THERE MAY EXIST EXACTLY κ P-POINT ULTRAFILTERS E79

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Abstract. This is a proof in the author's book on forcing. The point is proving the consistency of "there are exactly κ Ramsey ultrafilters" and more P-points. This was claimed but not proved there.
Debt: preservation in §2.

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Annotated Content

§ 0. Introduction

Here we prove prove the consistency of "there are exactly κ P-point ultrafilters up to isomorphism". In [She98, Ch.VI, §5] the case $k = 1$, and it had been stated that we can get, e.g. exactly two, or exactly κ and after a question of Fremlin, this is now explicitly proved. Halbeisen and Dzamonja ask in 2014 to clarify Lemma [She98, Ch.VI, 5.14], so its proof is expanded here.

Note that the numbers of the Definitions, Claims, etc., here are not the same as in [She98, Ch.VI,§5], because Remark 5.9A there becomes 1.12.

Using [She98, Ch. XVIII, §4] we can make those κ ultrafilters the unique P-point. We may use this [S$^+$a].

§ 1. Having exactly κ Ramsey ultrafilters

Usually it is significantly harder to prove that there is a unique object than to prove there is none. The proof is similar to the one in the previous section [She98, Ch.VI,§4], but here we are destroying other Ramsey ultrafilter (in fact "almost" all other P-points) while preserving our precious Ramsey ultrafilter. By a similar proof we can construct a forcing notion $\mathbb{P}$ such that e.g. in $\mathbf{V}^{\mathbb{P}}$ there are exactly two Ramsey ultrafilters (in both cases up to the equivalence induced by the Rudin-Keisler order) or any other number. In 2014 we rewrite the proof of Lemma 1.15 (after a request from Lorenz Halbeisen and Mirna Dzamonja) and write explicitly the case of κ Ramsey ultrafilters (following a question of David Fremlin).

More exactly we shall prove the consistency of "there is a unique Ramsey ultrafilter F_0 on ω, up to permutation of ω, moreover for every P-point $F, F_0 \leq_{\mathrm{RK}} F$".

Note that if there is a unique P-point it should be Ramsey; however, concerning the question of the existence of a unique P-point we return to it in Ch.XVIII, §4.

Our scheme is to start with a universe with a fixed Ramsey ultrafilter F_0, to preserve its being an ultrafilter and even a Ramsey ultrafilter. Our ultrafilter will be generated by $\aleph_1$ sets. Now in each stage we shall try to destroy a given P-point F such that $F_0 \leq_{\mathrm{RK}} F$. The forcing from [She98, Ch.VI,§4] does not work, but if we use a version of it in the direction of Sacks forcing it will work.

Claim 1.1. *(1) If F is a P-point in $\mathbf{V}, \mathbb{P}$ is a proper forcing notion and $\Vdash_{\mathbb{P}}$ "F generates an ultrafilter", <u>then</u> it (more exactly the one it generates) is a P-point in $\mathbf{V}^{\mathbb{P}}$.*

(2) If the ultrafilter F is Ramsey in $\mathbf{V}$, and P is ${}^{\omega}\omega$-bounding, proper and $\Vdash_{\mathbb{P}}$ "F generates an ultrafilter", <u>then</u> in $\mathbf{V}^{\mathbb{P}}, F$ still generates a Ramsey ultrafilter.

Proof. (1) As for being a P-filter, let $p \Vdash_{\mathbb{P}}$ "$\{\underset{\sim}{A}_n : n < \omega\}$ is included in the ultrafilter which F generates". So without loss of generality $p \Vdash_{\mathbb{P}}$ "$\underset{\sim}{A}_n \in F$", and by properness for some $q, p \leq q \in \mathbb{P}$, and $A_{n,m} \in F$ (for $n, m < \omega$) we have $q \Vdash_{\mathbb{P}}$ "for each $n, \underset{\sim}{A}_n \in \{A_{n,m} : m < \omega\}$". As F is a P-point in $\mathbf{V}$ and $\{A_{n,m} : n, m < \omega\} \subseteq F$ belong to $\mathbf{V}$, there is $A \in F$ which is almost included in every $A_{n,m}$, hence in each $\underset{\sim}{A}_n$; (note: e.g., if F is generated by $\aleph_1$ sets, then "P does not collapse $\aleph_1$" is sufficient instead of "P is proper").

(2) As by part (1), F generates a P-point in $\mathbf{V}^{\mathbb{P}}$, the following will suffice: let $0 = \underset{\sim}{n}_0 < \underset{\sim}{n}_1 < \underset{\sim}{n}_2 \ldots$ and $p \in \mathbb{P}$; then we can find $A \in F$ and $q \geq p$ such that $q \Vdash$ "$A \cap [\underset{\sim}{n}_i, \underset{\sim}{n}_{i+1})$ has at most one element for each i" (i.e. F is a so called Q-point). Remember $\mathbb{P}$ has the ${}^{\omega}\omega$-bounding property. So there are $h \in {}^{\omega}\omega \cap V$, and $q \geq p$ such that $q \Vdash_P$ "$(\forall i)\underset{\sim}{n}_i < h(i)$". Without loss of generality h is strictly increasing.

Define n_i^* (in V by induction on i): $n_0^* = 0, n_{i+1}^* = h(n_i^* + 1) + 1$. Now for no i, j we have $\underset{\sim}{n}_i[G] \leq n_j^* < n_{j+1}^* < \underset{\sim}{n}_{i+1}[G]$.

[Why? Assume this holds and, of course, $i < j$; as $\underset{\sim}{n}_\ell^* < \underset{\sim}{n}_{\ell+1}^*$, clearly $\ell \leq \underset{\sim}{n}_\ell^*[G]$, hence

$$n_{j+1}^* > h(n_j^* + 1) \geq h(j+1) \geq h(i+1) \geq \underset{\sim}{n}_{i+1}[G]$$

(remember h is strictly increasing), a contradiction].

Also F generates an ultrafilter in $\mathbf{V}[\mathbf{G}]$, by the assumption. As in $\mathbf{V}, F$ is a Ramsey ultrafilter and $\langle n_i^* : i < \omega\rangle \in V$, there is $A \in F$ such that $A \cap [n_i^*, n_{i+1}^*)$ has at most one element for each i. Let $\mathbf{G} \subseteq \mathbb{P}$ be generic over $\mathbf{V}$ be such that $q \in \mathbf{G}$. Checking carefully in $\mathbf{V}[\mathbf{G}]$ we see that, for every i we have $A \cap [\underset{\sim}{n}_i[\mathbf{G}], \underset{\sim}{n}_{i+1}[\mathbf{G}])$ has at most two elements and in this case they are necessarily successive members of A. Let $A_0 = \{k \in A\colon |A \cap k| \text{ is even}\}$, so either A_0 or $A \setminus A_0$ belong to the ultrafilter which F generates, and both are as required. $\square_{1.1}$

Lemma 1.2. *(1) "F generates an ultrafilter in $\mathbf{V}^{\mathbb{Q}}$ which is a P-point, $\mathbb{Q}$ is proper" is preserved by countable support iteration for F a P-point.*

(2) "F generates an ultrafilter in $\mathbf{V}^{\mathbb{Q}}$ which is Ramsey $+ Q$ is ${}^{\omega}\omega$-bounding $+ Q$ is proper" is preserved by countable support iteration for F a Ramsey ultrafilter.

Proof. (1) By [She98, Ch.VI,4.9] and see 1.1(1).

(2) Combine (1), 1.1(2) and [She98, Ch.VI,4.9]. $\square_{1.2}$

Definition 1.3. For F a filter on ω, let $\mathrm{SP}(F)$ be $\{T\colon T$ is a perfect tree $\subseteq {}^{\omega>}2$ so closed under initial segments and for some $A \in F$, for every $n \in A, \eta \in T \cap {}^{n}2$ implies $\eta\hat{\ }\langle 0\rangle \in T \wedge \eta\hat{\ }\langle 1\rangle \in T\}$. The order is the inverse inclusion. We denote the maximal such A by $\mathrm{spt}(T)$.

Remark 1.4. (1) So $\mathrm{SP}(F)$ is a "mixture" of $\mathbb{P}(F)$ and Sacks forcing and $\mathrm{SP}^*(F)$ (defined below) is half way between $\mathrm{SP}(F)$ and $\mathrm{SP}(F)^{\omega}$.

(2) Remember $T_{[\eta]} := \{\nu \in T\colon \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu\}$ for any $\eta \in T$ and $T^{[n]} := \{\eta \in T\colon \ell g(\eta) = n\}$ for any $n < \omega$.

Definition 1.5. (1) Let $T^{\otimes}_n := \prod_{\ell<n}({}^{\ell}2)$ and $T^{\otimes} := \bigcup\limits_{n<\omega} T^{\otimes}_n$ ordered by the being "initial segment".

