \times \left(\prod_{i=1}^k p_i^{1_i} \right) c_0 \\ &= m(*) \times \left(\prod_{i=1}^k p_i^{1_i} \right) a_0 + m(*) \times \left(\prod_{i=1}^k p_i^{1_i} \right) \sum_{\ell \leq k+1} \left(\prod_{i=1}^{\ell} i! \right) a_{\ell} \\ &\quad + m(*) \times \left(\prod_{i=1}^k p_i^{1_i} \right) \left(\prod_{i=1}^{k+1} i! \right) c_{k+2}. \end{aligned}$$

We have that $m(*) \times \left(\prod_{i=1}^k p_i^{1_i} \right) a_0 = 0$ and

$$m(*) \times \left(\prod_{i=1}^k p_i^{1_i} \right) \left(\prod_{i=1}^{\ell} i! \right) a_{\ell} = 0,$$

for all $\ell \leq k$, thus

$$0 = m(*) \times \left(\prod_{i=1}^k p_i^{1_i} \right) \left(\prod_{i=1}^{k+1} i! \right) a_{k+1} + m(*) \times \left(\prod_{i=1}^k p_i^{1_i} \right) \left(\prod_{i=1}^{k+1} i! \right) c_{k+2}.$$

Again, according to $(+)_b$, we know $a_{k+1} \mathbb{Z} \cap H_{k+2} = 0$, and by using $(**)$ along with the previous formula, we lead to the following vanishing formula

$$m(*) \times \left(\prod_{i=1}^k p_i^{1_i} \right) \left(\prod_{i=1}^{k+1} i! \right) a_{k+1} = 0.$$

As the order of a_{k+1} is a power of p_{k+1} and it is different from all p_{ℓ} 's, for $\ell \leq k$, we have

$$m(*) \times \left(\prod_{i=1}^{k+1} i! \right) a_{k+1} = 0.$$

So,

$$p_{k+1} < p_{k+1}^{1_{k+1}} = \text{ord}(a_{k+1}) \leq m(*) \times \left(\prod_{i=1}^{k+1} i! \right).$$

But this contradicts $(\dagger\dagger)$. The result follows. \square

To prove the endo-rigidity property, we first deal with the following special case, and then we reduce things to this situation:

Lemma 3.30. *Let $\mathbf{c} \in \text{AP}$ be full. Then every $h \in \text{Hom}(G_{\mathbf{c}}, K)$ is bounded.*

Proof. Towards a contradiction assume $h \in \text{Hom}(G_{\mathbf{c}}, K)$ is not bounded. In view of Lemma 3.29, this implies that there is a sequence

$$\langle a_n : n < \omega \rangle \in {}^{\omega} \text{Rang}(h)$$

such that the set of equations

$$\Gamma = \{z_n = a_n + n!z_{n+1} : n < \omega\}$$

has no solutions in $G_{\mathbf{c}}$. Let $\gamma = |K|$, and define $f : \Lambda_{<\omega}[\lambda] \rightarrow \gamma$ such that

$$f(\eta) = f(\nu) \iff h(x_\eta) = h(x_\nu) \quad (*)$$

Since $a_n \in \text{Rang}(h)$ there is b_n such that

$$\forall n < \omega, a_n = h(b_n) \quad (+)$$

As \mathbf{c} is full, we can find some η such that

- (1) $f(\eta \upharpoonright_L n) = f(\eta \upharpoonright_R n)$,
- (2) $a_{\eta, n}^{\mathbf{c}} = b_n$ for each n .

Combining (*) and (1) yields that

$$\forall n < \omega, h(x_{\eta \upharpoonright_L n}) = h(x_{\eta \upharpoonright_R n}) \quad (\dagger).$$

By applying h to the both sides of the equation

$$y_{\eta, n} = a_{\eta, n}^{\mathbf{c}} + (n!)y_{\eta, n+1} + (x_{\eta \upharpoonright_L n} - x_{\eta \upharpoonright_R n}),$$

we get

$$\begin{aligned} h(y_{\eta, n}) &= h(a_{\eta, n}^{\mathbf{c}}) + n!h(y_{\eta, n+1}) + (h(x_{\eta \upharpoonright_L n}) - h(x_{\eta \upharpoonright_R n})) \\ &\stackrel{(2)}{=} h(b_n) + n!h(y_{\eta, n+1}) + (h(x_{\eta \upharpoonright_L n}) - h(x_{\eta \upharpoonright_R n})) \\ &\stackrel{(\dagger)}{=} h(b_n) + (n!)h(y_{\eta, n+1}) \\ &\stackrel{(+)}{=} a_n + (n!)h(y_{\eta, n+1}). \end{aligned}$$

In other words, $h(y_{\eta, n})$ is a solution for

$$\Gamma = \{z_n = a_n + n!z_{n+1} : n < \omega\}.$$

This is a contradiction with the choice of the sequence $\langle a_n : n < \omega \rangle$. □

Notation 3.31. Suppose $\mathbf{c} \in \text{AP}$. For each $n < \omega$, we define

$$G_n := \frac{G_{\mathbf{c}}}{K + \left(\prod_{i=1}^n i!\right)G_{\mathbf{c}}}.$$

Also, the notation π_n stands for the natural projection $G_{\mathbf{c}} \twoheadrightarrow G_n$.

Fact 3.32. *Adopt the above notation, let $n < \omega$ and $g \in G_{\mathbf{c}}$.*

(a) *The abelian group G_n is a torsion abelian group with the following minimal generating set*

$$\{x_\rho : \rho \in \Lambda_{\mathbf{c}, < \omega}\} \cup \{y_{\eta, k} : \eta \in \Lambda_{\mathbf{c}, \omega} \text{ and } k \geq n + 2\}.$$

(b) *Similar to Definition 3.19, we can define $\text{supp}_o(\pi_n(g))$ with respect to generating set presented in clause (a).*

(c) *According to its definition, it is easy to see that $\text{supp}_o(\pi_n(g)) \subseteq \text{supp}_o(g)$.*

(d) *Recall from Lemma 3.25 that $G_{\mathbf{c}}/K$ is reduced. This in turns gives us an integer $m_n > n$ such that $\text{supp}_o(g) \subseteq \text{supp}_o(\pi_{m_n}(g))$.*

Proof. This is straightforward. \square

Lemma 3.33. *Suppose $\mathbf{c} \in \text{AP}$ is full and $h \in \text{End}(G_{\mathbf{c}})$. Then for some countable $\Lambda_h \subseteq \Omega_{\mathbf{c}}$ we have:*

$$r \in R, \nu \in \Omega_{\mathbf{c}} \setminus \Lambda_h \implies \text{supp}_o(h(rx_\nu)) \subseteq \{\nu\} \cup \Lambda_h.$$

Proof. Towards contradiction assume $h \in \text{End}(G_{\mathbf{c}})$ but there is no Λ_h as promised.

We define a sequence

$$\langle (\eta_i, Y_i, \nu_i, r_i) : i < \omega_1 \rangle,$$

by induction on $i < \omega_1$, such that

- (*) (a) $\eta_i \in \Omega_{\mathbf{c}}$ and $r_i \in R \setminus \{0\}$,
- (b) $Y_i = \bigcup \{\text{supp}_o(h(r_j x_{\eta_j})) : j < i\} \cup \{\eta_j : j < i\}$,
- (c) $\nu_i \in \text{supp}_o(h(r_i x_{\eta_i}))$ but $\nu_i \neq \eta_i, \nu_i \notin Y_i$.

To this end, suppose that $i < \omega_1$ and we have defined $\langle (\eta_j, Y_j, \nu_j, r_j) : j < i \rangle$. Set

$$Y_i = \bigcup \{\text{supp}_o(h(r_j x_{\eta_j})) : j < i\} \cup \{\eta_j : j < i\}.$$

Following its definition, we know Y_i is at most countable. Thus, due to our assumption, we can find some $\eta_i \in \Omega_{\mathbf{c}} \setminus Y_i$ and $r_i \in R \setminus \{0\}$ such that

$$\text{supp}_o(h(r_i x_{\eta_i})) \not\subseteq (\{\eta_i\} \cup Y_i).$$

This allow us to define ν_i , namely, it is enough to take ν_i be any element of $\text{supp}_o(h(r_i x_{\eta_i})) \setminus (\{\eta_i\} \cup Y_i)$. This completes the definition of $(\eta_i, Y_i, \nu_i, r_i)$.

As $\nu_i \in \text{supp}_o(h(r_i x_{\eta_i}))$ $\nu_i \notin (Y_i \cup \{\eta_i\})$ and that $\text{supp}_o(h(x_{\eta_i}))$ is finite, there is $W \subseteq \omega_1$ of cardinality ω_1 such that

(*) If $i \neq j \in W$ then $\nu_j \notin \text{supp}_o(h(r_i x_{\eta_i}))$.

Without loss of generality we may and do assume that $W = \omega_1$. Let $a_i = r_i x_{\eta_i}$. We can find $f : \Lambda_{\mathbf{c}, < \omega} \rightarrow |R| + \aleph_0 < \lambda$ such that if $b \in G_{\mathbf{c}}^3$, then from $f(b)$ we can compute

$$\langle n_b, \{(\ell, m_{b,\ell}, r_{b,\ell}) : \ell < n_b\} \rangle.$$

As \mathbf{c} is full, and that $\text{Rang}(f)$ has size less than λ , there is some $\eta \in \Lambda_{\mathbf{c}, \omega}$ such that

- (1) $f(\eta \upharpoonright_L n) = f(\eta \upharpoonright_R n)$, for $n < \omega$,
- (2) $a_{\eta, n}^{\mathbf{c}} = a_n$ for all $n < \omega$.

