LARGE SUBSETS OF THE COFINALITY SPECTRUM
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ABSTRACT. In this paper, we show that large subsets of $\mathrm{pcf}(\mathfrak{a})$ behave nicely, provided that they have no inaccessibles as accumulation points.

§ 0. INTRODUCTION

A set $\mathfrak{a}$ of regular cardinals is called *progressive* iff $|\mathfrak{a}| < \min(\mathfrak{a})$. This concept is central in pcf theory, and it appears as a needed assumption in may pcf theorems. Moreover, counterexamples of basic pcf properties are known to exist if $\mathfrak{a}$ is not progressive (see [blank]).

Suppose that $\mathfrak{a}$ is progressive. If $|\mathrm{pcf}(\mathfrak{a})| = |\mathfrak{a}|$ then $\mathrm{pcf}(\mathfrak{a})$ is progressive as well. However, [in general / in most cases] we do not know whether $|\mathrm{pcf}(\mathfrak{a})| = |\mathfrak{a}|$. If one considers the possibility that $|\mathrm{pcf}(\mathfrak{a})| > |\mathfrak{a}|$, then there might be a set $\mathfrak{b} \subseteq \mathrm{pcf}(\mathfrak{a})$ for which $|\mathfrak{b}| > \min(\mathfrak{b})$; that is, a non-progressive subset of $\mathrm{pcf}(\mathfrak{a})$.

It turns out that subsets of $\mathrm{pcf}(\mathfrak{a})$ behave nicely even when they are not progressive, provided that they do not possess inaccessible accumulation points. That is, if $\mathfrak{b} \subseteq \mathrm{pcf}(\mathfrak{a})$ and $\mu \in \mathrm{pcf}(\mathfrak{a})$, then $\mu > \sup(\mathfrak{b} \cap \mu)$. This assumption is not far-fetched. In fact, we do not know whether an inaccessible accumulation point is possible at all.

The content of this paper comes from [She94b, VIII,§3]. However, the presentation there is not complete. We thank E. Weitz for asking us to give more details. Our notation is consistent with [She94b].

§ 1. ON pcf

Definition 1.1 ([She94b, VIII 3.1])**.** (A) Let

$$J_*[\mathfrak{a}] = \big\{\mathfrak{b} \subseteq \mathfrak{a} : \text{for every inaccessible } \mu, \text{ we have } \mu > \sup(\mathfrak{b} \cap \mu)\big\}.$$

(B) $\mathrm{pcf}_*(\mathfrak{a}) = \big\{\mathrm{tcf}(\prod \mathfrak{a}/D) : D \text{ is an ultrafilter on } \mathfrak{a},\ D \cap J_*[\mathfrak{a}] \neq \varnothing\big\}$.

(C) If $\mathfrak{b} \subseteq \mathfrak{a}$ [and $\mathfrak{b} \in D$], then $\mathfrak{b}$ compels $\prod \mathfrak{a}$ to have cofinality $<\mu$ iff $\mathrm{tcf}(\prod \mathfrak{a}/D) < \mu$. The ideal $J_{<\mu}(\mathfrak{a})$ is the collection of $\mathfrak{b} \subseteq \mathfrak{a}$ which compel $\prod \mathfrak{a}$ to have cofinality $<\mu$.

(D) If $|\mathfrak{a}| < \min(\mathfrak{a})$, for $\mu \in \mathrm{pcf}(\mathfrak{a})$ let $\mathfrak{b}_\mu[\mathfrak{a}]$ be a subset of $\mathfrak{a}$ such that $J_{\leq\mu}[\mathfrak{a}] = J_{<\mu}[\mathfrak{a}] + \mathfrak{b}_\mu[\mathfrak{a}]$.

(Note that $\mathfrak{b}_\mu[\mathfrak{a}]$ exists by [She94b, VIII 2.6]; also, $\mathfrak{a}$ is a finite union of $\mathfrak{b}_\mu[\mathfrak{a}]$-s).

(E) If $|\mathfrak{a}| < \min(\mathfrak{a})$, let $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ be the ideal of subsets of $\mathrm{pcf}(\mathfrak{a})$ generated by $\{\mathrm{pcf}(\mathfrak{b}_\mu[\mathfrak{a}]) : \mu \in \lambda \cap \mathrm{pcf}(\mathfrak{a})\}$.