(2) For a filter F on ω, let $\mathrm{SP}^*(F)$ be

$$\{T:\ T \text{ is a perfect tree } \subseteq T^{\otimes} \text{ (so closed under initial segments)} \text{ and for every } k<\omega \text{ we have } \mathrm{spt}_k(T) \in F\},$$

where,

$$\mathrm{spt}_k(T) = \{n<\omega:\ \text{for every } \eta \in T^{[n]} (= T \cap T^{\otimes}_n) \text{ and } \nu \in {}^{k}2 \text{ there is } \rho \in {}^{n}2 \text{ such that } \eta\hat{\ }\langle\rho\rangle \in T^{\otimes}_{n+1} \cap T \text{ and } \rho{\restriction}k = \nu\}.$$

(3) We say $\mathbb{Q}$ is a finitarily closed subforcing of $\mathrm{SP}^*(F)$ where:

- (a) $\mathbb{Q} \subseteq \mathrm{SP}^*(F)$ as a partial order (so $\mathbb{Q} \neq \emptyset$)
- (b) if $u \subseteq T^{\otimes}_n$ is non-empty and $(p_\eta \in \mathbb{Q}) \wedge (\nu_\eta \in p_\eta \cap T^{\otimes}_n)$ for $\eta \in u$, <u>then</u> $q \in \mathbb{Q}$ where $q = \cup\{\rho$: for some $\eta \in u$ and ν we have $\nu_\eta\hat{\ }\nu \in p_\eta$ and $\rho \trianglelefteq \eta\hat{\ }\nu\}$.

Remark 1.6. (1) Part 1.5(3) is intended for use in the [S$^+$a] try to continue [She], i.e. for [S$^+$b].

(2) We can replace 1.5(3)(b) by:

- (b)$_1$ if $p \in \mathbb{Q}$ and $\eta \in p \cap T^{\otimes}_n$ then $p^{[\geq\eta]} = \{\nu \in p\colon \nu \trianglelefteq \eta \text{ or } \eta \trianglelefteq \nu\} \in \mathbb{Q}$,
- (b)$_2$ if $p \in \mathbb{Q}, p \cap T^{\otimes}_n = \{\eta_2\}$ and $\eta_2 \in T^{\otimes}_n$ then $p^{[\eta_1,\eta_2]} = \{\nu\colon$ for some $\rho \in p$ we have $\eta_1 \triangleleft \rho$ and $\nu \triangleleft \eta_2\hat{\ }\rho{\restriction}[n, \ell g(\rho)]\}$,
- (b)$_3$ if $u \subseteq T^{\otimes}_n$ is non-empty and $p_\eta \in \mathbb{Q}$ for $\eta \in u$ and $p_\eta \cap T^{\otimes}_n = \{\eta\}$ <u>then</u> $\cup\{p_\eta\colon \eta \in u\} \in \mathbb{Q}$.

The order is the inverse inclusion.

Claim 1.7. *Let F be a filter on ω and $\mathbb{Q}$ be $\mathrm{SP}(F)$ or $\mathrm{SP}^*(F)$.*

(1) If $T \in \mathbb{Q}$, $T^{[n]} = \{\eta_1, \ldots, \eta_k\}$ (with no repetition), $T_\ell = T_{[\eta_\ell]}, T^\dagger_\ell \in \mathbb{Q}$, $T_\ell \le T^\dagger_\ell$ (i.e. $T^\dagger_\ell \subseteq T_\ell$) <u>then</u> $T \le T^\dagger := \bigcup_{\ell=1}^{k} T^\dagger_\ell \in \mathbb{Q}$ and $T^\dagger \Vdash$ "for some $\ell \in \{1, \ldots, k\}$ we have $T^\dagger_\ell \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}}$" and $(T^\dagger)^{[m]} = T^{[m]}$ for every $m \le n$.

(2) If $\underset{\sim}{t}$ is a $\mathbb{P}$-name of an ordinal, $T \in \mathbb{Q}$ and $n < \omega$ <u>then</u> there are $T^\dagger, T \le T^\dagger \in \mathbb{Q}$ and A such that $T^\dagger \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{t} \in A$" and $|A| \le |T^{[n]}|$ and $\bigcup_{\ell \le n} T^{[\ell]} \subseteq T^\dagger$. Moreover for each $\eta \in T^{[n]}, T^\dagger_{[\eta]}$ determines $\underset{\sim}{t}$.

Proof. (1) Observe that $\mathrm{spt}_j(T^\dagger) \supseteq \bigcap_{1 \le \ell \le k} \mathrm{spt}_j(T_\ell) \setminus (n+1)$.

(2) For each $\eta \in T^{[n]}$ there is $T^\eta, T_{[\eta]} \le T^\eta$ such that T^η decides the value $\underset{\sim}{t}$. Now amalgamate the T^η together by applying part 1). $\square_{1.7}$

Lemma 1.8. *Let F be a P-point ultrafilter on ω. <u>Then</u>*

(1) $\mathrm{SP}(F)$ is proper, in fact α-proper for every $\alpha < \omega_1$, and has the strong PP-property; and so is $\mathrm{SP}(F)^\omega$.

(2) $\mathrm{SP}^*(F)$ is also proper, α-proper for every $\alpha < \omega_1$ and has the strong PP-property.

Proof. Similar to the proof of [She98, Ch.VI,4.4]. For its proof we shall use the following theorem, of Galvin and McKenzie, (but later we shall prove a similar theorem in detail (5.11)); note that we use only the "only if" direction. $\square_{1.8}$

Theorem 1.9. *Let F be an ultrafilter on ω. Then F is a P-point [Ramsey ultrafilter] iff in the following game player I has no winning strategy:*

In the n-th move:

- *player I chooses $A_n \in F$,*
- *player II choose $w_n \subseteq A_n$, w_n is finite [a singleton].*

In the end, player II wins if $\bigcup_{n<\omega} w_n \in F$.

Next we are going to prove Lemma 1.8, using Theorem 1.9:

Proof. We just have to define a strategy for player I, (in the game from 1.9): playing on the side with the conditions in the forcing. From the two forcing listed in the lemma we concentrate on proving only the properness of $\mathrm{SP}^*(F)$ (the other have similar proofs and this is the only one we shall use). Let $N \prec (\mathscr{H}(\chi), \in, <^*_\chi)$ be countable with $F \in N$, so $\mathrm{SP}^*(F) \in N$; and let $T \in \mathrm{SP}^*(F) \cap N$ and let $\langle \mathscr{I}_n \colon n < \omega \rangle$ be a list of the dense subsets of $\mathrm{SP}^*(F)$ which belong to N. We shall define now a strategy for player I. In the n'th move player I chooses "on the side" a condition $T_n \in \mathrm{SP}^*(F) \cap N$ in addition to choosing $A_n \in F$ and player II chooses finite $w_n \subseteq A_n$. For $n = 0$, player I chooses $T_0 = T$ and $A_0 = \omega$.

For $n > 0$, for the n'th step player I, using 1.7, chooses $T_n \in \mathrm{SP}^*(F) \cap N$ such that $T_{n-1} \le T_n, T^{[k_n]}_{n-1} = T^{[k_n]}_n$, where $k_n := \max[\bigcup\{w_{n'} \colon n' < n\} \cup \{n\}] + n + 1$ and $(\forall \eta \in T^{[k_n]}_n)\ ((T_n)_{[\eta]} \in \mathscr{I}_{n-1})$. Then player I plays $A_n = \mathrm{spt}_n(T_n)$. Note that whatever are the choices of player II, we have $T_n \in N$ and we can let player I choose T_n as the first one which is as required by the well ordering $<^*_\chi$.

As F is a P-point, by 1.9 there is a play in which he uses the strategy described above and player II wins the play; this will give us the desired sequence of conditions. Indeed, $T = \bigcap\limits_{n<\omega} T_n \in \mathrm{SP}^*(F)$ satisfies $\mathrm{spt}_n(T) \supseteq \bigcup\{w_k \colon k \in [n,\omega)\}$ (for each $n < \omega$) and hence T belongs to $\mathrm{SP}^*(F)$. $\square_{1.8}$

Similar argument is carried out in more detail in the proof of 1.15.

Lemma 1.10. *(1) If F is a P-point ultrafilter, $\mathrm{SP}(F)^\omega \lessdot \mathbb{Q}$ and $\mathbb{Q}$ has the* PP-*property* <u>*then*</u> *in $\mathbf{V}^{\mathbb{Q}}$, F cannot be extended to a P-point ultrafilter.*

(2) If F is a P-point ultrafilter, $\mathrm{SP}^*(F) \lessdot \mathbb{Q}$, $\mathbb{Q}$ has the PP-*property* <u>*then*</u> *in $\mathbf{V}^{\mathbb{Q}}$, F cannot be extended to a P-point ultrafilter.*

Proof. The proof is almost identical with the proof of [She98, Ch.VI,4.7], so we do not carry out it in detail. (In fact we get the variant with weaker assumption as proved in [She98, Ch.VI,4.7]).

This is particularly true for part (1). For part (2) copy the proof of [She98, Ch.VI,4.7], replacing $P(F)$ by $\mathrm{SP}^*(F)$ and defining $\underset{\sim}{r}_n$ as:

$$\underset{\sim}{r}_n(i) = \ell \Leftrightarrow i \le n \Rightarrow \ell = 0,$$

and

$$i > n \Rightarrow (\exists T \in \underset{\sim}{G}_{\mathrm{SP}^*(F)})(\exists \eta \in T^{\otimes}_{i+1})[T = T_{[\eta]} \ \&\ (\eta(i))(n) = \ell].$$

This is done up to and including the choice of p_2 (i.e. $(*)$ in the proof of [She98, Ch.VI,4.7]).