Now, we show that

$$\nu_i \in \text{supp}_0(h(y_{\eta_0})) \quad \forall i < \omega \quad (\boxplus)$$

This will be a contradiction, as $\text{supp}_0(h(y_{\eta_0}))$ is finite. By Lemma 3.26 we first observe that:

$$y_{\eta, 0} = \sum_{i=0}^n (\prod_{j<i} j!) r_i x_{\eta_i} + (\prod_{i=1}^n i!) y_{\eta, n+1} + \sum_{i=0}^n (\prod_{j<i} j!) (x_{\eta \upharpoonright_L i} - x_{\eta \upharpoonright_R i}).$$

Let ℓ be any integer. We are going to use the notation presented in Notation 3.31 for $n = m_\ell$. Applying $\pi_n h(-)$ to it, yields that

$$\begin{aligned} (3) \quad \pi_n(h(y_{\eta, 0})) &= \sum_{i=0}^n (\prod_{j<i} j!) \pi_n h(r_i x_{\eta_i}) + (\prod_{i=1}^n i!) \pi_n h(y_{\eta, n+1}) \\ &\quad + \sum_{i=0}^n (\prod_{j<i} j!) \pi_n h(x_{\eta \upharpoonright_L i} - x_{\eta \upharpoonright_R i}) \\ &= \sum_{i=0}^n (\prod_{j<i} j!) \pi_n h(r_i x_{\eta_i}) + \sum_{i=0}^n (\prod_{j<i} j!) \pi_n h(x_{\eta \upharpoonright_L i} - x_{\eta \upharpoonright_R i}), \end{aligned}$$

where the last equality follows by Definition 3.31. Now, we recall from the construction (*) that:

³Recall we have chosen

$$b - \sum_{\ell < n_b} r_{b,\ell} y_{\eta_{b,\ell}, m_{b,\ell}} \in \sum_{\rho \in \Lambda_{\mathbf{c}, < \omega}} R x_\rho + K.$$

$$(3.1) \quad \nu_i \in \text{supp}_o(h(r_i x_{\eta_i})),$$

$$(3.2) \quad \nu_i \neq \eta_i \text{ and } \nu_i \notin Y_i.$$

Thanks to Fact 3.32(d) we have

$$(4) \quad \nu_i \in \text{supp}_o(\pi_n h(r_i x_{\eta_i})).$$

By clause (1) above, $\text{supp}_o(h(x_{\eta \upharpoonright_L i} - x_{\eta \upharpoonright_R i})) = \emptyset$. In view of Fact 3.32(c), we deduce that

$$(5) \quad \text{supp}_o(\pi_n(h(x_{\eta \upharpoonright_L i} - x_{\eta \upharpoonright_R i}))) = \emptyset.$$

First, we plug items (4) and (5) in the clause (3), then we use (*). These enable us to observe that

$$\begin{aligned} \nu_i &\in \text{supp}_o \left(\sum_{i=0}^n (\prod_{j<i} j!) \pi_n h(r_i x_{\eta_i}) + \sum_{i=0}^n (\prod_{j<i} j!) \pi_n h(x_{\eta \upharpoonright_L i} - x_{\eta \upharpoonright_R i}) \right) \\ &= \text{supp}_o(\pi_n h(y_{\eta,0})). \end{aligned}$$

Another use of Fact 3.32(c), shows that $\nu_i \in \text{supp}_o(h(y_{\eta,0}))$. So, the proof of \boxplus is now completed, and the lemma follows. \square

The following lemma can be proved easily.

Lemma 3.34. *Let $\mathbf{c} \in \text{AP}$ be full and $h \in \text{End}(G_{\mathbf{c}})$. Let $Y_0 \subseteq \Omega_{\mathbf{c}}$ be the downward closure of Λ_h , where Λ_h is as in Lemma 3.33 and set*

$$K^+ := K + \sum_{\rho \in Y_0 \cap \Lambda_{\mathbf{c}, < \omega}} R x_{\rho} + \sum_{\rho \in Y_0 \cap \Lambda_{\mathbf{c}, \omega}^{n < \omega}} R y_{\rho, n}.$$

If $b \in G_{\mathbf{c}}$, then there are choices

- $\bar{r}_b := \langle r_{b, \rho}^2 : \rho \in \Lambda_b \rangle$, and
- $\Lambda_b \subseteq \Lambda_{\mathbf{c}, < \omega} \setminus Y_0$ finite

such that

$$b - \sum_{\rho \in \Lambda_b} r_{b, \rho}^2 x_{\rho} \in K^+.$$

Proof. This is straightforward. \square

Hypothesis 3.35. For the rest of this section, we fix a well-ordering \prec of the large enough part of the universe, and for each:

- $\mathbf{c} \in \text{AP}$ which is full,
- $h \in \text{End}(G_{\mathbf{c}})$, and
- $b \in G_{\mathbf{c}}$,

we let $\bar{r}_b := \langle r_{b,\rho}^2 : \rho \in \Lambda_b \rangle$ be the \prec -least sequence satisfying the conclusions of Lemma 3.34.

Notation 3.36. Suppose $\mathbf{c} \in \text{AP}$ and $\Lambda \subseteq \Lambda_{\mathbf{c}}$. By $G_{\mathbf{c},\Lambda}$ we mean

$$G_{\mathbf{c},\Lambda} := G_{\Lambda} := \left\langle \{rx_{\nu}, ry_{\eta,n} : r \in R, \nu \in \Lambda_{<\omega}, \eta \in \Lambda_{\omega} \text{ and } n < \omega\} \right\rangle.$$

We have the following lemma, but as we do not use it, we leave its proof.

Lemma 3.37. *Suppose $\Lambda \subseteq \Lambda[\lambda]$ is downward closed. Then $G_{\mathbf{c},\Lambda}$ is a pure subgroup of $G_{\mathbf{c}}$.*

Lemma 3.38. *Let $\mathbf{c} \in \text{AP}$ be full, and $h \in \text{End}(G_{\mathbf{c}})$. Then for some countable $\Lambda_h \subseteq \Lambda[\lambda]$ we have:*

$$r \in R, \nu \in \Omega_{\mathbf{c}} \setminus \Lambda_h \implies h(rx_{\nu}) \in G_{\mathbf{c},\Lambda_h \cup \{\nu\}} + K.$$

Proof. Suppose on the way of contradiction that the lemma fails. Let Y_0 be as Lemma 3.34. We define a sequence

$$\langle (Y_i, \nu_i, \rho_i, r_i) : i < \omega_1 \rangle,$$

by induction on $i < \omega_1$, such that

- (i) (a) $r_i \in R \setminus \{0\}$,
- (b) $Y_i = \bigcup \{\text{supp}(h(r_j x_{\nu_j})) : j < i\} \cup \{\rho_j : j < i\} \cup Y_0$,
- (c) $\nu_i \in \Omega_{\mathbf{c}} \setminus Y_i$,
- (d) $h(r_i \nu_i) \notin G_{\mathbf{c},Y_i \cup \{\nu_i\}} + K$,
- (e) let $b_i := h(r_i \nu_i)$, and let $\bar{r}_{b_i} := \langle r_{b_i,\rho}^2 : \rho \in \Lambda_i \rangle$ be as Lemma 3.34 applied to b_i . Then $\rho_i \in \Lambda_i \setminus (Y_i \cup \{\nu_i\})$, and even

$$r_{b_i,\rho_i}^2 x_{\rho_i} \notin G_{\mathbf{c},Y_i \cup \{\nu_i\}} + K.$$

To construct this, suppose $i < \omega$ and we have constructed the sequence up to i . Now, $(\natural)_b$ gives the definition of Y_i . Since we assume that the lemma fails, there is an $r_i \in R$ and $\nu_i \in \Omega_{\mathbf{c}} \setminus Y_i$ such that $h(r_i x_{\nu_i}) \notin G_{\mathbf{c}, \Lambda_h \cup \{\nu\}} + K$. Now, we define $b_i := h(r_i \nu_i)$. Thanks to Lemma 3.34, there is a finite set $\Lambda_i \subseteq \Lambda_{\mathbf{c}, < \omega} \setminus Y_i$ and a sequence $\langle r_{b_i, \rho}^2 : \rho \in \Lambda_i \rangle$ such that

$$b_i - \sum_{\rho \in \Lambda_i} r_{b_i, \rho}^2 x_{\rho} \in K^+.$$

As $b_i \notin G_{\mathbf{c}, Y_i \cup \{\nu_i\}} + K$ and due to the following containment

$$b_i - \sum_{\rho \in \Lambda_i} r_{b_i, \rho}^2 x_{\rho} \in K^+ \subseteq G_{\mathbf{c}, Y_i \cup \{\nu_i\}} + K,$$

there is $\rho_i \in \Lambda_i$ such that $\rho_i \notin (Y_i \cup \{\nu_i\})$, and indeed $r_{b_i, \rho_i}^2 x_{\rho_i} \notin G_{\mathbf{c}, Y_i \cup \{\nu_i\}} + K$.

This completes the proof of construction. By shrinking the sequence, we may assume that

- for all $i \neq j < \omega_1$, $\rho_j \notin \Lambda_i$.