Let $J^{\mathrm{pcf}}_{\leq\lambda}[\mathfrak{a}] = J^{\mathrm{pcf}}_{<\lambda^+}[\mathfrak{a}]$.

Claim 1.2 ([She94b, VIII 3.1A])**.** *(1) The ideal $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ depends on $\mathfrak{a}$ and λ only (and not on the choice of the $\mathfrak{b}_\mu[\mathfrak{a}]$-s).*

(2) If $\mathfrak{b} \subseteq \mathfrak{a}$ then $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{b}] = \mathcal{P}(\mathfrak{b}) \cap J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ and $J_*[\mathfrak{b}] = \mathcal{P}(\mathfrak{b}) \cap J_*[\mathfrak{a}]$.

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Proof. (1) Let $\langle \mathfrak{b}'_\mu[\mathfrak{a}] : \mu \in \lambda \cap \mathrm{pcf}(\mathfrak{a}) \rangle$, $\langle \mathfrak{b}''_\mu[\mathfrak{a}] : \mu \in \lambda \cap \mathrm{pcf}(\mathfrak{a}) \rangle$ both be as in 1.1(D). So for each θ, $\mathfrak{b}'_\theta[\mathfrak{a}] \subseteq \mathfrak{b}''_\theta[\mathfrak{a}] \cup \bigcup_{\ell<n} \mathfrak{b}''_{\theta_\ell}[\mathfrak{a}]$ for some $n < \omega$, $\theta_0, \ldots, \theta_n - 1 < \theta$.

[Is this good, or is it supposed to be θ_{n-1}? Either option is believable.]

Hence, if $\theta < \lambda$,

$$\mathrm{pcf}(\mathfrak{b}'_\theta[\mathfrak{a}]) \subseteq \mathrm{pcf}(\mathfrak{b}''_\theta[\mathfrak{a}]) \cup \bigcup_{\ell<n} \mathrm{pcf}(\mathfrak{b}''_{\theta_\ell}[\mathfrak{a}]),$$

and each is in $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$, as defined by $\langle \mathfrak{b}''_\sigma[\mathfrak{a}] : \sigma \in \lambda \cap \mathrm{pcf}(\mathfrak{a}) \rangle$ (as $\theta_\ell < \theta < \lambda$). As this holds for every $\theta < \lambda$, all generators of $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ as defined by $\langle \mathfrak{b}'_\sigma[\mathfrak{a}] : \sigma \in \lambda \cap \mathrm{pcf}(\mathfrak{a}) \rangle$ are in $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ as defined by $\langle \mathfrak{b}''_\sigma[\mathfrak{a}] : \sigma \in \lambda \cap \mathrm{pcf}(\mathfrak{a}) \rangle$. As the situation is symmetric we are done.

(2) Similar proof. The first phrase follows from part (1), and the reader may check the second. □

Lemma 1.3 ([She94b, VIII 3.2]). *Suppose* $|\mathfrak{a}|^+ < \min(\mathfrak{a})$, $\mathfrak{a} \subseteq \mathfrak{b} \in J_*[\mathrm{pcf}(\mathfrak{a})]$, $\mathfrak{b} \notin J := J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$, *and* $\lambda = \max \mathrm{pcf}(\mathfrak{a})$. *Then* $\mathrm{tcf}(\prod \mathfrak{b}/J)$ *is* λ.

Proof. Remember that (by [She94b, VIII 2.6]) there is $\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a}) \rangle$, a generating sequence for $\mathfrak{a}$. For $\mu \in \mathrm{pcf}(\mathfrak{a})$, let $\langle f^\mu_\alpha : \alpha < \mu \rangle$ exemplify $\mu = \mathrm{tcf}(\prod \mathfrak{b}_\mu[\mathfrak{a}], J_{<\mu}[\mathfrak{a}])$, with $f^\mu_\alpha \in \prod \mathfrak{a}$. By [She94a, 3.1], without loss of generality

$(*)_0$ $(\forall f \in \prod \mathfrak{a}) \Big[\bigvee_\alpha f \restriction \mathfrak{b}_\mu[\mathfrak{a}] \leq f^\mu_\alpha \Big]$