As $p_2 \in \mathbb{P}$ and $\mathrm{SP}^*(F) \lessdot \mathbb{P}$ clearly there is $q \in \mathrm{SP}^*(F)$ such that p_2 is compatible in $\mathbb{P}$ with any q' satisfying $q \le q' \in \mathrm{SP}^*(F)$. For $k < \omega$, as $q \in \mathrm{SP}^*(F)$ by Definition 1.5 we know that $\mathrm{spt}_k(q) \in F$, so as F is a P-point there is $B^* \in F$ such that $B^* \setminus \mathrm{spt}_k(q)$ is finite for every $k < \omega$. Choose by induction on $n < \omega, \alpha_n < \omega$ such that $\alpha_n < \alpha_{n+1}, \alpha_n > g(n)$ and $\alpha_n > j_n(k(n))$ and $B^* \setminus \mathrm{spt}_{j_n(k(n))+1}(q) \subseteq [0, \alpha_n)$. Define $q' \coloneqq \{\eta \colon \eta \in q$ and for every $m < \omega$ we have: if $\alpha_n \le m < \ell g(\eta)$, $m < \alpha_{n+1}$ and $m \in \mathrm{spt}_{j_n(k(n))+1}(q)$ then for each $\ell \le k(n)$ we have $(\eta(m))(i_n(\ell)) = 0$ and $(\eta(m))(j_n(\ell)) = 1\}$.

Now,

(a) $q' \subseteq T^{\otimes}$ is closed under initial segments and $\langle\rangle \in q'$.

[Why? Read the definition of q'.]

(b) q' has no $\vartriangleleft$-maximal element.

[Why? Assume $\eta \in q' \cap T^{\otimes}_m$. If $m < \alpha_0$ then any $\nu \in \mathrm{Suc}_q(\eta)$ belongs to q'. So let $\alpha_n \le m < \alpha_{n+1}$; if $m \notin \mathrm{spt}_{j_n(k(n))+1}(q)$ again any $\nu \in \mathrm{Suc}_q(\eta)$ belongs to q', so assume $m \in \mathrm{spt}_{j_n(k(n))+1}(q)$, which means

$$(\forall \eta' \in q \cap T^{\otimes}_m)(\forall \rho \in {}^{j_n(k(n))+1}2)(\exists \nu)[\eta'{}^\frown\langle\nu\rangle \in q \ \&\ \nu j_n(k(n)) + 1 = \rho].$$

Apply this for η' and for the $\rho^* \in {}^{j_n((k(n))+1}2$ defined by $\{\ell < j_n(k(n))+1 \colon \rho^*(\ell) = 1\} = \{j_n(\ell) : \ell \le k(n)\}$, and find ν satisfying $\rho^* \trianglelefteq \nu$ and such that $\eta{}^\frown\langle\nu\rangle \in \mathrm{Suc}_q(\eta)$ and even $\eta{}^\frown\langle\nu\rangle \in \mathrm{Suc}_{q'}(\eta)$.]

(c) If $\alpha_n \le m < \alpha_{n+1}, m \in \mathrm{spt}_{j_n(k(n))+1}(q)$ then $m \in \mathrm{spt}_{i_n(0)}(q')$.

[Why? Same proof as of clause (b) noting that for any $\rho_1 \in {}^{i_n(0)}2$ we can find ρ^* such that $\rho_1 \triangleleft \rho^* \in {}^{j_n(k(n))+1}2$, such that for $m \in [i_n(0), j_n(k(n))+1)$, we have $\rho^*(m) = 1 \Leftrightarrow m \in \{j_n(\ell)\colon \ell \le k(n)\}$.]

(d) Let $k < \omega$, then $\mathrm{spt}_k(q') \in F$.

[Why? Choose $n(*)$ such that $k < i_{n(*)}(0)$. Now if $m \in B^* \setminus \alpha_{n(*)}$ then for some $n, n(*) \le n < \omega$ and $\alpha_n \le m < \alpha_{n+1}$ hence $m \in \mathrm{spt}_{j_n(k(n))+1}(q)$ and so by clause (c) we have $m \in \mathrm{spt}_{i_n(0)}(q')$. But $\mathrm{spt}_\ell(q')$ decreases with ℓ and $k < i_{n(*)}(0) \le i_n(0)$, so $m \in \mathrm{spt}_k(q')$. Together $B^* \setminus \alpha_{n(*)} \subseteq \mathrm{spt}_k(q')$, but the former belongs to F.]

(e) $q' \Vdash_{\mathrm{SP}^*(F)}$ "$\bigcap_{n<\omega} (\underset{\sim}{A}_n \cup [0, g(n)))$ is disjoint to $B^* \setminus \alpha_0$".

[Why? Because if $\alpha_n \le m < \alpha_{n+1}$ and $m \in B^*$ then: by the definitions of $\underset{\sim}{r}_{i_n(\ell)}, \underset{\sim}{r}_{j_n(\ell)}$ $(\ell \le k(n))$ and $\underset{\sim}{A}_n$ (which is $\{\alpha < \omega : \text{for some } \ell \le k(n),\ r_{i_n(\ell)}(\alpha) = r_{j_n(\ell)}(\alpha)\}$) we know $m \notin \underset{\sim}{A}_n$, also $m \ge \alpha_n > g(n)$, together this suffices.]

Now q', p_2 are compatible members of $\mathbb{P}$ (see the choice of q and remember $q \le q' \in \mathrm{SP}^*(F)$), so let $p_3 \in P$ be such that $p_2 \le p_3, q' \le p_3$. So by clause (e) the condition p_3, being above q', forces that $\bigcap_{n<\omega} (\underset{\sim}{A}_n \cup [0, g(n)))$ is disjoint to a member of F. So as $p_2 \le p_3$ clearly p_2 cannot force $\bigcap_{n<\omega} (\underset{\sim}{A}_n \cup [0, g(n))) \ne \emptyset \mod F$. But this contradicts the choice of p_2. $\square_{1.10}$

We now state some well known basic facts on the Rudin-Keisler order on ultrafilters.

Definition 1.11. (1) Let F_1, F_2 be ultrafilters on I_1, I_2, respectively. We say $F_1 \le_{\mathrm{RK}} F_2$ <u>iff</u> there is a function f from I_2 to I_1 such that $f(I_2) = \{f(i) : i \in I_2\} \in F_1$ and: $A \in F_1$ iff $f^{-1}(A) \in F_2$.

(2) In this case we say $F_1 = f(F_2)$; if $|I_1| \le |I_2|$ we can assume without loss of generality f is onto I_1.

Remark 1.12. We shall use only ultrafilters on ω, which are not principal, i.e. in $\beta(\omega) \setminus \omega$ in topological notation.

It is known (see e.g. [Jec03]).

Theorem 1.13. *(1) $\le_{\mathrm{RK}}$ is a quasi-order.*

(2) An ultrafilter F on ω is minimal <u>iff</u> it is Ramsey (minimal means $F^\dagger \le_{\mathrm{RK}} F \Rightarrow F \le_{RK} F^\dagger$ (see part (4)).

(3) If F is a P-point, $F^\dagger \le_{\mathrm{RK}} F$ <u>then</u> $F^\dagger$ is a P-point.

(4) If $F^1 \le_{\mathrm{RK}} F^2 \le_{\mathrm{RK}} F^1$, <u>then</u> there is a permutation f of ω such that $F_2 = f(F_1)$.

Proof. Well known. $\square_{1.12}$

Lemma 1.14. *Suppose F_0, F_1 are ultrafilters on ω (non-principal, of course). <u>Then</u> the condition (A) and condition (B) below are equivalent.*

(A) F_1 is a P-point, F_0 is a Ramsey ultrafilter, and not $F_0 \le_{\mathrm{RK}} F_1$.
(B) In the following game, player I has no winning strategy:
In the n-th move, when n is even:

- *player I chooses $A_n \in F_0$,*
- *player II chooses $k_n \in A_n$.*

In the n-th move, when n is odd:

- *player I chooses $A_n \in F_1$*
- *player II chooses a finite set $w_n \subseteq A_n$.*

In the end, player II wins if

$$\{k_n \colon n < \omega \text{ even}\} \in F_0 \text{ and } \bigcup\{w_n \colon n < \omega \text{ odd }\} \in F_1.$$

Proof. $\neg(A) \Rightarrow \neg(B)$: If F_1 is not a P-point or F_0 is not Ramsey then player I can win by 1.9. (I.e., if F_1 is not a P-point, then are $B_n \in F_1$ for $n < \omega$ such that for no $B \in F_1$ do we have $B \setminus B_n$ is finite for every n, now player I has a strategy guaranteeing: for n odd, $A_n = \bigcap\limits_{\ell \leq (n-1)/2} B_\ell \setminus (\sup \bigcup\{w_\ell \colon \ell < n \text{ odd}) + 1)$ or just $A_n = B_{(n-1)/2}$, this is a winning strategy. If F_0 is not a Ramsey ultrafilter there are $B_n \in F_0$ for $n < \omega$ such that for no $k_n \in B_n$ (for $n < \omega$) do we have $\{k_n \colon n < \omega\} \in F_0$, now player I has a strategy guaranteeing $A_{2n} = B_n$, this is a winning strategy.) So we can assume F_1 is a P-point and F_0 is Ramsey, so by $\neg(A)$ necessarily $F_0 \leq_{\rm RK} F_1$, hence some $h \colon \omega \to \omega$ witnesses $F_0 \leq_{\rm RK} F_1$. Then player I can play such that $\bigcup\{h^{-1}(k_n) \colon n \in \omega\}$ and $\bigcup\{w_n : n \in \omega\}$ will be disjoint. So one of them is not in F_1. Now if $\cup\{h^{-1}(k_n) : n \in w\} \notin F_1$ then by the choice of h we have $\{k_n : n \in \omega\} \notin\in F_0$, thus player I wins.

