CATEGORICITY AND SOLVABILITY OF AEC, QUITE HIGHLY SH734

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ABSTRACT. We investigate in ZFC what can be the family of large enough cardinals μ in which an AEC $\mathfrak{K}$ is categorical or even just solvable. We show that for not few cardinals $\lambda < \mu$ there is a superlimit model in $\mathfrak{K}_\lambda$. Moreover, our main result is that we can find a good λ-frame $\mathfrak{s}$, categorical in λ, such that $\mathfrak{K}_\mathfrak{s} \subseteq \mathfrak{K}_\lambda$. We then show how to use [She09e] to get categoricity in every large enough cardinality if $\mathfrak{K}$ has cases of μ-amalgamation for enough μ and $2^\mu < 2^{\mu^{+1}} < \ldots < 2^{\mu^{+n}} \ldots$ for enough μ.

§ 0. INTRODUCTION

The hope which motivates this work is:

Conjecture 0.1. If $\mathfrak{K}$ is an AEC then either for every large enough cardinal μ, $\mathfrak{K}$ is categorical in μ or for every large enough cardinal μ, $\mathfrak{K}$ is not categorical in μ.

Why do we consider this a good dream? See [S$^+$a].

Our main result is 4.10, it says that if $\mathfrak{K}$ is categorical in μ (ignoring few exceptional μ-s) and $\lambda \in [\mathrm{LST}(\mathfrak{K}), \mu)$ has countable cofinality and is a fix point of the sequence of the $\beth_\alpha$-s, (moreover a limit of such cardinals) then there is a superlimit $M \in K_\lambda$ for which $\mathfrak{K}_{[M]} = \mathfrak{K}_\lambda \restriction \{M' : M' \cong M\}$ has the amalgamation property (and a good λ-frame $\mathfrak{s}$ with $\mathfrak{K}_\mathfrak{s} = \mathfrak{K}_{[M]}$). Note that [She09e] seems to give a strong indication that finding good λ-frames is a significant advance. This may be considered an unsatisfactory evidence of an advance, being too much phrased in the work's own terms. So we prove in §5 - §7 that for a restrictive context we make a clear cut advance: assuming amalgamation and enough instances of $2^\lambda < 2^{\lambda^+}$ occurs, much more than the conjecture holds, see [She] on background.

Note that as we try to get results on $\lambda = \beth_\lambda > \mathrm{LST}(\mathfrak{K})$, clearly it does not particularly matter if for $\kappa \in (\mathrm{LST}(\mathfrak{K}), \lambda)$ we use, e.g. $\kappa_1 = \kappa^+$ or $\kappa_1 = \beth_{(2^\kappa)^+}$ $(= \beth_{1,1}(\kappa))$ or even $\beth_{1,7}(\kappa)$.

After 4.10 the next natural step is to show that $\mathfrak{s}_\lambda$ has the better properties dealt with in [She09c], [She09e], see [S$^+$b]. Note that if we strengthen the assumption on μ in §4 (to $\mu = \mu^{<\lambda}$), then it relies on §1 only. Without this we need §2 (hence 5.1(1),(4)).

Originally we have used here categoricity assumptions but lately it seems desirable to use a weaker one: (variants of) solvability. About being solvable, see [She, §4(B)], [SV]. This seems better as it is a candidate for being an "outside" generalization of being superstable (rather than of being categorical).

Here we use solvable when it does not require much change; for more on it see [SV], [S$^+$c] and on material delayed from here see [S$^+$b].

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Note we can systematically use $K^{\mathrm{sc}(\theta)\text{-lin}}$, say with $\theta = \aleph_0$ or $\theta = \mathrm{LST}(\mathfrak{K})$ instead of K^{lin}; see Definition 0.14(8). In several respects this is better, but not enough to make us use it. Also working more it seemed we can get rid of "wide", "wide over", see Definition 0.14(1),(2),(3). If instead proving the existence of a good λ-frame it suffices for us to prove the existence of almost good λ-frame, then the assumption on λ can be somewhat weaker (fixed point instead limit of fix points of the sequence of the $\beth_\alpha$'s). In §7 we sometimes give alternative quotations in [She99a] but do not rely on it.

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We thank Will Boney and Sebastien Vasey for pointing out (in 10.2016) a gap in §2: in the proof of; we quote 2.9, however 2.9 speaks about $\mathbb{L}_{\infty,\theta}$-types whereas we speak on generic such types. However, we can use 5.1 is a stronger way: though the theorem is stated using $\lambda \times \theta_2$ (in $\mathrm{EM}(I^{\mathrm{lim}}_{\theta_2,\lambda\times\theta_2}, \psi)$) really we prove it for any $\zeta \in [\lambda, \lambda^+)$ of cofinality θ_2 as stated explicitly in the beginning of the proof; see details in the proof of 2.15 (also other minor changes were introduced).

Basic knowledge on infinitary logics is assumed, see e.g. [Dic85]; though the reader may just read the definition here in [She, §5] and believe some quoted results.

Notation 0.2. Let $\beth_{0,\alpha}(\lambda) = \beth_\alpha(\lambda) := \lambda + \sum\{\beth_\beta(\lambda) : \beta < \alpha\}$. Let $\beth_{1,\alpha}(\lambda)$ be defined by induction on α: $\beth_{1,0}(\lambda) = \lambda$, for limit β we let $\beth_{1,\beta} = \sum\limits_{\gamma<\beta} \beth_{1,\gamma}$ and $\beth_{1,\beta+1}(\lambda) = \beth_\mu$ where $\mu = (2^{\beth_{1,\beta}(\lambda)})^+$.

Remark 0.3. 1) For our purpose, usually $\beth_{1,\beta+1}(\lambda) = \beth_{\delta(\mu)}$ where $\mu = \beth_{1,\beta}(\lambda)$ suffice, see e.g. [She09g, §1] in particular on $\delta(-)$. Generally $\mu = (\beth_{1,\beta}(\lambda))^+$ is a more natural definition, but:

(A) the difference is not significant, e.g. for α limit we get the same value

(B) our use of omitting types makes our choice more natural.

2) We do not use but it is natural to define $\beth_{\gamma+1,0}(\lambda) = \lambda$, $\beth_{\gamma+1,\beta+1}(\lambda) = \beth_{\gamma,\mu}(\lambda)$ with $\mu = (2^{\beth_{\gamma+1,\beta}(\lambda)})^+$, $\beth_{\gamma+1,\delta}(\lambda) = \sum\limits_{\beta<\delta} \beth_{\gamma+1,\beta}(\lambda)$ and

$$\beth_{\delta,0}(\lambda) = \sup\{\beth_{\gamma,0}(\lambda) : \gamma < \delta\} = \lambda,$$

$\beth_{\delta,\beta+1}(\lambda) = \beth_{\delta,\beta}(\beth_{\delta,\beta}(\lambda))$, $\beth_{\delta,\delta_1} = \sup\{\beth_{\delta,\alpha}(\lambda) : \alpha < \delta_1\}$; this is used, e.g. in [She94, Ch.V].

Definition 0.4. Assume M is a model, $\tau = \tau_M$ is its vocabulary and Δ is a language (or just a set of formulas) in some logic, in the vocabulary τ.

For any set $A \subseteq M$ and set Δ of formulas in the vocabulary τ_M, let $\mathrm{Sfr}^\alpha_\Delta(A, M)$ (which we call the set of formal (Δ, α)-types over A in M)[1] be the set of p such that

(A) p a set of formulas of the form $\varphi(\bar{x}, \bar{a})$ where $\varphi(\bar{x}, \bar{y}) \in \Delta$, $\bar{x} = \langle x_i : i < \alpha\rangle$ and $\bar{a} \in {}^{\ell g(\bar{y})}A$

(B) if Δ is closed under negation (which is the case we use here) then for any $\varphi(\bar{x}, \bar{y}) \in \Delta$ with $\bar{x}$ as above and $\bar{a} \in {}^{\ell g(\bar{y})}A$ we have $\varphi(\bar{x}, \bar{a}) \in p$ or $\neg\varphi(\bar{x}, \bar{a}) \in p$.

Recall

[1]And we may omit A if $A = M$.

Definition 0.5. 1) For $\mathfrak{K}$ an AEC we say $M \in \mathfrak{K}_\theta$ is a superlimit (model in $\mathfrak{K}$ or in $\mathfrak{K}_\theta$) when:

(a) M is universal

(b) if δ is a limit ordinal $< \theta^+$ and $\langle M_\alpha : \alpha \leq \delta \rangle$ is $\leq_{\mathfrak{K}_\theta}$-increasing continuous and $\alpha < \delta \Rightarrow M_\alpha \cong M$ then $M_\delta \cong M$ (equivalently,

$$\mathfrak{K}_\theta^{[M]} = \mathfrak{K} \restriction \{N : N \cong M\}$$

is a θ-AEC)

(c) there is N such that $M <_{\mathfrak{K}} N \in \mathfrak{K}_\theta$ and N is isomorphic to M.

2) We say $M \in \mathfrak{K}_\theta$ is locally superlimit when we weaken clause (a) to

$(a)^-$ if $N \in \mathfrak{K}_\theta$ is a $\leq_{\mathfrak{K}}$-extension of M then N can be $\leq_{\mathfrak{K}}$-embedded into M.

3) We say that M is pseudo superlimit when in part (1) clauses (b),(c) hold (but we omit clause (a)); see 0.6(7) below.

3A) For $M \in K_\lambda$ let $\mathfrak{K}_{[M]} = \mathfrak{K}_\lambda^{[M]}$ be $\mathfrak{K} \restriction \{N : N \cong M\}$.

4) In (1) we may say 'globally superlimit'.

Observation 0.6. *Assume ($\mathfrak{K}$ is an AEC and) $\mathfrak{K}_\lambda \neq \emptyset$.*

1) If $\mathfrak{K}$ is categorical in λ and there are $M <_{\mathfrak{K}_\lambda} N$ <u>then</u> every $M \in \mathfrak{K}_\lambda$ is superlimit.

2) If every/some $M \in \mathfrak{K}_\lambda$ is superlimit <u>then</u> every/some $M \in K_\lambda$ is locally superlimit.

3) If every/some $M \in \mathfrak{K}_\lambda$ is locally superlimit <u>then</u> every/some $M \in \mathfrak{K}_\lambda$ is pseudo superlimit.

4) If some $M \in \mathfrak{K}_\lambda$ is superlimit <u>then</u> every locally superlimit $M' \in \mathfrak{K}_\lambda$ is isomorphic to M.

5) If M is superlimit in $\mathfrak{K}$ <u>then</u> M is locally superlimit in $\mathfrak{K}$. If M is locally superlimit in $\mathfrak{K}$, <u>then</u> M is pseudo superlimit in $\mathfrak{K}$. If M is locally superlimit in $\mathfrak{K}_\theta$ <u>then</u> $\mathfrak{K}_\theta$ has the joint embedding property <u>iff</u> M is superlimit.

6) In Definition 0.5(1), clause (c) follows from

$(c)^-$ $LST(\mathfrak{K}) \leq \theta$ and $K_{\geq \theta^+} \neq \emptyset$.

7) $M \in K_\lambda$ is pseudo-superlimit <u>iff</u> $\mathfrak{K}_{[M]}$ is a λ-AEC and $\leq_{\mathfrak{K}_{[M]}}$ is not the equality. Also Definition 0.5(3A) is compatible with [She09c, 0.33].

Definition 0.7. For an AEC $\mathfrak{K}$, let $\mathfrak{K}_\mu^{\rm sl}, \mathfrak{K}_\mu^{\rm ls}, \mathfrak{K}_\mu^{\rm pl}$ be the class of $M \in \mathfrak{K}_\mu$ which are superlimit, locally superlimit, pseudo superlimit respectively with the partial orders $\leq_{\mathfrak{K}_\mu^{\rm sl}}, \leq_{\mathfrak{K}_\mu^{\rm ls}}, \leq_{\mathfrak{K}_\mu^{\rm pl}}$ being $\leq_{\mathfrak{K}} \restriction K_\mu^{\rm sl}, \leq_{\mathfrak{K}} \restriction K_\mu^{\rm pl}$ respectively.

Definition 0.8. 1) Φ is proper for linear orders <u>when</u>:

(A) for some vocabulary $\tau = \tau_\Phi = \tau(\Phi), \Phi$ is an ω-sequence, the $n^{\rm th}$ element a complete quantifier free n-type in the vocabulary τ

(B) for every linear order I there is a τ-model M denoted by $\mathrm{EM}(I, \Phi)$, generated by $\{a_t : t \in I\}$ such that $s \neq t \Rightarrow a_s \neq a_t$ for $s, t \in I$ and $\langle a_{t_0}, \ldots, a_{t_{n-1}} \rangle$ realizes the quantifier free n-type from clause (a) whenever $n < \omega$ and $t_0 <_I \ldots <_I t_{n-1}$; so really M is determined only up to isomorphism but we may ignore this and use $I_1 \subseteq J_1 \Rightarrow \mathrm{EM}(I_1, \Phi) \subseteq \mathrm{EM}(I_2, \Phi)$. We call $\langle a_t : t \in I \rangle$ "the" skeleton of M; of course again "the" is an abuse of notation as it is not necessarily unique.

1A) If $\tau \subseteq \tau(\Phi)$ then we let $\mathrm{EM}_\tau(I,\Phi)$ be the τ-reduct of $\mathrm{EM}(I,\Phi)$.

2) $\Upsilon^{\rm or}_\kappa[\mathfrak{K}]$ is the class of Φ proper for linear orders satisfying clauses $(a)(\alpha),(b),(c)$ of Claim 0.9(1) below and $|\tau(\Phi)| \le \kappa$. The default value of κ is $\mathrm{LST}(\mathfrak{K})$ and then we may write $\Upsilon^{\rm or}_{\mathfrak{K}}$ or $\Upsilon^{\rm or}[\mathfrak{K}]$ and for simplicity always $\kappa \ge \mathrm{LST}(\mathfrak{K})$ (and so $\kappa \ge |\tau_{\mathfrak{K}}|$).

3) We define "Φ proper for K" similarly when in clause (b) of part (1) we demand $I \in K$, so K is a class of τ_K-models, i.e.

(a) Φ is a function, giving for a quantifier free n-type in τ_K, a quantifier free n-type in τ_Φ

(b)$'$ in clause (b) of part (1), the quantifier free type which $\langle a_{t_0},\dots,a_{t_{n-1}}\rangle$ realizes in M is $\Phi(\mathrm{tp}_{\rm qf}(\langle t_0,\dots,t_{n-1}\rangle,\varnothing,M))$ for $n<\omega$, $t_0,\dots,t_{n-1} \in I$.

Claim 0.9. *1) Let $\mathfrak{K}$ be an AEC and $M \in K$ be of cardinality $\ge \beth_{1,1}(\mathrm{LST}(\mathfrak{K}))$ recalling we naturally assume $|\tau_{\mathfrak{K}}| \le \mathrm{LST}(\mathfrak{K})$ as usual.*

Then there is a Φ such that Φ is proper for linear orders and:

(a) (α) $\tau_{\mathfrak{K}} \subseteq \tau_\Phi$,
(β) $|\tau_\Phi| = \mathrm{LST}(\mathfrak{K}) + |\tau_{\mathfrak{K}}|$

(b) for any linear order I the model $\mathrm{EM}(I,\Phi)$ has cardinality $|\tau(\Phi)|+|I|$ and we have $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi) \in K$

(c) for any linear orders $I \subseteq J$ we have $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi) \le_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(J,\Phi)$

(d) for every finite linear order I, the model $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi)$ can be $\le_{\mathfrak{K}}$-embedded into M.

2) If we allow $\mathrm{LST}(\mathfrak{K}) < |\tau_{\mathfrak{K}}|$ and there is $M \in \mathfrak{K}$ of cardinality $\ge \beth_{1,1}(\mathrm{LST}(\mathfrak{K}) + |\tau_{\mathfrak{K}}|)$, <u>then</u> there is $\Phi \in \Upsilon^{\rm or}_{\mathrm{LST}(\mathfrak{K})+|\tau(\Phi)|}[\mathfrak{K}]$ such that $\mathrm{EM}(I,\Phi)$ has cardinality $\le \mathrm{LST}(\mathfrak{K})$ for I finite. Hence $\mathcal{E}$ has $\le 2^{\mathrm{LST}(\mathfrak{K})}$ equivalence classes where $\mathcal{E} = \{(P_1,P_2) : P_1,P_2 \in \tau_\Phi$ and $P_1^{\mathrm{EM}(I,\Phi)} = P_2^{\mathrm{EM}(I,\Phi)}$ for every linear order $I\}$.

3) Actually having a model of cardinality $\ge \beth_\alpha$ for every $\alpha < (2^{\mathrm{LST}(\mathfrak{K})+|\tau(\mathfrak{K})|})^+$ suffice (in part (2)).

Proof. Follows from the existence of a representation of $\mathfrak{K}$ as a $\mathrm{PC}_{\mu,2^\mu}$-class when $\mu = \mathrm{LST}(\mathfrak{K}) + |\tau(\mathfrak{K})|$ in [She09a, 1.4(3),(4),(5)] and [She09a, 1.8] (or see [She99a, 0.6]). $\square_{0.9}$

Remark 0.10. Note that some of the definitions and claims below will be used only in remarks: $K^{\mathrm{sc}(\kappa)}_\theta$ from 0.14(8), in 1.7; and some only in §6,§7 (and part of §5 needed for it): $\Upsilon^{\rm lin}_\kappa[2]$ from 0.11(5) (and even less $\Upsilon^{\rm lin}_\kappa[\alpha(*)]$ from Definition 0.14(9)). Also, the use of $\le^{\otimes}_\kappa, \le^{\rm ie}_\kappa, \le^{\oplus}_\kappa$ is marginal.

Definition 0.11. We define partial orders $\le^{\oplus}_\kappa, \le^{\rm ie}_\kappa$ and $\le^{\otimes}_\kappa$ on $\Upsilon^{\rm or}_\kappa[\mathfrak{K}]$ (for $\kappa \ge \mathrm{LST}(\mathfrak{K})$) as follows:

1) $\Psi_1 \le^{\oplus}_\kappa \Psi_2$ <u>if</u> $\tau(\Psi_1) \subseteq \tau(\Psi_2)$ and $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Psi_1) \le_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(I,\Psi_2)$ and $\mathrm{EM}(I,\Psi_1) = \mathrm{EM}_{\tau(\Psi_1)}(I,\Psi_1) \subseteq \mathrm{EM}_{\tau(\Psi_1)}(I,\Psi_2)$ for any linear order I.

Again for $\kappa = \mathrm{LST}(\mathfrak{K})$ we may drop the κ.

2) For $\Phi_1,\Phi_2 \in \Upsilon^{\rm or}_\kappa[\mathfrak{K}]$, we say Φ_2 is an inessential extension of Φ_1 and write $\Phi_1 \le^{\rm ie}_\kappa \Phi_2$ <u>if</u> $\Phi_1 \le^{\oplus}_\kappa \Phi_2$ and for every linear order I, we have (note: there may be more function symbols in $\tau(\Phi_2)$!)

$$\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi_1) = \mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi_2).$$

3) Let $\Upsilon^{\rm lin}_\kappa$ be the class of Ψ proper for linear order and (producing a linear order

extending the original one, i.e.) such that:

(A) $\tau(\Psi)$ has cardinality $\le \kappa$ and the two-place predicate $<$ belongs to $\tau(\Psi)$

(B) $\mathrm{EM}_{\{<\}}(I,\Psi)$ is a linear order which is an extension of I in the sense that $\mathrm{EM}(I,\Phi) \models$ "$a_s < a_t$" iff $I \models$ "$s < t$"; in fact we usually stipulate $[t \in I \Rightarrow a_t = t]$.

4) $\Phi_1 \le^{\otimes}_{\kappa} \Phi_2$ iff there is Ψ such that

(A) $\Psi \in \Upsilon^{\rm lin}_\kappa$

(B) $\Phi_\ell \in \Upsilon^{\rm or}_\kappa[\mathfrak{K}]$ for $\ell = 1,2$

(C) $\Phi'_2 \le^{\rm ie}_\kappa \Phi_2$ where $\Phi'_2 = \Psi \circ \Phi_1$, i.e. for every linear order I we have

$$\mathrm{EM}(I,\Phi'_2) = \mathrm{EM}(\mathrm{EM}_{\{<\}}(I,\Psi),\Phi_1).$$

5) $\Upsilon^{\rm lin}_\kappa[2]$ is the class of Ψ proper for $K^{\rm lin}_{\tau^*_2}$ and producing structures from $K^{\rm lin}_{\tau^*_2}$ extending the originals, i.e.

(A) $\tau^*_2 = \{<, P_0, P_1\}$ where P_0, P_1 are unary predicates, $<$ a binary predicate

(B) $K^{\rm lin}_{\tau^*_2} = \{M : M \text{ a } \tau^*_2\text{-model}, <^M \text{ a linear order}, \langle P^M_0, P^M_1\rangle \text{ a partition of } M\}$

(C) the two-place predicate $<$ and the one place predicates P_0, P_1 belong to $\tau(\Psi)$

(D) if $I \in K^{\rm lin}_{\tau^*_2}$ then $M = \mathrm{EM}_{\tau^*_2}(I,\Phi)$ belongs to $K^{\rm lin}_{\tau^*_2}$, $<^M$ is a linear order, $I \models s < t \Rightarrow M \models a_s < a_t$, and $t \in P^I_\ell \Rightarrow a_\ell \in P^M_\ell$.

6) Similarly $\Upsilon^{\rm lin}_\kappa[\alpha(*)]$ using $K^{\rm lin}_{\tau^*_{\alpha(*)}}$ (see below in 0.14(9)).

Claim 0.12. *Assume $\Phi \in \Upsilon^{\rm or}_{\mathfrak{K}}$.*

1) If π is an isomorphism from the linear order I_1 onto the linear order I_2 then it induces a unique isomorphism $\hat{\pi}$ from $M_1 = \mathrm{EM}(I_1,\Phi)$ onto $M_2 = \mathrm{EM}(I_2,\Phi)$ such that:

(A) $\hat{\pi}(a_t) = a_{\pi(t)}$ for $t \in I$

(B) $\hat{\pi}(\sigma^{M_1}(a_{t_0},\dots,a_{t_{n-1}})) = \sigma^{M_2}(a_{\pi(t_0)},\dots,a_{\pi(t_{n-1})})$ where $\sigma(x_0,\dots,x_{n-1})$ is a τ_Φ-term and $t_0,\dots,t_{n-1} \in I_1$.

2) If π is an automorphism of the linear order I then it induces a unique automorphism $\hat{\pi}$ of $\mathrm{EM}(I,\Phi)$ (as above with $I_1 = I = I_2$).

Remark 0.13. 1) So in 0.11(2) we allow further expansion by functions definable from earlier ones (composition or even definition by cases), as long as the number is $\le \kappa$.

2) Of course, in 0.12 is true for trivial $\mathfrak{K}$.

So we may be interested in some classes of linear orders; below 0.14(1) is used much more than the others and also 0.14(5),(6) are used not so few times, in particular parts (8),(9) are not used till §5.

Definition 0.14. 1) A linear order I is κ-wide when for every $\theta < \kappa$ there is a monotonic sequence of length θ^+ in I.

2) A linear order I is κ-wider if $|I| \ge \beth_{1,1}(\kappa)$.

3) I_2 is κ-wide over I_1 if $I_1 \subseteq I_2$ and for every $\theta < \kappa$ there is a convex subset of I_2 disjoint to I_1 which is θ^+-wide. We say "I_2 is wide over I_1" if "I_2 is $|I_1|$-wide over I_2".

4) $K^{\rm lin}[K^{\rm lin}_\lambda]$ is the class of linear orders [of cardinality λ].

5) Let $K^{\rm flin}$ be the class of infinite linear order I such that every interval has cardinality $|I|$ and is with neither first nor last elements.

6) Let the two-place relation $\leq_{K^{\rm flin}}$ on $K^{\rm flin}$ be defined by: $I \leq_{K^{\rm flin}} J$ iff $I, J \in K^{\rm flin}$ and $I \subseteq J$ and either $I = J$ or $J \setminus I$ is a dense subset of J and for every $t \in J \setminus I$, I can be embedded into $J \restriction \{s \in J \setminus I : (\forall r \in I)(s <_J r \equiv t <_J r)\}$.

6A) Let the two-place relation $\leq^*_{K^{\rm flin}}$ on $K^{\rm lin}$ be defined similarly omitting "$I \in K^{\rm flin}$" (but not $J \in K^{\rm flin}$).

7) $K^{\rm flin}_\theta = \{I \in K^{\rm flin} : |I| = \theta\}$ and $\leq_{K^{\rm flin}_\theta} = \leq_{K^{\rm flin}} \restriction K^{\rm flin}_\theta$.

8) $K^{{\rm sc}(\kappa)-{\rm lin}}_\theta$ is the class of linear orders of cardinality θ which are the union of $\leq \kappa$ scattered linear orders (recalling I is scattered when there is no $J \subseteq I$ isomorphic to the rationals). If $\kappa = \aleph_0$ we may omit it (i.e. write $K^{\rm sc-lin}_\theta$).

9) Let $\tau^*_{\alpha(*)} = \{<\} \cup \{P_i : i < \alpha(*)\}$, P_i a monadic predicate, $K^{\rm lin}_{\tau^*_{\alpha(*)}} = \{I : I$ a $\tau^*_{\alpha(*)}$-model, $<^I$ a linear order and $\langle P^I_i : i < \alpha(*)\rangle$ a partition of $I\}$. If $\alpha(*) = 1$ we may omit P^I_0, so I is a linear order, so any ordinal can be treated as a member of $K^{\rm lin}_{\tau^*_1}$.

Observation 0.15. *1) If $|I| > 2^\theta$ <u>then</u> I is θ^+-wide.*

2) If $|I| \geq \lambda$ and λ is a strong limit cardinal <u>then</u> I is λ-wide.

3) $(K^{\rm flin}_\theta, \leq_{K^{\rm flin}_\theta})$ almost is a θ-AEC, only smoothness may fail.

4) If $I_1 \in K^{\rm lin}$ <u>then</u> for some $I_2 \in K^{\rm flin}$ we have: $|I_2| = |I_1| + \aleph_0$ and $I_1 \leq^*_{K^{\rm flin}} I_2$; and $(\forall I_0)[I_0 \subseteq I_1 \wedge I_0 \in K^{\rm flin} \Rightarrow I_0 \leq_{K^{\rm flin}} I_2]$.

5) If I_1 is κ-wide and $I_1 <_{K^{\rm flin}} I_2$ <u>then</u> I_2 is κ-wide over I_2.

Remark 0.16. If in the definition of $\leq_{K^{\rm flin}}$ in 0.14(6) we can add

$$\text{“}(\forall t \in I)(\exists t' \in J)\big[t' <_J t \wedge (\forall s \in I)(s <_I t \Rightarrow s <_J t')\big]\text{”}$$

(and its dual, i.e. inverting the order). So we can strengthen 0.14(6) by the demand above.

Proof. 1) By Erdős-Rado Theorem, i.e., by $(2^\theta)^+ \to (\theta^+)^2_2$.

2) Follows by part (1).

3),4),5) Easy. $\square_{0.15}$

Claim 0.17. *1) $(\Upsilon^{\rm or}_{\kappa[\mathfrak{K}]}, \leq^\otimes_\kappa), (\Upsilon^{\rm or}_\kappa[\mathfrak{K}], <^{\rm ie}_\kappa)$ and $(\Upsilon^{\rm or}_{\kappa[\mathfrak{K}]}, \leq^\oplus)$ are partial orders (and $\leq^\otimes_\kappa, \leq^{\rm ie}_\kappa \subseteq \leq^\oplus_\kappa$).*

2) If $\Phi_i \in \Upsilon^{\rm or}_\kappa[\mathfrak{K}]$ and the sequence $\langle \Phi_i : i < \delta\rangle$ is a $\leq^\otimes_\kappa$-increasing sequence, $\delta < \kappa^+$, <u>then</u> it has a $<^\otimes_\kappa$-l.u.b. $\Phi \in \Upsilon^{\rm or}_\kappa[\mathfrak{K}]$, and ${\rm EM}(I, \Phi) = \bigcup\limits_{i<\delta} {\rm EM}(I, \Phi_i)$ for every linear order I, i.e. $\tau(\Phi) = \bigcup\{\tau(\Phi_i) : i < \delta\}$ and for every $j < \delta$ we have ${\rm EM}_{\tau(\Phi_j)}(I, \Phi) = \bigcup\big\{{\rm EM}_{\tau(\Phi_i)}(I, \Phi) : i \in [j, \delta)\big\}$.

3) Similarly for $<^\oplus_\kappa$ and $\leq^{\rm ie}_\kappa$.

4) If $\Phi \in \Upsilon^{\rm lin}_\kappa$ and $I \in K^{\rm lin}$ <u>then</u> $I \subseteq {\rm EM}_{\{<\}}(I, \Phi)$ as linear orders stipulating (as in 0.11(3)) that $a_t = t$.

Proof. Easy. $\square_{0.17}$

Recall various well known facts on $\mathbb{L}_{\infty,\theta}$.

Claim 0.18. *1) If M, N are τ-models of cardinality λ, $\mathrm{cf}(\lambda) = \aleph_0$ and $M \equiv_{\mathbb{L}_{\infty,\lambda}} N$ then $M \cong N$.*

2) If M, N are τ-models then $M \equiv_{\mathbb{L}_{\infty,\theta}} N$ iff there is $\mathcal{F}$ such that

$\circledast$ *(a) (α) each $f \in \mathcal{F}$ is a partial isomorphism from M to N*
(β) $\mathcal{F} \neq \varnothing$
(γ) if $f \in \mathcal{F}$ and $A \subseteq \mathrm{dom}(f)$ then $f \restriction A \in \mathcal{F}$
(b) if $f \in \mathcal{F}, A \in [M]^{<\theta}$ and $B \in [N]^{<\theta}$ then for some $g \in \mathcal{F}$ we have $f \subseteq g$, $A \subseteq \mathrm{dom}(g)$, $B \subseteq \mathrm{rang}(g)$.

2A) If $M \subseteq N$ are τ-models, then $M \prec_{\mathbb{L}_{\infty,\theta}} N$ iff for some $\mathcal{F}$ clauses $\circledast(a),(b)$ hold together with

(c) if $A \in [M]^{<\theta}$ then for some $f \in \mathcal{F}$ we have $\mathrm{id}_A \subseteq f$.

2B) In part (2) (and part (2A)), we can omit subclause (γ) of clause (a), and if $\mathcal{F}$ satisfies $(a)(\alpha),(\beta) + (b)$ (and (c)), then also $\mathcal{F}' = \{f \restriction A : f \in \mathcal{F}$ and $A \subseteq \mathrm{dom}(f)\}$ satisfies the demands.

2C) Let M, N be τ-models and define $\mathcal{F} = \{f$: for some $\bar{a} \in {}^{\theta>}M$, f is a function from $\mathrm{rang}(\bar{a})$ to N such that $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty,\theta}} (N, f(\bar{a}))\}$ then $M \equiv_{\mathbb{L}_{\infty,\theta}} N$ iff $\mathcal{F} \neq \varnothing$ iff $\mathcal{F}$ satisfies clauses (a),(b) of $\circledast$.

3) If M is a τ-model, $\theta = \mathrm{cf}(\theta)$ and $\mu = \|M\|^{<\theta}$ then for some $\gamma < \mu^+$ and $\Delta \subseteq \mathbb{L}_{\mu^+,\theta}(\tau)$ of cardinality $\leq \mu$ such that each $\varphi(\bar{x}) \in \Delta$ is of quantifier depth $< \gamma$, we have

(A) for $\bar{a}, \bar{b} \in {}^{\theta>}M$ we have $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty,\theta}} (M, \bar{b})$ iff $\mathrm{tp}_\Delta(\bar{a}, \varnothing, M) = \mathrm{tp}_\Delta(\bar{a}, \varnothing, M)$
(B) for any τ-model N we have $N \equiv_{\mathbb{L}_{\infty,\theta}} M$ iff $\{\mathrm{tp}_\Delta(\bar{a}, \varnothing, N) : \bar{a} \in {}^{\theta>}N\} = \{\mathrm{tp}_\Delta(\bar{a}, \varnothing, M) : \bar{a} \in {}^{\theta>}M\}$.

4) Assume $\chi > \mu = \mu^{<\kappa}$ and $x \in \mathcal{H}(\chi)$. There is $\mathfrak{B}$ such that (in fact clauses (d)-(g) follow from clauses (a),(b),(c))

(a) $\mathfrak{B} \prec (\mathcal{H}(\chi), \in)$ has cardinality μ,
(b) $\mu + 1 \subseteq \mathfrak{B}$ and $[\mathfrak{B}]^{<\kappa} \subseteq \mathfrak{B}$ and $x \in \mathfrak{B}$
(c) $\mathfrak{B} \prec_{\mathbb{L}_{\kappa,\kappa}} (\mathcal{H}(\chi), \in)$
(d) if $\mathfrak{K}$ is an AEC with $\mathrm{LST}(\mathfrak{K}) + |\tau(\mathfrak{K})| \leq \mu$ and $\mathfrak{K} \in \mathfrak{B}$ (which means $\{(M,N) : M \leq_{\mathfrak{K}} N$ has universes $\subseteq \mathrm{LST}(\mathfrak{K})\} \in \mathfrak{B}$) then
(α) $M \in \mathfrak{K} \cap \mathfrak{B} \Rightarrow M \restriction \mathfrak{B} := M \restriction (\mathfrak{B} \cap M) \leq_{\mathfrak{K}} M$
(β) if $M \leq_{\mathfrak{K}} N$ belongs to $\mathfrak{B}$ then $M \restriction \mathfrak{B} \leq_{\mathfrak{K}} N \restriction \mathfrak{B}$
(e) if $\mathfrak{K}$ is as in (d), $\Phi \in \Upsilon^{\mathrm{or}}_{\leq\mu}[\mathfrak{K}] \cap \mathfrak{B}$ and $I \in \mathfrak{B}$ is a linear order and so $M = \mathrm{EM}(I, \Phi) \in \mathfrak{B}$ then $I' = I \restriction \mathfrak{B} \subseteq I$ and $M \restriction \mathfrak{B} = \mathrm{EM}(I', \Phi)$ so $(M \restriction \tau(\mathfrak{K})) \restriction \mathfrak{B} = \mathrm{EM}_{\tau(\mathfrak{K})}(I', \Phi) \leq_{\mathfrak{K}} M \restriction \tau(\mathfrak{K})$
(f) if $|\tau| \leq \mu, \tau \in \mathfrak{B}$ and $M, N \in \mathfrak{B}$ are τ-models, then
(α) $M \restriction \mathfrak{B} \prec_{\mathbb{L}_{\kappa,\kappa}[\tau]} M$
(β) $M \not\equiv_{\mathbb{L}_{\infty,\kappa}[\tau]} N$ iff $(M \restriction \mathfrak{B}) \not\equiv_{\mathbb{L}_{\infty,\kappa}[\tau]} (N \restriction \mathfrak{B})$
(γ) if $M \subseteq N$ then $(M \prec_{\mathbb{L}_{\infty,\kappa}(\tau)} N)$ iff $(M \restriction \mathfrak{B}) \prec_{\mathbb{L}_{\infty,\kappa}(\tau)} (N \restriction \mathfrak{B})$; this applies also to $(M, \bar{a}), (N, \bar{a})$ for $\bar{a} \in {}^{\kappa>}M$
(g) if $I \in K^{\mathrm{flin}}$ then $I_1 \cap \mathfrak{B} \in K^{\mathrm{flin}}$ and if $I_1 <^*_{K^{\mathrm{flin}}} I_2$ then $(I_1 \cap \mathfrak{B}) <^*_{K^{\mathrm{flin}}} (I_2 \cap \mathfrak{B})$.

Proof. 1)-3) and 4)(a),(b),(c) Well known, e.g. see [Dic85].

4) Clauses (d),(e),(f): as in 0.9(1), i.e. by absoluteness. Also clause (g) should be clear. $\square_{0.18}$

Remark 0.19. 1) We will be able to add, in 0.18(4):

(h) if $\mathfrak{K}$ is as in clause (d) and $\tau = \tau_{\mathfrak{K}}$ then in clause (f) we can replace $\mathbb{L}_{\infty,\kappa}(\tau)$ by $\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$ and $\mathbb{L}_{\kappa,\kappa}(\tau)$ by $\mathbb{L}_{\kappa,\kappa}[\mathfrak{K}]$, see Definition 1.10 and Fact 1.11(5).

2) We use part (4) in 1.27(3).

Definition 0.20. For a model M and for a set Δ of formulas in the vocabulary of M, $\bar{x} = \langle x_i : i < \alpha \rangle$, $A \subseteq M$ and $\bar{a} \in {}^{\alpha}M$, let the Δ-type of $\bar{a}$ over A in M be

$$\mathrm{tp}_\Delta(\bar{a}, A, M) = \{\varphi(\bar{x}, \bar{b}) : M \models \varphi[\bar{a}, \bar{b}] \text{ where } \varphi = \varphi(\bar{x}, \bar{y}) \in \Delta \text{ and } \bar{b} \in {}^{\ell g(\bar{y})}A\}.$$

§ 1. §1 Amalgamation in K^*_λ

Our aim is to investigate what is implied by 1.3 below but instead of assuming it we shall shortly assume only some of its consequences. For our purpose here, for $\theta \in [\mathrm{LST}(\mathfrak{K}), \lambda), \lambda = \beth_\lambda$ it does not really matter if we use $\kappa = \beth_{1,1}(\theta)$ or $\kappa = \beth_{1,1}(\beth_n(\theta))$ or $\beth_{1,n}(\theta)$, as we are trying to analyze models in K_λ.

Remark 1.1. 1) We can in our claims use only $\Phi \in \Upsilon^{\mathrm{or}}_{\mathfrak{K}} = \Upsilon^{\mathrm{or}}_{\mathrm{LST}(\mathfrak{K})}[\mathfrak{K}]$ because for every $\theta \geq \mathrm{LST}(\mathfrak{K})$ we can replace $\mathfrak{K}$ by $\mathfrak{K}_{\geq\theta}$ as $\mathrm{LST}(\mathfrak{K}_{\geq\theta}) = \theta$ when $\mathfrak{K}_{\geq\theta} \neq \emptyset$, of course.

2) As usual we assume $|\tau_{\mathfrak{K}}| \leq \mathrm{LST}(\mathfrak{K})$ just for convenience, otherwise we should just replace $\mathrm{LST}(\mathfrak{K})$ by $\mathrm{LST}(\mathfrak{K}) + |\tau_{\mathfrak{K}}|$.

Hypothesis 1.2. (A) $\mathfrak{K} = (K, \leq_{\mathfrak{K}})$ is an AEC with vocabulary $\tau = \tau(\mathfrak{K})$ (and we can assume $|\tau| \leq \mathrm{LST}(\mathfrak{K})$ for notational simplicity)

(B) $\mathfrak{K}$ has arbitrarily large models (equivalently has a model of cardinality $\geq \beth_{1,1}(\mathrm{LST}(\mathfrak{K}))$), not used, e.g. in 1.11, 1.12 but from 1.13 on it is used extensively.

Definition 1.3. We say (μ, λ) or really (μ, λ, Φ) is a weak/strong/pseudo $\mathfrak{K}$-candidate when (weak is the default value):

(a) $\mu > \lambda = \beth_\lambda > \mathrm{LST}(\mathfrak{K})$ (e.g. the first beth fix point $> \mathrm{LST}(\mathfrak{K})$, see 3.4; in the main case λ has cofinality $\aleph_0$)

(b) $\mathfrak{K}$ categorical in μ and $\Phi \in \Upsilon^{\mathrm{or}}_{\mathfrak{K}}$
   or just

(b)$^-$ $\mathfrak{K}$ is weakly/strongly/pseudo solvable in μ and $\Phi \in \Upsilon^{\mathrm{or}}_{\mathfrak{K}}$ witnesses it; see below.

Definition 1.4. 1) We say $\mathfrak{K}$ is weakly (μ, κ)-solvable <u>when</u> $\mu \geq \kappa \geq \mathrm{LST}(\mathfrak{K})$ and there is $\Phi \in \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$ witnessing it, which means that $\Phi \in \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$ and $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ is a locally superlimit member of $\mathfrak{K}_\mu$ for every linear order I of cardinality μ. We may say $(\mathfrak{K}, \Phi)$ is weakly (μ, κ)-solvable and we may say Φ witness that $\mathfrak{K}$ is weakly (μ, κ)-solvable.

If $\kappa = \mathrm{LST}(\mathfrak{K})$ we may omit it, saying $\mathfrak{K}$ or $(\mathfrak{K}, \Phi)$ is weakly μ-solvable in μ.

2) $\mathfrak{K}$ is strongly (μ, κ)-solvable <u>when</u> $\mu \geq \kappa \geq \mathrm{LST}(\mathfrak{K})$ and some $\Phi \in \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$ witness it which means that if $I \in K^{\mathrm{lin}}_\mu$ then $\mathrm{EM}_{\tau[\mathfrak{K}]}(I, \Phi)$ is superlimit (for $\mathfrak{K}_\mu$). We use the conventions from part (1).

3) We say $\mathfrak{K}$ is pseudo (μ, κ)-solvable when $\mu \geq \kappa \geq \mathrm{LST}(\mathfrak{K})$ and there is $\Phi \in \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$ witnessing it which means that for some μ-AEC $\mathfrak{K}'$ with no $\leq_{\mathfrak{K}'}$-maximal member, we have $M \in \mathfrak{K}'$ <u>iff</u> $M \cong \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ for some $I \in K^{\mathrm{lin}}_\mu$ iff $M \cong \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ for every $I \in K^{\mathrm{lin}}_\mu$. We use the conventions from part (1).

4) Let (μ, κ)-solvable mean weakly (μ, κ)-solvable, etc., (including 1.3)

Claim 1.5. *1) In Definition 1.3, clause (b) implies clause (b)$^-$. Also in Definition 1.4 "$\mathfrak{K}$ is strongly (μ, κ)-solvable" implies "$\mathfrak{K}$ is weakly (μ, κ)-solvable" which implies "$\mathfrak{K}$ is pseudo (μ, κ)-solvable". Similarly for $(\mathfrak{K}, \Phi)$.*

2) Assume $\Phi \in \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$; if clause (b)$^-$ of 1.3 or just $\dot{I}(\mu, \mathfrak{K}) < 2^\mu$, or just $2^\mu > \dot{I}(\mu, \{\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi) : I \in K^{\mathrm{lin}}_\mu\})$ for some μ satisfying $\mathrm{LST}(\mathfrak{K}) < \kappa^+ < \mu$ <u>then</u> we can deduce that

($*$) *Φ (really $(\mathfrak{K}, \Phi)$) has the κ-non-order property, where the κ-non-order property means that:*

if I is a linear order of cardinality $\kappa, \bar{t}^1, \bar{t}^2 \in {}^{\kappa}I$ form a Δ-system pair (see below) and $\langle \sigma_i(\bar{x}) : i < \kappa \rangle$ lists the $\tau(\Phi)$-terms (with the sequence $\bar{x}$ of variables being $\langle x_i : i < \kappa \rangle$) and $\langle a_t : t \in I \rangle$ is "the" indiscernible sequence generating $\mathrm{EM}(I, \Phi)$ (i.e. as usual $\langle a_t : t \in I \rangle$ is "the" skeleton of $\mathrm{EM}(I, \Phi)$, so generating it, see Definition 0.8) then for some $J \supseteq I$ there is an automorphism of $\mathrm{EM}_{\tau(\mathfrak{K})}(J, \Phi)$ which exchanges $\langle \sigma_i(\langle a_{t^1_i} : i < \kappa \rangle) : i < \kappa \rangle$ and $\langle \sigma_i(\langle a_{t^2_i} : i < \kappa \rangle) : i < \kappa \rangle$.

where

$\boxtimes$ *$\bar{t}^1, \bar{t}^2 \in {}^{\alpha}I$ is a Δ-system pair when for some $J \supseteq I$ there are $\bar{t}^\zeta \in {}^{\alpha}J$ for $\zeta \in \kappa \setminus \{1, 2\}$ such that $\langle \bar{t}^\alpha : \alpha < \kappa \rangle$ is an indiscernible sequence for quantifier free formulas in the linear order J.*

Proof. 1) The first sentence holds by Claim 0.9(1) and Definition 0.8 (and Claim 0.6). The second and third sentences follows by 0.6.

2) Otherwise we get a contradiction by [She87b, Ch.III] or better [Shear, III]. $\square_{1.4}$

Definition 1.6. 1) If $\mathcal{M}'$ is a class of linear orders and $\Phi \in \Upsilon^{\mathrm{or}}_{\kappa}[\mathfrak{K}]$ then we let $K[\mathcal{M}', \Phi] = \{\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi) : I \in \mathcal{M}'\}$.

2) Let $K^{u(\kappa)\text{-lin}}_{\theta}$ be the class of linear orders I of cardinality θ such that for some scattered[2] linear order J and Φ proper for K^{lin} such that $<$ belongs to τ_Φ and $|\tau_\Phi| \le \kappa$ we have I is embeddable into $\mathrm{EM}_{\{<\}}(J, \Phi)$. If we omit κ we mean $\mathrm{LST}(\mathfrak{K})$. If $\kappa = \aleph_0$ we may omit it.

Remark 1.7. 1) Note that in Definition 1.4(1) we can restrict ourselves to $I \in K^{\mathrm{sc}(\theta)\text{-lin}}_{\lambda}$, see 0.14(8) and even $I \in K^{u(\theta)\text{-lin}}$ see 1.6(2), i.e., assume $2^\mu > \dot{I}(\mu, K[\mathcal{M}', \Phi])$, for $\mathcal{M}' = K^{\mathrm{sc}(\theta)\text{-lin}}_{\lambda}$ or $\mathcal{M}' = K^{u(\theta)\text{-lin}}_{\lambda}$ and restrict the conclusion ($*$) to $I \in K^{\mathrm{sc}(\theta)\text{-lin}}$. A gain is that, if $\lambda > \theta$, every $I \in K^{\mathrm{sc}(\theta)\text{-lin}}_{\lambda}$ is λ-wide so later $K^* = K^{**}$, and being solvable is a weaker demand. But it is less natural. Anyhow we presently do not deal with this.

1A) Note that $K^{\mathrm{sc}(\theta)\text{-lin}}_{\lambda} \subseteq K^{u(\theta)\text{-lin}}_{\lambda}$.

2) An aim of 1.8 below is to show that: by changing Φ instead of assuming $I_1 \subset I_2 \wedge (I_2$ is κ-wide over $I_1)$ it suffices to assume $I_1 \subset I_2 \wedge (I_2$ is κ-wide).

Claim 1.8. *For every $\Phi_1 \in \Upsilon^{\mathrm{or}}_{\kappa}[\mathfrak{K}]$ there is Φ_2 such that*

- *(A) $\Phi_2 \in \Upsilon^{\mathrm{or}}_{\kappa}[\mathfrak{K}]$ and if Φ_1 witnesses $\mathfrak{K}$ is weakly/strongly/pseudo (λ, κ)-solvable then so does Φ_2*
- *(B) $\tau_{\Phi_1} \subseteq \tau_{\Phi_2}$ and $|\tau_{\Phi_2}| = |\tau_{\Phi_1}| + \aleph_0$*
- *(C) for any $I_2 \in K^{\mathrm{lin}}$ there are I_1 and h such that:*
  - *(α) $I_1 \in K^{\mathrm{lin}}$ and even $I_1 \in K^{\mathrm{flin}}$, see 0.14(5)*
  - *(β) h is an embedding of I_2 into I_1*
  - *(γ) there is an isomorphism f from $\mathrm{EM}_{\tau(\Phi_1)}(I_2, \Phi_2)$ onto $\mathrm{EM}(I_1, \Phi_1)$ such that $f(a_t) = a_{h(t)}$ for $t \in I_2$*

[2] i.e. one into which the rational order cannot be embedded

(δ) *if* $J_1 = I_1 \restriction \mathrm{rang}(h)$ *and we let*

$$\mathcal{E} = \{(t_1, t_2) : t_1, t_2 \in I_1 \setminus J_1 \text{ and } (\forall s \in J_1)(s < t_1 \equiv s < t_2)\}$$

then $\mathcal{E}$ *is an equivalence relation, each equivalence class has* $\geq |I_2|$ *members, and* $J_1 \leq_{K^{\mathrm{flin}}} I_1$ *(see 0.14(6)).*

(ε) ***[Not used]*** *if* $\varnothing \neq J_2 \subseteq I_2$,

$$J_1 = \big\{t \in I_1 : \text{for some } \tau(\Phi_2)\text{-term } \sigma(x_0, \dots, x_{n-1}) \text{ and } t_0, \dots, t_{n-1} \in J_2 \text{ we have } f^{-1}(a_t) = \sigma^{\mathrm{EM}(I_2,\Phi_2)}(a_{t_0}, \dots, a_{t_{n-1}})\big\}$$

and $J_1' \subseteq \mathrm{rang}(h) \setminus J_1$ *and* $t \in J_1'$ *then* $\{s \in t/\mathcal{E} : f^{-1}(a_s)$ *belongs to the Skolem hull of* $\{f^{-1}(a_r) : r \in J_1'\}$ *in* $\mathrm{EM}(I_2, \Phi)\}$ *has cardinality* $\geq |J_1'|$ *and* J_1' *and its inverse can be embedded into it; in fact,* I_1 *and its inverse are embeddable into any interval of* I_2.

Remark 1.9. 1) We can express it by $\leq^{\otimes}_{\kappa}$, see 0.11(4). So for some Ψ proper for linear orders such that τ_Ψ is countable, the two-place predicate $<$ belongs to τ_Ψ and above $\mathrm{EM}_{\{<\}}(I_2, \Psi)$ is I_1.

2) In fact, $J_2 \subset I_2 \Rightarrow \mathrm{EM}_{\{<\}}(J_2, \Psi) <_{K^{\mathrm{flin}}} \mathrm{EM}_{\{<\}}(I_2, \Psi)$ and $I_2 <^*_{K^{\mathrm{flin}}} \mathrm{EM}_{\{<\}}(I_2, \Phi)$ when we identify $t \in I_2$ with a_t.

Proof. For $I_2 \in K^{\mathrm{lin}}$ let the set of elements of I_1 be $\{\eta : \eta$ is a finite sequence of elements from $(\mathbb{Z} \setminus \{0\}) \times I_2\}$. For $\eta \in I_1$ let $(\ell_{\eta,k}, t_{\eta,k})$ be $\eta(k)$ for $k < \ell g(\eta)$.

Lastly, I_1 is ordered by: $\eta_1 < \eta_2$ iff for some n one of the following occurs

$\circledast$ (a) $\eta_1 \restriction n = \eta_2 \restriction n$, $\ell g(\eta_1) > n$, $\ell g(\eta_2) > n$, and $\ell_{\eta_1,n} < \ell_{\eta_2,n}$

(b) $\eta_1 \restriction n = \eta_2 \restriction n$, $\ell g(\eta_1) > n$, $\ell g(\eta_2) > n$, $\ell_{\eta_1,n} = \ell_{\eta_2,n} > 0$, and $t_{\eta_1,n} <_{I_2} t_{\eta_2,n}$

(c) $\eta_1 \restriction n = \eta_2 \restriction n$, $\ell g(\eta_1) > n$, $\ell g(\eta_2) > n, \ell_{\eta_1,n} = \ell_{\eta_2,n} < 0$, and $t_{\eta_2,n} <_{I_2} t_{\eta_1,n}$

(d) $\eta_1 \restriction n = \eta_2 \restriction n$, $\ell g(\eta_1) = n$, $\ell g(\eta_2) > n$, and $\ell_{\eta_2,n} > 0$

(e) $\eta_1 \restriction n = \eta_2 \restriction n$, $\ell g(\eta_1) > n$, $\ell g(\eta_2) = n$, and $\ell_{\eta_1,n} < 0$.

We identify $t \in I_1$ with the pair $(1, t)$. Now check. $\square_{1.8}$

Definition 1.10. 1) Let the language $\mathbb{L}_{\theta,\partial}[\mathfrak{K}]$ or $\mathbb{L}_{\theta,\partial,\mathfrak{K}}$ where $\theta \geq \partial \geq \aleph_0$ and θ is possibly ∞, be defined like the infinitary logic $\mathbb{L}_{\theta,\partial}(\tau_{\mathfrak{K}})$, except that we deal only with models from K and we add for $i^* < \partial$ the atomic formula "$\{x_i : i < i^*\}$ is the universe of a $\leq_{\mathfrak{K}}$-submodel", with obvious syntax and semantics. Of course, it is interesting normally only for $\partial > \mathrm{LST}(\mathfrak{K})$ and recall that any formula has $< \partial$ free variables.

2) For M a $\tau_{\mathfrak{K}}$-model and $N \in K$ let $M \prec_{\mathbb{L}_{\theta,\partial}[\mathfrak{K}]} N$ means that $M \subseteq N$ and if $\varphi(\bar{x}, \bar{y})$ is a formula from $\mathbb{L}_{\theta,\partial}[\mathfrak{K}]$ and $N \models (\exists \bar{x})\varphi(\bar{x}, \bar{b})$ where $\bar{b} \in {}^{\ell g(\bar{y})}M$, then for some $\bar{a} \in {}^{\ell g(\bar{x})}M$ we have $N \models \varphi[\bar{a}, \bar{b}]$.

Fact 1.11. 1) If $\theta \geq \partial > \mathrm{LST}(\mathfrak{K})$ and M, N are $\tau_{\mathfrak{K}}$-models and $N \in K$ and $M \prec_{\mathbb{L}_{\theta,\partial}[\mathfrak{K}]} N$, then $M \leq_{\mathfrak{K}} N$ and $M \in K$.

2) The relation $\prec_{\mathbb{L}_{\theta,\partial}[\mathfrak{K}]}$ can also be defined as usual: $M \prec_{\mathbb{L}_{\theta,\partial}[\mathfrak{K}]} N$ iff $M, N \in K, M \subseteq N$ and for every $\varphi(\bar{x}) \in \mathbb{L}_{\theta,\partial}[\mathfrak{K}]$ and $\bar{a} \in {}^{\ell g(\bar{x})}M$ we have $M \models \varphi[\bar{a}]$ iff $N \models \varphi[\bar{a}]$.

3) If $N \in \mathfrak{K}$ and M is a τ_K-model satisfying $M \prec_{\mathbb{L}_{\infty,\kappa}} N$ and $\kappa > \mathrm{LST}(\mathfrak{K})$ then $M \in K, M \leq_{\mathfrak{K}} N$ and $M \prec_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} N$.

4) If $N \in K, M$ a τ_K-model and $M \equiv_{\mathbb{L}_{\infty,\kappa}} N$ where $\kappa > \mathrm{LST}(\mathfrak{K})$ then $M \in K$ and $M \equiv_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} N$.

5) The parallel of 0.18(2) holds for $\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$, i.e. $M \equiv_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} N$ iff there is $\mathcal{F}$ satisfying clauses (a),(b) there and

- (d) if $f \in \mathcal{F}$ then
  - (α) $M \restriction \mathrm{dom}(f) \leq_{\mathfrak{K}} M$
  - (β) $N \restriction \mathrm{rang}(f) \leq_{\mathfrak{K}} M$.

6) Also the parallel of 0.18(2A) holds for $\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$.

7) The parallel of 0.18(4) holds for $\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$.

Proof. Part (1) is straight (knowing [She09a, §1] or [She87a, §1]). Part (2) is proved as in the Tarski-Vaught criterion and parts (5),(6),(7) are proved as in 0.18.

Toward proving parts (3),(4) we first assume just

$\boxtimes_1$ M, N are τ_K-models, $N \in K$ and $M \equiv_{\mathbb{L}_{\infty,\kappa}} N$ and $\kappa > \mathrm{LST}(\mathfrak{K})$ and $\lambda \in [\mathrm{LST}(\mathfrak{K}), \kappa)$

and we define:

$\boxdot$ (a) $I = I_\lambda =$

$$\big\{(f, M', N') : M' \subseteq M,\ N' \subseteq N,\ \|M'\| \leq \lambda,\ f : M' \to N' \text{ is an isomorphism, and } (M, \bar{a}) \equiv_{\mathbb{L}_{\infty,\kappa}} (N, f(\bar{a})),\ \text{where } \bar{a} \text{ lists } M'.\big\}$$

(Note that we do not require $M', N' \in K$.)

- (b) for $t \in I$ let $t = (f_t, M_t, N_t)$
- (c) for $\ell = 0,1,2$ we define the two-place relation $\leq^\ell_I$ on I as follows. Let $s \leq^\ell_I t$ hold iff:
  - (α) $\ell = 0$ and $M_s \subseteq M_t \wedge N_s \subseteq N_t$
  - (β) $\ell = 1$ and $(M_s \leq_{\mathfrak{K}} M_t \vee M_s = M_t) \wedge (N_s \leq_{\mathfrak{K}} N_t \vee N_s = N_t)$
  - (γ) $\ell = 2$ and $f_s \subseteq f_t$
- (d) $I_1 = I^1_\lambda := \{t \in I : N_t \leq_{\mathfrak{K}} N\}$ and let $\leq^\ell_{I_1} = \leq^\ell_I \restriction I_1$ for $\ell = 0,1,2$.

Now easily

$(*)_0$ (α) $I \neq \varnothing$ is partially ordered by $\leq^\ell_I$ for $\ell = 0,1,2$
(β) $s \leq^1_I t \Rightarrow s \leq^0_I t$
(γ) $s \leq^2_I t \Rightarrow s \leq^0_I t$.

[Why? Straightforward; e.g. $I \neq \varnothing$ by 0.18(2).]

$(*)_1$ if $t \in I_1$ then $M_t \in K_{\leq\lambda}$ and $N_t \in K_{\leq\lambda}$ hence for $r, s \in I_2$ we have $r_1 \leq^1_{I_s} s$ iff $M_r \leq_{\mathfrak{K}} M_s \wedge N_r \leq_{\mathfrak{K}} N_s$).

[Why? As $t \in I_1$ by the definition of I we have $N_t \in K_{\leq\lambda}$ (because $N_t \leq_{\mathfrak{K}} N$) and $M_t \in K_{\leq\lambda}$ as f_t is an isomorphism from M_t onto N_t.]

$(*)_2$ if $s \in I, A \in [M]^{\leq\lambda}$ and $B \in [N]^{\leq\lambda}$ then for some t we have $s \leq^2_I t$ and $A \subseteq M_t$ and $B \subseteq N_t$.

[Why? By the properties of $\equiv_{\mathbb{L}_{\infty,\kappa}}$, see 0.18(2C) as $\kappa > \lambda, M \equiv_{\mathbb{L}_{\infty,\kappa}} N$ and the definition of I.]

$(*)_3$ if $s \leq^2_{I_1} t$ then $s \leq^1_I t$, i.e. $M_s \leq_{\mathfrak{K}} M_t$ and $N_s \leq_{\mathfrak{K}} N_t$.

[Why? As $s, t \in I_1$ we know that $N_s \leq_{\mathfrak{K}} N$ and $N_t \leq_{\mathfrak{K}} N$ and as $s \leq^2_I t$ we have $f_s \subseteq f_t$ hence $N_s \subseteq N_t$. By axiom V of AEC it follows that $N_s \leq_{\mathfrak{K}} N_t$. Now $M_s \leq_{\mathfrak{K}} M_t$ as f_t is an isomorphism from M_t onto N_t mapping M_s onto N_s (as it extends f_s by the definition of $\leq^2_I$) and $\leq_{\mathfrak{K}}$ is preserved by any isomorphism. So by the definition of $\leq^1_I$ we are done.]

$(*)_4$ if $s \in I$ <u>then</u> for some $t \in I_1$ we have $s \leq^2_I t$ (hence $I_1 \neq \varnothing$).

[Why? First, choose $N' \leq_{\mathfrak{K}} N$ of cardinality $\leq \lambda$ such that $N_s \subseteq N'$, (possibly by the basic properties of AEC (see [She09a, §1] or [She09f])). Second, we can find $t \in I$ such that $N_t = N' \wedge f_s \subseteq f_t$ by the characterization of $\equiv_{\mathbb{L}_{\infty,\kappa}}$ as in the proof of $(*)_2$. So $s \leq^2_I t$ by the definition of $\leq^2_I$ and $N_t = N' \leq_{\mathfrak{K}} N$ hence $t \in I_1$ as required. Lastly, $I_1 \neq \varnothing$ as by $(*)_0(\alpha)$ we know that $I \neq \varnothing$ and apply what we have just proved.]

$(*)_5$ if $s \leq^0_{I_1} t$ then $N_s \leq_{\mathfrak{K}} N_t$.

[Why? As in the proof of $(*)_3$ by Ax.V of AEC we have $N_s \leq_{\mathfrak{K}} N_t$ (not the part on the M's!)]

$(*)_6$ if $s \in I_1, A \in [M]^{\leq \lambda}$ and $B \in [N]^{\leq \lambda}$ <u>then</u> for some t we have $s \leq^2_{I_1} t$ and $A \subseteq M_t, B \subseteq N_t$.

[Why? By $(*)_2$ there is t_1 such that $s \leq^2_I t_1, A \subseteq M_{t_1}$ and $B \subseteq N_{t_1}$. By $(*)_4$ there is $t \in I_1$ such that $t_1 \leq^2_I t$ hence by $(*)_0(\alpha)$ we have $s \leq^2_I t$. As $s, t \in I_1$ this implies $s \leq^2_{I_1} t$.]

Note that it is unreasonable to have "$(I_1, \leq^2_{I_1})$ is directed" but

$(*)_7$ $(I_1, \leq^1_{I_1})$ is directed.

[Why? Let $s_1, s_2 \in I_1$. We now choose t_n by induction on $n < \omega$ such that

- (a) $t_n \in I_1$
- (b) M_{t_n} includes $\cup\{M_{t_k} : k < n\} \cup M_{s_1} \cup M_{s_2}$ if $n \geq 2$
- (c) N_{t_n} includes $\cup\{N_{t_k} : k < n\} \cup N_{s_1} \cup N_{s_2}$ if $n \geq 2$
- (d) $t_0 = s_1$
- (e) $t_1 = s_2$
- (f) if $n = m + 1 \geq 2$ then $t_m \leq^0_{I_1} t_n$
- (g) if $n = m + 2$ then $t_m \leq^2_I t_n$ hence $t_m \leq^2_{I_1} t_n$.

For $n = 0, 1$ this is trivial. For $n = m + 2 \geq 2$, apply $(*)_6$ with

$$t_m, \ \bigcup\{M_{t_k} : k \leq m+1\}, \ \bigcup\{N_{t_k} : k \leq m+1\}$$

here standing for s, A, B there, getting t_n so we get $t_n \in I_1$. In particular, $t_m \leq^2_{I_1} t_n$, so clause (a) is satisfied by t_n. By the choice of t_n and as $s_1 = t_0$, $s_2 = t_1$, clauses (b) + (c) hold for t_n. By the choice of t_n, obviously also clause (g) holds. Now why does clause (f) holds (i.e. $t_{m+1} \leq^0_I t_n$)? It follows from clauses (a),(b),(c), so t_n is as required. Hence we have carried the induction. Let $N^* = \bigcup\{N_{t_n} : 2 \leq n < \omega\}$, so clearly by $(*)_5$ and clause (f) we have $N_{t_n} \leq_{\mathfrak{K}} N_{t_{n+1}}$ for $n \geq 1$, and clearly $M_{t_n} \subseteq M_{t_{n+1}}$ for $n \geq 1$. Let $M^* = \bigcup\{M_{t_n} : 2 \leq n < \omega\}$. Note that by $(*)_3$ and clause (g) we have $M_{t_n} \leq_{\mathfrak{K}} M_{t_{n+2}}$, so $\langle M_{t_{n+2}} : n < \omega\rangle$ is $\subseteq$-increasing, and for $\ell = 0, 1$ the sequence $\langle M_{t_{2n+\ell}} : n < \omega\rangle$ is $\leq_{\mathfrak{K}}$-increasing with union M^*, hence by the basic properties of AEC we have $M_{t_{2n+\ell}} \leq_{\mathfrak{K}} M^*$. So $M_{s_1} = M_{t_0} \leq_{\mathfrak{K}} M^*$ and $M_{s_2} = M_{t_1} \leq_{\mathfrak{K}} M^*$. Now $M_{s_1}, M_{s_2} \subseteq M_{t_2} \leq_{\mathfrak{K}} M^*$ hence $M_{s_1}, M_{s_2} \leq_{\mathfrak{K}} M_{t_2}$. Recall that $N_{s_1} = N_{t_0} \leq_{\mathfrak{K}} N_{t_2}$ was proved above and $N_{s_2} = N_{t_1} \leq_{\mathfrak{K}} N_{t_2}$ was also proved above so t_2 is a common $\leq^1_I$-upper bound of s_1, s_2 as required.]

$(*)_8$ if $s \leq^0_{I_1} t$ then $s \leq^1_{I_1} t$.

[Why? By $(*)_7$ there is $t_1 \in I_1$ which is a common $\leq^1_{I_1}$-upper bound of s, t. So $M_s \subseteq M_t$ (as $s \leq^0_{I_1} t$) and $M_s \leq_{\mathfrak{K}} M_{t_1}$ (as $s \leq^1_{I_1} t_1$) and $M_t \leq_{\mathfrak{K}} M_{t_1}$ (as $t \leq^1_{I_1} t_1$). Together by axiom V of AEC we get $M_s \leq_{\mathfrak{K}} M_t$ and by $(*)_5$ we have $N_s \leq_{\mathfrak{K}} N_t$. Together $s \leq^1_{I_1} t$ as required.]

$(*)_9$ $\langle M_s : s \in (I_1, \leq^1_{I_1})\rangle$ is $\leq_{\mathfrak{K}}$-increasing, $(I_1, \leq^1_{I_1})$ is directed and

$$\bigcup\{M_s : s \in I_1\} = M.$$

[Why? The first phrase by the definition of $\leq^1_{I_1}$ in clause $(c)(\beta)$ of $\boxdot$, the second by $(*)_7$ and the third by $(*)_6 + (*)_4$.]

By the basic properties of AEC (see [She09a, 1.6]) from $(*)_9$ we deduce

$\odot$ (a) $M \in K$
(b) $t \in I_1 \Rightarrow M_t \leq_{\mathfrak{K}} M$.

Now we strengthen the assumption $\boxtimes_1$ to

$\boxtimes_2$ The demands in $\boxtimes_1$ and $M \prec_{\mathbb{L}_{\infty,\kappa}[\tau_{\mathfrak{K}}]} N$.

We note

$\circledast_1$ (a) If $\bar{a} \in {}^\alpha M$, $|\alpha| + \text{LST}(\mathfrak{K}) \leq \lambda < \kappa$ then for some $t \in I_\lambda$, $f_t(\bar{a}) = \bar{a}$.
(b) If $M' \subseteq M$ and $\|M'\| \leq \lambda$ <u>then</u> $(\text{id}_{M'}, M', M') \in I_\lambda$.
(c) If $M_1 \subseteq N_1 \subseteq N$ and $M_1 \subseteq M$ and $\|N_1\| \leq \lambda$ <u>then</u> for some $t \in I$ we have $N_t = N_1$ and $\text{id}_{M_1} \subseteq f_t$.

[Why? Clause (a) is a special case of clause (b) and clause (b) is a special case of clause (c). Lastly, clause (c) follows from the assumption $M \prec_{\mathbb{L}_{\infty,\kappa}[\tau_{\mathfrak{K}}]} N$ and 0.18(2A),(2B).]

We next shall prove

$\circledast_2$ $M \leq_{\mathfrak{K}} N$.

By [She09a, 1.6] and $(*)_9$ above for proving $\circledast_2$ it suffices to prove:

$\circledast_3$ if $s \in I_1$ then $M_s \leq_{\mathfrak{K}} N$.

[Why $\circledast_3$ holds? As $M \subseteq N$ there is $N_* \leq_{\mathfrak{K}} N$ of cardinality $\leq \lambda$ such that $M_s \cup N_s \subseteq N_*$. By $\circledast_1(c)$ there is $t \in I$ such that $N_t = N_*$ and $\text{id}_{M_s} \subseteq f_t$. As $N_* \leq_{\mathfrak{K}} N$ it follows that $t \in I_1$. So by $\boxtimes_1 \Rightarrow \odot(b)$ applied to s and to t we can deduce $M_s \leq_{\mathfrak{K}} M$ and $M_t \leq_{\mathfrak{K}} M$. But as $\text{id}_{M_s} \subseteq f_t$ it follows that $M_s \subseteq \text{dom}(f_t) = M_t$ hence by Ax.V of AEC we know that $M_s \leq_{\mathfrak{K}} M_t$. But as $t \in I$ clearly f_t is an isomorphism from M_t onto N_t hence $f_t(M_s) \leq_{\mathfrak{K}} f_t(M_t) = N_t$, and as $\text{id}_{M_s} \subseteq f_t$ this means that $M_s = f_t(M_s) \leq_{\mathfrak{K}} N_t$. Recalling $N_t \leq_{\mathfrak{K}} N$ because $t \in I_1$ and $\leq_{\mathfrak{K}}$ is transitive it follows that $M_s \leq_{\mathfrak{K}} N$ as required.]

Let us check parts (3) and (4) of the Fact. Having proved $\boxtimes_1 \Rightarrow \odot(a)$, clearly in part (4) of the fact the first conclusion there, $M \in K$, holds. The second conclusion, $M \equiv_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} N$ holds by

$\circledast_4$ If $\varphi(\bar{x}) \in \mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$, $|\ell g(\bar{x})| + \text{LST}(\mathfrak{K}) \leq \lambda < \kappa$, $t \in I$, and $\bar{a} \in {}^{\ell g(\bar{x})}(M_t)$ then $M \models \varphi[\bar{a}] \Leftrightarrow N \models \varphi[f_t(\bar{a})]$.

[Why? We prove this by induction on the depth of φ for all λ simultaneously. For $\alpha = 0$, first for the usual atomic formulas this should be clear. Second, by $(*)_4$ there is t_1 such that $t \leq^2_I t_1 \in I_1$ hence by $\circledast_3$+ clause (d) of $\boxdot$+ clause (b) of $\odot$ we have $M_{t_1} \leq_{\mathfrak{K}} N \wedge N_{t_1} \leq_{\mathfrak{K}} N \wedge M_{t_1} \leq_{\mathfrak{K}} M$ respectively. So if $u \subseteq \ell g(\bar{x})$ then $M \restriction \text{rang}(\bar{a} \restriction u) \leq_{\mathfrak{K}} M \Leftrightarrow M \restriction \text{rang}(\bar{a} \restriction u) \leq_{\mathfrak{K}} M_{t_1} \Leftrightarrow N \restriction \text{rang}(f(\bar{a}) \restriction u) \leq_{\mathfrak{K}} N_{t_1} \Leftrightarrow N \restriction \text{rang}(f(\bar{a}) \restriction u) \leq_{\mathfrak{K}} N$. So we have finished the case of atomic formulas, i.e. $\alpha = 0$. For $\varphi(\bar{x}) = (\exists \bar{y})\psi(\bar{x}, \bar{y})$ use $(*)_2$, the other cases are obvious.] So part (4) holds. As for part (3), the first statement, "$M \in K$" holds by part (4), the second statement, $M \leq_{\mathfrak{K}} N$, holds by $\circledast_2$ and the third statement, $M \prec_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} N$ follows by $\circledast_1(b) + \circledast_4$. As we have already noted parts (1),(2),(5),(6) and part (7) is proved as $\circledast_4$ is proved, we are done. $\square_{1.11}$

Claim 1.12. *For a limit cardinal $\kappa > \text{LST}(\mathfrak{K})$:*

1) $M \prec_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} N$ provided that

(a) *if $\theta < \kappa$ and $\theta \in (\mathrm{LST}(\mathfrak{K}), \kappa)$ then $M \prec_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} N$*

(b) *for every $\partial < \kappa$ for some $\theta \in (\partial, \kappa)$ we have: if $\bar a, \bar b \in {}^{\partial}M$ and $(M, \bar a) \equiv_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} (M, \bar b)$ <u>then</u> $(M, \bar a) \equiv_{\mathbb{L}_{\infty,\theta_1}[\mathfrak{K}]} (M, \bar b)$ for every $\theta_1 \in [\theta, \kappa)$.*

1A) $M \equiv_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} N$ provided that

(a) *if $\mathrm{LST}(\mathfrak{K}) < \theta < \kappa$ then $M \equiv_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} N$*

(b) *as in part (1).*

2) In parts (1) and (1A) we can conclude

$(b)^+$ *for every $\partial < \kappa$ for some $\theta \in (\partial, \kappa)$ we have: if $\bar a, \bar b \in {}^{\partial}M$ and*

$$(M, \bar a) \equiv_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} (M, \bar b) \ \underline{\text{then}}\ (M, \bar a) \equiv_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} (M, \bar b).$$

3) If $\mathrm{cf}(\kappa) = \aleph_0$ <u>then</u> $M \cong N$ when

(a) *if $\theta < \kappa$ and $\theta \in (\mathrm{LST}(\mathfrak{K}), \kappa)$ then $M \equiv_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} N$*

(b) *as in part (1), i.e., for every $\partial \in (\mathrm{LST}(\mathfrak{K}), \kappa)$, for some $\theta \in (\partial, \kappa)$, we have: if $\bar a \in {}^{\partial}M$ and $\bar b \in {}^{\partial}N$ and $(M, \bar a) \equiv_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} (N, \bar b)$ <u>then</u> $(M, \bar a) \equiv_{\mathbb{L}_{\infty,\theta_1}[\mathfrak{K}]} (N, \bar b)$ for every $\theta_1 \in (\theta, \kappa)$.*

(c) *M, N have cardinality κ.*

Proof. 1) By 1.11(3) it suffices to prove $M \prec_{\mathbb{L}_{\infty,\kappa}} N$, for this it suffices to apply the criterion from 0.18(2A).

Let $\mathcal{F}$ be the set of functions f such that:

$\odot$ (α) $\mathrm{dom}(f) \subseteq M$ has cardinality $< \kappa$.

(β) $\mathrm{rang}(f) \subseteq N$.

(γ) If $\bar a$ lists $\mathrm{dom}(f)$ then for every $\theta \in (\ell g(\bar a), \kappa)$ we have $\mathrm{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar a, \varnothing, M) = \mathrm{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(f(\bar a), \varnothing, N)$.

1A) Similarly.

2) Similarly to part (1) using 1.11(4) and 0.18(2) instead 1.11(3),0.18(2A).

3) Recall 0.18(1). $\square_{1.12}$

Claim 1.13. *1) Assume 1.3(a) + (b), i.e. $\mathfrak{K}$ is categorical in $\mu > \mathrm{LST}(\mathfrak{K})$. If $\mu = \mu^{<\kappa}$ and $\kappa > \mathrm{LST}(\mathfrak{K})$ <u>then</u> for every $M \leq_{\mathfrak{K}} N$ from K_μ we have $M \prec_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} N$ (and there are such $M <_{\mathfrak{K}_\mu} N$).*

2) Assume $\mathfrak{K}$ is weakly or just pseudo μ-solvable as witnessed by Φ (see Definition 1.4 and Claim 1.5) and $M^* = \mathrm{EM}_{\tau(\mathfrak{K})}(\mu, \Phi)$ and $\mu = \mu^{<\kappa}$ and $\kappa > |\tau_\Phi|$. If $M \leq_{\mathfrak{K}} N$ are both isomorphic to M^* <u>then</u> $M \prec_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} N$.

Proof. 1) We prove by induction on γ that for any formula $\varphi(\bar x)$ from $\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$ of quantifier depth $\leq \gamma$ (and necessarily $\ell g(\bar x) < \kappa$) we have

$(*)$ if $M \leq_{\mathfrak{K}} N$ are from K_μ and $\bar a \in {}^{\ell g(\bar x)}M$ then $M \models \varphi[\bar a] \Leftrightarrow N \models \varphi[\bar a]$.

If $\varphi(\bar x)$ is atomic this is clear (for the "$\{x_i : i < i^*\}$ is the universe of a $\leq_{\mathfrak{K}}$-submodel", the implication $\Rightarrow$ holds as $\leq_{\mathfrak{K}}$ is transitive and the implication $\Leftarrow$ as $\mathfrak{K}$ satisfies Ax.V of AEC). If $\varphi(\bar x)$ is a Boolean combination of formulas for which the assertion was proved, clearly it holds for $\varphi(\bar x)$. So we are left with the case $\varphi(\bar x) = (\exists \bar y)\psi(\bar y, \bar x)$, so $\ell g(\bar y) < \kappa$. The implication $\Rightarrow$ is trivial by the induction hypothesis and so suppose that the other fails, say $N \models \psi[\bar b, \bar a]$ and $M \models \neg(\exists \bar y)\psi(\bar y, \bar a)$. We choose by induction on $i < \mu^+$ a model $M_i \in K_\mu$, $\leq_{\mathfrak{K}}$-increasing continuous, and for each i in addition we choose an isomorphism f_i from M onto M_i and if $i = j + 1$ we shall choose an isomorphism g_j from N onto M_{j+1}

extending f_j. For $i = 0$, let $M_0 = M$. For i limit let $M_i = \bigcup_{j<i} M_j$. For any i, if M_i was chosen, f_i exists as $\mathfrak{K}$ is categorical in μ. Now if $i = j+1$ then M_j, f_j are well defined and clearly we can choose $M_i = M_{j+1}$, g_j as required.