Without loss of generality, for $\theta \in \mathfrak{a}$, **[we can fix the following values]**: $f^\theta_\alpha(\theta) = \alpha$ if $\alpha < \theta$ and $f^\theta_\alpha(\theta') = 0$ if $\alpha < \theta < \theta' \in \mathfrak{a}$. For each $\alpha \in \lambda$, we define $f^{\lambda,\mathfrak{b}}_\alpha \in \prod \mathfrak{b}$ by:

$$f^{\lambda,\mathfrak{b}}_\alpha \restriction \mathfrak{a} = f^\lambda_\alpha,$$

and for $\theta \in \mathfrak{b} \setminus \mathfrak{a}$:

$$f^{\lambda,\mathfrak{b}}_\alpha(\theta) = \min\big\{ \beta : f^\lambda_\alpha \restriction \mathfrak{b}_\theta[\mathfrak{a}] \leq f^\theta_\beta \mod J_{<\theta}[\mathfrak{a}] \big\}.$$

Clearly

$(*)_1$ $f^\lambda_\alpha \leq f^\lambda_\beta \Rightarrow f^{\lambda,\mathfrak{b}}_\alpha \leq f^{\lambda,\mathfrak{b}}_\beta$.

Sub-fact 1.4 ([She94b, VIII 3.2A]).

$$\alpha < \beta < \lambda \Rightarrow f^{\lambda,\mathfrak{b}}_\alpha \leq f^{\lambda,\mathfrak{b}}_\beta \mod J.$$

Proof of the subfact. Let $\mathfrak{c} = \big\{ \theta \in \mathfrak{a} : f^\lambda_\alpha(\theta) > f^\lambda_\beta(\theta) \big\}$, so $\mathfrak{c} \in J_{<\lambda}[\mathfrak{a}]$, and hence for some $n < \omega$ and $\sigma_1 < \ldots < \sigma_n$ from $\lambda \cap \mathrm{pcf}(\mathfrak{a})$ (hence $< \lambda$), we have $\mathfrak{c} \subseteq \bigcup_{\ell=1}^n \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}]$. So by the definition of the $f^{\lambda,\mathfrak{b}}_\alpha$-s we have:

$(*)_2$ If $\mu \in \mathfrak{b}$ and $\mathfrak{b}_\mu[\mathfrak{a}] \cap \bigcap_{\ell=1}^n \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}] \in J_{<\mu}[\mathfrak{a}]$ then $f^{\lambda,\mathfrak{b}}_\alpha(\mu) \leq f^{\lambda,\mathfrak{b}}_\beta(\mu)$.

However,

$(*)_3$ $\mathfrak{d} := \Big\{ \mu \in \mathrm{pcf}(\mathfrak{a}) : \mathfrak{b}_\mu[\mathfrak{a}] \cap \bigcup_{\ell=1}^n \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}] \neq \varnothing \mod J_{<\mu}[\mathfrak{a}] \Big\}$
(for our fixed $\sigma_1, \ldots, \sigma_n \in \lambda \cap \mathrm{pcf}(\mathfrak{a})$) belongs to J.

[Why? As $\mu \in \mathfrak{d}$ implies $\mu \in \bigcup_{\ell=1}^n \mathrm{pcf}(\mathfrak{b}_{\sigma_\ell}[\mathfrak{a}])$ which is in J.]

Together we get subfact 1.4. □

Sub-fact 1.5 ([She94b, VIII 3.2B]). For any $f \in \prod \mathfrak{b}$, for some α, $f \leq f^{\lambda,\mathfrak{b}}_\alpha$.

Proof of the subfact. The family J_1 of sets $\mathfrak{c} \subseteq \mathfrak{b}$ for which this holds (i.e., for each $f \in \prod \mathfrak{c}$ there is $\alpha < \lambda$ such that $f \leq f^{\lambda,\mathfrak{b}}_\alpha \restriction \mathfrak{c}$) will satisfy the following conditions:

(A) $\{\theta\} \in J_1$ for $\theta \in \lambda \cap \mathrm{pcf}(\mathfrak{a})$.
(B) J_1 is an ideal of subsets of $\mathrm{pcf}(\mathfrak{a})$.
(C) If $\{\mathfrak{c}_i : i < \kappa\} \subseteq J_1$ and $\min(\mathfrak{c}_i) > \kappa^+$ for all $i < \kappa$, then $\bigcup_{i<\kappa} \mathfrak{c}_i$ is in J_1.