$(A) \Rightarrow (B)$: Suppose toward contradiction H is a wining strategy of player I. Let λ be big enough, $N \prec (\mathscr{H}(\lambda), \in), \{F_0, F_1, H\} \in N$ and N is countable. For $\ell = 0, 1$ as F_ℓ is a P-point there is $A^*_\ell \in F_\ell$ such that $A^*_\ell \subseteq_{\ae} B$ for every $B \in F_\ell \cap N$.

Now we can find an increasing sequence $\langle M_n \colon n < \omega\rangle$ of finite subsets of $N, N = \bigcup\limits_{n<\omega} M_n$ such that it increases rapidly enough; more exactly[1]:

- (α) $H, F_0, F_1 \in M_0, M_n \in M_{n+1}$; also can demand $x \in M_n$ & x finite $\Rightarrow x \subseteq M_n$; also $M_n \cap \omega$ is an initial segment of ω,
- (β) if $\varphi(x, a_0, \ldots)$ is a formula of length $\leq 1000 + |M_n|$ with parameters from $M_n \cup \{M_n\}$ satisfied by some $x \in N$, then it is satisfied by some $x \in M_{n+1}$,
- (γ) for $\ell = 0, 1$ if $B \in F_\ell \cap N, B \in M_n$ then $B \cup M_{n+1} \supseteq A^*_\ell$,
- (δ) $M_0 \cap \omega = \emptyset$.

Let $u_{n+1} = (M_{n+1} \setminus M_n) \cap \omega$. So $\langle u_n \colon n < \omega\rangle$ forms a partition of ω. As F_ℓ is an ultrafilter, there are $S_\ell \subseteq \omega$ such that $\bigcup\{u_n \colon n \in S_\ell\} \in F_\ell$, and $n < m$ & $\{n, m\} \subseteq S_\ell \Rightarrow m - n \geq 10$.

- ($*$) Without loss of generality $n \in S_0, m \in S_1$ implies the absolute value of $n - m$ is ≥ 5.

[Why? For the S_0, S_1 we have, for each $n \in S_0$ there is at most one $m \in S_1$ such that $|n - m| \leq 4$ and vice versa. By the previous sentence $\{(n, m) \colon n \in S_0, m \in S_1$ and $(n - m) \leq 4\}$ is the graph of a function, call if f, and f is a partial one-to-one function from S_0 into S_1.

[1] Do not try to understand the numbers 1000 and later 10,5 and clause (β) below: such demands in this direction are necessary, and no reason to check the exact demand. They are used in choosing a play of the game in the last paragraph of the proof.

Case 1: $\cup\{u_n : n \in \mathrm{dom}(f)\}$ belongs to F_0 and for every $S \subseteq \mathrm{dom}(f)$, we have $\cup\{u_n : n \in S\} \in F_0 \Leftrightarrow \cup\{u_n : n \in S\} \in F_1$.

For $\ell = 0, 1$ let $F'_\ell = \{A \subseteq \omega : \cup\{u_n : n \in A\} \in F_\ell\}$, so $F'_\ell \leq_{\mathrm{RK}} F_\ell$; as F^*_0 is a Ramsey ultrafilter it follows that $F_0 \leq_{\mathrm{RK}} F'_0$. By the assumption of the case $F'_0 = F'_1$, so we have $F_0 \leq_{\mathrm{RK}} F'_0 = F'_1 \leq_{\mathrm{RK}} F_1$, hence $F_0 \leq_{\mathrm{RK}} F_1$ contradicting the present assumption, clause (A) of the lemma.

Case 2: $\cup\{u_n : n \in \mathrm{dom}(f)\} \notin F_0$.

Let $S^\dagger_0 = S_0 \backslash \mathrm{dom}(f), S^\dagger_1 = S_1$; now $(S^\dagger_0, S^\dagger_1)$ are as required on (S_0, S_1) in $(*)$ and earlier.

Case 3: $\bigcup\{u_n : n \in \mathrm{dom}(f)\} \in F_0$, but there is $S^\dagger \subseteq \mathrm{dom}(f)$ such that $\cup\{u_n : n \in S^\dagger\} \in F_0$ but $\bigcup\{u_n : n \in S^\dagger\} \notin F_1$.

Let $S^\dagger_0 = S^\dagger, S^\dagger_1 = S_1 \backslash S^\dagger$ and continue as in case 2.

Clearly exactly one of the three cases holds so we are done.]

$(**)$ there are $k^*_n \in u_n \cap A^*_0$ (for $n \in S_0$) such that $\{k^*_n : n \in S_0\} \in F_0$.

[Why? Because F_0 is Ramsey.]

$(***)$ (a) Without loss of generality $\mathrm{Min}(S_0 \cup S_1) \geq 2$,

(b) for $n \in S_1$ letting $v_n := u_n \cap \bigcap\{A : A \in F_1 \cap M_{n-2}\}$ we have

$$\bigcup\{v_n : n \in S_1\} \in F_1,$$

(c) $k^*_\ell \in \bigcap\{A : A \in F_0 \cap M_{n-2}\}$.

[Why? Clause (a) holds as $S_0 \backslash \{0,1\}, S_1 \backslash \{0,1\}$ satisfies the requirements on S_0, S_1. For clause (b) recall that $B \in F_1$ and clause (γ) above, i.e. it implies $\cup\{\cap\{A : A \in F_1 \cap M_{n-2}\} \cap M_n : n < \omega\}$ include A^*_1 hence belongs to F_1. The proof of clause (c) is similar.]

Now there is no problem to define by induction on $\ell < \omega, n_\ell < \omega$ and an initial segment $\bar{t}^\ell$ of length ℓ of a play of the game (both increasing) such that: the initial segment belong to M_{n_ℓ}; and every k^*_n will appear among the k's which player II have chosen in the play if $n \leq n_\ell, n \in S_0$; and every v_n will appear among the w's player II have chosen in the play if $n \leq n_\ell, n \in S_1$; and n_ℓ has the form $n^* + 2$ with $n^* \in S_0 \cup S_1$; and player I uses his strategy. But in the play we produce player II wins, contradiction. $\square_{1.14}$

Main Lemma 1.15. *Suppose F_0 is a Ramsey ultrafilter (on ω), F is a P-point, and $\mathbb{Q} = \mathrm{SP}^*(F)$, and for some $T \in \mathbb{Q}$ we have $T \Vdash_{\mathbb{Q}}$ "F_0 is not an ultrafilter" then $F_0 \leq_{\mathrm{RK}} F$.*

Proof. Let $T_0 \in \mathbb{Q}, \underset{\sim}{A}$ be a $\mathbb{Q}$-name, $T_0 \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{A} \subseteq \omega$ and $\omega \setminus \underset{\sim}{A}, \underset{\sim}{A} \neq \emptyset \mod F_0$", and without loss of generality $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{A} \subseteq \omega$", (such $T_0, \underset{\sim}{A}$ exists as after forcing with $\mathbb{Q}, F_0$ will no longer generate an ultrafilter). Note that by the choice of $T_0, \underset{\sim}{A}$ for any $T \geq T_0$, the set

$$\{n < \omega : \text{ for some } T^\dagger \geq T, T^\dagger \Vdash_{\mathbb{Q}} \text{“}n \in \underset{\sim}{A}\text{” and for some} T^\dagger \geq T, T^\dagger \Vdash_{\mathbb{Q}} \text{“}n \notin \underset{\sim}{A}\text{”}\},$$

belongs to F_0.

Now we use the game defined in Lemma 1.14. We shall describe a winning strategy for player I. During the play, player I in his moves defines also $T_n \in \mathbb{Q}$ preserving the following:

$(*)$ (a) $T_{n+1} \geq T_n$,
(b) $T_n \Vdash_{\mathbb{Q}}$ "$k_\ell \in \underset{\sim}{A}$" for ℓ even, $\ell < n$,
(c) $T_{n+1}^{[m(n)]} = T_n^{[m(n)]}$ where $m(n) := 1 + \max[\bigcup\{w_\ell \colon \ell \text{ odd}, \ell < n\} \cup \{n\}]$,
(d) for $\ell < n$ odd we have: $w_\ell \subseteq \mathrm{spt}_\ell(T_n)$ (see Definition 1.5),
(e) for n even, for the play from 1.14 player I chooses

$$A_n \subseteq \{k\colon \text{ if } \eta \in T_n^{[m(n)]} \text{ then } (T_n)_{[\eta]} \nVdash \text{"}k \notin \underset{\sim}{A}\text{"}\},$$

(f) for n odd, for the play from 1.14 player I chooses $A_n = \mathrm{spt}_{m(n)}(T_n)$.