Let $a_n := r_n x_{\nu_n}$ and define

$$f : \Lambda_{\mathbf{c}, < \omega} \rightarrow |R| + |K| + \aleph_0 < \lambda$$

be such that for any $\rho \in \Lambda_{\mathbf{c}, < \omega}$, $f(\rho)$ codes

- $\langle r_{b, \rho}^2 : \rho \in \Lambda_b \rangle$, and
- $b - \sum_{\nu \in \Lambda_i} r_{b, \nu}^2 x_{\nu}$,

where $b := h(x_{\rho})$. To see such a function f exists, first we define:

- $f_1 : R^{< \omega} \times K^+ \rightarrow |R| + |K| + \aleph_0$ is a bijection, and
- $f_2 : \Lambda_{\mathbf{c}, < \omega} \rightarrow R^{< \omega} \times K^+$ is defined as

$$f_2(b) = (\langle r_{b, \rho}^2 : \rho \in \Lambda_b \rangle, b - \sum_{\nu \in \Lambda_i} r_{b, \nu}^2 x_{\nu}).$$

Then, we set $f := f_1 \circ f_2$. Suppose $\rho_1, \rho_2 \in \Lambda_{\mathbf{c}, < \omega}$ are such that $f(\rho_1) = f(\rho_2)$. We claim that $h(x_{\rho_1}) = h(x_{\rho_2})$. To see this, it is enough to apply $f(\rho_1) = f(\rho_2)$, and conclude that

- 1) $\langle r_{b_1, \nu}^2 : \nu \in \Lambda_{b_1} \rangle = \langle r_{b_2, \nu}^2 : \nu \in \Lambda_{b_2} \rangle$
- 2) $b_1 - \sum_{\nu \in \Lambda_{b_1}} r_{b_1, \nu}^2 x_{\nu} = b_2 - \sum_{\nu \in \Lambda_{b_2}} r_{b_2, \nu}^2 x_{\nu}$,

where $b_i = h(x_{\rho_i})$. But, then we have

$$\begin{aligned} b_1 &= b_1 - \sum_{\nu \in \Lambda_{b_1}} r_{b,\nu}^2 x_\nu + (\sum_{\nu \in \Lambda_{b_1}} r_{b,\nu}^2 x_\nu) \\ &\stackrel{(2)}{=} b_2 - \sum_{\nu \in \Lambda_{b_2}} r_{b,\nu}^2 x_\nu + (\sum_{\nu \in \Lambda_{b_2}} r_{b,\nu}^2 x_\nu) \\ &= b_2, \end{aligned}$$

i.e., $h(x_{\rho_1}) = h(x_{\rho_2})$.

Since \mathbf{c} is full, there is an $\eta \in \Lambda_{\mathbf{c},\omega}$ such that for all $n < \omega$

$$(3) \ a_n = a_{\eta,n}^{\mathbf{c}}, \text{ and}$$

$$(4) \ f(\eta \upharpoonright_L n) = f(\eta \upharpoonright_R n).$$

Thanks to the previous paragraph and clause (4) we deduce

$$h(x_{\eta \upharpoonright_L n}) = h(x_{\eta \upharpoonright_R n}) \quad (\#)$$

By applying h to the both sides of the following equation

$$y_{\eta,0} = \sum_{i=0}^n (\prod_{j<i} j!) r_i x_{\nu_i} + (\prod_{i=1}^n i!) y_{\eta,n+1} + \sum_{i=0}^n (\prod_{j<i} j!) (x_{\eta \upharpoonright_L i} - x_{\eta \upharpoonright_R i}),$$

we get

$$\begin{aligned} h(y_{\eta,0}) &= \sum_{i=0}^n (\prod_{j<i} j!) h(r_i x_{\nu_i}) + (\prod_{i=1}^n i!) h(y_{\eta,n+1}) \\ &\quad + (\prod_{j<i} j!) (h(x_{\eta \upharpoonright_L i}) - h(x_{\eta \upharpoonright_R i})) \\ &\stackrel{(\#)}{=} \sum_{i=0}^n (\prod_{j<i} j!) h(r_i x_{\nu_i}) + (\prod_{i=1}^n i!) h(y_{\eta,n+1}) \quad (+) \end{aligned}$$

For each $i < \omega_1$ let $b_i = h(r_i x_{\nu_i})$. Let also $b = h(y_{\eta,0})$ and let Λ_b be as in Lemma 3.34. As Λ_b is finite, for some large enough n , we have

$$\{\rho_i : i < n\} \setminus \Lambda_b \neq \emptyset.$$

Let $i < n$ be such that $\rho_i \notin \Lambda_b$. Here, we apply the argument presented in items (3)-(5) from Lemma 3.33 to the displayed formula (+). So, on the one hand, it turns out that

$$\rho_i \in \Lambda_i \subseteq \Lambda_b.$$

On the other hand by the choice of i , $\rho_i \notin \Lambda_b$. This is a contraction that we searched for it. \square

Lemma 3.39. *Let $\mathbf{c} \in \text{AP}$ be full, and $h \in \text{End}(G_{\mathbf{c}})$. Then for some $m_* \in R$ and some countable $\Lambda_h = \text{cl}(\Lambda_h) \subseteq \Lambda[\lambda]$ we have:*

$$r \in R, \nu \in \Omega_{\mathbf{c}} \setminus \Lambda_h \implies h(rx_\nu) - m_*x_\nu \in G_{\Lambda_h} + K.$$

Proof. In view of Lemma 3.38, there is some countable downward closed subset Λ of $\Lambda_{\mathbf{c}}$ such that for every $r \in R$ and $\eta \in \Omega_{\mathbf{c}} \setminus \Lambda$, we have $h(rx_\eta) \in G_{\Lambda \cup \{\nu\}} + K$. Thus, for such r and η , there are $m_\eta^r \in R$ and b_η^r satisfying the following two bold properties:

- $h(rx_\eta) = m_\eta^r x_\eta + b_\eta^r$,
- $b_\eta^r \in G_\Lambda + K$.

Suppose on the way of contradiction that the desired conclusion fails. By induction on $i < \omega_1$ we define a sequence

$$\langle Y_i, r_{i,1}, r_{i,2}, \eta_{i,1}, \eta_{i,2} : i < \omega_1 \rangle$$

such that:

- (†) (a) $Y_i = \Lambda \cup \{\eta_{j,\ell} : j < i, \ell \in \{1, 2\}\}$,
- (b) $r_{i,1}, r_{i,2} \in R \setminus \{0\}$,
- (c) $\eta_{i,\ell} \in \Omega_{\mathbf{c}} \setminus Y_i$, for $\ell \in \{1, 2\}$,
- (d) $m_{\eta_{i,1}}^{r_{i,1}} \neq m_{\eta_{i,2}}^{r_{i,2}}$.

The construction is easy, but we elaborate. Let us start with the case $i = 0$. We set $Y_0 = \Lambda$ and then choose $r_{0,1}, r_{0,2} \in R \setminus \{0\}$ and $\eta_{0,1}, \eta_{0,2} \in \Lambda_{<\omega}[\lambda] \setminus \Lambda_h$ such that $m_{\eta_{0,1}}^{r_{0,1}} \neq m_{\eta_{0,2}}^{r_{0,2}}$. Now suppose $i < \omega_1$ and we have define the sequence for all $j < i$. Define Y_i as in clause (†)(a). By our assumption, we can find $r_{i,1}, r_{i,2} \in R \setminus \{0\}$ and $\eta_{i,1}, \eta_{i,2} \in \Omega_{\mathbf{c}} \setminus Y_i$ such that $m_{\eta_{i,1}}^{r_{i,1}} \neq m_{\eta_{i,2}}^{r_{i,2}}$. This completes the induction construction.

Let

$$f : \Lambda_{\mathbf{c}, <\omega} \rightarrow |R| + |K| + \aleph_0 < \lambda$$

be such that if $r \in R$ and $\eta \in \Omega_{\mathbf{c}}$, then $f(rx_\eta)$ is defined in a way that one can compute m_η^r and b_η^r . Again we can define f as

$$f = f_1 \circ f_2 \circ f_3,$$

where:

- $f_1 : R \times (G_\Lambda + K) \rightarrow |R| + |K| + \aleph_0$ is a bijection,
- $f_2 : R \times \Lambda_{\mathbf{c}, < \omega} \rightarrow R \times (G_\Lambda + K)$ is defined as $f_2(r, \eta) = (m_\eta^r, b_\eta^r)$,
- $f_3 : \Lambda_{\mathbf{c}, < \omega} \rightarrow R \times \Lambda_{\mathbf{c}, < \omega}$ is a bijection.

For each $n < \omega$, we set $a_n = r_{n,1}x_{\eta_{n,1}} - r_{n,2}x_{\eta_{n,2}}$. Applying h to it yields:

$$h(a_n) = m_{\eta_{n,1}}^{r_{n,1}}x_{\eta_{n,1}} - m_{\eta_{n,2}}^{r_{n,2}}x_{\eta_{n,2}} + b_n \quad (+),$$

where $b_n := b_{\eta_{n,1}}^{r_{n,1}} - b_{\eta_{n,2}}^{r_{n,2}}$. Since \mathbf{c} is full, there is an $\eta \in \Lambda_{\mathbf{c}, \omega}$ such that

- 1) $a_n = a_{\eta, n}^{\mathbf{c}}$, and
- 2) $f(\eta \upharpoonright_L n) = f(\eta \upharpoonright_R n)$

for all $n < \omega$. By clause (2)

- 3) $\text{supp}_o(h(x_{\eta \upharpoonright_L n} - x_{\eta \upharpoonright_R n})) = \emptyset$ for all $n < \omega$.