We shall show their satisfaction below.
This suffices for 1.5, as $\mathfrak{b} \in J_*[\mathrm{pcf}(\mathfrak{a})]$. Why? just prove that

$$\mathfrak{c} \subseteq \mathfrak{b} \ \&\ \mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})] \ \Rightarrow\ \mathfrak{c} \in J_1$$

by induction on $\sup\{\mu^+ : \mu \in \mathfrak{c}\}$. For successor use (A)+(B). For singular, let $\langle \mu_i : i < \kappa\rangle$ be such that $\mu_0 > \kappa^+$ and μ_i is strictly increasing continuous with limit $\sup \mathfrak{c} = \sup\{\mu^+ : \mu \in \mathfrak{c}\}$; by the induction hypothesis $\mathfrak{c} \cap \mu_0$ and $\mathfrak{c} \cap [\mu_i, \mu_{i+1}]$ are in the ideal. By (C) we know that

$$\bigcup_{i<\kappa} \big(\mathfrak{c} \cap [\mu_i, \mu_{i+1})\big) = \mathfrak{c} \cap [\mu_0, \sup \mathfrak{c})$$

is in the ideal, and by the induction hypothesis, $\mathfrak{c} \cap \mu_0 \in J_1$. So by (B)

$$\mathfrak{c} = \big(\mathfrak{c} \cap \mu_0\big) \cup \big(\mathfrak{c} \cap [\mu_0, \sup \mathfrak{c})\big)$$

is in J_1; note $\sup(\mathfrak{c}) \notin \mathfrak{c}$ as $\sup \mathfrak{c}$ is singular. As $\mathfrak{b} \in J_*[\mathrm{pcf}(\mathfrak{a})]$, we have covered all cases.

Now, why do (A),(B),(C) hold? We shall use $(*)_1$ from above freely.

For (A): If $\theta \in \mathfrak{a}$ [this follows from] $f^{\lambda,\mathfrak{b}}_\alpha \restriction \mathfrak{a} = f^\lambda_\alpha$ and $(*)_0$; if $\theta \in \mathfrak{b} \setminus \mathfrak{a}$ (a subset of $\mathrm{pcf}(\mathfrak{a})$) and $\alpha < \theta$ then for some $\beta < \lambda$, $f^\theta_{\alpha+1} \le f^\lambda_\beta$, hence $f^{\lambda,\mathfrak{b}}_\beta(\theta) > \alpha$.
This shows $\{\theta\} \in J_1$.

For (B): Trivially, $\mathfrak{c} \subseteq \mathfrak{c}' \in J_1 \ \Rightarrow\ \mathfrak{c} \in J_1$. If $\mathfrak{c}_1, \mathfrak{c}_2 \in J_1$, $\mathfrak{c} = \mathfrak{c}_1 \cup \mathfrak{c}_2$, and $f \in \prod \mathfrak{c}$, then choose (for $\ell = 1,2$) $\alpha_\ell < \lambda$ such that $f \restriction \mathfrak{c}_\ell \le f^{\lambda,\mathfrak{b}}_{\alpha_\ell}$. Now let $f' \in \prod \mathfrak{a}$ be defined by $f'(\theta) = \max\big\{f^\lambda_{\alpha_1}(\theta), f^\lambda_{\alpha_2}(\theta)\big\}$, so by [our] assumption on $\langle f^\lambda_\alpha : \alpha < \lambda\rangle$ and $(*)_0$, $f' \le f^\lambda_\alpha$ for some α.
Now $f^{\lambda,\mathfrak{b}}_\alpha$ is as required by $(*)_1$.

For (3): Let $f \in \prod \mathfrak{c}$. By assumption, for each $i < \kappa$, for some $\alpha(i) < \lambda$, we have $f \restriction \mathfrak{c}_i \le f^{\lambda,\mathfrak{b}}_{\alpha(i)}$. Now $\big(\prod \mathfrak{a}, <_{J_{\le\kappa}[\mathfrak{a}]}\big)$ is κ^+-directed, hence for some $f' \in \prod \mathfrak{a}$,

$$\bigwedge_{i<\kappa} f^\lambda_{\alpha(i)} <_{J_{\le\kappa}[\mathfrak{a}]} f'.$$