By Fodor lemma, as $\mu = \mu^{<\kappa}$ and the set $\{\delta < \mu^+ : \mathrm{cf}(\delta) \geq \kappa\}$ is stationary, clearly for some $\alpha < \beta < \mu^+$ we have $f_\alpha(\bar{a}) = f_\beta(\bar{a})$. Now (by the choice of g_α) we have $M_{\alpha+1} \models \psi[g_\alpha(\bar{b}), g_\alpha(\bar{a})]$, hence by the induction hypothesis applied to the pair $(M_{\alpha+1}, M_\beta)$ we have $M_\beta \models \psi[g_\alpha(\bar{b}), g_\alpha(\bar{a})]$ so $M_\beta \models \varphi[g_\alpha(\bar{a})]$. But $g_\alpha(\bar{a}) = f_\alpha(\bar{a}) = f_\beta(\bar{a})$, in contradiction to $M \models \neg\varphi[\bar{a}]$.

2) The same proof but we restrict ourselves to models in $K_{[M^*]}$ so, e.g. in $(*)$ we have $M, N \in K_{[M^*]}$ recalling that $\mathfrak{K}_{[M^*]}$ is a μ-AEC, see Definition 0.5(3A) and Claim 0.6(7). $\square_{1.13}$

<u>Exercise</u>: 1) For the proof (of 1.13(1)) it suffices to assume "$S \subseteq \{\delta < \mu^+ : \mathrm{cf}(\delta) \geq \kappa\}$ is a stationary subset of μ^+ and $M^* \in K_\mu$ is locally S-weakly limit." (See [She09a, 3.1(5)].)

2) Similarly we can weaken the demands "$M^* = \mathrm{EM}_{\tau(\mathfrak{K})}(\mu, \Phi)$ and (K, Φ) is pseudo solvable" to: 'for every $M \leq_{\mathfrak{K}} N$ isomorphic to M^* (which $\in K_\mu$) there is a $\leq_{\mathfrak{K}}$-increasing sequence $\langle M_\alpha : \alpha < \mu^+\rangle$ such that

$$\big\{\delta < \mu^+ : \mathrm{cf}(\delta) \geq \kappa,\ (M_\delta, M_{\delta+1}) \cong (M, N),\ \text{and } M_\delta = \bigcup\{M_\alpha : \alpha < \delta\}\big\}$$

is a stationary subset of μ^+.'

Claim 1.14. *Assume $\Phi \in \Upsilon^{\mathrm{or}}_{<\kappa}[\mathfrak{K}]$ satisfies the conclusion of 1.13(2) for (μ, κ) and $\mathrm{LST}(\mathfrak{K}) < \kappa \leq \mu$ and J, I_1, I_2 are linear orders and I_1, I_2 are κ-wide, see Definition 0.14(1). <u>Then</u>*

- (a) *If $I_1 \subseteq I_2$ <u>then</u> $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \prec_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} \mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$*
- (b) *Assume $J \subseteq I_1, J \subseteq I_2$; if $\varphi(\bar{x}) \in \mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$ so $\ell g(\bar{x}) < \kappa$ and $\bar{a} \in {}^{\ell g(\bar{x})}(\mathrm{EM}(J, \Phi))$, <u>then</u> $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \models \varphi[\bar{a}] \Leftrightarrow \mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi) \models \varphi[\bar{a}]$*
- (c) *Assume $\bar{\sigma} = \langle \sigma_i(\ldots, x_{\alpha(i,\ell)}, \ldots)_{\ell<\ell(i)} : i < i(*)\rangle$ where $i(*) < \kappa$, each σ_i is a $\tau(\Phi)$-term, $\alpha(i,\ell) < \alpha(*) < \kappa$. If $\bar{t}^\ell = \langle t^\ell_\alpha : \alpha < \alpha(*)\rangle$ is a sequence of members of I_ℓ for $\ell = 1, 2$ and $\bar{t}^1, \bar{t}^2$ realizes the same quantifier free type in I_1, I_2 respectively and $\bar{a}^\ell = \langle \sigma_i(\ldots, a_{t^\ell_{\alpha(i,j)}}, \ldots)_{j<j(i)} : i < i(*)\rangle$ for $\ell = 1, 2$ <u>then</u> $\bar{a}^1, \bar{a}^2$ realize the same $\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$ -type in $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi)$, $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$ respectively.*

Proof. **Clause (a)**: We prove that for $\varphi(\bar{x}) \in \mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$ we have

$(*)_{\varphi(\bar{x})}$ if $I_1 \subseteq I_2$ are κ-wide linear orders of cardinality $\leq \mu$ and $\bar{a} \in {}^{\ell g(\bar{x})}(\mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi))$ <u>then</u> $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \models \varphi[\bar{a}] \Leftrightarrow \mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi) \models \varphi[\bar{a}]$.

This easily suffices as for any $I \in K^{\mathrm{lin}}$, the model $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ is the direct limit of $\big\langle \mathrm{EM}(I', \Phi) : I' \subseteq I,\ |I'| \leq \mu \big\rangle$, which is $\leq_{\mathfrak{K}}$-increasing and μ^+-directed and as we have:

- $\odot$ $M^1 \prec_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} M^2$ <u>when</u>:
  - (a) I is a κ-directed partial order
  - (b) $\overline{M} = \langle M_t : t \in I\rangle$
  - (c) $s <_I t \to M_s \prec_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} M_t$
  - (d) $M^2 = \bigcup\{M_t : t \in I\}$
  - (e) $M^1 \in \{M_t : t \in I\}$ or for some κ-directed $I' \subseteq I$ we have $M^1 = \bigcup\{M_t : t \in I'\}$.

We prove $(*)_{\varphi(\bar{x})}$ by induction on φ (as in the proof of 1.13 above). The only non-obvious case is $\varphi(\bar{x}) = (\exists \bar{y})\psi(\bar{y},\bar{x})$, so let $I_1 \subseteq I_2$ be κ-wide linear orders of cardinality $\le \mu$ and $\bar{a} \in {}^{\ell g(\bar{x})}(\mathrm{EM}_{\tau(\mathfrak{K})}(I_1,\Phi))$. Now if $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1,\Phi) \models \varphi[\bar{a}]$ then for some $\bar{b} \in {}^{\ell g(\bar{y})}(\mathrm{EM}_{\tau(\mathfrak{K})}(I_1,\Phi))$ we have $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1,\Phi) \models \psi[\bar{b},\bar{a}]$. Hence by the induction hypothesis $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi) \models \psi[\bar{b},\bar{a}]$ hence by the satisfaction definition $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi) \models \varphi[\bar{a}]$, so we have proved the implication $\Rightarrow$.

For the other implication assume that $\bar{b} \in {}^{\ell g(\bar{y})}(\mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi))$ and $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi) \models \psi[\bar{b},\bar{a}]$. Let $\theta = |\ell g(\bar{a}\,\hat{}\,\bar{b})| + \aleph_0$, so $\theta < \kappa$ and without loss of generality if κ is singular then $\theta \ge \mathrm{cf}(\kappa)$. Hence there is in I_1 a monotonic sequence $\bar{c} = \langle c_i : i < \theta^+ \rangle$: without loss of generality, it is increasing. Clearly there is I^* such that $\bar{a}\,\hat{}\,\bar{b} \in {}^{\ell g(\bar{x}\hat{}\bar{y})}(\mathrm{EM}(I^*,\Phi))$, $I^* \subseteq I_2$, $|I^*| \le \theta$ and $\bar{a} \in {}^{\ell g(\bar{x})}(\mathrm{EM}(I^* \cap I_1,\Phi))$ and without loss of generality $i < \theta^+ \Rightarrow [c_0,c_i]_{I_2} \cap I^* = \varnothing$.

Similarly without loss of generality

$(*)$ $I_1 \setminus \bigcup\{[c_0,c_i)_{I_1} : i < \theta^+\}$ is κ-wide or $\kappa = \theta^+$.

Let $J_0 = I_2$; we can find J_1 such that $J_0 = I_2 \subseteq J_1$ and $J_1 \setminus I_2 = \{d_\alpha : \alpha < \mu \times \theta^+\}$ with d_α being $<_{J_1}$-increasing with α and

$$(\forall x \in I_2)\Big(x <_{J_1} d_\alpha \equiv \bigvee_{i<\theta^+} x <_{J_1} c_i\Big).$$

As $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi) \models \psi[\bar{b},\bar{a}]$ and $I_2 = J_0 \subseteq J_1$, $|J_1| \le \mu$ and I_2 is κ-wide (and trivially J_1 is κ-wide). By the induction hypothesis $\mathrm{EM}_{\tau(\mathfrak{K})}(J_1,\Phi) \models \psi[\bar{b},\bar{a}]$ hence $\mathrm{EM}_{\tau(\mathfrak{K})}(J_1,\Phi) \models \varphi[\bar{a}]$. Let

$$J_2 = J_1 \restriction \Big\{x : x \in J_1 \setminus J_0 \text{ or } x \in I_1 \setminus \bigcup\big\{[c_0,c_i]_{I_1} : i < \theta^+\big\}\Big\}.$$

So $J_1 \supseteq J_2$, both linear orders have cardinality μ and are κ-wide as witnessed by $\langle d_\alpha : \alpha < \mu \times \theta^+ \rangle$ for both hence the conclusion of 1.13 holds, i.e. $\mathrm{EM}(J_2,\Phi) \prec_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} \mathrm{EM}(J_1,\Phi)$. Also, $I^* \cap I_1 \subseteq J_2$, and recall that $\bar{a} \in {}^{\ell g(\bar{x})}(\mathrm{EM}(I^* \cap I_1,\Phi))$ hence $\bar{a} \in {}^{\ell g(\bar{x})}(\mathrm{EM}(J_2,\Phi))$. However, $\mathrm{EM}_{\tau(\mathfrak{K})}(J_1,\Phi) \models \varphi[\bar{a}]$, see above, hence by the last two sentences $\mathrm{EM}_{\tau(\mathfrak{K})}(J_2,\Phi) \models \varphi[\bar{a}]$.

So there is $\bar{b}^* \in {}^{\ell g(\bar{y})}(\mathrm{EM}_{\tau(\mathfrak{K})}(J_2,\Phi))$ such that $\mathrm{EM}_{\tau(\mathfrak{K})}(J_2,\Phi) \models \psi[\bar{b}^*,\bar{a}]$. Let $J^* \subseteq J_2$ be of cardinality θ such that $\bar{b}^* \in {}^{\ell g(\bar{y})}(\mathrm{EM}_{\tau(\mathfrak{K})}(J^*,\Phi))$ and $I^* \cap I_1 \subseteq J^*$ recalling $I^* \cap [c_0,c_i)_{I_2} = \varnothing$ for $i < \theta^+$. Now let $u \subseteq \mu \times \theta^+$ be such that $J^* \setminus I_1 = \{d_\alpha : \alpha \in u\}$ so $|u| < \theta^+$. Let

$$J_3 = J_2 \restriction \big\{t : t \in J_2 \cap I_1,\ \text{ or } t = d_\alpha \text{ for } \alpha > \sup(u) \text{ or } \alpha \in u\big\}.$$

[I might be getting distracted from the main goal, but isn't this literally $J_2 \restriction (J_2 \cap I_1 \cup J^* \setminus I_1 \cup \{d_\alpha : \alpha > \sup u\}) = J_2 \restriction (I_1 \cup \{d_\alpha : \alpha > \sup u\})$?]

As $\mathrm{cf}(\mu \times \theta^+) = \theta^+ > |u|$, clearly $\sup(u) < \mu \times \theta^+$ hence $|J_3| = \mu$ and J_3 is κ-wide. So by the conclusion of 1.13 (or by the induction hypothesis) also $\mathrm{EM}_{\tau(\mathfrak{K})}(J_3,\Phi) \models \psi[\bar{b}^*,\bar{a}]$. Let $w = \{\alpha < \mu \times \theta^+ : \alpha \in u \text{ or } \alpha > \sup(u) \wedge (\alpha - \sup(u) < \theta^+)\}$, so $\mathrm{otp}(w) = \theta^+$.

Let $J_4 = (J_3 \cap I_1) \cup \{d_\alpha : \alpha \in w\}$, so J_4 is κ-wide as witnessed by

$$I_1 \setminus \bigcup\big\{[c_0,c_i) : i < \theta^+\big\}$$

or by $\{d_\alpha : \alpha \in w\}$ recalling $(*)$ above and $J_4 \subseteq J_3$ and $J^* \subseteq J_4$ hence $\bar{a}, \bar{b}^* \subseteq {}^{\kappa >}(\mathrm{EM}(J_4,\Phi))$ hence by the induction hypothesis $\mathrm{EM}_{\tau(\mathfrak{K})}(J_4,\Phi) \models \psi[\bar{b}^*,\bar{a}]$.

Let $J_5 = J_4 \cup \{c_i : i < \theta^+\} \setminus \{d_\alpha : \alpha \in w\}$; equivalently,

$$J_5 = (J_3 \cap I_1) \cup \{c_\alpha : \alpha < \theta^+\} = \Big(I_1 \setminus \bigcup\big\{[c_0,c_i)_{I_1} : i < \theta^+\big\}\Big) \cup \{c_i : i < \theta^+\}$$

so $J_5 \subseteq I_1$. Let $h : J_4 \to J_5$ be such that $h(d_\alpha) = c_{\mathrm{otp}(w \cap \alpha)}$ for $\alpha \in w$ and $h(t) = t$ for others, i.e. for $t \in J_3 \cap I_1$. So h is an isomorphism from J_4 onto J_5. Recalling

0.12 let $\hat{h}$ be the isomorphism from $\mathrm{EM}(J_4, \Phi)$ onto $\mathrm{EM}(J_5, \Phi)$ which h induces, so clearly $\hat{h}(\bar{a}) = \bar{a}$. Hence for some $\bar{b}^{**}$ we have $\bar{b}^{**} = \hat{h}(\bar{b}^*) \in {}^{\ell g(\bar{y})}(\mathrm{EM}_{\tau(\mathfrak{K})}(J_5, \Phi))$ and $\mathrm{EM}_{\tau(\mathfrak{K})}(J_5, \Phi) \models \psi[\bar{b}^{**}, \bar{a}]$. Note that by the choice of $\langle c_i : i < \theta^+ \rangle$, (see $(*)$ above), we know that J_5 is κ-wide. Also $J_5 \subseteq I_1$ so by the induction hypothesis applied to $\psi(\bar{y}, \bar{x}), J_5, I_1$ we have $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \models \psi[\bar{b}^{**}, \bar{a}]$ hence by the definition of satisfaction $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \models \varphi[\bar{a}]$, so we have finished proving the implication $\Leftarrow$ hence clause (a).

Clause (b): Without loss of generality for some linear order I we have $I_1 \subseteq I$, $I_2 \subseteq I$ and $\mathrm{EM}(I_\ell, \Phi) \subseteq \mathrm{EM}(I, \Phi)$ for $\ell = 1, 2$ and use clause (a) twice.

Clause (c): Easy by now, e.g. using a linear order I' extending I_1, I_2 which has an automorphism h such that $h(t^1_\alpha) = t^2_\alpha$ for $\alpha < \alpha(*)$. $\square_{1.14}$ $\square_{1.14}$

Definition 1.15. Fixing $\Phi \in \Upsilon^{\mathrm{or}}_{\mathfrak{K}}$.

1) For $\theta \geq \mathrm{LST}(\mathfrak{K})$ let K^*_θ, [let K^{**}_θ] [let $K^{*,*}_\theta$] be the family of $M \in K_\theta$ isomorphic to some $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ where I is a linear order of cardinality θ [which is θ-wide][which $\in K^{\mathrm{flin}}_\theta$]. More accurately we should write $K^*_{\Phi,\theta}, K^{**}_{\Phi,\theta}, K^{*,*}_{\Phi,\theta}$; similarly below.

2) Let K^* is the class $\bigcup\{K^*_\theta : \theta \text{ a cardinal} \geq \mathrm{LST}(\mathfrak{K})\}$, similarly $K^{*,*}, K^*_{\geq\lambda}, K^{**}_{\geq\lambda}$, etc.

3) Let $\mathfrak{K}^* = \mathfrak{K}^*_\Phi = (K^*, \leq_{\mathfrak{K}} \restriction K^*)$.

4) Let $\mathfrak{K}^*_\lambda = K^*_{\Phi,\lambda}$ be $(K^*_{\Phi,\lambda}, \leq_{\mathfrak{K}} \restriction K^*_{\Phi,\lambda})$.

Claim 1.16. *1) K^{**}_θ is categorical in θ if $\mathrm{LST}(\mathfrak{K}) < \theta \leq \mu$, $\mathrm{cf}(\theta) = \aleph_0$ and the conclusion of 1.13(2) hence of 1.14 holds for $\partial = \theta$ (and Φ), e.g. $\mathfrak{K}$ is pseudo solvable in μ as witnessed by Φ and $\mu = \mu^{<\theta}$.*

2) $K^{*,*}_\theta, K^{**}_\theta \subseteq K^*_\theta$.

3) If θ is strong limit $> \mathrm{LST}(\mathfrak{K})$ <u>then</u> $K^{**}_\theta = K^*_\theta$.

Proof. 1) By 1.14 and 0.18(1).

2) Read the definitions.

3) Recall 0.15(2). $\square_{1.16}$

Remark 1.17. 1) We will be specially interested in 1.16 in the case (μ, λ) is a $\mathfrak{K}$-candidate (see Definition [She09b, 11.0.3]) and $\theta = \lambda$.

2) Note that K^*_θ, in general, is not a θ-AEC.

3) If we strengthen 1.18(2) below, replacing (μ, λ) by (μ, λ^+) then categoricity of K^*_λ and in fact Claim 1.19(4) follows immediately from (or as in) Claim 1.16(1).

For the rest of this section we assume that the triple (μ, λ, Φ) is a pseudo $\mathfrak{K}$-candidate (see Definition 1.3) and rather than $\mu = \mu^\lambda$ we assume just the conclusion of 1.13, that is:

Hypothesis 1.18. 1) The pair (μ, λ) is a pseudo $\mathfrak{K}$-candidate and Φ witnesses this, so $|\tau_\Phi| \leq \mathrm{LST}(\mathfrak{K}) < \lambda = \beth_\lambda < \mu$ and $\Phi \in \Upsilon^{\mathrm{or}}_{\mathfrak{K}}$ is as in Definition 1.4 so $I \in K^{\mathrm{lin}}_\mu \Rightarrow \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi) \in K^{\mathrm{pl}}_\mu$.

2) For every $\kappa \in (\mathrm{LST}(\mathfrak{K}), \lambda)$ the conclusion of 1.13(2) holds hence also of 1.14 (if $\mu = \mu^{<\lambda}$ this follows from (1) even for $\kappa = \lambda^+$ as $\mu^{<\kappa} = \mu^\lambda = \mu$ by cardinal arithmetic).

Claim 1.19. *1) If $M_1 \leq_{\mathfrak{K}} M_2$ are from K^*_λ or just $K^*_{\geq \lambda}$ and* $\mathrm{LST}(\mathfrak{K}) < \theta < \lambda$ *<u>then</u> $M_1 \prec_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} M_2$; moreover $M_1 \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} M_2$.*

2) If $M_1 \leq_{\mathfrak{K}} M_2$ are from K^* and $\|M_1\| \geq \kappa := \beth_{1,1}(\theta)$ (recall that this is $\beth_{(2^\theta)^+}$) and $\lambda > \theta \geq$ $\mathrm{LST}(\mathfrak{K})$ *<u>then</u> $M_1 \prec_{\mathbb{L}_{\infty,\theta^+}[\mathfrak{K}]} M_2$.*

3) Assume $\mathrm{LST}(\mathfrak{K}) < \theta < \kappa = \beth_{1,1}(\theta) \leq \chi < \lambda$, $\chi_1 = \beth_{1,1}(\chi)$ *and $M \in K^*_{\geq \chi_1}$ and $\bar{a}, \bar{b} \in {}^\gamma M$ where $\gamma < \theta^+$ and $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} (M, \bar{b})$; i.e.*

$$\varphi(\langle x_\beta : \beta < \gamma\rangle) \in \mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}] \Rightarrow \big(M \models \varphi[\bar{a}] \Leftrightarrow M \models \varphi[\bar{b}]\big).$$

<u>Then</u> $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty,\chi}[\mathfrak{K}]} (M, \bar{b})$.

4) K^*_λ is categorical in λ provided that $\mathrm{cf}(\lambda) = \aleph_0$.

Remark 1.20. 1) What is the difference between say 1.19(3) and clause (a) of 1.14? Here there is no connection between the additional $\tau(\Phi)$-structures expanding M_1, M_2.

2) Note that Φ has the κ-non-order property (see 1.5(2)(*)) when $\kappa \geq \mathrm{LST}(\mathfrak{K})$, $\kappa^+ < \mu$ using 1.19(4).

3) Concerning 1.19(2), note that if $\|M_1\| \geq \mu$ it is easy to deduce this from 1.18(2), i.e, 1.13(2). But the whole point in this stage is to deduce something on cardinals $< \mu$.

4) Note that the proof of 1.19(2) gives:

- $\circledast$ assume $\mathrm{LST}(\mathfrak{K}) \leq \theta$ and $\delta(*) = \min\{(2^\theta)^+, \delta(2^{\mathrm{LST}(\mathfrak{K})} + \theta)\}$.[3] If $\beth_{\delta(*)} \leq \mu$ then for some $\alpha(*) < \delta(*)$ we have:
  - $\odot$ if $M_1 \leq_{\mathfrak{K}} M_2$ are from K^* and $\|M_1\| \geq \beth_{\alpha(*)}$ then $M_1 \prec_{\mathbb{L}_{\infty,\theta^+}[\mathfrak{K}]} M_2$.

5) Similarly for 1.19(3) so we can weaken the demand $M \in K^*_{\geq \chi_1}$

6) We use "λ has countable cofinality, i.e. $\mathrm{cf}(\lambda) = \aleph_0$" in the proof of part (4) of 1.19, but not in the proof of the other parts.

7) Recall that for notational simplicity we assume $\mathrm{LST}(\mathfrak{K}) \geq |\tau_{\mathfrak{K}}|$ hence $\theta \geq |\tau_\Phi|$.

8) Note that for 1.19(2),(3) we can omit λ from Hypothesis 1.18, i.e. we need it only for κ.

9) Note that we shall use not only 1.19 but also its proof.

Proof. 1) The first phrase holds by part (2) noting that $\kappa < \lambda$ if $\theta < \lambda$ as $\theta < \lambda = \beth_\lambda$. The second phrase holds by 1.12 as its assumption holds by parts (1) and (3).

2) We prove by induction on the ordinal γ that:

- $(*)$ if $M_1 \leq_{\mathfrak{K}} M_2$ are from $K^*_{\geq \kappa}$ and the formula $\varphi(\bar{x}) \in \mathbb{L}_{\infty,\theta^+}[\mathfrak{K}]$ has depth $\leq \gamma$ (so necessarily $\ell g(\bar{x}) < \theta^+$) and $\bar{a} \in {}^{\ell g(\bar{x})}(M_1)$ <u>then</u>
$$M_1 \models \varphi[\bar{a}] \Leftrightarrow M_2 \models \varphi[\bar{a}].$$

As in 1.13, the non-trivial case is to assume $\varphi(\bar{x}) = (\exists \bar{y})\psi(\bar{y}, \bar{x})$ where $\bar{a} \in {}^{\ell g(\bar{x})}(M_1)$ and $M_2 \models \varphi[\bar{a}]$. We shall prove $M_1 \models \varphi[\bar{a}]$, so necessarily $\ell g(\bar{x}) + \ell g(\bar{y}) < \theta^+$ and we can choose $\bar{b} \in {}^{\ell g(\bar{y})}(M_2)$ such that $M_2 \models \psi[\bar{b}, \bar{a}]$. For $\ell = 1, 2$ as $M_\ell \in K^*_{\geq \kappa}$ there is an isomorphism f_ℓ from $\mathrm{EM}_{\tau(\mathfrak{K})}(I_\ell, \Phi)$ onto M_ℓ for some linear order I_ℓ of cardinality $\geq \kappa$.

So we can find $J_\ell \subseteq I_\ell$ of cardinality θ for $\ell = 1, 2$ such that $\bar{a} \subseteq M_1^-$ where $M_1^- = f_1(\mathrm{EM}_{\tau(\mathfrak{K})}(J_1, \Phi))$, and $\bar{a}{}^\frown\bar{b} \subseteq M_2^-$ where $M_2^- = f_2(\mathrm{EM}_{\tau(\mathfrak{K})}(J_2, \Phi))$ and without loss of generality $M_1^- = M_2^- \cap M_1$. By 1.18(1), i.e. 0.9(1), clause (c) clearly $M_\ell^- \leq_{\mathfrak{K}} M_\ell$ and so by Ax.V of AEC (see Definition [She09c, 0.2]), we have $M_1^- \leq_{\mathfrak{K}} M_2^-$. First assume $\theta \geq 2^{\mathrm{LST}(\mathfrak{K})}$; in fact it is not a real loss to assume

[3] On the function $\delta(-)$, see [She09g, 1.2.3,1.2].

this. By renaming without loss of generality there is a transitive set B (in the set theoretic sense) of cardinality $\le \theta$ such that the following objects belong to it:

$\oplus(a)$ J_1, J_2

(b) Φ (i.e. τ_Φ and $\langle(\mathrm{EM}(n,\Phi), a_\ell)_{\ell<n} : n<\omega\rangle$)

(c) $\mathfrak{K}$, i.e., $\tau_{\mathfrak{K}}$ and $\{(M,N) : M \le_{\mathfrak{K}} N$ have universes included in $\mathrm{LST}(\mathfrak{K})\}$

(d) $\mathrm{EM}(J_\ell,\Phi)$ and $\langle a_t : t \in J_\ell\rangle$ for $\ell = 1,2$.

Let χ be large enough, $\mathfrak{B} = (\mathcal{H}(\chi), \in, <^*_\chi)$ and $\mathfrak{B}^+$ be $\mathfrak{B}$ expanded by the individual constants $M^+_\ell = \mathrm{EM}(I_\ell,\Phi)$, $\langle a^\ell_t : t \in I_\ell\rangle$ the skeleton, M_ℓ, M^-_ℓ and f_ℓ (all for $\ell = 1,2$), κ, B and x for each $x \in B$. By the assumption $\|M_1\| \ge \kappa = \beth_{1,1}(\theta)$, hence (see here [She09g, 1.2]) there is $\mathfrak{C}$ such that

$\odot$ (a) $\mathfrak{C}$ is a $\tau(\mathfrak{B}^+)$-model elementarily equivalent to $\mathfrak{B}^+$ (that is, in first order logic)

(b) $\mathfrak{C}$ omits the type $\{x \ne b$ and $x \in B : b \in B\}$ but

(c) $\big|\{b : \mathfrak{C} \models \text{“}b \in \kappa^{\mathfrak{C}}\text{”}\}\big| = \mu = \|\mathfrak{C}\|$.

Without loss of generality $b \in B \Rightarrow b^{\mathfrak{C}} = b$.

Now

$\circledast_1$ if $\mathfrak{C} \models$ “$M \in K$”, so M is just a member of the model $\mathfrak{C}$ <u>then</u> we can define a $\tau_{\mathfrak{K}}$-model $M^{\mathfrak{C}} = M[\mathfrak{C}]$ as follows:

(a) the set of elements of $M^{\mathfrak{C}}$ is $\{a : \mathfrak{C} \models$ “a is a member of the model M”$\}$

(b) if $R \in \tau_K$ is an n-place predicate then

$$R^{M[\mathfrak{C}]} = \big\{\langle a_\ell : \ell < n\rangle : \mathfrak{C} \models \text{“}\langle a_\ell : \ell<n\rangle \in R^M\text{”}\big\}$$

(c) if $F \in \tau_K$ is an n-place function symbol, $F^{M[\mathfrak{C}]}$ is defined similarly.

$\circledast_2$ (a) if $\mathfrak{C} \models$ “I is a linear order” then we define $I^{\mathfrak{C}}$ similarly

(b) similarly if $\mathfrak{C} \models$ “M is a $\tau(\Phi)$-model”

$\circledast_3$ if $\mathfrak{C} \models$ “I is a directed partial order, $\bar M = \langle M_s : s \in I\rangle$ satisfies $M_s \in K$ has cardinality $\mathrm{LST}(\mathfrak{K})$ and $s \le_I t \Rightarrow M_s \le_{\mathfrak{K}} M_t$” <u>then</u> also $\langle M^{\mathfrak{C}}_s : s \in I^{\mathfrak{C}}\rangle$ satisfies this.

By easy absoluteness (for clauses $(a)_1, (a)_2$ we use [She09a, 1.6-1.7] and $\circledast_3$):

$\boxtimes$ $(a)_1$ if $\mathfrak{C} \models$ “$M \in K$” then $M^{\mathfrak{C}} \in K$

$(a)_2$ if $\mathfrak{C} \models$ “$M \le_{\mathfrak{K}} N$” then $M^{\mathfrak{C}} \le_{\mathfrak{K}} N^{\mathfrak{C}}$

$(b)_1$ if $\mathfrak{C} \models$ “I is a linear order” then $I^{\mathfrak{C}} = I[\mathfrak{C}]$ is a linear order

$(b)_2$ if $\mathfrak{C} \models$ “$I \subseteq J$ as linear orders” then $I^{\mathfrak{C}} \subseteq J^{\mathfrak{C}}$

(c) similarly for τ_Φ-models

$(d)_1$ if $\mathfrak{C} \models$ “$M = \mathrm{EM}(I,\Phi)$” <u>then</u> there is a canonical isomorphism $f^{\mathfrak{C}}_I$ from $\mathrm{EM}(I^{\mathfrak{C}},\Phi)$ onto $M^{\mathfrak{C}}$ (hence it is also an isomorphism from $\mathrm{EM}_{\tau(\mathfrak{K})}(I^{\mathfrak{C}},\Phi)$ onto $M^{\mathfrak{C}} \restriction \tau(\mathfrak{K})$)

$(d)_2$ if $\mathfrak{C} \models$ “$I \subseteq J$ as linear orders” then $f^{\mathfrak{C}}_J$ extends $f^{\mathfrak{C}}_I$.

Now clearly $J^{\mathfrak{C}}_\ell = J_\ell$ and $I^{\mathfrak{C}}_\ell$ is a linear order of cardinality μ extending J_ℓ for $\ell = 1,2$. Let $M^*_\ell = (M^-_\ell)^{\mathfrak{C}}$ for $\ell = 1,2$.

So recalling clause (c) of $\odot$ we have: $M^{\mathfrak{C}}_1, M^{\mathfrak{C}}_2 \in K^*_\mu$, $M^{\mathfrak{C}}_1 \le_{\mathfrak{K}} M^{\mathfrak{C}}_2$, $M^*_\ell \le_{\mathfrak{K}} M^{\mathfrak{C}}_\ell$, $M^*_1 \le_{\mathfrak{K}} M^*_2$ and $f^{\mathfrak{C}_0}_\ell, f^{\mathfrak{C}}_{I_\ell}$ are isomorphisms from $\mathrm{EM}_{\tau(\mathfrak{K})}(I^{\mathfrak{C}}_\ell,\Phi)$ onto $M^{\mathfrak{C}}_\ell$, in fact, $f^{\mathfrak{C}}_{I_\ell}$ is the identity on $\mathrm{EM}_{\tau(\mathfrak{K})}(J^{\mathfrak{C}}_\ell,\Phi) = \mathrm{EM}_{\tau(\mathfrak{K})}(J_\ell,\Phi)$ and $f^{\mathfrak{C}}_\ell$ maps it onto M^*_ℓ for $\ell = 1,2$.

Now $M_2 \models \psi[\bar a,\bar b]$, (why? assumed above) hence $M^{\mathfrak{C}}_2 \models \psi[\bar a,\bar b]$

(why? By 1.14, clause (b) or (c) and the situation recalling 1.18(2), of course noting that $I_2, I^{\mathfrak{C}}_2$ are of cardinality $\ge \kappa = \beth_{1,1}(\theta)$ hence are θ^+-wide), hence $M^{\mathfrak{C}}_2 \models \varphi[\bar a]$ (by definition of satisfaction), hence $M^{\mathfrak{C}}_1 \models \varphi[\bar a]$. (Why? As $M^{\mathfrak{C}}_1, M^{\mathfrak{C}}_2 \in K^*_\mu$ hence $M^{\mathfrak{C}}_1 \prec_{\mathbb{L}_{\infty,\theta^+}[\mathfrak{K}]} M^{\mathfrak{C}}_2$ by $\boxtimes$ and 1.18(2) and recalling 1.13(2).) Hence $M_1 \models \varphi[\bar a]$ as required in 1.19(2). (Why? By clause (b) of 1.14 recalling 1.18(2))

So we are done except for a small debt: the case $\theta < 2^{\mathrm{LST}(\mathfrak{K})}$ and $f^{\mathfrak{C}}_\ell$ is an isomorphism from $\mathrm{EM}_{\tau(\mathfrak{K})}(I^{\mathfrak{C}}_\ell, \Phi)$.

In this case choose two sets B_1, B_2 such that $|B_1| = \theta$, $|B_2| = 2^{\mathrm{LST}(\mathfrak{K})}$, $B_1 \subseteq B_2$ and concerning the demands in $\oplus$ above the objects from (a),(b),(d) and $\tau_{\mathfrak{K}}$ belong to B_1, the objects from (c) belong to B_2.

Again, without loss of generality B_1, B_2 are transitive sets and B_1, B_2 serve as individual constants of $\mathfrak{B}^+$ as well as each member of B_1. Now concerning $\mathfrak{C}$ we demand that it is elementarily equivalent to $\mathfrak{B}^+$; omit $\{x \in B_1 \wedge x \neq b : b \in B_1\}$ and for some $\mathfrak{B}^+_1 \prec \mathfrak{B}^+$ of cardinality θ we have $\mathfrak{B}^+_1 \prec \mathfrak{C}$ and $\{b : \mathfrak{C} \models b \in B_2\} \subseteq \mathfrak{B}^+$. This influences just the proof of $\circledast_3$.

3) Without loss of generality $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ and $I \in K^{\mathrm{lin}}_{\geq \chi_1}$. As $\gamma < \theta^+$ and $\bar{a}, \bar{b} \in {}^\gamma M$ there is $I_1 \subseteq I$ of cardinality θ such that $\bar{a}, \bar{b} \in {}^\gamma M_1$ where $M_1 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi)$. As $(M, \bar{a}) \equiv_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]} (M, \bar{b})$ necessarily there is $I_2 \subseteq I$ of cardinality κ and automorphism f of $M_2 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$ mapping $\bar{a}$ to $\bar{b}$ such that $I_1 \subseteq I_2$. Why? Recalling 0.18(2), by the hence and forth argument as in the second part of the proof of 1.11(3).

Now as in the proof of part (2) there is a linear order I_3 extending I_1 of cardinality χ_1 and an automorphism g of $M_3 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$ mapping $\bar{a}$ to $\bar{b}$. Without loss of generality for some linear order I_4 we have $I \subseteq I_4$ and $I_3 \subseteq I_4$.

Let $M_4 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$, now $M \prec_{\mathbb{L}_{\infty,\chi^+}[\mathfrak{K}]} M_4$ by part (2), $M_3 \prec_{\mathbb{L}_{\infty,\chi^+}[\mathfrak{K}]} M_4$ by part (3) and $(M_3, \bar{a}) \equiv_{\mathbb{L}_{\infty,\chi^+}[\mathfrak{K}]} (M_3, \bar{b})$ by using the automorphism g of M_3 so together we are done.

4) So let $M, N \in K^*_\lambda$ (in fact, hence $\in K^{**}_\lambda$ recalling $K^*_\lambda = K^{**}_\lambda$ by 1.16(3) but not used). By parts (1),(3) the assumptions of 1.12(3) hold with λ here standing for κ there, hence its conclusion, i.e. $M \cong N$. $\square_{1.19}$

Note: here the types below are sets of formulas.

Definition 1.21. Assume $M \in K$, $\mathbf{I} \subseteq {}^\gamma M$ and $\mathscr{L}, \mathscr{L}_1, \mathscr{L}_2$ are languages in the vocabulary $\tau_{\mathfrak{K}}$.

1) We say that $\mathbf{I}$ is $(\mathscr{L}, \partial, {<}\kappa)$-convergent in M, <u>when</u>: $|\mathbf{I}| \geq \partial$ and for every $\bar{b} \in {}^{\kappa>}M$, for some $\mathbf{J} \subseteq \mathbf{I}$ of cardinality $< \partial$, for some[4] p we have:

($*$) for every $\bar{c} \in \mathbf{I} \setminus \mathbf{J}$, the $\mathscr{L}$-type of $\bar{c}\hat{\ }\bar{b}$ in M is p.

2) Let

$$\begin{aligned}\mathrm{Av}_{\mathscr{L},\partial,<\kappa}(\mathbf{I}, M) = \big\{\varphi(\bar{x}, \bar{b}) : \ & \varphi(\bar{x}, \bar{y}) \text{ is an } \mathscr{L}\text{-formula}, \ \ell g(\bar{y}) < \kappa, \\ & \bar{a} \in \mathbf{I} \Rightarrow \ell g(\bar{a}) = \ell g(\bar{x}), \ \bar{b} \in {}^{\ell g(\bar{y})}M, \text{ and} \\ & \text{for all but } < \partial\text{-many sequences } \bar{c} \in \mathbf{I} \\ & \bar{c} \text{ satisfies } \varphi(\bar{x}, \bar{b}) \text{ in } M\big\}\end{aligned}$$

If ∂ is missing, we mean $\partial = \kappa$. In parts (1) and (2) we may write "κ" instead of $< \kappa^+$; similarly below. (κ^+, κ)-convergent means $(\mathbb{L}_{\infty,\kappa^+}(\mathfrak{K}), \kappa^+, < \kappa^+)$-convergent.

3) We say that $\mathbf{I}$ is $(\mathscr{L}_1, \mathscr{L}_2, \partial, < \kappa)$-based[5] on A in M <u>when</u>:

(a) $A \subseteq M$

(b) $\mathbf{I}$ is $(\mathscr{L}_1, \partial, < \kappa)$-convergent,

(c) $\mathrm{Av}_{\mathscr{L}_1,\partial,<\kappa}(\mathbf{I}, M)$ does not $(\mathscr{L}_1, \mathscr{L}_2, < \kappa)$-split over A, see below.

4) We say that $p(\bar{x}) \in \mathrm{Sfr}^\alpha_{\mathscr{L}}(B, M)$ does not $(\mathscr{L}_1, \mathscr{L}_2, < \kappa)$-split over A <u>when</u>: if $\varphi(\bar{x}, \bar{y}) \in \mathscr{L}_1$, $\alpha = \ell g(\bar{x}) < \kappa$, $\ell g(\bar{y}) < \kappa$ and $\bar{b}, \bar{c} \in {}^{\ell g(\bar{y})}B$ realize the same $\mathscr{L}_2$-type

[4] We could have demanded it for every single formula, here this distinction is not important

[5] If $\mathscr{L}_1 = \mathscr{L} = \mathscr{L}_2$ we may write only $\mathscr{L}$.

in M over A <u>then</u> $\varphi(\bar{x},\bar{b}) \in p \Leftrightarrow \varphi(\bar{x},\bar{c}) \in p$; recalling that $\mathrm{Sfr}^{\alpha}_{\mathscr{L}}(A,M)$ is defined in 0.4 and normally $\mathscr{L}_1 = \mathscr{L}_2$ or at least $\mathscr{L}_1 \subseteq \mathscr{L}_2$.

5) Let $\mathrm{Av}_{<\kappa}(\mathbf{I},M)$ be $\mathrm{Av}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}],\kappa,\kappa}(\mathbf{I},M)$ and $\mathrm{Av}_{\kappa}(\mathbf{I},M)$ be $\mathrm{Av}_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}],\kappa^+,\kappa^+}(\mathbf{I},M)$.

Remark 1.22. 1) See definition of $\mathrm{Sav}^{\alpha}(M)$ in 1.37(2) below.

2) An alternative for clause (c) of 1.21(3) is:

- $(c)'$ the set $\{\mathrm{Av}_{\mathscr{L},\partial,<\kappa}(f(\mathbf{I}),M) : f$ an automorphism of M over $A\}$ has cardinality $\leq \beth_{1,1}(\mathrm{LST}(\mathfrak{K})+\theta+|A|) < \|M\|$.

Claim 1.23. *1) Assume that $M \in K$, $A \subseteq M$, $\mathbf{I} \subseteq {}^{\theta}M$, $|\mathbf{I}| \geq \partial = \mathrm{cf}(\partial) > \kappa \geq \theta + \mathrm{LST}(\mathfrak{K})$ and $\mathbf{I}$ is $(\mathscr{L},\partial,\kappa)$-convergent. <u>Then</u> the type $p = \mathrm{Av}_{\mathscr{L},\partial,\kappa}(\mathbf{I},M)$ belongs to $\mathrm{Sfr}^{\theta}_{\mathscr{L}}(M) = \mathrm{Sfr}^{\theta}_{\mathscr{L}}(M,M)$; i.e., it is complete, recalling Definition 0.4 (no demand that it is realized in some $N \geq_{\mathfrak{K}} M$!).*

2) Also, $\mathbf{I}$ is $(\mathscr{L},\partial,\kappa)$-based on some set of cardinality $\leq \partial$, even on $\bigcup \mathbf{J}$, for any $\mathbf{J} \subseteq \mathbf{I}$ of cardinality $\geq \partial$.

Proof. 1) By the definition.

2) By the definitions: if $\bar{b} \in {}^{\kappa^+>}M$, $\varphi = \varphi(\bar{x},\bar{y}) \in \mathscr{L}$ and $\ell g(\bar{b}) = \ell g(\bar{y})$, $\ell g(\bar{x}) = \theta$, then by the convergence

$$\begin{aligned}\varphi(\bar{x},\bar{b}) \in p \Leftrightarrow &\text{ for all but } < \partial \text{ members } \bar{a} \text{ of } \mathbf{I},\ M \models \varphi[\bar{a},\bar{b}] \Leftrightarrow\\ &\text{ for all but } < \partial \text{ members of } \mathbf{J},\ M \models \varphi[\bar{a},\bar{b}].\end{aligned}$$

So only $\mathrm{tp}_{\mathscr{L}}(\bar{b},\bigcup\mathbf{J},M)$ matters, hence the non-splitting required in clause (c) of Definition 1.21(3). $\square_{1.23}$

As in [She09g, 1.7], we deduce non-splitting over a small set from non-order.

Claim 1.24. *Assume $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi)$, $\theta + \mathrm{LST}(\mathfrak{K}) \leq \kappa < \lambda$, and $\beth_{1,1}(\partial) \leq |I|$ where $\partial = (2^{2^{\kappa}})^+$ or I is well ordered and $\partial = (2^{\kappa})^+$. If $M \prec_{\mathbb{L}_{\infty,\partial}[\mathfrak{K}]} N$ <u>then</u> for every $\bar{a} \in {}^{\theta\geq}N$ there is $B \subseteq M$ of cardinality $< \partial$ such that $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]}(\bar{a},M,N)$ does not $(\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}],\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}])$-split over B.*

Proof. Let $\bar{x} = \langle x_i : i < \ell g(\bar{a})\rangle$.

We try to choose $B_\alpha,\gamma_\alpha,\bar{a}_\alpha,\bar{b}_\alpha,\bar{c}_\alpha,\varphi_\alpha(\bar{x},\bar{y}_\alpha) \in \mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]$ by induction on $\alpha < \partial$ such that

$\circledast$
- (a) $B_\alpha = \bigcup\{\bar{a}_\beta : \beta < \alpha\}$
- (b) $\bar{b}_\alpha,\bar{c}_\alpha \in {}^{\gamma_\alpha}M$ and $\gamma_\alpha < \kappa^+$
- (c) $\varphi_\alpha(\bar{x},\bar{y}_\alpha) \in \mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]$ such that $\ell g(\bar{y}_\alpha) = \gamma_\alpha$
- (d) $N \models$ "$\varphi_\alpha[\bar{a},\bar{b}_\alpha] \equiv \neg\varphi_\alpha[\bar{a},\bar{c}_\alpha]$"
- (e) $\bar{a}_\alpha \in {}^{\ell g(\bar{a})}M$ realizes $\{\varphi_\beta(\bar{x},\bar{b}_\beta) \equiv \neg\varphi_\beta(\bar{x},\bar{c}_\beta) : \beta < \alpha\}$ in M
- (f) $M \models$ "$\varphi_\alpha[\bar{a}_\beta,\bar{b}_\alpha] \equiv \varphi_\alpha[\bar{a}_\beta,\bar{c}_\alpha]$" for $\beta \leq \alpha$.

If we are stuck at $\alpha(*) < \partial$ then we cannot choose $\gamma_\alpha,\bar{b}_\alpha,\bar{c}_\alpha,\varphi_\alpha(\bar{x},\bar{y}_\alpha)$ clauses (b),(c),(d), because then $\bar{a}_\alpha$ as required in clauses (e),(f) exists because $M \prec_{\mathbb{L}_{\infty,\partial}[\mathfrak{K}]} N$. Hence $B := \bigcup\{\bar{a}_\alpha : \alpha < \alpha(*)\}$ is as required. So assume that we have carried the induction. As $\gamma_\alpha < \kappa^+ < \partial = \mathrm{cf}(\partial)$, without loss of generality, $\gamma_\alpha = \gamma < \kappa^+$ for every $\alpha < \partial$.

Let $\partial_1 = (2^{\kappa})^+$.

Now by 1.25(5) below when I is not well ordered and by 1.25(4) below when I is well ordered (and part (1) of 1.25(1), recalling I is κ^+-wide as $\kappa < \partial$ and $\beth_{1,1}(\partial) \leq |I|$) clearly for some $S \subseteq \partial$ of order type ∂_1, the sequence $\langle \bar{a}_\alpha{}^\frown\bar{b}_\alpha{}^\frown\bar{c}_\alpha : \alpha \in S\rangle$ is $(\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}],\kappa^+,\kappa)$-convergent and $(\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}],<\omega)$-indiscernible in M hence without loss of generality $\alpha \in S \Rightarrow \varphi_\alpha = \varphi$. But as $\partial_1 > \kappa^+$ this contradicts (e) + (f) of $\circledast$ (if we use $\partial_1 = \kappa^+$, we can use a further conclusion of 1.25(1) stated in 1.25(2), i.e., $\langle \bar a_\alpha{}^\frown \bar b_\alpha{}^\frown \bar c_\alpha : \alpha \in S\rangle$ is a $(\mathbb{L}_{\infty,\kappa}[\mathfrak{K}], < \omega)$-indiscernible set – not just a sequence, in contradiction to (e) + (f) of $\circledast$). $\square_{1.24}$

Claim 1.25. *Assume $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi), I$ is κ^+-wide, $\kappa < \lambda$ and $\mathrm{LST}(\mathfrak{K}) + \theta \le \kappa < \partial$.*

1) Assume that $\mathscr{L} = \mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]$ and $\bar a_\alpha = \langle \sigma_i(\ldots, a_{t(\alpha,i,\ell)},\ldots)_{\ell<n_i} : i < \theta\rangle$ for $\alpha < \partial$ so σ_i is a $\tau(\Phi)$-term, and $\mathrm{cf}(\partial) > \kappa$. Assume further that letting $\bar t_\alpha = \langle t(\alpha,i,\ell) : i < \theta, \ell < n_i\rangle$, the sequence $\langle \bar t_\alpha : \alpha < \partial\rangle$ is indiscernible in I for quantifier free formulas (i.e. the truth values of $t(\alpha_1,i_1,\ell_1) < t(\alpha_2,i_2,\ell_2)$ depend only on i_1,ℓ_1,i_2,ℓ_2 and the truth value of $\alpha_1 < \alpha_2$, $\alpha_1 = \alpha_2$, $\alpha_1 > \alpha_2$). Then $\langle \bar a_\alpha : \alpha < \partial\rangle$ is $(\mathscr{L},\partial,\kappa)$-convergent in the model M.

2) In part (1), even dropping the assumption $\mathrm{cf}(\partial) > \kappa$, moreover, the sequence $\langle \bar a_\alpha : \alpha < \partial\rangle$ is $(\mathscr{L},\kappa^+,\kappa)$-convergent and $(\mathscr{L}, < \omega)$-indiscernible in M.

3) In part (1) and in part (2), letting

$$J_0 = \{t(0,i,\ell) : t(0,i,\ell) = t(1,i,\ell) \text{ and } i < \theta,\ \ell < n_i\}$$

assume $J_0 \subseteq J \subseteq I$, J is κ^+-wide (e.g. $J = \{t(\alpha,i,\ell) : \alpha < \kappa^+, i < \theta, \ell < n_i\}$), B is the universe of $\mathrm{EM}_{\tau(\mathfrak{K})}(J,\Phi)$, $i_1,i_2 < \theta$, $\ell_1 < n_{\ell_1}$, $\ell_2 < n_{i_2}$, and

$$\big[\alpha,\beta < \partial \Rightarrow t(\alpha,i_1,\ell_1) <_I t(\beta,i_2,\ell_2)\big] \Rightarrow$$
$$(\exists s \in J_0)\big[\alpha,\beta < \partial \Rightarrow t(\alpha,i_1,\ell_1) <_I s <_I t(\beta,i_2,\ell_2)\big]$$

then B is a (∂,κ)-base of $\{\bar a_\alpha : \alpha < \partial\}$.

[The conclusion did not depend on s anywhere, so I changed it.]

4) If I is well ordered (or just is $\mathrm{EM}_{\{<\}}(J,\Psi)$, $\Psi \in \Upsilon^{\mathrm{or}}$, J well ordered), $\mathrm{LST}(\mathfrak{K}) + \theta \le \kappa$, $2^\kappa < \partial$, $(\forall \alpha < \partial)\big[|\alpha|^\theta < \partial = \mathrm{cf}(\partial)\big]$ and $\bar b_\alpha \in {}^\theta M$ for $\alpha < \partial$, then for some stationary $S \subseteq \{\delta < \partial : \mathrm{cf}(\delta) \ge \theta^+\}$, the sequence $\langle \bar a_\alpha : \alpha \in S\rangle$ is as in part (1); hence it is (κ^+,κ)-convergent in M. Moreover, if $S_0 \subseteq \{\delta < \partial : \mathrm{cf}(\delta) \ge \theta^+\}$ is stationary we can demand $S \subseteq S_0$.

5) If in (4) we omit the assumption "I is well ordered", and add $\partial \to (\partial_1)^2_{2^\kappa}$, e.g. $\partial_1 = (2^\kappa)^+$, $\partial = (2^{2^\kappa})^+$ then we can find $S \subseteq \partial$, $|S| = \partial_1$ such that $\langle \bar a_\alpha : \alpha \in S\rangle$ is as in (1).

Remark 1.26. In fact the well order case always applies at least if $\partial < \mu$.

Proof. 1) Let $\bar b \in {}^\kappa M$, so $\bar b = \langle \sigma^*_j(\ldots, a_{s(j,\ell)},\ldots)_{\ell<m_j} : j < \kappa\rangle$ where σ^*_i is a $\tau(\Phi)$-term, $s(j,\ell) \in I$ and let $\bar s = \langle s(j,\ell) : \ell < m_j,\ j < \kappa\rangle$.

Now for each $i_1 < \theta$, $\ell_1 < n_{i_1}$ and $j_1 < \kappa$, $k_1 < m_{j_1}$ the sequence $\langle t(\alpha,i_1,\ell_1) : \alpha < \partial\rangle$ is monotonic (in I) hence there is $\alpha(i_1,\ell_1,j_1,k_1) < \partial$ such that

$(*)_1$ if $\beta,\gamma \in \partial \setminus \{\alpha(i_1,\ell_1,j_1,k_1)\}$ and $\beta < \alpha(i_1,\ell_1,j_1,k_1) \equiv \gamma < \alpha(i_1,\ell_1,j_1,k_1)$ then

$$\big(t(\beta,i_1,\ell_1) <_I s(j_1,k_1)\big) \equiv \big(t(\gamma,i_1,\ell_1) <_I s(j_1,k_1)\big)$$

and

$$\big(t(\beta,i_1,\ell_1) >_I s(j_1,k_1)\big) \equiv \big(t(\gamma,i_1,\ell_1) >_I s(j_1,k_1)\big).$$

Let

$$u := \{\alpha(i_1,\ell_1,j_1,k_1) : i_1 < \theta,\ \ell_1 < n_{i_1},\ j_1 < \kappa,\ k_1 < m_{j_1}\}.$$

It is a subset of ∂ of cardinality $\le \theta + \kappa = \kappa$.

Hence

$(*)_2$ if $\beta, \gamma \in \partial \setminus u$ and $\beta\ \mathcal{E}_u\ \gamma$ (which is defined by $(\forall \alpha \in u)\big[\alpha < \beta \equiv \alpha < \gamma\big]$) <u>then</u> $\bar{t}_\beta \hat{\ } \bar{s}$, $\bar{t}_\gamma \hat{\ } \bar{s}$ realize the same quantifier free type in I.

Now by clause (c) of 1.14 recalling I is κ^+-wide we have

$(*)_3$ if $\beta, \gamma \in \partial \setminus u$ and $\beta\ \mathcal{E}_u\ \gamma$ then $\bar{a}_\beta \hat{\ } \bar{b}$, $\bar{a}_\gamma \hat{\ } \bar{b}$ realize the same $\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]$-type in M.

As $\bar{b}$ was any member of ${}^\kappa M$ we have gotten

$(*)_4$ if $\bar{b} \in {}^{\kappa \geq}M$, then for some $u = u_{\bar{b}} \subseteq \partial$ of cardinality $\leq \kappa$ we have: if $\beta, \gamma \in \partial \setminus u$ and $\beta\ \mathcal{E}_u\ \gamma$ then $\bar{a}_\beta \hat{\ } \bar{b}$, $\bar{a}_\gamma \hat{\ } \bar{b}$ realize the same $\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]$-type in M.

As we are assuming $\mathrm{cf}(\partial) > \kappa (\geq \theta + \mathrm{LST}(\mathfrak{K}) \geq |\tau_\Phi|)$ we can conclude that

$(*)_5$ $\langle \bar{a}_\alpha : \alpha < \partial \rangle$ is $(\mathscr{L}, \partial, \kappa)$-convergent in M.

So we have proved 1.25(1).

2) We start as in the proof of part (1). However, after $(*)_3$ above letting for simplicity $u^+ = \{\alpha < \partial :$ for some $\beta \in u \cap \alpha$ we have $\alpha + \kappa = \beta + \kappa\}$ we have

$(*)_6$ if $\beta, \gamma \in \partial \setminus u^+$ and $\beta < \gamma, \neg(\beta E_{u^+} \gamma)$ <u>then</u> we can find $(\mu^+, I^+, \bar{s}', \bar{b}')$ such that
- (α) $I \subseteq I^+ \in K^{\mathrm{lin}}$
- (β) $M^+ = \mathrm{EM}_{\tau(\mathfrak{K})}(I^+, \Phi)$ hence $M \prec_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]} M^+$
- (γ) $\bar{s}' = \langle s'(j,k) : k < m_j,\ j < \kappa \rangle$ a sequence of elements of I^+
- (δ) $\bar{b}' = \langle \sigma^*_j(\ldots, a_{s'(j,\ell)}, \ldots)_{\ell < m_j} : j < \kappa \rangle \in {}^\kappa(M^+)$
- (ε) $\bar{b} \hat{\ } \bar{a}_\gamma$, $\bar{b}' \hat{\ } \bar{a}_\beta$ realize the same $\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]$-types in M^+ as $\bar{b} \hat{\ } \bar{a}_\gamma$, $\bar{b} \hat{\ } \bar{a}_\beta$ respectively
- (ζ) $\bar{s} \hat{\ } \bar{t}_\beta$, $\bar{s}' \hat{\ } \bar{t}_\beta$ form a Δ-system pair, i.e. they are as in $\boxtimes$ from 1.5(2).

Why?

Let $w^+ = \big\{(j,k) : k < m_j,\ j < \kappa,\ (\exists \ell < n_{i_1},\ i_1 < \theta)\big[\alpha(i_1, \ell_1, j, k) \in (\beta, \gamma)\big]\big\}$

$$w^- := \big\{(j,k) : j < \kappa,\ k < m_j \text{ and } (j,\kappa) \notin w^+\big\}.$$

We choose I^+ extending I and $\bar{s}_\varepsilon = \langle s_\varepsilon(j,k) : k < m_j,\ j < \kappa \rangle$ for $\varepsilon < \kappa$ such that

(a) the set of elements of I^+ is the disjoint union
$$I \cup \{s_\varepsilon(j,k) : (j,k) \in w,\ \varepsilon \in (0,\kappa)\}$$
(b) $\bar{s}_\varepsilon, \bar{s}$ realize the same quantifier-free type in I^+

(c) if $\varepsilon, \zeta < \kappa$ then $\bar{t}_{\gamma+\varepsilon} \hat{\ } \bar{s}_\zeta$ realizes in I^+ the quantifier-free type $\mathrm{tp}_{\mathrm{qf}}(\bar{t}_\beta \hat{\ } \bar{s}, \varnothing, I)$ if $\varepsilon < \zeta$ and $\mathrm{tp}_q(\bar{t}_\gamma \hat{\ } \bar{s}, \varnothing, I)$ if $\varepsilon \geq \zeta$

(d) $\langle \bar{t}_{\gamma+\varepsilon} \hat{\ } \bar{s}_\varepsilon : \varepsilon < \kappa \rangle$ is indiscernible for quantifier-free formulas on I^+

(e) $\bar{s}_0 = \bar{s}$.

This is straight. Using $\bar{s}' = \bar{s}_1$ we are done.

Now as Φ has the κ-non-order property (by Claim 1.5(2) which contains a definition, noting that the assumption of 1.5 holds by 1.18(1) and also 1.18(2)), repeating $(*)_4, (*)_5$ we get

$(*)_7$ for every $\bar{b} \in {}^{\kappa \geq}M$, for some $u = u^+_{\bar{b}} \in [\partial]^{\leq \kappa}$ if $\beta, \gamma \in \partial \setminus u^+$ then $\bar{a}_\beta \hat{\ } \bar{b}$, $\bar{a}_\gamma \hat{\ } \bar{b}$ realize the same $\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]$-type in M.

In other words

$(*)_8$ the sequence $\langle \bar{a}_\alpha : \alpha < \partial \rangle$ is $(\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}], \kappa^+)$-convergent.

The proof that it is a $(\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}], < \omega)$-indiscernible set is similar.

3) Not used; easy by 1.23(2) and convergence. [That is, note that we can find I^+ and $\bar{a}'_\alpha = \langle \sigma_i(\ldots, a_{t'(\alpha,i,\ell)}, \ldots)_{\ell_i < n_i} : i < \theta \rangle$ for $\alpha < \partial + \gamma$ such that:

(a) $I^+ \in K^{\rm lin}$ extend I
(b) $t'(\alpha, i, \ell) \in I^+$
(c) $\bar{t}'_\alpha = \langle t'(\alpha, i, \ell) : i < \theta,\ \ell < n_i \rangle$
(d) $\langle \bar{t}'_\alpha : \alpha < \partial + \gamma \rangle$ is indiscernible for quantifier-free formulas in I^+
(e) $\langle \bar{t}_\alpha : \alpha < \partial \rangle\char`^\langle \bar{t}'_\alpha : \alpha \in [\partial, \partial + \partial) \rangle$ is indiscernible for quantifier-free formulas in I'
(f) for each $i < \theta$, $\ell < n_i$ such that $t(0, i, \ell) = (j, i, t)$, the convex hull I_* of $\{t'(\alpha, i, \ell) : \alpha < \partial\}$ in I^+ is disjoint to I, and if $s_1 <_I s_2$ and $(s_1, s_2)_{I^*} \cap I_* = \varnothing$ then $[s_1, s_2]_{I^*} \cap J_0 \neq \varnothing$.

So we can average over $\langle \bar{a}'_\alpha : \alpha < \partial \rangle$ instead [of] averaging over $\langle \bar{a}_\alpha : \alpha < \partial \rangle$, and this implies the result. In fact we can weaken the assumption.]

4) Should be clear. [Still, let $\bar{t}_\alpha = \langle t_{\alpha,i} : i < \theta \rangle$ be such that

$$\bar{b}_\alpha = \langle \sigma_{\alpha,j}(\ldots, a_{t_{\alpha,i(j,\alpha,\ell)}}, \ldots)_{\ell < n(\alpha,j)} : j < \theta \rangle.$$

So as $(\mathrm{LST}(\mathfrak{K}) + |\tau_\Phi|)^\theta < \partial = \mathrm{cf}(\partial)$ for some stationary $S_1 \subseteq \{\delta < \partial\text{: } \mathrm{cf}(\delta) \geq \theta^+\}$ we have $\alpha \in S_1 \wedge j < \theta \Rightarrow \sigma_{\alpha,j} = \sigma_j$ (hence $j < \theta \Rightarrow n(\alpha, j) = n(j)$) and

$$\alpha \in S_1 \wedge j < \theta \wedge \ell < n(j) \Rightarrow i(j, \alpha, \ell) = i(j, \ell)$$

and for every $i_1, i_2 < \theta$ we have $t_{\alpha,i_1} <_I t_{\alpha,i_2} \equiv (i_1, i_2) \in W$ for some sequence $\bar{\sigma} = \langle \sigma_j : j < \theta \rangle$ of τ_Φ-terms and $W \subseteq \kappa \times \kappa$ and sequence $\big\langle \langle i(j, \ell) : \ell < n(j) \rangle : j < \theta \big\rangle$.

If I is well ordered, for $\delta \in S_1$ let

$\gamma_\delta = \min\{\gamma$: if $i < \theta$ and there are $\beta < \delta$, $j < \theta$ such that $t_{\delta,i} <_I t_{\beta,j}$ and <u>then</u> letting $(\beta_{\delta,i}, j_{\delta,i})$ be such a pair with $t_{\beta_{\delta,i},j_{\delta,i}}$ being $<_I$-minimal, we have $\beta_{\delta,i} < \gamma\}$.

[I tried to reformat this into {align*}, but I couldn't follow what was written. It'd be more readable if we broke up the definition over two sets. Even if you never use it anywhere else, define a dummy set like $D_{\delta,i} = \{t_{\beta,j} : \beta < \delta,\ j < \theta,\ t_{\delta,i} <_I t_{\beta,j}\}$. Then the real definition is a lot more digestible: $\gamma_\delta = \min\{\gamma : t_{\beta,j} \in D_{\delta,i}$ is $<_I$-minimal $\Rightarrow \beta < \gamma\}$. Not only that, but now you can specify exactly how β depends on i, which seems to be a sticking point both in the definition and in the following paragraph.]

Clearly γ_δ is well defined and $< \delta$ so by Fodor lemma, for some $\gamma_* < \partial$, the set $S_1 := \{\delta \in S_2 : \gamma_\delta = \gamma_*\}$ is stationary. As $|\gamma_*|^\theta < \partial$, for some $u \subseteq \theta$ and stationary $S_3 \subseteq S_2$ we have: if $\delta \in S_3$ then $j \in u \Leftrightarrow (\beta_{\delta,i}, j_{\delta,i})$ well defined and $j \in u \wedge \alpha \in S_3 \Rightarrow (\beta_{\delta,i}, j_{\delta,i}) = (\beta_i, j_i)$ and for each $i \in u$ the truth value of "$t_{\delta,i} = t_{\beta_i,j_i}$" is the same for all $\delta \in S_3$.

Now apply part (1) to $\langle \bar{b}_\alpha : \alpha \in S_3 \rangle$.]

5) By (1) and the definition of $\partial \rightarrow (\partial_1)^2_{2^\kappa}$. $\square_{1.25}$

Claim 1.27. *1) If $M \leq_{\mathfrak{K}} N$ are from K^*_λ, $\kappa \in [\mathrm{LST}(\mathfrak{K}), \lambda)$, $\kappa^+ < \partial = \mathrm{cf}(\partial) < \lambda$ and moreover $\theta \leq \kappa$ and $\bar{a} \in {}^\theta N$ <u>then</u> there is a (κ^+, κ)-convergent set $\mathbf{I} \subseteq {}^\theta M$ of cardinality ∂ such that $\mathrm{Av}_\kappa(\mathbf{I}, M)$ is realized in N by $\bar{a}$.*

2) In fact we can weaken $M, N \in K^*_\lambda$ to $M, N \in K^*_{\geq \beth_{1,1}(\partial')}$ where, e.g. $\partial' = \beth_5(\kappa)^+$.

3) Assume $\theta \leq \kappa$, $\kappa \in [\mathrm{LST}(\mathfrak{K}), \lambda)$, $\partial' = \beth_5(\kappa)^+$ and $M_1 \in K^*_{\geq \beth_{1,1}(\partial')}$. Assume further $M_1 \leq_{\mathfrak{K}} M_2 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$, $|\xi| = \theta$, and $\mathbf{I} \subseteq {}^\xi(M_1)$ is a (κ^+, κ)-convergent set[6] of cardinality ∂'. If $I_2 <^*_{K^{\rm flin}} I_3$ (or just I_3 is κ^+-wide over I_2, which follows as $|I_2| \geq |\mathbf{I}| = \partial'$) and $M_3 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$ <u>then</u>

[6] in M_1, see 1.12

(a) *We can find* $\bar d \in {}^\xi(M_3)$ *realizing* $\mathrm{Av}_\kappa(\mathbf{I}, M_2)$, *so [it is] well defined.*
(b) *If* $M_1 \le_{\mathfrak{K}} N \in K^*$ *and* $\bar d^* \in {}^\xi N$, $|\xi| \le \theta$ *then we can find* $\bar d \in {}^\xi(M_3)$ *realizing* $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]}(\bar d^*, M_1, N)$, *and* $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]}(\bar d, M_2, M_3)$ *is the average of some* (κ^+,κ)*-convergent* $\mathbf{I}' \subseteq {}^\alpha(M_1)$ *of cardinality* ∂'.

Remark 1.28. The exact value of ∂' has no influences for our purpose.

Proof. 1) Without loss of generality $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi)$. Let $\partial_0 = \partial$ and $\partial_{\ell+1} = \beth_2(\partial_\ell)^+$ for $\ell = 0,1$ so $\partial_\ell < \lambda$ and

$$\ell \in \{1,2\} \Rightarrow (\forall \alpha < \partial_\ell)\big[|\alpha|^{\kappa+\theta} < \partial_\ell = \mathrm{cf}(\partial_\ell) < \lambda\big].$$

If I is well ordered (which is O.K. by 1.19(4)) and $(\forall \alpha < \partial)\big[|\alpha|^\kappa < \partial\big]$ then we can use $\partial_\ell = \partial$.

By 1.24 there is $B_* \subseteq M$ of cardinality $< \partial_2$ (or just $\le 2^{2^\kappa} < \partial_2$) such that $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]}(\bar a, M, N)$ does not $(\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}], \mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}])$-split over B_*.

Now by 1.19(1) for every $B \subseteq M$, $|B| < \partial_2$ there is $\bar a' \in {}^\theta M$ realizing in M, equivalently in N (with $\ell g(\bar x) = \theta$, of course), the type

$$\mathrm{tp}_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]}(\bar a, B, N) = \big\{\varphi(\bar x, \bar b) : \bar b \in {}^{\kappa\ge}B,\ \varphi(\bar x,\bar y) \in \mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}],\ N \models \varphi[\bar a, \bar b]\big\}.$$

We can choose $J_\alpha, B_\alpha, \bar a_\alpha$ by induction on $\alpha < \partial_2$ such that

$$B_\alpha \supseteq \bigcup\{\bar a_\beta : \beta < \alpha\} \cup B_*$$

B_α is the universe of $\mathrm{EM}(J_\alpha,\Phi)$, $J_\alpha \subseteq I$, $|J_\alpha| < \partial_2$, J_α increasing with α and J_α is quite closed (e.g. is $\mathfrak{B}_\alpha \cap I$ where $\mathfrak{B}_\alpha \prec_{\mathbb{L}_{\kappa^+,\kappa^+}} (\mathcal{H}(\chi), \in, <^*_\chi)$ with

$$M, N, \mathrm{EM}(I,\Phi), \mathfrak{K}, \langle \bar a_\beta : \beta < \alpha\rangle, \mathfrak{K}, \kappa, \theta$$

belonging to $\mathfrak{B}_\alpha$, $\mathfrak{B}_\alpha$ has cardinality $< \partial_2$, and $\mathfrak{B}_\alpha \cap \partial_2 \in \partial_2$). Then choose $\bar a' = \bar a_\alpha$ as above, i.e. $\bar a_\alpha \in {}^\theta M$ realizes the same $\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]$-type as $\bar a$ over $B_\alpha = M \cap \mathfrak{B}_\alpha = \mathrm{EM}_{\tau(\mathfrak{K})}(J_\alpha,\Phi)$ in N; such $\bar a_\alpha$ exists by 1.19(1). So for some set $S_1 \subseteq \partial_2$ of order type ∂_1 the sequence $\mathbf{I} = \langle \bar a_\beta : \beta \in S_1\rangle$ is (κ^+,κ)-convergent (by 1.25(4),(5)).

It is enough to show that $\mathbf{I}$ is as required, toward contradiction assume that not. Then there is an appropriate formula $\varphi(\bar x,\bar y)$ with $\ell g(\bar x) = \theta$, $\ell g(\bar y) = \kappa$ and $\bar b \in {}^\kappa M$ such that $N \models \varphi[\bar a, \bar b]$ but $u := \{\alpha \in S_1 : M \models \varphi[\bar a_\alpha, \bar b]\}$ has cardinality $< \kappa^+$. Now for $\alpha \in S_1$ as J_α was chosen "closed enough", there is

$$\bar b_\alpha \in {}^\kappa(\mathrm{EM}_{\tau(\mathfrak{K})}(J_\alpha,\Phi)) \subseteq {}^\kappa M$$

realizing $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]}(\bar b, B_*, M)$ such that

$$\beta \in S_1 \cap \alpha \Rightarrow M \models \text{“}\varphi[\bar a_\beta, \bar b] \equiv \varphi[\bar a_\beta, \bar b_\alpha]\text{”}$$

(possible, e.g. as $|B_\alpha|^{|S\cap\alpha|} \le (2^{<\partial_1})^{<\partial_1} < \partial_2$).

So, again by 1.25(4),(5), for some $S_0 \subseteq S_1$ of order type $\partial = \partial_0$, the sequence $\langle \bar a_\alpha {}^\frown \bar b_\alpha : \alpha \in S_0\rangle$ is $(\mathbb{L}_{\infty,\kappa^+}, \kappa^+, \kappa)$-convergent in M and $(\mathbb{L}_{\infty,\kappa^+[\mathfrak{K}]}, < \omega)$-indiscernible. Let $\alpha \in S_0$ be such that $|S_0 \cap \alpha| > \kappa$, possible as $|S_0| = \partial_0 > \kappa^+$. So the set $\{\beta \in S_1 \cap \alpha : M \models \varphi[\bar a_\beta, \bar b_\alpha]\}$ has cardinality $\le \kappa$ (being equal to $\{\beta \in S_1 \cap \alpha : N \models \varphi[\bar a_\beta, \bar b]\}$) but $\alpha \in S_0 \subseteq S_1$ and $|S_0 \cap \alpha| > \kappa$, so for some $\beta < \alpha$ from S_0, $M \models \neg\varphi[a_\beta, \bar b_\alpha]$ hence by the indiscernibility $M \models \neg\varphi[\bar a_\beta, \bar b_\gamma]$ for every $\beta < \gamma$ from S_0.

On the other hand, if $\alpha < \beta$ are from S_0 then by the choice of $\bar b_\alpha$ the sequences $\bar b, \bar b_\alpha$ realize the same $\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]$-type over B_*. Now $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]}(\bar a, M, N)$ does not split over B_*, by the choice of B_*, so we have $N \models \text{“}\varphi[\bar a, \bar b] \equiv \varphi[\bar a, \bar b_\alpha]\text{”}$. But by the choice of $\bar{b}$ we have $N \models \varphi[\bar{a}, \bar{b}]$ hence $N \models \varphi[\bar{a}, \bar{b}_\alpha]$ hence $M \models \varphi[\bar{a}_\beta, \bar{b}_\alpha]$ by the choice of $\bar{a}_\beta$. Together this contradicts 1.5, i.e., 1.18(1).

2) Similarly (using 1.19(2) instead of 1.19(1).

3) **Clause (a)**:

By 1.14 and the LST argument (i.e. by 0.18(4)) without loss of generality $M_1 \in K^*_{<\lambda}$ and also $M_2 \in K^*_{<\lambda}$. Let $\partial_\ell = \beth_\ell(\kappa)^+$ for $\ell \leq 5$, so $\partial' = \partial_5$, and for notational simplicity assume $\theta \geq \aleph_0$.

Let $\{\bar{a}_\alpha : \alpha < \partial'\}$ list the members of $\mathbf{I}$, so for each $\alpha < \partial'$ there is $I_{2,\alpha} \subseteq I_2$ of cardinality θ such that $\bar{a}_\alpha$ is from $\mathrm{EM}_{\tau(\mathfrak{K})}(I_{2,\alpha}, \Phi)$.

For each $\alpha < \partial'$ let $\bar{t}^\alpha = \langle t^\alpha_i : i < \theta\rangle$ list $I_{2,\alpha}$ and so $\bar{a}_\alpha = \langle \sigma_{\alpha,\zeta}(\bar{t}^\alpha) : \zeta < \xi\rangle$ for some sequence $\langle \sigma_{\alpha,\zeta}(\bar{x}) : \zeta < \xi\rangle$ of τ_Φ-terms. We can find $S \subseteq \partial'$ of order type ∂_4 such that $\zeta < \xi \wedge \alpha \in S \Rightarrow \sigma_{\alpha,\zeta} = \sigma_\zeta$ and $\langle \bar{t}^\alpha : \alpha \in S\rangle$ is an indiscernible sequence (for quantifier free formulas, in I_2, of course).

By renaming $\kappa^+ \subseteq S$. We define a partition $\langle u_{-1}, u_0, u_1\rangle$ of ξ by

$$u_0 = \{i < \theta : t^\alpha_i = t^\beta_i \text{ for } \alpha, \beta \in S\}$$
$$u_1 = \{i < \theta : t^\alpha_i <_{I_2} t^\beta_i \text{ for } \alpha < \beta \text{ from } S\}$$
$$u_{-1} = \{i < \theta : t^\beta_i <_{I_2} t^\alpha_i \text{ for } \alpha < \beta \text{ from } S\}.$$

We define an equivalence relation e on $u_{-1} \cup u_1$

$\odot$ $i_1 \; e \; i_2$ iff for some $\ell \in \{1, -1\}$, $i_1, i_2 \in u_\ell$ and $(t^\alpha_{i_1} <_I t^\beta_{i_2}) \equiv (t^\alpha_{i_2} <_I t^\beta_{i_1})$ for every (equivalently, 'some') $\alpha < \beta$ from S.

There is a natural set of representatives:

$$W = \{\zeta < \theta : \zeta \in u_{-1} \cup u_1 \text{ and } \zeta = \min(\zeta/e)\}.$$

We now define a linear order I^+_2; its set of elements is

$$\{t : t \in I_2\} \cup \{t^*_i : i \in u_{-1} \cup u_1\}$$

where, of course, $t^*_i \in I^+_2$ are pairwise distinct and $\notin I_2$. The order is defined by the following: (or see $\circledast_2$ and think about what conditions are necessary)

$\circledast_1$ $s_1 <_{I^+_2} s_2$ iff
- (a) $s_1, s_2 \in I_2$ and $s_1 <_{I_2} s_2$
- (b) $s_1 \in I_2$, $s_2 = t^*_i$ and $s_1 <_{I_2} t^\alpha_i$ for every $\alpha < \kappa^+$ large enough
- (c) $s_1 = t^*_i$, $s_2 \in I_2$ and $t^\alpha_i <_{I_2} s_2$ for every $\alpha < \kappa^+$ large enough
- (d) $s_1 = t^*_i$, $s_2 = t^*_j$ and $t^\alpha_i <_{I_2} t^\alpha_j$ for every $\alpha < \kappa^+$.

Let $t^*_i = t^\alpha_i$ for $i \in u_0$ and any $\alpha < \kappa^+$. Let $M^+_2 = \mathrm{EM}_{\tau(\mathfrak{K})}(I^+_2, \Phi)$.

It is easy to check (by 1.14(a),(c)) that

$\circledast_2$
- (a) $I_2 \subseteq I^+_2$
- (b) $\bar{t}^* \in {}^\theta(I^+_2)$
- (c) If $J \subseteq I_2$ has cardinality $\leq \kappa$ <u>then</u> for every $\alpha < \kappa^+$ large enough, the sequences $\bar{t}^*, \bar{t}^\alpha$ realizes the same quantifier free type over J inside I^+_2.