More exactly, player I chooses T_{n+1} in the n-th move after player II's move (see below more).

If this is a well defined strategy, i.e. player I can make those choices, this is enough. Why? As if in the end $\bigcup\{w_\ell \colon \ell < \omega \text{ odd}\} \in F$, then $T := \bigcap_n T_n \in \mathbb{Q}$, because for each $\ell < \omega$, we have $n > \ell \Rightarrow \mathrm{spt}_\ell(T_{n+1}) \subseteq \mathrm{spt}_\ell(T_n)$ and $\mathrm{spt}_{\ell+1}(T_n) \subseteq \mathrm{spt}_\ell(T_n)$ so by clauses (c)+(d)

$(*)$ $\ell < m \leq k \Rightarrow w_k \subseteq \mathrm{spt}_\ell(T_m)$.

Hence $\mathrm{spt}_\ell(T) \supseteq \bigcap_{m>\ell} \mathrm{spt}_\ell(T_m) \supseteq \bigcup_{m\geq\ell} w_m \in F$ (as all cofinite subsets of ω belong to F). Now T forces $\{k_\ell : \ell < \omega \text{ even}\} \subseteq \underset{\sim}{A}$ (remember clause (b)), so $\{k_\ell \colon \ell < \omega \text{ even}\} \notin F_0$ by the hypothesis on $T_0, \underset{\sim}{A}$ (as $\{k_\ell \colon \ell < \omega\} \in V$, and $T_0 \leq T, T \Vdash_P$ "$\{k_\ell \colon \ell < \omega\} \subseteq \underset{\sim}{A}$" so $\{k_\ell \colon \ell < \omega\} \in F_0$ implies: $T \Vdash_{\mathbb{Q}}$ "$\omega \setminus \underset{\sim}{A} = \emptyset \mod F$", a contradiction). So the strategy defined above is a winning strategy for player I hence by Lemma 1.14, $F_0 \leq_{\mathrm{RK}} F$.

So it remains to show that player I can indeed carry out the strategy i.e. can preserve $(*)$. Note that T_0 is defined.

Case 1: when $n > 0$ is even.

Player I lets $m(n) < \omega$ be $\max[\bigcup\{w_\ell \colon \ell < n \text{ odd}\} \cup \{n\}] + 1$, and let $T_n^{[m(n)]} = \{\eta_0, \ldots, \eta_{s(n)}\}$ with no repetition. For each η_ℓ ($\ell \leq s(n)$) clearly $(T_n)_{[\eta_\ell]}$ is $\geq T_0$ and belongs to $\mathbb{Q}$, hence the set

$$A_\ell^n = \{k < \omega \colon \quad \text{there are } T'_{\ell,k}, T''_{\ell,k} \geq (T_n)_{[\eta_\ell]} \text{ such that } T'_{\ell,k} \Vdash_{\mathbb{Q}} \text{"}k \in \underset{\sim}{A}\text{", and } T''_{\ell,k} \Vdash_{\mathbb{Q}} \text{"}k \notin \underset{\sim}{A}\text{"}\}$$

belongs to F_0.

Now, player I plays $A_n = \bigcap_{\ell \leq s(n)} A_\ell^n$ which is clearly a legal move.

Player II chooses some $k_n \in A_n$.

Player I ("on the side") lets $T_{n+1} = \bigcup\limits_{\ell \leq s(n)} T'_{\ell,k_n}$ (it is as required in $(*)$).

Case 2: when n is odd.

Player I lets $A_n = \mathrm{spt}_{m(n)}(T_n)$ (note $\mathbb{Q} = \mathrm{SP}^*(F)$). Note T_n has just been chosen.

Player II chooses a finite $w_n \subseteq A_n$ and player I lets on the side $T_{n+1} = T_n$. $\square_{1.15}$

Theorem 1.16. *(1) It is consistent with* $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_2$ *that, up to a permutation on* ω*, there is a unique Ramsey ultrafilter on* ω*. Moreover any* P*-point is above it (in the Rudin-Keisler order).*

(2) If $\kappa \in [1, \aleph_2]$ then in some forcing extension of V we have $2^{\aleph_0} = \aleph_2$, up to permutation of ω there are exactly κ Ramsey ultrafilters. Moreover any P-point is $\leq_{\rm RK}$-above at least one Ramsey ultrafilter.

Proof. Without loss of generality we start with a universe satisfying $2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$ and $\diamondsuit_{\{\delta<\aleph_2 : \, {\rm cf}(\delta)=\aleph_1\}}$. There is in $\mathbf{V}$ a sequence $\langle F^*_\varepsilon : \varepsilon < \kappa\rangle$ of Ramsey ultrafilters such that $\varepsilon \neq \zeta \Rightarrow F^*_\varepsilon \not\leq_{\rm RK} F^*_\zeta$; for part (1) we use $\kappa = 1$.

We shall use a CS iterated forcing $\langle \mathbb{P}_i, \underset{\sim}{\mathbb{Q}}_i : i < \omega_2\rangle$ such that each $\mathbb{Q}_i$ is proper, has the PP-property (hence is ${}^\omega\omega$-bounding), has cardinality continuum and forces that F^*_ε still generates an ultrafilter. So by 1.1, 1.2, F^*_ε remains a Ramsey ultrafilter in $\mathbf{V}^{\mathbb{P}_i}$ for $i \leq \omega_2$ and also we can show by induction on $i < \omega_2$, that in $\mathbf{V}^{\mathbb{P}_i}$, CH holds and P_i has cardinality $\aleph_1$; so by [She98, Ch.VIII,§2] below, P_{ω_2} satisfies the $\aleph_2$-chain condition. If $F_1 \in \mathbf{V}[\mathbf{G}_{\omega_2}]$ ($\mathbf{G} \subseteq \mathbb{P}_{\omega_2}$ generic) is a P-point, not above any F^*_ε, then there is a $p \in P_{\omega_2}$ forcing $\underset{\sim}{F}_1$ is a name of such ultrafilter, and for a closed unbounded set of $\delta < \aleph_2, {\rm cf}(\delta) = \aleph_1$ implies that $\underset{\sim}{F}^1_\delta := \underset{\sim}{F}_1 \cap {\rm SP}(\omega)^{\mathbf{V}^{\mathbb{P}_\delta}} \in \mathbf{V}^{\mathbb{P}_\delta}$ and p forces that $\underset{\sim}{F}^1_\delta$ is a P-point not above F^*_ε for $\varepsilon < \kappa$ (in $\mathbf{V}^{\mathbb{P}_\delta}$).

Now, by the diamond $\diamondsuit_{\{\delta<\aleph_2 : \, {\rm cf}(\delta)=\aleph_1\}}$ we can assume that for some such $\delta, \underset{\sim}{\mathbb{Q}}_\delta = {\rm SP}^*(\underset{\sim}{F}^1_\delta)$.

Now by 1.15 forcing with $\mathbb{Q}_\delta$ (over $\mathbf{V}^{\mathbb{P}_\delta}$) preserves "$F^*_\varepsilon$ (generates) an ultrafilter", by 1.8(2) $\mathbb{Q}_\delta$ has the PP-property hence (by [She98, Ch.VI]) $\mathbb{Q}_\delta$ is ${}^\omega\omega$-bounding and trivially $\mathbb{Q}_\delta$ has cardinality continuum; so $\mathbb{Q}_\delta$ is as required. Now as each $\mathbb{Q}_j$ ($i < j < \omega_2$) has the PP-property, $\mathbb{P}_{\omega_2}/\mathbb{P}_\delta$ has the PP-property (by [She98, Ch.VI]). So by lemma 1.15 we know $\underset{\sim}{F}^1_\delta$ cannot be completed to a P-point in $\mathbf{V}^{\mathbb{P}_{\omega_2}}$. $\square_{1.16}$

§ 2. There may be a unique P-point

Theorem 2.1. *Assume* $\mathbf{V}$ *satisfies* $2^{\aleph_0} = \aleph_1$ *and* $\lambda = \lambda_1^{\aleph_0} > \kappa \geq 1$, F_α *for* $\alpha < \kappa$ *are Ramsey ultrafilters on* ω *pairwise non-isomorphic. Then for some* $\aleph_2$-*c.c. proper,* ${}^\omega\omega$-*bounding forcing notion* $\mathbb{P}$ *of cardinality* $\aleph_2$ *in* $\mathbf{V}^{\mathbb{P}}$, *there is a unique* P-*point, and it is* F_0 *(i.e. the filter it generates in* $\mathbf{V}^{\mathbb{P}}$*).*

Remark 2.2. In fact, in $\mathbf{V}^{\mathbb{P}}$, F_0 is a Ramsey ultrafilter (actually this follows).

Proof. By the proof of §1, it suffices to prove the following lemma: $\square_{2.1}$

Lemma 2.3. *Suppose*

$(*)_0$ F_0, F_1 *are ultrafilters on* ω, F_0 *is a Ramsey ultrafilter,* F_1 *is a* P-*point,* $F_0 \leq_{\mathrm{RK}} F_1$ *but not* $F_1 \leq_{\mathrm{RK}} F_0$.