Applying h to

$$y_{\eta, 0} = \sum_{i=0}^n a_i + \left(\prod_{i=1}^n i! \right) y_{\eta, n+1} + \sum_{i=0}^n \left(\prod_{j < i} j! \right) (x_{\eta \upharpoonright_L i} - x_{\eta \upharpoonright_R i}),$$

yields that

$$\begin{aligned} (\natural) \quad h(y_{\eta, 0}) &= \sum_{i=0}^n h(a_i) + \left(\prod_{i=1}^n i! \right) h(y_{\eta, n+1}) + \left(\prod_{j < i} j! \right) (h(x_{\eta \upharpoonright_L i}) - h(x_{\eta \upharpoonright_R i})) \\ &\stackrel{(3)}{=} \sum_{i=0}^n h(a_i) + \left(\prod_{i=1}^n i! \right) h(y_{\eta, n+1}) \\ &\stackrel{(+)}{=} \sum_{i=0}^n (m_{\eta_{n,1}}^{r_{n,1}}x_{\eta_{n,1}} - m_{\eta_{n,2}}^{r_{n,2}}x_{\eta_{n,2}} + b_n) + \left(\prod_{i=1}^n i! \right) h(y_{\eta, n+1}). \end{aligned}$$

Let $n < \omega$ be large enough. Here, we are going to apply the arguments taken from (3)-(5) in Lemma 3.33 to the displayed formula (\natural). Then, it turns out that

- 4) $\text{supp}_o(h(y_{\eta, 0})) \supseteq \text{supp}_o(h(a_n))$, and
- 5) $\text{supp}_o(h(a_n)) \cap \{\eta_{n,1}, \eta_{n,2}\} \neq \emptyset$.

Without loss of generality, let us assume that for each $n < \omega$, $\eta_{n,1} \in \text{supp}_o(h(a_n))$.

So,

$$\{\eta_{n,1} : n < \omega\} \subseteq \text{supp}_o(h(y_{\eta, 0})),$$

which is infinite. This is a contraction. \square

Lemma 3.40. *Assume $\Lambda = \text{cl}(\Lambda) \subseteq \Lambda_{\mathbf{c}}$ is countable and $h \in \text{Hom}(G_{\mathbf{c}}, G_{\Lambda} + K)$. Then h is bounded.*

Proof. Towards contradiction assume that h is unbounded. It follows from Lemma 3.30 that $\text{Rang}(h) \not\subseteq K$. Let $b_* \in \text{Rang}(h) \setminus K$. Then, for some $d_* \in K$, a finite set Λ_* and two sequences $\langle r_{\eta} \in R \setminus \{0\} : \eta \in \Lambda_* \rangle$ and $\langle m_{\eta} \in \omega : \eta \in \Lambda_* \rangle$, we can represent b_* as

$$b_* = \sum \{r_{\eta}x_{\eta} : \eta \in \Lambda_* \cap \Lambda_{<\omega}\} + \sum \{r_{\eta}y_{\eta, m(\eta)} : \eta \in \Lambda_* \cap \Lambda_{\omega}\} + d_*.$$

Let

- (1) $J_0 = G_{\Lambda} + K$,
- (2) $J_1 = J_0/K$, which is torsion free.

So, $b_* \in J_0$. Let $\pi : J_0 \rightarrow J_1$ be the natural map $\pi(b) = b + K$. Since $b_* \in \text{Rang}(h) \setminus K$, we have $\pi(b_*) \neq 0$. Suppose on the way of contradiction that for any sequence $\langle e_n : n < \omega \rangle \in {}^{\omega}\mathbb{Z}$ the following equations

$$\Gamma = \{y_n = n!y_{n+1} + e_n b_* : n < \omega\}$$

is solvable in J_1 . Say for example, $\{y_n\}$. Recall $M_{\mathbf{c}}$ is \aleph_1 -free as R -module. Since J_1 is countably generated, we can find a solution to

$$\Gamma = \{y_n = n!y_{n+1} + e_n \bar{b}_* : n < \omega\}$$

in R . Since R is cotorsion-free, a such system of equations has no solution the ring. So, there is a sequence $\langle e_n : n < \omega \rangle \in {}^{\omega}\mathbb{Z}$ the following equations

$$\Gamma = \{y_n = n!y_{n+1} + e_n b_* : n < \omega\}$$

is not solvable in J_1 .

Let $a_* \in G_{\mathbf{c}}$ be such that $b_* = h(a_*)$. Let also $f : \Lambda_{\mathbf{c}, <\omega} \rightarrow \omega$ be such that for all $\nu, \rho \in \Lambda_{\mathbf{c}, <\omega}$,

$$f(\nu) = f(\rho) \Leftrightarrow \pi \circ h(x_{\nu}) = \pi \circ h(x_{\rho}).$$

As \mathbf{c} is full, there is some $\eta \in \Lambda_{\mathbf{c}, \omega}$ such that:

- (3) $a_{\eta, n}^{\mathbf{c}} = e_n a_*$, for all $n < \omega$, and
- (4) $f(\eta \upharpoonright_L n) = f(\eta \upharpoonright_R n)$, for $n < \omega$.

Thanks to (4), one has

$$\forall n < \omega, \pi \circ h(x_{\eta \upharpoonright L n}) = \pi \circ h(x_{\eta \upharpoonright R n}) \quad (+)$$

By applying $\pi \circ h$ into the equation

$$y_{\eta, n} = a_{\eta, n}^{\mathbf{c}} + n!y_{\eta, n+1} + (x_{\eta \upharpoonright L n} - x_{\eta \upharpoonright R n}),$$

and using clause (3) and (+) we get

$$\pi \circ h(y_{\eta, n}) = e_n \pi(b_*) + n! \pi \circ h(y_{\eta, n+1}).$$

This clearly gives a contradiction, as then

$$J_1 \models \left(y_n = n!y_{n+1} + e_n b_* \right),$$

where $y_n = \pi \circ h(y_{\eta, n})$. □

Lemma 3.41. *Let \mathbf{c} be full and $h \in \text{End}(G_{\mathbf{c}})$. Then $\text{Rang}(h)$ is bounded.*

Proof. Suppose not, it follows that for some countable $\Lambda = \text{cl}(\Lambda) \subseteq \Lambda_{\mathbf{c}}$,

$$h \upharpoonright G \in \text{Hom}(G, G_{\Lambda} + K)$$

is unbounded, where G is the subgroup of $G_{\mathbf{c}}$ generated by $h^{-1}[G_{\Lambda} + K]$. This contradicts Lemma 3.40. □

Now, we are ready to prove:

Theorem 3.42. *There is some \mathbf{c} such that the abelian group $G_{\mathbf{c}}$ is boundedly rigid.*

In particular, there is an abelian group G equipped with the following properties

- (1) $\text{Tor}(G) = K$,
- (2) G is of size λ ,
- (3) the sequence

$$0 \longrightarrow R_{\mathbf{c}} \longrightarrow \text{End}(G) \longrightarrow \frac{\text{End}(G)}{E_b(G)} \longrightarrow 0$$

is exact.

Proof. According to Lemma 3.28, there is a full $\mathbf{c} \in \text{AP}$. This allow us to apply Lemma 3.41 to deduce $G := G_{\mathbf{c}}$ is boundedly rigid. By definition, this completes the proof. \square

§ 4. CO-HOPFIAN AND BOUNDEDLY ENDO-RIGID ABELIAN GROUPS

As stated in [15], it is difficult to construct an infinite Hopfian–co-Hopfian p -group. What about mixed groups? In this section, we answer the question. We start by recalling the definition of (co)-Hopfian groups.

Definition 4.1. Suppose G is a group.

- (i) G is called *Hopfian* if its surjective endomorphisms are automorphisms.
- (ii) G is called *co-Hopfian* if its injective endomorphisms are automorphisms.

Here, we introduce a useful criteria:

Definition 4.2. Let G be an abelian group of size λ and $m, n \geq 1$ be such that $m \mid n$. Then:

- (1) $\text{NQr}_{(m,n)}(G)$ means that there is an (m, n) -anti-witness h , which means:
 - (a) $h \in \text{End}(\text{Tor}_n(G))$,
 - (b) $\text{Rang}(h)$ is a bounded group,
 - (c) $h' := m \cdot \text{id}_{\text{Tor}_n(G)} + h \in \text{End}(\text{Tor}_n(G))$ is 1-to-1,
 - (d) h' is not onto or $m > 1$ and $G/\text{Tor}_n(G)$ is not m -divisible.
- (2) $\text{NQr}_m(G)$ means $\text{NQr}_{(m,n)}(G)$ for some $n \geq 1$.
- (3) $\text{NQr}(G)$ means $\text{NQr}_m(G)$ for some $m \geq 1$.
- (4) $\text{Qr}(G)$ means the negation of $\text{NQr}(G)$.
- (5) $\text{Qr}_*(G)$ means $\text{Qr}(G)$ and in addition:
 - (e) $\text{Tor}_p(G)$ is unbounded, for at least one $p \in \mathbb{P}$.

In items 4.3–4.7 we check $\text{NQr}_{(m,n)}(G)$ and its negation. This enables us to present some new classes of co-Hopfian and non co-Hopfian groups.

Lemma 4.3. *Let G be an abelian group such that the property $\text{NQr}(G)$ holds. Then G is not co-Hopfian. Furthermore, let $h \in \text{Hom}(G, \text{Tor}_n(G))$ be such that*

$h \upharpoonright \text{Tor}_n(G)$ is an (m, n) -anti-witness. Then $m \cdot \text{id}_G + h$ witnesses that G is not co-Hopfian.

Proof. Suppose that G admits an (m, n) -anti-witness $h_0 \in \text{End}(\text{Tor}_n(G))$ as in Definition 4.2. As h_0 is bounded, by Fact 2.14 we can extend h_0 to $h_1 \in \text{Hom}(G, \text{Tor}_n(G))$. So, the following diagram commutes:

$$\begin{array}{ccc} 0 & \longrightarrow & \text{Tor}_n(G) & \xrightarrow{\subseteq^*} & G \\ & & \downarrow h_0 & \swarrow \exists h_1 & \\ & & \text{Tor}_n(G) & & \end{array}$$

We claim that $f = m \cdot \text{id}_G + h_1 \in \text{End}(G)$ is 1-to-1 but not onto.