Let

$\circledast_3$ $\bar{d} := \langle \sigma_\zeta(\bar{t}^*) : \zeta < \xi\rangle \in {}^\xi(M^+_2)$.

Recall that $\|M_2\| < \lambda$ hence $|I_2| < \lambda$ and I_2 is κ^+-wide having cardinality $\geq \partial' > 2^\kappa$.

Note

$\circledast_4$ $\bar{t}^*$ realizes $\mathrm{Av}_{\mathrm{qf}}(\{\bar{t}^\alpha : \alpha \in S\}, I_2)$ in the linear order I^+_2.

Without loss of generality $I^+_2 \cap I_3 = I_2$, so we can find a linear order I_4 of cardinality λ such that $I^+_2 \subseteq I_4 \wedge I_3 \subseteq I_4$. As I_3 is κ^+-wide over I_2 (see the assumption and Definition 0.14(6)+(3)), there is a convex subset I'_3 of I_3 disjoint to I_2 which contains a monotonic sequence $\langle s_\alpha : \alpha < \kappa^+\rangle$. Without loss of generality there are elements s_α (with $\alpha \in [\kappa^+, \lambda \times \kappa^+)$) in I_4 such that $\langle s_\alpha : \alpha < \lambda \times \kappa^+ \rangle$ is monotonic (in I_4), and its convex hull is disjoint to I_2. Let $I_3^- = I_2 \cup \{s_\alpha : \alpha < \kappa^+\}$ and $I_3^\pm = I_2 \cup \{s_\alpha : \alpha < \lambda \times \kappa^+\}$.

Now we use 1.14 several times. First,

$$\mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi) \prec_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]} \mathrm{EM}_{\tau(\mathfrak{K})}(I_2^+, \Phi) \prec_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]} \mathrm{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$$

as $I_2 \subseteq I_2^+ \subseteq I_4$ are κ^+-wide, hence by $\circledast_4$ the sequence $\bar d$ realizes

$$q := \mathrm{Av}_\kappa\big(\{\langle \sigma_\zeta(\bar t^\alpha) : \zeta < \theta\rangle : \alpha < \kappa^+\}, M_2\big) = \mathrm{Av}_\kappa\big(\{\bar a_\alpha : \alpha < \kappa^+\}, M_2\big) = \mathrm{Av}_\kappa\big(\mathbf{I}, M_2\big)$$

in M_2^+ and also in $\mathrm{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$. Second, as $|I_2| < \lambda$, $I_2 \subseteq I_3^\pm \subseteq I_4$ and $|I_3^\pm| = |I_4| = \lambda$, by 1.19(1) we have $\mathrm{EM}_{\tau(\mathfrak{K})}(I_3^\pm, \Phi) \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} \mathrm{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$, so some $\bar d' \in {}^\xi(\mathrm{EM}_{\tau(\mathfrak{K})}(I_3^\pm, \Phi))$ realizes the type q in $\mathrm{EM}_{\tau(\mathfrak{K})}(I_3^\pm, \Phi)$. Let $w_1 \subseteq \lambda \times \kappa^+$ be of cardinality $\leq \theta \leq \kappa$ such that $\bar d'$ belongs to $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2 \cup \{s_\alpha : \alpha \in w_1\}, \Phi)$. Choose $w_2 \subseteq \lambda \times \kappa^+$ of order type κ^+ including w_1, so

$$\mathrm{EM}_{\tau((\mathfrak{K})}(I_2 \cup \{s_\alpha : \alpha \in w_2\}, \Phi) \prec_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]} \mathrm{EM}_{\tau(\mathfrak{K})}(I_3^\pm, \Phi)$$

and $\bar d'$ belongs to the former hence realizes q in it. But there is an isomorphism h from $I_2 \cup \{s_\alpha : \alpha \in w_2\}$ onto I_3^- over I_2, hence it induces an isomorphism $\hat h$ from $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2 \cup \{s_\alpha : \alpha \in w_2\}, \Phi)$ onto $\mathrm{EM}_{\tau(\mathfrak{K})}(I_3^-, \Phi)$ so $\hat h,(\bar d')$ realizes q in the latter. But $I_3^- \subseteq I_3$ are both κ^+-wide hence by 1.14 the sequence $\hat h(\bar d')$ realizes q in $M_3 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$ as required.

Clause (b):

By part (2) we can find appropriate $\mathbf{I}$ and then apply clause (a). $\square_{1.27}$

Remark 1.29. 1) In fact, in 1.24 we can choose B of cardinality κ, hence similarly in the proof of 1.27(1).

2) Also using solvability to get well ordered I we can prove: if $A \subseteq M = \mathrm{EM}_{\tau(\mathfrak{K})}(\lambda, \Phi)$ and $|A| < \lambda$ then the set of $\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]$-types realized in M over A is $\leq (|A| + 2)^\kappa$.

Claim 1.30. *1) If $M \in K^{**}_{\geq\kappa}$ and $\mathrm{LST}(\mathfrak{K}) \leq \theta$ and $\partial = \beth_{1,1}(\theta) \leq \kappa \leq \lambda$, <u>then</u> for $\bar a, \bar b \in {}^\theta M$ the following are equivalent: (the difference is using ∂ or κ)*

(a) $\bar a, \bar b$ realize the same $\mathbb{L}_{\infty,\partial}[\mathfrak{K}]$-type in M

(b) $\bar a, \bar b$ realize the same $\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$-type in M.

2) For $M, \theta, \partial, \kappa$ as above, the number of $\mathbb{L}_{\infty,\partial}[\mathfrak{K}]$-types of $\bar a \in {}^\theta M$ where $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$, $|I| \geq \partial$ is $\leq 2^\theta$.

[Can we say $\partial \leq |I| \leq 2^\theta$?]

Remark 1.31. Part (1) improves 1.19(3).

Proof. 1) Clearly $(b) \Rightarrow (a)$, so assume clause (a) holds. As $M \in K^{**}_{\geq\kappa}$, without loss of generality there is a κ-wide linear order I such that $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$; hence for some $J \subseteq I$, $|J| = \theta$ we have $\bar a, \bar b \in {}^\theta(\mathrm{EM}_{\tau(\mathfrak{K})}(J, \Phi))$. So for every $\alpha < (2^\theta)^+$, by the hence and forth argument for $\mathbb{L}_{\infty,\beth_\alpha^+}[\mathfrak{K}]$ there are J_α, f_α such that $J \subseteq J_\alpha \subseteq I$, $|J_\alpha| = \beth_\alpha$ and f_α is an automorphism of $\mathrm{EM}_{\tau(\mathfrak{K})}(J_\alpha, \Phi)$ which maps $\bar a$ to $\bar b$. Hence, as in the proof of 1.19, there is a linear order J^+ of cardinality μ extending J and an automorphism f of $M^+ = \mathrm{EM}_{\tau(\mathfrak{K})}(J^+, M)$ mapping $\bar a$ to $\bar b$. By clause (b) of Claim 1.14 we are done.

2) Easy by clause (c) of 1.14, i.e., by 1.18. $\square_{1.30}$

Claim 1.32. *Assume:*

(a) $I_1 \subseteq I_2$, $I_1 \neq I_2$. *Moreover,* $I_1 <_{K^{\rm flin}} I_2$, *see Definition 0.14(6)*
(b) $M_\ell = \mathrm{EM}_{\tau(\mathfrak{K})}(I_\ell, \Phi)$ *for* $\ell = 1, 2$
(c) $\bar b, \bar c \in {}^\alpha(M_2)$
(d) $\theta \geq |\alpha| + \mathrm{LST}(\mathfrak{K})$
(e) $\kappa = \beth_{1,1}(\theta_2) \leq \lambda$ *where* $\theta_1 = 2^\theta$, $\theta_2 = (2^{\theta_1})^+$
(f) $|I_1| \geq \kappa$
(g) $M_1 \leq_{\mathfrak{K}} M_2$ *(follows from (a) + (b))*

1) Assume that for every $\bar a \in {}^{\kappa>}(M_1)$ *the sequences* $\bar a \hat{\ } \bar b$, $\bar a \hat{\ } \bar c$ *realize the same* $\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$*-type in* M_2. *Then there are* I_3, M_3 *and* f *such that* $I_2 \leq_{K^{\rm flin}} I_3 \in K^{\rm flin}_\lambda$, $M_3 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$, *and* f *an automorphism of* M_3 *over* M_1 *mapping* $\bar b$ *to* $\bar c$.

2) Assume that for every $\bar a \in {}^{\kappa>}(M_1)$ *the sequences* $\bar a \hat{\ } \bar b$, $\bar a \hat{\ } \bar c$ *realize the same* $\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]$*-type in* M_2 *(as in part (a)) and* $\beth_{1,1}(\partial) \leq |I_1|$ *and* $\partial < \lambda$. *Then for every* $\bar a \in {}^{\kappa>}(M_1)$, *the sequences* $\bar a \hat{\ } \bar b$, $\bar a \hat{\ } \bar c$ *realize the same* $\mathbb{L}_{\infty,\partial}[\mathfrak{K}]$*-type in* M_2.

3) Assume that $\mathrm{cf}(\lambda) = \aleph_0$ *and* $|I_1| = \lambda$, *and recall* $\lambda = \beth_\lambda > \mathrm{LST}(\mathfrak{K})$. *If* $M_1 \leq_{\mathfrak{K}} M_2^* \in K^*_\lambda$ *then for some* I_3, *a linear order* $\leq_{K^{\rm flin}_\lambda}$*-extending* I_2 *the model* M_2^* *can be* $\leq_{\mathfrak{K}}$*-embedded into* $M_3 := \mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$ *over* M_1.

Remark 1.33. 1) Under mild assumptions with somewhat more work in 1.32(1),(3) we can choose $I_3 = I_2$ (but for this has to be more careful with the linear orders). Recall that for $I \in K^{\rm lin}_\lambda$ like I_2 in 1.8(c) we have $\alpha < \lambda^+ \Rightarrow I \times \alpha$ can be embedded into I and 1.4(1)(d).

Proof. 1) There is $J_2 \subseteq I_2$ of cardinality $\leq \theta$ such that $\bar b, \bar c \in {}^\alpha(\mathrm{EM}_{\tau(\mathfrak{K})}(J_2, \Phi))$. Let $J_1 = I_1 \cap J_2$.

We define a two-place relation $\mathcal{E}$ on $I_2 \backslash J_2$: $s \,\mathcal{E}\, t$ iff $(\forall x \in J_2)\big[x <_{I_2} s \equiv x <_{I_2} t\big]$. Clearly $\mathcal{E}$ is an equivalence relation. As $I_1 <_{K^{\rm flin}} I_2$ clearly

$\odot_1$ (α) any interval of I_1 has cardinality $|I_1| \geq \kappa$
(β) for every $t \in I_2 \setminus J_2$ the equivalence class $t/\mathcal{E}$ is a singleton or has $|I_2| \geq \kappa$ members,
(γ) for every $t \in I_1 \setminus J_1$, $(t/\mathcal{E}) \cap I_1$ is a singleton or has $|I_1| \geq \kappa$ members
(δ) $I_1 \setminus J_2$ has at least κ elements
(ε) $\mathcal{E}$ has $\leq 2^{|J_2|} \leq 2^\theta$ equivalence classes
(ζ) we may $\leq_{K^{\rm flin}}$-increase I_2, so without loss of generality
$(*)_1$ $t \in I_2 \setminus J_2 \Rightarrow |t/\mathcal{E}| = |I_2|$
$(*)_2$ For every $t \in I_1$ for some $s_1, s_2 \in I_2$ we have $s_1 <_{I_2} t <_{I_2} s_2$ and $(s_1, t_2)_{I_2}$, $(t, s_2)_{I_2}$ are disjoint to I_1.

Let $\langle \mathcal{U}_i : i < i(*)\rangle$ list the equivalence classes of $\mathcal{E}$, so without loss of generality $i(*) \leq 2^\theta$. For $\ell = 0, 1$ let $u_\ell = \{i < i(*) : \mathcal{U}_i \cap I_1$ has exactly ℓ members$\}$ and let $u_2 = i(*) \setminus u_0 \setminus u_1$, so by clause $\odot_1(\gamma)$ (i.e. the definition of $I_1 \in K^{\rm flin}$) we have $i \in u_2 \Rightarrow |\mathcal{U}_i \cap I_1| = |I_1| \geq \kappa$. For $i \in u_1$ let t^*_i be the unique member of $\mathcal{U}_i \cap I_1$.

Without loss of generality $u_1 = \{i : i \in [j^*_0, j^*_1)\}$

[Is there a type-theoretic reason why I can't just say $u_1 = [j^*_0, j^*_1)$?]

for some $j^*_0 \leq j^*_1 \leq i(*)$ and let $i'(*) = i(*) + (j^*_1 - j^*_0)$ and $u'_1 = [i(*), i'(*))$ and define $\mathcal{U}'_i$ for $i < i'(*)$ by

$\odot_2$ (a) $\mathcal{U}'_i = \mathcal{U}_i$ if $i \in u_0 \cup u_2$
(b) $\mathcal{U}'_i = \{t \in \mathcal{U}_i : t <_{I_2} t^*_i\}$ if $i \in u_1$ and
(c) $\mathcal{U}'_i = \{t \in \mathcal{U}_k : t^*_\iota <_{I_2} t\}$ if $i \in [i(*), i'(*)]$, $k \in (j^*_0, j^*_1)$ and $i - i(*) = k - j^*_0$.

[Mixing i-s and iotas in the same *paper* is never a good idea, much less in the same line. I'm changing them all to k.]

For $i < i'(*)$ let $\langle t_{i,\alpha} : \alpha < \kappa\rangle$ be a sequence of pairwise distinct members of $\mathcal{U}'_i$ such that $i \in u_2 \Rightarrow t_{i,\alpha} \in I_1$ and $i \in u_0 \Rightarrow t_{i,\alpha} \notin I_1$, this actually follows. By $\odot_1(\zeta)$ and $\odot_1(\beta), (\gamma)$ we can find such $t_{i,\alpha}$-s.

For $\zeta < \theta_2$ (see clause (e) of the assumption so $\beth_\zeta < \kappa$) let

$$J_{1,\zeta} = \{t_{i,\alpha} : i \in u_2,\ \alpha < \beth_\zeta\} \cup J_1 \cup \{t^*_i : i \in u_1\}.$$

Now by the hence and forth argument (or see 0.18(2)) for each $\zeta < \theta_2$, there are $J_{2,\zeta}$ and f_ζ such that $J_{2,\zeta} \subseteq I_2$ is of cardinality $\beth_\zeta$, it includes $J_{1,\zeta} \cup J_2$ and also $\{t_{i,\alpha} : i < i'(*)$ and $\alpha < \beth_\zeta\}$ and f_ζ is an automorphism of $\mathrm{EM}_{\tau(\mathfrak{K})}(J_{2,\zeta}, \Phi)$ over $\mathrm{EM}_{\tau(\mathfrak{K})}(J_{1,\zeta}, \Phi)$ mapping $\bar b$ to $\bar c$.

(Why? Let $\bar a_0$ list $\mathrm{EM}(J_{1,\zeta}, \Phi)$ so $\bar a_0{}^\frown \bar b$, $\bar a_0{}^\frown \bar c$ realize the same $\mathbb{L}_{\infty,\beth^+_\zeta}[\mathfrak{K}]$-type in M_2, and f be the mapping taking $\bar a_0{}^\frown \bar b$ to $\bar a_0{}^\frown \bar c$, etc.)

Now we shall imitate the proof of 1.19. By renaming without loss of generality there is a transitive set B (in the set theoretic sense) of cardinality $\le \theta_1 = 2^\theta$ which includes

$\oplus$(a) J_1, J_2
(b) Φ (i.e. τ_Φ and $\langle(\mathrm{EM}(n, \Phi), a_\ell)_{\ell<n} : n < \omega\rangle)$
(c) $\mathfrak{K}$, i.e., $\tau_{\mathfrak{K}}$ and $\{(M, N) : M \le_{\mathfrak{K}} N$ have universe included in $\mathrm{LST}(\mathfrak{K})\}$
(d) $\langle t^*_i : i \in u_1\rangle$ so each t^*_i for $i \in u_1$
(e) the ordinal $i(*)$.

Let χ be large enough, let $\mathfrak{B} = (\mathcal{H}(\chi), \in, <^*_\chi)$ and let $\mathfrak{B}^+_\zeta$ be $\mathfrak{B}$ expanded by

$\circledast_1$ (a) $Q^{\mathfrak{B}_\zeta} = \{\alpha : \alpha < \beth_\zeta\}$
(b) $P^{\mathfrak{B}_\zeta}_i = J_{2,\zeta} \cap \mathcal{U}'_i$ for $i < i'(*)$
(c) $F^{\mathfrak{B}_\zeta}_2(t) = a_t$ for $t \in I_2$
(d) $H^{\mathfrak{B}_\zeta} = f_\zeta$ and $Q^{\mathfrak{B}_\zeta}_1 = J_{1,\zeta}, Q^{\mathfrak{B}_\zeta}_2 = J_{2,\zeta}$
(e) for $i < i'(*)$, $H^{\mathfrak{B}_\zeta}_i$ is the function mapping $\alpha < \beth_\zeta$ to $t_{i,\alpha}$
(f) individual constants for B and for each $x \in B$, hence, e.g. for t^*_i (with $i \in u_1$), J_1, J_2, t for $t \in J_2$
(g) individual constants $J_{1,*}, J_{2,*}$ interpreted as the linear orders $J_{1,\zeta}, J_{2,\zeta}$, respectively, and individual constants for $M^+_\ell = \mathrm{EM}(J_{\ell,\zeta}, \Phi)$ and $\langle a_t : t \in I_\ell\rangle$ for $\ell = 1, 2$.

Clearly the vocabulary $\tau(\mathfrak{B}^+_\zeta)$ does not depend on ζ, so we call it τ^+. As in the proof of 1.19 there is a τ^+-model $\mathfrak{C}$, such that

$\boxtimes$ (a) for some unbounded $S \subseteq \theta_2$,
 (α) $\mathfrak{C}$ is a first order elementarily equivalent to $\mathfrak{B}^+_\zeta$ for every $\zeta \in S$
 (β) $\mathfrak{C}$ omits every type omitted by $\mathfrak{B}^+_\zeta$ for every $\zeta \in S$. In particular this gives
 (γ) $\mathfrak{C}$ omits the type $\{x \ne b \wedge x \in B : b \in B\}$ so
 (δ) without loss of generality $b \in B \Rightarrow b^{\mathfrak{C}} = b$
(b) $\mathfrak{C}$ is the Skolem hull of some infinite indiscernible sequence $\langle y_r : r \in I\rangle$, where I an infinite linear order and $y_r \in Q^{\mathfrak{C}}$ for $r \in I$.

Without loss of generality $I \in K^{\rm flin}$ and I_2 can be $\le_{K^{\rm flin}}$-embedded into I, say by the function g such that

$$(\forall t \in I_2)(\exists s_1, s_2 \in I)\Big[s_1 <_I g(t) <_I s_2 \wedge (\forall t' \in I_2)[t' <_{I_2} t \rightarrow g(t') <_I s_1] \wedge (\forall t' \in I_2)\big[t <_{I_2} t' \rightarrow s_2 <_I g(t')\big]\Big]$$

and also $\|\mathfrak{C}\| = |I|$. Hence for each $i < i'(*)$ there is an embedding h_i of the linear order $\mathcal{U}'_i$: i.e., $I_2 \restriction \mathcal{U}'_i$ into $(P^{\mathfrak{C}}_i, (<_{I_2})^{\mathfrak{C}})$ such that

$$t \in \mathcal{U}'_i \Rightarrow \big[t \in I_1 \Leftrightarrow h_i(t) \in Q^{\mathfrak{C}}_1\big].$$

Why?

Case 0: $i \in u_0$.
Trivial.

Case 1: $i \in u_1 \cup u'_1$.
Similar to Case 0 as $\mathcal{U}'_i \cap I_1 = \varnothing$, of course, we take care that

$$a = h_i(t) \wedge t \in \mathcal{U}'_i \wedge i \in u_1 \Rightarrow \mathfrak{C} \models \text{“}a <_{I_2} t^*_i\text{”}$$

and similarly for u_{-1}.

Case 2: $i \in u_2$.
First approximation is $h'_i = H^{\mathfrak{C}}_i \circ (g \restriction \mathcal{U}_i)$, so $t \in \mathcal{U}_i \Rightarrow h'_i(t) \in Q^{\mathfrak{C}}_1$. However by the choice of g we can find $\langle (s^-_t, s^+_t) : t \in \mathcal{U}_i\rangle$ such that:

- (α) $s^-_t, s^+_t \in Q^{\mathfrak{C}}_2$
- (β) $(s^-_t, s^+_t)_{I^{\mathfrak{C}}_2} \cap Q^{\mathfrak{C}}_2 = \{h'_i(t)\}$.

As I_2 is dense with no extremal members (being from $K^{\rm flin}$) clearly

$$t_1 <_{I_2 \restriction \mathcal{U}'_i} t_2 \Rightarrow s^+_{t_1} <_{(I_2)^{\mathfrak{C}}} s^-_{t_2}.$$

Now choose h_i by: $h_i(t) = h'_i(t)$ if $t \in I_1$ and is $s^+_{t_1}$ if $t \in I_1 \setminus I_2$.

Hence there is an embedding h of the linear order I_2 into $J^{\mathfrak{C}}_{1,*}$ such that:

- $\circledast_2$ $h(t)$ is:
  - (a) t if $t \in J_2 \cup \{t^*_i : i \in u_1\}$
  - (b) $h_i(t)$ if $t \in \mathcal{U}'_i$ and $i < i'(*)$.

Note

- $\circledast_3$ for every $t \in I_2 \setminus J_2$ for some $i < i(*) \le \theta_1$ we have

$$(\forall s \in J_2)\big[s <_{I_2} t \equiv s <_{I_2} h_i(t_{i,0})\big]$$

hence by the omitting type demand in $\boxtimes(a)(\beta)$:

- $\circledast'_3$ for $t \in I^{\mathfrak{C}}_2 \setminus J_2$, for some $i < i(*)$, we have

$$(\forall s \in J_2)\big[s <_{I^{\mathfrak{C}}_2} t \equiv s <_{I^{\mathfrak{C}}_2} h_i(t_{i,0})\big].$$

We can find a linear order I_3, $I_2 \subseteq I_3$ and an isomorphism h_* from I_3 onto $Q^{\mathfrak{C}}_2$ extending h, so clearly $I_3 \in K^{\rm flin}$ and without loss of generality $h(I_2) <_{K^{\rm flin}} I_3$. Now let $\hat{h}_*$ be the isomorphism which h_* induces from $\mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$ onto $(\mathrm{EM}_{\tau(\mathfrak{K})}(J^{\mathfrak{C}}_{2,*}, \Phi))^{\mathfrak{C}}$, so e.g., it maps for each $t \in I_2$, the member a_t of the skeleton to $F^{\mathfrak{C}}_2(h_*(t))$.

Note that h_* maps $\mathcal{U}_i \cap I_1$ into $Q^{\mathfrak{C}}_1 \subseteq I^{\mathfrak{C}}_1$ when $\mathcal{U}_i \subseteq I_1$ and is the identity on $J_1 \cup \{t^*_i : i \in u_1\}$, so recalling

$$Q^{\mathfrak{B}_\zeta} = J_{1,\zeta} = \{t_{i,\alpha} : i \in u_2,\ \alpha < \beth_\zeta\} \cup J_1 \cup \{t^*_i : i \in u_1\}$$

hence it maps I_1 into $Q_1^{\mathfrak{C}}$. However, $\mathfrak{B}_\zeta \models$ "H is a unary function, an automorphism of $\mathrm{EM}_{\tau(\mathfrak{K})}(J_{2,*}^{\mathfrak{C}}, \Phi)$ mapping $\bar b$ to $\bar c$ and is the identity on $\mathrm{EM}_{\tau(\mathfrak{K})}(J_{1,*}^{\mathfrak{C}}, \Phi)$". Now $(\hat h_*)^{-1} H^{\mathfrak{C}}(\hat h_*)$ is an automorphism of $\mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$ as required.

2) By part (1), i.e. choose I_3, M_3, f_3 as there; so as f is an automorphism of M_3 over M_1 mapping $\bar b$ to $\bar c$, clearly $\bar b, \bar c$ realize the same $\mathbb{L}_{\infty,\partial}[\mathfrak{K}]$-type over M_1 inside M_3. The desired result (the type inside M_2 rather than inside M_3) follows because

$$M_1 \prec_{\mathbb{L}_{\infty,\partial}[\mathfrak{K}]} M_2 \prec_{\mathbb{L}_{\infty,\partial}[\mathfrak{K}]} M_3$$

by 1.14(a).

3) Let $M_2^* = \bigcup_{n<\omega} M_{2,n}^*$ be such that $n < \omega \Rightarrow M_{2,n}^* \le_{\mathfrak{K}} M_{2,n+1}^*$ and $\|M_{2,n}^*\| < \lambda$. Let $\bar c_n$ list $M_{2,n}^*$ for $n < \omega$ (with no repetitions) and be such that $\bar c_n \triangleleft \bar c_{n+1}$. Let $\theta_n = \|M_{2,n}^*\| + \mathrm{LST}(\mathfrak{K})$ so without loss of generality $\theta_n = \ell g(\bar c_n)$ and let $\theta_n' = \beth_3(\theta_n)$, $\kappa_n = \beth_{1,1}(\theta_n')$, without loss of generality $\kappa_n < \theta_{n+1}$ and we choose, for each $n < \omega$, a sequence $\bar b_n \in {}^{\ell g(\bar c_n)}(M_2)$ realizing $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_n^+}[\mathfrak{K}]}(\bar c_n, M_1, M_2^*)$ in M_2. This is possible by 1.27(3), possibly after $<_{K^{\mathrm{flin}}}$-increasing I_2.

Now we choose $(I_{3,n}, f_n, M_{3,n}, \bar b_n')$ by induction on n such that

$(*)$ (a) $I_{3,0} = I_2$ and $I_{3,n} \in K_\lambda^{\mathrm{lin}}$
 (b) $n = m+1 \Rightarrow I_{3,m} <_{K^{\mathrm{flin}}} I_{3,n}$
 (c) $M_{3,n} = \mathrm{EM}_{\tau(\mathfrak{K})}(I_{3,n}, \Phi)$ (hence $n = m+1 \Rightarrow M_{3,m} \le_{\mathfrak{K}_\lambda} M_{3,n}$)
 (d) f_n is an automorphism of $M_{3,n}$ over M_1
 (e) $\bar b_n' \in {}^{\ell g(\bar b_n)}(M_{3,n})$ realizes $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_n^+}[\mathfrak{K}]}(\bar c_n, M_1, M_2^*)$
 (f) if $n = m+1$ then $\bar b_m' \trianglelefteq \bar b_n'$
 (g) if $n = m+1$ then f_n maps $\bar b_{n+1} \restriction \ell g(\bar b_n)$ to $\bar b_n'$ and f_0 maps $\bar b_0$ to $\bar b_0'$.

For $n = 0, I_{3,0}, M_{3,0}$ are defined in clauses (a),(c) of $(*)$ and we let $f_0 = \mathrm{id}_{M_2} = \mathrm{id}_{M_{3,n}}, \bar b_0' = \bar b_0$ this is trivially as required. For $n = m+1$ we apply part (1) with

$\boxdot$ $I_1, I_{3,m}, M_1, M_{3,m}, \bar b_{n+1} \restriction \ell g(\bar c_m), \bar b_m', \theta_m, \kappa_m$ here standing for $I_1, I_2, M_1, M_2, \bar b, \bar c, \theta, \kappa$ there.

Why does its assumptions hold? The main point is to check that for every $\bar a \in {}^{\kappa_m>}(M_1)$ the sequences $\bar a\hat{\ }(\bar b_{n+1} \restriction \theta_m)$, $\bar a\hat{\ }\bar b_m'$ realize the same $\mathbb{L}_{\infty,\kappa_m}[\mathfrak{K}]$-type in $M_{3,m}$. Now $\bar a\hat{\ }(\bar b_{m+1} \restriction \theta_m)$, $\bar a\hat{\ }\bar b_m'$ realize the same $\mathbb{L}_{\infty,\kappa_m}[\mathfrak{K}]$-type in $M_{3,m}$ by the induction hypothesis. Also, the sequences $\bar b_{n+1} \restriction \theta_m$, $\bar b_{m+1} \restriction \theta_m$ satisfy for any $\bar a \in {}^{\kappa_m>}(M_1)$ the sequences $\bar a\hat{\ }(\bar b_{n+1} \restriction \theta_m)$, $\bar a\hat{\ }(\bar b_{m+1} \restriction \theta_m)$ realize the same $\mathbb{L}_{\infty,\kappa_m}[\mathfrak{K}]$-type in $M_{3,m}$ because the $\mathbb{L}_{\infty,\kappa_m}[\mathfrak{K}]$-type which $\bar a\hat{\ }(\bar b_{n+1} \restriction \theta_m)$ realizes in $M_{3,m}$ is the same as the $\mathbb{L}_{\infty,\kappa_m}[\mathfrak{K}]$-type it realizes in $M_2 = M_{3,0}$ which (by the choice of $\bar b_{n+1}$) is equal to the $\mathbb{L}_{\infty,\kappa_m}[\mathfrak{K}]$-type which $\bar a\hat{\ }(\bar c_{n+1} \restriction \theta_m)$ realizes in M_2^* which is the same as the $\mathbb{L}_{\infty,\kappa_m}[\mathfrak{K}]$-type which $\bar a\hat{\ }(\bar c_{m+1} \restriction \theta_m)$ realizes in M_2^* which is equal to the $\mathbb{L}_{\infty,\kappa_m}[\mathfrak{K}]$-type which $\bar a\hat{\ }(\bar b_{m+1} \restriction \theta_m)$ realizes in $M_{3,m}$.

By the last two sentences for every $\bar a \in {}^{\kappa_m>}(M_1)$ the sequences $\bar a\hat{\ }(\bar b_{n+1} \restriction \theta_m)$, $\bar a\hat{\ }\bar b_m'$ realize the same $\mathbb{L}_{\infty,\kappa_m}[\mathfrak{K}]$-type in $M_{3,m}$, so indeed the assumptions of part (1) holds for the case we are trying to apply it (see $\boxdot$ above).

So we get the conclusion of part (1), i.e. we get $I_{3,n}, f_n$ here standing for I_3, f there so $I_{3,m} <_{K_\lambda^{\mathrm{flin}}} I_{3,n}$ and f_n is an automorphism of $M_{3,n} = \mathrm{EM}_{\tau(\mathfrak{K})}(I_{3,n}, \Phi)$ over M_1 mapping $\bar b_{n+1} \restriction \theta_m$ to $\bar b_m'$. Now we let $\bar b_n' = f_n(\bar b_{n+1} \restriction \theta_n)$ and can check all the clauses in $(*)$. Hence we have carried the induction. So we can satisfy $(*)$.

So $\bar b_n'$ satisfies the requirements on $\bar b_n$ and $\bar b_n' \triangleleft \bar b_{n+1}'$. Let $I_3 = \bigcup\{I_{3,n} : n < \omega\}$ and let $M_3 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$ and let $g : M_2^* \to M_3$ map $c_{n,i}$ to $b_{n,i}'$ for $i < \ell g(\bar c_n)$, $n < \omega$, easily it is as required. That is, $g(c_{n,i})$ is well defined as $c_{n,i} \mapsto b_{n,i}'$ (for $i < \ell g(\bar c_n)$) is a well defined mapping for each n and

$$i < \ell g(\bar c_n) \Rightarrow c_{n,i} = c_{n+1,i} \wedge b_{n,i}' = b_{n+1,i}'.$$

Also $g \restriction \{c_{n,i} : i < \ell g(\bar{c}_n)\}$ is a $\leq_{\mathfrak{K}}$-embedding of $M^*_{2,n}$ into M_3 and is the identity on $M^*_{2,n} \cap M_1$ as $\bar{c}_n$ list the elements of $M_{2,i}$ and

$$\mathrm{tp}_{\mathbb{L}_{\infty,\kappa^+_n}[\mathfrak{K}]}(\bar{c}_n, M_1, M^*_2) = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa^+_n}[\mathfrak{K}]}(\bar{b}'_n, M_1, M_3)$$

by clause (e) of $(*)$. But $\langle g \restriction M^*_{2,n} : n < \omega\rangle$ is $\subseteq$-increasing with union g so by Ax.V of AEC g is a $\leq_{\mathfrak{K}}$-embedding of M^*_2 into M_3. Lastly, obviously

$$g \supseteq \bigcup\{\mathrm{id}_{M^*_{2,n} \cap M_1} : n < \omega\} = \mathrm{id}_{M_1}$$

so we are done. $\square_{1.32}$

We arrive to the crucial advance:

Theorem 1.34. *The Amalgamation Theorem:*
If $\mathrm{cf}(\lambda) = \aleph_0$, *then* $\mathfrak{K}^*_\lambda$ *(i.e.* $(K^*_\lambda, \leq_{\mathfrak{K}} \restriction K^*_\lambda)$*) has amalgamation, even disjoint one.*

Proof. Assume $M_0 \leq_{\mathfrak{K}^*_\lambda} M_\ell$ for $\ell = 1, 2$. Choose $I_0 \in K^{\mathrm{flin}}_\lambda$ so

$$M'_0 := \mathrm{EM}_{\tau(\mathfrak{K})}(I_0, \Phi) \in K^*_\lambda$$

but K^*_λ is categorical (see 1.16 or 1.19(4)) hence $M'_0 \cong M_0$, so without loss of generality $M'_0 = M_0$. Choose $I_1 \in K^{\mathrm{flin}}_\lambda$ such that $I_0 <_{K^{\mathrm{flin}}} I_1$ and let $M'_1 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi)$ so $M_0 \leq_{\mathfrak{K}} M'_1$. By applying 1.32(3) with I_0, I_1, M_0, M'_1, M_1 here standing for $I_1, I_2, M_1, M_2, M^*_2$ there, we can find a pair (I_2, f_1) such that $I_1 <_{K^{\mathrm{flin}}_\lambda} I_2$ and f_1 is a $\leq_{\mathfrak{K}}$-embedding of M_1 into $M'_2 := \mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$ over M_0. Apply 1.32(3) again with $I_0, I_2, M_0, \mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi), M_2$ here standing for $I_1, I_2, M_1, M_2, M^*_2$ there. So there is a pair (I_3, f_2) such that $I_2 <_{K^{\mathrm{flin}}_\lambda} I_3$ and f_2 is $\leq_{\mathfrak{K}}$-embedding M_2 into $M_3 := \mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$ over $M_0 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$. Of course, $M_3 \in K^*_\lambda$ and we are done proving the "has amalgamation."

Why disjoint? Let (I_4, h) be such that $I_3 <_{K^{\mathrm{flin}}_\lambda} I_4$ and h is a $\leq_{K^{\mathrm{flin}}}$-embedding of I_3 into I_4 over I_0 such that $h(I_3) \cap I_3 = I_0$. Now h induces an isomorphism $\hat{h}$ from $\mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$ onto $\mathrm{EM}_{\tau(\mathfrak{K})}(h(I_3), \Phi) \leq_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$.

Lastly, by our assumptions on Φ if $J_1, J_2 \subseteq J$ are linear orders and $J_1 \cap J_2$ is a dense linear order (in particular with neither first nor last member, e.g. are from $K^{\mathrm{flin}}_\lambda$ as in our case) then

$$\mathrm{EM}_{\tau(\mathfrak{K})}(J_1, \Phi) \cap \mathrm{EM}_{\tau(\mathfrak{K})}(J_2, \Phi) = \mathrm{EM}_{\tau(\mathfrak{K})}(J_1 \cap J_2, \Phi).$$

So in particular, above

$$\mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi) \cap \mathrm{EM}_{\tau(\mathfrak{K})}(\hat{h}(I_3, \Phi) = \mathrm{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$$

and $f_1, \hat{h} \circ f_2$ are $\leq_{\mathfrak{K}}$-embeddings of M_1, M_2 respectively over $M_0 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$ into $\mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi) \leq_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$ and $\mathrm{EM}_{\tau(\mathfrak{K})}(h(I_3), \Phi) \leq_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(I_4, \Phi)$, respectively, so we are done. $\square_{1.34}$

Claim 1.35. *Assume* $\mathrm{cf}(\lambda) = \aleph_0$. *If* $\delta < \lambda^+$, *the sequence* $\langle M_i : i < \delta\rangle$ *is* $\leq_{\mathfrak{K}}$-*increasing continuous and* $M_i \in K^*_\lambda$ *for* $i < \delta$, *then* $M_\delta := \bigcup\{M_i : i < \delta\}$ *can be* $\leq_{\mathfrak{K}}$-*embedded into some member of* K^*_λ.

Proof. We choose $I_i \in K^{\mathrm{flin}}_\lambda$ by induction on $i \leq \delta$, which is $<_{K^{\mathrm{flin}}_\lambda}$-increasing continuous with i, and a $\leq_{\mathfrak{K}}$-embedding f_i of M_i into $N_i := \mathrm{EM}_{\tau(\mathfrak{K})}(I_i, \Phi)$, increasing continuous with i. For $i = 0$ choose $I_0 \in K^{\mathrm{flin}}_\lambda$, so $N_0 := \mathrm{EM}_{\tau(\mathfrak{K})}(I_0, M)$ is isomorphic to M_0 hence f_0 exists; for i limit use $I_i := \bigcup\{I_j : j < i\}$ and $f_i := \bigcup\{f_j : j < i\}$. So assume $i = j + 1$. Now we can find M'_i, f'_i satisfying: f'_i is an isomorphism from M_i onto M'_i extending f_j such that $f_j(M_j) \leq_{\mathfrak{K}} M'_i$ (actually this trivially follows) and $M'_i \cap N_j = f_j(M_j)$; so also M'_i belongs to K^*_λ. Now $f_j(M_j)$, $\mathrm{EM}_{\tau(\mathfrak{K})}(I_j,\Phi)$, M'_i can be disjointly amalgamated (by 1.34) in $(K^*_\lambda, \le_{\mathfrak{K}})$, so there is $M^*_i \in K^*_\lambda$ such that $N_j = \mathrm{EM}_{\tau(\mathfrak{K})}(I_j,\Phi) \le_{\mathfrak{K}} M^*_i$ and $M'_i \le_{\mathfrak{K}} M^*_i$. Now by 1.32(3) there are I_i, g_i such that $I_j <_{K^{\rm flin}_\lambda} I_i$ and g_i is a $\le_{\mathfrak{K}}$-embedding of M^*_i into $N_i := \mathrm{EM}_{\tau(\mathfrak{K})}(I_i,\Phi)$ over $\mathrm{EM}_\tau(I_j,\Phi)$. Let $f_i = g_i \circ f'_i$, clearly it is as required. Having carried the induction, f_δ is a $\le_{\mathfrak{K}}$-embedding of M_δ into $\mathrm{EM}_{\tau(\mathfrak{K})}(\bigcup\limits_{j<\delta} I_j,\Phi)$, as promised. $\square_{1.35}$

Claim 1.36. *1) Assume* $\mathrm{cf}(\lambda) = \aleph_0$. *For every* $M_0 \in K^*_\lambda$ *there is a* $\le_{\mathfrak{K}}$*-extension* $M_1 \in K^*_\lambda$ *of* M_0 *such that: if* $M_0 \le_{\mathfrak{K}_\lambda} M_2 \in K^*_\lambda$ *and* $\bar a \in {}^{\lambda>}(M_2)$ *<u>then</u> for some* (M_3, f) *we have:*

$$\boxed{M_1 \le_{\mathfrak{K}} M_3 \in K^*_\lambda, f \text{ is a } \le_{\mathfrak{K}}\text{-embedding of } M_2 \text{ into } M_3 \text{ over } M_0 \text{ and } f(\bar a) \in {}^{\lambda>}(M_2).}$$

2) Assume $\mathrm{cf}(\lambda) = \aleph_0$. *For every* $M_0 \in K^*_\lambda$ *there is a* $\le_{\mathfrak{K}}$*-extension* $M_1 \in K^*_\lambda$ *which is universal over* M_0 *for* $\le_{\mathfrak{K}_\lambda}$*-extensions.*

3) If (A) then (B), where

(A) $I_0 \le_{K^{\rm flin}_\lambda} I'_1 <_{K^{\rm flin}_\lambda} I_1$

(B) If $I_0 \subseteq I_2 \in K^{\rm flin}_\lambda$, $\beta \le \gamma < \lambda$, $\bar b_1 \in {}^\beta(\mathrm{EM}_{\tau(\mathfrak{K})}(I'_1,\Phi))$, $\bar c_2 \in {}^\gamma(\mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi))$, $\bar b_2 = \bar c_2 \restriction \beta$, *and for every* $\kappa < \lambda$ *we have*

$$\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar b_1, \mathrm{EM}_{\tau(\mathfrak{K})}(I_0,\Phi), \mathrm{EM}_{\tau(\mathfrak{K})}(I_1,\Phi)) = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar b_2, \mathrm{EM}_{\tau(\mathfrak{K})}(I_0,\Phi), \mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi))$$

<u>then</u> for some (I^+_1, f) *we have* $I_1 \le_{K^{\rm flin}} I^+_1 \in K^{\rm flin}_\lambda$ *and* f *is a* $\le_{\mathfrak{K}}$*-embedding of* $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi)$ *into* $\mathrm{EM}_{\tau(\mathfrak{K})}(I^+_1,\Phi)$ *over* $\mathrm{EM}_{\tau(\mathfrak{K})}(I_0,\Phi)$ *mapping* $\bar b_2$ *to* $\bar b_1$ *and* $\bar c_2$ *into* $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1,\Phi)$.

4) Assume $\mathrm{cf}(\lambda) = \aleph_0$. *If (C) then (D) (and moreover* $(D)^+$*) when*

(C) $\langle J_\alpha : \alpha \le \omega\rangle$ *is* $<_{K^{\rm flin}_\lambda}$*-increasing,* $I_0 = J_0$, $I_1 = J_\omega$.

(D) If $I_0 \subseteq I_2 \in K^{\rm flin}_\lambda$ *then some* f *is a* $\le_{\mathfrak{K}}$*-embedding of* $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi)$ *into* $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1,\Phi)$ *over* $\mathrm{EM}_{\tau(\mathfrak{K})}(I_0,\Phi)$.

$(D)^+$ $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1,\Phi)$ *is* $\le_{\mathfrak{K}^*_\lambda}$*-universal over* $\mathrm{EM}_{\tau(\mathfrak{K})}(I_0,\Phi)$.

Proof. Note that by 1.32(3) clearly (3) $\Rightarrow$ (1) and (4) $\Rightarrow$ (2). So we shall prove (3) and (4).

3) First assume $\beta = 0$, $\gamma = 1$ so $\bar c_2 = \langle c\rangle$. Toward contradiction assume $I_0 \subseteq I_2 \in K^{\rm lin}_\lambda$, $a \in M_2 := \mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi)$ but there is no pair (I^+_1, f) as required in clause (b). Without loss of generality for some I_3 we have $I_0 \le_{K^{\rm flin}_\lambda} I_2 \le_{K^{\rm flin}_\lambda} I_3$ and $I_0 \le_{K^{\rm flin}_\lambda} I_1 \le_{K^{\rm flin}_\lambda} I_3$.

Let $\mathrm{EM}(I_2,\Phi) \models$ "$c_2 = \sigma(a_{t^2_0},\dots,a_{t^2_{n-1}})$" where $\sigma(x_0,\dots,x_{n-1})$ a τ_Φ-term, $n < \omega$ and $I_2 \models$ "$t^2_0 < \dots < t^2_{n-1}$". Let $u = \{\ell < n : t^2_\ell \in I_0\}$. As $I_0 <_{K^{\rm flin}_\lambda} I_1$, we can find $\langle t^1_0,\dots,t^1_{n_1}\rangle$ such that:

⊛ (a) $t^1_\ell \in I_1$ for $\ell < n$
 (b) $t^1_0 <_{I_1} \dots <_{I_1} t^1_{n-1}$
 (c) if $\ell \in u$ then $t^2_\ell = t^1_\ell (\in I_0)$
 (d) if $\ell < n \wedge \ell \notin u$ then $t^1_\ell \in I_1 \setminus I_0$
 (e) if $\ell_1 \le \ell_2 < n$ and $[\ell_1,\ell_2] \cap u = \varnothing$ then $t^2_{\ell_2} <_{I_3} t^1_{\ell_1}$.

Let $M_\ell = \mathrm{EM}_{\tau(\mathfrak{K})}(I_\ell,\Phi)$ for $\ell = 0,1,2,3$ and let

$$c_2 = c,\ c_1 = \sigma^{\mathrm{EM}(I_1,\Phi)}(a_{t^1_0},\dots,a_{t^1_{n-1}}).$$

Let $\kappa < \lambda$ be large enough such that $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa^+}[\mathfrak{K}]}(c_\ell, M_0, M_\ell)$ for $\ell = 1,2$ be distinct (exists by 1.32(1) because its conclusion fails by the "toward contradiction"). We easily get contradiction to the non-order property (see $(*)$ of 1.5(2)).

Note that if in addition $\langle I_{1,\alpha} : \alpha \le \lambda\rangle$ is $<_{K^{\rm flin}_\lambda}$-increasing continuous, $I_{1,0} = I'_1$, $I_{1,\lambda} = I_1$ then by what we have just proved and the proof of [She09c, 4.2a] we can prove the general case (and part (4)). But we also give a direct proof.

In the general case, let $\theta = |\beta| + \aleph_0$, so we assume clause (a) and the assumptions of clause (b) and without loss of generality $I_1 \cap I_2 = I_0$ hence there is I_3 such that $I_\ell <_{K^{\rm flin}_\lambda} I_3$ for $\ell = 1,2$. Let $\kappa \in (\theta, \lambda)$ be large enough.

Hence

$$\mathrm{EM}_{\tau(\mathfrak{K})}(I_0,\Phi) \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} \mathrm{EM}_{\tau(\mathfrak{K})}(I_\ell,\Phi) \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} \mathrm{EM}_{\tau(\mathfrak{K})}(I_3,\Phi)$$

for $\ell = 1,2$. Applying 1.32(1) with $I_1, I_2, \bar b, \bar c$ there standing for $I_0, I_3, \bar b_1, \bar b_2$ here we can find a pair (I_4, f_4) such that $I_3 <_{K^{\rm flin}_\lambda} I_4$ and f_4 is an automorphism of $M_4 := \mathrm{EM}_{\tau(\mathfrak{K})}(I_4,\Phi)$ over $\mathrm{EM}_{\tau(\mathfrak{K})}(I_0,\Phi)$ mapping $\bar b_2$ to $\bar b_1$. Clearly

$$M_3 := \mathrm{EM}_{\tau(\mathfrak{K})}(I_3,\Phi) \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} \mathrm{EM}_{\tau(\mathfrak{K})}(I_4,\Phi).$$

So $f_4(\bar c_2) \in {}^\gamma(M_4)$, hence we can apply clause (b) of Claim 1.27(3) with

$$M_1, M_2, I_2, N, \xi, \bar d^*$$

there standing for

$$\mathrm{EM}_{\tau(\mathfrak{K})}(I'_1,\Phi), \mathrm{EM}_{\tau(\mathfrak{K})}(I_1,\Phi), I_1, \mathrm{EM}_{\tau(\mathfrak{K})}(I_4,\Phi), \gamma, f_4(\bar c_2)$$

here. Hence we can find $\bar c'_2 \in {}^\gamma(M_1)$ realizing in M_1 the type

$$\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}\big(f_4(\bar c_2), \mathrm{EM}_{\tau(\mathfrak{K})}(I'_1,\Phi), \mathrm{EM}_{\tau(\mathfrak{K})}(I_1,\Phi)\big)$$

Lastly, applying Claim 1.32(1) with $I_1, I_2, \bar b, \bar c$ there standing for $I'_1, I_4, f_4(\bar c_2), \bar c'_2$ here, clearly there is a pair (I_5, f_5) such that $I_4 <_{K^{\rm flin}_\lambda} I_5$ and f_5 is an automorphism of $\mathrm{EM}_{\tau(\mathfrak{K})}(I_5,\Phi)$ over $\mathrm{EM}(I'_1,\Phi)$ mapping to $f_4(\bar c_2)$ to $\bar c'_2$.

Let $I_1^+ := I_5$, $f = f'_5 \circ f'_4$ where $f'_5 = f_5 \restriction \mathrm{EM}_{\tau(\mathfrak{K})}(I_4,\Phi))$ and $f'_4 = f_4 \restriction \mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi)$. Now I_1^+, f are as required because $f_4(\bar b_2) = \bar b_1$ while $f_5(\bar b_1) = \bar b_1$.

4) Easy by part (3). First note that $(\mathrm{d})^+$ follows by (d) by 1.32(3), so we shall ignore clause $(\mathrm{d})^+$. Let $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi)$ be $\bigcup\{M_{2,n} : n < \omega\}$ where $M_{2,n} \in K_{<\lambda}$ and $n < \omega \Rightarrow M_{2,n} \le_{\mathfrak{K}} M_{2,n+1}$.

Let $\bar a_n$ list the elements of $M_{2,n}$ with no repetitions such that $\bar a_n \lhd \bar a_{n+1}$ for $n < \omega$. By induction on n, we choose $\bar b_n$ such that

⊛ (a) $\bar b_n \in {}^{\ell g(\bar a_n)}(\mathrm{EM}_{\tau(\mathfrak{K})}(J_{n+1},\Phi))$
(b) If $n = m+1$ then $\bar b_m \lhd \bar b_n$
(c) For every $\kappa < \lambda$, the type $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}\big(\bar b_n, \mathrm{EM}_{\tau(\mathfrak{K})}(I_0,\Phi), \mathrm{EM}_{\tau(\mathfrak{K})}(I_{n+1},\Phi)\big)$ is equal to the type $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}\big(\bar a_n, \mathrm{EM}_{\tau(\mathfrak{K})}(I_0,\Phi), \mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi)\big)$.

The induction step is by part (3). Let f_n be the unique function mapping $\bar a_n$ to $\bar b_n$ (with domain $\mathrm{rang}(\bar a_n)$). So $f_n \subseteq f_{n+1}$ and f_n is a $\le_{\mathfrak{K}}$-embedding of $M_{2,n}$ into $\mathrm{EM}_{\tau(\mathfrak{K})}(J_{n+1},\Phi)$ but $J_{n+1} \subseteq I_1$ hence into $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1,\Phi)$. So $f := \bigcup\{f_n : n < \omega\}$ is a $\le_{\mathfrak{K}}$-embedding of $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2,\Phi)$ into $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1,\Phi)$. Also, f_n is the identity on $\mathrm{rang}(\bar a_n) \cap \mathrm{EM}_{\tau(\mathfrak{K})}(I_0,\Phi)$ hence f is the identity on

$$\bigcup_n \mathrm{rang}(\bar a_n) \cap \mathrm{EM}_{\tau(\mathfrak{K})}(I_0,\Phi) = \mathrm{EM}_{\tau(\mathfrak{K})}(I_0,\Phi)$$

so f is as required. $\square_{1.36}$

Exercise: 1) Assume $\mathfrak{K}_\lambda = (K_\lambda, \le_{\mathfrak{K}_\lambda})$ satisfies axioms I, II (and 0, presented below) and amalgamation. Then $\mathbf{tp}(a, M, N)$ for $M \le_{\mathfrak{K}_\lambda} N$ and $a \in N$ and $\mathcal{S}_{\mathfrak{K}_\lambda}(M)$ are well defined and has the basic properties of types from [She09c, §1].

2) If in addition $\mathfrak{K}_\lambda$ satisfies Ax.III$^\odot$ below and $\mathfrak{K}_\lambda$ is stable (i.e. $|\mathcal{S}_{\mathfrak{K}_\lambda}(M)| \le \lambda$ for $M \in K_\lambda$) then every $M \in \mathfrak{K}_\lambda$ has a $\le_{\mathfrak{K}}$-universal extension N which means $M \le_{\mathfrak{K}_\lambda} N$ and

$$(\forall N')\Big(M \le_{\mathfrak{K}_\lambda} N' \Rightarrow (\exists f)\big[f \text{ is a } \le_{\mathfrak{K}_\lambda}\text{-embedding of } N' \text{ into } N \text{ over } M\big]\Big).$$

3) Ax.III (see [She09c, 0.2]) implies Ax.III$^\odot$
where:

- Ax.0: K is a class of $\tau_{\mathfrak{K}}$-models, $\le_{\mathfrak{K}}$ a two place relation of K_λ, both preserved under isomorphisms
- Ax.I: if $M \le_{\mathfrak{K}_\lambda} N$ then $M \subseteq N$ (are $\tau(\mathfrak{K}_\lambda)$-models of cardinality λ
- Ax.II: $\le_{\mathfrak{K}_\lambda}$ is a partial order (so $M \le_{\mathfrak{K}_\lambda} M$ for $M \in K_\lambda$)
- Ax.III$^\odot$: In following game the player COM has a winning strategy. A play lasts λ moves, and the players take turns to construct a $\le_{\mathfrak{K}_\lambda}$-increasing continuous sequence $\langle M_\alpha : \alpha \le \lambda\rangle$. In the α^{th} move, M_α is chosen by INC if α is even or by COM is α is odd. Now COM wins if INC always has a legal move.
- Ax.IV$^\odot$: For each $M \in K_\lambda$, in the following game, INC has no winning strategy: a play lasts $\lambda + 1$ moves; in the α^{th} move $f_\alpha, M_\alpha, N_\alpha$ are chosen such that f_α is a $\le_{\mathfrak{K}}$-embedding of M_α into N_α, both are $\le_{\mathfrak{K}_\lambda}$-increasing continuous, f_α is $\subseteq$-increasing continuous, $M_0 = M$ and in the α^{th} move, M_α is chosen by INC, and the pair is chosen by the player INC if α is even and by the player COM if α is odd. The player COM wins if INC has always a legal move (the player COM always has: he can choose $N_\alpha = M_\alpha$)

Definition 1.37. 1) Let $<^*_\lambda = <^*_{\mathfrak{K}^*_\lambda}$ be the following two-place relation on K^*_λ (so $M \le^*_{\mathfrak{K}^*_\lambda} N$ mean $M = N \in \mathfrak{K}^*_\lambda$ or $M <^*_{\mathfrak{K}^*_\lambda} N$):

$$\boxed{M_1 <^*_\lambda M_2 \text{ iff } M_1 \le_{\mathfrak{K}_\lambda} M_2 \text{ are from } K^*_\lambda \text{ and } M_2 \text{ is } \le_{\mathfrak{K}_\lambda}\text{-universal over } M_1.}$$

2) For $\alpha < \lambda$, $\kappa = \beth_{1,1}(|\alpha| + \mathrm{LST}(\mathfrak{K}))$ and $M \in K^*_\lambda$ let $\mathrm{Sav}^{\mathrm{bs},\alpha}(M)$ be the set of $\{\mathrm{Av}_\kappa(\mathbf{I}, M) : \mathbf{I}$ is a $((2^\kappa)^+, \kappa)$-convergent subset of ${}^\alpha M\}$. We define $\mathrm{tp}_*(\bar a, M, N)$ when $M \le_{\mathfrak{K}} N$ are from K^*_λ and $\bar a \in {}^\alpha N$, as $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a, M, N) \in \mathrm{Sav}^{\mathrm{bs},\alpha}(M)$ naturally.

3) Let $\mathfrak{K}^*_\lambda = (K^*_\lambda, \le_{\mathfrak{K}} \restriction \mathfrak{K}^*_\lambda, \le^*_{\mathfrak{K}^*_\lambda})$ (see 1.38 below) but if $(K^*_\lambda, \le_{\mathfrak{K}} \restriction K^*_\lambda)$ is a λ-AEC then we omit $\le^*_{\mathfrak{K}^*_\lambda}$.

Remark 1.38. 1) Note that the relation $<^*_\lambda = <^*_{\mathfrak{K}_\lambda}$ seemingly depends on the choice of Φ. However, assuming μ-solvability, by 1.40(2) below it does not depend.

2) The proof of 1.40 is like [She09c, 0.22(3)].

3) So $\mathfrak{K}^*_\lambda$ is a semi-λ-AEC (see [She]) but we do not use this notion here.

Claim 1.39. *Assume* $\mathrm{cf}(\lambda) = \aleph_0$.

0) If $M \in K^*_\lambda$ then for some $N \in K^*_\lambda$, $M <^*_{\mathfrak{K}^*_\lambda} N$.

1) If $M \le_{\mathfrak{K}} N$ are from K^*_λ, $\alpha < \lambda$ and $\bar a \in {}^\alpha N \setminus {}^\alpha M$ then $\bar a$ realizes some $p \in \mathrm{Sav}^{\mathrm{bs},\alpha}(M)$.

2) If $M_0 \le_{\mathfrak{K}} M_1 <^*_{\mathfrak{K}^*_\lambda} M_2 \le_{\mathfrak{K}} M_3$ and $M_\ell \in K^*_\lambda$ for $\ell < 4$, then $M_0 <^*_{\mathfrak{K}^*_\lambda} M_3$.

Proof. 0) As K^*_λ is categorical (by 1.16(1)) this follows by 1.36(2).

1) A proof of this is included in the proof of 1.32(2), i.e. by 1.27(1).

2) Easy, recalling amalgamation. $\square_{1.39}$

Claim 1.40. *Assume* $\mathrm{cf}(\lambda) = \aleph_0$.

1) Assume $\langle M_i : i \le \delta\rangle$ is $\le_{\mathfrak{K}_\lambda}$-increasing continuous and $M_{2i+1} <^*_{\mathfrak{K}^*_\lambda} M_{2i+2}$ for $i < \delta$. Then $M_\delta \in K^*_\lambda$.

2) Assume that $\langle M^\ell_i : i \le \delta\rangle$ is an $\le_{\mathfrak{K}^*_\lambda}$-increasing continuous sequence such that $M^\ell_{2i+1} <^*_{\mathfrak{K}^*_\lambda} M^\ell_{2i+2}$ for $i < \delta$ all for $\ell = 1, 2$. Any isomorphism f from M^1_0 onto M^2_0 (or just a $\le_{\mathfrak{K}_\lambda}$-embedding) can be extended to an isomorphism from M^1_δ onto M^2_δ.

Proof. 1) We prove this by induction on δ, hence without loss of generality $i < \delta \Rightarrow M_i \in K^*_\lambda$.

Let $M^1_\alpha = M_\alpha$ for $\alpha \le \delta$ and let $\langle I_\alpha : \alpha \le \delta\rangle$ be $<_{K^{\mathrm{flin}}_\lambda}$-increasing. Let $M^2_\alpha = \mathrm{EM}_{\tau(\mathfrak{K})}(I_\alpha, \Phi)$. Now there is an isomorphism f from M^1_0 onto M^2_0 as K^*_λ is categorical, so by part (2) there is an isomorphism g from M^1_α onto M^2_α, but $M^2_\alpha \in K^*_\lambda$ so we are done.

2) Note

$\boxtimes_2$ without loss of generality

$\boxdot$ $M^2_i <^*_\lambda M^2_{i+1}$.

[Why? We can find $\langle M^3_i : i \le \delta\rangle$ which is $\le^*_{\mathfrak{K}^*_\lambda}$-increasing continuous and $M^3_0 = M^2_0$ and $M^3_i <^*_\lambda M^3_{i+1}$. Now apply the restricted version (i.e., with the assumption $\boxdot$) twice.]

By induction on $i \le \delta$ we choose (f_i, N^1_i, N^2_i) such that

$\circledast$ (a) N^1_i, N^2_i belong to K^*_λ
(b) f_i is an isomorphism from N^1_i onto N^2_i
(c) N^1_i, N^2_i, f_i are increasing continuous with i
(d) For $i = 0$, $N^1_i = M^1_i$, $f_i = f$ and N^2_i is $f(M^1_i) = M^2_i$
(e) If $i > 0$ is a limit ordinal then $N^1_i = M^1_i$ and $N^2_i = M^2_i$
(f) When $i = \omega\alpha + 2n < \delta$ we have
(α) $N^1_{\omega\alpha+2n+1} = M^1_{\omega\alpha+2n+1}$
(β) $N^2_{\omega\alpha+2n+1} \le_{\mathfrak{K}} M^2_{\omega\alpha+2n+1}$
(γ) $N^1_{\omega\alpha+2n+2} \le_{\mathfrak{K}} M^1_{\omega\alpha+2n+2}$
(δ) $N^2_{\omega\alpha+2n+2} = M^2_{\omega\alpha+2n+2}$.

Case 1: $i = 0$.

This is trivial by clause (d) and the assumption of the claim on f.

Case 2: $i = \omega\alpha + 2n + 1$.

Note that $N^2_{\omega\alpha+2n} = M^2_{\omega\alpha+2n}$. [Why? If $i = 0$ (i.e. $\alpha = 0 = n$) by $\circledast(d)$, and if i is a limit ordinal (i.e. $\alpha > 0 \wedge n = 0$) by clause (e) of $\circledast$, and if $n > 0$ by clause $(f)(\delta)$ of $\circledast$.]

Now we let $N^1_i = N^1_{\omega\alpha+2n+1} := M^1_{\omega\alpha+2n+1}$ and hence satisfying clause $(f)(\alpha)$ of $\circledast$. So

$$N^1_{i-1} = N^1_{\omega\alpha+2n} \le_{\mathfrak{K}} M^1_{\omega\alpha+2n} \le_{\mathfrak{K}} M^1_{\omega\alpha+2n+1} = N^1_{\omega\alpha+2n+1} = N^1_i.$$

Note that $N^2_{i-1} = N^2_{\omega\alpha+2n} <^*_\lambda M^2_{\omega\alpha+2n}$ by $\boxdot$ above hence we can apply Definition 1.37(1) and find an extension f_i of f_{i-1} to $\le_{\mathfrak{K}}$-embedding of $N^1_i = M^1_{\omega\alpha+2n+1}$ into $M^2_{\omega\alpha+2n+1}$ and let $N^2_i := f_i(N^1_i)$.

Case 3: $i = \omega\alpha + 2n + 2$.

Note that $N^1_{\omega\alpha+2n+1} = M^1_{\omega\alpha+2n+1}$ by clause $(f)(\alpha)$ of $\circledast$ hence by the assumption of the claim $N^1_{\omega\alpha+2n+1} <^*_{\mathfrak{K}^*_\lambda} M^1_{\omega\alpha+2n+2}$. We choose $N^2_{\omega\alpha+2n+2} := M^2_{\omega\alpha+2n+2}$ hence

$$N^2_{i-1} = N^2_{\omega\alpha+2n+1} \le_{\mathfrak{K}} M^2_{\omega\alpha+2n+1} \le_{\mathfrak{K}} M^2_{\omega\alpha+2n+2} = N^2_{\omega\alpha+2n+2} = N^2_i.$$

Now we apply Definition 1.37(1) to find a $\le_{\mathfrak{K}}$-embedding g_i of $N^2_{\omega\alpha+2n+2}$ into $M^1_{\omega\alpha+2n+2}$ extending f^{-1}_{i-1}.

Lastly, let $f_i = g_i^{-1}$ and $N^1_i = M^1_i \restriction \mathrm{dom}(f_i)$. So we can carry the induction, hence we can prove the claim. $\square_{1.40}$

Note that now we use more than in Hypothesis 1.18.

Claim 1.41. *Assume*

$\boxtimes$ *(a)* $\langle \lambda_n : n < \omega \rangle$ *is increasing,* $\lambda = \lambda_\omega = \sum\limits_{n<\omega} \lambda_n$ *satisfying* $\lambda_n = \beth_{\lambda_n} > \mathrm{LST}(\mathfrak{K})$ *and* $\mathrm{cf}(\lambda_n) = \aleph_0$ *for* $n < \omega$.

(b) $\Phi \in \Upsilon^{\mathrm{or}}_{\mathfrak{K}}$, *and each* λ_n *and* $\lambda = \lambda_\omega$ *is as in Hypothesis 1.18, or just satisfies all its conclusions so far.*

1) K^*_λ *is closed under unions* $\le_{\mathfrak{K}}$*-increasing chains (of length* $< \lambda^+$*).*

2) If $M_n \in K^*_{\lambda_n}, M_n \le_{\mathfrak{K}} M_{n+1}$ *and* $M = \bigcup\limits_{n<\omega} M_n$ *then* $M \in K^*_\lambda$.

3) If $M \in K_\lambda$ *and* $\theta < \lambda \Rightarrow M \equiv_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} \mathrm{EM}_{\tau(\mathfrak{K})}(\lambda, \Phi)$ *then* $M \in K^*_\lambda$.

4) K^*_λ *is categorical.*

Proof. 1) We rely on part (2) which is proven below.

So let $\langle M_i : i < \delta \rangle$ be $\le_{\mathfrak{K}}$-increasing in K^*_λ with $\delta < \lambda^+$. Without loss of generality $\delta = \mathrm{cf}(\delta)$, hence $\delta < \lambda$. Call it θ, and we prove this by induction on θ. Without loss of generality $\langle M_i : i < \theta \rangle$ is $\le_{\mathfrak{K}}$-increasing continuous such that $M_i \in K^*_\lambda$ for $i < \theta$, and let $M_\theta = \bigcup\limits_{i<\theta} M_i$. By renaming, without loss of generality $\theta < \lambda_0$.

Let I_n, I'_n be such that:

$\odot_1$ (a) I_n is a linear order of cardinality λ_n from K^{flin}

(b) I'_n is a linear order of cardinality 2^{λ_n} from K^{flin}

(c) I'_n is λ^+_n-saturated. (This means that its cofinality is $> \lambda_n$, the cofinality of its inverse is $> \lambda_n$ and if $I'_n \models$ "$s_{\alpha_1} < s_{\beta_1} < t_{\beta_2} < t_{\alpha_2}$" where $\alpha_1 < \beta_1 < \gamma_1$, $\alpha_1 < \beta_2 < \gamma_2$ and $|\gamma_1| + |\gamma_2| < \lambda^+_n$ then for some r we have $I'_n \models$ "$s_{\alpha_1} < r < t_{\alpha_2}$" for $\alpha_1 < \gamma_1$, $\alpha_2 < \gamma_2$.)

(d) $I_n <_{K^{\mathrm{flin}}} I'_n <_{K^{\mathrm{flin}}} I_{n+1}$ for $n < \omega$.

Let $I = \bigcup\{I_n : n < \omega\}$, so I is a universal member of K^{lin}_λ. Let $M^* = \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$, so for every $i < \theta$ there is an isomorphism f_i from M^* onto M_i, which exists as K^*_λ is categorical by 1.19(4) as $\mathrm{cf}(\lambda) = \aleph_0$.

Now

$\odot_2$ (a) Every interval of I is universal in K^{lin}_λ.

(b) If $n < \omega$, $J \subseteq I$, $\chi = |J| < \lambda$, and

$$\mathcal{E}_{J,I} = \{(t_1, t_2) \in (I \setminus J)^2 : s \in J \Rightarrow [s <_I t_1 \equiv s <_J t_2]\}$$

then for at most χ elements t of $J \setminus I$ the set $t/\mathcal{E}_{J,I}$ is a singleton.

[Why? Clause (a) is obvious. For clause (b) assume $\langle t_\alpha : \alpha < \chi^+ \rangle$ are pairwise distinct members of $J \setminus I$ such that $t_\alpha/\mathcal{E}_{J,I}$ is a singleton for each $\alpha < \chi^+$. Without loss of generality for some $k < \omega$ we have $\alpha < \chi^+ \Rightarrow t_\alpha \in I_k$ hence $\chi \le \lambda_k$. For each $\alpha < \chi^+$ we can choose $s_\alpha \in I'_k$ such that $s_\alpha <_{I'_k} t_\alpha$ and $(s_\alpha, t_\alpha)_{I'_k} \cap J = \varnothing$. Clearly

$$\alpha < \beta < \chi^+ \Rightarrow (t_\alpha <_I s_\beta \vee t_\beta <_I s_\alpha)$$

hence $\langle (s_\alpha, t_\alpha)_I : \alpha < \chi^+ \rangle$ are pairwise disjoint intervals of I, so for every $\alpha < \chi^+$ large enough, $(s_\alpha, t_\alpha)_I \cap J = \varnothing$, but then $(s_\alpha, t_\alpha)_I \subseteq t_\alpha / \mathcal{E}_{J,I}$: a contradiction.]

Now by induction on $n < \omega$ and for each n by induction on $\varepsilon \le \theta$ and for each $n < \omega$ and $\varepsilon \le \theta$ for $i \le \theta$, we choose $J_{n,\varepsilon,i} \in K^{\rm flin}_{\lambda_n}$ such that:

$\odot_3$ (a) $J_{n,\varepsilon,i} \subseteq I$

(b) $J_{n,\varepsilon,i}$ has cardinality λ_n

(c) $I_n <_{K^{\rm flin}} J_{n,0,i}$

(d) If $\zeta < \varepsilon \le \theta$ and $i \le \theta$ then $J_{n,\zeta,i} \subseteq J_{n,\varepsilon,i}$. Moreover, if for some ξ, $\zeta = 2\xi + 1$ and $\varepsilon = 2\xi + 2$, then there is a $<_{K^{\rm flin}_{\lambda_n}}$-increasing continuous sequence of length ω with first member $J_{n,\zeta,i}$ and union $J_{n,\varepsilon,i}$.

(e) For ε limit, $J_{n,\varepsilon,i} = \bigcup\limits_{\zeta<\varepsilon} J_{n,\zeta,i}$.

(f) If ε is odd and $i < j < \theta$ then
$$f_i\big(\mathrm{EM}_{\tau(\mathfrak{K})}(J_{n,\varepsilon,i}, \Phi)\big) = M_i \cap f_j\big(\mathrm{EM}_{\tau(\mathfrak{K})}(J_{n,\varepsilon,j}, \Phi)\big).$$

(g) $J_{n,\theta,i} \subseteq J_{n+1,0,i}$

(h) For every $k < \omega$ and $s <_I t$ from $J_{n,\varepsilon,i}$, if $[s,t]_I \cap I'_k \neq \varnothing$ then
$$[s,t]_I \cap I'_k \cap J_{n,\varepsilon,i} \neq \varnothing.$$

(i) If ζ is odd and $\varepsilon = \zeta+1$, then $\mathrm{EM}_{\tau(\mathfrak{K})}(J_{n,\zeta,i}, \Phi) <^*_{\mathfrak{K}^*_{\lambda_n}} \mathrm{EM}_{\tau(\mathfrak{K})}(J_{n,\varepsilon,i}, \Phi)$.

There is no problem to carry the definition, for $\varepsilon = 2\xi + 2$ recalling $\odot_2$ above; the only non-trivial point is clause (i), which follows by 1.36(4) and clause (d) of $\odot_3$. Clearly $\langle J_{n,\varepsilon,i} : \varepsilon \le \theta \rangle$ is $\subseteq$-increasing continuous by $\odot_3(d) + (e)$.

Let $M^*_{n,\varepsilon,i} = f_i\big(\mathrm{EM}_{\tau(\mathfrak{K})}(J_{n,\varepsilon,i}, \Phi)\big)$ and $M^*_{n,\varepsilon} = M^*_{n,2\varepsilon,\varepsilon}$. So clearly $M^*_{n,\varepsilon,i} \in K^*_{\lambda_n}$ by $\odot_3(b)$ and the choice of $M^*_{n,\varepsilon,i}$ the sequence $\langle M^*_{n,\varepsilon} : \varepsilon < \theta \rangle$ is $\le_{\mathfrak{K}}$-increasing continuous with all of its members in $K^*_{\lambda_n}$.

Now

$\odot_4$ $\langle M^*_{n,\varepsilon} : \varepsilon < \theta \rangle$ is $<^*_{\mathfrak{K}^*_{\lambda_n}}$-increasing.

[Why? As
$$\begin{aligned}\zeta < \varepsilon < \theta \Rightarrow M^*_{n,\zeta} = M_{n,2\zeta,\zeta} &\le_{\mathfrak{K}^*_{\lambda_n}} M_{n,2\zeta+1,\zeta} \le_{\mathfrak{K}^*_{\lambda_n}} M_{n,2\zeta+1,\varepsilon} \\ &<^*_{\mathfrak{K}^*_{\lambda_n}} M_{n,2\zeta+2,\varepsilon} \le_{\mathfrak{K}^*_{\lambda_n}} M_{n,2\varepsilon,\varepsilon} = M^*_{n,\varepsilon}\end{aligned}$$
by the choice of $M^*_{n,\zeta}$, by $\odot_3(d)$ and Ax.V of AEC, by $\odot_3(f)$ and Ax.V of AEC, by $\odot_3(i)$, by $\odot_3(d)$ + Ax.V of AEC(e), by the choice of $M^*_{n,\varepsilon}$ respectively). Now by 1.39(2) this argument shows that $\zeta < \varepsilon < \theta \Rightarrow M^*_{n,\zeta} <^*_{\mathfrak{K}^*_{\lambda_n}} M^*_{n,\varepsilon}$.]

We can conclude, by using 1.40(1) for $\mathfrak{K}^*_{\lambda_n}$, that $M^*_n := \bigcup\limits_{\varepsilon<\theta} M^*_{n,\varepsilon}$ belongs to $K^*_{\lambda_n}$. Also as $M^*_{n,\varepsilon} \le_{\mathfrak{K}} M_\varepsilon \le_{\mathfrak{K}} M_\delta$ for $\varepsilon < \theta = \delta$ by Ax.IV of AEC, we have $M^*_n \le_{\mathfrak{K}} M_\delta$ and similarly $M^*_n \le_{\mathfrak{K}} M^*_{n+1}$, and obviously for each $i < \theta$ we have
$$\bigcup_{n<\omega} M^*_n \supseteq \bigcup\{M^*_{n,\varepsilon} : n < \omega,\ \varepsilon < \theta\} = \bigcup\{M^*_{n,2,\varepsilon,\varepsilon} : n < \omega,\ \varepsilon < \theta\} =$$
$$\bigcup\{M^*_{n,2\varepsilon,i} : n < \omega,\ i < \theta,\ \varepsilon < \theta\} = \bigcup_{n<\omega} M^*_{n,0,i}$$
which recalling the choice of $M^*_{n,0,i}$ includes
$$\bigcup_n f_i(\mathrm{EM}_{\tau(\mathfrak{K})}(J_{n,0,i}, \Phi)) \supseteq \bigcup_{n<\omega} f_i(\mathrm{EM}_{\tau(\mathfrak{K})}(I_n, \Phi)) = f_i(\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)) = M_i.$$
As this holds for every $i < \theta$ we get $\bigcup\limits_{n<\omega} M^*_n = M_\delta$. So by part (2) we are done.

2) We choose I_n by induction on n such that:

$\odot_5$ (a) $I_n \in K^{\rm flin}_{\lambda_n}$
(b) $I_m <_{K^{\rm flin}} I_n$ if $n = m+1$.

Let $N_n = {\rm EM}_{\tau(\mathfrak{K})}(I_n, \Phi)$.

We now choose $(g_n, I'_n, I''_n, M'_n, M''_n, N'_n, N''_n)$ by induction on $n < \omega$ such that:

$\odot_6$ (a) g_n is an isomorphism from N''_n onto M''_n
(b) $I_n \subseteq I'_n \subseteq I''_n \subseteq I_{n+2}$ and $|I'_n| = \lambda_n$, $|I''_n| = \lambda_{n+1}$, and $I_{n+1} \subseteq I''_n$
(c) $N'_n = {\rm EM}_{\tau(\mathfrak{K})}(I'_n, \Phi)$ and $N''_n = {\rm EM}_{\tau(\mathfrak{K})}(I''_n, \Phi)$
(d) $M_n \le_{\mathfrak{K}^*_{\lambda_n}} M'_n \le_{\mathfrak{K}^*} M''_n \le_{\mathfrak{K}^*} M_{n+2}$ and $M_{n+1} \le_{\mathfrak{K}^*_{\lambda_{n+1}}} M''_n$
(e) g_n maps $N'_n = {\rm EM}_{\tau(\mathfrak{K})}(I'_n, \Phi)$ onto M'_n
(f) g_n extends $g_m \restriction N'_m$ if $n = m+1$
(g) $I'_n \subseteq I'_{n+1}$.

<u>Case 1</u>: For $n = 0$.

First, let $M''_n = M_1$, $I''_n = I_1$ so also N''_n is defined. Second, choose g_n satisfying $\odot_6$(a) by 1.16(1), i.e. 1.19(4), categoricity in $K^*_{\lambda_n}$. Third, choose $I^*_n \subseteq I''_n = I_1$ of cardinality λ_n such that $g_n({\rm EM}_{\tau(\mathfrak{K})}(I^*_n, \Phi))$ includes M_0. Fourth, let $I'_n = I^*_n \cup I_n$ and $N'_n = {\rm EM}_{\tau(\mathfrak{K})}(I'_n, \Phi)$ and let $M'_n = g_n(N'_n)$.