Then there is a forcing notion $\mathbb{Q}$ *such that:*

(a) $\mathbb{Q}$ *has the* PP-*property, (hence it is* ${}^\omega\omega$-*bounding) and it is of cardinality* $2^{\aleph_0}$ *and,*
(b) $\Vdash_{\mathbb{Q}}$ *"*F_0 *is an ultrafilter", but*
(c) *if* $\mathbb{Q} \lessdot \mathbb{Q}'$ *and* $\mathbb{Q}'$ *has the* PP-*property then, in* $\mathbf{V}^{\mathbb{Q}'}$ *we have:* F_1 *cannot be extended with to a* P-*point (ultrafilter),*
(d) *if in* $\mathbf{V}$, D_* *is a Ramsey ultrafilter not isomorphic to* F_0 *then* $\Vdash_{\mathbb{Q}}$ *"*D_* *is (= generates) an ultrafilter".*

Remark 2.4. During the proof of Theorem 2.1 we use the forcing notions $\mathrm{SP}^*(F)$ from Definition IV.5.4 to kill P-points with $F_0 \not\leq_{\mathrm{RK}} F$.

The rest of this section is dedicated to the proof of this Lemma.

Proof. Since $F_0 \leq_{\mathrm{RK}} F_1$ and F_1 is a P-point, there is a function $h\colon \omega \to \omega$ such that

$(*)_1$ $h(F_1) = F_0$ and for each $\ell < \omega$ the set $I(\ell) = I_\ell := h^{-1}(\{\ell\})$ is finite. Note that then $[A \subseteq \omega \wedge \bigwedge_\ell 1 \geq |I_\ell \cap A| \Rightarrow A \notin F_1]$ because $F_1 \not\leq_{\mathrm{RK}} F_0$. Now, in Definition 2.7 below, we define a forcing notion $Q = \mathrm{SP}^*(F_0, F_1, h)$ and then prove in 2.5-2.12 that it has all the required properties thus finishing the proof of Lemma 2.3 and therefore, of Theorem 2.1.

$\square$

Claim 2.5. *In the following game, player I has no winning strategy: in the* n-*th move player I chooses* $A_n \in F_0$ *and* $B_n \in F_1$*; player II chooses* $k_n \in A_n$ *(*$k_n < k_\ell$ *for* $\ell < n$*) and* $w_n \subseteq B_n \cap I_{k_n}$. *In the end, player II wins the play if* $\{k_n : n < \omega\} \in F_0$ *and* $\bigcup\{w_n : n < \omega\} \in F_1$ *(the first demand follows from the second).*

Remark 2.6. Clearly player II has no better choice than $w_n = B_n \cap I_k$. Remember $I_{k_n} = h^{-1}(\{k_n\})$ is finite.

Proof. Towards contradiction, suppose that H is a wining strategy of player I. Let λ be big enough, $N \prec (\mathscr{H}(\lambda), \in, <^*_\lambda)$ be such that $\{F_0, F_1, h, H\} \in N$ and N is countable. As F_ℓ is a P-point there are, for $\ell \in \{0, 1\}$ sets $A^*_\ell \in F_\ell$ such that $A^*_\ell \subseteq_{ae} B$ (i.e. $A^*_\ell \setminus B$ finite) for every $B \in F_\ell \cap N$.

Now we can find an increasing sequence $\langle M_n : n < \omega \rangle$ of finite subsets of N, $N = \bigcup_{n<\omega} M_n$ such that it increases rapidly enough; more exactly:

- (α) $H, F_0, F_1, h \in M_0$ and $M_n \in M_{n+1}$,
- (β) if $\varphi(x, a_0, \ldots)$ is a formula of length $\leq 1000 + |M_n|$ with parameters from $M_n \cup \{M_n\}$ satisfied by some $x \in N$, *then* it is satisfied by some $x \in M_{n+1}$,
- (γ) if $\ell \in \{0,1\}$, $B \in F_\ell \cap N$, $B \in M_n$ then $B \cup M_{n+1} \supseteq A^*_\ell$,
- (δ) $M_0 \cap \omega = \emptyset$,
- (ε) if $\ell \in M_n$ then $I(\ell) \subseteq M_{n+1}$ and M_n is closed under h (we can demand $m \in M_n \Leftrightarrow h(m) \in M_n$ if we make the domains of F_0, F_1 disjoint).

Let $u_{n+1} = (M_{n+1} \setminus M_n) \cap \omega$. So $\langle u_n : n < \omega \rangle$ forms a partition of ω into finite sets. As F_0 is Ramsey, we can find $A \in F_0$ such that $\bigwedge_n |u_n \cap A| \leq 1$ and $A \subseteq A^*_0$ and

$$u_n \cap A \neq \emptyset \;\&\; u_m \cap A \neq \emptyset \;\&\; n < m \;\Rightarrow m - n \geq 10.$$

Let $A = \{i_\zeta : \zeta < \omega\}$ (increasing), $i_\zeta \in u_{n_\zeta}$. Now we define by induction on ζ, A_ζ, B_ζ, k_ζ, w_ζ such that:

- (a) $\langle A_\xi, B_\xi, k_\xi, w_\xi : \xi < \zeta \rangle$ is an initial segment of a play of the game in which Player I uses his winning strategy,
- (b) $\langle A_\xi, B_\xi, k_\xi, w_\xi : \xi \leq \zeta \rangle$ belongs to $M_{n_\zeta + 3}$,
- (c) $k_\zeta = i_\zeta$ and $w_\zeta = B_\zeta \cap I(k_\zeta) \cap A^*_1$.

There is no problem to carry out the definition, and clearly Player II wins because not only $\{k_\zeta : \zeta < \omega\} = \{i_\zeta : \zeta < \omega\} = A \subseteq A^*_0$ but also

$$\bigcup_{\zeta<\omega} w_\zeta = A^*_1 \cap \bigcup_{\zeta<\omega} w_\zeta = A^*_1 \cap \{j < \omega : h(j) = i_\zeta \text{ for some } \zeta < \omega\}$$
$$= A^*_1 \cap \{j : h(j) \in A\} \in F_1.$$

[Why? As respectively: $w_\zeta \subseteq A^*_1$; as $A^*_1 \setminus A_\xi \subseteq \bigcup \{w_\zeta : \zeta \leq i_\xi + 4\}$ by clause (γ) above; as $A = \{i_\zeta : \zeta < \omega\}$; as $A^*_1 \in F_1$ and $A \in F_0$ hence $\{j : h(j) \in A\} \in F_1$.]

Contradiction. $\square_{2.5}$

Definition 2.7. Let $T^h_n = \times_{\ell<n} {}^{I(\ell)\times\ell}2$ and let $T^h = \bigcup_{n<\omega} T^h_n$. Note that T^h is a perfect tree with finite branching ordered by $\triangleleft$ (being initial segment). Let $\mathbb{Q} := \mathrm{SP}^*(F_0, F_1, h) = \{T : T$ is a perfect subtree of T^h and for each $k < \omega$ for some $A_k \in F_0$ and $B_k \in F_1$ we have: if $\ell \in A_k$ and $\eta \in T^{[\ell]} := T \cap T^h_\ell$ and $\rho \in {}^{(B_k \cap I(\ell))\times k}2$ then for some $\nu \in {}^{I(\ell)\times\ell}2$ we have $\rho \subseteq \nu$ and $\eta {}^\frown \langle \nu \rangle \in T\}$ endowed with the inverse inclusion.

Claim 2.8. *(1) If $T \in \mathbb{Q}$, $T^{[n]} = \{\eta_1, \ldots, \eta_k\}$ (with no repetition) $T_\ell = T_{[\eta_\ell]} := \{\nu \in T : \eta_\ell \trianglelefteq \nu$ or $\nu \trianglelefteq \eta_\ell\}$, $T^\dagger_\ell \in \mathbb{Q}$, $T_\ell \leq T^\dagger_\ell$ (i.e. $T^\dagger_\ell \subseteq T_\ell$) <u>then</u> $T \leq T^\dagger := \bigcup_{\ell=1}^k T_\ell \in \mathbb{Q}$.*

(2) If $\underset{\sim}{\tau}$ is a $\mathbb{Q}$-name of an ordinal and $n < \omega$ <u>then</u> there is $T^\dagger$, $T \leq T^\dagger \in \mathbb{Q}$ such that $T^\dagger \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{\tau} \in A$" for some A satisfying $|A| \leq |T^{[n]}|$, and $T \cap \bigcup_{\ell \leq n} T^{[\ell]} = T^\dagger \cap \bigcup_{\ell \leq n} T^{[\ell]}$. Moreover for each $\eta \in T^{[n]}, T^\dagger_{[\eta]}$ determines $\underset{\sim}{\tau}$.

Proof. Same as in the proof of VI 5.5. $\square_{2.8}$

Claim 2.9. *$\mathbb{Q}$ is proper, in fact α-proper for every $\alpha < \omega_1$, and has the strong* PP*-property (see VI 2.12E(3)).*

Proof. First we prove properness. Let λ be regular $> 2^{\aleph_1}$, $N \prec (H(\lambda), \in, <^*_\lambda)$ be countable, $\{\mathbb{Q}, F_0, F_1, h\} \in N$ and $T \in N \cap \mathbb{Q}$.