(*)₁ f is one-to-one.

To see this, suppose $x \in G$ in non-zero and we show that $f(x) \neq 0$. If $x \in \text{Tor}_n(G) \setminus \{0\}$, then by clause (c) of Definition 4.2(1) we have

$$f(x) = mx + h_1(x) = m \cdot \text{id}_{\text{Tor}_n(G)}(x) + h_0(x) \Rightarrow f(x) \neq 0.$$

Now, suppose that $x \in G \setminus \text{Tor}_n(G)$. As $m \mid n$, we have $mx \in G \setminus \text{Tor}_n(G)$. If $f(x) = 0$, we have $mx + h_1(x) = 0$, thus

$$h_1(x) = -mx \in G \setminus \text{Tor}_n(G).$$

But, $\text{Rang}(h_1) \subseteq \text{Tor}_n(G)$, which is impossible. Thus f is 1-to-1, as wanted.

(*)₂ f is not onto.

For this, we consider two cases:

Case 1) h_0 is not onto:

By the case assumption, there is

$$y \in \text{Tor}_n(G) \setminus \text{Rang}(\text{id}_{\text{Tor}_n(G)} + (h_0 \upharpoonright \text{Tor}_n(G)))$$

and it is easy to see that such a y is also a witness for f to be not onto.

Case 2) h_0 is onto:

By Definition 4.2(1)(d), we must have $m > 1$ and $G/\text{Tor}_n(G)$ is not m -divisible.

Let $z \in G$ be such that $z + \text{Tor}_n(G)$ is not divisible by m in $G/\text{Tor}_m(G)$. Clearly, z does not belong to $\text{Rang}(f)$.

The lemma follows. \square

Lemma 4.4. *Let K be an abelian p -group of size λ . The following claims are valid:*

- (i) *If $\text{NQr}(K)$ holds, then K is infinite.*
- (ii) *If K is unbounded, then K is not co-Hopfian.*

Proof. (i) By definition, there are m and n such that $m \mid n$ and that $\text{NQr}_{(m,n)}(K)$ holds. Thanks to Definition 4.2(1), there is $h \in \text{End}(\text{Tor}_n(G))$ satisfying the following properties:

- (a) $\text{Rang}(h)$ is a bounded group,
- (b) $h' := m \cdot (\text{id}_{\text{Tor}_n(K)}) + h \in \text{End}(\text{Tor}_n(K))$ is 1-to-1,
- (c) h' is not onto or $m > 1$ and $K/\text{Tor}_n(K)$ is not m -divisible.

We have two possibilities: 1) $p \nmid n$ and 2) $p \mid n$.

- (1) Suppose first that $p \nmid n$. As K is a p -group, $\text{Tor}_n(K) = \{0\}$. This means that h is constantly zero and is onto, as well as h' . Thanks to clause (c) it follows that $m > 1$ and K is not m -divisible. Since $m \mid n$ we deduce that $p \nmid m$. Now, we consider the map $m \cdot \text{id}_K : K \rightarrow K$. Since K is not m -divisible, this map is not surjective. Let us show that it is 1-to-1. To this end, let $x \in K$ be such that $mx = 0$. Let ℓ be the order of x so that $p^\ell x = 0$. As $(p^\ell, m) = 1$, we can find r, s such that $rp^\ell + sm = 1$. By multiplying both sides with x , we obtain

$$x = rp^\ell x + smx = 0 + 0 = 0.$$

It follows that $m \cdot \text{id}_K : K \rightarrow K$ is 1-to-1 and not onto, hence K is infinite.

- (2) Suppose $p \mid n$. As K is a p -group, this implies that $\text{Tor}_n(K) = K$. Therefore, in the above item (c), the case “ $K/\text{Tor}_n(K)$ is not m -divisible” does not occur. This in turn implies that h' is not onto K . We proved that the map $h' \in \text{End}(K)$ is 1-to-1 and not onto. Hence K is infinite.

The argument of clause (i) is now complete.

(ii) Suppose K is unbounded. We want to show that K is not co-Hopfian. Let K_2 be the maximal divisible subgroup of K . Since it is injective, we know K_2 is a directed summand. Let us write K as $K = K_1 \oplus K_2$. Due to the maximality of K_2 one may know that K_1 is reduced. We show that K_1 is not co-Hopfian, and hence by [22, Claim 2.15(1)], K is not co-Hopfian. Thus by replacing K by K_1 if necessary, we may assume without loss of generality that K is reduced. For $\ell < \omega$, we choose by induction H_ℓ, y_ℓ and z_ℓ such that:

- (a) $H_0 = K$,
- (b) if $\ell = k + 1$, then $H_k = H_\ell \oplus \mathbb{Z}z_\ell$,
- (c) $z_\ell \in (\mathbb{Z}y_\ell)_*$ recall that $(\mathbb{Z}y_\ell)_*$ denotes the pure closure of $\mathbb{Z}y_\ell$,
- (d) $y_{\ell+1} \in H_\ell$,
- (e) The order of z_i is $\geq p^\ell$.

[Why? For $\ell = 0$, we set $H_0 = K$ and let $y_0 \in K$ be arbitrary. Then $(\mathbb{Z}y_0)_*$ is a pure subgroup of K of bounded exponent. Thanks to Fact 2.5 we know $(\mathbb{Z}y_0)_*$ is a direct summand of K . In view of Fact 2.7 we can find z_0 such that $\mathbb{Z}z_0$ is a direct summand of $(\mathbb{Z}y_0)_*$. In other words, $\mathbb{Z}z_0$ is a direct summand of $H_0 = K$ as well. Consequently, we have $H_0 = H_1 \oplus \mathbb{Z}z_0$ for some H_1 . Having defined inductively $\{H_\ell, y_\ell, z_\ell\}$, let $y_{\ell+1} \in H_\ell$. Let χ be a regular cardinal, large enough, so that $H_\ell \in \mathcal{H}(\chi)$. The notation \mathcal{B} stands for $(\mathcal{H}(\chi), \in)$. Let \mathcal{B}_ℓ be countable such that $H_\ell \in \mathcal{B}_\ell$. Also, let $\mathcal{L}_\ell := \mathcal{B}_\ell \cap H_\ell$. So, easily \mathcal{L}_ℓ is an unbounded countable abelian p -group. Hence it is of the form $\bigoplus_i \mathbb{Z}z_{\ell,i}$ where $z_{\ell,i}$ is of order $p^{m(\ell,i)}$. As \mathcal{L}_ℓ is unbounded, we may and do assume that $m(\ell,i) > \ell$. This implies that $\mathbb{Z}z_{\ell,i}$ is a pure subgroup of \mathcal{L}_ℓ , and hence H_ℓ . Consequently, $\mathbb{Z}z_{\ell,i}$ is a direct summand of H_ℓ as well. By definition, we have $H_\ell = H_{\ell+1} \oplus \mathbb{Z}z_{\ell+1}$ for some abelian subgroup $H_{\ell+1}$ of H_ℓ .]

For each $i < \omega$, we let $\ell(i) > 1$ be such that z_i is of order $p^{\ell(i)}$. Following clause (e), e.g. K is unbounded, clearly we can find some infinite $u \subseteq \omega$ such that the sequence $\langle \ell(i) : i \in u \rangle$ is increasing. For any $j < \omega$, we clearly have

$$\bigoplus_{i \in u \cap j} \mathbb{Z}z_i \subseteq_* K,$$

and hence

$$\bigoplus_{i \in u} \mathbb{Z}z_i \subseteq_* K.$$

In the light of [16, Theorem 7], $\bigoplus_{i \in u} \mathbb{Z}z_i$ is a direct summand of K , thus there is some K_1 such that

$$K = \bigoplus_{i \in u} \mathbb{Z}z_i \oplus K_1.$$

Let $\langle j(k) : k < \omega \rangle$ be lists u in an increasing order, and define $h \in \text{End}(K)$ be such that

- $h \upharpoonright K_1 = \text{id}_{K_1}$,
- $h(z_{j(k)}) = p^{\ell(k+1)-1} z_{j(\ell+1)}$.

It is easy to check that h is a well-defined endomorphism of K and it satisfies the following properties:

- h is injective,
- h is not surjective.

In sum, h witnesses that K is not co-Hopfian. □

Lemma 4.5. *Let G be an abelian group of size λ and $m \geq 1$. Suppose there is a bounded $h \in \text{End}(G)$ such that $f := m \cdot \text{id}_G + h \in \text{End}(G)$ is 1-to-1 not onto ⁴. Then for some $n \geq 1$ we have:*

- (i) $\text{NQR}_{(m,n)}(G)$,
- (ii) Letting $h_0 = h \upharpoonright \text{Tor}_n(G)$, h_0 is an (m,n) -anti-witness for $\text{Tor}_n(G)$.

Proof. Let f and h be as above. As $\text{Rang}(h)$ is bounded, for some $n \geq 1$ we have $\text{Rang}(h) \leq \text{Tor}_n(G)$ and without loss of generality $m \mid n$ (possibly replacing n with nm , which is possible as $n_1 \mid n_2$ implies that $\text{Tor}_{n_1}(G) \leq \text{Tor}_{n_2}(G)$). Notice now that:

- (*)₁ (a) f maps $\text{Tor}_n(G)$ into itself.
- (b) if $x \in G \setminus \text{Tor}_n(G)$, then $f(x) \notin \text{Tor}_n(G)$.

Clause (a) clearly holds as by the choice of n we have $\text{Rang}(h) \leq \text{Tor}_n(G)$. To see clause (b) holds, suppose by contradiction that $f(x) = mx + h(x) \in \text{Tor}_n(G)$.