<u>Case 2</u>: For $n = m+1$.

Let $k = n+2$, let $\bar a \in {}^{\lambda_m}(M'_m)$ list M'_m (with no repetitions).

Now

$(*)_1$ If $\theta < \lambda_n$ then ${\rm tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar a, \varnothing, N_k) = {\rm tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar a, \varnothing, N''_m)$.

[Why? As ${\rm EM}_{\tau(\mathfrak{K})}(I''_m, \Phi) \prec_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} {\rm EM}_{\tau(\mathfrak{K})}(I_k, \Phi)$ by 1.14(a) as $I''_m \subseteq I_k$.]

$(*)_2$ if $\theta < \lambda_n = \lambda_{m+1}$ <u>then</u> ${\rm tp}_{\mathbb{L}_{\infty,\theta}}(\bar a, \varnothing, N''_m) = {\rm tp}_{\mathbb{L}_{\infty,\theta}}(g_m(\bar a), \varnothing, M''_m)$.

[Why? As g_m is an isomorphism from N''_m onto M''_m by $\odot_6(a)$, i.e. the induction hypothesis.]

$(*)_3$ if $\theta < \lambda_n$ then ${\rm tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(g_m(\bar a), \varnothing, M''_m) = {\rm tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(g_m(\bar a), \varnothing, M_k)$.

[Why? This follows from $M'_m \prec_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} M_k$ which we can deduce from 1.19(1), as $M''_m \in K^*_{\lambda_{m+1}} = K^*_{\lambda_n}$ by clause (d) of $\odot_6$, $M_k \in K^*_k$ by an assumption of the claim, $M''_m \le_{\mathfrak{K}_\lambda} M_k$ by clause (d) of $\odot_6$.]

$(*)_4$ if $\theta < \lambda_n$ then ${\rm tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar a, \varnothing, N_k) = {\rm tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(g_m(\bar a), \varnothing, M_k)$.

[Why? By $(*)_1 + (*)_2 + (*)_3$.]

$(*)_5$ ${\rm tp}_{\mathbb{L}_{\infty,\lambda^+_{n+1}}[\mathfrak{K}]}(\bar a, \varnothing, N_k) = {\rm tp}_{\mathbb{L}_{\infty,\lambda^+_{n+1}}[\mathfrak{K}]}(g_m(\bar a), \varnothing, M_k)$.

[Why? Clearly $N_k, M_k \in K^*_{\lambda_k}$, hence by 1.19(4) there is an isomorphism f_n from N_k onto M_k, so obviously ${\rm tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar a, \varnothing, N_k) = {\rm tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(f_n(\bar a), \varnothing, N_k)$, so by $(*)_4$ we have

$${\rm tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(g_m(\bar a), \varnothing, M_k) = {\rm tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar a, \varnothing, N_k) = {\rm tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(f_n(\bar a), \varnothing, M_k)$$

so by 1.19(3) we have ${\rm tp}_{\mathbb{L}_{\infty,\lambda^+_{n+1}}[\mathfrak{K}]}(g_n(\bar a), \varnothing, M_k) = {\rm tp}_{\mathbb{L}_{\infty,\lambda^+_{n+1}}[\mathfrak{K}]}(f_n(\bar a), \varnothing, M_k)$. But as f_n is an isomorphism from N_k onto M_k and the previous sentence we get ${\rm tp}_{\mathbb{L}_{\infty,\lambda_{n+1}}[\mathfrak{K}]}(\bar a, \varnothing, N_k) = {\rm tp}_{\mathbb{L}_{\infty,\lambda^+_{n+1}}[\mathfrak{K}]}(f_n(\bar a), \varnothing, M_k) = {\rm tp}_{\mathbb{L}_{\infty,\lambda}}(g_n(\bar a), \varnothing, M_k)$ as required.]

$(*)_6$ there are g_n, I''_n, N''_n, M''_n as required in the relevant parts of $\odot_6$ (ignoring I'_n, N'_n, M'_n), i.e. clauses (a),(f) and the relevant parts of (b),(c),(d):
$(b)'$ $I_n \subseteq I''_n \subseteq I_{n+2} = I_k$ and $|I''_n| = \lambda_{n+1}$ and $I_{n+1} \subseteq I'_n$
$(c)'$ $N''_n = {\rm EM}_{\tau(\mathfrak{K})}(I''_n, \Phi)$
$(d)'$ $M_n \le_{\mathfrak{K}^*} M''_n \le_{\mathfrak{K}^*} M_{n+2}$ and $M_{n+1} \le_{\mathfrak{K}^*_{\lambda_{n+2}}} M''_n$.

[Why? By the hence and forth argument, but let us elaborate.

First, let $\bar{a}'$ be a sequence of length λ_{n+1} listing (without repetitions) the set of elements of M_{n+1} and without loss of generality $g(\bar{a}) \lhd \bar{a}'$. Note that $\mathrm{rang}(g_m) \subseteq M_{m+2} = M_{n+1}$.

Second, let g' be a function from $\mathrm{rang}(\bar{a}')$ into N_k extending $(g_m \restriction N'_m)^{-1} = (g_m \restriction \mathrm{rang}(\bar{a}))^{-1}$ such that $\mathrm{tp}_{\mathbb{L}_{\infty,\lambda^+_{n+1}}[\mathfrak{K}]}(g'(\bar{a}'), \emptyset, N_k) = \mathrm{tp}_{\mathbb{L}_{\infty,\lambda^+_{n+1}}[\mathfrak{K}]}(\bar{a}', \emptyset, M_k)$; this exists by $(*)_5$. Let $I''_n \subseteq I_k$ of cardinality λ_{n+1} be such that $\mathrm{rang}(g') \subseteq \mathrm{EM}(I''_n, \Phi)$ and $I_{n+1} \subseteq I''_n$. Let $\bar{a}''$ list the elements of $\mathrm{EM}_{\tau(\mathfrak{K})}(I''_n, \Phi) \subseteq N_k$ and without loss of generality $g'(\bar{a}') \lhd \bar{a}''$. Let g_n be a function from $\mathrm{EM}_{\tau(\mathfrak{K})}(I''_n, \Phi)$ to M_k extending $(g')^{-1}$ such that

$$\mathrm{tp}_{\mathbb{L}_{\infty,\lambda^+_{n+1}}[\mathfrak{K}]}(\bar{a}'', \emptyset, N_k) = \mathrm{tp}_{\mathbb{L}_{\infty,\lambda^+_{n+1}}[\mathfrak{K}]}(g_n(\bar{a}''), \emptyset, M_k).$$

Lastly, let $N''_n = \mathrm{EM}_{\tau(\mathfrak{K})}(I''_n, \Phi)$ and $M''_n = g_n(N'_n)$ so we are done.]

$(*)_7$ there are I'_n, N'_n, M'_n as required.

[Why? By the LST argument we can choose I'_n and define N'_n, M'_n accordingly.]

So we can carry the induction. Now $N'_n \leq_{\mathfrak{K}} N'_{n+1}$ (by clauses (g),(c) of $\odot_6$) and $g_n \restriction N'_n \subseteq g_{n+1} \restriction N'_{n+1}$ (by clause (f) + the previous statement). Hence $g = \bigcup\{g_n \restriction N'_n : n < \omega\}$ is an isomorphism from $\bigcup\{N'_n : n < \omega\}$ onto $\bigcup\{M'_n : n < \omega\}$. But

$$N = \bigcup\{N_n : n < \omega\} \subseteq \bigcup\{N'_n : n < \omega\} \subseteq \mathrm{dom}(g) \subseteq N$$

and

$$M = \bigcup\{M_n : n < \omega\} \subseteq \bigcup\{M'_n : n < \omega\} \subseteq \mathrm{rang}(g) \subseteq M.$$

Together g is an isomorphism from N onto M but obviously $N \in K^*_\lambda$ hence $M \in K^*_\lambda$ is as required.

3), 4) Should be clear; just depends on 1.19(4). $\square_{1.41}$

Conclusion 1.42. *Let λ be as in $\boxtimes$ of 1.41. 1) $\mathfrak{K}^*_\lambda$ is a λ-AEC (with $\leq_{\mathfrak{K}} \restriction K^*_\lambda$) and it has amalgamation and is categorical.*

2) $\mathfrak{K}^\oplus_{\geq\lambda}$ is an AEC, $\mathrm{LST}(\mathfrak{K}^\oplus_{\geq\lambda}) = \lambda$ and $(\mathfrak{K}^*_\lambda)^{\mathrm{up}} = K^\oplus_{\geq\lambda}$ and $(\mathfrak{K}^\oplus_{\geq\lambda})_\lambda = \mathfrak{K}^*_\lambda$, see Definition 1.43 below.

Definition 1.43. Let $\mathfrak{K}^\oplus_{\geq\lambda} = \mathfrak{K} \restriction K^\oplus_{\geq\lambda}$ where

$$K^\oplus_{\geq\lambda} = \{M \in K_\lambda : M \equiv_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} \mathrm{EM}_{\tau(\mathfrak{K})}(\lambda, \Phi)\}.$$

Proof. 1) It was clear defining $(K^*_\lambda, \leq_{\mathfrak{K}} \restriction K^*_\lambda)$ that it is of the right form and "$M \in K^*_\lambda$", "$M \leq_{\mathfrak{K}^*_\lambda} N$" are preserved by isomorphisms. Obviously "$\leq_{\mathfrak{K}} \restriction K^*_\lambda$ is a partial order", so Ax.I, Ax.II hold, and obviously Ax.V holds (see [She09c, 0.2]). The missing point was Ax.III (about $\leq_{\mathfrak{K}}$-increasing union) and it holds by 1.41(1). Then Ax.IV becomes easy by the definition of $\leq_{\mathfrak{K}^*_\lambda} = \leq_{\mathfrak{K}} \restriction K^*_\lambda$, and lastly the amalgamation holds by 1.34.

2) By [She09c, §1] we can "lift $\mathfrak{K}^*_\lambda$ up", the result is $\mathfrak{K}^\oplus_{\geq\lambda}$ (see [She09c, 0.31,0.32]). $\square_{1.42}$

Let us formulate a major conclusion in ways less buried inside our notation.

Conclusion 1.44. *Assume $(\mathfrak{K}, \Phi)$ is pseudo solvable in μ. <u>Then</u> $(\mathfrak{K}, \Phi)$ is pseudo solvable in λ provided that $\mathrm{LST}(\mathfrak{K}) < \lambda$, $\mu = \mu^{<\lambda}$ (or just the hypothesis 1.18 holds), $\mathrm{cf}(\lambda) = \aleph_0$, and λ is an accumulation point of the class of the fixed points of the sequence of the $\beth$-s.*

Proof. By 1.42(1). $\square_{1.44}$

Remark 1.45. About [weak] solvability, see [S$^+$b].

§ 2. §2 Trying to Eliminate $\mu = \mu^{<\lambda}$

There was one point in §1 where we use $\mu = \mu^\lambda$ (i.e. in 1.13; more accurately, in justifying hypothesis 1.18(1)). In this section we try to eliminate it. So we try to prove $M_1 \le_{\mathfrak{K}_\mu} M_2 \Rightarrow M_1 \prec_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} M_2$ for $\theta < \lambda$, hence we fix $\mathfrak{K}, \mu, \theta$. We succeed to do it with "few exceptions".

Hypothesis 2.1. (We shall mention $(b)_\mu$ or $(b)^-_\mu$, $(c), (d)$ when used! but not clause (a).)

(a) $\mathfrak{K}$ is an AEC and $\Phi \in \Upsilon^{\rm or}_{\mathfrak{K}}$

$(b)_\mu$ $\mathfrak{K}$ categorical in μ and $\Phi \in \Upsilon^{\rm or}_{\mathfrak{K}}$, or at least

$(b)^-_\mu$ $\mathfrak{K}$ is pseudo μ-solvable as witnessed by $\Phi \in \Upsilon^{\rm or}_{\mathfrak{K}}$ (see Definition 1.4). In particular, $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \mu)$ is pseudo superlimit for $I \in K^{\rm lin}_\lambda$,

(c) $\mu \ge \beth_{1,1}(\mathrm{LST}(\mathfrak{K}))$

(d) $\mu > \mathrm{LST}(\mathfrak{K})$.

Convention 2.2. $K^*_\lambda = K^*_{\Phi,\lambda}$, etc., see Definition 1.15.

Definition 2.3. Assume

$\boxdot$ $\mu \ge \chi \ge \theta > \mathrm{LST}(\mathfrak{K})$

1) We let

$$K^1_{\mu,\chi} = \big\{(M, N) : N \le_{\mathfrak{K}} M,\ N \in K_\chi,\ M \in K_\mu \text{ and } \mu = \chi \Rightarrow M = N\big\}$$

and let $\le_{\mathfrak{K}} = \le_{\mathfrak{K},\mu,\chi}$ be the following partial order on $K_{\mu,\chi}$:

$$(M_0, N_0) \le_{\mathfrak{K}} (M_1, N_1) \quad \text{iff} \quad M_0 \le_{\mathfrak{K}} M_1,\ N_0 \le_{\mathfrak{K}} N_1$$

(formally we should have written $\le_{\mathfrak{K},\mu,\chi}$). Note that each pair $(M, N) \in K_{\mu,\chi}$ determine μ, χ. So if $\chi = \mu$, $K_{\mu,\chi}$ is essentially $\mathfrak{K}_\mu$. Let $K^1_\mu = K_\mu$ and let $\bigcup\{(M_i, N_i) : i < \delta\} = \big(\bigcup\{M_i : i < \delta\}, \bigcup\{N_i : i < \delta\}\big)$ for any $\le_{\mathfrak{K}}$-increasing sequence $\big\langle (M_i, N_i) : i < \delta\big\rangle$.

1A) Let $K_{\mu,\chi} = K^2_{\mu,\chi} = \big\{(M, N) \in K^1_{\mu,\chi} : M \in K^*_\mu\big\}$ and $K^2_\mu = K^*_\mu$ but we use them only when Φ witnesses $\mathfrak{K}$ is pseudo μ-solvable: i.e. $(b)^-_\mu$ from Hypothesis 2.1 holds.

2) For $k \in \{1, 2\}$, a formula $\varphi(\bar{x}) \in \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ (so $\ell g(\bar{x}) < \theta$), cardinal $\kappa \ge \theta$ (the main case being $\kappa = \mu$), and $M \in K^k_\kappa$, $\bar{a} \in {}^{\ell g(\bar{x})}M$ <u>we define</u> when $M \Vdash_k \varphi[\bar{a}]$ by induction on the depth of $\varphi(\bar{x}) \in \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$, so the least obvious case is:

$(*)$ $M \Vdash_k (\exists \bar{y})\psi(\bar{y}, \bar{a})$ <u>when</u> for every $M_1 \in K^k_\kappa$ such that $M \le_{\mathfrak{K}} M_1$ there is $M_2 \in K^k_\kappa$ satisfying $M_1 \le_{\mathfrak{K}} M_2$ and $\bar{b} \in {}^{\ell g(\bar{y})}M_2$ such that $M_2 \Vdash_k \psi[\bar{b}, \bar{a}]$.

(We may omit k if $k = 2$.)

Of course

(α) for φ atomic, $M \Vdash_k \varphi[\bar{a}]$ <u>iff</u> $M \models \varphi[\bar{a}]$

(β) for $\varphi(\bar{x}) = \bigwedge\limits_{i<\alpha} \varphi_i(\bar{x})$ let $M \Vdash_k \varphi[\bar{a}]$ iff $M \Vdash_k \varphi_i[\bar{a}]$ for each $i < \alpha$

(γ) $M \Vdash_k \neg\varphi[\bar{a}]$ iff for no N do we have $M \le_{\mathfrak{K}} N \in K^k_\kappa$ and $N \Vdash_k \varphi[\bar{a}]$.

3) Let $k \in \{1, 2\}$, $\Lambda \subseteq \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ (each formula with $< \theta$ free variables, of course):

(a) Λ is downward closed if it is closed under subformulas

(b) Λ is (μ, χ)-model^k complete (when μ is clear from the context we may write χ-model^k complete) <u>if</u> $|\Lambda| < \mu$, and for every $(M_0, N_0) \in K^k_{\mu,\chi}$ we can find $(M, N) \in K^2_{\mu,\chi}$ above (M_0, N_0) which is Λ-generic, where:

(c) $(M,N) \in K^k_{\mu,\chi}$ is Λ-generic^k when: if $\varphi(\bar{x}) \in \Lambda$ and $\bar{a} \in {}^{\ell g(\bar{x})}N$ then $M \Vdash_k \varphi[\bar{a}] \Leftrightarrow N \models \varphi[\bar{a}]$. (Yes! Neither $(M,N) \Vdash_k \varphi[\bar{a}]$, which was not defined, nor "$M \models \varphi[\bar{a}]$"!)

(d) Λ is called $(\mu, <\mu)$-model^k complete when $|\Lambda| + \theta_\Lambda < \mu$ and for every χ: if $|\Lambda| + \theta_\Lambda \leq \chi < \mu$ then Λ is χ-model^k complete, where

$$\theta_\Lambda := \min\big\{\partial > \mathrm{LST}(\mathfrak{K}) : \Lambda \subseteq \mathbb{L}_{\infty,\partial}[\mathfrak{K}]\big\}.$$

We say Λ is model^k complete if it is $(\mu, <\mu)$-model^k complete and μ is understood from the context.

(e) Above, if Φ or $(\mathfrak{K}, \Phi)$ is not clear from the context, we may replace Λ by (Λ, Φ) or by $(\Lambda, \Phi, \mathfrak{K})$.

4) For $M \in K^k_\kappa$, $\bar{a} \in {}^{\theta >}M$ and $\Lambda \subseteq \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$, let

$$\mathrm{gtp}^k_\Lambda(\bar{a}, \varnothing, M) = \big\{\varphi[\bar{a}] : M \Vdash_k \varphi[\bar{a}]\big\}.$$

If we write θ instead of Λ we mean $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$. (Note: this type is <u>not *a priori* complete</u>!) and we say that $\bar{a}$ materializes this type in M. To stress κ we may write $\mathrm{gtp}^{\kappa,k}_\Lambda(\bar{a}, \varnothing, M)$ or $\mathrm{gtp}^{\kappa,k}_\theta(\bar{a}, \varnothing, M)$, even though M determines κ.

5) We say $M \in K_\kappa$ is Λ-generic^k <u>when</u> for every $\varphi(\bar{x}) \in \Lambda$ and $\bar{a} \in {}^{\ell g(\bar{x})}M$ we have $M \Vdash_k \varphi[\bar{a}] \Leftrightarrow M \models \varphi[\bar{a}]$. So $M \in K^k_\mu$ is Λ-generic^k iff $(M,M) \in K^k_{\mu,\mu}$ is Λ-generic^k. We say Λ is κ-model^k complete <u>when</u> every $M \in K^k_\kappa$ has a Λ-generic $\leq_{\mathfrak{K}}$-extension in K^k_κ (so depend on $\mathfrak{K}$ and if $k = 2$ also on Φ).

6) In all cases above, if $k = 2$ we may omit it.

Claim 2.4. *Assume that* $\mathrm{LST}(\mathfrak{K}) < \theta \leq \chi < \mu$, $\kappa > \theta$, *and* $k \in \{1,2\}$ *(so if* $k = 2$ *then 2.1(b)*$^-_\mu$ *holds; see 2.3(1A)).*

1) $(K^k_{\mu,\chi}, \leq_{\mathfrak{K}})$ *is a partial order and chains of length* $\delta < \chi^+$ *have a* $\leq_{\mathfrak{K}}$*-l.u.b: this is the union, see 2.3(1). If* $\mathrm{EM}_{\tau(\mathfrak{K})}(\mu, \Phi)$ *is superlimit (not just pseudo superlimit)* <u>*then*</u> $K^2_{\mu,\chi}$ *is a dense subclass of* $K^1_{\mu,\chi}$ *under* $\leq_{\mathfrak{K}}$.

2) If $M_1 \Vdash_k \varphi(\bar{a})$ *and* $M_1 \leq_{\mathfrak{K}} M_2$ *are from* K^k_κ <u>*then*</u> $M_2 \Vdash_k \varphi[\bar{a}]$.

3) If $(M_\ell, N_\ell) \in K^k_{\mu,\chi}$ *are* Λ*-generic*^{k} *for* $\ell = 1,2$ *and* $(M_1,N_1) \leq_{\mathfrak{K}} (M_2,N_2)$ <u>*then*</u> $N_1 \prec_\Lambda N_2$.

4) If $M_i \in K^k_\kappa$ *for* $i < \delta$ *is* $\leq_{\mathfrak{K}}$*-increasing,* $\delta < \kappa^+$, $\mathrm{cf}(\delta) \geq \theta$, $\Lambda \subseteq \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$, *and each* M_i *is* Λ*-generic*^{k}, <u>*then*</u> $M_\delta := \bigcup\limits_{i<\delta} M_i$ *is* Λ*-generic*^{k} *and* $i < \delta \Rightarrow M_i \prec_\Lambda M_\delta$.

5) If $(M_i, N_i) \in K^k_{\mu,\chi}$ *for* $i < \delta$ *is* $\leq_{\mathfrak{K}}$*-increasing,* $\delta < \chi^+$, $\mathrm{cf}(\delta) \geq \theta$, $\Lambda \subseteq \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ *and each* (M_i, N_i) *is* Λ*-generic*^{k}, <u>*then*</u> $\big(\bigcup\limits_{i<\delta} M_i, \bigcup\limits_{i<\delta} N_i\big)$ *is* Λ*-generic*^{k} *and* $N_j \prec_\Lambda \bigcup\limits_{i<\delta} N_i$ *for each* $j < \delta$.

Proof. Should be clear; in part (1), for $k = 2$, we use clause $(b)^-_\mu$ of 2.1. In part (5) note that $\bigcup\{M_i : i < \delta\} \in K^*_\mu$ by Clause $(\mathrm{b})^-_\mu$ of 2.1. $\square_{2.4}$

<u>Exercise</u>: If (M,N) is Λ-generic^k and $(M,N) \leq_{\mathfrak{K}} (M',N) \in K^k_{\mu,\chi}$ <u>then</u> (M',N) is Λ-generic^k.

Claim 2.5. *Assume that* $\mu \geq \chi \geq \theta > \mathrm{LST}(\mathfrak{K})$ *and* $k \in \{1,2\}$.

1) The set of quantifier free formulas in $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ *is* (μ,χ)*-model*^{k} *complete.*

2) If $\Lambda_\varepsilon \subseteq \mathbb{L}_{\infty,\theta}(\tau_{\mathfrak{K}})$ *is downward closed,* (μ,χ)*-model*^{k} *complete for* $\varepsilon < \varepsilon^*$, *and* $\Lambda := \bigcup\limits_{\varepsilon<\varepsilon^*} \Lambda_\varepsilon$, $\theta = \mathrm{cf}(\theta) \leq \chi \vee \theta < \chi$, *and* $\varepsilon^* < \chi^+$ *(and* $\mu > \theta \vee \mu = \theta = \mathrm{cf}(\theta)$*)* <u>*then*</u> Λ *is* (μ,χ)*-model*^{k} *complete.*

Proof. 1) Easy.

2) Given $(M,N) \in K^k_{\mu,\chi}$ let θ_r be $\min\{\partial : \partial \geq \theta$ is regular$\}$. Clearly $\theta_r \leq \chi$, and we choose $(M_i,N_i) \in K^k_{\mu,\chi}$ for $i \leq \varepsilon^* \times \theta_r$ such that

⊛ (a) $\langle M_i : i \leq \varepsilon^* \times \theta_r \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous
(b) $\langle N_i : i \leq \varepsilon^* \times \theta_r \rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous
(c) If $i = \varepsilon^* \times \gamma + \varepsilon$ and $\varepsilon < \varepsilon^*$ then (M_{i+1},N_{i+1}) is Λ_ε-generic^k.
(d) $(M_0,N_0) = (M,N)$.

There is no problem to do this.

Now for each $\varepsilon < \varepsilon^*$ the sequence $\big\langle (M_{\varepsilon^*\times\gamma+\varepsilon+1}, N_{\varepsilon^*\times\gamma+\varepsilon+1}) : \gamma < \theta_r \big\rangle$ is $\leq_{\mathfrak{K},\mu,\chi}$-increasing with union $(M_{\varepsilon^*\times\theta_r}, N_{\varepsilon^*\times\theta_r})$, and each member of the sequence is Λ_ε-generic^k; hence by 2.4(5) we know that the pair $(M_{\varepsilon^*\times\theta_r}, N_{\varepsilon^*\times\theta_r})$ is Λ_ε-generic^k. As this holds for each Λ_ε it holds for Λ, so $(M_{\varepsilon^*\times\theta_r}, N_{\varepsilon^*\times\theta_r})$ is as required. $\square_{2.5}$

From now on in this section

Hypothesis 2.6. We assume (a) + (b)$^-_\mu$ +(d) of 2.1 and we omit k using Definition 2.3 meaning $k=2$.

Claim 2.7. *1) For $M \in K^*_\mu$ and $\mathrm{LST}(\mathfrak{K}) < \theta < \mu$ the number of complete $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$-types realized by sequences from ${}^{\theta>}M$ is $\leq 2^{<\theta}$. Moreover, the relation*

$$\mathcal{E}^{<\theta}_M := \big\{(\bar a,\bar b) : \bar a,\bar b \in {}^{\theta>}M \text{ and some automorphism of } M \text{ maps } \bar a \text{ to } \bar b\big\}$$

is an equivalence relation with $\leq 2^{<\theta}$ equivalence classes.

2) Hence there is a set $\Lambda_* = \Lambda^*_\theta = \Lambda^*_{\mathfrak{K},\Phi,\mu,\theta} \subseteq \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ such that:

(a) $|\Lambda_*| \leq 2^{<\theta}$ and $\Lambda_* \subseteq \mathbb{L}_{(2^{<\theta})^+,\theta}[\mathfrak{K}]$
(b) Λ_* is closed under sub-formulas and finitary operations
(c) Each $\varphi(\bar x) \in \Lambda_*$ has quantifier depth $< \gamma^*$ for some $\gamma^* < (2^{<\theta})^+$.
(d) For $\alpha < \theta$, $M \in K^*_\mu$, and $\bar a \in {}^\alpha M$, the Λ_*-type which $\bar a$ realizes in M determines the $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$-type which $\bar a$ realizes in M. Moreover, one formula in the type determines it.
(e) Similarly for materialize in $M \in K^*_\mu$; see Definition 2.3(4).
(f) If $\mathrm{LST}(\mathfrak{K}) < \theta \leq \chi < \mu$ and $(M,N) \in K_{\mu,\chi}$ is Λ_*-generic <u>then</u> it is $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$-generic.
(g) if $M \in K^2_\mu$ is Λ_*-generic <u>then</u> it is $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$-generic.

Remark 2.8. Part (1) can also be proved using just $(\lambda+1)\times I_*$ with I_* a θ-saturated dense linear order with neither first nor last element, but this is not clear for 2.11(1).

Proof. 1) By 5.1(1) and categoricity of K^*_λ.

2) Follows, but we elaborate.

Let $\{\bar a_\alpha : \alpha < \alpha^* \leq 2^{<\theta}\}$ be a set of representatives of the $\mathcal{E}^{<\theta}_M$-equivalence classes. For each $\alpha \neq \beta$ such that $\ell g(\bar a_n) = \ell g(\bar a_\beta)$, let $\bar x_\alpha = \big\langle x_i : i < \ell g(\bar a_\alpha)\big\rangle$ and choose $\varphi_{\alpha,\beta}(\bar x_\alpha)$, $\psi_{\alpha,\beta}(\bar x_\alpha) \in \mathbb{L}_{(2^{<\theta})^+,\theta}[\mathfrak{K}]$ such that, if possible, we have

$$M \models \varphi_{\alpha,\beta}[\bar a_\alpha] \wedge \neg\varphi_{\alpha,\beta}[\bar a_\beta]$$

and under this, if possible,

$$M \Vdash \text{“}\psi_{\alpha,\beta}(\bar a_\alpha) \wedge \neg\psi_{\alpha,\beta}(\bar a_\beta).$$

But in any case, $M \models \varphi_{\alpha,\beta}[\bar a_\alpha]$ and $M \Vdash \psi_{\alpha,\beta}[\bar a_\alpha]$. Let

$$\varphi_\alpha(\bar x) = \bigwedge\{\varphi_{\alpha,\beta}(\bar x_\alpha) : \beta < \alpha^*,\ \beta \neq \alpha \text{ and } \ell g(\bar a_\beta) = \ell g(\bar a_\alpha)\}$$

and similarly define $\psi_\alpha(\bar x_\alpha)$. Let Λ_* be the closure of $\{\varphi_{\alpha,\beta}, \psi_{\alpha,\beta}, \varphi_\alpha, \psi_\alpha : \alpha \neq \beta < \alpha^*\}$ under subformulas and finitary operations. Obviously, clauses (a),(b) hold hence the existence of $\gamma^* < (2^{<\theta})^+$, as required in clause (c), follows. Clause (d) holds as

$$\bar a\ \mathcal{E}^{<\theta}_M\ \bar b \Rightarrow \mathrm{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar a, \varnothing, M) = \mathrm{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar b, \varnothing, M)$$

using the automorphisms. For $\alpha, \beta < \alpha_*$ such that $\ell g(\bar a_\alpha) = \ell g(\bar a_\beta)$ we have

$$M \models (\forall \bar x_\alpha)\big[\varphi_\alpha(\bar x_\alpha) = \varphi_\beta(\bar x_\beta)\big]$$

implies $\mathrm{tp}_{\mathbb{L}_{(2^{<\theta})^+,\theta}[\mathfrak{K}]}(\bar a_\alpha, \varnothing, M) = \mathrm{tp}_{\mathbb{L}_{(2^{<\theta})^+,\theta}[\mathfrak{K}]}(\bar a_\beta, \varnothing, M)$ and even

$$\mathrm{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar a_\alpha, \varnothing, M) = \mathrm{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar a_\beta, \varnothing, M)$$

recalling the choice of the $\varphi_{\alpha,\beta}$-s.

Clause (e) holds similarly by the choice of the $\psi_{\alpha,\beta}$-s. Clauses (f),(g) should also be clear. (The proof is similar to the proof of the classical 0.18(3).) $\square_{2.7}$

Observation 2.9. *Assume 2.1(b)$^-_\mu$ of course, $\Lambda \subseteq \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$, $\mu > 2^{<\theta}$, and $\theta >$ LST$(\mathfrak{K})$.*

1) The number of complete $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$-types realized in some $M \in K^*_\mu$, by a sequence of length $< \theta$ of course, is $\le 2^{<\theta}$. Hence every formula in $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ is equivalent, for models from K^*_μ to a formula of quantifier depth $< (2^{<\theta})^+$, even from $\Lambda_* \subseteq \mathbb{L}_{(2^{<\theta})^+,\theta}[\mathfrak{K}]$ where Λ_* is in 2.7(2).

2) Assume that $I_1 \subseteq I_2$ are well ordered, $\mathrm{cf}(I_1), \mathrm{cf}(I_2) > 2^{<\theta}$,

$$t \in I_2 \setminus I_1 \Rightarrow 2^{<\theta} < \mathrm{cf}(I_1 \restriction \{s \in I_1 : s <_{I_2} t\})$$

and

$$t \in I_2 \setminus I_1 \Rightarrow 2^{<\theta} < \mathrm{cf}\big(I_2 \restriction \{s \in I_2 : (\forall r \in I_1)[r <_{I_2} t \equiv r <_{I_2} s]\}\big).$$

Then $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi) \prec_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} \mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$.

3) If $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$, $|I| = \mu$, I well ordered of cofinality $> 2^{<\theta}$, $\bar a \in {}^\alpha M$ where $\alpha < \theta$ and $a_i = \sigma_i(\ldots, a_{t_{i,\ell}}, \ldots)_{\ell < n(i)}$ for $i < \alpha$ then $\mathrm{tp}_{\Lambda_*}(\bar a, \varnothing, M)$ is determined by $\langle \sigma_i(x_0, \ldots, x_{n(\ell)-1}) : i < \ell g(\bar a)\rangle$ and the essential θ-type of $\langle t_{i,\ell} : i < \ell g(\bar a),\ \ell < n(i)\rangle$; see Definition 2.10 below.

Before proving 2.9:

Definition 2.10. 1) For $\bar t = \langle t_i : i < \alpha\rangle \in {}^\alpha I$, I well ordered, let the essential θ-type of $\bar t$ in I be shorthand for the essential $(\theta, (2^{<\theta})^+)$-type.

By this we mean: for an ordinal γ, let the essential (θ, γ)-type of $\bar t$ in I, $\mathrm{estp}_{\theta,\gamma}(\bar t, \varnothing, I)$, be the following information stipulating $t_\alpha = \infty$:

(a) The truth value of $t_i < t_j$ (for $i, j < \alpha$).

(b) $\mathrm{otp}([r_i, t_i)_I)$ for $i < \alpha$, where for $i \le \alpha$ we let r_i be the minimal member r of I such that $\mathrm{otp}([r, t_i)_I) < \theta \times \gamma$ and $r \le_I t_i$ and

$$(j < \alpha \wedge t_j < t_i) \Rightarrow t_j \le r.$$

(c) $\min\{\theta \times \gamma, \mathrm{otp}[s_i, r_i)_I\}$ for $i \le \alpha$, where we let s_i be the minimal member of I such that $(\forall j < \alpha)[t_j <_I t_i \Rightarrow t_j <_I s_i]$.

(d) $\min\big\{\theta, \mathrm{cf}(I \restriction \{s : s <_I r_i\})\big\}$ for $i \le \alpha$, which may be zero.

2) Let the function implicit in 2.9(3) be called $\mathbf{t}^\mu_\Lambda = \mathbf{t}^\mu_{\mathfrak{K},\Lambda} = \mathbf{t}^\mu_{\mathfrak{K},\Phi,\Lambda}$, i.e., $\mathbf{t}^\mu_\Lambda(\mathbf{s},\bar\sigma) = \mathrm{tp}_\Lambda(\bar a,\varnothing,M)$ when

$$\bar a = \big\langle \sigma_i(\ldots,a_{t_{\beta(i,\ell)}},\ldots)_{\ell<n_i} : i<\ell g(\bar a)\big\rangle,$$
$$\bar\sigma = \big\langle \sigma_i(\ldots,x_{\beta(i,\ell)},\ldots)_{\ell<n} : i<\ell g(\bar a)\big\rangle,$$

and $\mathbf{s}$ is the essential θ-type of $\langle t_{i,\ell} : i<\ell g(\bar a),\ \ell<n_i\rangle$ in I.

If $\Lambda = \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ we may write just θ.

Proof. 1) By 2.7(1) this holds for each $M \in K^*_\mu$.

2) It is known by Kino [Kin66] that $I_1 \prec_{\mathscr{L}} I_2$ if

$$\mathscr{L} \subseteq \{\varphi \in \mathbb{L}_{\infty,\theta}(\{<\}) : \varphi \text{ has quantifier depth} < (2^{<\theta})^+\}.$$

From this the result follows by part (1).

More fully, let θ_r be the first regular cardinal $\geq \theta$, and we say that the pair (I_1,I_2) is γ-suitable when we replace the assumption "of cofinality $> 2^{<\theta}$" by "of cofinality $\geq \theta$ and of order type divisible by $\theta \times \gamma$". Now we prove by induction on γ that:

$\odot_1$ Assume that for $\alpha<\theta$ and for $\ell=1,2$ we have that I_ℓ is a well ordering, $\bar t^\ell = \langle t^\ell_i : i<\alpha\rangle$ is $<_{I_\ell}$-increasing, and t^ℓ_0 is the first element of I_ℓ. We stipulate $t^\ell_\alpha = \infty$ and $\mathrm{otp}([t^\ell_i,t^\ell_{i+1})_{I_0}) = \theta_r\gamma\alpha^\ell_i + \beta_i$ where $\beta_i < \theta\gamma$ and

$$\Big(\mathrm{cf}(\alpha^1_i) = \mathrm{cf}(\alpha^1_i)\Big) \vee \Big(\mathrm{cf}(\alpha^1_i) \geq \theta \wedge \mathrm{cf}(\alpha^2_i) \geq \theta\Big).$$

<u>Then</u> for any formula $\varphi(\langle x_i : i<\alpha\rangle) \in \mathbb{L}_{\infty,\theta}(\{<\})$ of quantifier depth $\leq \gamma$ we have $I_1 \models \varphi[\bar t^1] \Leftrightarrow I_2 \models \varphi[\bar t^2]$.

Hence

$\odot_2$ if $\vartheta(\bar x) \in \mathbb{L}_{\infty,\theta}(\{<\})$ has quantifier depth $<\gamma$ and (I_1,I_2) is γ-suitable and $\bar t \in {}^{\ell g(\bar x)}(I_1)$ then $I_1 \models \varphi[\bar t] \Leftrightarrow I_2 \models \theta[\bar t]$.

3) Follows by part (2). $\square_{2.9}$

Claim 2.11. *Assume*

$\boxdot$ *(a)* $M \in K^*_\mu$

(b) $\Lambda \subseteq \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ *is downward closed,* $|\Lambda| \leq \chi$, $\mathrm{LST}(\mathfrak{K}) < \theta \leq \chi < \mu$ *and* $2^{<\theta} \leq \chi$ *and* $\theta = \mathrm{cf}(\theta) \vee \theta < \chi$ *so* $\Lambda = \Lambda_*$ *from 2.7 is O.K.*

(c) In part (3),(4),(5) we assume $(\chi^{<\theta} \leq \mu) \vee (\mathrm{cf}(\mu) \geq \theta)$.

(d) For part (6) we assume $\mathrm{cf}(\mu) \geq \theta$ *(hence the demand in clause (c) holds).*

1) If $M \in K^*_\mu$ *then* $\{\mathrm{gtp}_\Lambda(\bar a,\varnothing,M) : \bar a \in {}^{\theta>}M\}$ *has cardinality* $\leq 2^{<\theta}$.

2) If $(M,N) \in K_{\mu,\chi}$ <u>*then*</u> *we can find* N', $(M,N) \leq_{\mathfrak{K}} (M,N') \in K_{\mu,\chi}$ *such that*

$(*)$ *if* $\alpha<\theta$ *and* $\bar b \in {}^\alpha M$ *and* $\Lambda \subseteq \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ *then for some* $\bar b' \in {}^\alpha(N')$ *we have: for every* $\bar a \in {}^{\theta>}N$, $\mathrm{gtp}_\Lambda(\bar a\hat{}\bar b,\varnothing,M) = \mathrm{gtp}_\Lambda(\bar a\hat{}\bar b',\varnothing,M)$.

3) If $(M,N) \in K_{\mu,\chi}$, <u>*then*</u> *we can find* (M_1,N_1) *such that* $(M,N) \leq_{\mathfrak{K}} (M_1,N_1) \in K_{\mu,\chi}$ *and: (note that* $\bar y$ *may be the empty sequence)*

$(*)$ *if* $(\exists\bar y)\varphi(\bar y,\bar x) \in \Lambda$ *and* $\bar a \in {}^{\ell g(\bar x)}N$ *then* $M_1 \Vdash \neg\exists\bar y\varphi(\bar y,\bar x)$ *or for some* $\bar b \in {}^{\ell g(\bar y)}(N_1)$ *we have* $M_1 \Vdash \varphi[\bar b,\bar a]$.

4) In part (3) we can demand

$(*)^+$ *if* $(\exists\bar y)\varphi(\bar y,\bar x) \in \Lambda$ *and* $\bar a \in {}^{\ell g(\bar x)}(N_1)$ *then* $M_1 \Vdash \neg(\exists\bar y)\varphi(\bar y,\bar x)$ *or for some* $\bar b \in {}^{\ell g(\bar y)}(N_1)$ *we have* $M_1 \models \varphi[\bar b,\bar a]$.

5) In part (4) it follows that the pair (M_1,N_1) *is* Λ*-generic (most interesting for* Λ_**; see 2.7).*

6) If $M_1 \in K^*_\mu$ *then it is* Λ*-generic.*

Proof. 1) Proved just like 2.7(1).

2) First assume θ is a successor cardinal.[7] As $M \in K^*_\mu$, without loss of generality $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$ for some linear order I of cardinality μ as in 5.1(1),(4) with $\theta^-, \theta, \chi^+, \mu$ here standing for $\mu, \theta_1, \theta_2, \lambda$ there. It follows that for some $J \subseteq I$ of cardinality χ we have $N \subseteq \mathrm{EM}_{\tau(\mathfrak{K})}(J, \Phi)$, and let $J^+ \subseteq I$ be such that $J \subseteq J^+$, $|J^+ \setminus J| = \chi$ and for every $\bar{t} \in {}^{\theta>}I$ there is an automorphism f of I over J which maps $\bar{t}$ to some member of ${}^{\ell g(\bar{t})}(J^+)$.

Lastly, let $N' = \mathrm{EM}_{\tau(\mathfrak{K})}(J^+, \Phi)$. It is easy to check (see 1.4) that $(*)$ holds. If θ is a limit ordinal it is enough to prove for each $\partial < \theta$, a version of $(*)$ with $\alpha < \partial$; and this gives N'_∂. Now we choose N' such that $\partial < \theta \Rightarrow N'_\partial \le_{\mathfrak{K}} N'$ and $(M, N') \in K_{\mu,\chi}$.

3),4),5),6) We prove by induction on γ that if we let

$$\Lambda_\gamma = \{\varphi(\bar{x}) \in \Lambda : \varphi(\bar{x}) \text{ has quantifier depth } < 1 + \gamma\}$$

then parts (3),(4),(5),(6) hold for Λ_γ. For all four parts, $|\Lambda| \le \chi$ hence $|\Lambda_\gamma| \le \chi$ and it suffices to consider $\gamma < \chi^+$. For $\gamma = 0$ they are trivial and for γ limit also easy (let θ_r be the first regular $\ge \theta$ and extend $|\gamma|^+ \times \theta_r$ times taking care of Λ_β in stage $\gamma \times \zeta + \beta$ for each $\beta < \gamma$). So let $\gamma = \beta + 1$.

We first prove (3), but we have two cases (see clause (c)) of the assumption. If $\chi^{<\theta} \le \mu$ this is straight by bookkeeping. So assume $\mathrm{cf}(\mu) \ge \theta$. Given $(M, N) \in K_{\mu,\chi}$ we try to choose, by induction on $i < \chi^+$, a pair (M_i, N_i) and also $\psi_i(\bar{y}_i, \bar{x}_i), \bar{a}_i, \bar{b}_i$ for i odd such that

$\circledast_1$ (a) $(M_0, N_0) = (M, N)$
- (b) $(M_i, N_i) \in K_{\mu,\chi}$ is $\le_{\mathfrak{K}}$-increasing continuous
- (c) M_{i+1} is Λ_β-generic for i even
- (d) for i odd $\psi_i(\bar{y}_i, \bar{x}_i) \in \Lambda_\beta$ and $\bar{a}_i \in {}^{\theta>}N$ and $\bar{b}_i \in {}^{\theta>}(N_{i+1})$ are such that $\ell g(\bar{a}_i) = \ell g(\bar{x}_i)$, $\ell g(\bar{b}_i) = \ell g(\bar{y}_i)$ and
  - (α) $\bar{b} \in {}^{\ell g(\bar{y}_i)}(M_i) \Rightarrow M_i \nVdash \psi_i[\bar{b}_i, \bar{a}]$ but
  - (β) $M_{i+1} \Vdash \psi_i[\bar{b}_i, \bar{a}_i]$.
  - (γ) For every $\bar{b} \in {}^{\theta>}(M_{i+1})$ there is an automorphism of M_{i+1} over N_i mapping $\bar{b}$ into N_{i+1}.

If we succeed, by part (2) applied to the pair of models $\big(\bigcup\limits_{i<\chi^+} M_i, N\big)$ as $\chi^+ \le \mu$, this pair belongs to $K_{\mu,\chi}$ we get N' as there, hence for some odd $i < \chi^+$, $N' \subseteq M_i$. Let $\zeta = i + 2$, and this gives a contradiction to the choice of $(\psi_\zeta, \bar{a}_\zeta, \bar{b}_\zeta)$.

[Why? There is an automorphism f of $M := \bigcup\{M_j : j < \chi^+\}$ over N mapping $\bar{b}_\zeta$ into N' hence into M_i hence $f(\bar{b}_\zeta) \in {}^{\theta>}(M_\zeta)$. We know (by clause $(d)(\beta)$ above) that $M_{\zeta+1} \Vdash \psi_\zeta[\bar{b}_\zeta, \bar{a}_\zeta]$ but $M_{\zeta+1} \le_{\mathfrak{K}_\mu} M$, hence $M \Vdash \psi_\zeta[f(\bar{b}_\zeta), \bar{a}_\zeta]$. Recall that f is an automorphism of M over N hence $M \Vdash \psi_\zeta[f(\bar{b}_\zeta), f(\bar{a}_\zeta)]$, but $\bar{a}_\zeta \in {}^{\theta>}N$ so $f(\bar{a}_\zeta) = \bar{a}_\zeta$ hence $M \Vdash \psi_\zeta[\bar{b}_\zeta, f(\bar{a}_\zeta)]$. But $M_\zeta \le_{\mathfrak{K}_\mu} M$ and $\bar{a}, f(\bar{b}_\zeta)$ are from M_ζ hence $M_\zeta \nVdash \neg\psi_\zeta[f(\bar{b}_\zeta), \bar{a}_\zeta]$. However by clause (d)($\alpha$) of $\circledast_1$ we have $M_\zeta \nVdash \psi_\zeta[f(\bar{b}_\zeta), \bar{a}_\zeta]$. But as i (hence ζ) is an odd ordinal the last two sentences contradict clause (c) of $\circledast_1$ applied to $i + 1$.]

Hence we are stuck for some $i < \chi^+$. Now for $i = 0$ clause $\circledast(a)$ gives a permissible value and for i limit take unions noting that clauses (c),(d) required nothing. So $i = j + 1$; if j is even we apply the induction hypothesis for the pair (M_i, N_i). Hence j is odd so we cannot choose $\psi_j(\bar{y}, \bar{x}), \bar{a}_j, \bar{b}_j$, recalling part (2) so the pair (M_j, N_j) is as required thus proving the induction step for part (3), i.e. (3) for Λ_γ.

Second, we prove part (4) still for $\gamma = \beta + 1$. We can now again try to choose by induction on $i < \chi^+$ a pair (M_i, N_i) satisfying

[7] Not a real loss to assume this, as it suffices to deal with arbitrary large $\theta < \beth_{1,1}(\mathrm{LST}(\mathfrak{K}))$.

$\circledast_2$ (a) $(M_0, N_0) = (M, N)$
(b) $(M_i, N_i) \in K_{\mu,\chi}$ is $\leq_{\mathfrak{K}}$-increasing continuous
(c) If $i = 2j+1$, <u>then</u> (M_{i+1}, N_{i+1}) is as in part (3) for Λ_γ with (M_i, N_i), (M_{i+1}, N_{i+1}) here standing for $(M, N), (M_1, N_1)$ there.
(d) if $i = 2j$ <u>then</u> for some $\psi_i(\bar{y}_i, \bar{x}_i) \in \Lambda_\beta$ and $\bar{a}_i \in {}^{\ell g(\bar{x}_i)}(N_i)$ and $\bar{b}_i \in {}^{\ell g(\bar{y}_i)}(N_{i+1})$ we have $M_{i+1} \Vdash \psi_i(\bar{b}_i, \bar{a}_i)$ but
$$\bar{b} \in {}^{\ell g(\bar{y}_i)}(M_i) \Rightarrow M_i \nVdash \psi_i[\bar{b}, \bar{a}_i].$$

If we succeed, let $S_0 = \{\delta < \chi^+ : \mathrm{cf}(\delta) \geq \theta\}$, so by an assumption S is a stationary subset of χ^+, i.e. as by clause $\boxdot(b)$ we have $\theta = \mathrm{cf}(\theta) \leq \chi \vee \theta < \chi$. Also, for $\delta \in S_0$, as $\langle N_i : i < \delta\rangle$ is increasing with union N_δ and $\delta = 2\delta$, clearly $\bar{a}_\delta$ is well defined, so for some $i(\delta) < \delta$ we have $\bar{a}_\delta \in {}^{\theta>}(N_{i(\delta)})$ and without loss of generality $i(\delta) = 2j(\delta) + 1$ for some $j(\delta)$ hence by clause (c) of $\circledast_2$ the pair $(M_{i(\delta)+1}, N_{i(\delta)+1})$ is as required there: contradiction, as in the proof for part (3). Hence for some i we cannot choose (M_i, N_i).

For $i = 0$ let $(M_i, N_i) = (M, N)$ so only clauses (a) + (b) of $\circledast_2$ apply and are satisfied. For i limit take unions. So $i = j + 1$. If $j = 1$ mod 2, clause (d) of $\circledast_2$ is relevant and we use part (3) for Λ_β which holds as we have just proved it.

Lastly, if $j = 2$ mod 2 and we are stuck then the pair (M_j, N_j) is as required.

Third, Part (5) should be clear but we elaborate.

We prove by induction on γ' that if $\varphi(\bar{x}) \in \Lambda_\gamma$ has quantifier depth $< 1 + \gamma'$ then for every $\bar{a} \in {}^{\ell g(\bar{x})}(N_1)$ we have $M_1 \models \varphi[\bar{a}] \Leftrightarrow N_1 \models \varphi[\bar{a}]$. For atomic φ this is obvious and for $\varphi = \bigwedge\limits_{i<\alpha} \varphi_i$ should be clear. If $\varphi(\bar{x}) = \neg\psi(\bar{x})$ note that in $(*)^+$ of part (4) we can use empty $\bar{y}$ so $\neg(\exists\bar{y})\psi(\bar{x}) = \neg\psi(\bar{x})$. Also for $\varphi(\bar{x}) = (\exists\bar{y})\varphi'(\bar{y}, \bar{x})$ we apply part (4).

Fourth, we deal with part (6), so (see clause (d) of the assumption) we have $\mathrm{cf}(\mu) \geq \theta$. Let $\chi = \langle \chi_i : i < \mathrm{cf}(\mu)\rangle$ be constantly μ^- (so $\mu = \chi_i^+$) if μ is a successor cardinal, and be increasing continuous with limit μ. $2^{<\theta} < \chi_i < \mu$ if μ is a limit cardinal recalling $2^{<\theta} < \mu$ by $\boxdot(b)$. Consider
$$K_{\mu,\bar{\chi}} = \big\{\overline{M} = \big\langle M_i : i \leq \mathrm{cf}(\mu)\big\rangle : \overline{M} \text{ is } \leq_{\mathfrak{K}}\text{-increasing continuous,}$$
$$M_i \in K_{\chi_i} \text{ for } i < \mathrm{cf}(\mu),\ M_{\mathrm{cf}(\mu)} \in K^*_\mu\big\}$$
ordered by $\overline{M}^1 \leq_{\mathfrak{K}} \overline{M}^2$ <u>iff</u> $i \leq \mathrm{cf}(\mu) \Rightarrow M^1_i \leq_{\mathfrak{K}} M^2_i$.

By 2.11 and part (5) for Λ_γ which we proved we can easily find $\overline{M} \in K_{\mu,\bar{\chi}}$ such that $i < \mathrm{cf}(\mu) \Rightarrow$ '$(M_{\mathrm{cf}(\mu)}, M_{i+1})$ is Λ_γ-generic'. Such $\overline{M}$ we call Λ_*-generic.

Next

$\boxtimes$ if $\varphi(\bar{x}) \in \Lambda_\gamma$ and $\overline{M}$ is Λ_γ-generic, $\bar{a} \in {}^{\theta>}(M_i)$, i successor, $\varphi(\bar{x}) \in \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ and $\ell g(\bar{x}) = \ell g(\bar{a})$ <u>then</u> $M_{\mathrm{cf}(\mu)} \models \varphi[\bar{a}] \Leftrightarrow M_{\mathrm{cf}(\mu)} \Vdash \varphi[\bar{a}]$.

[Why? Recalling $\mathrm{cf}(\mu) \geq \theta$, we prove this by induction on the quantifier depth of φ.]

By the definition of "M is Λ-generic" and categoricity of K^*_μ we are done. $\square_{2.11}$

Conclusion 2.12. *If $\mu \geq (2^{<\theta})^+$, $\theta > \mathrm{LST}(\mathfrak{K})$ and $\mathrm{cf}(\mu) \geq \theta > \mathrm{LST}(\mathfrak{K})$ <u>then</u> every $M \in K^*_\mu$ is $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$-generic, hence if $M_1 \leq_{\mathfrak{K}} M_2$ are from K^*_μ <u>then</u> $M_1 \prec_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} M_2$.*

Remark 2.13. 1) With a little more care, if $\mu = \mu_0^+$ also $\theta = \mu$ is O.K. but here this is peripheral.

2) $\theta \leq \mathrm{LST}(\mathfrak{K})$ is not problematic, so we just ignore it.

3) So 2.12 improves 1.13; i.e., we need $\mathrm{cf}(\mu) \geq \lambda$ $(> \mathrm{LST}(\mathfrak{K}))$ instead of $\mu = \mu^{<\lambda}$, but still there is a class of μ which are not covered.

Proof. Let Λ_* be as in 2.7(2), so in particular $|\Lambda_*| \le 2^{<\theta}$. Now 2.11(6) and clause (g) of 2.7 proves the first assertion in 2.12. For the second assume that $M_1 \le_{\mathfrak{K}_\mu} M_2$ and we shall prove that $M_1 \prec_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} M_2$.

By the categoricity of $\mathfrak{K}$ in μ, or clause $(b)^-_\mu$ of Hypothesis 2.1, K^* is categorical in μ hence $M_1, M_2 \in K^*_\mu$ are Λ_*-generic. Suppose $\bar a \in {}^{\ell g(\bar x)}(M_1)$, $\varphi(\bar x) \in \Lambda_*$, so by M'_1 being Λ_*-generic (or $\boxtimes$ from the end of the proof of 2.11 applied to $\overline{M}^2$) we have

$(*)_1$ $M_1 \models \varphi[\bar a] \Rightarrow M_1 \Vdash \varphi[\bar a] \Rightarrow M_1 \models \varphi[\bar a]$

and by M_2 being Λ_*-generic (or $\boxtimes$ from the end of the proof of 2.11 applied to $\overline{M}^2$) we have

$(*)_2$ $M_2 \models \varphi[\bar a] \Rightarrow M_2 \Vdash \varphi[\bar a] \Rightarrow M_2 \models \varphi[\bar a]$

and by the definition of "$M \Vdash \varphi[\bar a]$" recalling $M_1 \le_{\mathfrak{K}_\mu} M_2$,

$(*)_3$ if $M_1 \Vdash \varphi'[\bar a]$ then $M_2 \Vdash \varphi'[\bar a]$ for $\varphi'(\bar x) \in \{\varphi(\bar x), \neg\varphi(\bar x)\}$.

So both M_1 and M_2 satisfy $\varphi[\bar a]$ if M_1 satisfies it, but this applies to $\neg\varphi[\bar a]$ too; so we are done. $\square_{2.12}$

Claim 2.14. *If K is also categorical in μ^* (or just Hypothesis 2.6 applies also to μ^*, with the same Φ) and $\mu^* \ge \mu^{<\theta} > \mu > \theta > \mathrm{LST}(\mathfrak{K})$ and $(*)$ below, then every $M \in K^*_\mu$ is $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$-generic and*

$$M_1 \in K^*_\mu \wedge M_2 \in K^*_\mu \wedge M_1 \le_{\mathfrak{K}_\mu} M_2 \Rightarrow M_1 \prec_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} M_2,$$

i.e. the conclusions of 1.13, 2.12 hold where

$(*)$ *if $M \in K^*_{\mu^*}$ and $A \in [M]^\mu$ then we can find $N \le_{\mathfrak{K}} M$ such that $A \subseteq N \in K^*_\mu$ and for every $\varphi(\bar x) \in \mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ and $\bar a \in {}^{\ell g(\bar x)}N$ we have*

$$M \Vdash \varphi[\bar a] \Leftrightarrow N \Vdash \varphi[\bar a].$$

Proof. We shall choose $(M_i, N_i) \in K_{\mu^*,\mu}$ by induction on $i \le \theta^+$ such that not only $M_i \in K^*_{\mu^*}$ (see the definition of $K_{\mu^*,\mu}$) but also $N_i \in K^*_\mu$ and this sequence of pairs is $\le_{\mathfrak{K}}$-increasing continuous. For $i = 0$ use any pair; e.g. $M_0 = \mathrm{EM}_{\tau(\mathfrak{K})}(\mu^*, \Phi)$ and $N_0 = \mathrm{EM}_{\tau(\mathfrak{K})}(\mu, \Phi)$.

For i limit take unions, recalling M_j, N_j are pseudo superlimit for $j < i$.

For $i = j + 1$, let $N^+_j \le_{\mathfrak{K}} M_j$ be such that $N_j \subseteq N^+_j \in K_\mu$ and (M_j, N^+_j) satisfies $(*)$ of the claim (standing for (M, N)). Let Λ_* be as in 2.7 for μ^*. Then by 2.11(5) with (μ^*, μ, θ) here standing for (μ, χ, θ) there (noting that in $\boxdot(c)$ there we use the case $\chi^{<\theta} \le \mu$ which here means $\mu^* = \mu^{<\theta}$) we can choose a Λ_*-generic pair $(M_i, N_i) \in K_{\mu^*,\mu}$ above (M_j, N^+_j). Hence by 2.7(2)(g) it is also a $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$-generic pair. Now for $j < \theta^+$, for $\bar a \in {}^{\theta>}(N_j)$, we can read $\mathrm{gtp}^{\mu^*}_\theta(\bar a, \varnothing, M_{j+1})$ and it is complete, but as by our use of $(*)$ it is the same as $\mathrm{gtp}^\mu_\theta(\bar a, \varnothing, N^+_{j+1})$. So $\mathrm{gtp}^\mu_\theta(\bar a, \varnothing, N^+_{j+1})$ is complete for every $\bar a \in {}^{\theta>}(N_j)$, so also $\mathrm{gtp}^\mu(\bar a, \varnothing, N_{\theta^+})$ is complete by monotonicity.

Now if $\bar a \in {}^{\theta>}(N_{\theta^+})$ then for some $j < \theta^+$ we have $\bar a \in {}^{\theta>}(N_j)$, so by the above $p_{\bar a} := \mathrm{gtp}^{\mu^*}_\theta(\bar a, \varnothing, M_{j+1}) = \mathrm{gtp}^\mu_\theta(\bar a, \varnothing, N^+_{j+1}) = \mathrm{gtp}^\mu_\theta(\bar a, \varnothing, N_{\theta^+})$ is complete and does not depend on j as long as j is large enough.

Now we prove that if $\bar a \in {}^{\theta>}(N_{\theta^+})$ then $\varphi(\bar x) \in p_{\bar a} \Rightarrow N_{\theta^+} \models \varphi[\bar a]$, and we prove this by induction on the quantifier depth of $\varphi(\bar x)$. As usual, the real case is $\varphi(\bar x) = (\exists \bar y)\varphi(\bar y, \bar x)$. Let $j < \theta^+$ be such that $\bar a \in {}^{\ell g(\bar x)}(N_j)$, so $p_{\bar a} = \mathrm{gtp}^{\mu^*}_\theta(\bar a, M_{j+1})$ so $M_{j+1} \Vdash \varphi[\bar a]$ and by the choice of (M_{j+1}, N_{j+1}) it follows that $N_{j+1} \models \varphi[\bar a]$. Hence for some $\bar b \in {}^{\ell g(\bar y)}(N_{j+1})$ we have $N_{j+1} \models \psi[\bar b, \bar a]$ hence $M_{j+1} \Vdash \psi(\bar b, \bar a)$, hence $\psi(\bar{y},\bar{x}) \in p_{\bar{b}\hat{}\bar{a}}$ hence by the induction hypothesis $N_{\theta^+} \models \psi[\bar{b},\bar{a}]$ hence $N_{\theta^+} \models \varphi[\bar{a}]$. $\square_{2.14}$

Conclusion 2.15. *1) For each $\theta \geq \mathrm{LST}(\mathfrak{K})$, the family of $\mu > 2^{<\theta}$ in which K is categorical but some (equivalently, every) $M \in K_\mu$ is not $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$-generic is $\subseteq \{[\mu_i, \mu_i^{<\theta}] : i < 2^{2^\theta}\}$ for some sequence $\langle \mu_i : i < 2^{2^\theta}\rangle$ of cardinals.*

2) Similarly for pseudo solvable: i.e. for each $\theta \geq \mathrm{LST}(\mathfrak{K})$ and $\Phi \in \Upsilon^{\mathrm{or}}_\theta$, for at most $\beth_2(\theta)$ cardinals $\mu > 2^{<\theta}$, we have $(\forall \alpha < \mu)\big[|\alpha|^{<\theta} < \mu\big]$ and for some $\mu^* \in [\mu, \mu^{<\theta}]$ the pair $(\mathfrak{K}, \Phi)$ is pseudo μ^*-solvable but some (equivalently, every) $M \in K^*_{\Phi,\mu^*}$ is not $\mathbb{L}_{\infty,\theta^+}[\mathfrak{K}]$-generic.

Proof. Straightforward. Note that it is enough to prove this for each Φ separately.

Toward contradiction, assume $\langle \mu_\varepsilon : \varepsilon < (\beth_2(\theta))^+\rangle$ is an increasing sequence of such cardinals, satisfying $(\mu_\varepsilon)^{<\theta} < \mu_{\varepsilon+1}$.

$(*)_1$ for a linear order I let

(a) $\Omega_{I,\theta} = \{(\bar{t},\bar{\sigma}) : \bar{t} \in {}^{\theta>}I,\ \bar{\sigma} = \langle \sigma_i(a_{\bar{t}_{\eta_i}}) : i < i_*\rangle\}$, where $\eta_i \in {}^{\omega>}\ell g(\bar{t})$, $\bar{t}_{\eta_i} = \langle t_{\eta_i(\ell)} : \ell < \ell g(\eta_i)\rangle$, and σ is a $\tau(\Phi)$-term.

(b) $\mathcal{E}_{I,\theta}$ is the following equivalence relation on $\Omega_{I,\theta}$: $(\bar{t}^1,\bar{\sigma}^1)\ \mathcal{E}_{I,\theta}\ (\bar{t}^2,\bar{\sigma}^2)$ iff

(α) $\ell g(\bar{t}^1) = \ell g(\bar{t}^2)$

(β) $\bar{\sigma}^1 = \bar{\sigma}^2$

(γ) $\{(t^1_i : t^2_i) : i\}$ is a partial automorphism of I.

[No idea what this means; I've been fixing typos freely, but I can't even guess at the intention.]

For transparency assume θ is regular. Let Ψ be as in 5.1(3) so for a linear order I', $\mathrm{EM}_{\{<\}}(I,\psi)$ is a linear ordinal (of cardinality (I)).

[I assume $|I|$?]

Now for each ε and

$$\zeta \in w_\varepsilon = \{\zeta \in [\mu_\varepsilon, \mu_{\varepsilon+1}) : \zeta \text{ has cofinality } \theta \text{ and is divisible by } \mu_\varepsilon\}$$

let $I_\mu = \mathrm{EM}_{\{<\}}(I^{\mathrm{lim}}_{\theta,\zeta},\psi)$, hence (in the statement of 5.1), instead of ζ we have $\lambda \times \theta_2$ which here will be $\mu_\varepsilon \times \theta$; but in the proof of 5.1 we start it for any $\zeta \in [\lambda, \lambda]$ **[of]** cofinality θ_2, we have

$(*)_3$ (a) $I_{\varepsilon,\zeta} = I_\zeta$ is a linear order of cardinality μ_ε.

(b) $I_{\varepsilon,\zeta}$ is increasing with ε and for a fixed ε increasing with ζ.

(c) let $M_{\varepsilon,\zeta} = M_\zeta$, $\mathrm{EM}_{\tau(\mathfrak{K})}(I_{\varepsilon,\zeta},\Phi)$, so $\leq_{\mathfrak{K}}$-increasing

(d) If h is a partial automorphism of $(\zeta,<)$ of cardinality $< \theta$ then $\hat{h}$, the partial automorphism of $I_{\varepsilon,\zeta}$ which induces an automorphism of $\mathrm{EM}(I_{\varepsilon,\zeta},\Phi)$

[Sentence ends here. Does this bleed into $(*)_4$?]

$(*)_4$ we define[8] an equivalence relation on $\mathcal{E}_{\varepsilon,\zeta} = \mathcal{E}_\zeta$ on ${}^{\theta>}(M_{\varepsilon,\zeta})$ as follows:

$\bar{a}\ \mathcal{E}_{\varepsilon,\zeta}\ \bar{b}$ iff there is a partial automorphism h of $(\zeta,<)$ such that the partial automorphism $\hat{h}$ it induces on $I_{\varepsilon,\zeta}$ satisfies that the partial automorphism $\hat{h}$ it induces on $M_{\varepsilon,\zeta}$ maps $\bar{a}$ to $\bar{b}$.

[$\hat{h}$ is used twice. ". . . such that the induced partial automorphism on $M_{\varepsilon,\zeta}$ maps $\bar{a}$ to $\bar{b}$?"]

[8]For being an equivalence relation it is better to assume the following on Φ: if $\bar{t}_1,\bar{t}_2 \in {}^{\omega>}I$, $\mathrm{EM}(I,\Phi) \models \sigma_1(\bar{a}_{\bar{t}_1}) = \sigma_2(\bar{a}_{\bar{t}_2})$, $\bar{t} \in {}^{\omega>}I$, $\mathrm{rang}(\bar{t}) = \mathrm{rang}(\bar{t}_1) \cap \mathrm{rang}(\bar{t}_2)$, then for some σ, $\mathrm{EM}(I,\Phi) \models \sigma(\bar{a}_{\bar{t}}) = \sigma_\ell(\bar{a}_{\bar{t}_\ell})$ for $\ell = 1,2$.

$(*)_5$ If $\zeta_1 < \zeta_2$ then any equivalence class of ${}^{\theta>}(M_{\zeta_2})$ is represented in M_{ζ_1}. (Recall $\zeta_\ell \geq \mu_0 > \theta > |\tau(\Phi)|$.)

$(*)_6$ for any $(\bar{t}, \bar{\sigma}) \in \Omega_{I_\zeta,\theta}$, the generic type $\mathrm{gps}(\langle \sigma_i(\bar{a}_{t_{\eta_i}}) : i < i_* \rangle, \varnothing, M_\zeta)$ is determined by ζ and $(\bar{t}, \bar{\sigma})/\mathcal{E}_{I_\zeta,\theta}$.

As $\mathcal{E}_{\varepsilon,\zeta}$ has $\leq \beth_2(\theta) \leq \mu_\varepsilon$ (even $\leq 2^{<\theta} \leq \mu_\varepsilon$) equivalence classes, for each ε there is $w^*_\varepsilon \subseteq w_\varepsilon$, unbounded in μ^+_ε, such that the function implicit in $(*)_6$ is constant for $\zeta \in W_\varepsilon$.

Similarly there is $S \subseteq \beth_2(\theta)$, unbounded in it, such that the above function is constant on $\bigcup\{W^*_\varepsilon : \varepsilon \in S\}$. For any $\varepsilon_1 < \varepsilon_2$ in S and $\zeta_2 \in W^*_\varepsilon$, let $(\mu^*, \mu) := (\mu_{\varepsilon_2}, \mu_{\varepsilon_1})$ and we verify condition $(*)$ in 2.14. Let $M \in K_{\mu^*}$, so without loss of generality $M = M_{\varepsilon_2,\varepsilon_2}$ and suppose $A \in [M]^\mu$, then there is $J_0 \subseteq \zeta_1$ of cardinality μ such that $A \subseteq \mathrm{EM}(J_0, \Phi \circ \Psi)$.

Let $\zeta_1 \in w_{\varepsilon_1}$ be $> \mathrm{otp}(J_0)$. We can find $J_1 \subseteq \zeta_2$ extending J_0 of order type ζ_1 (because $\mathrm{cf}(\zeta_1) = \mathrm{cf}(\zeta_2) = \theta$ and μ_{ε_2} divides ζ_2). So there is an isomorphism f from $M_{\varepsilon_1,\zeta_1}$ onto $\mathrm{EM}_{\tau(\mathfrak{K})}(J_1, \Phi \circ \Psi)$. Choosing the choices / With the appropriate choices of $S, W_{\varepsilon_1}, W_{\varepsilon_2}$ we are done. $\square_{2.15}$

* * *

For the rest of this section we note some basic facts on the dependency on Φ (not used here).

Definition 2.16. 1) We define a two-place relation $\mathcal{E}_\kappa = \mathcal{E}^{\mathrm{or}}_\kappa[\mathfrak{K}]$ on $\Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$, so $\kappa \geq \mathrm{LST}(\mathfrak{K})$: $\Phi_1 \; \mathcal{E}_\kappa \; \Phi_2$ iff for every linear orders I_1, I_2 there are linear orders J_1, J_2 extending I_1, I_2 respectively such that $\mathrm{EM}_{\tau(\mathfrak{K})}(J_1, \Phi)$, $\mathrm{EM}_{\tau(\mathfrak{K})}(J_2, \Phi)$ are isomorphic.

2) We define $\leq^{\mathrm{or}}_\kappa = \leq^{\mathrm{or}}_{\kappa,[\mathfrak{K}]}$, a two-place relation on $\Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$ as in part (1); only in the end, $\mathrm{EM}_{\tau(\mathfrak{K})}(J_1, \Phi_1)$ can be $\leq_{\mathfrak{K}}$-embedded into $\mathrm{EM}_{\tau(\mathfrak{K})}(J_2, \Phi_2)$.

[The highlighted relation was originally typeset as $\leq^{\mathrm{or}}_\kappa [\mathfrak{K}]$ throughout; it and $\mathcal{E}^{\mathrm{or}}_\kappa[\mathfrak{K}]$ look horrific when actually used in an expression.]

Claim 2.17. *1) The following conditions on $\Phi_1, \Phi_2 \in \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$ are equivalent:*

(a) $\Phi_1 \; \mathcal{E}_\kappa \; \Phi_2$

(b) There are $I_1, I_2 \in K^{\mathrm{lin}}$ of cardinality $\geq \beth_{1,1}(\kappa)$ such that $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi_1)$, $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$ are isomorphic.

(c) there are Φ'_1, Φ'_2 satisfying $\Phi_\ell \leq^{\otimes} \Phi'_\ell \in \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$ for $\ell = 1, 2$ such that Φ'_1, Φ'_2 are essentially equal (see Definition 2.18 below).

2) The following conditions are equivalent

(a) $\Phi_1 \leq^{\mathrm{or}}_\kappa \Phi_2$ (recall $\leq_\kappa = \leq^{\mathrm{or}}_\kappa [\mathfrak{K}]$).

(b) There are $I_1, I_2 \in K^{\mathrm{lin}}$ of cardinality $\geq \beth_{1,1}(\kappa)$ such that $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi_1)$ can be $\leq_{\mathfrak{K}}$-embedded into $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi_2)$.

(c) for every $I_1 \in K^{\mathrm{lin}}$ there is $I_2 \in K^{\mathrm{lin}}$ such that $\mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi_1)$ can be $\leq_{\mathfrak{K}}$-embedded into $\mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi_2)$.

Definition 2.18. $\Phi_1, \Phi_2 \in \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$ are *essentially equal* when for every linear order I there is an isomorphism f from $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi_1)$ onto $\mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi_2)$ such that for any τ_{Φ_1}-term $\sigma_1(x_0, \ldots, x_{n-1})$ there is a τ_{Φ_2}-term $\sigma_2(x_0, \ldots, x_{n-1})$ such that: $t_0 <_I \ldots <_I t_{n-1} \Rightarrow f(a_1) = a_2$, where a_ℓ is $\sigma_\ell(a_{t_0}, \ldots, a_{t_{n-1}})$ as computed in $\mathrm{EM}(I, \Phi_\ell)$ for $\ell = 1, 2$.

Proof. Straightforward (particularly recalling such proof in 1.32(1)). $\square_{2.17}$

Claim 2.19. *1) $\mathcal{E}_\kappa = \mathcal{E}^{\rm or}_\kappa[\mathfrak{K}]$ is an equivalence relation and*

$$\Phi_1 \mathcal{E}^{\rm or}_\kappa[\mathfrak{K}]\Phi_2 \Rightarrow \Phi_1 \le^{\rm or}_\kappa [\mathfrak{K}]\Phi_2.$$

[See what I mean?]

1A) In fact, if $\langle \Phi_\varepsilon : \varepsilon < \varepsilon(*)\rangle$ are pairwise $\mathcal{E}_\kappa$-equivalent and $\varepsilon(*) \le \kappa$ then we can find $\langle \Phi'_\varepsilon : \varepsilon < \kappa\rangle$ satisfying $\Phi'_\varepsilon \le^{\otimes} \Phi'_\varepsilon$ for $\varepsilon < \varepsilon(*)$ such that the Φ'_ε for $\varepsilon < \varepsilon(*)$ are pairwise essentially equal.

2) $\le^{\rm or}_\kappa$ is a partial order.

3) If $\Phi_1, \Phi_2 \in \Upsilon^{\rm or}_\kappa[\mathfrak{K}]$ are essentially equal then $(\mathfrak{K}, \Phi_1)$ is pseudo/weakly/strongly (μ, κ)-solvable iff $(\mathfrak{K}, \Phi_2)$ is pseudo/weakly/strongly (μ, κ)-solvable.

4) If $\Phi_1 \in \Upsilon^{\rm or}_\kappa[\mathfrak{K}]$ is strongly (μ, κ)-solvable and Φ_2 exemplifies $\mathfrak{K}$ is (μ, κ)-solvable then Φ_1 $\mathcal{E}_\kappa$ Φ_2.

5) If $\mathfrak{K}$ is categorical in μ and $\mu > \kappa \ge \mathrm{LST}(\mathfrak{K})$ then every $\Phi \in \Upsilon^{\rm or}_\kappa[\mathfrak{K}]$ is strongly (μ, κ)-solvable.

6) Assume $(\mathfrak{K}, \Phi_\ell)$ is pseudo (μ, κ)-solvable and $\mu \ge \beth_{1,1}(\kappa)$ for $\ell = 1, 2$. Then Φ_1 $\mathcal{E}_\kappa$ Φ_2 iff $\Phi_1 \le^{\rm or}_\kappa \Phi_2 \wedge \Phi_2 \le^{\rm or}_\kappa \Phi_1$.

7) If $\Phi_1 \le^{\rm or}_\kappa \Phi_2$ and Φ_1 is strongly (μ, κ)-solvable or just pseudo (μ, κ)-solvable then Φ_1, Φ_2 are $\mathcal{E}^{\rm or}_\kappa[\mathfrak{K}]$-equivalent.

Proof. Easy, use 1.32(1) and its proof. $\square_{2.19}$

§ 3. §3 Categoricity for cardinals on a club

We draw here an easy conclusion from §2, getting that, on a closed unbounded class of cardinals which is $\aleph_0$-closed, we get a constant answer to being categorical. This is, of course, considerably weaker than conjecture 0.1 but is still progress; e.g. it shows that the categoricity spectrum is not totally chaotic.

We concentrate on the case the results of §1 hold (e.g. $\mu = \mu^\lambda$) for the λ-s with which we deal. To eliminate this extra assumption we need §2. This section is not used later. Note that 3.3 is continued (and improved) in [S$^+$c] and Exercise 3, [S$^+$b] improve 3.5; similarly 3.6.

In the claims below we concentrate on fixed points of the sequence of $\beth_\alpha$-s.

Hypothesis 3.1. As in Hypothesis 1.2, (i.e. $\mathfrak{K}$ is an AEC with models of arbitrarily large cardinality).

Definition 3.2. 1) Let $\mathrm{Cat}_{\mathfrak{K}}$ be the class of cardinals in which $\mathfrak{K}$ is categorical.

1A) Let $\mathrm{Sol} = \mathrm{Sol}_{\mathfrak{K},\Phi} = \mathrm{Sol}^1_{\mathfrak{K},\Phi}$ be the class of $\mu > \mathrm{LST}[\mathfrak{K}]$ such that $(\mathfrak{K},\Phi)$ is pseudo μ-solvable. Let $\mathrm{Sol}^2_{\mathfrak{K},\Phi}$ $[\mathrm{Sol}^3_{\mathfrak{K},\Phi}]$ be the class of $\mu > \mathrm{LST}(\mathfrak{K})$ such that $(\mathfrak{K},\Phi)$ is weakly [strongly] μ-solvable.

2) Let $\mathrm{mod}{-}\mathrm{com}_{\mathfrak{K},\Phi}$ be the class of pairs (μ,θ) such that: $\mu > \theta \geq \mathrm{LST}(\mathfrak{K})$ and $\mathbb{L}_{\infty,\theta^+}[\mathfrak{K}]$ is μ-model complete. (On $K^*_{\Phi,\mu}$ see Definition 2.3(3)(b), 2.3(5).)