Let $\{\underset{\sim}{\tau}_\ell \colon \ell < \omega\}$ list the $\mathbb{Q}$-names of ordinals from N. We now define a strategy for player I in the game from Claim 4.3. In the n-th move player I chooses $A_n \in F_0 \cap N$, $B_n \in F_1 \cap N$ and player II chooses $k_n \in A_n$ and $w_n := B_n \cap I_{k_n}$ (remember 4.3A); on the side player I chooses $T_n \in N \cap \mathbb{Q}$ and m_n such that $T_0 = T$, $T_n \le T_{n+1}$, $T_n^{[m_{n+1}]} = T_{n+1}^{[m_{n+1}]}$ and $m_n > \max\{m_{n-1}, k_{n'} \colon n' < n\}$ and $m_0 = 1$.

In the $(n+1)$'th move, player I first chooses m_{n+1} as above then he chooses $T_{n+1} \in \mathbb{Q}$, $T_n \le T_{n+1}$, $T_{n+1}^{[m_{n+1}]} = T_n^{[m_{n+1}]}$ such that for every $\eta \in T_n^{[m_{n+1}]}$, $(T_{n+1})_{[\eta]}$ forces a value to $\underset{\sim}{\tau}_\ell$ for $\ell \le m_{n+1}$. This is possible by 4.5. Then as $T_{n+1} \in \mathbb{Q} \cap N$ there are sets $A_{n+1} \in F_0 \cap N$, $B_{n+1} \in F_1 \cap N$ such that for every $k \in A_{n+1}$, $\eta \in (T_{n+1})^{[k]}$ and $\rho \in {}^{(B_{n+1} \cap I(k)) \times n}2$ for some $\nu \in {}^{I(k) \times k}2$, we have: $\rho \subseteq \nu$ and $\eta ^\frown \langle \nu \rangle \in T_{n+1}$ and for simplicity $A_{n+1} \cap m_n = A_n \cap m_n$. Note that the amount of free choice player II retains is in N.

So by 4.3 for some such play, player II wins. Now $T^* := \bigcap_{n<\omega} T_n \in \mathbb{Q}$ as $\{k_n : n < \omega\} \in F_0$ and $\bigcup\limits_{n<\omega} B_n \cap I(k_n) \in F_1$ witness; of course $T_n \le T^*$ for each n hence $T = T_0 \le T^*$ and $T^* \Vdash$ "$\underset{\sim}{\tau}_\ell[\underset{\sim}{G}_{\mathbb{Q}}] \in N \cap \mathbb{Q}_n$" (as $T_{\ell+1} \le T^*$, see its choice).

So $\mathbb{Q}$ is proper. The proof also shows that $\mathbb{Q}$ has the strong PP-property (see VI 2.12: for more details see the proof of VI 4.4.). The proof of α-properness is as in VI 4.4 (and anyhow it is not used). $\square_{2.9}$

Lemma 2.10. *Suppose $((*)_0$ of Lemma 2.3, $\mathbb{Q} = \mathrm{SP}^*(F_0, F_1, h)$ as defined in Definition 2.7 of course and) $\mathbb{Q} \lessdot \mathbb{P}$ and P has the PP-property. Then in $\mathbf{V}^{\mathbb{P}}$, F_1 cannot be extended to a P-point.*

Proof. Suppose $p \in P$ forces that $\underset{\sim}{E}$ is an extension of F_1 to a P-point (in $\mathbf{V}^{\mathbb{P}}$). Let $\langle \underset{\sim}{r}_n \colon n < \omega \rangle$ be the sequence of reals which $\mathbb{Q}$ introduces, i.e. $r_n(i) = \ell \in \{0, 1\}$ is defined as follows: clearly for a unique $k < \omega$, $i \in I_k$; now $\underset{\sim}{r}_n(i) = \ell$ iff: $n \ge k$, $\ell = 0$ or for some $T \in \underset{\sim}{G}_{\mathbb{Q}}$, $T^{[k+1]} = \{\eta\}$ and $(\eta(k))(i, n) = \ell$ (remember that $\eta(k)$ is a function from $I(k) \times k$ to $\{0, 1\}$). Define a P-name $\underset{\sim}{h}$:

- $\underset{\sim}{h}(n)$ is 1 if $\{i < \omega \colon \underset{\sim}{r}_n(i) = 1\} \in \underset{\sim}{E}$ and,
- $\underset{\sim}{h}(n)$ is 0 if $\{i < \omega : \underset{\sim}{r}_n(i) = 0\} \in \underset{\sim}{E}$

So $p \Vdash$ "$\underset{\sim}{h} \in {}^\omega 2$". Now as P has the PP-property, by VI 2.12D, there are $p_1 \ge p$, $(p_1 \in P)$, and $\langle \langle \langle k(n), \langle i_n(\ell), j_n(\ell) \rangle : \ell \le k(n) \rangle \colon n < \omega \rangle$ in V such that $k(n) < \omega$, $i_n(0) < j_n(0) < i_n(1) < j_n(1) < \cdots < i_n(k(n)) < j_n(k(n))$, and $j_n(k(n)) < i_{n+1}(0)$ such that:

$$p_1 \Vdash_P \text{`` for every } n < \omega \text{ for some} \ell \le k(n) \text{ we have } \underset{\sim}{h}(i_n(\ell)) = \underset{\sim}{h}(j_n(\ell))\text{''}$$

Now define the following P-names:

$$\underset{\sim}{A}_n = \{m < \omega : \text{ for some } \ell \le \underset{\sim}{k}(n),\ \underset{\sim}{r}_{i_n(\ell)}(m) = \underset{\sim}{r}_{j_n(\ell)}(m)\}.$$

We can conclude as in the proofs of VI 4.7, $\square_{2.10}$

Claim 2.11. *In $\mathbf{V}^{\mathbb{Q}}$, F_0 still generates an ultrafilter.*

Proof. If not, then for some $T_0 \in \mathbb{Q}$, and $\mathbb{Q}$-name $\underset{\sim}{A}$ we have $T_0 \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{A} \subseteq \omega$ and $\underset{\sim}{A}, \omega \setminus \underset{\sim}{A}$ are $\neq \emptyset \mod F_0$".

By the proof of Claim 2.9 without loss of generality, for some $A_0 \in F_0$ we have: for $k \in A_0$ and $\eta \in T_0^{[k+1]}$, $(T_0)_{[\eta]}$ forces a truth value to "$k \in \underset{\sim}{A}$" which we call $\mathbf{t}(T_0, \eta)$; without loss of generality for $\eta \in T_0^{[k]}, k \notin A_0 \Rightarrow |\mathrm{suc}_{T_0}(\eta)| = 1$.

Now for every $T \geq T_0$ and $\ell < \omega$ there are $A(T,\ell), B(T,\ell)$ as in Definition 2.7. For every $\ell < \omega$, $T \geq T_0$ and $k \in A(T,\ell)$ fix an arbitrary $\eta(T,\ell,k) \in T^{[k]}$.

Then, by Observation 2.12 below, there are $m_{T,\ell,k} \in I(k) \cap B(T,\ell)$ and a partition $\langle u_i(T,\ell,k) \colon i < 3\rangle$ of $I(k) \cap B(T,\ell)$ and a triple $\langle \mathbf{t}_i(T,\ell,k) \colon i < 3\rangle$ of truth values and $j_k(T,\ell) \in \{0,1\}$ and truth value $\mathrm{bs}_k(T,\ell)$ such that:

$(*)$ (a) if $j_k(T,\ell) = 0$ then for $i < 3$, for every $\rho \in {}^{u_i(T,\ell,k)\times\ell}2$ there is $\nu \in {}^{I(k)\times k}2$ such that $\rho \subseteq \nu$ and $\eta(T,\ell,k)^\frown\langle\nu\rangle \in T$ and

$$T_{[\eta^\frown\langle\nu\rangle]} \Vdash_{\mathbb{Q}} \text{“}k \in \underset{\sim}{A} \text{ iff } \mathbf{t}_i(T,\ell,k)\text{”}.$$

(Clearly $\mathbf{t}_i(T,\ell,k) = \mathbf{t}(T_0, \eta^\frown\langle\nu\rangle)$),

(b) if $j_k(T,\ell) = 1$ then for every $\rho \in {}^{(I(k)\cap B(T,\ell)\setminus\{m_{T,\ell,k}\})\times\ell}2$ there is $\nu \in {}^{(I(k)\times k)}2$ such that: $\rho \subseteq \nu$ and $(\eta(T,\ell,k))^\frown\langle\nu\rangle \in T$ and $T_{[\eta^\frown\langle\nu\rangle]} \Vdash_{\mathbb{Q}}$ "$k \in \underset{\sim}{A}$ iff $\mathrm{bs}_k(T,\ell)$".

So for some $j(T,\ell) < 2$ and $i(T,\ell) < 3$ and truth value $\mathbf{t}(T,\ell)$ we have:

(α) if $j(T,\ell) = 0$, then

$$\bigcup\{u_{i(T,\ell)}(T,\ell,k) \colon j_k(T,\ell) = 0,\ k \in A(T,\ell), \mathbf{t}_{i(T,\ell)}(T,\ell,k) = \mathbf{t}(T,\ell) \in F_1.$$

(β) if $j(T,\ell) = 1$ then $\{k \in A(T,\ell) \colon j_k(T,\ell) = 1, \mathrm{bs}_k(T,\ell) = \mathbf{t}(T,\ell)\} \in F_0$.