⁴Thus f witnesses non co-Hopfianity of G .

It follows that $mx = f(x) - h(x) \in \text{Tor}_n(G)$, and hence as $m \mid n$, $x \in \text{Tor}_n(G)$, a contradiction.

Let now $h_0 = h \upharpoonright \text{Tor}_n(G)$. Then we have:

- (*)₂ (a) $h_0 \in \text{End}(\text{Tor}_n(G))$,
 (b) h_0 is bounded,
 (c) Since f is 1-to-1, so is $f_0 = m \cdot \text{id}_{\text{Tor}_n(G)} + h_0 \in \text{End}(\text{Tor}_n(G))$.

We are left to show that h_0 is an (m, n) -anti-witness. By (*)₂ it suffices show that f_0 is not onto or $G/\text{Tor}_n(G)$ is not m -divisible. Suppose on the contrary that f_0 is onto and $G/\text{Tor}_n(G)$ is m -divisible. We are going to show that f is onto, which contradicts our assumption. To this end, let $x \in G$. Then as $G/\text{Tor}_n(G)$ is m -divisible, we can find some $y \in G$ such that

$$x - my \in \text{Tor}_n(G).$$

We look at

$$w := x - my - h_0(y) \in \text{Tor}_n(G).$$

As f_0 is onto, we can find some $z \in \text{Tor}_n(G)$ such that $f_0(z) = w$. So,

$$x - my - h_0(y) = w = f_0(z) = mz + h_0(z).$$

Thus

$$x = m(y + z) + h_0(y + z) = f(y + z).$$

In other words, f is onto. This is a contradiction. □

Lemma 4.6. *Let G be a reduced abelian group of size λ such that*

- (1) $\lambda > 2^{\aleph_0}$,
- (2) G is co-Hopfian.

Then the following claims hold:

- i) $\text{Qr}_*(G)$,
- ii) G has no infinite bounded pure subgroup.

Proof. (i). Thanks to Lemma 4.3, $\text{Qr}(G)$ is satisfied, so it is enough to show that for some prime p , $\text{Tor}_p(G)$ is not bounded. Towards contradiction suppose that for every prime $p \in \mathbb{P}$ we have that $\text{Tor}_p(G)$ is bounded.

Here, we are going to show the pure subgroup $\text{Tor}_p(G)$ is finite. Suppose on the way of contradiction that $\text{Tor}_p(G)$ is infinite. Recall that p -torsion subgroups are pure. According to Fact 2.4 $\text{Tor}_p(G)$ is a direct summand of G . Also, following Fact 2.7 we know that $\text{Tor}_p(G)$ is a direct summand of cyclic groups. In sum, $\text{Tor}_p(G)$ has a direct summand K which is a countably infinite p -group. In view of Fact 2.6(i), we observe that K is not co-Hopfian. Recall that any direct summand of co-Hopfian, is co-Hopfian. This means that G is not co-Hopfian as well, which contradicts our assumption. Thus, it follows that for every $p \in \mathbb{P}$, $\text{Tor}_p(G)$ is finite and therefore a direct summand of G , hence there is a projection h_p from G onto $\text{Tor}_p(G)$.

Let $p \in \mathbb{P}$. Then $h_p \upharpoonright \text{Tor}_p(G) \in \text{End}(\text{Tor}_p(G))$ is essentially equal to the identity map, so is one-to-one, and hence onto, as $\text{Tor}_p(G)$ is fine. Since $\text{Qr}(G)$ is satisfied, it follows from Definition 4.2(1)(d) that $G/\text{Tor}_p(G)$ is p -divisible.

Now, we take χ be a regular cardinal, large enough, such that $G \in \mathcal{H}(\chi)$ and let

$$M \prec_{\mathbb{L}_{\aleph_1, \aleph_1}} (\mathcal{H}(\chi), \in)$$

be such that:

- M has cardinality 2^{\aleph_0} ,
- $G, \text{Tor}(G) \in M$,
- $2^{\aleph_0} + 1 \subseteq M$.

In the light of Fact 2.6(ii), we may and do assume that $|\text{Tor}(G)| = \mu \leq 2^{\aleph_0}$. Recall that $2^{\aleph_0} + 1 \subseteq M$ and $\text{Tor}(G) \in M$. These imply that $\text{Tor}(G) \subseteq M$. Now, as $G/\text{Tor}_p(G)$ is p -divisible, then so is

$$\frac{G/\text{Tor}_p(G)}{(G \cap M)/\text{Tor}_p(G)},$$

which by the Third Isomorphism Theorem, is canonically isomorphic to $G/G \cap M$. As $\text{Tor}(G) \subseteq M$, $G/(G \cap M)$ is torsion-free, it is divisible. Let $x \in G \setminus M$ and define the sequence $(x_n : n < \omega)$ such that:

- $x_0 = x$,
- If $n = m + 1$ then

$$G/(G \cap M) \models \left(n!x_n + (G \cap M) = x_m + (G \cap M) \right).$$

So, letting $a_0 = 0$ and for $n = m + 1 < \omega$,

$$a_n = n!x_n - x_m \in G \cap M,$$

we have that $(a_n : n < \omega) \in M^\omega \subseteq M$ and so, as

$$M \prec_{\mathbb{L}_{\aleph_1, \aleph_1}} (\mathcal{H}(\chi), \in),$$

we can find

$$\bar{y} = (y_n : n < \omega) \in (G \cap M)^\omega$$

such that $a_n = n!y_n - y_m$, but then for every $m < \omega$:

$$G \models \left(m!(x_{m+1} - y_{m+1}) = x_m - y_m \right).$$

Hence,

$$\bigcup \{ \mathbb{Z}(x_m - y_m) : m < \omega \}$$

is a non-trivial divisible subgroup of G , contradicting the assumption that G is reduced. So we have proved item (i).

(ii). According to the first item we know the property $\text{Qr}_*(G)$ is valid. In view of Definition 4.2(5), there are some $p_0 \in \mathbb{P}$ such that $\text{Tor}_{p_0}(G)$ is unbounded. Since $\text{Tor}_{p_0}(G)$ is an unbounded p_0 -group, then it has pure cyclic subgroups of arbitrary large finite order. For such a p_0 , without loss of generality, we may and do assume that

$$G_{p_0} := \bigoplus_{n > n_0} \frac{\mathbb{Z}}{p_0^n \mathbb{Z}} \subseteq_* \text{Tor}_{p_0}(G)$$

for some n_0 . We apply this along with

$$\text{Tor}_{p_0}(G) \subseteq_* \text{Tor}(G) \subseteq_* G,$$

and deduce that $G_{p_0} \subseteq_* G$. Now, suppose on the way of contradiction that G has an infinite bounded pure subgroup K . In the light of Fact 2.4, K is a direct summand of G . Let

$$T := K \oplus G_{p_0} \subseteq_* G,$$

which is pure in G . Thanks to Fact 2.5, T is a direct summand of G . Consequently, G_{p_0} is a direct summand of G . Recall that direct summand of co-Hopfian is again co-Hopfian. From this, we deduce that G_{p_0} is co-Hopfian. Since G_{p_0} is a countable p_0 -group, and in the light of Fact 2.6(i), we conclude that G_{p_0} is finite. This contradiction shows that G has no infinite bounded pure subgroup. \square

Proposition 4.7. *Let $G \in \mathcal{B}$ be a boundedly endo-rigid abelian group. The following assertions are valid:*

- (1) G is co-Hopfian iff $\text{Qr}(G)$,
- (2) If $|G| > 2^{\aleph_0}$, then G is co-Hopfian iff $\text{Qr}_*(G)$.

Proof. (1). If G is co-Hopfian, then by Claim 4.3, $\text{Qr}(G)$ holds. For the other direction, suppose that G is boundedly rigid and $\text{Qr}(G)$ holds. Let $f \in \text{End}(G)$ be 1-to-1, we want to show that f is onto. As G is boundedly rigid we have m , h and L such that the following items hold:

- (a) $m \in \mathbb{Z}$, $h \in \text{End}(G)$,
- (b) $f(x) = mx + h(x)$,
- (c) $L = \text{Rang}(h)$ is a bounded subgroup of G (and so of $\text{Tor}(G)$).

If f is not onto, then by Lemma 4.5, there is $n \geq 1$ such that $\text{NQr}_{(m,n)}(G)$ holds, which is not possible (as we are assuming $\text{Qr}(G)$). Thus f is onto as required.