3) Let $\mathrm{Cat}'_{\mathfrak{K}}$ be the class of $\mu \in \mathrm{Cat}_{\mathfrak{K}}$ such that: $\mu \geq \beth_{1,1}(\mathrm{LST}(\mathfrak{K}))$ and if $\mathrm{LST}(\mathfrak{K}) \leq \theta$ and $\beth_{1,1}(\theta) \leq \mu$ <u>then</u> $\mathbb{L}_{\infty,\theta^+}[\mathfrak{K}]$ is μ-model complete.

3A) For $\Phi \in \Upsilon^{\mathrm{or}}_{\mathfrak{K}}$ let $\mathrm{Sol}^{k,*}_{\mathfrak{K},\Phi}$ be the class of $\mu \in \mathrm{Sol}^k_{\mathfrak{K},\Phi}$ such that $\mu \geq \beth_{1,1}(\mathrm{LST}(\mathfrak{K}))$ and: if $\mathrm{LST}(\mathfrak{K}) \leq \theta$ and $\beth_{1,1}(\theta) \leq \mu$ then the pair $(\mathbb{L}_{\infty,\theta^+}[\mathfrak{K}],\Phi)$ is μ-model complete.

Let $\mathrm{Sol}^{\ell,<\theta}_{\mathfrak{K},\Phi}$ be the class of $\lambda \in \mathrm{Sol}^\ell_{\mathfrak{K},\Phi}$ such that $\mathbb{L}_{\infty,\theta}[\mathfrak{K}]$ is μ-model complete (see [She09b, §2]).

Let $\mathrm{Sol}'_{\mathfrak{K},\Phi} = \mathrm{Sol}^{1,*}_{\mathfrak{K},\Phi}$. Instead of $k,*$ we may write $3+k$.

4) Let $\mathbf{C} = \{\lambda : \lambda = \beth_\lambda \text{ and } \mathrm{cf}(\lambda) = \aleph_0\}$.

<u>Exercise</u>: 1) The conclusion of 1.13(1) (equivalently, 1.13(2)) means that $\theta \leq \lambda \Rightarrow (\mu,\theta) \in \mathrm{mod}{-}\mathrm{com}_{\mathfrak{K},\Phi}$.

2) Write down the obvious implications.

Claim 3.3. *If $\mu > \lambda = \beth_\lambda > \kappa \geq \mathrm{LST}(\mathfrak{K})$ and $\Phi \in \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$, $\mathrm{cf}(\lambda) = \aleph_0$ <u>then</u>*

$$\mu = \mu^{<\lambda} \Rightarrow \mu \in \mathrm{Sol}'_{\mathfrak{K},\Phi} \Rightarrow \lambda \in \mathrm{Sol}'_{\mathfrak{K},\Phi}.$$

Proof. The first implication holds by 1.13(2) and 3. The assumption of the second implication implies Hypothesis 1.18 (see 3(1)) hence its conclusion holds by 1.44. $\square_{3.3}$

Observation 3.4. *K_λ is categorical in λ (hence Hypothesis 1.18 holds), <u>if</u>:*

$\circledast_\lambda$ $\lambda = \beth_\lambda = \sup(\lambda \cap \mathrm{Cat}'_{\mathfrak{K}}) > \mathrm{LST}(\mathfrak{K})$ *and* $\aleph_0 = \mathrm{cf}(\lambda)$.

Proof. Fix $\Phi \in \Upsilon^{\mathrm{or}}_{\mathfrak{K}}$; now clearly $\mathrm{Sol}'_{\mathfrak{K},\Phi} \supseteq \mathrm{Cat}'_{\mathfrak{K}}$ by their definitions.

By the assumptions we can find $\langle \mu_n : n < \omega\rangle$ such that $\lambda = \sum\{\mu_n : n < \omega\}$, $\mathrm{LST}(\mathfrak{K}) < \mu_n \in \mathrm{Cat}'_{\mathfrak{K}}$, and $\beth_{1,1}(\mu'_n) < \mu_{n+1}$ where $\mu'_n = \beth_{1,1}(\mu_n)$. As every $M \in K_{\mu_{n+1}}$ is $\mathbb{L}_{\infty,\mu'_n}[\mathfrak{K}]$-generic (as $K_{\mu_{n+1}} \subseteq K_{\Phi,\mu_{n+1}}$ and $\mu_{n+1} \in \mathrm{Cat}'_{\mathfrak{K}}$), easily

$(*)_0$ if $M \leq_{\mathfrak{K}} N$ are from $K^*_{\Phi,\geq\mu_{n+1}}$ then $M \prec_{\mathbb{L}_{\infty,\mu'_n}[\mathfrak{K}]} N$.

Let $M^\ell \in K_\lambda$ for $\ell \in \{1,2\}$, so we can find a $\le_{\mathfrak{K}}$-increasing sequence $\langle M^\ell_n : n < \omega\rangle$ such that $M^\ell_n \in K_{\mu_n}$, $M^\ell_n \le_{\mathfrak{K}} M^\ell_{n+1} \le_{\mathfrak{K}} M^\ell$, and $M^\ell = \bigcup\{M^\ell_n : n < \omega\}$. Now

$(*)_1$ $M^\ell_n \in K^*_{\Phi,\mu_n}$.

[Why? As $\mathfrak{K}$ is categorical in $\mu_n = \|M^\ell_n\|$.]

$(*)_2$ if $\alpha \le \mu_n$, $n < m < k$, and $\bar a, \bar b \in {}^\alpha(M^\ell_m)$ <u>then</u>:

  (a) $\mathrm{tp}_{\mathbb{L}_{\infty,\mu'_n}[\mathfrak{K}]}(\bar a, \varnothing, M^\ell_m) = \mathrm{tp}_{\mathbb{L}_{\infty,\mu'_n}}(\bar b, \varnothing, M^\ell_m)$ <u>iff</u> $\mathrm{tp}_{\mathbb{L}_{\infty,\mu'_n}[\mathfrak{K}]}(\bar a, \varnothing, M^\ell_k) = \mathrm{tp}_{\mathbb{L}_{\infty,\mu'_n}}(\bar b, \varnothing, M^\ell_k)$.

  (b) If $\mathrm{tp}_{\mathbb{L}_{\infty,\mu'_n}[\mathfrak{K}]}(\bar a, \varnothing, M^\ell_k) = \mathrm{tp}_{\mathbb{L}_{\infty,\mu'_n}[\mathfrak{K}]}(\bar b, \varnothing, M^\ell_k)$ <u>then</u> $\mathrm{tp}_{\mathbb{L}_{\infty,\mu'_m}[\mathfrak{K}]}(\bar a, \varnothing, M^\ell_k) = \mathrm{tp}_{\mathbb{L}_{\infty,\mu'_m}[\mathfrak{K}]}(\bar b, \varnothing, M^\ell_k)$.

[Why? Clause (a) by $(*)_0$, clause (b) by 1.19(3).]

$(*)_3$ $M^1_n \cong M^2_n$.

[Why? As $\mathfrak{K}$ is categorical in μ_n.]

We now proceed as in the proof of 1.41. Let

$$\begin{aligned}\mathcal{F}_n = \big\{f :\ & \text{for some } \bar a_1, \bar a_2 \text{ and } \alpha < \mu_n \text{ we have } \bar a_\ell \in {}^\alpha(M^\ell_{n+2}) \text{ for } \ell = 1,2,\\ & \mathrm{tp}_{\mathbb{L}_{\infty,\mu_{n+1}}[\mathfrak{K}]}(\bar a_1, \varnothing, M^1_{n+2}) = \mathrm{tp}_{\mathbb{L}_{\infty,\mu_{n+1}}[\mathfrak{K}]}(\bar a_2, \varnothing, M^2_{n+1}),\\ & \text{and } f \text{ is the function which maps } \bar a_1 \text{ into } \bar a_2\big\}\end{aligned}$$

(Actually, we can use $\alpha = \mu_n$.) By the hence and forth argument we can find $f_n \in \mathcal{F}_n$ by induction on $n < \omega$ such that $M^1_n \subseteq \mathrm{dom}(f_{2n+2})$, $M^2_n \subseteq \mathrm{rang}(f_{2n+2})$, and $f_n \subseteq f_{n+1}$; hence $\bigcup\{f_n : n < \omega\}$ is an isomorphism from M^1 onto M^1. $\square_{3.3}$

Claim 3.5. $\mathfrak{K}$ *is categorical in* λ <u>*when*</u>*:*

$\circledast^+_\lambda$ $\lambda = \beth_\lambda > \mathrm{LST}(\mathfrak{K})$ *and* $\lambda = \mathrm{otp}(\mathrm{Cat}_{\mathfrak{K}} \cap \lambda \cap \mathbf{C})$ *and* $\mathrm{cf}(\lambda) = \aleph_0$.

Proof. Fix Φ as in the proof of 3.3. Let $\langle \theta_n : n < \omega\rangle$ be increasing such that $\lambda = \Sigma\{\theta_n : n < \omega\}$ and $\mathrm{LST}(\mathfrak{K}) < \theta_0$. For each n, by 2.15 we know

$$\{\mu \in \mathrm{Cat}_{\mathfrak{K}} : \mu > \theta_n \text{ and the } M \in K_\mu \text{ is not } \mathbb{L}_{\infty,\theta^+_n}\text{-generic}\}$$

is "not too large"; i.e. it is included in the union of at most $\beth_2(\theta_n)$ intervals of the form $[\chi, \chi^{\theta_n}]$. Now we choose $(n(\ell), \mu_\ell)$ by induction on $\ell < \omega$ such that

$\circledast$ (a) $n(\ell) < \omega$ and $\mu_\ell \in \mathrm{Cat}_{\mathfrak{K}} \cap \lambda$

  (b) If $\ell = k+1$ then $n(\ell) > n(k)$, $\theta_{n(\ell)} > \mu_k$, $\mu_\ell \in \mathrm{Cat}_{\mathfrak{K}} \cap \lambda \setminus \theta^+_{n(\ell)}$ and the $M \in K_{\mu_\ell}$ is $\mathbb{L}_{\infty,\theta_{n(\ell)}}[\mathfrak{K}]$-generic (hence $\mathbb{L}_{\infty,\mu^+_k}[\mathfrak{K}]$-generic).

This is easy and then continue as in 3.4. $\square_{3.5}$

We have essentially proved

Theorem 3.6. *In 3.4, 3.5 we can use* $\mathrm{Sol}_{\mathfrak{K},\Phi}$, $\mathrm{Sol}'_{\mathfrak{K},\Phi}$ *instead of* $\mathrm{Cat}_{\mathfrak{K}}$, $\mathrm{Cat}'_{\mathfrak{K}}$.

<u>Exercise</u>: For Claim 1.41(2), Hypothesis 1.18 suffices.

[Hint: The proof is similar to the existing one using 1.19.]

§ 4. §4 Good Frames

Here comes the main result of [She09b]: from categoricity (or solvability) assumptions we derive the existence of good λ-frames.

Our assumption is such that we can apply §1.

Hypothesis 4.1. 1)

(a) $\mathfrak{K}$ is an AEC.

(b) $\mu > \lambda = \beth_\lambda > \mathrm{LST}(\mathfrak{K})$ and $\mathrm{cf}(\lambda) = \aleph_0$.

(c) $\Phi \in \Upsilon^{\mathrm{or}}_{\mathfrak{K}}$

(d) $\mathfrak{K}$ is categorical in μ or just

(d)$^-$ $(\mathfrak{K}, \Phi)$ is pseudo superlimit in μ (this means $\Phi \in \mathrm{Sol}^1_{\mathfrak{K},\Phi}$; so 1.18(1) holds)

(e) Also, 1.18(2)(a) holds; i.e. the conclusion of 1.13(2) holds.

2) In addition we may use some of the following, but then we mention them and we add superscript * when used. (Note that (g) $\Rightarrow$ (f) by 1.42.)

(f) K^*_λ is closed under $\le_{\mathfrak{K}}$-increasing unions (justified by 1.41)

(g) $\langle \lambda_n : n < \omega \rangle$ is increasing, $\lambda_0 > \mathrm{LST}(\mathfrak{K})$, $\lambda = \Sigma\{\lambda_n : n < \omega\}$ and the assumptions of 1.41 hold.

Observation 4.2. *1) $\mathfrak{K}^*_\lambda$ is categorical.*

2) $\mathfrak{K}^*_\lambda$ has amalgamation.

3)* (We assume (f) of 4.1(2)). $\mathfrak{K}_\lambda$ is a λ-AEC.

Proof. 1) By 1.16(1) or 1.19(4) as $\mathrm{cf}(\lambda) = \aleph_0$.

2) By 1.34(1).

3) As in 1.42, (i.e. as $\le_{\mathfrak{K}^*_\lambda} = \le_{\mathfrak{K}} \restriction \mathfrak{K}$, closure under unions of $\le_{\mathfrak{K}}$-increasing chains is the only problematic point and it holds by (f) of 4.1(2)). $\square_{4.2}$

Remark 4.3. 1) Why do we not assume 4.1(1),(2) all the time? The main reason is that for proving some of the results assuming 4.1(1),(2) we use some such results on smaller cardinals on which we use 4.1(1) only.

2) Note that it is not clear whether improvement by using 4.1(1) only will have any affect when (or should we say if) we succeed to have the parallel of [She09e, §12].

Claim 4.4. *1) Assume $M_0 \le_{\mathfrak{K}^*_\lambda} M_\ell$, $\alpha < \lambda$, $\bar a_\ell \in {}^\alpha(M_\ell)$ for $\ell = 1,2$, and $\kappa := \beth_{1,1}(\beth_2(\theta)^+)$ where $\theta := |\alpha| + \mathrm{LST}(\mathfrak{K})$ (so $\kappa < \lambda$). If*

$$\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a_1, M_0, M_1) = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a_2, M_0, M_2)$$

<u>then</u>

$$\mathbf{tp}_{\mathfrak{K}^*_\lambda}(\bar a_1, M_0, M_1) = \mathbf{tp}_{\mathfrak{K}^*_\lambda}(\bar a_2, M_0, M_2).$$

2) If $M_1 \le_{\mathfrak{K}^*_\lambda} M_2$ <u>then</u> $M_1 \prec_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]} M_2$ for every $\theta < \lambda$, and moreover

$$M_1 \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} M_2.$$

2A) If $M_0 \le_{\mathfrak{K}^*_\lambda} M_\ell$ for $\ell = 1,2$ and $\mathbf{tp}_{\mathfrak{K}^*_\lambda}(\bar a_1, M_0, M_1) = \mathbf{tp}_{\mathfrak{K}^*_\lambda}(\bar a_2, M_0, M_2)$ and $\bar a_\ell \in {}^\alpha(M_0)$, $\alpha < \kappa \le \lambda$ <u>then</u> $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a_1, M_0, M_1) = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a_2, M_0, M_2)$.

2B) In part (1), if $M_\ell \le_{\mathfrak{K}^*_\lambda} M'_\ell$ for $\ell = 1,2$ then

$$\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a_1, M, M'_1) = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a_2, M, M'_2).$$

3) Assume that $M_0 \le_{\mathfrak{K}^*_\lambda} M_1 \le_{\mathfrak{K}^*_\lambda} M_2 \le_{\mathfrak{K}^*_\lambda} M_3$, $\bar a \in {}^\alpha(M_2)$, $\alpha < \lambda$ and $\kappa = \beth_{1,1}(|\alpha| + \mathrm{LST}(\mathfrak{K})) < \theta < \lambda$. Then

- (a) *From $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a, M_1, M_2)$ we can compute $\mathrm{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar a, M_1, M_2)$ and $\mathrm{tp}_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]}(\bar a, M_0, M_3)$.*
- (b) *From $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a, \varnothing, M_2)$ we can compute $\mathrm{tp}_{\mathbb{L}_{\infty,\theta}[\mathfrak{K}]}(\bar a, \varnothing, M_2)$ and even $\mathrm{tp}_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]}(\bar a, \varnothing, M_2)$.*
- (c) *From $\mathbf{tp}_{\mathfrak{K}^*_\lambda}(\bar a, M_1, M_2)$ we can compute $\mathrm{tp}_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]}(\bar a, M_1, M_2)$ and $\mathbf{tp}_{\mathfrak{K}^*_\lambda}(\bar a, M_0, M_3)$.*

4) If $M_1 \le_{\mathfrak{K}^*_\lambda} M_2$ and $\alpha < \kappa^* < \lambda$, $\mathbf{I}_\ell \subseteq {}^\alpha(M_1)$, $|\mathbf{I}_\ell| > \kappa$, $\mathbf{I}_\ell$ is $(\mathbb{L}_{\infty,\theta}[\mathfrak{K}], \kappa^*)$-convergent in M_1 for $\ell = 1,2$ and $\mathrm{Av}_{<\kappa}(\mathbf{I}_1, M_1) = \mathrm{Av}_{<\kappa}(\mathbf{I}_1, M_1)$ then $\mathbf{I}_\ell$ is $(\mathbb{L}_{\infty,\kappa}[\mathfrak{K}], \kappa^*)$-convergent in M_ℓ for $\ell = 1,2$ and $\mathrm{Av}_{<\kappa}(\mathbf{I}_1, M_\ell) = \mathrm{Av}_{<\kappa}(\mathbf{I}_1, M_2)$.

Proof. 1) Without loss of generality $M_0 = \mathrm{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$ and $I_0 \in K^{\mathrm{flin}}_\lambda$. By 1.32(3) for $\ell = 1,2$ there is a pair (I_ℓ, f_ℓ) such that $I_0 \le_{K^{\mathrm{flin}}} I_\ell \in K^{\mathrm{flin}}_\lambda$ and f_ℓ is a $\le_{\mathfrak{K}}$-embedding of M_ℓ into $M'_\ell = \mathrm{EM}_{\tau(\mathfrak{K})}(I_\ell, \Phi)$ over M_0. By renaming, without loss of generality f_ℓ is the identity on M_ℓ hence $M_\ell \le_{\mathfrak{K}} M'_\ell$. By 1.19(1) we know that $M_\ell \prec_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]} M'_\ell$ hence

$$\begin{aligned}&\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a_1, M_0, M'_1) = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a_1, M_0, M_1) = \\ &\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a_2, M_0, M_2) = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(\bar a_2, M_0, M'_2).\end{aligned}$$

By 1.32(1) we can find (I_3, g_1, g_2, h) such that $I_0 \le_{K^{\mathrm{flin}}} I_3 \in K^{\mathrm{flin}}_\lambda$, g_ℓ is a $\le_{\mathfrak{K}}$-embedding of M'_ℓ into $M_4 := \mathrm{EM}_{\tau(\mathfrak{K})}(I_3, \Phi)$ over M_0 for $\ell = 1,2$, and h is an automorphism of M_4 over M_0 mapping $g_1(\bar a_1)$ to $g_2(\bar a_2)$. By the definition of orbital types, this gives $\mathbf{tp}_{\mathfrak{K}^*_\lambda}(\bar a_1, M_0, M_1) = \mathbf{tp}_{\mathfrak{K}^*_\lambda}(\bar a_2, M_0, M_2)$ as required.

2) This holds by 1.19(1) for $\theta \in (\mathrm{LST}(\mathfrak{K}), \lambda)$, hence by 1.12(1) also for $\theta = \lambda$ (the assumptions of 1.12 hold as clause (a) there holds by the case above $\theta < \lambda$ and clause (b) there holds by 1.30(1)).

2A) Should be clear:

- (a) By part (2), this holds if $\bar a_1 = \bar a_2$ and $M_1 \le_{\mathfrak{K}} M_2$.
- (b) Trivially, it holds if there is an isomorphism from M_1 onto M_2 over M_0 mapping $\bar a_1$ to $\bar a_2$.
- (c) by the definition of $\mathbf{tp}$ we are done.

2B) Should be clear by part (2).

3) **Clause (a)**:

By parts (1) + (2).

Clause (b): By 1.30(1).

Clause (c): By part (2A) and the definition of $\mathbf{tp}$.

4) Easy, too. $\square_{4.4}$

Definition 4.5. Assume $M_0 \le_{\mathfrak{K}^*_\lambda} M_1 \le_{\mathfrak{K}^*_\lambda} M_2$, $\alpha < \lambda$, $\bar a \in {}^\alpha(M_2)$, and $p = \mathbf{tp}_{\mathfrak{K}^*_\lambda}(\bar a, M_1, M_2)$. We say that p *does not fork* over M_0 (for $\mathfrak{K}^*_\lambda$) when, letting $\theta_0 = |\alpha| + \mathrm{LST}(\mathfrak{K})$, $\theta_1 = \beth_{1,1}(\beth_2(\theta_0)^+)$, $\theta_2 = 2^{\theta_1}$, $\theta_2 = \beth_2(\theta_1)$, we have:

- $(*)$ for some $N \le_{\mathfrak{K}^*} M_0$ satisfying $\|N\| \le \theta_2$ we have $\mathrm{tp}_{\mathbb{L}_{\infty,\theta_1}[\mathfrak{K}]}(\bar a, M_1, M_2)$ does not split over N.

We now would like to show that there is $\mathfrak{s}_\lambda$ which fits [She09c] and [She09e] and $\mathfrak{K}_{\mathfrak{s}_\lambda} = \mathfrak{K}^*_\lambda$.

Observation 4.6. *Assume that $M_0 \le_{\mathfrak{K}^*_\lambda} M_1 \le_{\mathfrak{K}^*_\lambda} M_2$, $\bar a \in {}^\alpha(M_2)$, $\alpha < \lambda$, $\lambda > \kappa_0 \ge |\alpha| + \mathrm{LST}(\mathfrak{K})$, $\kappa_1 = \beth_{1,1}(\beth_2(\kappa_0)^+)$, and $\kappa_2 = \beth_2(\kappa_1)$. Then the following conditions are equivalent*

(a) $\mathbf{tp}_{\mathfrak{K}^*_\lambda}(\bar a, M_1, M_2)$ does not fork over M_0

(b) For some (κ_1^+, κ_1)-convergent $\mathbf{I} \subseteq {}^\alpha(M_0)$ of cardinality $> \kappa_2$ we have $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(\bar a, M_1, M_2) = \mathrm{Av}_{<\kappa_1}(\mathbf{I}, M_1)$ hence this type does not split over $\bigcup \mathbf{I}'$ for any $\mathbf{I}' \subseteq \mathbf{I}$ of cardinality $> \kappa_1$.

(c) for every $N \le_{\mathfrak{K}} M_0$ of cardinality $\le \kappa_2$, if $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(\bar a, M_0, M_2)$ does not split over N then the type $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(\bar a, M_1, M_2)$ does not split over N.

Remark 4.7. 1) See verification of axiom (E)(c) in the proof of Theorem 4.10.

2) Note that have we used $\beth_7(\kappa_1)^+$ instead of κ_1 in 4.5, 4.6: the difference would be small.

3) We could in clause (c) of 4.6 use "for some $N \le_{\mathfrak{K}} M_0$ of cardinality $< \kappa_1$, $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]} \ldots$" The proof is the same.

4) We can allow [something] below $M_0 \le_{\mathfrak{K}} M_1$ if $M_0 \in K_{\ge \kappa_2}$.

Proof. $\underline{(a) \Rightarrow (b)}$

Let $\theta_0, \theta_1, \theta_2$ be as in Definition 4.5. By Definition 4.5 there is $N \le_{\mathfrak{K}} M_0$ of cardinality $\le \theta_2$ such that

$(*)_1$ the type $\mathrm{tp}_{\mathbb{L}_{\infty,\theta_1}[\mathfrak{K}]}(\bar a, M_1, M_2)$ does not split over N.

By Claim 1.27(1) there is a (κ_1^+, κ_1)-convergent set $\mathbf{I} \subseteq {}^\alpha(M_0)$ of cardinality κ_2^+ (convergence in M_0, of course) such that $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(\bar a, M_0, M_2) = \mathrm{Av}_{<\kappa_1}(\mathbf{I}, M_0)$. So as $M_0 \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} M_1 \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} M_2$, by Claim 4.4(2), clearly $\mathbf{I}$ is (κ_1^+, κ_1)-convergent also in M_1 and in M_2, hence $\mathrm{Av}_{<\kappa_1}(\mathbf{I}, M_1)$ is well defined. Hence, by Claims 1.23(2), 1.21(3) the type $\mathrm{Av}_{<\kappa_1}(\mathbf{I}, M_1)$ does not split over $\bigcup \mathbf{I}$ but $\theta_2 \le \kappa_2$ and $\bigcup \mathbf{I} \subseteq \bigcup \mathbf{I} \cup N$ hence

$(*)_2$ $\mathrm{Av}_{<\theta_1}(\mathbf{I}, M_1)$ does not split over $\bigcup \mathbf{I} \cup N$.

But also

$(*)_3$ $\mathrm{tp}_{\mathbb{L}_{\infty,\theta_1}[\mathfrak{K}]}(\bar a, M_1, M_2)$ does not split over N (by the choice of N) hence over $\bigcup \mathbf{I} \cup N$.

As $M_0 \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} M_1$ and $|\bigcup \mathbf{I} \cup N| < \lambda$ and $\mathrm{tp}_{\mathbb{L}_{\infty,\theta_1}[\mathfrak{K}]}(\bar a, M_0, M_2) = \mathrm{Av}_{<\theta_1}(\mathbf{I}, M_0)$ clearly, by $(*)_2 + (*)_3$ we have $\mathrm{tp}_{\mathbb{L}_{\infty,\theta_1}[\mathfrak{K}]}(\bar a, M_1, M_2) = \mathrm{Av}_{<\theta_1}(\mathbf{I}, M_1)$. Now there is a pair $(M_2', \bar a')$ satisfying that $M_1 \le_{\mathfrak{K}} M_2' \in K^*_\lambda$ and $\bar a' \in {}^\alpha(M_2')$ such that $\mathrm{tp}_{\mathbb{L}_{\infty,\theta_1}[\mathfrak{K}]}(\bar a', M_1, M_2') = \mathrm{Av}_{<\theta_1}(\mathbf{I}, M_1)$ hence by the previous sentence

$$\mathrm{tp}_{\mathbb{L}_{\infty,\theta_1}[\mathfrak{K}]}(\bar a', M_1, M_2') = \mathrm{tp}_{\mathbb{L}_{\infty,\theta_1}[\mathfrak{K}]}(\bar a, M_1, M_2).$$

Now by 4.4(1) and then 4.4(2A) it follows that $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(\bar a, M_1, M_0) = \mathrm{Av}_{<\kappa_1}(\mathbf{I}, M_1)$ as required.

$\underline{(b) \Rightarrow (c)}$

Let $\mathbf{I}$ be as in clause (b), so $\mathbf{I}$ is (κ_1^+, κ_1)-convergent in M_0 and is of cardinality $> \kappa_1$. We know that $M_0 \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} M_1$, so by the previous sentence, $\mathbf{I}$ is (κ_1^+, κ_1)-convergent in M_1. To prove clause (c), assume that $N \leq_{\mathfrak{K}} M_0$ is of cardinality κ_2 and $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(\bar{a}, M_0, M_2)$ does not split over N. Hence

$$\mathrm{Av}_{<\kappa_1}(\mathbf{I}, M_0) = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(\bar{a}, M_0, M_2)$$

does not split over N. Again, as $M_0 \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} M_1$, we can deduce that $\mathrm{Av}_{<\kappa_1}(\mathbf{I}, M_1)$ does not split over N but by the choice of $\mathbf{I}$ it is equal to $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$, so we are done.

<u>$(c) \Rightarrow (a)$</u>

By Claim 1.24 there is $B \subseteq M_0$ of cardinality $\leq \kappa_2$ such that $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(\bar{a}, M_0, M_2)$ does not split over B.

As we can increase B as long as we preserve "of cardinality $\leq \kappa_2$", without loss of generality $B = |N|$ where $N \leq_{\mathfrak{K}} M_0$. So the antecedent of clause (c) holds, but we are assuming clause (c) so the conclusion of clause (c) holds, that is $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$ does not split over N.

Also by 1.27(1) there is $\mathbf{I}_1 \subseteq {}^{\alpha}(M_0)$ of cardinality κ_2^+ which is (κ_1^+, κ_1)-convergent and $\mathrm{Av}_{<\kappa_1}(\mathbf{I}_1, M_0) = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(\bar{a}, M_0, M_1)$. Clearly $\kappa_1 \geq \theta_1$ hence $\kappa_2 = (\kappa_2)^{\theta_1}$. Now as K_λ^* is categorical clearly $M_0 \cong \mathrm{EM}_{\tau(\mathfrak{K})}(\lambda, \Phi)$ hence applying 1.25(4) we can find $\mathbf{I}_2 \subseteq \mathbf{I}_1$ of cardinality κ_2^+ which is (θ_1^+, θ_1)-convergent. As above $M_0 \prec_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]} M_1$ so we deduce that $\mathbf{I}_2$ is (θ_1^+, θ_1)-convergent and (κ_1^+, κ_1)-convergent also in M_1.

As above we have $M_0 \prec_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]} M_1$ by 1.19(1) hence $\mathrm{Av}_{<\kappa_1}(\mathbf{I}_2, M_1)$ is well defined and does not split over N hence is equal to $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$. This implies that $\mathrm{Av}_{<\theta_1}(\mathbf{I}_2, M_1) = \mathrm{tp}_{\mathbb{L}_{\infty,\theta_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$.

Now choose $\mathbf{I}_3 \subseteq \mathbf{I}_2 \subseteq M_0$ of cardinality θ_2 and $N_3 \leq_{\mathfrak{K}} M_0$ of cardinality θ_2 such that $\mathbf{I}_3 \subseteq {}^{\alpha}(N_3)$. Now by 1.23(2) we know that $\mathrm{tp}_{\mathbb{L}_{\infty,\theta_1}[\mathfrak{K}]}(\bar{a}, M_1, M_2)$ does not split over $\mathbf{I}_3$ hence it does not split over N_3, so N_3 witnesses clause (a). $\square_{4.6}$

Definition 4.8. We define a pre-frame $\mathfrak{s}_\lambda = (\mathfrak{K}_{\mathfrak{s}_\lambda}, \bigcup_{\mathfrak{s}_\lambda}, \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}_\lambda})$ as follows:

(a) $\mathfrak{K}_{\mathfrak{s}_\lambda} = \mathfrak{K}_\lambda^*$

(b) $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}_\lambda}$ is defined by $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s},\lambda}(M) := \{\mathbf{tp}_{\mathfrak{K}_\lambda^*}(a, M, N) : M \leq_{\mathfrak{K}_\lambda^*} N,\ a \in N \setminus M\}$,

(c) $\bigcup_{\mathfrak{s}_\lambda} = \{(M_0, M_1, a, M_3) : M_0 \leq_{\mathfrak{K}_\lambda^*} M_1 \leq_{\mathfrak{K}_\lambda^*} M_2$ and $\mathbf{tp}_{\mathfrak{K}_\lambda^*}(a, M_1, M_3)$ does not fork over $M_0\}$ (see Definition 4.5).

Remark 4.9. 1) Recall $\leq_{\mathfrak{s}_\lambda} = \leq_{\mathfrak{K}} \restriction K_{\mathfrak{s}_\lambda} = \leq_{\mathfrak{K}_\lambda^*}$.

2) Concerning the proof of 4.10 below, we mention a variant which the reader may ignore. This variant, from weaker assumptions gets weaker conclusions. In detail, define the weak version (f)$^-$ of 4.1(2)(f); see Definition 1.37 and Claim 1.40(1).

(f)$^-$ if $\langle M_\alpha : \alpha \leq \delta\rangle$ is $\leq_{\mathfrak{K}}$-increasing continuous and

$$\alpha < \delta \Rightarrow M_{2\alpha+1} <^*_{\mathfrak{K}_\lambda^*} M_{2\alpha+2}$$

(e.g. $M_{2\alpha+2}$ is $\leq_{\mathfrak{K}_\lambda^*}$-universal over $M_{2\alpha+1}$) hence both are from K_λ^* <u>then</u> $M_\delta \in K_\lambda^*$.

Assuming only 4.1(1) + (f)$^-$ we do not know whether $\mathfrak{K}_\lambda^*$ is a λ-AEC but still $(K_\lambda^*, \leq_{\mathfrak{K}} \restriction K_\lambda^*, <^*_{\mathfrak{K}_\lambda^*})$, see Definition 1.37, is a so-called semi λ-AEC, see [She].

If clause (f) from 4.1(2) holds (i.e., $K_{\mathfrak{s}_\lambda}$ is closed under unions), we can omit "$<^*_{\mathfrak{s}_\lambda}$".

3) It will be less good but not a disaster if we have assumed below

$$\lambda = \sup(\mathrm{Cat}'_{\mathfrak{K}} \cap \lambda).$$

4) It will be better to have $\mathfrak{K}_{\mathfrak{s}_\lambda} = K_\lambda$; of courses, this follows from categoricity so by §3 is not unreasonable for conjecture 0.1.

5) But we can ask only for $M \in K_{\mathfrak{s}_\lambda}$ to be universal in $\mathfrak{K}_\lambda$,

6) We can ask that for every $\mu > \lambda$ large enough, for every $M \in K_\mu$, for a club of $N \in K_\lambda$ satisfying $N \leq_{\mathfrak{K}} M$, we have $N \in K_{\mathfrak{s}_\lambda}$.

Theorem 4.10. *(Assume 4.1(2)(g), hence (f)).*
$\mathfrak{s}_\lambda$ is a good λ-frame categorical in λ and is full.

Proof. We check the clauses in the definition [She09c, 1.1].

Clause (A):
By observation 4.2(3), [in the weak version using (f)$^-$ from 4.9(1)].

Clause (B):
Categoricity holds by 1.16 (or 4.2(1)) and this implies "there is a superlimit model", the non-maximality by $\leq_{\mathfrak{K}^*_\lambda}$ holds by the choice of Φ.

Clause (C):
Observation 4.2(2) guarantee amalgamation, categoricity (of $\mathfrak{K}^*_\lambda$ by 4.2(1)) implies the JEP and "no-maximal model" holds by clause (B).

Clause (D)(a),(b):
Obvious by the definition.

(D)(c) (density).
Assume $M <_{\mathfrak{K}^*_\lambda} N$, then there are $a \in N \setminus M$ and for any such a the type $\mathbf{tp}_{\mathfrak{K}^*_\lambda}(a, M, N)$ belongs to $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}_\lambda}(M)$. In fact

⊛ $\mathfrak{s}_\lambda$ is type-full

(D)(d) (bs-stability).
The demand means $M \in K^*_\lambda \Rightarrow |\mathcal{S}^1_{\mathfrak{K}^*_\lambda}(M)| \leq \lambda$.
This holds by 1.36(2) (and amalgamation).

(E)(a),(b). By the definition.

(E)(c) (local character)
This says that if $\langle M_i : i \leq \delta + 1\rangle$ is $\leq_{\mathfrak{s}_\lambda}$-increasing continuous and

$$p = \mathbf{tp}_{\mathfrak{s}_\lambda}(a, M_\delta, M_{\delta+1}) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}_\lambda}(M_\delta)$$

then for some $i < \delta$ the type p does not fork over M_i (for $\mathfrak{s}_\lambda$).
From now on (in the remainder of this proof) we use 4.6 freely and let (noting $\mathrm{cf}(\delta) < \lambda$ as λ is singular)

⊙ $\kappa_0 = \mathrm{LST}(\mathfrak{K}) + \mathrm{cf}(\delta)$, $\kappa_1 = \beth_{1,1}(\beth_2(\kappa_0))^+$, $\kappa_2 = \beth_2(\kappa_1)$.

Now by 4.6 there is a (κ_1^+, κ_1)-convergent $\mathbf{I} \subseteq M_\delta$ with

$$\mathrm{Av}_{<\kappa_1}(\mathbf{I}, M_\delta) = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(a, M_\delta, M_{\delta+1})$$

such that $\mathbf{I}$ is of cardinality $> \kappa_2$. For some $i(*) < \delta, |\mathbf{I} \cap M_{i(*)}| > \kappa_2$, so without loss of generality $\mathbf{I} \subseteq M_{i(*)}$, so by 4.6 we are done.

(E)(d) Transitivity of non-forking.
We are given $M_0 \leq_{\mathfrak{s}_\lambda} M_1 \leq_{\mathfrak{s}_\lambda} M_2 \leq_{\mathfrak{K}_\mathfrak{s}} M_3$ and $a \in M_3$ such that $\mathbf{tp}_{\mathfrak{s}_\lambda}(a, M_{\ell+1}, M_3)$ does not fork over M_ℓ for $\ell = 0, 1$. So for $\ell = 0, 1$ there is $\mathbf{I}_\ell \subseteq M_\ell$ which is (κ_1^+, κ_1)-convergent in $M_{\ell+1}$ of cardinality κ_2^+ such that $\mathrm{Av}_{<\kappa_1}(\mathbf{I}_\ell, M_{\ell+1}) =$

$\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(a, M_{\ell+1}, M_3)$. As $\mathrm{Av}_{<\kappa_1}(\mathbf{I}_0, M_1) = \mathrm{Av}_{<\kappa_1}(\mathbf{I}_1, M_1)$ (being both realized by a) because $M_1 \prec_{\mathbb{L}_{\infty,\lambda}[\mathfrak{K}]} M_2$ by 4.4(4) clearly we have

$$\mathrm{Av}_{<\kappa_1}(\mathbf{I}_0, M_2) = \mathrm{Av}_{<\kappa_1}(\mathbf{I}_1, M_2) = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(a, M_2, M_3)$$

all well defined. So $\mathbf{I}_0$ witnesses (by 4.6) that $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(a, M_2, M_3)$ does not fork over M_0, which means that $\mathbf{tp}_{\mathfrak{K}^*_\lambda}(a, M_2, M_3)$ does not fork over M_0 as required.

(E)(e) Uniqueness.
Recalling 4.4(1), the proof is similar to $(E)(d)$; the two witnesses are now in M_0.

(E)(f) Symmetry.
Toward contradiction, recalling [She09c, 1.16E] assume

$$M_0 \leq_{\mathfrak{K}^*_\lambda} M_1 \leq_{\mathfrak{K}^*_\lambda} M_2 \leq_{\mathfrak{K}^*_\lambda} M_3$$

and $a_\ell \in M_{\ell+1} \setminus M_\ell$ for $\ell = 0, 1, 2$ are such that $p_\ell = \mathbf{tp}_{\mathfrak{K}^*_\lambda}(a_\ell, M_\ell, M_{\ell+1})$ does not fork over M_0 for $\ell = 0, 1, 2$ and $\mathbf{tp}_{\mathfrak{K}^*_\lambda}(a_0, M_0, M_1) = \mathbf{tp}_{\mathfrak{K}^*_\lambda}(a_2, M_0, M_3)$ but $\mathbf{tp}_{\mathfrak{K}^*_\lambda}(\langle a_0, a_1\rangle, M_0, M_3) \neq \mathbf{tp}_{\mathfrak{K}^*_\lambda}(\langle a_2, a_1\rangle, M_0, M_3)$.

By 4.6 we can deal with $p_\ell = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(a_\ell, M_\ell, M_{\ell+1})$ for $\ell = 0, 1, 2$. For each $\ell \leq 2$, we can find a convergent $\mathbf{I}_\ell = \{a^\ell_\alpha : \alpha < \kappa_2^+\} \subseteq M_0$ which is (κ_1^+, κ_1)-convergent such that $\mathrm{Av}_{<\kappa_1}(\mathbf{I}_\ell, M_\ell) = p_\ell$.

So as $M_0 \prec_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]} M_k$ we deduce the set $\mathbf{I}_\ell$ is (κ_1^+, κ_1)-convergent in M_k for $\ell, k = 0, 1, 2$. Also, $\mathrm{Av}_{<\kappa_1}(\mathbf{I}_0, M_0) = \mathrm{Av}_{<\kappa_1}(\mathbf{I}_2, M_0)$ hence $\mathrm{Av}_{<\kappa_1}(\mathbf{I}_0, M_2) = \mathrm{Av}_{<\kappa_1}(\mathbf{I}_2, M_2)$ so without loss of generality $\mathbf{I}_0 = \mathbf{I}_2$.

Now use the non-order property to get symmetry.

(E)(g) Existence.
Assume $M \leq_{\mathfrak{s}_\lambda} N$ and $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}_\lambda}(M)$. So we can find a pair (M', a) such that $M \leq_{\mathfrak{s}_\lambda} M'$, $a \in M_1$, and $p = \mathbf{tp}_{\mathfrak{s}_\lambda}(a, M, M')$. By 1.27(1) there is a (κ_1^+, κ_1)-convergent $\mathbf{I} \subseteq M$ of cardinality κ_2^+ such that $\mathrm{Av}_{<\kappa_1}(M, \mathbf{I}) = \mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}[\mathfrak{K}]}(a, M, M')$. By 1.27(3) + 4.6 there is a pair (N', a') such that $N \leq_{\mathfrak{s}_\lambda} N'$, $a' \in N'$, and

$$\mathrm{tp}_{\mathbb{L}_{\infty,\kappa_1}}(a', N, N') = \mathrm{Av}_{<\kappa_1}(\mathbf{I}, N).$$

So by 4.6 the type $\mathbf{tp}_{\mathfrak{s}_\lambda}(a', N, N')$ is easily $\in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}_\lambda}(N)$, does not fork over N, and extends p, as required.

(E)(h) Continuity.
Follows by [She09c, 1.16A]. Alternatively, assume $\langle M_i : i \leq \delta + 1\rangle$ is $\leq_{\mathfrak{s}_\lambda}$-increasing continuous, $a \in M_{\delta+1} \setminus M_\delta$, and $\mathbf{tp}_{\mathfrak{s}_\lambda}(a, M_i, M_{\delta+1})$ does not fork over M_0 for $i < \delta$. So there is a convergent $\mathbf{I}_i \subseteq M_0$ such that

$$i < \delta \Rightarrow \mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(a, M_i, M_{\delta+1}) = \mathrm{Av}_\kappa(\mathbf{I}, M_i).$$

As above, without loss of generality $\mathbf{I}_i = \mathbf{I}_0$. We can find a convergent $\mathbf{I} \subseteq M_\delta$ of cardinality $> \mathrm{cf}(\delta) + \kappa$ (recall $\mathrm{cf}(\delta) < \lambda$!) such that $\mathrm{tp}_{\mathbb{L}_{\infty,\kappa}[\mathfrak{K}]}(a, M_0, M_{\delta+1}) = \mathrm{Av}_\kappa(\mathbf{I}, M_\delta)$. So for some $i(*) < \delta$ we have $|\mathbf{I} \cap M_{i(*)}| > \kappa$, so without loss of generality (by equivalence) $\mathbf{I} \subseteq M_{i(*)}$. We finish as in (E)(f).

Axiom (E)(i):
Follows by [She09c, 1.15]. $\square_{4.10}$

Exercise: Replace $\mathrm{Av}_{<\kappa_1}(\mathbf{I}, M)$ above by $\bigcup\{\mathrm{Av}_{\beth_\zeta(\kappa_0)}(\mathbf{I}, M) : \zeta < (2^{\kappa_0})^+\}$.

§ 5. Homogeneous enough linear orders

Claim 5.1. *Assume $\mu^+ = \theta_1 = \mathrm{cf}(\theta_1) < \theta_2 = \mathrm{cf}(\theta_2) < \lambda$.*

1) Then there is a linear order I of cardinality λ such that the following equivalence relation $\mathcal{E} = \mathcal{E}^{\mathrm{aut}}_{I,\mu}$ on ${}^{\mu}I$ has $\le 2^\mu$ equivalence classes, where $\eta_1 \;\mathcal{E}\; \eta_2$ iff there is an automorphism of I mapping η_1 to η_2.

2) Moreover, if $I' \subseteq I$ has cardinality $< \theta_2$, and $n < \omega$ then the following equivalence relation $\mathcal{E}$ on ${}^{n}I$ has $\le \mu + |I'|$ equivalence classes:

- *$\bar{s} \;\mathcal{E}\; \bar{t}$ iff there is an automorphism h of I over I' mapping $\bar{s}$ to $\bar{t}$.*

3) Moreover, there is Ψ proper for $K^{\mathrm{lin}}_{\tau_2^*}$ (i.e. $\Psi \in \Upsilon^{\mathrm{lin}}_{\aleph_0}[2]$; see Definitions 0.11(5), 0.14(9)) with $\tau(\Psi)$ countable such that $I = \mathrm{EM}_{\{<\}}(I^{\mathrm{lin}}_{\theta_2,\lambda\times\theta_2}, \Phi)$ where $I^{\mathrm{lin}}_{\theta_2,\zeta} = (\zeta, <, P_0, P_1)$, $P_\ell = \{\alpha < \zeta : \text{“}\mathrm{cf}(\alpha) < \theta_2\text{”} \equiv \text{“}\ell = 0\text{”}\}$.

4) If $I^*_0 \subseteq I$ has cardinality $< \theta_2$ then for some $I^*_1 \subseteq I$ of cardinality $\le \mu^+ + |I^*_0|$, for every $J \subseteq I$ of cardinality $\le \mu$, there is an automorphism of I over I^*_0 mapping J into I^*_1.

5) If $I^*_1, I^*_2 \subseteq I^{\mathrm{lin}}_{\mu,\lambda\times\mu^+}$ have cardinality $\le \mu$ and h is an isomorphism from I^*_1 onto I^*_2 then there is an automorphism $\hat{h}$ of the linear order $I = \mathrm{EM}_{\{<\}}(I^{\mathrm{lin}}_{\theta,\lambda}, \Psi)$ extending the natural isomorphism $\check{h}$ from $\mathrm{EM}_{\{<\}}(I^*_1, \Psi)$ onto $\mathrm{EM}_{\{<\}}(I^*_2, \Psi)$.

Remark 5.2. 1) Of course, if $\lambda = \lambda^{<\theta_2}$ and I is a dense linear order of cardinality λ which is θ-strongly saturated (hence θ-homogeneous) then the demand in 5.1(1) is satisfied (and in part (2) of 5.1 the number of $\mathcal{E}$-equivalence classes is $\le 2^\chi$ for every $\chi \in [\aleph_0, \theta_2)$). Also, if $\lambda = \sum\limits_{i<\delta} \lambda_i$, $\delta < \theta_2$, and $i < \delta \Rightarrow \lambda_i^{<\theta_2} = \lambda$ then we have such an order.

2) Laver [Lav71, §2] deals with related linear orders, but for his aims I_1, I_2 are equivalent if each is embeddable into the other; see more in [Shear, AP,§2]. For a cardinal ∂ and linear order I let

$$\Theta_{I,\partial} = \big\{\mathrm{cf}(J) : \text{for some } <_I\text{-decreasing sequence } \langle t_i : i < \partial\rangle \\ \text{we have } J = I \restriction \{t \in I : t <_I t_i \text{ for every } i < \partial\}\big\}.$$

So if $\partial \le \mu$ then $({}^{\mu}I)/E^{\mathrm{aut}}_{I,\mu}$ has $\ge |\Theta_{I,\partial}|$. So we have to be careful to make $\Theta_{I,\partial}$ small. We chose a very concrete construction, which leads quickly to defining I and the checking is straight. We thought it would be easy, but *a posteriori* the checking is lengthy; [Shear, AP,§2] is an antithetical approach.

3) We can replace $\theta_1 = \mu^+$ by $\theta_1 = \mathrm{cf}(\theta_1) > \aleph_0$ and “of cardinality $\le \mu$” by “of cardinality $< \theta_1$”.

4) In 2.7(1), 2.11(2) we use parts (1),(1)+(4) respectively. Also, we use 5.1 in the proof of 7.9.

5) The case $2^\mu \ge \lambda$ in 5.1(1) says nothing; in fact, if $2^\mu \ge \lambda$ then $2^\mu = \lambda^\mu = ({}^{\mu}M)/\mathcal{E}^{\mathrm{aut}}_{I,\mu}$ for any model M of cardinality ≥ 2 and $\le 2^\mu$, for any vocabulary τ_M.

6) Claim 5.1(1),(2) holds also if we replace μ by $\chi \in [\mu, \theta_2)$.

[We got an fifteen-page proof coming up. Of these five distinct claims, (3) and (5) are one-liners that don't reference anything else in 5.1, (2) is a one-page addendum to (1), and (4) is a half-page that references (1) four times.]

[(1) is 'organized' by *five* categories of bullets ($\circledast, (*), \boxtimes, \boxdot$, and $\odot$), each with their own independent numbering system. $\boxtimes$ seems to be reserved for high-level lemmas, but other than that I don't see any rhyme or reason regarding how or why these guys are used.]

[The longest multi-case proofs need to be moved to an appendix, as proofs to independent, labeled lemmas that can be cited with \ref{}s. $\boxtimes_3$ is 2.5 pages, and $\boxtimes_4$, $(*)_7$ are three full pages each.]

Proof. **5.1(1)**

Fix an ordinal ζ, $\lambda \le \zeta < \lambda^+$ such that $\mathrm{cf}(\zeta) = \theta_2$: e.g., $\zeta = \lambda \times \theta_2$. (Almost always, $\mathrm{cf}(\zeta) \ge \theta_2$ will suffice.)

Let I_1 be the following linear order. Its set of elements is

$$\big\{(\ell,\alpha) : \ell \in \{-2,-1,1,2\},\ \alpha < \zeta + \omega\big\}$$

ordered by $(\ell_1,\alpha_1) <_{I_1} (\ell_2,\alpha_2)$ iff $\ell_1 < \ell_2$ or $\ell_1 = \ell_2 \in \{-1,2\} \wedge \alpha_1 < \alpha_2$ or $\ell_1 = \ell_2 \in \{-2,1\} \wedge \alpha_1 > \alpha_2$.

For $t \in I_1$ let $t = (\ell^t, \alpha^t)$.

Let I^*_2 be the set $\{\eta : \eta$ is a finite sequence of members of $I_1\}$ ordered by

$$\eta_1 <_{I_2} \eta_2 \ \underline{\text{iff}}\ (\exists n)\big[n < \ell g(\eta_1) \wedge n < \ell g(\eta_2) \wedge \eta_1 \restriction n = \eta_2 \restriction n \text{ and } \eta_1(n) <_{I_1} \eta_2(n)\big]$$

or $\eta_1 \triangleleft \eta_2 \wedge \ell^{\eta_2(\ell g(\eta_1))} \in \{1,2\}$ or $\eta_2 \triangleleft \eta_1 \wedge \ell^{\eta_1(\ell g(\eta_2))} \in \{-2,-1\}$.

Let I_2 be I^*_2 restricted to the set of $\eta \in I^*_2$ satisfying $\circledast$ where

$\circledast$ For no $n < \omega$ do we have:
- (a) $\ell g(\eta) > n+1$ **[Read literally, this is identical to '$\ell g(\eta) = 0$,' correct?]**
- (b) $\alpha^{\eta(n)}$ is a limit ordinal of cofinality $\ge \theta_1$
- (c) $\alpha^{\eta(n+1)} \ge \zeta$
- (d) $\ell^{\eta(n)} \in \{-1,2\}$, $\ell^{\eta(n+1)} = -2$ $\underline{\text{or}}$ $\ell^{\eta(n)} \in \{-2,1\}$, $\ell^{\eta(n+1)} = 2$.

Let M_0 be the following ordered field:

$(*)_1$ (a) M_0, as a field, is $\mathbb{Q}(a_t : t \in I_2)$, the field of rational functions with $\{a_t : t \in I_2\}$ algebraically independent.
(b) The order of M_0 is determined by
  - (α) If $t \in I_2$, $n < \omega$ then $M_0 \models$ "$n < a_t$".
  - (β) If $s <_{I_2} t$ and $n < \omega$ then $M_0 \models$ "$(a_s)^n < a_t$".

(c) let M be the real[9] (algebraic) closure of M_0 (i.e. the elements algebraic over M_0 in the closure by adding elements realizing any Dedekind cut of M_0).

Now we shall prove that I, which is M as a linear order, is as requested.

$\boxtimes_1$ each of I_1, I^*_2, and I_2 is anti-isomorphic to itself.

[Why? Let $g : I_1 \to I_1$ be $g(t) = (-\ell^t, \alpha^t)$. Clearly it is an anti-isomorphism of I_1. Let $\hat g : I^*_2 \to I^*_2$ be defined by $\hat g(\eta) = \big\langle g(\eta(m)) : m < \ell g(\eta)\big\rangle$; it is an anti-isomorphism of I^*_2. Lastly, $\hat g$ maps I_2 onto itself: in particular by the character of clause (d) of $\circledast$, i.e. the two cases are interchanged by $\hat g$.]

$\boxtimes_2$ (a) I_1, I^*_2, I_2 have cofinality $\aleph_0$.
(b) if $t \in I_2$ then $I_{2,<t} := I_2 \restriction \{s : s <_{I_2} t\}$ has cofinality $\aleph_0$.

[Why? For clause (a), $\{(2, \lambda+n) : n < \omega\}$ is a cofinal subset of I_1 of order type ω and $\{\langle t\rangle : t \in I_1\}$ is a cofinal subset of I^*_2 (and of I_2) of order type the same as I_1. For clause (b) for $\eta \in I_2$ the set $\{\eta\hat{\ }\langle(-1, \lambda+n)\rangle : n < \omega\}$ is a cofinal subset of $I_{2,<\eta}$ of order type ω by $\boxdot$ below.]

Now

$\boxdot$ If η satisfies $\circledast$ and $\ell \in \{1,-1\}$ then also $\eta\hat{\ }\langle(\ell,\alpha)\rangle$ satisfies $\circledast$ for any $\alpha < \lambda + \omega$.

[9] In fact, we could just use M_0.

[Why? By clause (d) of $\circledast$ as the only value of n there which is not obvious is $n = \ell g(\eta) - 1$, but to be problematic we should have $\ell^{(\eta\hat{\ }\langle(\ell,\alpha)\rangle)(n+1)} \in \{-2, 2\}$ whereas $\ell = -1$.]

$\boxtimes_3$ If $\partial = \mathrm{cf}(\partial)$ (so ∂ is $0, 1$, or an infinite regular cardinal), $\bar\eta = \langle \eta_i : i < \partial \rangle$ is a $<_{I_2}$-decreasing sequence, and we let $J_{\bar\eta} = \{s \in I_2 : (\forall i < \partial)[s <_{I_2} \eta_i]\}$ <u>then</u> (clearly) exactly one of the following clauses applies:

(a) If $J_{\bar\eta} = \varnothing$ then $\partial = \aleph_0$.

(b) If $\mathrm{cf}(J_{\bar\eta}) = 1$ then $\partial = \aleph_0$.

(c) If $\mathrm{cf}(J_{\bar\eta}) = \aleph_0$ then $\partial < \theta_1$.

(d) If $\aleph_1 \le \mathrm{cf}(J_{\bar\eta}) < \theta_1$ then $\partial = \aleph_0$, and for some $\ell \in \{-1, 2\}$, $\nu \in I_2$, and ordinal $\delta < \zeta$ of cofinality $\mathrm{cf}(J_{\bar\eta})$, the set $\langle \nu\hat{\ }\langle(\ell, \alpha)\rangle : \alpha < \delta \rangle$ is an unbounded subset of $J_{\bar\eta}$.

(e) If $\theta_1 \le \mathrm{cf}(J_{\bar\eta})$ then $\partial \ge \theta_1$ and moreover $\partial = \theta_2 \vee \mathrm{cf}(J_{\bar\eta}) = \theta_2$.

[Why does $\boxtimes_3$ hold? The proof is split into cases, and finishing a case we can then assume it does not occur.

Clearly we can replace $\bar\eta$ by $\langle \eta_i : i \in u \rangle$ for any unbounded subset u of ∂, and modify it further to $\langle \nu_i : i \in u \rangle$ provided $\eta_{\zeta_{2i+1}} \le_{I_2} \nu_i \le_{I_2} \eta_{\zeta_{2i}}$ and $\langle \zeta_i : i < \partial \rangle$ is an increasing sequence of ordinals $< \partial$. We shall use this freely.

Case 0: $\partial = 0$ or $\partial = 1$.

By $\boxtimes_2$ clearly clause (c) of $\boxtimes_3$ holds.

Case 1: $\partial = \aleph_0$ and there is $\nu \in {}^\omega(I_1)$ such that $(\forall n < \omega)(\exists i < \partial)[\eta_i \restriction n \lhd \nu]$.

Let $n_i = \ell g(\eta_i \cap \nu)$. It is impossible that $\{i : n_i = k\}$ is infinite for any k, so without loss of generality $\langle n_i : i < \omega \rangle$ is an increasing sequence and $n_0 > 0$.

For every $i < \omega$ we have $\nu \restriction (n_i + 1) \trianglelefteq \eta_{i+1}$ and $\eta_{i+1} <_{I_2} \eta_i$, so by the definition of $<_{I_2}$ also $\nu \restriction (n_i + 1) <_{I_2} \eta_i$. We choose $\beta_{n_i} < \zeta + \omega$ so that $(-2, \beta_{n_i}) <_{I_1} \nu(n_i)$, hence letting $\rho_i = \nu \restriction n_i\hat{\ }\langle(-2, \beta_{n_i})\rangle$ we have $\rho_i \in I_2$. This can be done, e.g. because we can choose β_{n_i} such that $\beta_{n_i} = \alpha^{\nu(n_i)} + 1$ if $\ell^{\nu(n_i)} = -2$ and $\beta_{n_i} = 0$ otherwise.

For every $i, j < \omega$ we have $\rho_i <_{I_2} \rho_{i+1} <_{I_2} \eta_{i+1} <_{I_2} \eta_i$, so if $i \le j$ then $\rho_i <_{I_2} \rho_j <_{I_2} < \eta_j$, and if $i > j$ then $\rho_i <_{I_2} \eta_i <_{I_2} \eta_j$, so $\rho_i \in J_{\bar\eta}$.

Now $\langle \rho_i : i < \omega \rangle$ is $<_{I_2}$-increasing; also, it is cofinal in $J_{\bar\eta}$, for if $\rho \in J_{\bar\eta}$ let $n = \ell g(\rho \cap \nu)$, so for $i < \omega$ such that $n_i \le n < n_{i+1}$ we have $\rho <_{I_2} \eta_{i+1}$ so $\rho(n) <_{I_1} \eta_{i+1}(n) = \rho_{i+1}(n)$ and as $\rho \restriction n = \nu \restriction n = \rho_{i+1} \restriction n$ we have $\rho <_{I_2} \rho_{i+1}$.

As $\langle \rho_i : i < \omega \rangle$ is of order type ω, clearly $\mathrm{cf}(J_{\bar\eta}) = \aleph_0 = \partial$, hence clause (c) of $\boxtimes_3$ applies and we are done.

So from now on assume that Case 1 fails.

As $\ell g(\eta_i) < \omega$ and Case 1 fails, without loss of generality, for some n we have $i < \partial \Rightarrow \ell g(\eta_i) = n$. Similarly, without loss of generality for some m and $\nu \in I_2$ we have $i < \partial \Rightarrow \eta_i \restriction m = \nu$ and $\langle \eta_i(m) : i < \partial \rangle$ with no repetitions so $m < n$. Without loss of generality $i < \partial \Rightarrow \ell^{\eta_i(m)} = \ell^*$ and so $\langle \alpha^{\eta_i(m)} : i < \partial \rangle$ has no repetitions; without loss of generality it is monotonic as $\partial \ge \aleph_0$ is an increasing sequence of ordinals. As $\bar\eta$ is $<_{I_2}$-decreasing, necessarily $\ell^* \in \{-2, 1\}$. Let $\delta = \bigcup\{\alpha^{\eta_i(m)} : i < \partial\}$, so clearly $\mathrm{cf}(\delta) = \partial$ and δ is a limit ordinal $\le \zeta + \omega$. Now those ℓ^*, δ will be used until the end of the proof of $\boxtimes_3$. For the rest of the proof we are assuming

$\odot$ (a) $i < \partial \Rightarrow \eta_i \restriction m = \nu$

(b) $\langle \eta_i(m) : i < \partial \rangle$ is (strictly) increasing with limit δ.

(c) $\ell^{\eta_i(m)} = \ell^* \in \{-2, 1\}$

(d) $\mathrm{cf}(\delta) = \partial$ and $\delta \le \zeta + \omega$.

Also note by $\circledast$ that $\nu\hat{}\ \langle(\ell^*,\delta)\rangle \notin I_2 \Rightarrow \delta \in \{\zeta+\omega,\zeta\}$ and if $\delta=\zeta \wedge \nu\hat{}\langle(\ell^*,\delta)\rangle \notin I_2$ then $\ell g(\nu)>0$ and the ordinal $\alpha^{\nu(\ell g(\nu)-1)}$ is limit of cofinality $\geq \theta_1$ (and more).

Case 2: $J_{\bar\eta}=\varnothing$.

Clearly $m=0 \wedge \ell^*=-2 \wedge \delta=\zeta+\omega$ hence $\partial=\aleph_0$ so clause (a) of $\boxtimes_3$ holds.

Case 3: $\ell^*=1$ and $\nu\hat{}\langle(\ell^*,\delta)\rangle \notin I_2$.

As $\ell^*=1$, clearly we cannot have $\delta=\zeta$ by clause (d) of $\circledast$, so $\delta=\zeta+\omega$ and recalling $\partial=\mathrm{cf}(\delta)$ we have $\partial=\aleph_0$. Now clearly $J_{\bar\eta}$ has a last element, ν, so case (b) of $\boxtimes_3$ applies.

Case 4: $\ell^*=-2$, $\partial=\aleph_0$ and $\nu\hat{}\langle(\ell^*,\delta)\rangle \notin I_2$.

Again $\delta=\zeta+\omega$ as $\aleph_0=\partial=\mathrm{cf}(\delta)$ and $\mathrm{cf}(\zeta)=\theta_2>\mu\geq\aleph_0$ making $\delta=\zeta$ impossible; now $\ell g(\nu)>0$ (as we have discarded the case $J_{\bar\eta}=\varnothing$, i.e. Case 2); and let $k=\ell g(\nu)-1$. Now we prove case 4 by splitting to several subcases.

Subcase 4A: $\ell^{\nu(k)} \in \{-2,1\}$.

Let $\nu_1=(\nu\restriction k)\hat{}\langle(\ell^{\nu(k)},\alpha^{\nu(k)}+1)\rangle$. Note that $\nu_1\in I_2$ as $\nu \in I_2 \wedge (\alpha^{\nu(k)}<\zeta \equiv \alpha^{\nu(k)}+1<\zeta)$ and (as $\ell^{\nu(k)}\in\{-2,1\}$) clearly $\{\rho:\nu_1 \trianglelefteq \rho \in I_2\}$ is a cofinal subset of $J_{\bar\eta}$ even an end segment. Now for $n<\omega$ we have $\nu_1\hat{}\langle(2,\zeta+n)\rangle \in I^*_2$ and it satisfies $\circledast$. (Why? As $\nu_1\in I_2$, only $n=k$ may be problematic, but $\alpha^{\nu(k)}+1=\alpha^{\nu_1(k)}$ here stands for $\alpha^{\eta(n)}$ there hence clause (b) of $\circledast$ does not apply), so by the definition of I_2, clearly $\{\nu_1\hat{}\langle(2,\zeta+n)\rangle : n<\omega\}$ is $\subseteq I_2$ and is a cofinal subset of $J_{\bar\eta}$ so $\partial=\aleph_0=\mathrm{cf}(J_{\bar\eta})$ and clause (c) of $\boxtimes_3$ holds.

Subcase 4B: $\ell^{\nu(k)}\in\{-1,2\}$ and $\alpha^{\nu(k)}$ is a successor ordinal.

Let $\nu_1=(\nu\restriction k)\hat{}\langle(\ell^{\nu(k)},\alpha^{\nu(k)}-1)\rangle$, of course $\nu_1\in I^*_2$ and as $\nu\in I_2$ clearly $\nu_1\in I_2$ so the set $\{\rho:\nu_1\trianglelefteq\rho\in I_2\}$ is an end segment of $J_{\bar\eta}$ and has cofinality $\aleph_0$ because $n<\omega \Rightarrow \nu_1\hat{}\langle(2,\zeta+n)\rangle \in I_2$. (Why? It $\in I^*_2$ and as $\nu_1\in I_2$ checking $\circledast$ only $n=k$ may be problematic, but $(\ell^{\nu(k)},2)$ here stand for $(\ell^{\eta(n)},\ell^{\eta(n+1)})$ there but presently $\ell^{\nu(k)}\in\{-1,2\}$ contradicting clause (d) of $\circledast$). So clause (c) of $\boxtimes_3$.

Subcase 4C: $\ell^{\nu(k)}\in\{-1,2\}$ and $\alpha^{\nu(k)}=0$.

Then let $\nu_1=(\nu\restriction k)\hat{}\langle(\ell^{\nu(k)}-1,0)\rangle$. Now $\nu_1\in I_2$ as $\nu\restriction k\in I_2$ and for $n=k-1$ clause (c) of $\circledast$ fails and $\nu_1\hat{}\langle(2,\zeta+n)\rangle\in I_2$ because of $\nu_1\in I_2$ and for $n=k$ the failure of clause (b) of $\circledast$ so continue as in Subcase 4B above.

Lastly,

Subcase 4D: $\ell^{\nu(k)}\in\{-1,2\}$ and $\alpha^{\nu(k)}$ is a limit ordinal.

Then $\{(\nu\restriction k)\hat{}\langle(\ell^{\nu(k)},\alpha)\rangle : \alpha<\alpha^{\nu(k)}\}$ is $\subseteq I_2$ and is an unbounded subset of $J_{\bar\eta}$ hence $\mathrm{cf}(J_{\bar\eta})=\mathrm{cf}(\alpha^{\nu(k)})$. If $\mathrm{cf}(\alpha^{\nu(k)})=\aleph_0$, then clause (c) in $\boxtimes_3$ holds, and if $\mathrm{cf}(\alpha^{\nu(k)})\in[\aleph_1,\theta_1)$ then necessarily $\alpha^{\nu(k)}\neq\zeta$ so being a limit ordinal $<\zeta+\omega$ clearly $\alpha^{\nu(k)}<\zeta$ so clause (d) from $\boxtimes_3$ holds. To finish this subcase note that $\mathrm{cf}(\alpha^{\nu(k)})\geq\theta_1$ is impossible.

[Why "impossible"? Clearly for large enough $i<\partial$ we have $\eta_i(m)\geq\zeta$ (because $\delta=\zeta+\omega$ as said in the beginning of the case) and recall $\nu\triangleleft\eta_i\in I_2$. We now show that clauses (a)-(d) of $\circledast$ hold with η_i,k here standing for η,n there. For clause (a) recall $\ell g(\eta_i)\geq\ell g(\nu)+1$ and $m=\ell g(\nu)=k+1$. Now $\ell^{\eta_i(k+1)}=\ell^{\eta_i(m)}=\ell^*=-2$ as $\ell^*=-2$ is part of the case, $\ell^{\eta_i(k)}=\ell^{\nu(k)}\in\{-1,2\}$ in this subcase, so clause (d) of $\circledast$ holds. Also $\alpha^{\eta_i(k+1)}=\alpha^{\eta_i(m)}\geq\zeta$ as said above so clause (c) of $\circledast$ holds and $\mathrm{cf}(\alpha^{\eta_i(k)})=\mathrm{cf}(\alpha^{\nu(k)})\geq\theta_1$ (as we are trying to prove "impossible"), so clause (b) of $\circledast$ holds. Together we have proved (a)-(d) of $\circledast$. But $\eta_i\in I_2$, contradiction.]

Now subcases 4A,4B,4C,4D cover all the possibilities, hence we are done with case 4.

Case 5: $\ell^* = -2$, $\partial > \aleph_0$, and $\nu\hat{}\langle(\ell^*,\delta)\rangle \notin I_2$.

Recalling δ is the limit of the increasing sequence $\langle\alpha^{\eta_i(m)} : i < \partial\rangle$ hence $\mathrm{cf}(\delta) = \partial > \aleph_0$ and $\nu\hat{}\langle(-2,\delta)\rangle \notin I_2$, necessarily $\delta = \zeta$ so $\partial = \theta_2$. As $\nu\hat{}\langle(-2,\delta)\rangle \notin I_2$ necessarily clauses (a) - (d) of $\circledast$ hold for some n and as $\nu \in I_2$, clearly $n = \ell g(\nu)-1$ (see clause (a) of $\circledast$) so we have $\ell g(\nu) > 0$, and letting $k = \ell g(\nu) - 1$, by clause (d) of $\circledast$ the $\ell^{\eta(n+1)}$ there stands for $\ell^* = -2$ here so we have $\ell^{\nu(k)} \in \{-1,2\}$ and by clause (b) of $\circledast$ we have $\mathrm{cf}(\alpha^{\nu(k)}) \geq \theta_1$. Hence $\{(\nu \restriction k)\hat{}\langle(\ell^{\nu(k)},\beta)\rangle : \beta < \alpha^{\nu(k)}\}$ is cofinal in $J_{\bar\eta}$ and its cofinality is $\mathrm{cf}(\alpha^{\nu(k)})$ as $(\nu \restriction k)\hat{}\langle(\ell^{\nu(k)},\beta)\rangle$ increase (by $\leq_{I_2}$) with β as $\ell^{\nu(k)} \in \{-1,2\}$. But $\mathrm{cf}(\alpha^{\nu(k)}) \geq \theta_1$ and $\partial = \theta_2$ (see first sentence of the present case), so clause (e) of $\boxtimes_3$ holds.

Case 6: $\nu\hat{}\langle(\ell^*,\delta)\rangle \in I_2$.

Subcase 6A: $\nu\hat{}\langle(\ell^*,\delta),(2,\zeta)\rangle \in I_2$.

Note that for $m = \ell g(\nu)$ and the pair $(\nu\hat{}\langle(\ell^*,\delta),(2,\zeta)\rangle, m)$ standing for (η,n) in $\circledast$, clauses (a),(c),(d) of $\circledast$ hold (recall $\ell^* \in \{-2,1\}$, see the discussion after case 1) so necessarily clause (b) of $\circledast$ fails hence $\mathrm{cf}(\delta) < \theta_1$ but $\partial = \mathrm{cf}(\delta)$ so $\partial < \theta_1$. Now as $\nu\hat{}\langle(\ell^*,\delta),(2,\zeta)\rangle \in I_2$ clearly if $\ell < \omega$, then $\nu\hat{}\langle(\ell^*,\delta),(2,\zeta+\ell)\rangle$ belongs to I_2 hence $\{\nu\hat{}\langle(\ell^*,\delta),(2,\zeta+\ell)\rangle : \ell < \omega\}$ is a cofinal subset of $J_{\bar\eta}$ by the choice of I_2 hence $\mathrm{cf}(J_{\bar\eta}) = \aleph_0$ so clause (c) of $\boxtimes_3$ applies.

Subcase 6B: $\nu\hat{}\langle(\ell^*,\delta),(2,\zeta)\rangle \notin I_2$.

As $\nu\hat{}\langle(\ell^*,\delta)\rangle \in I_2$, necessarily clauses (a)-(d) of $\circledast$ hold with $(\nu\hat{}\langle(\ell^*,\delta),(2,\zeta)\rangle, m)$ here standing for (η,n) there, recalling $m = \ell g(\nu)$ so by clause (b) of $\circledast$ we know that $\mathrm{cf}(\delta) \geq \theta_1$ but $\partial = \mathrm{cf}(\delta)$ hence $\partial \geq \theta_1$. Also $\{\nu\hat{}\langle(\ell^*,\delta),(2,\alpha)\rangle : \alpha < \zeta\}$ is a subset of I_2 and cofinal in $J_{\bar\eta}$ and is increasing with α so $\mathrm{cf}(J_{\bar\eta}) = \theta_2$ so clause (e) of $\boxtimes_3$ applies.

As the two subcases 6A,6B are complimentary case 6 is done.

Finishing the proof of $\boxtimes_3$:

It is easy to check that our cases cover all the possibilities (as after discarding cases 0,1, if not case (6) then $\nu\hat{}\langle(\ell^*,\delta)\rangle \notin I_2$, as not case (3), $\ell^* \neq 1$ but (see clause $\odot(c)$ before case 2), $\ell^* \in \{-2,1\}$ so necessarily $\ell^* = -2$, so case (4),(5) cover the rest). Together we have proved $\boxtimes_3$.]

$\boxtimes_4$ Recall $\aleph_0 \leq \mu < \theta_1 < \theta_2$; if $X \subseteq I_2$ with $|X| < \theta_2$ then we can find Y such that $X \subseteq Y \subseteq I_2$, $|Y| = \mu + |X|$, Y is unbounded in I_2 from below and from above, and for every $\nu \in I_2 \setminus Y$ the following linear orders have cofinality $\aleph_0$:
- (a) $J^2_{Y,\nu} := I_2 \restriction \big\{\eta \in I_2 \setminus Y : (\forall \rho \in Y)[\rho <_{I_2} \nu \equiv \rho <_{I_2} \eta]\big\}$
- (b) The inverse of $J^2_{Y,\nu}$.
- (c) $J^-_{Y,\nu} = I_2 \restriction \big\{\eta \in I_2 : (\forall \rho \in J^2_{Y,\nu})[\eta <_{I_2} \rho]\big\}$
- (d) The inverse of $J^+_{Y,\nu} := I_2 \restriction \big\{\eta \in I_2 : (\forall \rho \in J^2_{Y,\nu})[\rho <_{I_2} \eta]\big\}$.

[Why? Let $\mathcal{U} = \{\alpha^{\eta(\ell)} : \eta \in X \text{ and } \ell < \ell g(\eta)\}$.

We choose W_n by induction on $n < \omega$ such that

$\boxdot_1$
- (a) $\mathcal{U} \subseteq W_n \subseteq \zeta + \omega$
- (b) W_n has cardinality $\mu + |\mathcal{U}| = \mu + |X|$ and $m < n \Rightarrow W_m \subseteq W_n$.
- (c) $\mu \subseteq W_0$ and $\zeta + n \in W_0$ for $n < \omega$.
- (d) $\alpha \in W_n \Rightarrow \alpha + 1 \in W_{n+1}$
- (e) $\alpha + 1 \in W_n \Rightarrow \alpha \in W_{n+1}$
- (f) If $\delta \in W_n$ is a limit ordinal of cofinality $< \theta_1$ then $\delta = \sup(\delta \cap W_{n+1})$.
- (g) if $\delta \in W_n$ and $\mathrm{cf}(\delta) \geq \theta_1$ (or just $\mathrm{cf}(\delta) \leq \mu + |X|$) then $\sup(\delta \cap W_n) + 1 \in W_{n+1}$.

This is straight. Let $W = \bigcup\{W_n : n < \omega\}$, so

$\boxdot_2$ $\mathcal{U} \subseteq W$ and $|W| = \mu + |X|$ and W satisfies

(a) $W \subseteq \zeta + \omega$

(b) $|W| < \theta_2$

(c) $0 \in W$ and $\{\zeta + m : m < \omega\} \subseteq W$

(d) $\alpha \in W \Leftrightarrow \alpha + 1 \in W$

(e) If $\delta \in W$ and $\aleph_0 < \mathrm{cf}(\delta) \le \mu$ <u>then</u> $\delta = \sup(W \cap \delta)$

(f) If $\delta \in W$ and $\mathrm{cf}(\delta) \ge \theta_1$ or $\mathrm{cf}(\delta) = \aleph_0$ then $\mathrm{cf}(\mathrm{otp}(W \cap \delta)) = \aleph_0$.

Let $Y = \{\eta \in I_2 : \alpha^{\eta(\ell)} \in W$ for every $\ell < \ell g(\eta)\}$. Clearly $X \subseteq Y$ and $|Y| = \aleph_0 + |W| = \mu + |\mathcal{U}| < \theta_2$. It suffices to check that Y is as required in $\boxtimes_4$. From now on we shall use only the choice of Y and clauses (a)-(f) of $\boxdot_2$. By $\boxdot_2(c)$ and the choice of Y clearly Y is unbounded in I_2 from above and from below.