Note:

$\otimes$ for (T,ℓ) as above there are $A = A^*(T,\ell) \in F_0$, $B = B^*(T,\ell) \in F_1$ satisfying: for every $k \in A$ there is $\eta \in T$, $\lg(\eta) = k$ such that: every $\rho \in {}^{((I(k)\cap B)\times\ell)}2$ can be extended to $\nu \in {}^{I(k)\times k}2$ satisfying: $\eta^\frown\langle\nu\rangle \in T$, $T_{[\eta^\frown\langle\nu\rangle]} \Vdash_{\mathbb{Q}}$ "$k \in \underset{\sim}{A}$ iff $\mathbf{t}(T,\ell)$".

[Why? If $j(T,\ell) = 0$ let

$$B = \bigcup\{u_{i(T,\ell)}(T,\ell,k) \colon j_k(T,\ell) = 0,\ k \in A(T,\ell), \mathbf{t}_{i(T,\ell)}(T,\ell,k) = \mathbf{t}(T,\ell)\},$$

and $A = \{k \colon I(k) \cap B \neq \emptyset\}$. Check the demand by clauses $(*)(a)$ and (α) above. So assume $j(T,\ell) = 1$ and let $B = \bigcup\{I(k) \cap B(T,\ell) \setminus \{m_{T,k,\ell}\} \colon k \in A(T,\ell)$ and $j_k(T,\ell) = 1,$ and $\mathrm{bs}_k(T,\ell) = \mathbf{t}(T,\ell)\}$

[why $B \in F_1$? because $F_1 \not\leq_{\mathrm{RK}} F_0$!). Put $A = \{k \colon I_k \cap B \neq \emptyset\}$ and check the demand by clauses $(*)$(b) and (β) above].

Note that we have been dealing with fixed T,ℓ.

As we can increase T_0 without loss of generality: for some truth value $\mathbf{t}^*$ for a dense set of $T' \geq T_0$ for the F_0-majority of $\ell < \omega$ we have and $\mathbf{t}(T',\ell) = \mathbf{t}^*$.

Now we can define a strategy for player I in the game from 4.3. So in the n'th move player I chooses A_n, B_n and player II chooses k_n, w_n; but we let player I play "on the side" also T_n, ℓ_n (chosen in the n'th move) such that:

(A) $T \leq T_n \leq T_{n+1}$, $T_n^{[k_n+1]} = T_{n+1}^{[k_n+1]}$, $\omega > \ell_{n+1} > \ell_n$, and $\mathbf{t}^* = \mathbf{t}((T_n)_{[\eta]}, \ell_n)$ for $n > 0$ and $\eta \in T_n^{[k_n+1]}$.

(B) For every $k \in A_{n+1}$ and $\eta \in T_n^{[k_n+1]}$ there is η_1, $\eta \lhd eq\eta_1 \in T_n^{[k]}$ such that for every $\rho \in {}^{(B_{n+1}\cap I(k))\times \ell_{n+1}}2$ there is ν, $\rho \subseteq \nu$, $\eta_1{}^\frown\langle\nu\rangle \in T_n$, $\mathbf{t}(T_n,\ell_n,k_n) = \mathbf{t}(T_n,\ell_n) = \mathbf{t}^*$, (note T_{n+1} is chosen only after k_{n+1}, w_{n+1} were chosen).

We should prove that player I can carry out his strategy. For stage $n+1$ let $\{\eta_0^n,\dots,\eta_{m(n)}^n\}$ list $T_n^{[k_n+1]}$, so for some $\ell_{n+1} > \ell_n$, for each $\zeta \le m(n)$ there is $T_{n,\zeta} \ge (T_n)_{[\eta_\zeta^n]}$ such that $\mathbf{t}(T_{n,\zeta},\ell_{n+1}) = \mathbf{t}^*$. Let $B_{n+1} = \bigcap_{\zeta\le m(n)} B^*(T_{n,\zeta},\ell_{n+1})$ and $A_{n+1} = \{k \in A_n : k > k_n \text{ and } I(k)\cap B_{n+1} \neq \emptyset\}$.

By clause (B) above, after player II moves, we can choose T_{n+1} as required. As this is a strategy, by Claim 2.5 for some play in which player I uses it he looses. For this play $\{k_n : n<\omega\} \in F_0$, $\bigcup_{n<\omega} w_n \in F_1$, so $T \coloneqq \bigcap_{n<\omega} T_n \in \mathbb{Q}$. By tracing the demands on the $\mathbf{t}$'s:

$\oplus$ for $n<\omega$, $\eta \in T$, $\lg(\eta) = k_n+1$ we have $T_{[\eta]} \Vdash$ "$k_n \in \underset{\sim}{A}$ iff $\mathbf{t}^*$".

We conclude: $T \Vdash$ "$\{k_n : n<\omega\}\cap \underset{\sim}{A}$ is $\emptyset$ or is $\underset{\sim}{A}$" as $\{k_n : n<\omega\} \in F_0$ we get the desired conclusion. $\square_{2.11}$

Observation 2.12. *Suppose $\mathbf{t}$ is a function from $X^* = \prod_{t\in u} A_t$ to $\{0,1\}$, u finite. Then at least one of the following holds:*

(α) *we can find u_i, X_i $(i<3)$ such that:*
  (a) *$\langle u_i i<3\rangle$ is a partition of u,*
  (b) *$X_i \subseteq X^*$,*
  (c) *$\mathbf{t}\restriction X_i$ is constant,*
  (d) *for every $i<3$ and $\rho \in \prod_{t\in u_i} A_t$ there is $\nu \in X_i$, $\rho \subseteq \nu$,*

(β) *for some $x \in u$, there is $X \subseteq X^*$ such that $\mathbf{t}\restriction X$ is constant and for every $\rho \in \prod_{t\in u\setminus\{x\}} A_t$ there is $\nu \in X$, $\rho \subseteq \nu$.*

Proof. Let for $j \in \{0,1\}, P_j = \{v\colon v \subseteq u$ and there is $X \subseteq X^*$ such that $\mathbf{t}\restriction X$ is constantly j and, for every $\rho \in \prod_{t\in v} A_t$ there is $\nu \in X$, $\rho\subseteq\nu\}$. Clearly

(A) $u_1 \in P_j, u_0 \subseteq u_1$ implies $u_0 \in P_j$.

[Why? Same X witnesses this.]

(B) $u_1 \subseteq u$ & $u_1 \notin P_j$ implies $u\setminus u_1 \in P_{1-j}$

[Why? As $u_1 \notin P_j$, for some $\rho \in \prod_{t\in u_1} A_t$ for no $\nu \in \prod_{t\in u\setminus u_1} A_t$ does $\mathbf{t}(\rho\cup\nu) = j$; let $X \coloneqq \{\nu \in \prod_{t\in u} A_t \colon \rho\subseteq\nu\}$, it is as required for $u\setminus u_1$.]

(C) $\emptyset \in P_0 \cup P_1$.

[Why? Trivially.]

Case (i): $P_0\cup P_1$ is not an ideal.

So there are $u_0,u_1 \in P_0\cup P_1$ with $v \coloneqq u_0\cup u_1 \notin P_0\cup P_1$. By (A) without loss of generality $u_0\cap u_1 = \emptyset$. Let $u_2 = u\setminus v$, so $\langle u_0,u_1,u_2\rangle$ is a partition of u. Now by clause (B) we know that $u_2 \in P_0$ (and to P_1) as $v = u\setminus u_2$ does not belong to P_1 (and to P_0). Now we know $u_0,u_1,u_2 \in P_0\cup P_1$, so for some $\langle j_\ell\colon \ell<3\rangle$ we have $u_\ell \in P_{j_\ell}$ for $\ell<3$, and let X_ℓ be a witness. Now check that clause (α) in the conclusion holds.

Case (ii): $P_0\cup P_1$ is an ideal.

If $u \in P_0 \cup P_1$, then $\mathbf{t}$ is constant so conclusion (α) is trivial, so assume not. By (B) above the ideal is a maximal ideal so it is principal (because u is finite), i.e. for some $x \in u$, $u \setminus \{x\} \in P_0 \cup P_1$, $\{x\} \notin P_0 \cup P_1$ so we have finished. (Reflection shows we get more than required in (β): reread the proof of (B)). $\square_{2.12}$

References

[Jec03] Thomas Jech, *Set theory*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003, The third millennium edition, revised and expanded.

[S+a] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F2217.

[S+b] ______, *Tba*, In preparation. Preliminary number: Sh:F1622.

[She] Saharon Shelah, *Nice $\aleph_1$ generated non-P-points, I*, arXiv: 1112.0819.

[She98] ______, *Proper and improper forcing*, 2nd ed., Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1998. MR 1623206

[She20] ______, *On* con($\mathfrak{d}_\lambda > \mathrm{cov}_\lambda$(meagre)), Trans. Amer. Math. Soc. **373** (2020), no. 8, 5351–5369, arXiv: 0904.0817. MR 4127879

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