- (2). It follows from clause (1) and Lemma 4.6(i). \square

Theorem 4.8. *Let $K = \bigoplus \{ \frac{\mathbb{Z}}{p^n \mathbb{Z}} : p \in \mathbb{P} \text{ and } 1 \leq n < \omega \}$. If G is a boundedly endo-rigid abelian group of size λ and $K = \text{Tor}(G)$, then G is co-Hopfian.*

Proof. For any $p_1 \in \mathbb{P}$ and $n_1 < \omega$, let us define

$$(x_{(p_1, n_1)})_{(p, n)} = \begin{cases} 1 + p^n \mathbb{Z} & \text{if } (p, n) = (p_1, n_1) \\ 0 & \text{otherwise} \end{cases}$$

For simplicity, we abbreviate it by $x_{(p_1, n_1)} \in K$. Assume towards a contradiction that there exists $f \in \text{End}(G)$ such that f is 1-to-1 and not onto. As G is boundedly endo-rigid, there are $m \in \mathbb{Z}$ and $h \in E_b(G)$ such that $f = m \cdot \text{id}_G + h$. As f is 1-to-1 and K has no infinite bounded subgroup, we can conclude that $m \neq 0$.

$$(*)_1 \quad m \in \{1, -1\}.$$

To see $(*)_1$, suppose on the contrary that there is $p \in \mathbb{P}$ such that $p|m$ and let m_1 be such that $m = m_1 p$. Now, as $\text{Rang}(h)$ is bounded, there is $k \geq 1$ such that

$$p^k(\text{Rang}(h)) \cap \text{Tor}_p(G) = \{0\}.$$

Let $n \geq k + 1$, then:

$$\begin{aligned} f(p^{n-1}x_{(p,n)}) &= mp^{n-1}x_{(p,n)} + h(p^{n-1}x_{(p,n)}) \\ &= m_1 p p^{n-1}x_{(p,n)} + p^k h(p^{n-1-k}x_{(p,n)}) \\ &= 0, \end{aligned}$$

which contradicts the fact that f is 1-to-1. This completes the argument of $m \in \{1, -1\}$ and without loss of generality we may assume that $m = 1$. Thus $f = \text{id}_G + h$.

$$(*)_2 \quad f \text{ maps } G \setminus \text{Tor}(G) \text{ into itself.}$$

This is because f is 1-to-1. Indeed let $x \in G \setminus \text{Tor}(G)$. If $f(x) \in \text{Tor}(G)$, then for some k , $f(kx) = kf(x) = 0$, thus $kx = 0$, i.e., $x \in \text{Tor}(G)$ which contradicts $x \in G \setminus \text{Tor}(G)$.

$$(*)_3 \quad f \upharpoonright \text{Tor}(G) \in \text{End}(\text{Tor}(G)) \text{ is 1-to-1 not onto.}$$

Clearly $f \upharpoonright \text{Tor}(G) \in \text{End}(\text{Tor}(G))$, and since f is 1-to-1, $f \upharpoonright \text{Tor}(G)$ is 1-to-1 as well. Now, suppose by contradiction that $f \upharpoonright \text{Tor}(G)$ is onto. Then

- (1) $\text{Tor}(G) \subseteq \text{Rang}(f)$,
- (2) $x \in G \Rightarrow f(x) = x + h(x) \in \text{Tor}(G)$.

Recall that $h(x) \in \text{Tor}(G)$. Apply this along with (1), we deduce that $h(x) \in \text{Rang}(f)$. Also, recall that $\text{Rang}(f)$ is a group. Now, let $x \in G$. Thanks to (2), we observe that

$$x = f(x) - h(x) \in \text{Rang}(f).$$

In other words, f is onto, a contradiction. So, $f \upharpoonright \text{Tor}(G)$ is not onto.

- (*)₄ (a) for every $p \in \mathbb{P}$, f maps $\text{Tor}_p(G)$ into itself and so $f \upharpoonright \text{Tor}_p(G)$ is 1-to-1,
 (b) for some $p \in \mathbb{P}$, $f \upharpoonright \text{Tor}_p(G)$ is not onto.

Item (a) above is simply because f is 1-to-1. To see (b) holds, note that if $f \upharpoonright \text{Tor}_p(G)$ is onto for all prime number p , then so is $f \upharpoonright \text{Tor}(G)$, which contradicts (*)₃.

Thus, let us fix some prime $p \in \mathbb{P}$ such that $f \upharpoonright \text{Tor}_p(G)$ is not onto and let $h_p = h \upharpoonright \text{Tor}_p(G)$. Then by the above observations, it equipped with the following properties:

- (*)₅ (a) $h_p \in \text{End}(\text{Tor}_p(G))$,
 (b) $\text{Rang}(h_p)$ is bounded,
 (c) $h'_p = m \cdot \text{id}_{\text{Tor}_p(G)} + h_p = \text{id}_{\text{Tor}_p(G)} + h_p$ is 1-to-1,
 (d) h'_p is not onto.

In the light of Definition 4.2 and (*)₅ we observe that

- (*)₆ h_p is a $(1, p)$ -anti-witness for $\text{Tor}_p(G)$ and so $\text{NQR}(\text{Tor}_p(G))$.

Thanks to Lemma 4.4, $\text{Tor}_p(G)$ is finite. But,

$$\text{Tor}_p(G) = \bigoplus \left\{ \frac{\mathbb{Z}}{p^n \mathbb{Z}} : n \geq 1 \right\}.$$

In particular, $\text{Tor}_p(G)$ is infinite. Thus we get a contradiction, and hence f is onto. It follows that G is co-Hopfian and the lemma follows. \square

By the same method one can prove the following variation of Theorem 4.8.

Observation 4.9. *Let $K = \bigoplus \left\{ \frac{z_{p,n} \mathbb{Z}}{p^n \mathbb{Z}} : p \in \mathbb{P} \text{ and } 1 \leq n < \omega \right\}$ where $z_{p,n} \in \mathbb{Z}$ is chosen so that $z_{p,n} \mathbb{Z} \neq p^n \mathbb{Z}$. If G is a boundedly endo-rigid abelian group of size λ and $K = \text{Tor}(G)$, then G is co-Hopfian.*

Notation 4.10. For each group G , we set

$$S := S_G := \{p \in \mathbb{P} : G/\text{Tor}_p(G) \text{ is not } p\text{-divisible}\}.$$

Now, we are ready to present the following promised criteria:

Theorem 4.11. *Let $\lambda > 2^{\aleph_0}$, and suppose G is a boundedly endo-rigid abelian group of size λ . Then G is co-Hopfian if and only if:*

- (a): S_G is a non-empty set of primes,
- (b): (b₁) $\text{Tor}(G) \neq G$,
 (b₂) if $p \in S$, then $\text{Tor}_p(G)$ is not bounded,
 (b₃) if $\text{Tor}_p(G)$ is bounded, then it is finite (and $p \notin S_G$).

Proof. Let $K = \text{Tor}(G)$ and for each prime number p set $K_p = \text{Tor}_p(G)$.

First, we assume that G is co-Hopfian, and we are going to show items (a) and (b) are valid. As G is co-Hopfian, and according to [4] we know $|\text{Tor}(G)| \leq 2^{\aleph_0}$. Recall that $|G| = \lambda > 2^{\aleph_0}$, so $K = \text{Tor}(G) \neq G$, as claimed by (b₁).

To prove (b₂), let $p \in S$ and suppose by contradiction that K_p is bounded. As K_p is pure in G , and following Fact 2.4, the boundedness assumption guarantees that K_p is a direct summand of G . By definition, there is G_p such that $G = K_p \oplus G_p$. Now, we look at $\text{id}_{K_p} + p \cdot \text{id}_{G_p} \in \text{End}(G)$. Let

$$(k, g) \in \text{Ker}(\text{id}_{K_p} + p \cdot \text{id}_{G_p}).$$

Following definition,

$$(0, 0) = (\text{id}_{K_p} + p \cdot \text{id}_{G_p})(k, g) = (k, pg).$$

In other words, $k = 0$ and as G_p is p -torsion-free, $g = 0$. This means that

$$\text{Ker}(\text{id}_{K_p} + p \cdot \text{id}_{G_p}) = 0,$$

and hence $\text{id}_{K_p} + p \cdot \text{id}_{G_p}$ is 1-to-1. Since $p \in S$, $G_p = G / \text{Tor}_p(G)$ is not p -divisible, thus there is g in G_p such that $g \notin \text{Rang}(p \cdot \text{id}_{G_p})$. Consequently, $\text{id}_{K_p} + p \cdot \text{id}_{G_p}$ is 1-to-1 not onto. This is in contradiction with the co-Hopfian assumption, so K_p is not bounded and (b₂) follows.

In order to check (b₃), suppose $K_p = \text{Tor}_p(G)$ is bounded. Then it is a direct summand of G , say $G = K_p \oplus G_p$. Since G is co-Hopfian, so is K_p . Thanks to Lemma 4.4 K_p is finite.

Lastly, we check clause (a). Suppose on the way of contradiction that S is empty. Let $G_1 \prec_{\mathbb{L}_{\aleph_1, \aleph_1}} G$ be of cardinality 2^{\aleph_0} containing $\text{Tor}(G)$, recalling $|\text{Tor}(G)| \leq 2^{\aleph_0}$, so G/G_1 is divisible of cardinality λ .

As $G_1 \neq G$, there is $x_0 \in G \setminus G_1$, and note that $x \notin \text{Tor}(G)$. Now as $G/\text{Tor}(G)$ is divisible, we can choose the sequence $\langle x_n : n \geq 1 \rangle$ of elements of G , by induction on n , such that $x_0 = x$ and for each n ,

$$G/\text{Tor}(G) \models \left(n!x_{n+1} + \text{Tor}(G) = x_n + \text{Tor}(G) \right).$$

Set $a_n = n!x_{n+1} - x_n \in \text{Tor}(G)$. Note that $\langle a_n : n < \omega \rangle \in G_1$, thus as $G_1 \prec_{\mathbb{L}_{\aleph_1, \aleph_1}} G$, we can find elements $y_n \in G_1$ for $n < \omega$ such that

$$n!y_{n+1} = y_n + a_n.$$

Subtracting the last two displayed formulas, shows that the group

$$L = \bigcup \{ \mathbb{Z}(x_n - y_n) : n < \omega \}$$

is a non-zero divisible subgroup of G . Since L is an injective group, the sequence

$$0 \longrightarrow L \xrightarrow{g} G \longrightarrow \text{Coker}(g) \longrightarrow 0,$$

splits. As the property of boundedly endo-rigid behaves well with respect to direct sum, it obviously implies G is not boundedly endo-rigid. This contradiction implies that S is not empty. So clause (a) holds. Together we are done proving the left-right implication.