So let $\nu \in I_2 \setminus Y$; as $\nu \restriction 0 \in Y$ there is $n < \ell g(\nu)$ such that $\nu \restriction n \in Y$ and $\nu \restriction (n+1) \notin Y$, so $\alpha^{\nu(n)} < \zeta + \omega$ and $\alpha^{\nu(n)} \notin W$. But by clause (c) of $\boxdot_2$ we have $\{\zeta + m : m < \omega\} \subseteq W$ hence $\alpha^{\nu(n)} < \zeta$ and so $\alpha_1 := \min(W \setminus \alpha^{\nu(n)})$ is well defined and is found in the interval $(\alpha^{\nu(n)}, \zeta]$. As clearly $0 \in W$ and $\beta \in W \Leftrightarrow \beta + 1 \in W$ by the choice of W, obviously α_1 is a limit ordinal. By clause (e) of $\boxdot_2$ clearly α_1 is of cofinality $\aleph_0$ or $\ge \theta_1 = \mu^+$. So clearly

$$\alpha_0 := \sup(W \cap \alpha^{\nu(n)}) = \sup(W \cap \alpha_1) = \min\{\alpha : W \cap \alpha = W \cap \alpha^{\nu(n)}\}$$

is a limit ordinal $\le \alpha^{\nu(n)}$ and $\alpha_0 \notin W$ so $\mathrm{cf}(\alpha_0) \le |W| < \theta_2$. But by the assumption on W, (see clause (f) of $\boxdot_2$) we have $\mathrm{cf}(\alpha_0) = \aleph_0$. So $(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \alpha_0)\rangle \in J^2_{Y,\nu}$; moreover

$\boxdot_3$ $\rho \in J^2_{Y,\nu}$ <u>iff</u> $\rho \in I_2$ satisfies one of the following:

(a) $\bullet_1$ $\nu \restriction n = \rho \restriction n$ and $\ell^{\nu(n)} = \ell^{\rho(n)}$.
$\bullet_2$ $\alpha^{\rho(n)} \in [\alpha_0, \alpha_1)$

(b) $\bullet_1$ $\nu \restriction n = \rho \restriction n$ and $\ell^{\nu(n)} = \ell^{\rho(n)}$.
$\bullet_2$ $\alpha^{\rho(n)} = \alpha_1$ and $\alpha^{\rho(n+1)} \in [\sup(W \cap \zeta), \zeta)$.
$\bullet_3$ $(\ell^{\rho(n+1)}, \ell^{\rho(n)}) = (\ell^{\rho(n_1)}, \ell^{\nu(n)}) \in \big\{(2,-2), (2,1), (-2,-1), (-2,2)\big\}$

(c) $\bullet_1$ $\alpha_1 = \zeta$ and $n > \theta$ and $(\nu \restriction n)\hat{\ }(\ell^{\nu(n)}, \alpha_1) \notin I_2$.
$\bullet_2$ $(\ell^{\nu(n)}, \ell^{\nu(n-1)}) \in \big\{(2,-2), (2,1), (-2,2), (-2,-1)\big\}$
$\bullet_3$ $\mathrm{cf}(\nu(n)) \ge \theta_1$ and $\nu(n) > \sup(W \cap \nu(n))$.
$\bullet_4$ $\rho \restriction (n-1) = \nu \restriction (n-1)$, $\ell^{\rho(n-1)} = \ell^{\nu(n-1)}$
$\bullet_5$ $\alpha^{\rho(n-1)} \in \big[\sup(\nu(n-1) \cap W), \nu(n-1)\big)$

[Why? First note that if $\rho \in J^2_{Y,\nu}$, $\rho \restriction k = \nu \restriction k$, $\rho(k) \ne \nu(k)$, and $k \le n$, then necessarily $k = n \wedge \ell^{\rho(k)} = \ell^{\nu(k)}$. We now proceed to check "if".

Let $f : \{-2,-1,1,2\} \to \{2,-2\}$ be such that $f^{-1}[2] = \{-2,1\}$ and $f^{-1}[-2] = \{-1,2\}$. Case (a) is obvious. In case (b), in order for $\eta \in Y$ to separate between ν and ρ, it is necessary that $\eta \restriction (n+1) = \rho \restriction (n+1)$, $\ell^{\eta(n+1)} = \ell^{\rho(n+1)} = f(\ell^{\rho(n)})$ and $\alpha^{\eta(n+1)} \ge \zeta$, but then $\eta \notin I_2$. In case (c), in order to separate between ρ and ν by $\eta \in Y$, there are two possibilities. Either $\eta \restriction n = \nu \restriction n$ and then

$$\ell^{\eta(n)} = \ell^{\nu(n)} = f(\ell^{\nu(n-1)})$$

(recall that $\nu \restriction n\hat{\ }\big\langle(\ell^{\nu(n)}, \alpha_1)\big\rangle \notin I_2$), and $\alpha^{\eta(n)} \ge \zeta$, but then also $\eta \notin I_2$. The other possibility is that $\eta \restriction (n-1) = \nu \restriction (n-1)$, $\ell^{\eta(n-1)} = \ell^{\nu(n-1)}$, $\alpha = \alpha^{\eta(n-1)}$ is such that $\alpha \in W$, and $\alpha^{\rho(n-1)} < \alpha < \alpha^{\nu(n-1)}$, which is also impossible by the choice of $\alpha^{\rho(n-1)}$. Showing that these are the only cases (the "only if" direction) is similar and is actually done below.]

Now we proceed to check that clauses of $\boxtimes_4$ hold.

Clause (a):

First assume $\ell^{\nu(n)} \in \{-2, 1\}$, and let

$$J = \big\{\nu \restriction n\hat{\ }\big\langle(\ell^{\nu(n)}, \alpha_0), (2, \zeta + m)\big\rangle : m < \omega\big\}.$$

Now $J \subseteq I_2$.

[Why? clearly if $\rho \in J$ then $\rho \restriction (n+1) \in I_2$ so we only need to check $\circledast$ for n, recall that $\mathrm{cf}(\alpha_0) = \aleph_0 < \theta_1$, hence clause (b) of $\circledast$ fails].

Now by clause (a) of $\boxdot_3$ we have that $J \subseteq J^2_{Y,\nu}$, and we claim that it is also cofinal in it.

[Why? Note that as $\ell^{\nu(n)} \in \{-2, 1\}$ then $\nu \restriction n\hat{\ }\langle(\ell^{\nu(n)}, \alpha_0)\rangle <_{I_2} \nu \restriction (n+1)$, and if $\rho \in J^2_{Y,\nu}$ is as in clauses (a) or (b) of $\boxdot_3$ then for every m large enough

$$\rho <_{I_2} \nu \restriction n\hat{\ }\big\langle(\ell^{\nu(n)}, \alpha_0), (2, \zeta + m)\big\rangle.$$

If $\rho \in J^2_{Y,\nu}$ is as in clause (c) of $\boxdot_3$ then $\ell^{\nu(n)} \in \{-2, 2\}$ by (ii) there, and as in this case $\ell^{\nu(n)} \in \{-2, 1\}$, necessarily $\ell^{\nu(n)} = -2$ and so by (ii) of (c) of $\boxdot_3$ we have $\ell^{\nu(n-1)} \in \{-1, 2\}$, but then $\rho <_{I_2} \nu$ and so it is below every element in J.]

Second, assume $\ell^{\nu(n)} \in \{-1, 2\}$ and $\nu \restriction n\hat{\ }\langle(\ell^{\nu(n)}, \alpha_1)\rangle \in I_2$; let $\delta^* = \sup(W \cap \zeta)$, so as above $\delta^* \notin W$ and has cofinality $\aleph_0$ (which is less than θ_1). Recall also that $\mathrm{cf}(\alpha_1) \geq \theta_1$. So (for $\ell \in \{-2, -1, 1, 2\}$) by $\circledast$ we have

$$(\nu \restriction n)\hat{\ }\big\langle(\ell^{\nu(n)}, \alpha_1), (\ell, \beta)\big\rangle \in I_2$$

iff

$$\Big(\beta < \zeta \text{ and } \ell \in \{-2, -1, 1, 2\}\Big) \text{ or } \Big(\zeta \leq \beta < \zeta + \omega \text{ and } \ell \neq -2\Big).$$

Hence we have $(\nu \restriction n)\hat{\ }\big\langle(\ell^{\nu(n)}, \alpha_1), (-2, \beta)\big\rangle \in I_2 \Leftrightarrow \beta < \zeta$. Also

$$(\nu \restriction n)\hat{\ }\big\langle(\ell^{\nu(n)}, \alpha_1), (-2, \beta)\big\rangle \in Y \Leftrightarrow \beta \in W,$$

and as $\nu(n) < \alpha_1 \wedge \ell^{\nu(n)} \in \{-1, 2\}$ clearly $\nu <_{I_2} (\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \alpha_1), (-2, \beta)\rangle$. Easily $\big\{(\nu \restriction n)\hat{\ }\big\langle(\ell^{\nu(n)}, \alpha_1), (-2, \varepsilon)\big\rangle : \varepsilon \in W \cap \zeta\big\}$ is a subset of $\{\eta \in Y : \nu <_{I_2} \eta\}$ unbounded from below in it.

So $\big\{(\nu \restriction n)\hat{\ }\big\langle(\ell^{\nu(n)}, \alpha_1), (-2, \delta^*), (2, \alpha)\big\rangle : \zeta < \alpha < \zeta + \omega\big\}$ is included in I_2 (recalling clause (b) of $\circledast$ as $\mathrm{cf}(\delta^*) = \aleph_0$) and moreover is a cofinal subset of $J^2_{Y,\nu}$ of order type ω, so $\mathrm{cf}(J^2_{Y,\nu}) = \aleph_0$ as required.

Third, assume $\rho^{\nu(n)} \in \{-1, 2\}$ and $(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \alpha_1)\rangle \in I_2$ and $\mathrm{cf}(\alpha_1) < \theta_1$, equivalently $\mathrm{cf}(\alpha_1) = \aleph_0$ by clause (e) of $\boxdot_2$. In this case

$$\big\{(\nu \restriction n)\hat{\ }\big\langle(\ell^{\nu(n)}, \alpha), (-2, \beta)\big\rangle : \zeta \leq \beta < \zeta + \omega\big\}$$

is included in I_2 (recalling clause (b) of $\circledast$) and in Y. Hence, recalling $\boxdot_3(a)$, the set $\big\{(\nu \restriction n)\hat{\ }\big\langle(\ell^{\nu(n)}, \alpha)\big\rangle : \alpha \in [\alpha_0, \alpha_1)\big\}$ is a cofinal subset of $J^2_{Y,\nu}$ hence its cofinality is $\mathrm{cf}(\alpha_1) = \aleph_0$ as required.

Fourth, we are left with the case $\ell^{\nu(n)} \in \{-1, 2\}$ and $(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \alpha_1)\rangle \notin I_2$ so necessarily $n > 0$ and clauses (a)-(d) of $\circledast$ hold for it for $n - 1$; then by clause (c) of $\circledast$ (recalling $\alpha_1 \leq \zeta$ as shown before $\boxdot_3$) necessarily $\alpha_1 = \zeta$. Clearly $k := n - 1 \geq 0$ and as clause (d) of $\circledast$ holds and it says there "$\ell^{\eta(n+1)} \in \{2, -2\}$" which means here $\ell^{\nu(n)} \in \{2, -2\}$ but we are assuming presently $\ell^{\nu(n)} \in \{-1, 2\}$ hence $\ell^{\nu(n)} = \ell^{\nu(k+1)} = 2$ so using clause (d) of $\circledast$, see above, it follows that $\ell^{\nu(k)} \in \{-2, 1\}$ and by clause (b) of $\circledast$ we have $\mathrm{cf}(\alpha^{\nu(k)}) \geq \theta_1$. Let $\delta_* = \sup(W \cap \alpha^{\nu(k)})$. Now if $\delta_* < \alpha^{\nu(k)}$ then by clause (f) of $\boxdot_2$ we know $\mathrm{cf}(\delta_*) = \aleph_0$ and

$$\big\{(\nu \restriction k)\hat{\ }\big\langle(\ell^{\nu(k)}, \delta_*), (2, \zeta + m)\big\rangle : m < \omega\big\}$$

is included in I_2 (as $\nu \in I_2$ and $\delta_* \le \alpha^{\nu(k)}$ we have only to check $\circledast$, with $k+1$ here standing for n there, but $\mathrm{cf}(\delta_*) = \aleph_0$ so clause (b) there fails) and so recalling $\boxdot_3(c)$ this set is a cofinal subset of $J^2_{Y,\nu}$ exemplifying that its cofinality is $\aleph_0$.

Lastly, if $\delta_* = \alpha^{\nu(k)}$ then $\langle (\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \alpha)\rangle : \alpha \in W \cap \zeta\rangle$ is $<_{I_2}$-increasing with α, all members in Y, and in $J^2_{Y,\nu}$, cofinal in it and has order type $\mathrm{otp}(W \cap \zeta)$ which has cofinality $\aleph_0$ so also $J^2_{Y,\nu}$ has cofinality $\aleph_0$ as required.

Clause (b): What about the cofinality of the inverse? Recall that I_2 is isomorphic to its inverse by the mapping $(\ell, \beta) \mapsto (-\ell, \beta)$, but this isomorphism maps Y onto itself hence it maps $J^2_{Y,\nu}$ onto $J^2_{Y,\nu'}$ for some $\nu' \in I_2 \setminus Y$, but clause (a) was proved also for ν', so this follows.

Clause (c): As Y is unbounded from below in I_2 (containing $\{\langle(-2, \zeta+n)\rangle : n < \omega\}$) it follows that $J^-_{Y,\nu}$ is non-empty, hence $\mathrm{cf}(J^-_{Y,\nu}) \neq 0$, but what is $\mathrm{cf}(J^-_{Y,\nu})$?

First, if $\ell^{\nu(n)} \in \{-1, 2\}$ then $\{(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \alpha)\rangle : \alpha < \alpha_0\}$ is an unbounded subset of $J^-_{Y,\nu}$ of order type α_0 hence $\mathrm{cf}(J^-_{Y,\nu}) = \mathrm{cf}(\alpha_0) = \aleph_0$ (see the assumption on W and the choice of α_0).

Second, if $\ell^{\nu(n)} = \{-2, 1\}$ and $(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \alpha_1)\rangle \in I_2$ and $\mathrm{cf}(\alpha_1) \ge \theta_1$ then as in the proof of clause (a) we have $\{(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \alpha_1), (2, \zeta+m)\rangle \notin I_2$ for $m < \omega$ and again letting $\delta^* = \sup(W \cap \zeta)$ we have $\{(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \alpha_1), (2, \beta)\rangle : \beta \in W \cap \zeta\}$ is included in I_2 and in $J^-_{Y,\nu}$ and even is an unbounded subset of $J^-_{Y,\nu}$ of order type $\mathrm{otp}(W \cap \delta^*)$ which has the same cofinality as δ^* which is $\aleph_0$.

Third, if $\ell^{\nu(n)} \in \{-2, 1\}$ and $(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \alpha_1)\rangle \in I_2$ and $\mathrm{cf}(\alpha_1) < \theta_1$ (equivalently $\mathrm{cf}(\alpha_1) = \aleph_0$) <u>then</u> $\{(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \alpha_1), (2, \zeta+m)\rangle : m < \omega\}$ is a subset of I_2 (as $\mathrm{cf}(\alpha_1) = \aleph_0$) is included in $J^-_{Y,\nu}$, unbounded in it and has cofinality $\aleph_0$, so we are done.

Fourth and lastly, if $\ell^{\nu(n)} \in \{-2, 1\}$ and $(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \alpha_1)\rangle \notin I_2$ then as in the proof of clause (a) we have $\alpha_1 = \zeta$. Again letting $\delta^* = \sup(W \cap \zeta)$ we have $\mathrm{cf}(\delta^*) = \aleph_0$, $(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \delta^*)\rangle \in I_2$, and

$$\{(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \delta^*), (2, \zeta+m)\rangle : m < \omega\}$$

is a subset of I_2; moreover, it is a subset of $J^-_{Y,\nu}$ unbounded in it, and

$$(\nu \restriction n)\hat{\ }\langle(\ell^{\nu(n)}, \delta^*), (2, \zeta+m)\rangle$$

is $<_{I_2}$–increasing with m. So indeed $J^-_{Y,\nu}$ has cofinality $\aleph_0$.

Clause (d): As in clause (b) we use the anti-isomorphism.

So $\boxtimes_4$ holds.]

$\boxtimes_5$ if $I' \subseteq I_2$ then the number of cuts of I' induced by members of $I_2 \setminus I'$ (that is, $\{\{s \in I' : s <_{I_2} t\} : t \in I_2 \setminus I'\}$) is $\le |I'| + 1$.

[Why? Let $\mathcal{U} := \{\alpha^{\eta(\ell)} : \ell < \ell g(\eta) \text{ and } \eta \in I'\}$. It belongs to $[\zeta + \omega]^{\le \mu}$. Now (by inspection) $\eta_1, \eta_2 \in I_2 \setminus I'$ realizes the same cut of I' <u>when</u>:

(a) $\ell g(\eta_1) = \ell g(\eta_2)$

(b) $\ell^{\eta_1(n)} = \ell^{\eta_2(n)}$ for $n < \ell g(\eta_1)$.

(c) $\alpha^{\eta_1(n)} \in \mathcal{U} \Leftrightarrow \alpha^{\eta_2(n)} \in \mathcal{U} \Rightarrow \alpha^{\eta_1(n)} = \alpha^{\eta_2(n)}$ for $n < \omega$.

(d) $\beta < \alpha^{\eta_1(n)} \equiv \beta < \alpha^{\eta_2(n)}$ for $\beta \in \mathcal{U}$ and $n < \omega$.

[Why? Clauses (a)-(d) define an equivalence relation on $I_2 \setminus I'$ which refines "inducing the same cut" and has $\le |\mathcal{U}| + \aleph_0 = |I'| + \aleph_0$ equivalence classes. As the case 'I' is finite' is trivial, we are done proving $\boxtimes_5$.]

$\boxtimes_6$ if ∂ is regular uncountable, $n^* < \omega$, $t_{\varepsilon,\ell} \in I_2$ for $\varepsilon < \partial$, $\ell < n^*$, and $t_{\varepsilon,0} <_{I_2} \ldots <_{I_2} t_{\varepsilon,n^*-1}$ for $\varepsilon < \partial$ then for some unbounded (and even stationary) set $S \subseteq \partial$, $m \le n^*$, $0 = k_0 < k_1 < \ldots < k_m = n^*$ stipulating $t_{\varepsilon,k_m} = \infty$, and letting $\varepsilon(*) = \min(S)$, we have:

(a) for each $i < m$ [exactly one / at least one] of the following hold:
- $\bullet_1$ If $\varepsilon < \xi$ are from S and $\ell_1, \ell_2 \in [k_i, k_{i+1})$ then $t_{\varepsilon,\ell_1} <_{I_2} t_{\xi,\ell_2}$.
- $\bullet_2$ if $\varepsilon < \xi$ are from S and $\ell_1, \ell_2 \in [k_i, k_{i+1})$ then $t_{\xi,\ell_2} <_{I_2} t_{\varepsilon,\ell_1}$.
- $\bullet_3$ $k_{i+1} = k_i + 1$ and for every $\varepsilon \in S$ we have $t_{\varepsilon,k_i} = t_{\varepsilon(*),k_i}$.

(b) There is a sequence $\langle s_i^-, s_i^+ : i < m\rangle$ such that
- $\bullet_1$ $i < m \Rightarrow s_i^- <_{I_2} s_i^+$
- $\bullet_2$ If $i < m-1$ then $s_i^+ < s_{i+1}^-$ (except possibly when $\langle t_{\varepsilon,k_i} : \varepsilon < \partial\rangle$ is $<_{I_2}$-decreasing and there is no $t \in I_2$ such that $\varepsilon < \partial \Rightarrow t_{\varepsilon,k_i} <_{I_2} t <_{I_2} t_{\varepsilon,k_{i+1}}$, hence (by $\boxtimes_3$) we have $\partial \ge \theta_2$).
- $\bullet_3$ For each $i < m$ the set $\{t_{\varepsilon,\ell} : \varepsilon \in S,\ \ell \in [k_i, k_{i+1})\}$ is included in the interval $(s_i^-, s_i^+)_{I_2}$.

[Why? Straight. For some stationary $S_1 \subseteq \partial$ and $\langle n_k : k < n^*\rangle$ we have

$$\varepsilon \in S_1 \wedge k < n^* \Rightarrow \ell g(t_{\varepsilon,k}) = n_k.$$

Also, without loss of generality $\langle \ell^{t_{\varepsilon,k}(i)} : i < n_k\rangle$ does not depend on $\varepsilon \in S_1$. By $\sum\limits_{k<n^*} n_k$ application of $\partial \to (\partial,\omega)^2$, without loss of generality for each $k < n^*$ and $i < n_k$ the sequence $\langle \alpha^{t_{\varepsilon,k}(i)} : \varepsilon \in S_1\rangle$ is constant or increasing. Cleaning a little more we are done. So $\boxtimes_6$ holds.]

Lastly, recall that we chose I to be $(|M|, <^M)$, where M was the real closure of M_0 (see $(*)_1$), M_0 the ordered field generated over $\mathbb{Q}$ by $\{a_t : t \in I_2\}$ as described in $(*)_1$ above, and for every $u \subseteq \zeta$ let:

$(*)_2$ (a) $I_u^1 = \{(\ell,\beta) \in I_1 : \beta \in u \text{ or } \beta \in [\zeta, \zeta+\omega)\}$

(b) $I_u^{*,2} = \{\eta \in I_2^* : \alpha^{\eta(\ell)} \in I_u^1 \text{ for every } \ell < \ell g(\eta)\}$

(c) $I_u^2 = \{\eta \in I_2 : \alpha^{\eta(\ell)} \in I_u^1 \text{ for every } \ell < \ell g(\eta)\}$

(d) I_u is the real closure of $\mathbb{Q}(a_t : t \in I_u^2)$ in M

(e) For $t \in I_2 \setminus I_u^2$, let $I_{u,t}^2 = I_2 \restriction \{s \in I_2 : s \notin I_u^2$ and for every $r \in I_u^2$ we have $r <_{I_2} t \equiv r <_{I_2} s\}$.

(f) For $x \in I \setminus I_u$ let

$$I_{u,x} = I \restriction \big\{y \in I \setminus I_u : (\forall a \in I_u)[a <_I y \equiv a <_I x]\big\}.$$

(g) Let $\hat{I}_u$ be the set $I_u \cup \{I_{u,a} : a \in I \setminus I_u\}$ ordered by: $x <_{\hat{I}_u} y$ iff one of the following holds:
- $\bullet_1$ $x, y \in I_u$ and $x <_{I_u} y$
- $\bullet_2$ $x \in I_u$, $y = I_{u,b}$ and $x <_{I_u} b$
- $\bullet_3$ $x = I_{u,a}$, $y \in I_u$ and $a <_{I_u} y$
- $\bullet_4$ $x = I_{u,a}$, $y = I_{u,b}$ and $a <_{I_u} b$ (can use it more!)

(Note that by $\boxtimes_5$, $|u| \le \mu \Rightarrow |\hat{I}_u| \le \mu$.)

Now observe

$(*)_3$ for $u \subseteq \zeta$, I_u^2 is unbounded in I_2 from below and from above.

We define **[the following property.]**

$(*)_4$ We say[10] that u is *μ-reasonable* if:

(a) $u \subseteq \zeta$, $|u| < \theta_2$, and $\mu \subseteq u$.

(b) $\alpha \in u \equiv \alpha + 1 \in u$ for every α.

[10] We may in clauses (e) + (c) replace μ by $\mu + |\mathcal{U}|$; there's no harm and it makes (c)(β) of $(*)_1$ redundant.

(c) If $\delta \in u$ and $\aleph_0 \leq \mathrm{cf}(\delta) \leq \mu$ then $\delta = \sup(u \cap \delta)$,
(d) If $\delta \leq \zeta$ and $\mathrm{cf}(\delta) > \mu$ then $\mathrm{cf}(\mathrm{otp}(\delta \cap u)) = \aleph_0$.

Now we note

$(*)_5$ if $X \subseteq I$ has cardinality $< \theta_2$ and $u_* \subseteq \zeta$ has cardinality $< \theta_2$ then we can find a μ-reasonable u such that $X \subseteq I_u$, $u_* \subseteq u$, and $|u| = \mu + |X| + |u_*|$.

[Why? By the proof of $\boxtimes_4$.]

$(*)_6$ if u is μ-reasonable then $Y := I^2_u$ satisfies the conclusions of $\boxtimes_4$.

[Why? By the proof of $\boxtimes_4$. That is, if $u^+ := u \cup \{\zeta + n : n < \omega\}$ then Y as defined in the proof there using u^+ for W, is I^2_u from $(*)_2(c)$, and it satisfies demands (a)-(f) from $\boxdot_2$ so the proof there applies.]

$(*)_7$ if u is μ-reasonable and $x \in I \setminus I_u$ then $\mathrm{cf}(I_{u,x}) \leq \aleph_0$.

Why? The proof takes awhile. Toward contradiction assume $\partial = \mathrm{cf}(I_{u,x})$ is $> \aleph_0$ and let $\langle b_\varepsilon : \varepsilon < \partial \rangle$ be an increasing sequence of members of $I_{u,x}$ unbounded in it. So for each $\varepsilon < \partial$ there is a definable[11] function $f_\varepsilon(x_0, \dots, x_{n(\varepsilon)-1})$ and $t_{\varepsilon,0} <_{I_2} t_{\varepsilon,1} <_{I_2} \dots <_{I_2} t_{\varepsilon,n(\varepsilon)-1}$ from I_2 such that $M \models$ "$b_\varepsilon = f_\varepsilon(a_{t_{\varepsilon,0}}, \dots, a_{t_{\varepsilon,n(\varepsilon)-1}})$" and $n(\varepsilon)$ is minimal. As $\mathrm{Th}(\mathbb{R})$ is countable and $\aleph_0 < \partial = \mathrm{cf}(\partial)$, without loss of generality $\varepsilon < \partial \Rightarrow f_\varepsilon = f_*$ so $\varepsilon < \partial \Rightarrow n(\varepsilon) = n(*)$.

Apply $\boxtimes_6$ to $\langle \bar{t}^\varepsilon = \langle t_{\varepsilon,\ell} : \ell < n(*) \rangle : \varepsilon < \partial \rangle$, and get $S \subseteq \partial$, $0 = k_0 < k_1 < \dots < k_m = n(*)$, $\langle (s^-_i, s^+_i) : i < m \rangle$, and $\varepsilon(*) = \min(S)$ as there. Without loss of generality the truth value of "$t_{\varepsilon,\ell} \in I^2_u$", for $\varepsilon \in S$, depends only on ℓ. Let $w_1 = \{i < m : (\forall \varepsilon \in S)[t_{\varepsilon,k_i} = t_{\varepsilon(*),k_i}]\}$ and $w_2 = \{\ell < n(*) : t_{\varepsilon(*),\ell} \in I^2_u\}$; clearly for every $\ell < n(*)$ we have

$$(\forall \varepsilon \in S)[t_{\varepsilon,\ell} = t_{\varepsilon(*),\ell}] \Leftrightarrow \ell \in \{k_i : i \in w_1\}$$

and $i \in w_1 \Rightarrow k_i + 1 = k_{i+1}$.

Let $t^*_{k_i} = t_{\varepsilon,k_i}$ for ($\varepsilon < \partial$ and $i \in w_1$). [By] renaming, without loss of generality $S = \partial$ and $\varepsilon(*) = 0$.

We have some free choice in choosing $\langle b_\varepsilon : \varepsilon < \partial \rangle$ (as long as it is cofinal in $I_{u,x}$), so without loss of generality we choose it such that $n(*)$ is minimal and then $|w_1|$ is maximal and then $|w_2|$ is maximal.

Now does the exceptional case in (b)$\bullet_2$ of $\boxtimes_6$ occur? This is an easier case and we delay it to the end.

As I_2 (and $I_{2,<t}$ for $t \in I_2$) have cofinality $\aleph_0$ (see $\boxtimes_2(a),(b)$) and $\boxtimes_3$ and this holds for the inverse of I_2, too, while $\partial = \mathrm{cf}(\partial) > \aleph_0$ and we can replace $\langle b_\varepsilon : \varepsilon < \partial \rangle$ by $\langle b_{n(*)+\varepsilon} : \varepsilon < \partial \rangle$ we can find $t_{\partial,\ell}$ for $\ell < n(*)$ such that

$\odot$ (a) $t_{\partial,0} <_{I_2} t_{\partial,1} <_{I_2} \dots <_{I_2} t_{\partial,n(*)-1}$
(b) If $\varepsilon < \xi < \partial$ and $\ell_1, \ell_2 < n(*)$ then $(t_{\varepsilon,\ell_1} <_{I_2} t_{\partial,\ell_2}) \equiv (t_{\varepsilon,\ell_1} <_{I_2} t_{\xi,\ell_2})$ and $(t_{\partial,\ell_1} <_{I_2} t_{\varepsilon,\ell_2}) \equiv (t_{\xi,\ell_1} < t_{\varepsilon,\ell_2})$.
(c) If $\ell \in [k_i, k_{i+1})$ then $t_{\partial,\ell} \in (s^-_i, s^+_i)_{I_2}$.

Case 0: $\{0, \dots, m-1\} = w_1$.

This implies $i < m \Rightarrow k_i + 1 = k_{i+1}$ hence $m = n$ hence $\ell < n \Rightarrow t_{\xi,\ell} = t^*_\ell$ and so contradicts "$\langle b_\varepsilon : \varepsilon < \partial \rangle$ is increasing" (as it becomes constant).

Case 1: $[0, m) \setminus w_1$ is not a singleton.

It cannot be empty by Case 0. Choose $i(*) \in \{0, \dots, m-1\} \setminus w_1$ and for $\varepsilon, \xi < \partial$ let $\bar{t}^{\varepsilon,\xi} = \langle t^{\varepsilon,\xi}_\ell : \ell < n(*) \rangle$ be defined by: $t^{\varepsilon,\xi}_\ell$ is $t_{\varepsilon,\ell}$ if $\ell \in [k_{i(*)}, k_{i(*)+1})$ and $t_{\xi,\ell}$ otherwise. Let $b_{\varepsilon,\xi} = f_*(a_{t^{\varepsilon,\xi}_0}, \dots, a_{t^{\varepsilon,\xi}_{n(*)-1}}) \in M$. Clearly

[11] where 'definable,' of course, means "in the theory of real closed fields"

$\circledast_0$ for any $\varepsilon_1, \varepsilon_2, \xi_1, \xi_2 \leq \partial$ the truth value of $b_{\varepsilon_1,\xi_1} < b_{\varepsilon_2,\xi_2}$ depends just on the inequalities which $\langle \varepsilon_1, \varepsilon_2, \xi_1, \xi_2 \rangle$ satisfies, and even just on the inequalities which the $t_{\varepsilon_1,\ell}, t_{\varepsilon_2,\ell}, t_{\xi_1,\ell}, t_{\xi_2,\ell}$ (for $\ell < n(*)$) satisfy.

[Why? Recall $\big\langle \langle t_{\varepsilon,\ell} : \ell < n(*)\rangle : \varepsilon \in S \big\rangle$ is an indiscernible sequence in the linear order I_2 (for quantifier free formulas) and M has elimination of quantifiers.]

$\circledast_1$ $\bigwedge\limits_{\ell=1,2} \varepsilon(0) < \varepsilon_\ell < \varepsilon(1) < \partial \Rightarrow b_{\varepsilon(0)} <_I b_{\varepsilon_1,\varepsilon_2} <_I b_{\varepsilon(1)}$.

[Why? By $\circledast_0$, the desired statement $b_{\varepsilon(0)} <_I b_{\varepsilon_1,\varepsilon_2} <_I b_{\varepsilon(1)}$ is equivalent to $b_{\varepsilon(0)} < b_{\varepsilon_1,\varepsilon_1} < b_{\varepsilon(1)}$, which means $b_{\varepsilon(0)} < b_{\varepsilon_1} < b_{\varepsilon(1)}$, which holds.]

$\circledast_2$ $b_{0,2} <_I b_1$.

[Why? Otherwise $b_1 \leq_I b_{0,2}$ hence $\varepsilon \in (0,\partial) \Rightarrow b_\varepsilon <_I b_{0,\varepsilon+1} <_I b_{\varepsilon+2}$ (by $\circledast_0 + \circledast_1$) so $\langle b_{0,\varepsilon} : \varepsilon \in (1,\partial)\rangle$ is also an increasing sequence unbounded in $I_{u,x}$ contradicting "w_1 maximal".]

$\circledast_3$ $b_{0,2} < b_{1,2}$.

[Why? By $\circledast_0 + \circledast_2$ we have $b_{0,4} < b_1$ and by $\circledast_1$ we have $b_1 < b_{2,4}$ together $b_{0,4} < b_{2,4}$ so by $\circledast_0$ we have $b_{0,2} < b_{1,2}$.]

But then $\langle b_{\varepsilon,\partial} : \varepsilon < \partial\rangle$ increases (by $\circledast_3 + \circledast_0$) and $\varepsilon < \partial \Rightarrow b_\varepsilon = b_{\varepsilon,\varepsilon} < b_{\varepsilon+1,\partial} < b_{\varepsilon+2}$ (by $\circledast_1$ and $\circledast_2$ respectively) hence is an unbounded subset of $I_{u,x}$ contradiction to the maximality of $|w_1|$.

Case 2: $m \setminus w_1 = \{0,\dots,m-1\} \setminus w_1$ is $\{i(*)\}$.

Subcase 2A: For some $i < m$, $i \neq i(*)$ and $j := k_i \notin w_2$.

Choose such i with $|i - i(*)|$ maximal. For any s let $t_{\varepsilon,\ell,s}$ be $t_{\varepsilon,\ell}$ if $\ell \neq j$ and be s if $\ell = j$.

Let

$$I' = \big\{ s \in I^2_{u,t_{\varepsilon(*),j}} : s \text{ and } t_{\varepsilon(*),j} \text{ realize the same cut of } \{t_{\varepsilon,\ell} : \varepsilon < \partial,\ \ell \neq j\}\big\}.$$

Note that $k_{j+1} = k_j + 1$. Recalling $\boxtimes_2(b)$, the cofinality of $I_{2,<t_{\varepsilon(*),j}}$ is $\aleph_0$ and also the cofinality of the inverse of $I_{2,>t_{\varepsilon(*),j}}$ is $\aleph_0$. Recalling the choice of $\big\langle (s^-_\iota, s^+_\iota) : \iota < m\big\rangle$, there is an open interval[12] of I_2 around $t_{\varepsilon(*),j}$ which is $\subseteq I'$. Note that I' is dense in itself and has neither a first nor last member by $\boxtimes_2 + \boxtimes_4(a),(b)$.

As f_* is definable, by the choice of M_0, M, and of $I' \subseteq I^2_{u,t_{\varepsilon(*),j}}$ we have: if $\varepsilon < \partial$ and $s \in I'$ then $t_{\varepsilon(*),j}$ and s realize the same cut of

$$I^2_u \cup \{t_{\varepsilon,\ell} : \varepsilon < \partial, j \neq \ell\}$$

hence $f^M_*(\dots, a_{t_{\varepsilon,\ell,s}}, \dots)_{\ell<n}$ and b_ε realize the same cut of I_u, which means that $f_*(\dots, a_{t_{\varepsilon,\ell,s}}, \dots)_{\ell<n} \in I_{u,x}$, hence by the choice of $\langle b_\varepsilon : \varepsilon < \partial\rangle$ we have

$$(\exists \xi < \partial)\big[f_*(\dots, a_{t_{\varepsilon,\ell,s}}, \dots) < b_\xi\big].$$

So again by the definability (and indiscernibility)

$\circledast_4$ $\varepsilon < \partial \wedge s \in I' \Rightarrow f^M_*(\dots, a_{t_{\varepsilon,\ell,s}}, \dots) < b_{\varepsilon+1}$.

As I' is dense in itself, what we say on the pair $(s, t_{\varepsilon(*),j})$ when $s \in I' \wedge s <_{I_2} t_{\varepsilon(*),j}$ holds for the pair $(t_{\varepsilon(*),j}, s)$ when $s \in I' \wedge t_{\varepsilon(*),j} <_I s$, so

$\circledast_5$ $\varepsilon < \partial \wedge s \in I' \Rightarrow b_\varepsilon < f^M_*(\dots, a_{t_{\varepsilon+1,\ell,s}}, \dots)$

More fully, let $s_1 <_{I_2} t_{\varepsilon(*),j} <_{I_2} s_2$ and $s_1, s_2 \in I'$. Then the sequences

$$\big\langle t_{\varepsilon,\ell} : \ell \neq j,\ \ell < n(*)\big\rangle{}^\frown\big\langle s_1\big\rangle{}^\frown\big\langle t_{\varepsilon+1,\ell} : \ell \neq j,\ \ell < n(*)\big\rangle{}^\frown\big\langle t_{\varepsilon(*),j}\big\rangle$$

and $\langle t_{\varepsilon,\ell} : \ell \neq j,\ \ell < n(*)\rangle{}^\frown\langle t_{\varepsilon(*),j}\rangle{}^\frown\langle t_{\varepsilon+1,\ell} : \ell \neq j,\ \ell < n(*)\rangle{}^\frown\langle s_2\rangle$ realize the same quantifier free type in I_2 (recalling $t_{\varepsilon,j} = t_{\varepsilon(*),j}$).

[12] if we allow $+\infty, -\infty$ as end points

By $\circledast_4 + \circledast_5$ and indiscernibility we can replace $t_{\varepsilon(*),j}$ by any $t' \in I'$ which realizes the same cut as $t_{\varepsilon(*),j}$ of $\{t_{\varepsilon,\ell} : \varepsilon < \partial, \ell \neq j\}$. But if $j > i(*)$ then $\{t^*_{j+1}, \dots, t^*_{n(*)-1}\} \subseteq I^2_u$ by the choice of j, and the set

$$I'' = \{t \in I_2 : \text{if } \varepsilon < \partial,\ \ell \neq j \text{ then } t \neq t_{\varepsilon,\ell} \text{ and } t_{\varepsilon,\ell} <_{I_2} t \equiv t_{\varepsilon,\ell} <_{I_2} t^*_j\}$$

includes an initial segment of $J^+_{I^2_u, t_{\varepsilon(*),j}}$ (see $\boxtimes_4(d)$) i.e. $(*)_6$, so its inverse has cofinality $\aleph_0$. Say $\langle s^*_n : n < \omega\rangle$ exemplifies this, so $n < \omega \Rightarrow s^*_{n+1} <_{I_2} s^*_n$. So for every $\varepsilon < \partial$ for some $n < \omega$, $f^M_*(\dots, a_{t_{\varepsilon+1,\ell}, s^*_n}, \dots) \in (b_\varepsilon, b_{\varepsilon+1})_I$. So for some $n_* < \omega$ this holds for unboundedly many $\varepsilon < \partial$, contradictory to "$|w_2|$ is maximal". Similarly if $j < i(*)$.

Subcase 2B: For every $\varepsilon < \partial$, for some $\xi \in (\varepsilon, \partial)$, the interval of I_2 which is defined by $t_{\varepsilon,k_{i(*)}}, t_{\xi,k_{i(*)}}$ is not disjoint to I^2_u (so without loss of generality it has $\geq k_{i(*)+1} - k_{i(*)}$ members of I^2_u).

In this case, as in case 1, without loss of generality $\{k_{i(*)}, \dots, k_{i(*)+1}\} \subseteq w_2$ so as $|w_2|$ is maximal this holds. Because subcase 2A is ruled out, $\{t_{\varepsilon,\ell} : \varepsilon < \partial,\ \ell < n\} \subseteq I^2_u$ hence $\{b_\varepsilon : \varepsilon < \partial\} \subseteq I_u$, a contradiction.

Subcase 2C: None of the above.

As subcase 2B is ruled out, without loss of generality

$$\big\{t_{\varepsilon,\ell} : \varepsilon < \partial \ell \in [k_{i(*)}, k_{i(*)+1})\big\} \subseteq I^2_{u, t_{\varepsilon(*),k_{i(*)}}}.$$

Then, as in subcase 2A, the sequence $\langle t_{\varepsilon,k_{i(*)}} : \varepsilon < \partial\rangle$ is increasing/decreasing and is unbounded from above/below in $I^2_{u,t_{\varepsilon(*),k_{i(*)}}}$ contradiction to $(*)_6$.

In more detail, $I' := I^2_{u,t_{0,k_{i(*)}}}$ includes all $\{t_{\varepsilon,\ell} : \varepsilon < \partial \text{ and } \ell \in [k_{i(*)}, k_{i(*)+1})\}$. Also I' and its inverse are of cofinality $\aleph_0$ by $(*)_6$, hence without loss of generality we can find (new) $\big\langle t_{\partial,\ell} : \ell \in [k_{i(*)}, k_{i(*)+1})\big\rangle$ such that $t_{\partial,\ell} <_{I_2} t_{\partial,\ell+1}$, $t_{\partial,\ell} \in (s^-_{i(*)}, s^+_{i(*)})_{I_2}$, $\varepsilon < \partial \Rightarrow t_{\varepsilon,\ell_1} <_{I_2} t_{\partial,\ell} \equiv t_{\varepsilon,\ell_1} < t_{\varepsilon+1,\ell_2}$, and the convex hull in I_2 of $\big\{t_{\zeta,\ell} : \zeta \leq \partial \text{ and } \ell \in [k_{i(*)}, k_{i(*)+1}]\big\}$ is disjoint to I^2_u. Let $t_{\partial,\ell} = t_{\partial,\ell}$ for $\ell \notin [k_{i(*)}, k_{i(*)+1}]$, $\ell < m$, $b_\partial = f_*(a_{t_{\partial,0}}, \dots, a_{t_{\partial,n-1}})$.

Easily $\varepsilon < \partial \Rightarrow b_\varepsilon <_I b_\partial$. As $\varepsilon < \xi < \partial \Rightarrow (b_\varepsilon, b_\xi)_{I_2} \cap u = \varnothing$, easily $\varepsilon < \partial \Rightarrow (b_\varepsilon, b_\partial)_{I_2} \cap u = 0$, in contradiction to $\langle b_\varepsilon : \varepsilon < \partial\rangle$ being cofinal in $I_{u,x}$.

To finish proving $(*)_7$, we have to consider the possibility that when applying $\boxtimes_6$, the exceptional case in (b)$\bullet_2$ of $\boxtimes_6$ occurs for some $i < m$; say, for $i(*)$ (see $\odot$).

Also, without loss of generality $\partial \geq \theta_2$ and so without loss of generality $\ell \in w_2 \Rightarrow t_{\varepsilon,\ell} = t_{\varepsilon(*),\ell}$ and for each $\ell < n(*)$ we have

$$(\forall \varepsilon, \zeta < \partial)(\forall s \in I^2_u)\big[s <_{I_2} t_{\varepsilon,\ell} \equiv s <_{I_2} t_{\zeta,\ell}\big].$$

Now we can define $\bar t^{\varepsilon,\xi} = \langle t^{\varepsilon,\xi}_\ell : \ell < n(*)\rangle$ as in case 1 and prove $\circledast_0$ - $\circledast_3$ there.

Clearly all members of $\big\{t_{\varepsilon,\ell} : \varepsilon < \partial, \ell \in [k_{i(*)}, k_{i(*)+2})\big\}$ realize the same cut of I^2_u and we get an easy contradiction.

As we can use only $\langle t_{n(*),\varepsilon} : \varepsilon < \partial\rangle$ and add dummy variables to f_*, without loss of generality $k_{i(*)+1} - k_{i(*)} = k_{i(*)+2} - k_{i(*)+1}$. Let J be $\{1, -1\} \times \partial$ ordered by $(\ell_1, \varepsilon_1) <_J (\ell_2, \varepsilon_2)$ iff $\ell_1 = 1 \wedge \ell_2 = -1$ or $\ell_1 = 1 = \ell_2 \wedge \varepsilon_1 < \varepsilon_2$ or $\ell_1 = -1 = \ell_2 \wedge \varepsilon_1 > \varepsilon_2$.

For $\iota \in J$ let $\iota = (\ell^\iota, \varepsilon^\iota) = (\ell[\iota], \varepsilon[\iota])$. For $\zeta < \partial$ and $\iota_1, \iota_2 \in J$ we define $\bar t_{\zeta,\iota_1,\iota_2} = \langle t_{\zeta,\iota_1,\iota_2,n} : n < n(*)\rangle$ by $t_{\zeta,\iota_1,\iota_2,n}$ is $t_{\varepsilon[\iota_1],n}$ if $n \in [k_{i(*)}, k_{i(*)+1})$, $t_{\varepsilon[\iota_2],n}$ if $n \in [k_{i(*)+1}, k_{i(*)+2})$, and $t_{\zeta,n}$ otherwise. Now, letting $b_{\zeta,\iota_1,\iota_2} = f_*(\bar t_{\zeta,\iota_1,\iota_2})$,

$\circledast_6$ All $b_{\zeta,\iota_1,\iota_2}$ realize the same cut of I^2_u.

Now

$\circledast_7$ Indiscernibility as in $\circledast_0$ holds.

$\circledast_8$ $\neg(b_{\zeta,(1,\varepsilon),(1,\varepsilon+1)} \leq_{I_*} b_{\zeta,(1,\varepsilon+2),(1+\varepsilon+3)})$.

[Why? Otherwise by indiscernibility, if $\zeta \in (6,\partial)$ then $b_{\zeta,(1,\zeta),(-1,3)} <_I b_{\zeta,(-1,5),(-1,4)}$. Hence $\langle b_{\zeta,(-1,5),(-1,4)} : \zeta \in (6,\partial)\rangle$ is monotonic in I_*, all members realizing the fixed cut of I^2_u and is unbounded in it (by the inequality above), contradicting the maximality of $|w_j|$.]

$\circledast_9$ $\neg(b_{\zeta,(1,\varepsilon+2),(1,\varepsilon+3)} <_I b_{\zeta,(1,\varepsilon),(1,\varepsilon+1)})$.

[Similarly, as otherwise if $\zeta \in (6,\partial)$ then $b_{\zeta,(1,\zeta),(-1,\zeta)} <_I b_{\zeta,(1,4),(1,5)}$. Hence $\langle b_{\zeta,(1,4),(1,5)} : \zeta \in (6,\partial)\rangle$ contradicts the maximality of (w_1).]

So we have proved $(*)_7$.

$(*)_8$ if u is μ-reasonable, $x \in I \setminus I_u$ then $\mathrm{cf}(I_{u,x}) = \aleph_0$.

[Otherwise by $(*)_7$ it has a last element; say $b = f_*(a_{t_0},\dots,a_{t_{n-1}})$, where $t_0,\dots,t_{n-1} \in I_2$ and f_* a definable function (without loss of generality, with n minimal). Hence $\{a_{t_0},\dots,a_{t_{n-1}}\}$ is transcendentally independent with no repetitions and b is not algebraic over $\{a_{t_0},\dots,a_{t_{n-1}}\} \setminus \{a_{t_\ell}\}$ for $\ell < n$. So $\{t_0,\dots,t_{n-1}\} \nsubseteq I^2_u$, and let $\ell < n$ be such that $t_\ell \notin I^2_u$, hence there are $s_0 <_{I_2} s_1$ such that $t_\ell \in (s_0,s_1)_{I_2}$ and $(s_0,s_1)_{I_2} \cap I^2_u = \varnothing$. (Recall $\boxtimes_4$(a),(b) and $(*)_6$ about cofinality $\aleph_0$ and I_2 being dense.) Also without loss of generality $\{t_0,\dots,t_{n-1}\} \cap (s_0,s_1)_{I_2} = \{t_\ell\}$; now the function $c \mapsto f^M_*(a_{t_0},\dots,a_{t_{\ell-1}},c,a_{t_{\ell+1}},\dots,a_{t_{n-1}})$ for $c \in (a_{s_0},a_{s_1})_I$ is increasing or decreasing (cannot be constant by the minimality on n and the elimination of quantifiers for real closed fields and the transcendental independence of $\{t_0,\dots,t_{n-1}\}$). So we can find s'_0, s'_1 such that $s_0 <_{I_2} s'_0 <_{I_2} t_\ell <_{I_2} s'_1 <_{I_2} s_1$ such that

$$X := \{f^M_*(a_{t_0},\dots,a_{t_{\ell-1}},c,a_{t_{\ell+1}},\dots,a_{t_{n-1}}) : c \in (a_{s'_0},a_{s'_1})_I\}$$

is included in $I_{u,x}$. Again as the function defined above is monotonic on $(a_{s'_0},a_{s'_1})_I$ so for some value $b' \in (a_{s'_0},a_{s'_1})$ we have $b <_I b'$. But b is last in $I_{u,x}$ by our assumption toward contradiction hence $(b,b')_{I_u} \cap I_u = \varnothing$. But this is impossible as all members of $\{f(a_{t_0},\dots,a_{t_{\ell-1}},c,a_{t_{\ell+1}},\dots,a_{t_{n-1}}) : c \in (a_{s'_1},a_{s'_2})_I\}$ realize the same cut of I_u so $(*)_8$ holds.]

$(*)_9$ if u is μ-reasonable, $x \in I \setminus I_u$ then also the inverse of $I_{u,x}$ has cofinality $\aleph_0$.

[Why? Similarly to the proof of $(*)_7 + (*)_8$, or note that the mapping $y \mapsto -y$ (defined in M) maps I_u onto itself and is an isomorphism from I onto its inverse.]

$(*)_{10}$ if u is μ-reasonable, then I_u is unbounded in I from below and from above.

[Why? Easy.]

$(*)_{11}$ if h, u_1, u_2 are as in clauses (a),(b),(c) below <u>then</u> the function h_4 defined below is (well defined and) is, recalling $(*)_2(g)$, an order preserving function from $\hat{I}_{u_1}$ onto $\hat{I}_{u_2}$ mapping u_1 onto u_2. Also, the functions $h_0, h_1, h^*_2, h_2, h_3$ are as stated, where:

(a) $u_1, u_2 \subseteq \zeta$ are μ-reasonable

(b) h is an order preserving function from u_1 onto u_2

(c) (α) For $\alpha \in u_1$, we have $\mathrm{cf}(\alpha) \geq \theta_1 \Leftrightarrow \mathrm{cf}(h(\alpha)) \geq \theta_1$.
(β) If $\gamma \in u_1$ then $(\forall\alpha < \gamma)(\exists\beta \in u_1)[\alpha \leq \beta < \gamma]$ iff $(\forall\alpha < h(\gamma))(\exists\beta \in u_2)[\alpha \leq \beta < h(\gamma)]$

(d) (α) h_1 is the [induced-order preserving / induced order-preserving] function from $I^1_{u_1}$ onto $I^1_{u_2}$, i.e., $h_1((\ell,\beta')) = (\ell,\beta'')$ when $h(\beta') = \beta'' < \zeta$ or $\beta' = \beta'' \in [\zeta,\zeta+\omega)$.
(β) Let h_0 be the partial function from $\zeta+\omega$ into $\zeta+\omega$ such that $h_0(\alpha) = \beta \Leftrightarrow (\exists\ell)[h_1((\ell,\alpha)) = (\ell,\beta)]$

(e) h^*_2 is the order preserving function from $I^{*,2}_{u_1}$ onto $I^{*,2}_{u_2}$ defined by: for $\eta \in I^{*,2}_{\eta_1}$,

$$h^*_2(\eta) = \langle h_1(\eta(\ell)) : \ell < \ell g(\eta)\rangle = \langle (\ell^{\eta(\ell)}, h_0(\alpha^{\eta(\ell)})) : \ell < \ell g(\eta)\rangle,$$

recalling (d).

(f) $h_2 = h^*_2 \restriction I^2_{u_1}$ is an order preserving function from $I^2_{u_1}$ onto $I^2_{u_2}$.

(g) h_3 is the unique isomorphism from the real closed field $M_{I^2_{u_1}}$ onto the real closed field $M_{I^2_{u_2}}$ mapping a_t to $a_{h_2(t)}$ for $t \in I^2_{u_1}$, where for $I' \subseteq I_2$ we let $M_{I'} \subseteq M$ be the real closure of $\{a_t : t \in I'\}$ inside M.

(h) h_4 is the map defined by: $h_4(x) = y$ iff $(\alpha) \vee (\beta)$, where
 (α) $x \in I_{u_1} \wedge y = h_3(x)$
 (β) For some $a \in I \setminus I_{u_1}$, $b \in I \setminus I_{u_2}$ we have $x = I_{u,a}$, $y \in I_{u,b}$, and $(\forall c \in I_u)[c <_I a \equiv h_3(c) <_I b]$.

(i) $\hat{I}_{u_1} = \mathrm{dom}(h_4)$ and $\hat{I}_{u_2} = \mathrm{rang}(h_4)$ ordered naturally.

[Why? Trivially, h_1 is an order preserving function from $I^1_{u_1}$ onto $I^1_{u_2}$. Recall $I^{2,*}_{u_\ell} = \{\eta \in I^*_2 : \eta(\ell) \in I^1_{u_\ell}$ for $\ell < \ell g(\eta)\}$. So obviously h^*_2 is an order preserving function from $I^{*,2}_{u_1}$ onto $I^{*,2}_{u_2}$. Now $h_2 = h^*_2 \restriction I^2_{u_1}$, but does it map $I^2_{u_1}$ onto $I^2_{u_2}$? We have excluded some members of $I^{*,2}_{u_2}$ by ⊛ above.

But by clauses (c) and (d)(α) of the assumption being excluded/not excluded is preserved by the natural mapping, i.e., h^*_2 maps $I^2_{u_1}$ onto $I^2_{u_2}$ hence $h_2 = h^*_2 \restriction I^1_{u_1}$ is an isomorphism from $I^1_{u_1}$ onto $I^1_{u_2}$. Also by $(*)_1$ being the real closure of the ordered field M_0, and the uniqueness of "the real closure" h_3 is the unique isomorphism from the real closed field $M_{I^2_{u_1}}$ onto $M_{I^2_{u_2}}$ mapping a_t to $a_{h_2(t)}$ for $t \in I^2_{u_1}$.

Let $\langle (\mathcal{U}^1_\varepsilon, \mathcal{U}^2_\varepsilon) : \varepsilon < \varepsilon^*\rangle$ list the pairs $(\mathcal{U}_1, \mathcal{U}_2)$ such that:

$\circledast_{10}$ (a) $\mathcal{U}_\ell$ has the form $I_{u_\ell,x}$ for some $x \in I \setminus I_{u_\ell}$ for $\ell = 1, 2$
 (b) for every $a \in I_{u_1}$, $(\exists y \in \mathcal{U}_1)[a <_I y] \Leftrightarrow (\exists y \in \mathcal{U}_2)[h_2(a) <_I y]$.

Now

$\circledast_{11}$ $\langle \mathcal{U}^\ell_\varepsilon : \varepsilon < \varepsilon^*\rangle$ is a partition of $I \setminus I_{u_\ell}$ for $\ell = 1, 2$.

[Why? First, note the parallel claim for I_1. For this, note that $h_1((\ell, 0)) = (\ell, 0)$ as $0 \in u_1 \cap u_2$ as u_1, u_2 are μ-reasonable (see clause (e) of $(*)_4$) and $h_1((\ell, \alpha)) = (\ell, \beta) \Leftrightarrow h_1((\ell, \alpha+1)) = (\ell, \beta+1)$, by clause (b) of $(*)_4$ and if $h((\ell, \delta_1)) = (\ell, \delta_2)$, δ_1 is a limit (equivalently δ_2 is limit) then

$$\delta_1 = \sup\{\alpha < \delta : (\ell, \alpha) \in I^1_{u_1}\} \Leftrightarrow \delta_2 = \sup\{\alpha < \delta : (\ell, \alpha) \in I^1_{u_2}\}.$$

Second, note the parallel claim for $h_2, I^{*,2}_{u_\ell}, h^*_2$.

Third, note the parallel claim for $I^2_{u_\ell}, h_2$.

Fourth, note the parallel claim for I_{u_ℓ}, h_3 (which is the required one).]

So it follows that

$\circledast_{12}$ h_4 is as promised.

So we are done proving $(*)_{11}$.

[Why? By clauses (b),(c) of $(*)_{11}$.]

$(*)_{12}$ If u_1, u_2 are μ-reasonable, h is an order preserving mapping from $\hat{I}_{u_1}$ onto $\hat{I}_{u_2}$ which maps I_{u_1} onto I_{u_2} then there is an automorphism h^+ of the linear order I extending $h \restriction I_{u_1}$.

[Why? Let $\langle \mathcal{U}^1_\varepsilon : \varepsilon < \varepsilon^*\rangle$ list $\hat{I}_{u_1} \setminus I_{u_1}$ and $\mathcal{U}^2_\varepsilon = h(\mathcal{U}^1_\varepsilon)$. Now for every ε we choose $\langle a^\ell_{\varepsilon,n} : n \in \mathbb{Z}\rangle$ such that

$\circledast_{13}$ (a) $a^\ell_{\varepsilon,n} \in \mathcal{U}^\ell_\varepsilon$
 (b) $a^\ell_{\varepsilon,n} <_I a^\ell_{\varepsilon,n+1}$ for $n \in \mathbb{Z}$.
 (c) $\{a^\ell_{\varepsilon,n} : n \in \mathbb{Z},\ n \geq 0\}$ is unbounded from above in $\mathcal{U}^\ell_\varepsilon$.

(d) $\{a^\ell_{\varepsilon,n} : n \in \mathbb{Z},\ n < 0\}$ is unbounded from below in $\mathcal{U}^\ell_\varepsilon$.

This is justified by u_ℓ being μ-reasonable by $(*)_6$, $\boxtimes_4$. Now define $h_5 : I \to I$ by:
$h_5(x) = h_4(x)$ $\underline{\text{if}}$ $x \in I_{u_1}$ and otherwise
$h_5(x) = a^2_{\varepsilon,n} + (a^2_{\varepsilon,n+1} - a^2_{\varepsilon,n})(x - a^1_{\varepsilon,n})/(a^1_{\varepsilon,n+1} - a^1_{\varepsilon,n})$ $\underline{\text{if}}$ $a^1_{\varepsilon,n} \leq_{I_2} x < a^1_{\varepsilon,n+1}$ and $n \in \mathbb{Z}$. Now check using linear algebra.]

$(*)_{13}$ $({}^\mu I)/\mathcal{E}^{\text{aut}}_{I,\mu}$ has $\leq 2^\mu$ members, recalling that $f_1 \mathcal{E}^{\text{aut}}_{I,h} f_2$ $\underline{\text{iff}}$ f_1, f_2 are functions from μ into I and for some automorphism h of I we have

$$(\forall \alpha < \mu)[h \circ f_1(\alpha) = f_2(\alpha)].$$

[Why? Should be clear recalling $|I^1_u| \leq \mu$, recalling $(*)_5, (*)_{11}, (*)_{12}$.]

So we have finished proving part (1) of 5.1. $\square_{5.1(1)}$

Proof. **5.1(2)**

Really the proof is included in the proof of part (1). That is, given $I' \subseteq I$ of cardinality $< \theta_2$ by $(*)_5$ there is a μ-reasonable $u \subseteq \zeta$ such that $I' \subseteq I_u$ and $|u| = \mu + |I'|$. Now clearly

$(*)_{14}$ For μ-reasonable $u \subseteq \zeta$, the family $\{I^2_{u,x} : x \in I_2 \setminus I^2_u\}$ has $\leq \mu + |u|$ members.

[Why? By $\boxtimes_5$.]

$(*)_{15}$ for a μ-reasonable $u \subseteq \zeta$, the family $\{I_{u,x} : x \in I \setminus I_u\}$ has $\leq \mu$ members.

[Why? By $(*)_{16}$ below.]

$(*)_{16}$ if u is μ-reasonable then $I_{u,b_1} = I_{u,b_2}$ when
- (a) $b_k = f(a_{t_{k,0}}, \dots, a_{t_{k,n-1}})$ for $k = 1, 2$.
- (b) f a definable function in M.
- (c) $t_{k,0} <_{I_2} \dots <_{I_2} t_{k,n-1}$ for $k = 1, 2$.
- (d) $t_{1,\ell} \in I^2_u \vee t_{2,\ell} \in I^2_u \Rightarrow t_{1,\ell} = t_{2,\ell}$
- (e) if $t_{1,\ell} \notin I^2_u$ then $I^2_{u,t_{1,\ell}} = I^2_{u,t_{2,\ell}}$ for $\ell = 0, \dots, n-1$.

[Why? Use the proof of $(*)_{11}$, for $u_1 = u = u_2$, $h = \mathrm{id}_{u_2}$ so $\mathcal{U}^1_\varepsilon = \mathcal{U}^2_\varepsilon$ for $\varepsilon < \varepsilon^*$.

By the assumptions, for each ℓ there is ε such that $a_{t_{\varepsilon,1,\ell}}, a_{t_{2,\ell}} \in \mathcal{U}^1_\varepsilon = \mathcal{U}^2_\varepsilon$. Now for each $\varepsilon < \varepsilon^*$ there is an automorphism π_ε of $\mathcal{U}^1_\varepsilon$ as a linear order mapping $t_{1,\ell}$ to $t_{2,\ell}$ if $t_{1,\ell} \in \mathcal{U}^1_\varepsilon$. Let $\pi = \bigcup\{\pi_\varepsilon : \varepsilon < \varepsilon^*\} \cup \mathrm{id}_{I_u}$.]

$(*)_{17}$ If $n < \omega$, $t^\ell_0 <_I t^\ell_1 <_I \dots <_I t^\ell_{n-1}$ for $\ell = 1, 2$, and $I_{u,t^1_k} = I_{u,t^2_k}$ for $k = 0, 1, \dots, n-1$ <u>then</u> for some automorphism g of I over I_u we have $k < n \Rightarrow g(t^1_k) = t^2_k$.

[Why? We shall use g such that $g \restriction I_u = \mathrm{id}_{I_u}$ and $g \restriction I_{u,x}$ is an automorphism of $I_{u,x}$ for each $x \in I \setminus I_u$. Clearly it suffices to deal with the case

$$\{t^\ell_k : \ell < n \text{ and } \ell \in \{1, n\}\} \subseteq I_{u,x}$$

for one $x \in I \setminus I_u$.

[Obviously one of those is supposed to be a k.]

We choose $s_1 < s_2$ from $I_{u,x}$ such that $s_1 <_I t^\ell_k < s_2$ for $\ell = 1, 2$. We choose $g \restriction I_{u,x}$ such that it is the identity on $\{s \in I_{u,x} : s \leq_I s_1 \text{ or } s_2 \leq_I s\}$. Now stipulate $t_{-1} = s_1$, $t_n = s_2$ and $[g \restriction I_{u,x}]$ maps $(t^1_k, t^1_{k+1})_I$ onto $(t^2_k, t^2_{k+1})_I$ for $k = -1, 0, \dots, n-1$ as in the definition above.]

So we have completed the proof of part (2) of 5.1. $\square_{5.1(2)}$

Proof. **5.1(3)** Obvious from the Definition (0.14(9)) and the construction.

5.1(4) First

$\odot_1$ There is $J_1^* \subseteq I$ of cardinality μ^+ such that for every $J_2^* \subseteq I$ of cardinality $\leq \mu$, there is an automorphism π of I which maps J_2^* into J_1^*.

[Why? Let $u = \mu^+ \times \mu^+ \subseteq \zeta$ and let $J_1^* = I_u$. Clearly u has cardinality μ^+ and so does $J_1^* = I_u$. So suppose $J_2^* \subseteq I$ has cardinality $\leq \mu$. There is $u_2 \subseteq \zeta$ of cardinality μ such that $J_2^* \subseteq I_{u_2}$ and without loss of generality u_2 is reasonable. We define an increasing function h from u_2 into u_1, by defining $h(\alpha)$ by induction on α:

$(*)_{17}$ If $\mathrm{cf}(\alpha) \leq \mu$ then $h(\alpha) = \bigcup\{h(\beta) + 1 : \beta \in u_2 \cap \alpha\}$.

$(*)_{18}$ If $\mathrm{cf}(\alpha) > \mu$ then $h(\alpha) = \bigcup\{h(\beta) + 1 : \beta \in u_2 \cap \alpha\} + \mu^+$.

Let $u_1 := \{h(\alpha) : \alpha \in u_2\}$ so $u_1 \subseteq u$. Now h, u_1, u_2 satisfies clauses (a),(b),(c) of $(*)_{11}$ hence $h_1, h_2^*, h_2, h_3, h_4, \hat{I}_{u_1}, \hat{I}_{u_2}$ are as there.

By $(*)_{12}$ there is an isomorphism h^+ of I which extends h_4; now does h^+ map J_2^* into J_1^*? Yes, as $J_2^* \subseteq I_{u_2}$ and $h^+ \restriction I_{u_2}$ is an isomorphism from I_{u_2} onto I_{u_1} but $I_{u_1} \subseteq I_u$ and $I_u = J_1^*$, so we are done proving $\odot_1$.]

Finally

$\odot_2$ Part (4) of 5.1 holds. I.e., if $I_0^* \subseteq I$ with $|I_0^*| < \theta_2$ <u>then</u> for some $I_1^* \subseteq I$ of cardinality $\leq \mu^+ + |I_0^*|$, for every $J \subseteq I$ of cardinality $\leq \mu$, there is an automorphism of I over I_0^* mapping J into I_1^*.

Why? Given $I_0^* \subseteq I$ of cardinality $< \theta_2$ we can find $u_1 \subseteq \zeta$ of cardinality $\mu + |I_0^*|$ such that $I_0^* \subseteq I_{u_1}$. By $(*)_5$ we can find a μ-reasonable set $u_2 \subseteq \zeta$ of cardinality $\mu + |u_1|$ such that $u_1 \subseteq u_2$.

Let $\langle \mathcal{U}_\varepsilon : \varepsilon < \varepsilon^* \rangle$ list the sets of the form $I_{u_2,x}$, $x \in I_2 \backslash I_{u_1}$, so by $\boxdot_5$, $\varepsilon^* \leq \mu + |I_0^*|$. For each ε we choose $\langle a_{\varepsilon,n} : n \in \mathbb{Z} \rangle$ as in $\circledast_{13}$ from the proof of $(*)_{12}$. For each $\varepsilon < \varepsilon^*$ and $n \in \mathbb{Z}$ let $\pi_{\varepsilon,n}$ be an isomorphism from I onto $(a_{\varepsilon,n}, a_{\varepsilon,n+1})_I$; it exists by the properties of ordered fields. Let $J_1^* \subseteq I$ be as in $\odot_1$ above and let

$$I_2^* = I_1^* \cup \{a_{\varepsilon,n} : \varepsilon < \varepsilon^* \text{ and } n < \omega\} \cup \{\pi_{\varepsilon,n}(J_1^*) : \varepsilon < \varepsilon^* \text{ and } n \in \mathbb{Z}\}.$$

Easily, I_2^* is as required.

5.1(5) By 0.12. $\square_{5.1(3)-(5)}$

Remark 5.3. Concerning $(*)_{11}$, we could have used more time.

$(*)'_{11}$ $h_2 : I_{u_1}^2 \to I_{u_2}^2$ is an order preserving function and onto, $h_3 : I_{u_1} \to I_{u_2}$ is an isomorphism, and $h_1 : \hat{I}_{u_2} \to \hat{I}_{u_2}$ is order preserving and onto.

§ 6. Linear orders and equivalence relations

This section deals with a relative of the stability spectrum. We ask: what can be the number of equivalence classes in ${}^{\mu}I$ for an equivalence relation on ${}^{\mu}I$ which is so called "invariant": in fact definable (essentially by a quantifier free infinitary formula, mainly for well ordered I).

It is done in a very restricted context, but via EM-models has useful conclusions, for AEC and also for AEC with amalgamation; i.e. it is used in 7.9.

There are two versions; one for well ordering and one for the class of linear orders both expanded by unary relations.

On $\tau^*_{\alpha(*)}, K^{\rm lin}_{\tau^*_{\alpha(*)}}$ see 0.14(4). We may replace sequences, i.e. ${\rm inc}_J(I)$, by subsets of I of cardinality $|J|$, this may help to eliminate $2^{|J|}$ later, but at present it seems not to help in the final bounds in §7. We do here only enough for §7.

Context 6.1. We fix $\alpha(*)$, $\bar{u}^* = (u^-, u^+)$ such that

(a) $\alpha(*)$ is an ordinal ≥ 1

(b) $u^- \subseteq \alpha(*)$

(c) $u^+ \subseteq \alpha(*)$.

Remark 6.2. 1) The main cases are

(A) $\alpha(*) = 1$, so $K^{\rm lin}_{\tau^*_{\alpha(*)}}$ is the class of linear orders

(B) $\alpha(*) = 2$, $u^+ = \varnothing$, $u^- = \{0\}$.

2) Usually the choice of the parameters does not matter.

Definition 6.3. 1) For $I, J \in K^{\rm lin}_{\tau^*_{\alpha(*)}}$, i.e. both linear orders expanded by a partition $P_\alpha(\alpha < \alpha(*))$, pedantically the interpretation of the P_α's, let ${\rm inc}'_J(I)$ be the set of embedding of J into I; see below, we denote members by h.

2) Recalling $\bar{u}^* = (u^-, u^+)$ where $u^- \cup u^+ \subseteq \alpha(*)$ let ${\rm inc}^{\bar{u}^*}_J(I)$ be the set of h such that

(a) h is an embedding of J into I, i.e. a one-to-one, order preserving function mapping P^J_α into P^I_α for $\alpha < \alpha(*)$.

(b) If $\alpha \in u^-$, $t \in P^J_\alpha$, and $s <_I h(t)$ <u>then</u> for some $t_1 <_J t$ we have $s \leq_I h(t_1)$.

(c) If $\alpha \in u^+$, $t \in P^J_\alpha$, and $h(t) <_I s$ <u>then</u> for some t_1 we have $t <_J t_1$ and $h(t_1) \leq_I s$.

Concerning $\bar{u}^*$

Observation 6.4. *1) For any $h \in {\rm inc}^{\bar{u}^*}_J(I)$:*

(A) If t is the successor of s in J (i.e. $s <_J t$ and $(s,t)_J = \varnothing$) and $t \in P^J_\alpha$, $\alpha \in u^-$ <u>then</u> $h(t)$ is the successor of $h(s)$ in I.

(B) if $\langle t_i : i < \delta\rangle$ is $<_J$-increasing with limit $t_\delta \in J$ (i.e. $i < \delta \Rightarrow t_i <_J t_\delta$ and $\varnothing = \bigcap\{(t_i, t_\delta)_J : i < \delta\}$) and $t_\delta \in P^J_\alpha$, $\alpha \in u^-$ <u>then</u> $\langle h(t_i) : i < \delta\rangle$ is $<_I$-increasing with limit $h(t_\delta)$ in I.

(C) If t is the first member of J and $t \in P^J_\alpha$, $\alpha \in u^-$ <u>then</u> $h(t)$ is the first member of I.

2) If $h_1, h_2 \in {\rm inc}^{\bar{u}^*}_J(I)$ <u>then</u>

(A) If t is the successor of s in J and $t \in P^J_\alpha$, $\alpha \in u^-$ <u>then</u> $h_1(s) = h_2(s) \Leftrightarrow h_1(t) = h_2(t)$ and $h_1(s) <_I h_2(s) \Leftrightarrow h_1(t) <_I h_2(t)$ and $h_1(s) >_I h_2(s) \Leftrightarrow h_1(t) >_I h_2(t)$.

(B) If $\langle t_i : i < \delta \rangle$ *is* $<_J$*-increasing with limit* t_δ *and* $t_\delta \in P^J_\alpha$, $\alpha \in u^-$, *<u>then</u>*

$$(\forall i < \delta)[h_1(t_i) = h_2(t_i)] \Rightarrow h_1(t_\delta) = h_2(t_\delta).$$

Moreover,

$$(\forall i < \delta)(\exists j < \delta)\big[h_1(t_i) <_I h_2(t_j) \wedge h_2(t_i) <_I h_1(t_j)\big] \Rightarrow h_1(t_\delta) = h_2(t_\delta)$$

and also $(\exists j < \delta)(\forall i < \delta)(h_1(t_i) <_I h_2(t_j)) \Rightarrow h_1(t_\delta) <_I h_2(t_\delta)$.

3) Similar to parts (1) + (2) for $\alpha \in u^+$ *(inverting the orders of course).*

4) $\mathrm{inc}'_I(J) = \mathrm{inc}^{(\varnothing,\varnothing)}_I(J)$.

Proof. Straight (and see the proof of 6.7). $\square_{6.4}$

Convention 6.5. 1) $\alpha(*), \bar{u}^*$ will be constant, so usually we shall not mention them (e.g. we will write $\mathrm{inc}_J(I)$ for $\mathrm{inc}^{\bar{u}^*}_I(I)$). Pedantically, below we should have written $\mathbf{e}^{\bar{u}^*}(J, I)$ and $\mathbf{e}^{\bar{u}^*}_*(J)$, and also in notions like 'reasonable' and 'wide' in Definition 6.10 which mention $\bar{u}^*$.

2) I, J will denote members of $K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$.

Below we use mainly "e-pairs" (and weak e-pairs and the reasonable case).

Definition 6.6. 1) let $\mathbf{e}(J)$ be the set of equivalence relations on some subset of J such that each equivalence class is a convex subset of J.

2) For $h_1, h_2 \in \mathrm{inc}_J(I)$ we say that (h_1, h_2) is a strict e-pair (for (I, J)) <u>when</u> $e \in \mathbf{e}(J)$ and (h_1, h_2) satisfies

(a) $s \in J \setminus \mathrm{dom}(e)$ iff $h_1(s) = h_2(s)$.

(b) If $s <_J t$ and $s/e \neq t/e$ (so $s, t \in \mathrm{dom}(e)$) <u>then</u> $h_1(s) <_I h_2(t)$ and $h_2(s) <_I h_1(t)$.

(c) If $s <_J t$ and $s/e = t/e$ (so $s, t \in \mathrm{dom}(e)$) <u>then</u> $h_1(t) <_I h_2(s)$.

2A) We say that (h_1, h_2) is a strict $(e, \mathcal{Y})$-pair, where $e \in \mathbf{e}(J)$ and $\mathcal{Y} \subseteq \mathrm{dom}(e)/e$, <u>when</u> clauses (a)+(b) from part (2) hold and

(c)′ if $s <_J t$ and $s/e = t/e$ (so $s, t \in \mathrm{dom}(e)$) <u>then</u>

$$[(h_1(t) <_I h_2(s)] \equiv [s/e \in \mathcal{Y}] \equiv [h_1(s) < h_2(t)].$$

2B) We say that (h_1, h_2) is an e-pair when (h_1, h_2) is a strict $(e, \mathcal{Y})$-pair for some $\mathcal{Y}$. (This relation is symmetric, see below.)

3) We say that (h_1, h_2) is a weak e-pair where $h_1, h_2 \in \mathrm{inc}_J(I)$ <u>when</u> clauses (a),(b) hold. (This, too, is symmetric!)

4) For $h_1, h_2 \in \mathrm{inc}_J(I)$, let $e = \mathbf{e}(h_1, h_2)$ be the (unique) $e \in \mathbf{e}(J)$ such that (see 6.8(1) below)

(a) $\mathrm{dom}(e) = \{s \in J : h_1(s) \neq h_2(s)\}$

(b) (h_1, h_2) is a weak e-pair.

(c) If $e' \in \mathbf{e}(J)$ and (h_1, h_2) is a weak e'-pair <u>then</u> $\mathrm{dom}(e) \subseteq \mathrm{dom}(e')$ and e refines $e' \restriction \mathrm{dom}(e)$.

5) If $e \in \mathbf{e}(J)$ and $\mathcal{Y} \subseteq \mathrm{dom}(e)/e$ <u>then</u> we let $\mathrm{set}(\mathcal{Y}) = \{s \in J : s/e \in \mathcal{Y}\}$ and $e \restriction \mathcal{Y} = e \restriction \mathrm{set}(\mathcal{Y})$.

6) Let $\mathbf{e}(J, I)$ be the set of $e \in \mathbf{e}(J)$ such that there is an e-pair.

7) Let $\mathbf{e}_*(J) = \bigcup\{\mathbf{e}(J, I) : I \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}\}$.

Concerning $\bar{u}^*$

Observation 6.7. *Assume that* $e \in \mathbf{e}(J, I)$.

0) (a) If t *is the first member of* J *and* $t \in P^J_\alpha$, $\alpha \in u^-$ *<u>then</u>* $t \notin \mathrm{dom}(e)$.

(b) If $t \in \mathrm{dom}(e)$ *and* t *is the first member of* t/e *and* $t \in P^J_\alpha$ *<u>then</u>* $\alpha \notin u^-$.

1) *If t is the $<_J$-successor of s and $t \in P^J_\alpha$, $\alpha \in u^-$ <u>then</u> $s \in \mathrm{dom}(e) \Leftrightarrow t \in \mathrm{dom}(e)$ and $s \in \mathrm{dom}(e) \Rightarrow s \in t/e$.*
2) *If $\langle t_i : i < \delta\rangle$ is $<_J$-increasing with limit t_δ and $t_\delta \in P^J_\alpha$ and $\alpha \in u^-$ <u>then</u>:*
   (a) *If $(\forall i < \delta)[t_i \notin \mathrm{dom}(e)]$ <u>then</u> $t_\delta \notin \mathrm{dom}(e)$.*
   (b) *If $(\forall i < \delta)(\neg t_i\ e\ t_{i+1})$ or just $(\forall i < \delta)(\exists j < \delta)[i < j \wedge \neg t_i\ e\ t_j]$ <u>then</u> $t_\delta \notin \mathrm{dom}(e)$.*
   (c) *If $(\forall i < \delta)(t_i \in t_0/e)$ <u>then</u> $t_\delta \in t_0/e$.*
3) *Similar to parts (0),(1),(2) when $\alpha \in u^+$ (inverting the order, of course).*
4) *$\mathbf{e}_*(J)$ is the family of $e \in \mathbf{e}(J)$ satisfying the requirements in parts (0),(1),(2),(3) above so if $\bar u^* = (\varnothing, \varnothing)$ then $\mathbf{e}_*(J) = \mathbf{e}(J)$.*

Proof. Easy by 6.4, e.g.