For the right-left implication, assume items (a) and (b) hold, and we show that G is co-Hopfian. Suppose on the way of contradiction that there exists $f \in \text{End}(G)$ such that f is 1-to-1 and not onto. As G is boundedly endo-rigid, there are $m \in \mathbb{Z}$ and $h \in E_b(G)$ such that $f = m \cdot \text{id}_G + h$.

$$(*)_1 \quad m \neq 0.$$

To see $(*)_1$, suppose $m = 0$. Then $f = h$, and since $\text{Rang}(h)$ is bounded and f is 1-to-1, we can conclude that G is bounded and therefor $G = \text{Tor}(G)$. This contradicts clause (b_1) .

$$(*)_2 \quad \text{If } \text{Tor}_p(G) \text{ is infinite, then } p \nmid m.$$

In order to see $(*)_2$, first note that $\text{Tor}(G)$ is unbounded, as otherwise $\text{Tor}_p(G)$ is also bounded, hence by (b_3) it is finite, contradicting our assumption. Suppose on the way of contradiction that $p \mid m$. Then there is m_1 such that $m = m_1 p$. Now, as $\text{Rang}(h)$ is bounded, there exists $k \geq 1$ such that

$$p^k \left(\text{Rang}(h) \upharpoonright \text{Tor}_p(G) \right) = \{0\}.$$

Recall that K_p is unbounded. This gives us an element $x \in \text{Tor}_p(G)$ of order p^n for some $n \geq k + 1$. But then

$$\begin{aligned} f(p^{n-1}x) &= mp^{n-1}x + h(p^{n-1}x) \\ &= m_1 p p^{n-1}x + p^k h(p^{n-1-k}x) \\ &= 0, \end{aligned}$$

which contradicts the fact that f is 1-to-1. As before, we have the following property:

- $(*)_3$ f maps $G \setminus \text{Tor}(G)$ into itself.
- $(*)_4$ $f \upharpoonright \text{Tor}(G) \in \text{End}(\text{Tor}(G))$ is 1-to-1 not onto.
- $(*)_5$ (a) for every $p \in \mathbb{P}$, f maps $\text{Tor}_p(G)$ into itself and so $f \upharpoonright \text{Tor}_p(G)$ is 1-to-1,
 (b) for some $p \in \mathbb{P}$, $f \upharpoonright \text{Tor}_p(G)$ is not onto.

Fix $p \in \mathbb{P}$ such that $f \upharpoonright \text{Tor}_p(G)$ is not onto. Then $h_p := h \upharpoonright \text{Tor}_p(G)$ is equipped with the following properties:

- $(*)_6$ (a) $h_p \in \text{End}(\text{Tor}_p(G))$,
- (b) $\text{Rang}(h_p)$ is bounded,
- (c) $h'_p = m \cdot \text{id}_{\text{Tor}_p(G)} + h_p = \text{id}_{\text{Tor}_p(G)} + h_p$ is 1-to-1,
- (d) h'_p is not onto.

In the light of its definition, h_p is a $(1, p)$ -anti-witness and so $\text{NQR}(\text{Tor}_p(G))$ holds.

Thanks to Lemma 4.4:

- $(*)_7$ $\text{Tor}_p(G)$ is finite.

But now $h'_p = h_p \upharpoonright \text{Tor}_p(G) : \text{Tor}_p(G) \rightarrow \text{Tor}_p(G)$ is one-to-one and since $\text{Tor}_p(G)$ is finite, we have h'_p is onto, which contradicts $(*)_6(d)$. \square

Combining the previous results with the mentioned theorem of Paolini-Shelah [21], we observe the following:

Corollary 4.12. *For any cardinals $\lambda > 2^{\aleph_0}$, there is a co-Hopfian group of size λ iff $\lambda = \lambda^{\aleph_0}$.*

REFERENCES

- [1] Reinhold Baer, *Types of elements and characteristic subgroups of abelian groups*, Proc. London Math. Soc. **39** (1935), 481-514.
- [2] Reinhold Baer, *Automorphism rings of primary abelian operator groups*, Ann. Math. **44** (1943), 192-227.
- [3] Reinhold Baer, *Groups without proper isomorphic quotient groups*. Bull. Amer. Math. Soc. **50** (1944), 267-278.
- [4] R. A. Beaumont and R. S. Pierce, *Partly transitive modules and modules with proper isomorphic submodules*. Trans. Amer. Math. Soc. **91** (1959), 209-219.
- [5] G. Braun and L. Strüingmann, *The independence of the notions of Hopfian and co-Hopfian abelian p -groups*. Proc. Amer. Math. Soc. **143** (2015), no. 8, 3331-3341.
- [6] Peter Crawley, *An infinite primary abelian group without proper isomorphic subgroups*, Bull. Amer. Math. Soc. **68** (1962), no. 5, 463-467.
- [7] A. L. S. Corner and Rüdiger Göbel, *Prescribing endomorphism algebras, a unified treatment*, Proceedings of the London Mathematical Society. Third Series **50** (1985), 447-479.
- [8] A. L. S. Corner, *On endomorphism rings of primary abelian groups*, Quart. J. Math. Oxford Ser., **20**, (1969) 277-296.
- [9] M. A. Dickman, *Larger infinitary languages*, *Model Theoretic Logics*, (J. Barwise and S. Feferman, eds.), Perspectives in Mathematical Logic, Springer-Verlag, New York Berlin Heidelberg Tokyo, 1985, 317-364.
- [10] Dugas, M. and Rüdiger Göbel, *Endomorphism rings of separable torsion-free abelian groups*, Houston J. Math. **11**, (1985) 471-483.
- [11] Paul C. Eklof and Alan Mekler, *Almost free modules: Set theoretic methods*, Revised Edition, North-Holland Publishing Co., North-Holland Mathematical Library, **65**, 2002.
- [12] B. Franzen, B. Goldsmith, *On endomorphism algebras of mixed modules* J. Lond. Math. Soc. **31**, 468-472 (1985).
- [13] L. Fuchs. *Infinite abelian groups. Vol. II*. Pure and Applied Mathematics, Vol. 36-II Academic Press, New York-London 1973.
- [14] L. Fuchs. *Infinite abelian groups. Vol. I*. Pure and Applied Mathematics, Vol. 36 Academic Press, New York-London 1970.

- [15] Laszlo Fuchs, *Abelian groups*, Springer Monographs in Mathematics. Springer, Cham, 2015.
- [16] Irving Kaplansky, *Infinite abelian groups*, University of Michigan Press, Ann Arbor, 1954.
- [17] Rüdiger Göbel and Jan Trlifaj, *Approximations and endomorphism algebras of modules*, Vols. 1, 2, de Gruyter Expositions in Mathematics, Vol. **41**, Walter de Gruyter, Berlin, 2012.
- [18] R. S. Pierce, *Homomorphisms of primary abelian groups*, (1963) Topics in abelian Groups (Proc. Sympos., New Mexico State Univ., 1962) pp. 215-310.
- [19] M. Kojman and S. Shelah, *A ZFC Dowker space in $\aleph_{\omega+1}$: an application of PCF theory to topology*, Proc. Amer. Math. Soc., **126**(8), (1998) 2459-2465.
- [20] G. Paolini and S. Shelah, *Torsion free abelian groups are Borel complete and a solution to some (co-)Hopfian problems*, Preprint. arXiv: 2102.12371.
- [21] G. Paolini and S. Shelah, *On the existence of uncountable Hopfian and co-Hopfian abelian groups*, to appear Israel J. Math. arXiv: 2107.11290
- [22] G. Paolini and S. Shelah, *Co-Hopfian groups are complete co-analytic and a solution to some other (co-)Hopfian problems*, Preprint.
- [23] Saharon Shelah, *A combinatorial principle and endomorphism rings. I. On p -groups*, Israel J. Math., **49**(1-3) (1984), 239-257.
- [24] Saharon Shelah, *Infinite abelian groups, Whitehead problem and some constructions*. Israel J. Math. **18** (1974), 243-256.
- [25] Saharon Shelah, *Existence of rigid-like families of abelian p -groups*, In Model theory and algebra (A memorial tribute to Abraham Robinson), Vol. **498**, (1975). Springer, Berlin, 384-402.
- [26] Saharon Shelah, *Categoricity in \aleph_1 of sentences in $L_{\omega_1, \omega}(Q)$* , Israel J. Math., **20**(2), (1975). 127-148.
- [27] Saharon Shelah, *Constructions of many complicated uncountable structures and Boolean algebras*, Israel J. Math. **45** (1983), no. 2-3, Lecture Notes in Math., **1292**, Springer, Berlin, 1987. 100-146.
- [28] Saharon Shelah, *A combinatorial principle and endomorphism rings. I*, On p -groups, Israel Journal of Mathematics **49** (1984), 239-257, Proceedings of the 1980/1 Jerusalem Model Theory year.
- [29] Saharon Shelah, *A combinatorial theorem and endomorphism rings of abelian groups. II*, abelian groups and modules (Udine, 1984), CISM Courses and Lectures, vol. **287**, Springer, Vienna, 1984, Proceedings of the Conference on abelian Groups, Udine, April 9-14, 1984); ed. Goebel, R., Metelli, C., Orsatti, A. and Solce, L., 37-86.
- [30] Saharon Shelah, *Universal classes. Classification theory* (Chicago, IL, 1985), 264-418, Lecture Notes in Math., **1292**, Springer, Berlin, 1987.
- [31] Saharon Shelah, *Black Boxes*, Preprint. arXiv: 0812.0656.
- [32] Saharon Shelah, *Building complicated index models and Boolean algebras*, To appear.

- [33] B. Thome, \aleph_1 -separable groups and Kaplansky's test problems, *Forum Math.* **2** (1990), 203–212.

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