Part (1): We are assuming $e \in \mathbf{e}(J, I)$ hence by Definition 6.6 there is an e-pair (h_1, h_2) where $h_1, h_2 \in \mathrm{inc}_J(I)$. Now for $\ell = 1, 2$, clearly $h_\ell(s), h_\ell(t) \in I$ and as $s <_J t$ we have $h_\ell(s) < h_\ell(t)$. Now if $h_\ell(t)$ is not the $<_I$-successor of $h_\ell(s)$ then there is $s'_\ell \in (h_\ell(s), h_\ell(t))_I$ hence by clause (b) of Definition 6.3(2) there is $s^*_\ell \in [s, t)_J$ such that $s'_\ell \le_I h_\ell(s^*_\ell) <_I h_\ell(t)$ so as $h_\ell(s) <_I s'_\ell$ we have $h_\ell(s) <_I h_\ell(s^*_\ell) <_I h_\ell(t)$ hence $s <_I s^*_\ell <_J t$, contradiction to the assumption "t is the successor of s in J". So indeed $h_\ell(t)$ is the successor of $h_\ell(s)$ in I.

As this holds for $\ell = 1, 2$, clearly $h_1(s) = h_2(s) \Leftrightarrow h_1(t) = h_2(t)$ but by Definition 6.3(2) we know $s \in \mathrm{dom}(e) \Leftrightarrow [h_1(s) \neq h_2(s)]$ and similarly for t hence $s \in \mathrm{dom}(e) \Leftrightarrow t \in \mathrm{dom}(e)$. Lastly, assume $s, t \in \mathrm{dom}(e)$, but s, t are nor e-equivalent so by Definition 6.6(2) clause (b) we have $h_1(s) <_I h_2(t) \wedge h_2(s) <_I h_1(t)$ clear contradiction.

Part (2): We leave clauses (a),(b) to the reader.

For clause (c) of part (2), if $t_\delta \notin t_0/e$ then choose $h_1, h_2 \in \mathrm{inc}^{\bar u^*}_J(I)$ such that (h_1, h_2) is an e-pair, hence an $(e, \mathcal{Y})$-pair for some $\mathcal{Y} \subseteq \mathrm{dom}(e)/e$. If $(t_0/e) \in \mathcal{Y}$ then $h_2(t_0)$ is above $\{h_1(t_i) : i < \delta\}$ by $<_I$ so we have $h_1(t_\delta) \le_I h_2(t_0)$ but if $t_\delta \notin t_0/e$ this contradicts clause (b) in Definition 6.6(2),(2A). The proof when $t_0/e \notin \mathcal{Y}$ is similar. $\square_{6.7}$

Observation 6.8. *Let $h_1, h_2 \in \mathrm{inc}_J(I)$ and $e \in \mathbf{e}(J)$.*

1) *$\mathbf{e}(h_1, h_2)$ is well defined.*

2) *(h_1, h_2) is a strict $(e, \mathcal{Y}_1)$-pair iff (h_2, h_1) is a strict $(e, \mathcal{Y}_2)$-pair when $(\mathcal{Y}_1, \mathcal{Y}_2)$ is a partition of $\mathrm{dom}(e)/e$.*

3) *(h_1, h_2) is a strict e-pair <u>iff</u> (h_2, h_1) is a strict $(e, \varnothing)$-pair.*

4) *(h_1, h_2) is an e-pair <u>iff</u> (h_2, h_1) is an e-pair.*

5) *(h_1, h_2) is a weak e-pair iff (h_2, h_1) is a weak e-pair.*

6) *If (h_1, h_2) is a strict e-pair <u>then</u> (h_1, h_2) is an e-pair which implies (h_1, h_2) being a weak e-pair.*

7) *If $e_\alpha \in \mathbf{e}(J)$ for $\alpha < \alpha^*$, <u>then</u>*

$$e := \bigcap\{e_\alpha : \alpha < \alpha^*\} = \{(s, t) : s, t \text{ are } e_\alpha\text{-equivalent for every } \alpha < \alpha^*\}$$

belongs to $\mathbf{e}(J)$ with $\mathrm{dom}(e) = \bigcap\{\mathrm{dom}(e_\alpha) : \alpha < \alpha^*\}$.

8) *If $e \in \mathbf{e}(J, I)$ <u>then</u> for every $\mathcal{Y} \subseteq \mathrm{dom}(e)/e$ also $e \restriction \mathrm{set}(\mathcal{Y})$ belongs to $\mathbf{e}(J, I)$ and there is a strict $(e \restriction \mathrm{set}(\mathcal{Y}))$-pair (h'_1, h'_2); moreover, for every $\mathcal{Y}_1 \subseteq \mathcal{Y}$ there is a strict $(e \restriction \mathrm{set}(\mathcal{Y}), \mathcal{Y}_1)$-pair.*

Proof. Easy, e.g.:

1) Let

$$e = \Big\{(s_1, s_2) : h_1(s_\ell) \neq h_2(s_\ell) \text{ for } \ell = 1,2 \text{ and if } s_1 \neq s_2 \text{ then for some } t_1 <_J t_2 \text{ we have } \{s_1, s_2\} = \{t_1, t_2\} \text{ and there is no initial segment } J' \text{ of } J \text{ such that } J' \cap \{t_1, t_2\} = \{t_1\} \text{ and } (\forall t' \in J')(\forall t'' \in J \setminus J')\big[h_1(t') <_I h_2(t'') \wedge h_2(t') <_I h_1(t'')\big]\Big\}.$$

Clearly e is an equivalence relation on $\{t \in J : h_1(t) \neq h_2(t)\}$ and each equivalence class is convex hence $e_1 \in \mathbf{e}(J)$, so clauses (a),(b) of 6.6(1),(4) holds. Easily e is as required.

8) Let (h_1, h_2) be an e-pair and $\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3$ be a partition of $\mathrm{dom}(e)/e$. We define $h'_1, h'_2 \in \mathrm{inc}_J(I)$ as follows, for $\ell \in \{1,2\}$

(a) If $t \in J \setminus \mathrm{dom}(e)$ then $h'_\ell(t) = h_1(t)\,(= h_2(t))$.
(b) If $t \in \mathrm{set}(\mathcal{Y}_1)$ then $h'_\ell(t) = h_1(t)$.
(c) If $t \in \mathrm{set}(\mathcal{Y}_2)$ then $h'_\ell(t)$ is $\min\{h_1(t), h_2(t)\}$ if $\ell = 1$, and is $\max\{h_1(t), h_2(t)\}$ if $\ell = 2$.
(d) If $t \in \mathrm{set}(\mathcal{Y}_3)$ then $h'_\ell(t)$ is $\max\{h_1(t), h_2(t)\}$ if $\ell = 1$ and is $\min\{h_1(t), h_2(t)\}$ if $\ell = 2$.

Now (h'_1, h'_2) is a strict $(e \restriction (\mathrm{set}(\mathcal{Y}_2) \cup \mathrm{set}(\mathcal{Y}_3)), \mathcal{Y}_2)$-pair, so we are done. $\square_{6.8}$

Definition 6.9. 1) For a subset u of $J \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$ we define $e = e_{J,u} \in \mathbf{e}(J)$ on $J \setminus u$ as follows:

$$s_1 \; e \; s_2 \Leftrightarrow (\forall t \in u)[t <_J s_1 \equiv t <_J s_2].$$

2) For $I, J \in K^{\mathrm{lin}}_{\alpha(*)}$, we say that the pair (I, J) is non-trivial <u>when</u> $\mathbf{e}(J, I) \neq \varnothing$.

Definition 6.10. 1) For $h_0, \ldots, h_{n-1} \in \mathrm{inc}_J(I)$ let

$$\mathrm{tp}^J_{\mathrm{qf}}\big(\langle h_0, \ldots, h_{n-1}\rangle, I\big) = \big\{(\ell, m, s, t) : s, t \in J \text{ and } h_\ell(s) < h_m(t)\big\}.$$

We may write $\mathrm{tp}^J_{\mathrm{qf}}(h_0, \ldots, h_{n-1}; I)$ and we usually omit J as it is clear from the context.

2) For $h_1, h_2 \in \mathrm{inc}_J(I)$ let $\mathrm{eq}(h_1, h_2) = \{s \in J : h_1(s) = h_2(s)\}$.

3) We say that the pair (I, J) is a *reasonable* $(\mu, \alpha(*))$*-base* <u>when</u>:

(a) $I, J \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$, $|J| \leq \mu$, and the pair (I, J) is non-trivial.
(b) If $e \in \mathbf{e}(J, I)$, $h_1, h_2 \in \mathrm{inc}_J(I)$, and (h_1, h_2) is an e-pair then we can find $h'_1, h'_2, h'_3 \in \mathrm{inc}_J(I)$ and $\mathcal{Y} \subseteq \mathrm{dom}(e)/e$ such that
  (α) $\mathrm{tp}_{\mathrm{qf}}((h'_1, h'_2), I) = \mathrm{tp}_{\mathrm{qf}}((h_1, h_2), I)$
  (β) (h'_1, h'_3) and (h'_2, h'_3) are strict $(e, \mathcal{Y})$-pairs.

4) We say that the pair (I, J) is a *wide* $(\lambda, \mu, \alpha(*))$-base <u>when</u>:

(a) $I, J \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$, $|J| \leq \mu$, and the pair (I, J) is non-trivial.
(b) for every $e \in \mathbf{e}(J, I)$ there is a sequence $\bar{h} = \langle h_\alpha : \alpha < \lambda\rangle$ such that
  (α) h_α is an embedding of J into I.
  (β) If $\alpha < \beta < \lambda$ then (h_α, h_β) is an e-pair.

5) We say that the pair (I, J) is a *strongly wide* $(\lambda, \mu, \alpha(*))$-base <u>when</u>:

(a) $I, J \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$, the pair (I, J) is non-trivial, and J has cardinality $\leq \mu$.
(b) For every $e \in \mathbf{e}(J, I)$ and $\mathcal{Y} \subseteq \mathrm{dom}(e)/e$ there is $\bar{h} = \langle h_\alpha : \alpha < \lambda\rangle$ such that
  (α) $h_\alpha \in \mathrm{inc}_J(I)$
  (β) If $\alpha < \beta$ then (h_α, h_β) is a strict $(e, \mathcal{Y})$-pair.

6) Above we may omit μ (meaning $\mu = |J|$) and we may omit $\alpha(*)$, as it is determined by J (and by I), and then may omit "base." So in part (3) we say (I, J) is reasonable, in part (4) we say λ-wide, and in part (5) say strongly λ-wide.

Observation 6.11. *1) If (I, J) is a reasonable $(\mu, \alpha(*))$-base <u>then</u> (I, J) is a reasonable $(\mu', \alpha(*))$-base for $\mu' \geq \mu$.*

2) If (I, J) is a wide $(\lambda, \mu, \alpha(*))$-base and $\lambda' \leq \lambda$, $\mu' \geq \mu$ <u>then</u> (I, J) is a wide $(\lambda', \mu', \alpha(*))$-base.

3) If (I, J) is a strongly wide $(\lambda, \mu, \alpha(*))$-base, <u>then</u> (I, J) is a wide $(\lambda, \mu, \alpha(*))$-base.

Proof. Obvious. $\square_{6.11}$

Claim 6.12. *1) If $\alpha(*) = 1$ and $\mu \leq \zeta(*) < \mu^+ \leq \lambda$, <u>then</u> the pair $(\lambda \times \zeta(*), \zeta(*))$ is a reasonable $(\mu, \alpha(*))$-based which is a wide $(\lambda, \mu, \alpha(*))$-base.*

2) If $\alpha(*) = 2$, $\bar{u}^* = (\{0\}, \varnothing)$ as in 6.2, $\mu \leq \zeta(*) < \mu^+ < \lambda$, $\zeta'(*) = \zeta(*) \times 3$, and $w \subseteq \zeta(*)$, $w \neq \zeta(*)$ <u>then</u> the pair $(I^{\rm lin}_{\mu, \lambda \times \zeta(*)}, I^{\rm lin}_{\mu, \zeta(*), w})$ is a reasonable $(\mu, \alpha(*))$-base [and] a wide $(\lambda, \mu, \alpha(*))$-base where

- $(*)$ *For any ordinal β and $w \subseteq \beta$ we define $I = I^{\rm lin}_{\mu, \beta, w}$, a $\tau^*_{\alpha(*)}$-model. (If $w = \varnothing$ we may omit it.)*
  - (α) *Its universe is β.*
  - (β) *The order is the usual one.*
  - (γ) *$P^I_1 = \{\alpha < \beta : {\rm cf}(\alpha) > \mu$ or $\alpha \in w\}$.*
    (If we write $I^{\rm lin}_{\geq \mu, \beta, w}$ we mean here ${\rm cf}(\alpha) \geq \mu$.)

Proof. 1) **First:** <u>$(I, J) = (\lambda \times \zeta(*), \zeta(*))$ **is a wide** $(\lambda, \mu, \alpha(*))$**-base**</u>

Easily, $\mathbf{e}(J, I) \neq \varnothing$, $|J| \leq \mu$ and $I, J \in K^{\rm lin}_{\tau^*_{\alpha(*)}}$, so clause (a) of Definition 6.10(4) holds (recalling Definition 6.9(2)), so it suffices to deal with clause (b).

Let $e \in \mathbf{e}(J, I)$ and define

$$u = \big\{\zeta < \zeta(*) : \zeta \in {\rm dom}(e) \text{ is minimal in } \zeta/e \text{ or } \zeta \in \zeta(*) \setminus {\rm dom}(e)\big\}.$$

Now for every $\alpha < \lambda$ we define $h_\alpha \in {\rm inc}_J(I)$ as follows:

(a) If $\zeta \in \zeta(*) \setminus {\rm dom}(e)$ then $h_\alpha(\zeta) = \lambda \times \zeta$.

(b) If $\zeta \in {\rm dom}(e)$ and $\varepsilon = \min(\zeta/e)$ then $h_\alpha(\zeta) = \lambda \times \varepsilon + \zeta(*) \times \alpha + \zeta$.

Second: <u>$(I, J) = (\lambda \times \zeta(*), \zeta(*))$ **is a reasonable** $(\mu, \alpha(*))$**-base**</u>

Again, clause (a) of Definition 6.10(3) holds so we deal with clause (b).

So assume $e \in \mathbf{e}(J, I)$, $h_1, h_2 \in {\rm inc}_J(I)$, (h_1, h_2) is just a weak e-pair, and $\mathcal{Y} \subseteq {\rm dom}(e)/e$. Let $u = {\rm rang}(h_1) \cup {\rm rang}(h_2)$. For $\ell = 1, 2$ let $h^*_\ell \in {\rm inc}_J(I)$ be $h^*_\ell(\zeta) = {\rm otp}(u \cap h_\ell(\zeta))$, so ${\rm rang}(h^*_\ell) \subseteq \xi(*) := {\rm otp}(u) \leq \zeta(*) \times 3$.

[Why? If $\zeta(*)$ is finite this is trivial, so assume $\zeta(*) \geq \omega$. Let $n < \omega$ and α be such that $\omega^\alpha n \leq \zeta(*) < \omega^\alpha(n+1)$, so $\alpha \geq 1$ and $n \geq 1$. As ω^α is additively indecomposable, ${\rm otp}(u) \leq \omega^\alpha(2n+1)$: alternatively, use natural sums [MR65] which give a better bound $\zeta(*) \oplus \zeta(*)$. **[Actually, $< \mu^+$ suffices using $\zeta(*) < \mu^+$ large enough below, still.]**]

For $\ell = 1, 2, 3$ we define $h'_\ell \in {\rm inc}_J(I)$ as follows:

(a) If $\zeta \in \zeta(*) \setminus {\rm dom}(e)$ then $h'_\ell(\zeta) = (\zeta(*) \times 4) \times \zeta$.

(b) If $\zeta \in {\rm dom}(e)$ and $\varepsilon = \min(\zeta/e)$ and $\zeta/e \in \mathcal{Y}$ then:
   - (α) If $\ell = 3$ then $h'_\ell(\zeta) = (\zeta(*) \times 4) \times \varepsilon + \zeta(*) \times 3 + \zeta$.
   - (β) If $\ell = 1, 2$ then $h'_\ell(\zeta) = (\zeta(*) \times 4) \times \varepsilon + h^*_\ell(\zeta)$.

(c) If $\zeta \in {\rm dom}(e)$ and $\varepsilon = \min(\zeta/e)$ and $\zeta/e \notin \mathcal{Y}$ then:

(α) If $\ell = 3$ then $h'_\ell(\zeta) = (\zeta(*) \times 4) \times \varepsilon + \zeta$.

(β) If $\ell = 1, 2$ then $h'_\ell(\zeta) = (\zeta(*) \times 4) \times \varepsilon + \zeta(*) + h^*_\ell(\zeta)$.

Now check.

2) **First:** $\underline{(I, J) = (I^{\rm lin}_{\mu,\lambda\times\zeta(*)}, I^{\rm lin}_{\mu,\zeta(*),w})}$ **is a wide** $(\lambda, \mu, \alpha(*))$**-base.**

Note that $P^J_1 = w$ because $\zeta(*) < \mu^+$ and $P^I_1 = \{\alpha \in I : {\rm cf}(\alpha) > \mu\}$. As above, clause (a) of the Definition 6.10 holds so we deal with clause (b).

Let

$$u = \big\{\zeta < \zeta(*) : \zeta \in {\rm dom}(e) \text{ is minimal in } \zeta/e \text{ or } \zeta \in \zeta(*) \setminus {\rm dom}(e)\big\}.$$

Clearly u is a closed subset of $\zeta(*)$ and $0 \in u$.

Given $\zeta < \zeta(*)$, let $\varepsilon_\zeta := \max(u \cap (\zeta + 1))$; clearly this is well defined by the choice of u and $\varepsilon_\zeta \le \zeta$.

For every $\alpha < \lambda$ we define $h_\alpha \in {\rm inc}_J(I)$ as follows:

We define $h_\alpha(\zeta)$ by induction on $\zeta < \zeta(*)$ such that $h_\alpha(\zeta) < \lambda \times (\varepsilon_\zeta + 1)$.

Case A: for $\zeta \in \zeta(*) \setminus {\rm dom}(e)$.

Subcase A1: $\zeta \in P^J_1$
Let $h_\alpha(\zeta)$ be $\lambda \times \varepsilon_\zeta + \mu^+$.

Subcase A2: $\zeta \in P^J_0$ and $\zeta = 0$.
Let $h_\alpha(\zeta) = 0$.

Subcase A3: $\zeta \in P^J_0$, $\zeta = \xi + 1$.
Let $h_\alpha(\zeta) = h_\alpha(\xi) + 1$.

Subcase A4: $\zeta \in P^J_0, \zeta$ is a limit ordinal, $\zeta = \sup(u \cap \zeta)$.
Let $h_\alpha(\zeta) = \lambda \times \varepsilon_\zeta$ which is equal to $\bigcup\{h_\alpha(\zeta') : \zeta' < \zeta\}$.

Subcase A5: $\zeta \in P^J_0$, ζ is a limit ordinal, and $\xi = \sup(u \cap \zeta) < \zeta$.
So $(\xi + 1)/e$ is an end-segment of ζ, but this is impossible by 6.7(2)(c).

Case B: $\zeta \in {\rm dom}(e)$.

Subcase B1: $\zeta = \min(\zeta/e)$ hence $\zeta \in P^J_1$ (see 6.7(0)(b)).
Let $h_\alpha(\zeta) = \lambda \times \varepsilon_\zeta + \mu^+ \times \zeta(*) \times \alpha + \mu^+$.

Subcase B2: $\zeta \in P^J_0$ hence $\zeta > \min(\zeta/e)$.
Let $h_\alpha(\zeta) = \bigcup\{h_\alpha(\zeta') + 1 : \zeta' < \zeta\}$.

Subcase B3: $\zeta \in P^J_1$ and $\zeta > \min(\zeta/e)$.
Let $h_\alpha(\zeta) = \bigcup\{h_\alpha(\zeta') : \zeta' < \zeta\} + \mu^+$.

So clearly we can show by induction on $\zeta < \zeta(*)$ that

$$h_\alpha(\zeta) < \lambda \times \varepsilon_\zeta + \mu^+ \times \zeta(*) \times (\alpha 2 + 2).$$

Also, recalling $\mu^+ < \lambda$, clearly for $\alpha < \lambda$ and $\zeta < \zeta(*)$ we have $h_\alpha(\zeta) < \lambda \times \varepsilon_\zeta + \lambda$.

Now check.

Second: $\underline{(I^{\rm lin}_{\mu,\lambda\times\zeta(*)}, I^{\rm lin}_{\mu,\zeta(*),w})}$ **is a reasonable** $(\mu, \alpha(*))$**-base.**

Combine the proof of "First" with the parallel proof in part (1). $\square_{6.12}$

Definition 6.13. 1) Let $I, J \in K^{\rm lin}_{\tau^*_{\alpha(*)}}$. We say that $\mathcal{E}$ is an invariant (I, J)-equivalence relation $\underline{\text{when}}$:

(a) $\mathcal{E}$ is an equivalence relation on ${\rm inc}_J(I)$, so $\mathcal{E}$ determines I and J,

(b) If $h_1, h_2, h_3, h_4 \in {\rm inc}_J(I)$ and ${\rm tp}_{\rm qf}(h_1, h_2; I) = {\rm tp}_{\rm qf}(h_3, h_4; I)$ then

$$h_1 \ \mathcal{E}\ h_2 \Leftrightarrow h_3 \ \mathcal{E}\ h_4.$$

2) We say it is also non-trivial $\underline{\text{when}}$:

(c) If ${\rm eq}(h_1, h_2) = \{t \in J : h_1(t) = h_2(t)\}$ is co-finite then $h_1 \ \mathcal{E}\ h_2$.

(d) There are $h_1, h_2 \in {\rm inc}_J(I)$ such that $\neg(h_1 \ \mathcal{E}\ h_2)$.

3) Let $J, I_1, I_2 \in K^{\rm lin}_{\tau^*_{\alpha(*)}}$. $\underline{\text{Then}}$ $I_1 \le^1_J I_2$ means that:

(a) $I_1 \subseteq I_2$

(b) For every $h_1, h_2, h_3 \in \mathrm{inc}_J(I_2)$ we can find $h'_1, h'_2, h'_3 \in \mathrm{inc}_J(I_1)$ such that $\mathrm{tp}_{\mathrm{qf}}(h'_1, h'_2, h'_3; I_1) = \mathrm{tp}_{\mathrm{qf}}(h_1, h_2, h_3; I_2)$.

Claim 6.14. *Assume $J, I_1, I_2 \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$.*

1) If $I_1 \subseteq I_2, \mathcal{E}$ is an invariant (I_2, J)-equivalence relation <u>then</u> $\mathcal{E} \restriction \mathrm{inc}_J(I_1)$ is an invariant (I_1, J)-equivalence relation.

2) If $I_1 <^1_J I_2$ and $\mathcal{E}_1$ is an invariant (I_1, J)-equivalence relation <u>then</u> there is one and only one invariant (I_2, J)-equivalence relation $\mathcal{E}_2$ such that $\mathcal{E}_2 \restriction \mathrm{inc}_J(I_1) = \mathcal{E}_1$.

3) Assume $e \in \mathbf{e}(J)$ and $\mathcal{Y} \subseteq \mathrm{dom}(e)/e$. If (h'_1, h'_2) is a strict $(e, \mathcal{Y})$-pair for (I_1, J) and (h''_1, h''_2) is a strict $(e, \mathcal{Y})$-pair for (I_2, J) <u>then</u>

$$\mathrm{tp}_{\mathrm{qf}}(h'_1, h'_2; I_1) = \mathrm{tp}_{\mathrm{qf}}(h''_1, h''_2; I_2).$$

4) Assume $\alpha(*) = 1$, $J = \zeta(*)$, $I_\ell = \beta_\ell$ with the usual order (for $\ell = 1, 2$), $\mu \leq \zeta(*) < \mu^+$, and $\mu^+ \leq \beta_1 \leq \beta_2$. <u>Then</u> $I_1 <^1_J I_2$ (see Definition 6.13(3)).

5) Assume $\alpha(*) = 2$, $J = I^{\mathrm{lin}}_{\mu,\zeta(*),w}$, $I_\ell = I^{\mathrm{lin}}_{\mu,\beta_\ell}$ for $\ell = 1, 2$, and $\mu^{++} \leq \beta_1 \leq \beta_2$. <u>Then</u> $I_1 <^1_J I_2$ (see Definition 6.13(3)).

Proof. 1) Obvious.

2) We define

$$\begin{aligned} \mathcal{E}^*_2 = \big\{(h_1, h_2) : \ & h_1, h_2 \in \mathrm{inc}_J(I_2), \text{ and for some} \\ & h'_1, h'_2 \in \mathrm{inc}_J(I_1) \text{ we have} \\ & \mathrm{tp}_{\mathrm{qf}}(h'_1, h'_2; I_1) = \mathrm{tp}_{\mathrm{qf}}(h_1, h_2; I_2) \\ & \text{and } h'_1 \ \mathcal{E}_1 \ h'_2\big\}. \end{aligned}$$

Now

$(*)_1$ $\mathcal{E}^*_2$ is a set of pairs of members of $\mathrm{inc}_J(I_2)$.

[Why? By its definition.]

$(*)_2$ $h_1 \ \mathcal{E}^*_2 \ h_1$ if $h_1 \in \mathrm{inc}_J(I_2)$.

[Why? Let $h' \in \mathrm{inc}_J(I_1)$ so clearly $h' \ \mathcal{E}_1 \ h'$ and $\mathrm{tp}_{\mathrm{qf}}(h', h'; I_1) = \mathrm{tp}_{\mathrm{qf}}(h, h; I_2)$]

$(*)_3$ $\mathcal{E}^*_2$ is symmetric.

[Why? As $\mathcal{E}_1$ is.]

$(*)_4$ $\mathcal{E}^*_2$ is transitive.

[Why? Assume $h_1 \ \mathcal{E}^*_2 \ h_2$ and $h_2 \ \mathcal{E}^*_2 \ h_3$; let $h'_1, h'_2 \in \mathrm{inc}_J(I_1)$ witness $h_1 \ \mathcal{E}^*_2 \ h_2$ and $h''_2, h''_3 \in \mathrm{inc}_J(I_1)$ witness $h_2 \ \mathcal{E}^*_2 \ h_3$.

Apply clause (b) of part (3) of Definition 6.13 to (h_1, h_2, h_3) so there are $g_1, g_2, g_3 \in \mathrm{inc}_J(I_1)$ such that $\mathrm{tp}_{\mathrm{qf}}(g_1, g_2, g_3; I_1) = \mathrm{tp}_{\mathrm{qf}}(h_1, h_2, h_3; I_2)$. Now $h'_1 \ \mathcal{E}_1 \ h'_2$ by the choice of (h'_1, h'_2) and $\mathrm{tp}_{\mathrm{qf}}(g_1, g_2; I_1) = \mathrm{tp}_{\mathrm{qf}}(h_1, h_2; I_2) = \mathrm{tp}_{\mathrm{qf}}(h'_1, h'_2; I_1)$ so as $\mathcal{E}_1$ is invariant we get $g_1 \ \mathcal{E}_1 \ g_2$. Similarly, $g_2 \ \mathcal{E}_1 \ g_3$, so as $\mathcal{E}_1$ is transitive we have $g_1 \ \mathcal{E}_1 \ g_3$. But clearly $\mathrm{tp}_{\mathrm{qf}}(g_1, g_3; I_1) = \mathrm{tp}_{\mathrm{qf}}(h_1, h_3; I_2)$ hence g_1, g_2 witness that $h_1 \ \mathcal{E}_2 \ h_3$ is as required.]

$(*)_5$ $\mathcal{E}^*_2$ is invariant.

[Why? See its definition.]

$(*)_6$ $\mathcal{E}^*_2 \restriction \mathrm{inc}_I(I_1) = \mathcal{E}_1$.

[Why? By the way $\mathcal{E}^*_2$ is defined and by $\mathcal{E}_1$ being invariant.]

So together $\mathcal{E}^*_2$ is as required. The uniqueness (i.e. if $\mathcal{E}_2$ is an invariant equivalence relation on $\mathrm{inc}_J(I)$ such that $\mathcal{E}_2 \restriction \mathrm{inc}_J(I_1) = \mathcal{E}_1$ then $\mathcal{E}_2 = \mathcal{E}^*_2$) is also easy.

3) Straight.

4) See[13] the proof of "Second" in the proof of 6.12(1).
5) Combine[14] the proof of part (4) and of "First" in the proof of 6.12(2). $\square_{6.14}$

Below mostly it suffices to consider $\mathcal{D}_{\mathcal{E},e}$.

Definition 6.15. 1) Let $\mathcal{E}$ be an invariant (I,J)-equivalence relation: we define

$$\mathcal{D}_{\mathcal{E}} = \{u \subseteq J : \text{if } h_1, h_2 \in \text{inc}_J(I) \text{ satisfies } \text{eq}(h_1,h_2) \supseteq u \text{ then } h_1 \ \mathcal{E}\ h_2\}$$

recalling

$$\text{eq}(h_1,h_2) := \{t \in J : h_1(t) = h_2(t)\}.$$

2) If, in addition, $e \in \mathbf{e}(J,I)$ then we let

$$\mathcal{D}_{\mathcal{E},e} = \big\{u \subseteq \text{dom}(e)/e : \text{if } h_1,h_2 \in \text{inc}_J(I) \text{ and } (h_1,h_2) \text{ is an } (e \restriction (\text{dom}(e) \setminus \text{set}(u)))\text{-pair then } h_1 \ \mathcal{E}\ h_2\big\}.$$

Claim 6.16. *Assume $I, J \in K^{\text{lin}}_{\tau^*_{\alpha(*)}}$, (I,J) is reasonable (see Definition 6.10(3),(6)), and $\mathcal{E}$ is an invariant (I,J)-equivalence relation.*

1) For $u \subseteq J$ such that $e_{J,u} \in \mathbf{e}(J,I)$ we have: $u \in \mathcal{D}_{\mathcal{E}}$ iff $h_1 \ \mathcal{E}\ h_2$ for every $e_{J,u}$-pair (h_1,h_2) iff $h_1 \ \mathcal{E}\ h_2$ for some $e_{J,u}$-pair (h_1,h_2); see Definition 6.9(1).

2) Assume $e \in \mathbf{e}(J,I)$. Then, for any $u \subseteq \text{dom}(e)/e$ we have $u \in \mathcal{D}_{\mathcal{E},e}$ iff $h_1 \ \mathcal{E}\ h_2$ for any $(e \restriction \text{set}(u))$-pair iff $h_1 \ \mathcal{E}\ h_2$ for some $(e \restriction \text{set}(u))$-pair.

3) If $e \in \mathbf{e}(J,I)$ and $u_1, u_2 \subseteq \text{dom}(e)/e$ then we can find $h_1,h_2,h_3 \in \text{inc}_J(t)$ such that (h_1,h_2) is a strict $(e \restriction \text{set}(u_1))$-pair, (h_2,h_3) is a strict $(e \restriction \text{set}(u_2))$ pair, and (h_1,h_3) is a strict $(e \restriction (\text{set}(u_1 \cup u_2))$-pair.

4) Assume $e \in \mathbf{e}(J,I)$ and that in clause (b) of Definition 6.10(3) we allow (h_1,h_2) to be a weak e-pair. Then, for any $u \subseteq \text{dom}(e)/e$ we have $\text{dom}(e) \setminus u \in \mathcal{D}_{\mathcal{E},e}$ iff $h_1 \ \mathcal{E}\ h_2$ for every weak e-pair (h_1,h_2).

Proof. 1) Like part (2).

2) In short, this follows by transitivity of equivalence and the definitions + mixing, but we elaborate.

The "first implies the second" holds by Definition 6.15(2) and "the second implies the third" holds trivially as there is such a pair (h_1,h_2) by the assumption $e \in \mathbf{e}(J,I)$. So it is enough to prove "the third implies the first"; hence suppose that $g_1 \ \mathcal{E}\ g_2$ where (g_1,g_2) is an $e_1 := e \restriction \text{set}(u)$-pair (recalling that $e_1 \in \mathbf{e}(J,I)$ by 6.8(8)), and let (h_1,h_2) be an e_1-pair, we need to show that $h_1 \ \mathcal{E}\ e_2$. By Definition 6.6(2B), for some sets $\mathcal{Y}_g, \mathcal{Y}_h \subseteq \text{dom}(e_1)/e_1$, the pair (g_1,g_2) is a strict $(e_1,\mathcal{Y}_g)$-pair and the pair (h_1,h_2) is a strict $(e_1,\mathcal{Y}_h)$-pair. Recalling clause (b) of 6.10(3) there are g'_1,g'_2,g'_3 and $\mathcal{Y}$ such that:

$(*)_1$ (a) $g'_\ell \in \text{inc}_J(I)$ for $\ell = 1,2,3$.
(b) $\text{tp}_{\text{qf}}(g_1,g_2) = \text{tp}_{\text{qf}}(g'_1,g'_2)$
(c) $\mathcal{Y} \subseteq \text{dom}(e_1)/e_1$
(d) (g'_1,g'_3) and (g'_2,g'_3) are strict $(e_1,\mathcal{Y})$-pairs.

[13] Actually, instead of "$\mu^+ \leq \beta_1$" it suffices to have $\zeta(*) \times 4 \leq \beta_1$, because if $\zeta(*) = \sum_{i<\gamma} \zeta_i$ then $\sum_{i<\gamma} \zeta_i \times 4 \leq \zeta(*) \times 4$ or just the natural sum $\zeta(*) \oplus \zeta(*) \oplus \zeta(*)$.

[14] Here $(\mu^+ + 1) \times (\zeta(*) \times 4)$ will suffice.

Now for each $s \in \mathrm{dom}(e_1)$, we can find a permutation $\bar\ell_s = (\ell_{s,1}, \ell_{s,2}, \ell_{s,3})$ of $\{1,2,3\}$ such that $I \models$ '$g'_{\ell_{s,1}}(s) < g'_{\ell_{s,2}}(s) < g'_{\ell_{s,3}}(s)$'. By $(*)_1(d)$ and $(*)_1(b)$ and (g_1, g) being an e_1-pair, $\bar\ell_s$ clearly depends only on s/e_1, and every member of $\{(g'_{\ell_{s,1}}(t) : t \in s/e_1\}$ is below every member of $\{g'_{\ell_{s,2}}(t) : t \in s/e_1\}$ (and similarly for the pair $(g'_{\ell_{s,2}}, g'_{\ell_{s,3}})$). Now we can find (g''_1, g''_2, g''_3) such that:

$(*)_2$ (a) $g''_\ell \in \mathrm{inc}_J(I)$ for $\ell = 1,2,3$.
 (b) (g''_1, g''_2) is a strict $(e_1, \mathcal{Y}_h)$**[-pair.]**
 (c) (g''_1, g''_3) and (g''_2, g''_3) are strict $(e_1, \mathcal{Y}_g)$-pairs.

[Why? We do the choice for each s/e_1 separately such that

$$\{g''_1 \restriction (s/e_1), g''_2 \restriction (s/e_1), g''_3 \restriction (s/e_1)\} = \{g'_1 \restriction (s/e_1), g'_2 \restriction (s/e_1), g'_3 \restriction (s/e_1)\}.]$$

Clearly $\mathrm{tp}_{\mathrm{qf}}(g''_1, g''_3; I) = \mathrm{tp}_{\mathrm{qf}}(g_1, g_2; I) = \mathrm{tp}_{\mathrm{qf}}(g''_2, g''_3; I)$, so as $\mathcal{E}$ is invariant and $g_1 \ \mathcal{E} \ g_2$ clearly $g''_1 \ \mathcal{E} \ g''_3 \wedge g''_2 \ \mathcal{E} \ g''_3$, which implies $g''_1 \ \mathcal{E} \ g''_2$. For $\mathcal{Y}' = \mathcal{Y}_h$, by clause (b) of $(*)_2$ we conclude that $\mathrm{tp}_{\mathrm{qf}}(g''_1, g''_2; I) = \mathrm{tp}_{\mathrm{qf}}(h_1, h_2; I)$, so as $\mathcal{E}$ is invariant we are done.

3),4) Similarly. $\square_{6.16}$

Claim 6.17. *Assume $I, J \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$ and $\mathcal{E}$ is an invariant (I,J)-equivalence relation.*

0) If $e \in \mathbf{e}(J,I)$ and $\mathcal{E}$ is non-trivial <u>then</u> $\mathcal{D}_{\mathcal{E},e}$ contains all co-finite subsets of $\mathrm{dom}(e)/e$.
1) If the pair (I,J) is reasonable and $e \in \mathbf{e}(I,J)$ <u>then</u> $\mathcal{D}_{\mathcal{E},e}$ is a filter on $\mathrm{dom}(e)/e$ (but possibly $\varnothing \in \mathcal{D}_{\mathcal{E},e}$).
2) (a) $\mathcal{D}_{\mathcal{E}}$ is a filter on J.
 (b) If $\mathcal{E}$ is non-trivial, then all cofinite subsets of J belong to $\mathcal{D}_{\mathcal{E}}$ but $\varnothing \notin \mathcal{D}_{\mathcal{E}}$.

Proof. 0) Easy, see Definition 6.13(2).
1) By 6.16(2) and 6.16(3).
2) Trivial by Definition 6.15(1). $\square_{6.17}$

Claim 6.18. *Assume*

(a) $I, J \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$
(b) $\mathcal{E}$ is an invariant (I,J)-equivalence relation.
(c) (I,J) is a reasonable $(\mu, \alpha(*))$-base which is a wide $(\lambda, \mu, \alpha(*))$-base.
(d) $e \in \mathbf{e}(J,I)$
(e) g is a function from $\mathrm{dom}(e)/e$ into some cardinal θ.
(f) $\mathcal{D}^* = \{Y \subseteq \theta : g^{-1}(Y) \in \mathcal{D}_{\mathcal{E},e}\}$ is a filter; i.e. $\varnothing \notin \mathcal{D}^*$.

<u>Then</u> $\mathcal{E}$ has at least $\chi := \lambda^\theta/\mathcal{D}^*$ equivalence classes.

Proof. Let $\langle f_\alpha : \alpha < \chi \rangle$ be a set of functions from θ to λ exemplifying $\chi := \lambda^\theta/\mathcal{D}^*$, so $\alpha \neq \beta \Rightarrow \{i < \theta : f_\alpha(i) = f_\beta(i)\} \notin \mathcal{D}^*$.

Let $\langle h_\zeta : \zeta < \lambda \rangle$ exemplify the pair (I,J) being a wide $(\lambda, \mu, \alpha(*))$-base (see Definition 6.10(4)), so $h_\zeta \in \mathrm{inc}_J(I)$.

Lastly, for each $\alpha < \chi$ we define $h^\alpha \in \mathrm{inc}_J(I)$ as follows:

$$h^\alpha(t) = \begin{cases} h_0(t) & \text{if } t \in J \setminus \mathrm{dom}(e) \\ h_{f_\alpha(g(t/e))}(t) & \text{if } t \in \mathrm{dom}(e) \end{cases}$$

Now

$(*)_1$ h^α is a function from J to I.

[Why? Trivially; recalling each h_ζ is as well.]

$(*)_2$ h^α is increasing.

[Why? Let $s <_J t$, and we split the proof to cases.

If $s, t \in J \setminus \mathrm{dom}(e)$ use "$h_0 \in \mathrm{inc}_J(I)$".

If $s \in J \setminus \mathrm{dom}(e)$ and $t \in \mathrm{dom}(e)$, then $h^\alpha(t) = h_{f_\alpha(g(t/e))}(t)$ and $h^\alpha(s) = h_0(s) = h_{f_\alpha(g(t/e))}(s)$ because $\langle h_\alpha \restriction (J \setminus \mathrm{dom}(e)) : \alpha < \lambda \rangle$ is constant (recalling (h_0, h_α) is an e-pair for $\alpha > 0$), so as $h_{f_\alpha(g(t/e))} \in \mathrm{inc}_J(I)$ we are done.

If $s \in \mathrm{dom}(e)$, $t \in J \setminus \mathrm{dom}(e)$, the proof is similar.

If $s, t \in \mathrm{dom}(e)$, $s/e \neq t/e$, we again use Definition 6.6(2B) and clause (b)(β) of Definition 6.10(4).

Lastly, if $s, t \in \mathrm{dom}(e)$, $s/e = t/e$ then we get $g(s/e) = g(t/e)$, hence $f_\alpha(g(s/e)) = f_\alpha(g(t/e))$ (call this γ). So $h^\alpha(s) = h_\gamma(s)$, $h^\alpha(t) = h_\gamma(t)$, and of course $h_\gamma \in \mathrm{inc}_J(I)$ hence $h_\gamma(s) <_I h_\gamma(t)$ so necessarily $h^\alpha(s) <_I h^\alpha(t)$ as required. So $(*)_2$ holds.]

$(*)_3$ $h^\alpha \in \mathrm{inc}_J(I)$.

[Why? Clearly if $i < \alpha(*)$ and $t \in P^J_i$ then $(\forall \beta < \lambda)[h_\beta(t) \in P^J_i]$ hence

$$\alpha < \chi \Rightarrow h_{f_\alpha(g(t/e))}(t) \in P^J_i$$

which means $\alpha < \chi \Rightarrow h^\alpha(t) \in P^J_i$; so recalling $(*)_2$, clause (a) of Definition 6.3(2) holds. We should check clauses (b),(c) of Definition 6.3(2) which is done as in the proof of 6.7 and of $(*)_2$ above.]

$(*)_4$ if $\alpha < \beta$ and we let

$$u = u_{\alpha,\beta} := \bigcup \{g^{-1}(\zeta) : \zeta < \theta \text{ and } f_\alpha(\zeta) \neq f_\beta(\zeta)\}$$

so $u \subseteq \mathrm{dom}(e)/e$ <u>then</u> (h^α, h^β) is a $(e \restriction \mathrm{set}(u))$-pair.

[Why?

Case 1: If $s \in J \setminus \mathrm{dom}(e)$ then $h^\alpha(s) = h_0(s) = h^\beta(s)$.

Case 2: If $s \in \mathrm{dom}(e) \setminus \mathrm{set}(u)$ then $h^\alpha(s) = h_{f_\alpha(g(s/e))}(s) = h_{f_\beta(g(s/e))}(s) = h^\beta(s)$.

Case 3: If $s, t \in \mathrm{set}(u)$, $s/e \neq t/e$, and $s <_J t$ <u>then</u> $h^\alpha(s) <_I h^\beta(t) \wedge h^\beta(s) <_I h^\alpha(t)$ because

Subcase 3A: If $f_\alpha(g(s/e)) = f_\beta(g(t/e))$ we use $h_{f_\alpha(g(t/e))} \in \mathrm{inc}_J(I)$ hence

$$h^\alpha(s) = h_{f_\alpha(g(s/e))}(s) <_I h_{f_\alpha(g(s/e))}(t) = h_{f_\beta(g(t/e))}(t) = h^\beta(t)$$

and similarly $h^\beta(s) <_I h^\alpha(t)$.

Subcase 3B: If $f_\alpha(g(s/e)) \neq f_\beta(g(t/e))$ we use "$(h_{f_\alpha(g(s/e))}, h_{f_\beta(g(t/e))})$ is an e-pair".

Case 4: And lastly, if $s, t \in \mathrm{set}(u), s/e = t/e$ and $s <_J t$ <u>then</u>

$$h^\alpha(t) <_I h^\beta(s) \equiv (s/e \in u) \equiv h^\alpha(s) <_I h^\beta(t).$$

Why? Recalling $f_\alpha(g(s/e)) \neq f_\beta(g(t/e))$ as $s, t \in \mathrm{set}(u)$ by the definition of u, see $(*)_4$ and we just use "$(h_{f_\alpha(g(s/e))}, h_{f_\beta(g(s/e))})$ is an e-pair" and clause $(c)'$ of Definition 6.6.]

$(*)_5$ If $\alpha < \beta$ then $u_{\alpha,\beta} \neq \emptyset \mod \mathcal{D}_{\mathcal{E},e}$.

[Why? By the choice of $\langle f_\alpha : \alpha < \lambda \rangle$.]

$(*)_6$ if $\alpha < \beta$ then h^α, h^β are not $\mathcal{E}$-equivalent.

[Why? By $(*)_4 + (*)_5$ and 6.16(2).]

Together we are done. $\square_{6.18}$

Claim 6.19. *Assume $\mathcal{E}$ is an invariant (I,J)-equivalence relation, I, J are well ordered and $|\mathrm{inc}_J(I)/\mathcal{E}| \geq \lambda = \mathrm{cf}(\lambda) > \mu = |I| > |2 + \alpha(*)|^{|J|}$. <u>Then</u> for some $e \in \mathbf{e}(I,J)$ there is an ultrafilter $\mathcal{D}$ on $\mathrm{dom}(e)/e$ extending $\mathcal{D}_{\mathcal{E},e}$ which is not principal.*

Remark 6.20. This is close to [She99b, §7].

Proof. Without loss of generality, as linear orders J is $\zeta(*)$ and I is $\xi(*) \in [\mu, \mu^+)$.

Toward contradiction assume the conclusion fails. Let g be a one-to-one function from μ onto $[\xi(*)]^{<\aleph_0}$, χ be large enough, $\kappa = |J|$, and $\partial = |2 + \alpha(*)|^{|J|}$ so $\partial^\kappa = \partial$.

We now choose $\langle N_\eta : \eta \in {}^n\mu\rangle$ by induction on $n < \omega$ such that

$\circledast_1$ (a) $N_\eta \prec (\mathcal{H}(\chi), \in)$
  (b) $\|N_\eta\| = \partial$ and $\partial + 1 \subseteq N_\eta$.
  (c) $A \subseteq N_\eta \wedge |A| \leq \kappa \Rightarrow A \in N_\eta$
  (d) I, J and g as well as η belong to N_η.
  (e) $\nu \triangleleft \eta \Rightarrow N_\nu \in N_\eta$ (hence $N_\nu \subseteq N_\eta$ so $N_\nu \prec N_\eta$).

There is no problem to do this. Now it suffices to prove that for every $h \in \mathrm{inc}_J(I)$, for some $h' \in \bigcup\{N_\eta : \eta \in {}^{\omega>}\mu\} \cap \mathrm{inc}_J(I)$, we have $h \;\mathcal{E}\; h'$.

Fix $h_* \in \mathrm{inc}_J(I)$ such that $h_* \notin \bigcup\{h/\mathcal{E} : h \in \mathrm{inc}_J(I) \cap N_\eta$ for some $\eta \in {}^{\omega>}\mu\}$ and for each $\eta \in {}^{\omega>}\mu$ we define $\bar\alpha_\eta, e_\eta$ as follows:

$\circledast_2$ (a) $\bar\alpha_\eta = \langle \alpha_{\eta,t} : t \in J\rangle$
  (b) $\alpha_{\eta,t} = \min\big((\xi(*)+1) \cap N_\eta \setminus h_*(t)\big)$
  (c) $e_\eta := \big\{(s,t) : s,t \in J,\ \alpha_{\eta,s} = \alpha_{\eta,t},\ \alpha_{\eta,s} > h_*(s),$ and $\alpha_{\eta,t} > h_*(t)\big\}$.
  (d) For $\alpha \in N_\eta$ let $X_{\eta,\alpha} := \big\{t \in J : \alpha_{\eta,t} = \alpha > h_*(t)\big\}$.

Note

$(*)_1$ $\bar\alpha_\eta \in N_\eta$.

[Why? As $[N_\eta]^{\leq\kappa} \subseteq N_\eta$, $|J| = \kappa$, and $\alpha_{\eta,t} \in N_\eta$ for every $t \in J$.]

$(*)_2$ (a) $e_\eta \in \mathbf{e}(J)$; i.e. e_η is an equivalence relation on some subset of J, with each equivalence class a convex subset of J (see Definition 6.6(1)).
  (b) *(b)* $\big\langle X_{\eta,\alpha} : \alpha \in \{\alpha_{\eta,t} : t \in \mathrm{dom}(e)\}\big\rangle$ list the e_η-equivalence classes. (Note that $X_{\eta,\alpha} \neq \varnothing$.)

[Why? Think.]

$(*)_3$ $h_\eta := h_* \restriction (J \setminus \mathrm{dom}(e_\eta)) \in N_\eta$.

[Why? By the definition of e_η we have $t \in J \wedge t \notin \mathrm{dom}(e_\eta) \Rightarrow h_*(t) \in N_\eta$, and recall $[N_\eta]^{\leq\kappa} \subseteq N_\eta$.]

$(*)_4$ If $t \in \mathrm{dom}(e_\eta)$ then $\mathrm{cf}(\alpha_{\eta,t}) > \partial$.

[Why? As $\alpha_{\eta,t} \in N_\eta \prec (\mathcal{H}(\chi), \in)$ if $\mathrm{cf}(\alpha_{\eta,t}) = \theta \leq \partial$ then there is a cofinal set B of $\alpha_{\eta,t}$ of cardinality θ in N_η but $\theta \leq \partial + 1 \subseteq N_\eta$ therefore $B \subseteq N_\eta$. In particular, as $h_*(t) < \alpha_{\eta,t}$, there is $\beta \in B$ so that $h_*(t) < \beta$, but this contradicts the choice of $\alpha_{\eta,t}$.]

$(*)_5$ $e_\eta \in \mathbf{e}(J, I)$.

[Why? Choose $h' \in \mathrm{inc}_J(I) \cap N_\eta$ similar enough to h_*. Specifically,

$$t \in J \setminus \mathrm{dom}(e_\eta) \Rightarrow h'(t) = h_*(t)$$

and

$$t \in \mathrm{dom}(e_\eta) \Rightarrow \sup\{\alpha_{\eta,s} : s \in J,\ s <_J t \text{ and } s \notin t/e_\eta\} < h'(t) < \alpha_{\eta,t}$$

(the point being that $\sup\{\alpha_{\eta,s} : s \in J, s <_J t$ and $s \notin t/e_n\} \in N_\eta$). Now (h', h_*) is a strict e-pair.]

$(*)_6$ There is $\ell_\eta < \omega$ and a finite sequence $\langle \beta_{\eta,\ell} : \ell < \ell_\eta\rangle$ of members of $\mathrm{rang}(\bar\alpha_\eta \restriction \mathrm{dom}(e_\eta))$ [with] $X_{\eta,\beta_{\eta,\ell}} \in \mathrm{dom}(e_\eta)/e_\eta$ for $\ell < \ell_\eta$ such that

$$\bigcup_{\ell < \ell_\eta} X_{\eta,\beta_{\eta,\ell}} \in \mathcal{D}_{\mathcal{E},e_\eta}.$$

[Why? Otherwise there is an ultrafilter as desired, but toward contradiction we have assumed this does not occur; in trying to get generalizations we should act differently.]

Now we choose (η_n, h_n) by induction on $n < \omega$ such that

$\boxdot$ (a) $\eta_n \in {}^n\mu$
(b) If $n = m + 1$ then $\eta_m = \eta_n \restriction m$.
(c) $h_n \in \mathrm{inc}_J(I)$
(d) $h_0 = h_*$
(e) If $n = m + 1$ then:
  (α) $h_n \ \mathcal{E} \ h_m$ hence $h_n \ \mathcal{E} \ h_*$ and $\mathrm{dom}(e_{\eta_n}) \subseteq \mathrm{dom}(e_{\eta_m})$.
  (β) $h_m \restriction (J \setminus \mathrm{dom}(e_{\eta_m})) \subseteq h_n$
  (γ) $(h_m \restriction \bigcup\{X_{\eta_m, \beta_{\eta_m,\ell}} : \ell < \ell_{\eta_m}\}) \subseteq h_n$
  (δ) $h_n \restriction \big(\mathrm{dom}(e_{\eta_m}) \setminus \bigcup\{X_{\eta_m, \beta_{\eta_m,\ell}} : \ell < \ell_{\eta_m}\}\big)$ belongs to N_{η_m}.
  (ε) Moreover, $t \in \mathrm{dom}(e_{\eta_m}) \setminus \bigcup\{X_{\eta_m,\beta_{\eta_m,\ell}} : \ell < \ell_{\eta_m}\}$ implies $h_n(t) < h_m(t)$.
  (ζ) $\ell_{\eta_m} > 0$
(f) $Y_{m+1} \subseteq Y_m$, where $Y_m := \bigcup\{X_{\eta_m, \beta_{\eta_m,\ell}} : \ell < \ell_\eta\}$.

Why can we carry out the construction? For $n = 0$ we obviously can: choose $h_0 = h_*$. For $n = m + 1$, first choose $h'_m \in N_{\eta_m}$ as in the proof of $(*)_5$. Now, recalling $\langle X_{\eta_m}, \beta_{\eta_m,\ell} : \ell < \ell_{\eta_m}\rangle$ was chosen in $(*)_6$, define h_n by

$$h_n \restriction \Big(\mathrm{dom}(e_{\eta_m}) \setminus \bigcup\{X_{\eta_m,\beta_{\eta_m,\ell}} : \ell < \ell_{\eta_m}\}\Big) = h'_m \restriction \Big(\mathrm{dom}(e_{\eta_m}) \setminus \bigcup\{X_{\eta_m,\beta_{\eta_m,\ell}} : \ell < \ell_\eta\}\Big),$$

$$h_n \restriction (J \setminus \mathrm{dom}(e_{\eta_m})) = h_m \restriction (J \setminus \mathrm{dom}(e_{\eta_m}),$$

and

$$h_n \restriction \Big(\bigcup\{X_{\eta_m,\beta_{\eta_m,\ell}} : \ell < n_{\eta_m}\}\Big) = h_m \restriction \Big(\bigcup\{X_{\eta_m,\beta_{\eta_m,\ell}} : \ell < \ell_{\eta_m}\}\Big).$$

Why $h_n \ \mathcal{E} \ h_m$? Because

(i) as in the proof of $(*)_5$, (h_n, h_m) form a strict ℓ_η-pair,
(ii) they agree on $\bigcup\{X_{\eta_m,\beta_{\eta_m,\ell}} : \ell < \ell_\eta\}$,
(iii) and $\{X_{\eta_m,\beta_{\eta_m,\ell}} : \ell < n\} \in \mathcal{D}_{\mathcal{E},\mathbf{e}_\eta}$.

Lastly, choose $\eta_n = \eta_m{}^\frown\langle\gamma_m\rangle$ where γ_m is chosen such that

$$g(\gamma_m) = \Big\{\sup\big(\beta_{\eta_m,\ell} \setminus \sup\{h_m(t) : t \in X_{\beta_{\eta_m,\ell}}\}\big) : \ell < \ell_{\eta_m}\Big\}$$

recalling that g is a function from μ onto $[\xi(*)]^{<\aleph_0} = [I]^{<\aleph_0}$.

Now check that η_n, h_n are as required.

Note that this induction never stops (in the sense that $h_n \notin N_{\eta_n}$) recalling the choice of h_* and $h_n \ \mathcal{E} \ h_*$. Now $\mathcal{U}_n := \{\beta_{\eta_m,\ell} : \ell < n_\eta\}$ is a finite non-empty set of ordinals, and if $n = m + 1$, then easily

$$(\forall \ell < \ell_{\eta_n})(\exists k < \ell_{\eta_m})[\beta_{\eta_n,\ell} < \beta_{\eta_m,k}]$$

because for $\ell < \ell_{\eta_n}$ letting $t \in X_{\eta_n,\ell}$ we know that for some $k \leq \ell_{\eta_m}$ we have $t \in X_{\eta_m,k}$ and $\eta_n(m)$ was chosen above such that as γ_m, now $h_*(t) \leq \gamma_n \in N_{\eta_n}$, $\gamma_m \leq \alpha_{\eta_m,t}$ and the inequality is strict as $\mathrm{cf}(\alpha_{\eta_m,t}) > 0$. So $\langle \max(\mathcal{U}_n) : n < \omega\rangle$ is a decreasing sequence of ordinals, a contradiction, so we are done. $\square_{6.19}$

Example: For $e \in \mathbf{e}(J, I)$, $J \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$ and $I \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$ we define $\mathcal{E}^*_e = \mathcal{E}^*_{e,I}$, an invariant equivalent relation on $\mathrm{inc}_J(I)$, by the following.

$h_1 \ \mathcal{E}^*_{e,I} \ h_2$ iff:

(a) If $t \in J \setminus \mathrm{dom}(e)$ then $h_1(t) = h_2(t)$.

(b) If $t \in \mathrm{dom}(e)$ then $\mathrm{cnv}_{I,h_1}(t) = \mathrm{cnv}_{I,h_2}(t)$, where $\mathrm{cnv}_{I,h}(s)$ is the convex hull (in I) of the set

$$\{h_1(s)\} \cup \bigcup\{[h_1(s), h_1(t)]_I : s <_J t \text{ and } t \in s/e\} \cup \bigcup\{[h(t), h(s)]_I : t <_J s \text{ and } t \in s/e\}.$$

1) If $J, I \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$ are well ordered and $e = J \times J$ then $\mathcal{E}^*_{e,I}$ from part (1) has $\leq |I| + \aleph_0$ equivalence classes.

2) If $J \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$ and e as in part (2), $\theta = \mathrm{cf}(J)$ and $|J| < \lambda = \lambda^{<\theta} < \lambda^\theta$ <u>then</u> there is $I \in K^{\mathrm{lin}}_{\tau^*_{\alpha(*)}}$ of cardinality λ such that $\mathcal{E}^*_{e,I}$ has λ^θ equivalence classes.

Remark 6.21. We can define the stability spectrum for some classes; essentially this is done in §7, and generally we intend to look at it in [S$^+$b].

§ 7. Categoricity for AEC with bounded amalgamation

Recall that 4.10 is the main result of this chapter; we think that it will lead to understanding the categoricity spectrum of an AEC. In particular, we hope to eventually prove that this spectrum contains, or is disjoint to, some end segments of the class of cardinals. Still, here we would like to show that we at least have enough for sufficiently restricted families of AEC $\mathfrak{K}$-s: those definable by $\mathbb{L}_{\kappa,\omega}$ for κ a measurable cardinal, or with enough amalgamation. (Concerning them and earlier results, see [She].) We could have relied on[15] [She99a], but though we mention connections we do not rely on it, preferring self-containment.

We can say much even if we replace categoricity by strong solvability, but do this only when it is cheap; we can work with weak and even pseudo-solvability, but will not do so here.

Hypothesis 7.1. 1) $\mathfrak{K}$ is an AEC, so $\mathcal{S}(M) = \mathcal{S}_{\mathfrak{K}_\lambda}(M)$ for $M \in K_\lambda$; see [She09c, 0.12].

2) Let K^x_μ be the class K_μ if K is categorical in μ, and be the class of superlimit models in $\mathfrak{K}_\mu$ if there is one. (The two definitions are compatible.)

The following is a crucial claim because lack of locality is the problem in [She99a].

Claim 7.2. *Assume*

(a) $\mathrm{cf}(\mu) > \kappa \geq \mathrm{LST}(\mathfrak{K})$

(b) $\mathfrak{K}_{<\mu}$ *has amalgamation*

(c) $\Phi \in \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$ *satisfies: if I is θ-wide and $\theta \in (\kappa,\mu)$ <u>then</u> $\mathrm{EM}_{\tau(\mathfrak{K})}(I,M)$ is θ-saturated (see 0.14(1),* [She09c, 0.15(2)] *and* [She09c, 0.19]*).*

<u>*Then*</u>

(α) *For some $\mu_* < \mu$, the class $\{M \in K_{<\mu} : M$ is saturated$\}$ is $[\mu_*,\mu)$-local (see Definition 7.4(3) below).*

$(\alpha)^+$ *This applies not only to $\mathcal{S}(M) = \mathcal{S}^1(M)$ but also for $\mathcal{S}^\partial(M)$ if $\mathrm{cf}(\mu) > \kappa^\partial$.*

Recall

Definition 7.3. $\mathfrak{K}$ is μ-stable if $\mu \geq \mathrm{LST}(\mathfrak{K})$ and $M \in K_{\leq\mu} \Rightarrow |\mathcal{S}(M)| \leq \mu$.

Recall [She99a, Def.1.8=1.6tex(1),(2)].

Definition 7.4. 1) For $M \in \mathfrak{K}$, $\mu \geq \mathrm{LST}(\mathfrak{K})$ satisfying $\mu \leq \|M\|$ and α **[what about α?]**, let $\mathbb{E}_{M,\mu,\alpha}$ be the following equivalence relation on $\mathcal{S}^\alpha(M)$:

$p_1 \; \mathbb{E}_{M,\mu,\alpha} \; p_2$ iff for every $N \leq_{\mathfrak{K}} M$ of cardinality μ we have $p_1 \restriction N = p_2 \restriction N$. We may suppress α if it is 1, similarly below; let $\mathbb{E}_{\mu,\alpha}$ be $\bigcup\{\mathbb{E}_{M,\mu,\alpha} : M \in K\}$ and so $\mathbb{E}_\mu = \mathbb{E}_{\mu,1}$.

2) We say that $M \in \mathfrak{K}$ is $\mu - \alpha$-local when $\mathbb{E}_{M,\mu,\alpha}$ is the equality; we say that $p \in \mathcal{S}^\alpha(M)$ is μ-local if $p/\mathbb{E}_{M,\mu,\alpha}$ is a singleton and we say, e.g., $K' \subseteq \mathfrak{K}$ is $\mu-\alpha$-local (in $\mathfrak{K}$, if not clear from the context) when every $M \in K'$ is.

[Is this supposed to be $(\mu - \alpha)$-local, μ-α-local, or (μ,α)-local?]

3) We say $K' \subseteq \mathfrak{K}$ is $[\mu_*,\mu) - \alpha$-local if every $M \in K' \cap \mathfrak{K}_{[\mu^*,\mu)}$ is $\mu_* - \alpha$-local.

4) We say that $\bar{a} \in N$ realizes $\mathbf{p} \in \mathcal{S}^\alpha_{\mathfrak{K}}(M)/\mathbb{E}_{\mu,\alpha}$ if $M \leq_{\mathfrak{K}} N$ and for every $M' \leq_{\mathfrak{K}} N$ of cardinality μ the sequence $\bar{a}$ realizes $\mathbf{p} \restriction M'$ in N (or pedantically, it realizes $q \restriction M'$ for some – equivalently, *every* – $q \in \mathbf{p}$).

[15] In the references to [She99a], e.g. 1.6tex is the definition labelled 1.6 in the published version and 1.8 in the e-version.

Remark 7.5. If $M \in \mathfrak{K}_\mu$, then M is $\mu - \alpha$-local.

Proof. Recall $\Phi \in \Upsilon^{\rm or}_\kappa[\mathfrak{K}]$; see Definition 0.8(2) and Claim 0.9. Easily, there exists $\langle I_\theta : \theta \in [\kappa, \mu) \rangle$ an increasing sequence of wide linear orders which are strongly $\aleph_0$-homogeneous (that is dense with neither first nor last element such that if $n < \omega$ and $\bar{s}, \bar{t} \in {}^n(I_\theta)$ are $<_I$-increasing then some automorphism of I_θ maps $\bar{s}$ to $\bar{t}$; e.g. the order of any real closed field, or just [of any] ordered field) satisfying $|I_\theta| = \theta$.

Recalling $\mathbb{Q}$ here is the rational order, we let $J_\theta = \mathbb{Q} + I_\theta$, $M_\theta = \mathrm{EM}_{\tau(\mathfrak{K})}(I_\theta, \Phi)$, and $N_\theta = \mathrm{EM}_{\tau(\mathfrak{K})}(J_\theta, \Phi)$. So

⊛ (a) $M_\theta \le_{\mathfrak{K}_\theta} N_\theta$
(b) $M_{\theta_1} \le_{\mathfrak{K}} M_{\theta_2}$ and $N_{\theta_1} \le_{\mathfrak{K}} N_{\theta_2}$ when $\kappa \le \theta_1 < \theta_2 < \mu$.
(c) M_θ is saturated (for $\mathfrak{K}$, of course) when $\theta > \kappa$.
(d) Every type from $\mathcal{S}(M_\theta)$ is realized in N_θ.
(e) if $n < \omega$, $\bar{a} \in {}^n(N_\theta)$ then for some $\bar{a}' \in {}^n(N_\kappa)$ and automorphism π of N_θ, $\pi(\bar{a}) = \bar{a}'$ and π maps M_θ onto itself.

[Why? Clauses (a),(b) hold by clause (c) of Claim 0.9(1), recalling Definition 0.8(2).

Clause (c) holds by Clause (c) of the assumption of 7.2; you may note [She99a, 6.7=6.4tex(2)].

Clause (d) holds as $\mathrm{EM}_{\tau(\mathfrak{K})}(\theta^+ + J_\theta, \Phi) \in \mathfrak{K}_{\theta^+}$ is saturated, and use the definition of a type (or, like the proof of clause (e) below, using appropriate $I' + I_\theta$ instead of $\theta^+ + J_\theta$); you may note [She99a, 6.8=6.5tex].

Clause (e) holds as for every finite sequence $\bar{t}$ from J_θ there is an automorphism π of J_θ such that: π is the identity on $\mathbb{Q}$, it maps I_θ onto itself and it maps $\bar{t}$ to a sequence from $J_\kappa = \mathbb{Q} + I_\kappa$. Such π exists as I_θ is strongly $\aleph_0$-homogeneous and $I_\kappa \subseteq I_\theta$ is infinite.]

For any $a \ne b$ from N_κ let

$$\mu(a,b) = \min\big\{\theta : \theta \ge \kappa \text{ and if } \theta < \mu \text{ then } \mathbf{tp}_{\mathfrak{K}}(a, M_\theta, N_\theta) \ne \mathbf{tp}_{\mathfrak{K}}(b, M_\theta, N_\theta)\big\}.$$

So $\mu(a,b) \le \mu$. Let

$$\mu_* = \sup\{\mu(a,b) : a, b \in N_\kappa \text{ and } \mu(a,b) < \mu\}.$$

So μ_* is defined as the supremum on a set of $\le \kappa \times \kappa$ cardinals $< \mu$, which is a cardinal of cofinality $\mathrm{cf}(\mu) > \kappa$, hence clearly $\mu_* < \mu$. Also $\mu_* \ge \kappa$ as there are $a \ne b$ from M_κ hence $\mu(a,b) = \kappa$. Now suppose that $\theta \in [\mu_*, \mu)$, $M \in \mathfrak{K}_\theta$ is saturated, and $p_1 \ne p_2 \in \mathcal{S}(M)$, and we shall find $M' \le_{\mathfrak{K}} M$ and $M' \in \mathfrak{K}_{\mu_*}$ such that $p_1 \restriction M' \ne p_2 \restriction M'$: this will suffice.

Clearly $M_\theta \in K_\theta$ is saturated (by clause (c) of ⊛) hence the models M, M_θ are isomorphic, so without loss of generality $M = M_\theta$. But by clause (d) of ⊛ every type from $\mathcal{S}(M_\theta)$ is realized in N_θ, so let b_ℓ be such that $p_\ell = \mathbf{tp}_{\mathfrak{K}}(b_\ell, M_\theta, N_\theta)$ for $\ell = 1, 2$. Now there is an automorphism π of N_θ which maps M_θ onto itself and maps b_1, b_2 into N_κ (by clause (e) of ⊛). Let $a_\ell = \pi(b_\ell)$ for $\ell = 1, 2$, so $a_1, a_2 \in N_\kappa$.

Now

$$\begin{aligned}\mathbf{tp}(a_1, M_\theta, N_\theta) &= \mathbf{tp}(\pi(b_1), \pi(M_\theta), \pi(N_\theta)) = \pi(\mathbf{tp}(b_1, M_\theta, N_\theta)) \\ &\ne \pi(\mathbf{tp}(b_2, M_\theta, N_\theta)) = \mathbf{tp}(\pi(b_2), \pi(M_\theta), \pi(N_\theta)) = \mathbf{tp}(a_2, M_\theta, N_\theta).\end{aligned}$$

Hence by the definition of $\mu(a_1, a_2)$ we have $\mu(a_1, a_2) \le \theta < \mu$. Hence by the definition of μ_* we have $\mu(a_1, a_2) \le \mu_*$ which implies that

$$\mathbf{tp}_{\mathfrak{K}}(a_1, M_{\mu_*}, N_{\mu_*}) \ne \mathbf{tp}_{\mathfrak{K}}(a_2, M_{\mu_*}, N_{\mu_*}).$$

As π is an automorphism of N_θ and $M_{\mu^*} \le_{\mathfrak{K}} M_\theta$ it follows that

$$\mathbf{tp}_{\mathfrak{K}}\big(\pi^{-1}(a_1), \pi^{-1}(M_{\mu^*}), \pi^{-1}(N_\theta)\big) \ne \mathbf{tp}_{\mathfrak{K}}\big(\pi^{-1}(a_2), \pi^{-1}(M_{\mu^*}), \pi^{-1}(N_\theta)\big)$$

which means

$$\mathbf{tp}_{\mathfrak{K}}(b_1, \pi^{-1}(M_{\mu^*}), N_\theta) \ne \mathbf{tp}_{\mathfrak{K}}(b_2, \pi^{-1}(M_{\mu^*}), N_\theta)$$

but $\pi^{-1}(M_{\mu^*}) \le_{\mathfrak{K}} M_\theta$, as π maps M_θ onto itself. Recall that $p_\ell = \mathbf{tp}_{\mathfrak{K}}(b_\ell, M_\theta, N_\theta)$ so $p_\ell \restriction \pi^{-1}(M_{\mu^*})$ is well defined for $\ell = 1, 2$. Hence $p_1 \restriction \pi^{-1}(M_{\mu^*}) \ne p_2 \restriction \pi^{-1}(M_{\mu^*})$ and clearly $\pi^{-1}(M_{\mu^*})$ has cardinality μ^* and is $\le_{\mathfrak{K}} M_\theta$, so we are done proving clause (α). The proof of clause $(\alpha)^+$ is the same except that

- $(*)_1$ if $\theta \in [\kappa, \mu)$, $\bar t \in {}^{\partial}(I_\theta)$ then some automorphism π of I_θ maps $\bar t$ to some $\bar t' \in {}^{\partial}(I_\kappa)$; this is justified by 5.1.
- $(*)_2$ We replace $\mathbb{Q}$ by ∂^+.
- $(*)_3$ ${}^{\partial}(N_\kappa)$ has cardinality $\le (\partial^+ + \kappa)^\partial \le \kappa^\partial < \mathrm{cf}(\mu)$. $\square_{7.2}$

Implicit in non-μ-splitting is

Definition 7.6. Assume $\alpha < \mu^+$, $N \in K_{\le\mu}$, $N \le_{\mathfrak{K}} M$, and $p \in \mathcal{S}^\alpha(M)$ does not μ-split over N (see Definition [She09e, gr.1(1)]). The scheme of the non-μ-splitting, $\mathfrak{p} = \mathrm{sch}_\mu(p, N)$, is

$$\big\{(N'', c, \bar b)_{c\in N}/\cong \,:\, N \le_{\mathfrak{K}} N' \le_{\mathfrak{K}} M \text{ and } N' \le_{\mathfrak{K}} N'',\ \{N', N''\} \subseteq K_\mu, \\ \text{and the sequence } \bar b \text{ realizes } p \restriction N' \text{ in the model } N''\big\}.$$

Definition 7.7. For a cardinal μ and model M let

1)

$$\text{ps-}\mathcal{S}_\mu(M) = \mathcal{S}_{\mathfrak{K},\mu}(M) = \big\{\mathbf{p} : \mathbf{p} \text{ is a function with domain } \{N \in K_\mu : N \le_{\mathfrak{K}} M\} \\ \text{such that } \mathbf{p}(N) \in \mathcal{S}(N) \text{ and} \\ N_1 \le_{\mathfrak{K}} N_2 \in \mathrm{dom}(\mathbf{p}) \Rightarrow \mathbf{p}(N_1) = \mathbf{p}(N_2) \restriction N_1\big\}.$$

2) For $p \in \mathcal{S}(M)$ let $p \restriction (\le \mu)$ be the function $\mathbf{p}$ with domain $\{N \in K_\mu : N \le_{\mathfrak{K}} M\}$ such that $\mathbf{p}(N) = p \restriction N$.

Observation 7.8. *1) The function $p \mapsto p \restriction (\le \mu)$ is a function from $\mathcal{S}(M)$ into* ps-$\mathcal{S}_\mu(M)$ *such that for $p_1, p_2 \in \mathcal{S}(M)$ we have $p_1 \restriction (\le \mu) = p_2 \restriction (\le \mu) \Leftrightarrow p_1\ \mathbb{E}_\mu\ p_2$.*

2) The subset $\{p \restriction (\le \mu) : p \in \mathcal{S}(M)\}$ of ps-$\mathcal{S}_\mu(M)$ *has cardinality $|\mathcal{S}(M)/\mathbb{E}_\mu|$.*

Proof. Should be clear. $\square_{7.8}$

Claim 7.9. *Every (equivalently,* some*) $M \in K^x_\mu$ is λ^+-saturated when:*

- *(a) (α) $\mathfrak{K}$ is categorical in μ, or just*
  - *(β) $\mathfrak{K}$ is strongly solvable in μ.*
- *(b) $\mathrm{LST}(\mathfrak{K}) \le \lambda < \chi \le \mu$ and $2^{2^\lambda} \le \mu$ (actually, $2^\lambda \le \mu$ will suffice).*
- *(c) (α) $\aleph_{\lambda^{+4}} = \lambda^{+\lambda^{+4}} \le \chi$, or at least*
  - *(β) If $\theta = \mathrm{cf}(\theta) \le \lambda$ is $\aleph_0$ or a measurable cardinal then for some $\partial \in (\lambda, \chi)$ we have $\partial = \partial^{<\theta} < \partial^\theta$ or at least $\partial^{\langle\theta\rangle_{\mathrm{tr}}} > \partial$. (I.e. there is a tree $\mathcal{T}$ with θ levels, ∂ nodes and the number of θ-branches of $\mathcal{T}$ is $> \chi$; see* [She00].*)*
- *(d) $\mathfrak{K}_{\ge\partial} \ne \varnothing$ for every ∂. Equivalently, $K_{\ge\theta} \ne \varnothing$ for arbitrarily large $\theta < \beth_{1,1}(\mathrm{LST}(\mathfrak{K}))$.*
- *(e) (α) $\mathfrak{K}_{<\mu}$ has amalgamation and JEP, or just*
  - *(β) If $\mathrm{LST}(\mathfrak{K}) \le \partial < \chi$ then*
    - *(i) $\mathfrak{K}_\partial$ has amalgamation and JEP, and*

(ii) $\mathfrak{K}$ *has* $(\partial, \le \partial^+, \mu)$*-amalgamation*[16] *(see* [She09a, 2.5(2)]*) hence*[17]

(iii) Every $M \in K_{\partial+}$ *has a* $\le_{\mathfrak{K}}$*-extension in* K^x_μ*. (Actually, (i) + (iii) suffices.)*

Remark 7.10. 1) M is λ^+-saturated is well defined as $\mathfrak{K}_{\le\lambda}$ has amalgamation.

2) We assume $2^{2^\lambda} \le \mu$ because the proof is simpler with not much loss (at least as long as other parts of the analysis are not much tighter).

3) We can weaken the assumptions. In particular using solvability instead categoricity, but for non-essential reasons this is delayed; similarly in 7.13.

4) If $\mu = \mu^\lambda$ the claim is easy (as in §1).

Proof. Note that by [She94, IX,§2], [She94, II,3.1] if clause $(c)(\alpha)$ holds then clause $(c)(\beta)$ holds, hence we can assume $(c)(\beta)$.

Let $\Phi \in \Upsilon^{\rm or}_{\mathfrak{K}}$ (see Definition 0.8(2)); [it exists] by 0.9 and clause (d) of the assumption and $I \in K^{\rm lin}_\mu \Rightarrow {\rm EM}_{\tau(\mathfrak{K})}(I, \Phi) \in K^x_\mu$ (trivially if K is categorical in μ, otherwise by the definition of solvable).

Clearly

$(*)_0$ If $\partial \in [{\rm LST}(\mathfrak{K}), \chi)$ <u>then</u> $\mathfrak{K}$ is stable in ∂.

[Why? We prove assuming clause $(e)(\beta)$, as the case of clause $(e)(\alpha)$ is easier. Otherwise, as $\mathfrak{K}_\partial$ has amalgamation, there are $M_0 \le_{\mathfrak{K}} M_1$ such that $M_0 \in K_\partial$, $M_1 \in K_{\partial+}$ and $\{\mathbf{tp}_{\mathfrak{K}}(a, M_0, M_1) : a \in M_1\}$ has cardinality ∂^+. By assumption (e)(β)(iii) there is N_1 such that $M_1 \le_{\mathfrak{K}} N_1 \in \mathfrak{K}_\mu$ and without loss of generality $N_1 \in K^x_\mu$. Let I be as in 5.1 with $(\lambda, \theta_2, \theta_1, \mu)$ there standing for $(\mu, \partial^{++}, \partial^+, \partial)$ here and $N_2 := {\rm EM}_{\tau(\mathfrak{K})}(I, \Phi)$. Now by 5.1(2), $N_1 \not\cong N_2$, contradicting "$\mathfrak{K}$ categorical in μ". Or you may see [She99a, 1.7=1.5tex].]

The proof now splits to two cases.

Case 1: For every $M \in K^x_\mu$ we have $\mu \ge |\mathcal{S}(M)/\mathbb{E}_\lambda|$.

For every $M \in K^x_\mu$ there is M' such that: $M \le_{\mathfrak{K}} M' \in K_\mu$ and for every $\mathbf{p} \in \mathcal{S}(M)/\mathbb{E}_\lambda$ either $\mathbf{p}$ is realized in M' <u>or</u> there are no M'' or a such that $M' \le_{\mathfrak{K}} M'' \in K_\mu$ and $a \in M''$ realizes p in M''.

[Why? Let $\langle p_i/\mathbb{E}_\lambda : i < \mu\rangle$ list $\mathcal{S}(M)/\mathbb{E}_\lambda$ (this exists by the assumptions) and choose M_i for $i \le \mu$, $\le_{\mathfrak{K}_\mu}$-increasing continuous, such that M_{i+1} satisfies the demand for $\mathbf{p} = p_i/\mathbb{E}_\lambda$, possibly no $p \in p_i/\mathbb{E}_\lambda$ has an extension in $\mathcal{S}(M_{i+1})$ (hence is not realized in it), so then the desired demand holds trivially; note that it is not unreasonable to assume $\mathfrak{K}_\mu$ has amalgamation and it clarifies matters, but it is not necessary.]

Also without loss of generality $M' \in K^x_\mu$ as any model M from K_μ has a $\le_{\mathfrak{K}}$-extension in K^x_μ (at least if M does $\le_{\mathfrak{K}}$-extend some $M' \in K^x_\mu$).

Now we can choose by induction on $i \le \lambda^+$ a model $M_i \in K^x_\mu$, $\le_{\mathfrak{K}}$-increasing continuous with i, such that for every $p \in \mathcal{S}(M_i)$ <u>either</u> there is $q \in \mathcal{S}(M_i)$ realized in M_{i+1} which is $\mathbb{E}_\lambda$-equivalent to p <u>or</u> there is no $\le_{\mathfrak{K}}$-extension of M_{i+1} satisfying this. Now we shall prove that M_{λ^+} is λ^+-saturated, recalling Definition [She09c, 0.15]. Now if $N \le_{\mathfrak{K}} M_{\lambda^+}$, $\|N\| \le \lambda$, and $p \in \mathcal{S}(N)$ then there is $i < \lambda^+$ such that $N \le_{\mathfrak{K}} M_i$ and we can find $p' \in \mathcal{S}(M_{\lambda^+})$ extending p. (Why? If clause

[16] It suffices to have: if $M_0 \le_{\mathfrak{K}} M_1 \in K_{\partial^+}$, $M_1 \le_{\mathfrak{K}} M_2 \in K^x_\mu$, and $M_0 \in K_\partial$ then M_1 can be $\le_{\mathfrak{K}}$-embedded into some $M_3 \in K^x_\mu$. Similarly in 7.13.

[17] Why? Assume $M \in K_{\partial+}$. Let $M_2 \in K^x_\mu$, let $M_0 \le_{\mathfrak{K}} M_2$ be of cardinality ∂, let $M_1 \in K_{\partial+}$ be a $\le_{\mathfrak{K}}$-extension of M_0 which there is an $\le_{\mathfrak{K}}$-embedding f of M into M_1 (exists as $\mathfrak{K}_\partial$ has amalgamation and JEP). Lastly, use "$\mathfrak{K}$ has $(\partial, \le \partial^+, \mu)$-amalgamation

$(e)(\alpha)$ holds then this follows by $\mathfrak{K}_{<\mu}$ having amalgamation; see [She09a, 2.8]. If clause $(e)(\beta)$ holds, use "$\mathfrak{K}$ has the $(\lambda, \leq \lambda^+, \mu)$-amalgamation property," recalling $\mathrm{LST}(\mathfrak{K}) \leq \lambda < \chi$.) Hence there is $a \in M_{i+1}$ such that $\mathbf{tp}(a, M_i, M_{i+1})\ \mathbb{E}_\lambda\ (p' \restriction M_i)$, hence a realizes p in M_{i+1}, hence in M_{λ^+}.

Case 2: Not Case 1.

Let I be as in 5.1 with $(\lambda, \theta_2, \theta_1, \mu)$ there standing for $(\mu, \lambda^{++}, \lambda^+, \lambda)$ here, so $|I| = \mu$. Let $M = \mathrm{EM}_{\tau(\mathfrak{K})}(I, \Phi)$, so by 'not Case 1' we can find $p_i \in \mathcal{S}(M)$ for $i < \mu^+$ pairwise non-$\mathbb{E}_\lambda$-equivalent. As $\mathfrak{K}_\lambda$ is a λ-AEC with amalgamation and is stable in λ (by $(*)_0$) we can deduce (see [She09e, gr.6(2)]) that: if $p \in \mathcal{S}(M)$ then for some $N \leq_{\mathfrak{K}} M$ of cardinality λ the type p does not λ-split over N (or see [She99a, 3.2 = 3.2tex(1)]). For each i choose $N_i \leq_{\mathfrak{K}} M$ of cardinality λ such that p_i does not μ-split over N_i. As there is no loss in increasing N_i (as long as it is $\leq_{\mathfrak{K}} M$ and has cardinality λ) without loss of generality,

$(*)_1$ $N_i = \mathrm{EM}_{\tau(\mathfrak{K})}(I_i, \Phi)$ where $I_i \subseteq I$ and $|I_i| = \lambda$, and let $\bar{t}_i = \langle t^i_\varepsilon : \varepsilon < \lambda \rangle$ list I_i with no repetitions.

As $2^\lambda \leq \mu$, without loss of generality the I_i-s are pairwise isomorphic, so without loss of generality for $i, j < \mu^+$, the mapping $t^i_\varepsilon \mapsto t^j_\varepsilon$ is such an isomorphism. Moreover, without loss of generality

$(*)_2$ For every $i, j < \mu^+$ there is an automorphism $\pi_{i,j}$ of I mapping t^i_ε to t^j_ε for $\varepsilon < \lambda$.

[Why? By 5.1(1) as we can replace $\langle p_i : i < \mu^+ \rangle$ by $\langle p_i : i \in \mathcal{U} \rangle$ for every unbounded $\mathcal{U} \subseteq \mu^+$.]

Let $\mathfrak{p}_i$ be the non-λ-splitting scheme of p over N_i (see Definition 7.6). Without loss of generality:

$(*)_3$ For $i, j < \mu^+$, the isomorphism $h_{i,j}$ from $N_j = \mathrm{EM}_{\tau(\mathfrak{K})}(I_j, \Phi)$ onto $N_i = \mathrm{EM}_{\tau(\mathfrak{K})}(I_i, \Phi)$ induced by the mapping $t^j_\zeta \mapsto t^i_\zeta$ (for $\zeta < \lambda$) satisfies
 (i) It is an isomorphism from N_j onto N_i.
 (ii) It maps $\mathfrak{p}_j$ to $\mathfrak{p}_i$.

[Why? For (i) this holds by the definition of $\mathrm{EM}(I_i, \Phi)$. For (ii) let $h_{i,0}$ map $\mathfrak{p}_i$ to $\mathfrak{p}'_i$. The number of schemes is $\leq 2^{2^\lambda}$, so if $\mu \geq 2^{2^\lambda}$ then without loss of generality $i < \mu^+ \Rightarrow \mathfrak{p}'_i = \mathfrak{p}'_1$ hence we are done (with no real loss). If we weaken the assumption $\mu \geq 2^{2^\lambda}$ to $\mu \geq 2^\lambda$ (or even $\mu > \lambda$, so waive $(*)_2$) using 5.1(4) we can find I^+_i such that $I_i \subseteq I^+_i \subseteq I$, $|I^+_i| \leq \lambda^+$, and for every $J \subseteq I$ of cardinality $\leq \lambda$ there is an automorphism of I over I_i mapping J into I^+_i. So only

$$\Big\langle \mathfrak{p}'_i\big((\mathrm{EM}_{\tau(\mathfrak{K})}(I^+_0, \Phi), c, \bar{b})_{c \in \mathrm{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)}/\cong\big) : \bar{b} \in {}^\lambda\big(\mathrm{EM}_{\tau(\mathfrak{K})}(I^+_0, \Phi)\big)\Big\rangle$$

matters (an overkill) but this is determined by $p_i \restriction \mathrm{EM}_{\tau(\mathfrak{K})}(I^+_0, \Phi))$ which $\in \mathcal{S}(\mathrm{EM}_{\tau(\mathfrak{K})}(I^+_0, \Phi))$ by $(*)_0$, and as $\mathfrak{K}$ is stable in λ^+, without loss of generality $\mathfrak{p}'_{1+i} = \mathfrak{p}'_1$ and we are done.]

Now we translate our problem to one on expanded (by unary predicates) linear orders which was treated in §6. Recall that by 5.1(3), we can use $I = \mathrm{EM}_{\{<\}}(I^*, \Psi)$ where $\Psi \in \Upsilon^{\mathrm{lin}}_{\aleph_0}[2]$ (see Definition 0.11(5)) and $I^* = I^{\mathrm{lin}}_{\lambda, \mu \times \lambda^+}$ from 6.12(2) with $\alpha(*) = 2$. Recall that $I^* = I^{\mathrm{lin}}_{\lambda, \mu \times \lambda^{++}}$ is $\mu \times \lambda^{++}$ expanded by

$$P_1 = \{\alpha \in I^* : \mathrm{cf}(\alpha) \geq \lambda^+\},$$

$P_0 = I_* \setminus P_0$ so I^* is a well ordered τ^*_2-model, i.e. $\in K^{\mathrm{lin}}_{\tau^*_2}$, see Definition 0.11(5). Without loss of generality $I_i = \mathrm{EM}_{\{<\}}(I^*_i, \Psi)$ where $I^*_i \subseteq I^*$ has cardinality λ and the pair (I^*, I^*_i) is a reasonable $(\lambda, \alpha(*))$-base which is a wide $(\mu, \lambda, \alpha(*))$-base; see Definition 6.10(3)(4), Claim 6.12(2). Without loss of generality, for every $i < \mu^+$

there is h_i an isomorphism from I^*_0 onto I^*_i such that (see below) the induced function $h^{[1]}_1$ maps $\bar t_0$ to $\bar t_i$. Let $J^* = I^*_0$ and $J = I_0$. We would like to apply §6 for J^*, I^* fixing $\alpha(*) = 2$, $\bar u^* = (u^-, u^+) = (\{0\}, \varnothing)$. So, recalling Definition 6.3(2), for every $h \in \mathrm{inc}^{\bar u^*}_{J^*}(I^*)$ we can naturally define the function $h^{[1]}$ by

$$h^{[1]}\big(\sigma^{\mathrm{EM}(J^*,\Psi)}(t_0,\dots,t_{n-1})\big) = \sigma^{\mathrm{EM}(J^*,\Psi)}(a_{h(t_0)},\dots,a_{h(t_{n-1})})$$

whenever $\sigma(x_0,\dots,x_{n-1})$ is a $\tau(\Psi)$-term and $J^* \models$ "$t_0 < \dots < t_{n-1}$". It is an isomorphism from $\mathrm{EM}_{\{<\}}(J^*,\Psi)$ onto $\mathrm{EM}_{\{<\}}(I^* \restriction \mathrm{rang}(h),\Psi)$ so, as $J^* \subseteq I^*$, by 5.1(5) there is an automorphism $h^{[2]}$ of I extending $h^{[1]}$ and so there is an automorphism $h^{[3]}$ of $\mathrm{EM}(I,\Phi)$ such that $h^{[3]}(a_t) = a_{h^{[2]}(t)}$ for $t \in I$ and

$$h^{[3]}\big(\sigma^{\mathrm{EM}(I,\Phi)}(a_{t_0},\dots,a_{t_{n-1}})\big) = \sigma^{\mathrm{EM}(I,\Phi)}(a_{h^{[2]}(t_0)},\dots,a_{h^{[2]}(t_n)})$$

where $t_0 <_I \dots <_I t_{n-1}$ and $\sigma(x_0,\dots,x_{n-1})$ is a $\tau(\Phi)$-term.

Note that

$(*)_4$ If h', h'' are automorphisms of $\mathrm{EM}_{\tau[\mathfrak{K}]}(I,\Phi)$ extending $h^{[3]} \restriction \mathrm{EM}_{\tau[\mathfrak{K}]}(I_0)$ then $h'(p_0/\mathbb{E}_\lambda) = h''(p_0/\mathbb{E}_\lambda)$.

[Why? Because p_0 does not λ-split over $\mathrm{EM}_{\tau[\mathfrak{K}]}(I_0,\Phi)$.]

We define a two-place relation $\mathcal{E}$ on $\mathrm{inc}_{J^*}(I^*)$ by

$$h_1 \ \mathcal{E}\ h_2 \text{ if } h^{[3]}_1(p_0/\mathbb{E}_\lambda) = h^{[3]}_2(p_0/\mathbb{E}_\lambda).$$

(Note that $h \mapsto h^{[3]}$ is a function, so this is well defined, and $h^{[3]}$ is an automorphism of $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi)$.) By $(*)_4$ clearly $\mathcal{E}$ is an invariant equivalence relation on $\mathrm{inc}^{\bar u^*}_{J^*}(I^*)$ with $> \mu$ equivalence classes as exemplified by $\langle h_i : i < \mu^+\rangle$.

By 6.19 there is $e \in \mathbf{e}(J^*, I^*)$ such that (recalling Definition 6.16) the filter $\mathcal{D}_{\mathcal{E},e}$ has an extension to a non-principal ultrafilter $\mathcal{D}$; so for some regular $\theta \le \lambda$ there is a function g from $\mathrm{dom}(\mathbf{e})/e$ onto θ which maps $\mathcal{D}$ to a uniform ultrafilter $g(\mathcal{D})$ on θ, so $\partial^{\langle\theta\rangle_{\mathrm{tr}}} \le \partial^{\mathrm{dom}(e)/e}/\mathcal{D}_{\mathcal{E},e}$ for every cardinal ∂. Choose such a pair (g,θ) with minimal θ so $\mathcal{D}$ is θ-complete hence $\theta = \aleph_0$ or θ is a measurable cardinal $\le \lambda$. By clause $(c)(\beta)$ of our assumption (justified in the beginning of the proof) there is $\partial \in (\lambda^+, \chi)$ such that $\partial < \partial^{\langle\theta\rangle_{\mathrm{tr}}}$ hence $\partial^+ \le \partial^{\langle\theta\rangle_{\mathrm{tr}}} \le \partial^{\mathrm{dom}(e)/e}/\mathcal{D}_{\mathcal{E},e}$. So, letting $I^0_\partial = I^{\mathrm{lin}}_{\lambda,\partial\times\lambda^{++}} \subseteq I^*$, the set $\{\bar t/\mathcal{E} : \bar t \in \mathrm{inc}_{J^*}(I^*) \text{ and } \mathrm{rang}(\bar t) \subseteq I^0_\partial\}$ has cardinality $> \partial$. Now for each $\bar t \in \mathrm{inc}_{J^*}{}^{\bar u^*}(I^*)$ let $\pi_{\bar t} \in \mathrm{Aut}(I)$ be such that $\pi_{\bar t}(\bar t_0) = \bar t$ and let $\hat\pi_{\bar t}$ be the automorphism of $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi)$ induced by $\pi_{\bar t}$, and let $p_t = \hat\pi_{\bar t}(p_0) \in \mathcal{S}(M)$. Hence

$$\big\{\hat\pi_{\bar t}(p_0) \restriction \mathrm{EM}_{\tau(\mathfrak{K})}(I^{\mathrm{lin}}_{\lambda,\partial\times\lambda^+},\Phi) : \bar t \in \mathrm{inc}_{J^*}{}^{\bar u^*}(I^*) \text{ and } \mathrm{rang}(\bar t) \subseteq I^{\mathrm{lin}}_{\lambda,\partial\times\lambda^{++}}\big\}$$

is of cardinality $> \partial$, contradicting "$\mathfrak{K}$ stable in ∂" from $(*)_0$. $\square_{7.9}$

We note, but we shall not use

Conclusion 7.11. *1) Under the assumptions of 7.9 we have $\kappa(\mathfrak{K}_\mu) = \aleph_0$, see below.*
2) Moreover, $\kappa_{\mathrm{st}}(\mathfrak{K}_\mu) = \varnothing$.

Recall

Definition 7.12. If $\mathfrak{K}_\mu$ is an μ-AEC with amalgamation which is stable, then:

(a) $\kappa(\mathfrak{K}_\mu) = \aleph_0 + \sup\{\kappa^+ : \kappa \text{ regular } \le \mu$ and there is an $\le_{\mathfrak{K}_\mu}$-increasing continuous sequence $\langle M_i : i \le \kappa\rangle$ and $p \in \mathcal{S}(M_\kappa)$ such that M_{2i+2} is universal over M_{2i+1} and $p \restriction M_{2i+2}$ does μ-split over $M_{2i+1}\}$

(b) $\kappa_{\mathrm{sp}}(\mathfrak{K}_\mu) := \{\kappa \le \mu : \kappa$ regular and there is an $\le_{\mathfrak{K}_\mu}$-increasing continuous sequence $\langle M_i : i \le \kappa\rangle$ and $p \in \mathcal{S}(M_\kappa)$ which μ-splits over M_i for each $i < \kappa$ and M_{2i+2} is universal over $M_{2i+1}\}$.

Proof. By playing with $\mathrm{EM}(I,\Phi)$, (or see Claim [She99a, 5.7=5.7tex] and Definition [She99a, 4.9=4.4tex]). $\square_{7.11}$

Question: Can we omit assumption 7.9(c) (see below so $\chi = \mathrm{LST}(\mathfrak{K})$)?

Theorem 7.13. *For some cardinal $\lambda_* < \chi$ and a cardinal $\lambda_{**} < \beth_{1,1}(\lambda_*^{+\omega})$ above λ_*, $\mathfrak{K}$ is categorical in every cardinal $\lambda \geq \lambda_{**}$ but in no $\lambda \in (\lambda_*, \lambda_{**})$, provided that:*

$\circledast^{\mu,\chi}_{\mathfrak{K}}$ *(a) K is an AEC categorical in μ.*
(b) $\mathfrak{K}$ has amalgamation and JEP in every $\lambda < \aleph_\chi$, $\lambda \geq \mathrm{LST}(\mathfrak{K})$.
(c) χ is a limit cardinal, $\mathrm{cf}(\chi) > \mathrm{LST}(\mathfrak{K})$, and for arbitrarily large $\lambda < \chi$ the sequence $\langle 2^{\lambda^{+n}} : n < \omega\rangle$ is increasing.
(d) $\mu > \beth_{1,1}(\lambda)$ for every $\lambda < \chi$ hence $\mu \geq \aleph_\chi$.
(e) Every $M \in K_{<\aleph_\chi}$ has a $\leq_{\mathfrak{K}}$-extension in K_μ.

Remark 7.14. 1) Concerning [She99a] note

(a) There the central case was $\mathfrak{K}$ with full amalgamation (not just below $\chi \ll \mu$!), trying to concentrate on the difficulty of lack of localness,
(b) When we use clause (e), this is just to get the "$M \in K_\mu$ is λ-saturated"; this is where we use 7.9.
(c) We demand "$\mathrm{cf}(\chi) > \mathrm{LST}(\mathfrak{K})$" to prove locality.

2) We rely on [She09c] and [She09e] in the end.

3) The assumption (e) of 7.13 follows if $\mathfrak{K}$ has amalgamation in every $\lambda' \leq \beth_{1,1}(\lambda)$ for $\lambda < \chi$, which is a reasonable assumption.

4) Most of the proof works even if we weaken assumption (a) to "$\mathfrak{K}$ is strongly solvable in μ" and even to weakly solvable; i.e. up to $\boxdot_7$. We continue in [S$^+$b]; see more there.

5) Theorem 7.13 also continues Kolman-Shelah [KS96], [She01], as its assumptions are proved there.

Proof. Let $\kappa = \mathrm{LST}(\mathfrak{K})$, and let $\Phi \in \Upsilon^{\mathrm{or}}_\kappa[\mathfrak{K}]$ be as guaranteed by 0.9(1), hence

$(*)_1$ If $I \in K^{\mathrm{lin}}_\lambda$ then $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi)$ belongs to K_λ for $\lambda \geq \mathrm{LST}(\mathfrak{K})$ (and in the strongly solvable case, $I \in K^{\mathrm{lin}}_\mu \Rightarrow \mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi) \in K^x_\mu$).

and

$(*)_2$ If $I \subseteq J$ are from K^{lin} then $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi) \leq_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(J,\Phi)$.

Also

$(*)_3$ $\langle \mathcal{S}_{\mathfrak{K}}(M) : M \in \mathfrak{K}_{<\aleph_\chi}\rangle$ has the reasonable basic properties.

[Why? See [She09c, 0.12] and [She09c, 0.12A]; because $\mathfrak{K}_{<\aleph_\chi}$ has the amalgamation property by clause (b) of the assumption $\circledast^{\mu,\chi}_{\mathfrak{K}}$.]

$(*)_4$ If $M \in K_\mu$ then M is χ-saturated (hence χ-model homogeneous).

[Why? We shall prove that if $\mathrm{LST}(\mathfrak{K}) \leq \lambda < \chi$ and $M \in K^x_\mu$ <u>then</u> M is λ^+-saturated. We shall show that all the assumptions of 7.9, with (μ,χ,λ) there standing for $(\mu,\aleph_\chi,\lambda)$ here, hold. Let us check: clause (a) of 7.9 means "$\mathfrak{K}$ is categorical in μ" (or is strongly solvable) which holds by clause (a) of $\circledast^{\mu,\chi}_{\mathfrak{K}}$. Clause (b) of 7.9 says that $\mathrm{LST}(\mathfrak{K}) \leq \lambda < \aleph_\chi \leq \mu$ and $2^{2^\lambda} \leq \mu$; the first holds because of the way λ was chosen above and the second holds as clause (d) of $\circledast^{\mu,\chi}_{\mathfrak{K}}$ says that $\mu > \beth_{1,1}(\lambda)$ and $\mu \geq \aleph_\chi$. Clause $(c)(\alpha)$ of 7.9 holds as $\lambda^{+\lambda^{+4}} < \aleph_{\lambda^{+5}}$ which is $< \aleph_\chi$ as χ is a limit cardinal and $\aleph_\chi$ here plays the role of χ there. Clause (d) of 7.9 says $\mathfrak{K}_{\geq\partial} \neq \varnothing$ for every cardinal ∂, holds by $(*)_1$ above. Lastly, clause (e) of 7.9

holds: more exactly, clauses (e)(β)(i)+(iii) hold by clauses (b) + (e) of $\circledast^{\mu,\chi}_{\mathfrak{K}}$ and they suffice.

We have shown that all the assumptions of 7.9 hold, hence its conclusion, which says (as $M \in K_\mu$) that M is λ^+-saturated. The "χ-model homogeneous" holds by [She09c, 0.19].]

$(*)_5$ If $M \le_{\mathfrak{K}} N$ are from K^x_μ then $M \prec_{\mathbb{L}_{\infty,\chi}[\mathfrak{K}]} N$.

[Why? Obvious by $(*)_4$.]

$(*)_6$ If $\lambda \in (\kappa,\chi)$ and $I \in K^{\rm lin}_{\ge\lambda}$ is λ-wide then $\mathrm{EM}_{\tau(\mathfrak{K})}(I,\Phi)$ is λ-saturated; moreover, if $I^+ \in K^{\rm lin}_\lambda$ is wide over I then every $p \in \mathcal{S}(\mathrm{EM}_{\tau(K)}(I,\Phi))$ is realized in $\mathrm{EM}_{\tau(\mathfrak{K})}(I^+,\Phi)$.

[Why? By 1.14, its assumption "Φ satisfies the conclusion of 1.13" holds by $(*)_5$, (or as in [She99a, 6.8=6.5tex]). The "moreover" is immediate by $(*)_4$ as in the proof of $\circledast(d)$ inside the proof of 7.2 above, or see the proof of $(*)_{10}$ below.]

$(*)_7$ $\mathfrak{K}$ is stable in λ when $\kappa \le \lambda < \chi$.

[Why? Recalling clause (e) of the assumption of 7.13, by Claim 7.9 (or more accurately, $(*)_0$ in its proof) as we have proved (in the proof of $(*)_4$) that the assumptions of 7.9 hold with (μ,χ,λ) there standing for $(\mu,\aleph_\chi,\lambda)$ here.]

$(*)_8$ If $\lambda \in [\kappa,\chi)$ and $M \in K^x_\lambda$ <u>then</u> there is $N \in \mathfrak{K}_\lambda$ which is $(\lambda,\aleph_0)$-brimmed over M.

[Why? By $(*)_7$ and [She09c, 0.22(1)(b)] remembering the amalgamation, clause (b) of the assumption of the theorem.]

$(*)_9$ If $\langle M_\alpha : \alpha \le \lambda\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous, $\kappa \le \|M_\lambda\| \le \lambda < \chi$, <u>then</u> no $p \in \mathcal{S}_{\mathfrak{K}}(M_\lambda)$ satisfies "$p \restriction M_{i+1}$ does λ-split over M_i for every $i < \lambda$."

[Why? Otherwise we get a contradiction to stability in λ, i.e. $(*)_7$, see in [She09e, gr.6](1B), using amalgamation (using the tree ${}^{\theta>}2$ when $\theta = \min\{\partial : 2^\partial > \lambda\}$; also we can prove it as in the proof of case 2 inside the proof of 7.9.]

We could use more

$(*)_{10}$ If I_1, I_2 are wide linear orders of cardinality $\lambda \in (\kappa,\chi)$ and I_2 is wide over I_1 (so $I_1 \subseteq I_2$) and $M_\ell = \mathrm{EM}_{\tau(\mathfrak{K})}(I_\ell,\Phi)$,<u>then</u> M_2 is universal over M_1 and even brimmed over I_1, even (λ,∂)-brimmed for any regular $\partial < \lambda$.

[Why? As I_2 is wide over I_1, we can find a sequence $\langle J_\gamma : \gamma < \lambda\rangle$ of pairwise disjoint subsets of $I_2 \setminus I_1$ such that each J_γ is a convex subset of I_2 and in J_γ there is a monotonic sequence $\langle t_{\gamma,n} : n < \omega\rangle$ of members. Let $\langle \gamma_\varepsilon : \varepsilon < \lambda \times \partial\rangle$ list λ, and let $I_{2,0} = I_1$, $I_{2,1+\varepsilon} = I_2 \setminus \bigcup\{J_{\gamma_\zeta} : \zeta \in [1+\varepsilon, \lambda\times\partial)\}$, and $M'_\zeta = \mathrm{EM}_{\tau(\mathfrak{K})}(I_{2,\varepsilon},\Phi)$. So $\langle M'_\zeta : \zeta \le \lambda\times\partial\rangle$ is $\le_{\mathfrak{K}}$-increasing continuous sequence of members of K_λ; the first member is M_1, the last member M_2.

By [She09c, 0.22(4)(b)] it is enough to prove that if $\varepsilon < \lambda\times\partial$ and $p \in \mathcal{S}(M_\varepsilon)$ then p is realized in $M_{\varepsilon+1}$. As I_1 is wide of cardinality λ, so is $I_{2,\varepsilon}$, hence M'_ε is saturated. Also, for each ε we can find a linear order $I^+_{2,\varepsilon}$ of cardinality λ such that $I_{2,\varepsilon+1} \subseteq I^+_{2,\varepsilon}$ and $J^+_\varepsilon = I^+_{2,\varepsilon+1} \setminus I_{2,\varepsilon}$ is a convex subset of $I^+_{2,\varepsilon+1}$ and is a wide linear order of cardinality λ which is strongly $\aleph_0$-homogeneous. (Recall $J_{\gamma_\varepsilon} \subseteq J^+_{\gamma_\varepsilon}$ is infinite.) So in $M^+_{\varepsilon+1} = \mathrm{EM}_{\tau(\mathfrak{K})}(I^+_{2,\varepsilon+2},\Phi)$ every $p \in \mathcal{S}(M^1_\varepsilon)$ is realized (as $I^+_{2,\varepsilon+1}$ is wide over $I_{2,\varepsilon}$, as J^+_ε is wide of cardinality λ); moreover, [they are] realized in $M'_{\varepsilon+1}$.

(Why? By the strong $\aleph_0$-homogeneous [linear order] every element, and even finite sequence, from $M^+_{\varepsilon+1}$ can be mapped by some automorphism of $M^+_{\varepsilon+1}$ over M_ε into $M_{\varepsilon+1}$.) As said above, this suffices.]

$\circledast_1$ χ_* is well defined and exists in the interval (κ, χ), where

$$\chi_* = \min\{\theta : \kappa < \theta < \chi, \text{ and for every saturated } M \in \mathfrak{K}, \\ \text{if } \theta \le \|M\| < \chi, \text{ every } p \in \mathcal{S}(M) \text{ is } \theta\text{-local}\}.$$

(see Definition 7.4(2)).

[Why? By 7.2, which we apply with (μ, κ) there standing for (χ, κ) here, recalling $\kappa = \mathrm{LST}(\mathfrak{K})$. This is OK as: clause (a) in 7.2 holds by clause (c) of the assumption here, and clause (b) in 7.2 holds by clause (b) of the assumption here, as $\chi \le \aleph_\chi$. Lastly, clause (c) in 7.2 easily follows by $(*)_6$ above.]

$\circledast_2$ If $\lambda \in (\kappa, \chi)$ and $\langle M_i : i \le \delta\rangle$ is $\le_{\mathfrak{K}_\lambda}$-increasing continuous and M_{i+1} is $\le_{\mathfrak{K}}$-universal over M_i for $i < \delta$ then M_δ is saturated and moreover every $p \in \mathcal{S}(M_\delta)$ does not λ-split over M_α for some $\alpha < \delta$.

[Why? For $i \le \delta$ let I_i be the linear order $\lambda \times \lambda \times (1 + i)$ and $M'_i = \mathrm{EM}_{\tau(\mathfrak{K})}(I_i, \Phi)$. So $\langle M'_i : i \le \delta\rangle$ is $\le_{\mathfrak{K}_\lambda}$-increasing continuous. Also, for $i \le \delta$, $\zeta \le \lambda$ let

$$I_{i,\zeta} = \lambda \times \lambda \times (1 + i) + \lambda \times \zeta$$

and $M'_{i,\zeta} = \mathrm{EM}_{\tau(\mathfrak{K})}(I_{i,\zeta}, \Phi)$, so for each $i < \delta$ the sequence $\langle M'_{i,\zeta} : \zeta \le \lambda\rangle$ is $\le_{\mathfrak{K}_\lambda}$-increasing continuous, $M'_{i,0} = M'_i$, and $M'_{i,\lambda} = M'_{i+1}$. Now for $i < \delta$, $\zeta < \lambda$ every $p \in \mathcal{S}(M_{i,\zeta})$ is realized in $M'_{i,\zeta+1}$ by $(*)_6$ and the definition of type, varying the linear order. By [She09c, 0.22(4)(b)] the model M'_{i+1} is $\le_{\mathfrak{K}_\lambda}$-universal over M'_i and by Definition [She09c, 0.21] the models M'_δ and M_δ are $(\lambda, \mathrm{cf}(\delta))$-brimmed, hence by [She09c, 0.22(3)] they are isomorphic. But M'_δ is saturated by $(*)_6$, hence M_δ must be as well.

What about the "moreover"? (Note that if $\lambda = \lambda^{\mathrm{cf}(\delta)}$ then $(*)_9$ does not cover it.) We can easily find $\langle I''_\alpha : \alpha \le \lambda \times \delta + 1\rangle$ such that:

(a) I''_α is a linear order of cardinality λ into which λ can be embedded.
(b) I''_α is increasing continuous with α.
(c) I''_α is an initial segment of I''_β for $\alpha < \beta \le \delta + 1$.
(d) $I''_{\alpha+1}$ has a subset of order types $\lambda \times \lambda$ whose convex hull is disjoint to I''_α.
(e) If $\alpha \le \beta < \lambda \times \delta$ and $s \in I''_{\lambda\times\delta+1} \setminus I''_{\lambda\times\delta}$ then there is an automorphism $\pi_{\alpha,\beta,s}$ of $I''_{\lambda\times\delta+1}$ mapping $I''_{\beta+1}$ onto $I''_{\lambda\times\delta}$ and is over

$$I''_\alpha \cup \big\{t \in I''_{\lambda\times\delta+1} : s \le_{I''_{\lambda\times\delta+1}} t\big\}.$$

Let $M''_\alpha = \mathrm{EM}_{\tau(\mathfrak{K})}(I''_\alpha, \Phi)$, so $\langle M''_{\lambda\times\alpha} : \alpha \le \delta\rangle$ has the properties of $\langle M'_\alpha : \alpha \le \delta\rangle$, i.e. every $p \in \mathcal{S}(M''_\alpha)$ is realized in $M''_{\alpha+1}$, hence $M''_{\alpha+\lambda}$ is $\le_{\mathfrak{K}_\lambda}$-universal over M''_α. So (easily, or see [She09c, 0.22,0.21]) there is an isomorphism f from M_δ onto $M''_{\lambda\times\delta}$ such that $M''_{\lambda\alpha} \le_{\mathfrak{K}} f(M_{\alpha+1}) \le M''_{\lambda\alpha+2}$. So it suffices to prove the "moreover" for $\langle M''_{\lambda\times\alpha} : \alpha \le \delta\rangle$, equivalently for $\langle M''_\alpha : \alpha \le \lambda \times \delta\rangle$. Let $p \in \mathcal{S}(M''_{\lambda\times\delta})$, so some $a \in M''_{\lambda\times\delta+1}$ realizes it, hence for some $t_0 < \ldots < t_{n-1}$ from $I''_{\lambda\times\delta+1}$ and τ_Φ-term $\sigma(x_0, \ldots, x_{n-1})$ we have $a = \sigma^{\mathrm{EM}(I''_{\lambda\times\delta+1},\Phi)}(a_{t_0}, \ldots, a_{t_{n-1}})$. It follows that for some $m \le n$ we have $t_\ell \in I''_{\lambda\times\delta} \Leftrightarrow \ell < m$. Let $\alpha < \lambda \times \delta$ be such that $\{t_\ell : \ell < m\} \subseteq I''_\alpha$; if $m = n$ choose any $t_n \in I''_{\lambda\times\delta+1} \setminus I''_{\lambda\times\delta}$. If $\beta \in (\alpha, \lambda \times \delta)$ and $\mathbf{tp}_{\mathfrak{K}}(a, M''_\delta, M''_{\delta+1})$ does λ-split over M''_β then $\pi' := \pi_{\beta,\beta,t_m}$ is an automorphism of $I''_{\lambda\times\delta+1}$ mapping $I''_{\beta+1}$ onto $I''_{\lambda\times\delta}$ and is over $I''_\beta \cup \{s \in I''_{\lambda\times\delta+1} : t_m \le_{I''_{\lambda\times\delta+1}} s\}$ hence it is the identity on $\{t_\ell : \ell < n\}$. Now π' induces an automorphism $\hat{\pi}'$ of $\mathrm{EM}_{\tau(\mathfrak{K})}(I''_{\lambda\times\delta+1}, \Phi)$, so clearly it maps a to itself, maps $\mathbf{tp}_{\mathfrak{K}}(a, M''_{\beta+1}, M''_{\lambda\times\delta+1})$ to $\mathbf{tp}_{\mathfrak{K}}(a, M''_{\lambda\times\delta}, M''_{\lambda\times\delta+1})$, and it maps M''_β onto itself, hence also $\mathbf{tp}_{\mathfrak{K}}(a, M''_{\beta+1}, M''_{\delta+1})$ does λ-split over M''_β. So if for some $\beta \in (\alpha, \lambda \times \delta)$, the type $\mathbf{tp}_{\mathfrak{K}}(a, M''_\delta, M''_{\delta+1})$ does not λ-split over M''_β we get the desired conclusion, but otherwise this contradicts $(*)_9$.]

$\circledast_3$ If $\lambda \in [\chi_*, \chi)$, $M \in K_\lambda$ is saturated, and $p \in \mathcal{S}(M)$ then for some N we have:

(a) $N \leq_{\mathfrak{K}} M$

(b) $N \in K_{\chi_*}$ is saturated.

(c) p does not χ_*-split over N.

(d) p does not λ-split over N (follows by (a),(b),(c)).

[Why does $\circledast_3$ hold? For clauses (a),(b),(c) use $\circledast_2$ or just $(*)_9$; for clause (d) use localness, i.e. recall $\circledast_1$ and Definition 7.4.]

$\circledast_4$ Assume $\lambda \in [\kappa, \chi)$ and $M_1 \leq_{\mathfrak{K}} M_2 \leq_{\mathfrak{K}} M_3$ are members of K, M_2 is λ^+-saturated and $p \in \mathcal{S}(M_3)$. If $N_\ell \leq_{\mathfrak{K}} M_\ell$ is from $K_{\leq\lambda}$ and $p \restriction M_{\ell+1}$ does not λ-split over N_ℓ for $\ell = 1,2$ then p does not λ-split over N_1.

[Why? Easy manipulations. Without loss of generality, $N_1 \leq_{\mathfrak{K}} N_2$ as we can increase N_2. So for some pair (M_4, a) we have $M_3 \leq_{\mathfrak{K}} M_4$, $a \in M_4$, and $p = \mathbf{tp}_{\mathfrak{K}}(a, M_3, M_4)$. Assume $\alpha < \lambda^+$ and let $\bar b, \bar c \in {}^\alpha(M_3)$ be such that $\mathbf{tp}_{\mathfrak{K}}(\bar b, N_1, M_3) = \mathbf{tp}_{\mathfrak{K}}(\bar c, N_1, M_3)$. As M_2 is λ^+-saturated and $N_2 \leq_{\mathfrak{K}} M_2 \leq_{\mathfrak{K}} M_3$ we can find $\bar b', \bar c' \in {}^\alpha(M_2)$ such that $\mathbf{tp}_{\mathfrak{K}}(\bar b' {}^\frown \bar c', N_2, M_3) = \mathbf{tp}_{\mathfrak{K}}(\bar b {}^\frown \bar c, N_2, M_3)$ using [She09c, 0.19]. Hence

$$\mathbf{tp}_{\mathfrak{K}}(\bar b', N_1, M_3) = \mathbf{tp}_{\mathfrak{K}}(\bar b, N_1, M_3) = \mathbf{tp}_{\mathfrak{K}}(\bar c, N_1, M_3) = \mathbf{tp}_{\mathfrak{K}}(\bar c', N_1, M_3).$$

By the choice of (M_4, a), and the assumption on N_1 that $p \restriction M_2$ does not λ-split over N_1, we get

$$\mathbf{tp}_{\mathfrak{K}}\big(\langle a\rangle {}^\frown \bar b', N_1, M_4\big) = \mathbf{tp}_{\mathfrak{K}}\big(\langle a\rangle {}^\frown \bar c', N_1, M_4\big).$$

Clearly $\mathbf{tp}_{\mathfrak{K}}(\bar b', N_2, M_3) = \mathbf{tp}_{\mathfrak{K}}(\bar b, N_2, M_3)$ hence by the choice of (M_4, a) and the assumption on N_2 that p does not λ-split over N_2 we have $\mathbf{tp}_{\mathfrak{K}}(\langle a\rangle {}^\frown \bar b', N_2, M_4) = \mathbf{tp}(\langle a\rangle {}^\frown \bar b, N_2, M_4)$ hence by monotonicity

$$\mathbf{tp}_{\mathfrak{K}}(\langle a\rangle {}^\frown \bar b', N_1, M_4) = \mathbf{tp}_{\mathfrak{K}}(\langle a\rangle {}^\frown \bar b, N_1, M_4).$$

Similarly

$$\mathbf{tp}_{\mathfrak{K}}(\langle a\rangle {}^\frown \bar c', N_1, M_4) = \mathbf{tp}_{\mathfrak{K}}(\langle a\rangle {}^\frown \bar c, N_1, M_4).$$

As equality of types is transitive

$$\mathbf{tp}_{\mathfrak{K}}(\langle a\rangle {}^\frown \bar c, N_1, M_4) = \mathbf{tp}_{\mathfrak{K}}(\langle a\rangle {}^\frown \bar c', N_1, M_4) = \mathbf{tp}_{\mathfrak{K}}(\langle a\rangle {}^\frown \bar b', N_1, M_4) = \mathbf{tp}_{\mathfrak{K}}(\langle a\rangle {}^\frown \bar b, N_1, M_4)$$

as required.]

$\circledast_5$ Assume $I_3 = I_0 + I_1' + I_2'$ are wide linear orders of cardinality λ, where $\chi > \lambda > \kappa$, and let $I_\ell = I_0 + I'_\ell$ for $\ell = 1,2$ and $M_\ell = \mathrm{EM}_{\tau(\mathfrak{K})}(I_\ell, \Phi)$ for $\ell = 0,1,2,3$. If $\ell \in \{1,2\}$ and $\bar a \in {}^{\lambda>}(M_\ell)$ then $\mathbf{tp}_{\mathfrak{K}_\lambda}(\bar a, M_{3-\ell}, M_3)$ does not λ-split over M_0. (Moreover, if $\mathbf{tp}_{\mathfrak{K}_\lambda}(\bar a, M_0, M_3)$ does not λ-split over $N \in K_{\leq\lambda}$ then also $\mathbf{tp}_{\mathfrak{K}_\lambda}(\bar a, M_{3-\ell}, M_3)$ does not λ-split over N).

[Why? For $\ell = 2$, if the desired conclusion fails we get a contradiction as in the proof of $\circledast_2$, so for $\ell = 2$ we get the conclusion. For $\ell = 1$ if the desired conclusion fails (but it holds for $\ell = 2$) we get a contradiction to categoricity in μ by the order property (by 1.5).]

$\circledast_6$ If $\lambda \in (\chi_*, \chi)$, $\delta < \lambda^+$, $\langle M_i : i \leq \delta\rangle$ is $\leq_{\mathfrak{K}_\lambda}$-increasing continuous, and $i < \delta \Rightarrow M_i$ saturated then M_δ is saturated.

[Why? Let $N \leq_{\mathfrak{K}} M_\delta$, $\|N\| < \lambda$, and $p \in \mathcal{S}(N)$. If $\mathrm{cf}(\delta) > \|N\|$ this is easy so assume $\mathrm{cf}(\delta) \leq \|N\|$, hence $\mathrm{cf}(\delta) < \lambda$ and without loss of generality $\delta = \mathrm{cf}(\delta)$. Choose a cardinal θ such that

$$\mathrm{LST}(\mathfrak{K}) < \chi_* + |\mathrm{cf}(\delta)| + \|N\| \leq \theta < \lambda$$

and $\|N\|^+ < \lambda \Rightarrow \|N\| < \theta$, and let $q \in \mathcal{S}(M_\delta)$ extend p; this exists as $\mathfrak{K}_{\leq\lambda}$ has amalgamation.

Now for every $X \subseteq M_\delta$ of cardinality $\le \theta$, we can choose $N_i \le_{\mathfrak{K}} M_i$ by induction on $i \le \delta$ such that $N_i \in K_\theta$ is saturated, is $\le_{\mathfrak{K}}$-increasing continuous with i, N_i is $\le_{\mathfrak{K}}$-universal over N_j, and includes $(X \cup N) \cap M_i$ when $i = j+1$. So by $\circledast_2$ (we justify the choice of N_i for limit i and) the model N_δ is saturated, so if $\|N\|^+ < \lambda$ then $N \le_{\mathfrak{K}} N_\delta$, N_δ is saturated of cardinality $\theta > \|N\|$ so we are done as $N_\delta \le_{\mathfrak{K}} M_\delta$. So without loss of generality $\lambda = \|N\|^+$ hence $\lambda = \theta^+$.

Also, for some $\alpha_* < \delta$ and $N_* \le_{\mathfrak{K}} M_{\alpha_*}$ of cardinality θ, the type q does not θ-split over N_*.

[Why? Otherwise we choose (N_i, N_i^+) by induction on $i \le \delta$ such that $N_i \le_{\mathfrak{K}} N_i^+$ are from K_θ, $N_i \le_{\mathfrak{K}} M_i$, $N_i^+ \le_{\mathfrak{K}} M_\delta$, N_i is $\le_{\mathfrak{K}}$-increasing continuous, N_i is $\le_{\mathfrak{K}}$-universal over N_j if $i = j+1$, $q \restriction N_i^+$ does θ-split over N_i, and

$$\bigcup\{N_j^+ \cap M_i : j < i\} \subseteq N_i.$$

In the end we get a contradiction to $\circledast_2$.]

We can find $N' \le_{\mathfrak{K}} M_{\alpha_*}$ from K_{χ_*} such that $q \restriction M_{\alpha_*}$ does not θ-split over N', (why? by $\circledast_3$) and without loss of generality $N' \le_{\mathfrak{K}} N_*$ and $N' \le_{\mathfrak{K}} N$. Also, q does not θ-split over N' (why? by applying $\circledast_4$, with $\theta, N_*, M_{\alpha_*}, M_\delta$ here standing for $\lambda, M_1, M_2, M_3, N_1, N_2$ there; or use $N' = N_*$).

By $(*)_6$ as M_{α_*} is saturated without loss of generality $M_{\alpha_*} = \mathrm{EM}_{\tau(\mathfrak{K})}(\lambda, \Phi)$ and for $\varepsilon < \lambda$ let $M_{\alpha_*,\varepsilon} = \mathrm{EM}_{\tau(\mathfrak{K})}(\theta \times \theta \times (1+\varepsilon), \Phi)$, so $M_{\alpha_*,\varepsilon} \in K_\theta$ is saturated and is brimmed over $M_{\alpha^*,\zeta}$ when $\varepsilon = \zeta+1$ by $(*)_{10}$. So for each $\varepsilon < \lambda$ there is $a_\varepsilon \in M_{\alpha^*,\varepsilon+1}$ realizing $q \restriction M_{\alpha_*,\varepsilon}$. Also without loss of generality, $M_\delta \le_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(\lambda+\lambda, \Phi)$ as in the proof of $\circledast_2$ or by $(*)_{10}$, now for some $\varepsilon(*) < \lambda$ we have $N \le_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(I_2, \Phi)$ and $N_* \le_{\mathfrak{K}} \mathrm{EM}_{\tau(\mathfrak{K})}(I_0, \Phi)$ where

$$I_0 = \theta \times \theta \times (1+\varepsilon(*)) \text{ and } I_2 = [\lambda, \lambda+\varepsilon(*)) \cup I_0.$$

Let $I_1 = \theta \times \theta \times \zeta(*)$, where $\zeta(*) \in (\varepsilon(*), \lambda)$ is large enough such that $a_{\varepsilon(*)} \in \mathrm{EM}_{\tau(\mathfrak{K})}(I_1, \Phi)$, e.g. $\zeta(*) = 1+\varepsilon(*)+1$ and let $I_3 = I_1 \cup I_2 \subseteq \lambda+\lambda$. Let $M'_\ell = \mathrm{EM}_{\tau(\mathfrak{K})}(I_\ell, \Phi)$ for $\ell = 0,1,2,3$.

Now we apply $\circledast_5$, the "moreover" with $\theta, I_0, I_1, I_2, I_1 \setminus I_0, I_2 \setminus I_0, a_{\varepsilon(*)}, N'$ here standing for $\lambda, I_0, I_1, I_2, I'_1, I'_2, \bar{a}, N$ there, and we conclude that $\mathbf{tp}_{\mathfrak{K}_\lambda}(a_{\varepsilon(*)}, M'_2, M'_3)$ does not θ-split over N'.

As $N' \le_{\mathfrak{K}} M'_0 \le_{\mathfrak{K}} M'_2$ also the type $q' := \mathbf{tp}_{\mathfrak{K}_\lambda}(a_{\varepsilon(*)}, M'_2, M'_3)$ does not θ-split over N'. Let us sum up: $q \restriction M'_2$ and q' belong to $\mathcal{S}_{\mathfrak{K}_\lambda}(M'_2)$, [something] does not θ-split over N', $N' \in K_{\chi_*}$ and $\chi_* \le \theta$. Also $N' \le_{\mathfrak{K}_*} M'_0 \le_{\mathfrak{K}_*} M'_2$, the model M'_0 is θ-saturated, and $q \restriction M_{\alpha_*} = q' \restriction M_{\alpha_*}$. By the last two sentences obviously $q = q'$ (it may be more transparent to consider $q \restriction (\le \chi_*) = q' \restriction (\le \chi_*)$), so we are done proving $\circledast_6$.]

$\circledast_7$ If $\lambda \in (\chi_*, \chi)$ then the saturated $M \in \mathfrak{K}_\lambda$ is superlimit.

[Why? By $\circledast_6$ (existence by $(*)_6$, the non-maximality by $(*)_6$ + uniqueness; you may look at [She99a, 6.7=6.4tex(1)].]

Now we have arrived to the main point:

$\odot_1$ If $\lambda \in (\chi_*, \chi)$ then $\mathfrak{s}_\lambda$ is a full good λ-frame, $K_{\mathfrak{s}_\lambda}$ categorical, where $\mathfrak{s}_\lambda$ is defined by

(a) $\mathfrak{K}_{\mathfrak{s}_\lambda} = \mathfrak{K}_\lambda \restriction \{M \in K_\lambda : M \text{ saturated}\}$

(b) $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}_\lambda}(M) = \mathcal{S}^{\mathrm{na}}_{\mathfrak{s}_\lambda}(M) := \{\mathbf{tp}_{\mathfrak{s}}(a, M, N) : M \le_{\mathfrak{K}_\lambda} N \text{ and } a \in N \setminus M\}$ for $M \in K_{\mathfrak{s}_\lambda}$.

(c) $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}_\lambda}(M_2)$ does not fork over M_1 when $M_1 \le_{\mathfrak{s}_\lambda} M_2$ and for some $M \le_{\mathfrak{K}} M_1$ of cardinality χ_*, the type p does not χ_*-split over N.

[Why? We check the clauses of Definition [She09c, 1.1].

$K_{\mathfrak{s}_\lambda}$ is categorical:

By [She09c, 0.34](1) and $\circledast_7$.

Clause (A),Clause (B):
By $\circledast_7$, recalling that there is a saturated $M \in K_{\mathfrak{s}_\lambda}$ (and it is not $<_{\mathfrak{s}_\lambda}$-maximal) by $(*)_6$ and trivially recalling [She09c, 0.34], of course.

Clause (C):
By categoricity and $(*)_6$ clearly no $M \in K_{\mathfrak{s}_\lambda}$ is maximal; amalgamation and JEP holds by clause (b) of the assumption of the claim.

Clause (D)(a),(b): By the definition.

Clause (D)(c): Density is obvious; in fact $\mathfrak{s}_\lambda$ is full.

Clause (D)(d): (bs - stability).
Easily $\mathcal{S}_{\mathfrak{s}_\lambda}(M) = \mathcal{S}_{\mathfrak{K}_\lambda}(M)$ which has cardinality $\leq \lambda$ by the moreover in $(*)_6$.

Clause (E)(a): By the definition.

Clause (E)(b): Monotonicity (of non-forking).
By the definition of "does not χ_*-split".

Clause (E)(c): Local character.
Why? Let $\langle M_\alpha : \alpha \leq \delta \rangle$ be $\leq_{\mathfrak{s}_\lambda}$-increasing continuous, $\delta < \lambda^+$ and $q \in \mathcal{S}^{\rm bs}_{\mathfrak{s}_\lambda}(M_\delta)$. Using the third paragraph of the proof of $\circledast_6$ for $\theta = \chi_*$, for some $\alpha_* < \delta$ and $N_* \leq_{\mathfrak{s}_\lambda} M_{\alpha_*}$ of cardinality θ the type q does not θ-split over N_*. So clearly q does not fork over M_{α_*} (for $\mathfrak{s}_\lambda$), as required.

Clause (E)(d): Transitivity of non-forking.
By $\circledast_4$.

Clause (E)(e): Uniqueness.
Holds by the choice of χ_*, i.e. by $\circledast_1$.

Clause (E)(f): Symmetry.
Why? Let M_ℓ for $\ell \leq 3$ and a_0, a_1, a_2 be as in (E)(f)$'$ in [She09c, 1.16E]. We can find a $\leq_{\mathfrak{K}}$-increasing continuous sequence $\langle M_{0,\alpha} : \alpha \leq \lambda^+ \rangle$ such that $M_{0,0} = M_0$, $M_{0,\alpha+1}$ is $\leq_{\mathfrak{s}_\lambda}$-universal over $M_{0,\alpha}$, and without loss of generality $M_{0,\alpha} = {\rm EM}_{\tau(\mathfrak{K})}(\gamma_\alpha, \Phi)$ so it is $\leq_{\mathfrak{K}}$-increasing continuous, and λ divides γ_α.

By (E)(g) proved below we can find $a^\ell_\alpha \in M_{0,\alpha+1}$ realizing $\mathbf{tp}_{\mathfrak{s}_\lambda}(a_\ell, M_0, M_{\ell+1})$ such that $\mathbf{tp}_{\mathfrak{s}_\lambda}(a^\ell_\alpha, M_{0,\alpha}, M_{0,\alpha+1})$ does not fork over $M_0 = M_{0,0}$ for $\ell = 1, 2$. We can find $N_* \leq_{\mathfrak{K}} M_0$ of cardinality χ_* such that $\mathbf{tp}_{\mathfrak{s}_\lambda}(\langle a_1, a_2 \rangle, M_0, M_3)$ does not χ_*-split over N_* so $N_* \leq_{\mathfrak{K}} M_{0,0}$.

Then as in 1.5 we get a contradiction (recalling [She09c, 1.16E]).

Clause (E)(g): Extension existence.
If $M \leq_{\mathfrak{s}_\lambda} N$ and $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}_\lambda}(M) = \mathcal{S}^{\rm na}_{\mathfrak{K}}(M)$, then p does not χ_*-split over M_* for some $M_* \leq_{\mathfrak{K}} M$ of cardinality χ_* by $\circledast_3$. Let $M^* \in K_{\chi^*}$ be such that $M_* \leq_{\mathfrak{K}} M^* \leq_{\mathfrak{K}} M$ and M^* is $\leq_{\mathfrak{K}}$-universal over M_*. As $M, N \in K_{\mathfrak{s}_\lambda} \subseteq K_\lambda$ are saturated there is an isomorphism π from M onto N over M^* and let $q = \pi(p)^+$.

Now $q \restriction M = p$ by $\circledast_1$ as both are from $\mathcal{S}^{\rm na}_{\mathfrak{K}}(M)$, does not χ_*-split over M_* and has the same restriction to M^*.

Clause (E)(h): Follows by [She09c, 1.16A(3),(4)] recalling $\mathfrak{s}_\lambda$ is full.

Clause (E)(i): Follows by [She09c, 1.15].
So we have finished proving "$\mathfrak{s}_\lambda$ is a good λ-frame.]

$\odot_2$ If $\lambda \in (\chi_*, \chi)$ then $\mathfrak{K}^{\mathfrak{s}_\lambda}$ is $\mathfrak{K} \restriction \{M : M \text{ is } \lambda\text{-saturated}\}$.

[Why? Should be clear.]

$\odot_3$ λ_* is well defined, where

$$\lambda_* = \min\{\lambda \in (\chi_*, \chi) : 2^{\lambda^{+n}} < 2^{\lambda^{+n+1}} \text{ for every } n < \omega\}.$$

[Why? By clause (c) of the assumption.]

Let $\Theta = \{\lambda_*^{+n} : n < \omega\}$.

$\odot_4$ $\mathfrak{s}_\lambda$ is weakly successful for $\lambda \in \Theta$.

[Why? Recalling that $\mathfrak{s}_\lambda$ is categorical by Definition [She09e, stg.0A], Definition [She09c, nu.1] and Observation [She09c, nu.13.1(b)], if $(M, N, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}_\lambda}$ then for some $(M_1, N_1, a) \in K^{3,\mathrm{uq}}_{\mathfrak{s}_\lambda}$ we have $(M, N, a) \leq^{\mathrm{bs}}_{\mathfrak{s}_\lambda} (M_1, N_1, a)$ (see Definition [She09c, nu.1A]). Toward contradiction, assume that this fails. Let $\langle M_\alpha : \alpha < \lambda^+\rangle$ be $\leq_{\mathfrak{s}_\lambda}$-increasing continuous, $M_{\alpha+1}$ is brimmed over M_α for $\alpha < \lambda^+$ such that $M_0 = M$. Now directly by the definitions (as in [She09c, §5], see more in [She09d]) we can find $\langle M_\eta, f_\eta : \eta \in {}^{\lambda^+ >}2\rangle$ such that:

(a) If $\eta \triangleleft \nu \in {}^{\lambda^+ >}2$ then $M_\eta \leq_{\mathfrak{s}_\lambda} M_\nu$.

(b) If $\eta \in {}^{\lambda^+ >}2$ then f_η is a one-to-one function from $M_{\ell g(\eta)}$ to M_η over $M_0 = M$ such that $\rho \triangleleft \eta \Rightarrow f_\rho \subseteq f_\eta$ and $f_\eta(M_{\ell g(\eta)}) \leq_{\mathfrak{s}_\lambda} M_\eta$. In fact, $f_0 = \mathrm{id}_M$ and
$$(M, N, a) \leq^{\mathrm{bs}}_{\mathfrak{s}_\lambda} (f_\eta(M_{\ell g(\eta)}), M_\eta, a) \in K^{\mathrm{bs}}_{\mathfrak{s}}.$$

(c) If $\nu = \eta\hat{\ }\langle \ell\rangle \in {}^{\lambda >}2$ then M_ν is brimmed over M_η.

(d) If $\eta \in {}^{\lambda^+ >}2$ then $f_{\eta\hat{\ }\langle 0\rangle}(M_{\ell g(\eta)+1}) = f_{\eta\hat{\ }\langle 1\rangle}(M_{\ell g(\eta)+1})$.

(e) If $\eta \in {}^{\lambda >}2$ then there is no triple (N, f_0, f_1) such that $f_{\eta\hat{\ }\langle 1\rangle}(M_{\ell g(\eta)+1}) \leq_{\mathfrak{s}} N$, and f_ℓ is a $\leq_{\mathfrak{s}_\lambda}$-embedding of $M_{\eta\hat{\ }\langle \ell\rangle}$ into N over $f_{\eta\hat{\ }\langle \ell\rangle}(M_{\ell g(\eta)+1})$ for $\ell = 0, 1$ and $f_0 \restriction M_\eta = f_1 \restriction M_\eta$.

Having carried the induction by renaming, without loss of generality $\eta \in {}^{\lambda^+ >}2 \Rightarrow f_\eta = \mathrm{id}_{M_{\ell g(\eta)}}$. Now $M_* := \bigcup\{M_\alpha : \alpha < \lambda^+\}$; it belongs to $\mathfrak{s}_{\lambda^+}$ and is saturated. For $\eta \in {}^{\lambda^+}2$ let $M_\eta := \bigcup\{M_{\eta\restriction\alpha} : \alpha < \lambda^+\}$ so $M_* \leq_{\mathfrak{s}_{\lambda^+}} M_\eta \in K_{\mathfrak{s}_{\lambda^+}}$. But χ is a limit cardinal so also $\lambda^+ \in (\kappa, \chi)$ so let $N_* \in K_{\mathfrak{s}_{\lambda^+}}$ be $\leq_{\mathfrak{s}_{\lambda^+}}$-universal over M_*, so for every $\eta \in {}^{\lambda^+}2$ there is an $\leq_{\mathfrak{s}^+}$-embedding h_η of M_η into N_* over M_*. But $2^\lambda < 2^{\lambda^+}$ by the choice of λ_*, so by [She09a, 0.wD] we get a contradiction to clause (e).]

$\odot_5$ For $\lambda \in \Theta$, if $M \in K^{\mathfrak{s}_\lambda}_{\lambda^+}$ is saturated above λ for $K^{\mathfrak{s}_\lambda}$, <u>then</u> M is saturated for $\mathfrak{K}$.

[Why? Should be clear and implicitly was proved above.]

$\boxdot_1$ $\mathrm{NF}_{\mathfrak{s}_\lambda}$ is well defined and is a non-forking relation on $\mathfrak{K}_{\mathfrak{s}_\lambda}$ respecting $\mathfrak{s}_\lambda$ (for $\lambda \in \Theta$).

[Why? By [She09c, §6] as $\mathfrak{s}_\lambda$ is a weakly successful good λ frame.]

$\boxdot_2$ $\mathfrak{s}_\lambda$ is a good$^+$ λ-frame (for $\lambda \in \Theta$).

[Recalling Definition [She09e, stg.1], assume that this fails, so there are

$$\langle M_i, N_i : i < \lambda^+\rangle \text{ and } \langle a_{i+1} : i < \lambda^+\rangle$$

as there; i.e. $a_{i+1} \in M_{i+2} \setminus M_{i+1}$, $\mathbf{tp}_{\mathfrak{s}_\lambda}(a_{i+1}, M_{i+1}, M_{i+2})$ does not fork over M_0 for $\mathfrak{s}_\lambda$, but $\mathbf{tp}_{\mathfrak{s}_\lambda}(a_{i+1}, N_0, M_{i+1})$ forks over M_0. Also, recalling Definition [She09e, stg.1] the model $M = \bigcup\{M_i : i < \lambda^+\}$ is saturated for $\mathfrak{K}^{\mathfrak{s}_\lambda}_{\lambda^+}$ hence by $\odot_5$ for $\mathfrak{K}$, so it belongs to $K_{\mathfrak{s}_{\lambda^+}}$.

We can find an isomorphism f_0 from M onto $\mathrm{EM}_{\tau(\mathfrak{K})}(\lambda^+, \Phi)$, by $(*)_6$. By the "moreover" from $(*)_6$ (more exactly, by $(*)_{10}$) we can find a $\leq_{\mathfrak{K}}$-embedding f_1 of $N =: \bigcup\{N_i : i < \lambda^+\}$ into $\mathrm{EM}_{\tau(\mathfrak{K})}(\lambda \times \lambda, \Phi)$ extending f_0. As we can increase the N_i-s, without loss of generality f_1 is onto $\mathrm{EM}_{\tau(\mathfrak{K})}(\lambda \times \lambda, \Phi)$. We can find $\delta < \lambda^+$

such that $N_\delta = \mathrm{EM}_{\tau(\mathfrak{K})}(u, \Phi)$, where $u = \{\lambda\alpha + \beta : \alpha, \beta < \delta\}$. By $a_{\delta+1}$ we get a contradiction to $\circledast_5$.]

$\boxdot_3$ Let $\lambda \in \Theta$.

(α) $\leq^*_{\mathfrak{s}_\lambda}$ is a partial order on $K^{\mathrm{nice}}_{\lambda^+}[\mathfrak{s}_\lambda] = K_{\mathfrak{s}_{\lambda^+}}$, and $(K_{\mathfrak{s}_{\lambda^+}}, \leq^*_{\mathfrak{s}_\lambda})$ satisfies the demands on AEC, except possibly smoothness. (See [She09c, §7]).

(β) If $M \in K_{\lambda^+}$ is saturated and $p \in \mathcal{S}_{\mathfrak{K}}(M)$ then for some pair (N, a) we have $M \leq^*_{\mathfrak{s}_\lambda} N$ and $a \in N$ realizes p.

(γ) If $M \in K_{\lambda^+}$ is saturated then some N satisfies:
  - (a) $N \in K_{\lambda^+}$ is saturated.
  - (b) N is $\leq_{\mathfrak{K}}$-universal over M.
  - (c) $M \leq^*_{\mathfrak{s}_\lambda} N$

(δ) $\mathfrak{s}_\lambda$ is successful.

[Why? **Clause** (α):

We know that both $K^{\mathrm{nice}}_{\lambda^+}[\mathfrak{s}_\lambda]$ and $K_{\mathfrak{s}_{\lambda^+}}$ are the class of saturated $M \in K_\lambda$. The rest holds by [She09c, §7,§8].

Clause (β):

By $\circledast_3$ we can find $M_* \leq_{\mathfrak{K}} M$ of cardinality χ_* such that p does not χ_*-split over it (equivalently, does not λ^+-split over it).

Let $\langle M_\alpha : \alpha < \lambda^+\rangle$ be $\leq_{\mathfrak{s}_\lambda}$-increasing continuous such that $M_{\alpha+1}$ is brimmed over M_α for $\mathfrak{s}_\lambda$ for every $\alpha < \lambda^+$ and $M_* \leq_{\mathfrak{K}} M_0$ (so $\|M_*\| < \|M_0\|$; otherwise we would require that M_0 is brimmed over M_*). Hence $\bigcup\{M_\alpha : \alpha < \lambda^+\} \in K_{\lambda^+}$ is saturated (by $\odot_5$) so without loss of generality it is equal to M. We can choose $a_*, N_\alpha(\alpha < \lambda)$ such that $\langle N_\alpha : \alpha < \lambda^+\rangle$ is $\leq_{\mathfrak{s}_\lambda}$-increasing continuous, $M_\alpha \leq_{\mathfrak{s}_\lambda} M_\alpha$, $\mathrm{NF}_{\mathfrak{s}_\lambda}(M_\alpha, N_\alpha, M_\beta, M_\beta)$ for $\alpha < \beta < \lambda^+$, $N_{\alpha+1}$ is brimmed over $M_{\alpha+1} \cup N_\alpha$, and $\mathbf{tp}_{\mathfrak{s}_\lambda}(a, N_0, M_0) = p \restriction M_0$ so $a \in N_0$. Let $N = \bigcup\{N_\alpha : \alpha < \lambda^+\}$ so again $N \in K_{\lambda^+}$ is saturated (equivalently $N \in K^{\mathrm{nice}}_{\lambda^+}[\mathfrak{s}_\lambda]$) and $M \leq_{\mathfrak{K}} N$ and even $M \leq^*_{\mathfrak{s}_\lambda} N$ (by the definition of $\leq^*_{\mathfrak{s}_\lambda}$). For each $\alpha < \lambda^+$ we have $\mathrm{NF}_{\mathfrak{s}_\lambda}(M_0, N_0, M_\alpha, N_\alpha)$ but $\mathrm{NF}_{\mathfrak{s}_\lambda}$ respects $\mathfrak{s}_\lambda$, hence $\mathbf{tp}_{\mathfrak{s}_\lambda}(a, M_\alpha, N_\alpha)$ does not fork over M_0. Hence by the definition of $\mathfrak{s}_\lambda$, the type $\mathbf{tp}_{\mathfrak{s}_\lambda}(a, M_\alpha, N_\alpha)$ does not λ-split over M_*, hence $\mathbf{tp}_{\mathfrak{s}_\lambda}(a, M_\alpha, N_\alpha) = p \restriction M_\alpha$. As this holds for every $\alpha < \lambda^+$, by the choice of χ_* (i.e. by $\circledast_1$) clearly a realizes p.

Clause (γ):

By clause (β) as in the proofs in [She09c, §4]; that is, we choose $N \in K_{\lambda^+}$ which is $\leq_{\mathfrak{K}_\lambda}$-universal over M. We now try to choose $(M_\alpha, f_\alpha, N_\alpha)$ by induction on $\alpha < \lambda^+$ such that: $M_0 = M$, $N_0 = N$, $f_0 = \mathrm{id}_M$, M_α is $\leq^*_{\mathfrak{s}_\lambda}$-increasing continuous, N_α is $\leq_{\mathfrak{K}}$-increasing continuous, f_α is a $\leq_{\mathfrak{K}}$-embedding of M_α into N_α, f_α is $\subseteq$-increasing continuous with α, and $\alpha = \beta + 1 \Rightarrow f_\alpha(M_\alpha) \cap N_\beta \neq f_\beta(M_\beta)$.

For $\alpha = 0$, α limit there are no problems. If $\alpha = \beta + 1$ and $f_\alpha(M_\alpha) = N_\alpha$ we are done, and otherwise we use clause (β). But by Fodor lemma we cannot carry the induction for every $\alpha < \lambda^+$, so we are done proving (γ).

Clause (δ):

We should verify the conditions in Definition [She09e, stg.0A]. Now clause (a) there, being weakly successful, holds by $\odot_4$. As for clause (b) there, it suffices to prove that if $M_1, M_2 \in K^{\mathrm{nice}}_{\lambda^+}[\mathfrak{s}_\lambda] = K_{\mathfrak{s}^+_\lambda}$ and $M_1 \leq_{\mathfrak{K}} M_2$ then $M_1 \leq^*_{\mathfrak{s}_\lambda} M_2$, which means: if $\langle M^\ell_\alpha : \alpha < \lambda^+\rangle$ is $\leq_{\mathfrak{s}_\lambda}$-increasing continuous, $M^\ell_{\alpha+1}$ is brimmed over M^ℓ_α with $M_\ell = \bigcup\{M^\ell_\alpha : \alpha < \lambda^+\}$, then for some club E of λ^+, for every $\alpha < \beta$ from E, $\mathrm{NF}_{\mathfrak{s}_\lambda}(M^1_\alpha, M^2_\alpha, M^1_\beta, M^2_\delta)$.

By clause (γ) there is $N \in K_{\mathfrak{s}^+_\lambda}$ such that $M_1 \leq^*_{\mathfrak{s}_{\lambda^+}} N$ (hence $M_1 \leq_{\mathfrak{K}} N$) and N is $\leq_{\mathfrak{K}^{\mathfrak{s}_\lambda}}$-universal over M_1. So without loss of generality $M_2 \leq_{\mathfrak{K}} N$, but by [She09c, ne.3](3) all of this implies $M_1 \leq^*_{\lambda^+} M_2$. So we are done proving $\boxdot_3$.

$\boxdot_4$ $\mathfrak{s}_{\lambda^+}$ is the successor of $\mathfrak{s}_\lambda$ for $\lambda \in \Theta$.

[Why? Now by $\boxdot_3$ the good frame $\mathfrak{s}_\lambda$ is successful; by [She09e, stg.3] we know that $\mathfrak{s}^+_\lambda$ is a well defined good λ^+-frame. Clearly $K_{\mathfrak{s}_\lambda(+)}$ is the class of saturated $M \in \mathfrak{K}_{\lambda^+}$ (by $\odot_5$; see the definitions in [She09c, ne.1], [She09c, rg.7(5)]). But $\mathfrak{s}_\lambda$ is good$^+$ by $\boxdot_2$, so by [She09e, stg.3B] we know that $\leq_{\mathfrak{s}_\lambda(+)} = <^*_{\lambda^+,[\mathfrak{s}_\lambda]}$ is equal to $\leq_{\mathfrak{K}} \restriction K_{\mathfrak{s}_\lambda(+)}$, so $\mathfrak{K}_{\mathfrak{s}_\lambda(+)} = \mathfrak{K}_{\mathfrak{s}_{\lambda^+}}$. As both $\mathfrak{s}_\lambda(+)$ and $\mathfrak{s}_{\lambda^+}$ are full, clearly $\mathcal{S}^{\rm bs}_{\mathfrak{s}_\lambda(+)} = \mathcal{S}^{\rm bs}_{\mathfrak{s}_{\lambda^+}}$. For $M_1 \leq_{\mathfrak{s}_\lambda(+)} M_2 \leq_{\mathfrak{s}_\lambda(+)} M_3$ and $a \in M_3 \setminus M_2$, comparing the two definitions of "$\mathbf{tp}_{\mathfrak{K}_{\mathfrak{s}_\lambda(+)}}(a, M_2, M_1)$ does not fork over M_1," they are the same. So we are done.]

$\boxdot_5$ $\mathfrak{s}_{\lambda^{+\omega}_*}$ is the limit of $\langle \mathfrak{s}^{+n}_{\lambda_*} : n < \omega \rangle$.

[Why? Should be clear.]

$\boxdot_6$ $\mathfrak{s}_\lambda$ satisfies the hypothesis [She09e, 12.1] of [She09e, §12] if $\lambda \in \Theta \setminus \lambda^{+3}_*$ holds.

[Why? By $\boxdot_2, \boxdot_3, \boxdot_4$ and [She09e, 12D.1].]

Hence

$\boxdot_7$ $\mathfrak{s}_{\lambda_*}$ is beautiful $\lambda^{+\omega}_*$-frame.

[Why? By [She09e, 12b.14] and [She09e, 12f.16A].]

$\boxdot_8$ $K[\mathfrak{s}_{\lambda^{+\omega}_*}]$ is categorical in one $\chi > \lambda^{+\omega}_*$ $\underline{\text{iff}}$ it is categorical in every $\chi > \lambda^{+\omega}$.

[Why? By [She09e, 12f.16A(d),(e)].]

$\boxdot_9$ If $\lambda \geq \beth_{1,1}(\lambda^{+\omega}_*)$ then $\mathfrak{K}_\lambda = \mathfrak{K}_\lambda[\mathfrak{s}_{\lambda^{+\omega}_*}]$.

[Why? The conclusion $\supseteq$ is obvious. For the other inclusion let $M \in K_\lambda$, now by the definition of class in the left, it is enough to prove that M is $(\lambda^{+\omega}_*)^+$-saturated. But otherwise, by the omitting type theorem for AEC (i.e. by 0.9(1)(d), or see [She99a, 8.6=X1.3A]) there is such a model $M' \in K_\mu$, in contradiction to $(*)_4$.]

By $\boxdot_8 + \boxdot_9$ we are done. $\square_{7.13}$

References

[Dic85] M. A. Dickman, *Larger infinitary languages*, Model Theoretic Logics (J. Barwise and S. Feferman, eds.), Perspectives in Mathematical Logic, Springer-Verlag, New York Berlin Heidelberg Tokyo, 1985, pp. 317–364.

[Kin66] Akiko Kino, *On definability of ordinals in logic with infinitely long expressions*, Journal of Symbolic Logic **31** (1966), 365–375.

[KS96] Oren Kolman and Saharon Shelah, *Categoricity of theories in $L_{\kappa\omega}$, when κ is a measurable cardinal. I*, Fund. Math. **151** (1996), no. 3, 209–240, arXiv: math/9602216. MR 1424575

[Lav71] Richard Laver, *On fraissé's order type conjecture*, Annals of Mathematics **93** (1971), 89–111.

[MR65] Eric Milner and Richard Rado, *The pigeon-hole principle for ordinal numbers*, Proc. London Math. Soc. **15** (1965), 750–768.

[S$^+$a] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F604.

[S$^+$b] ______, *Tba*, In preparation. Preliminary number: Sh:F782.

[S$^+$c] ______, *Tba*, In preparation. Preliminary number: Sh:F820.

[She] Saharon Shelah, *Introduction and Annotated Contents*, arXiv: 0903.3428 introduction of [Sh:h].

[She87a] ______, *Classification of nonelementary classes. II. Abstract elementary classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Math., vol. 1292, Springer, Berlin, 1987, pp. 419–497. MR 1033034

[She87b] ______, *Universal classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Math., vol. 1292, Springer, Berlin, 1987, pp. 264–418. MR 1033033

[She94] ______, *Cardinal arithmetic*, Oxford Logic Guides, vol. 29, The Clarendon Press, Oxford University Press, New York, 1994. MR 1318912

[She99a] ______, *Categoricity for abstract classes with amalgamation*, Ann. Pure Appl. Logic **98** (1999), no. 1-3, 261–294, arXiv: math/9809197. MR 1696853

[She99b] ______, *Special subsets of* ${}^{\mathrm{cf}(\mu)}\mu$*, Boolean algebras and Maharam measure algebras*, Topology Appl. **99** (1999), no. 2-3, 135–235, arXiv: math/9804156. MR 1728851

[She00] ______, *Applications of PCF theory*, J. Symbolic Logic **65** (2000), no. 4, 1624–1674, arXiv: math/9804155. MR 1812172

[She01] ______, *Categoricity of theories in* $L_{\kappa^*,\omega}$*, when* κ^* *is a measurable cardinal. II*, Fund. Math. **170** (2001), no. 1-2, 165–196, arXiv: math/9604241. MR 1881375

[She09a] ______, *Abstract elementary classes near* $\aleph_1$, Classification theory for abstract elementary classes, Studies in Logic (London), vol. 18, College Publications, London, 2009, arXiv: 0705.4137 Ch. I of [Sh:h], pp. vi+813.

[She09b] ______, *Categoricity and solvability of A.E.C., quite highly*, 2009, arXiv: 0808.3023 Ch. IV of [Sh:h].

[She09c] ______, *Categoricity in abstract elementary classes: going up inductively*, 2009, arXiv: math/0011215 Ch. II of [Sh:h].

[She09d] ______, *Non-structure in* λ^{++} *using instances of WGCH*, 2009, arXiv: 0808.3020 Ch. VII of [Sh:i].

[She09e] ______, *Toward classification theory of good* λ *frames and abstract elementary classes*, 2009, arXiv: math/0404272 Ch. III of [Sh:h].

[She09f] ______, *Universal Classes: Axiomatic Framework [Sh:h]*, 2009, Ch. V (B) of [Sh:i].

[She09g] ______, *Universal Classes: Stability theory for a model*, 2009, Ch. V of [Sh:i].

[Shear] ______, *Non-structure theory*, Oxford University Press, to appear.

[SV] Saharon Shelah and Sebastien Vasey, *Categoricity and multidimensional diagrams*, arXiv: 1805.06291.

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