By $(*)_0$, for some $\beta < \lambda$ we have $f' \le f^\lambda_\beta$ and $f \restriction (\mathfrak{a} \cap \mathfrak{c}) \le f^\lambda_\beta$. (Necessarily, $\bigwedge_{i<\kappa} \alpha(i) < \beta$.) Let $\theta \in \bigcup_{i<\kappa} \mathfrak{c}_i$. If $\theta \in \mathfrak{a}$ then trivially $f(\theta) \le f^\lambda_\beta(\theta)$, so assume $\theta \notin \mathfrak{a}$. Now $\theta \in \mathfrak{c}_i$ for some i, so $\theta > \kappa$ and $f^\lambda_{\alpha(i)} <_{J_{\le\kappa}[\mathfrak{a}]} f^\lambda_\beta$, hence $f^\lambda_{\alpha(i)} <_{J_{<\theta}[\mathfrak{a}]} f^\lambda_\beta$, hence by their definitions $f^{\lambda,\mathfrak{b}}_{\alpha(i)}(\theta) \le f^{\lambda,\mathfrak{b}}_\beta(\theta)$.
So β is as required; i.e. we have proved subfact 1.5. $\square$

Now 1.3 follows from 1.4, 1.5.
[Using 1.5 for $f+1$ we can get there $f < f^{\lambda,\mathfrak{b}}_\alpha$, so (by 1.4) for some club C of λ,

$$\alpha < \beta \in C \ \Rightarrow\ f^{\lambda,\mathfrak{b}}_\alpha < f^{\lambda,\mathfrak{b}}_\beta \mod J.$$

Together $\langle f^{\lambda,\mathfrak{b}}_\alpha : \alpha \in C\rangle$ witness that $\mathrm{tcf}(\prod \mathfrak{b}, <_J)$ is λ, as required]. $\square$

Theorem 1.6 ([She94b, VIII 3.3])**.** *Assume* $\min(\mathfrak{a}) > |\mathfrak{a}|$.

(1) For an ultrafilter D on $\mathrm{pcf}(\mathfrak{a})$ not disjoint to $J_*[\mathrm{pcf}(\mathfrak{a})]$,

$$\begin{aligned}\mathrm{tcf}\big(\textstyle\prod \mathrm{pcf}(\mathfrak{a})/D\big) &= \min\{\lambda \in \mathrm{pcf}(\mathfrak{a}) : \mathrm{pcf}(\mathfrak{b}_\lambda[\mathfrak{a}]) \in D\} \\ &= \min\big\{\lambda \in \mathrm{pcf}(\mathfrak{a}) : D \cap J^{\mathrm{pcf}}_{\le\lambda}[\mathfrak{a}] \ne \varnothing\big\}.\end{aligned}$$

(2) For $\mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})]$, $\mathrm{pcf}(\mathfrak{c})$ is a subset of $\mathrm{pcf}(\mathfrak{a})$ and has a maximal element.

(3) For $\mathfrak{b} \in J_*[\mathrm{pcf}(\mathfrak{a})]$, $\prod \mathfrak{b}/J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ *is* λ*-directed.*
(4) $\mathrm{pcf}_*(\mathfrak{a}) = \mathrm{pcf}(\mathfrak{a}) = \mathrm{pcf}_*(\mathrm{pcf}(\mathfrak{a}))$.
(5) If $\mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})]$ *and* $\mathfrak{c} \in J^{\mathrm{pcf}}_{\le\lambda}[\mathfrak{a}]$ *then* $\prod \mathfrak{c}$ *has cofinality* $\le \lambda$.
(6) If $\mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})]$ *and* $\mathfrak{c} \in J^{\mathrm{pcf}}_{\le\lambda}[\mathfrak{a}] \setminus J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$ *then* $\lambda = \mathrm{tcf}\big(\prod \mathfrak{c}, <_{J^{\mathrm{pcf}}_{<\lambda}}\big)$.

Proof. **(1)** Trivially, the second and third terms are equal (see Definition 1.1(5)). Let λ be defined as in the second term, so $\mathrm{pcf}(\mathfrak{b}_\lambda[\mathfrak{a}]) \in D \cap J^{\mathrm{pcf}}_{\le\lambda}[\mathfrak{a}]$. So by 1.2(2) without loss of generality $\mathfrak{a} = \mathfrak{b}_\lambda[\mathfrak{a}]$, so $\lambda = \max\mathrm{pcf}(\mathfrak{a})$. Using 1.3's notation, $\langle f^{\lambda,\mathfrak{b}}_\alpha : \alpha < \lambda\rangle$ exemplifies $\lambda = \mathrm{tcf}(\prod \mathfrak{a}/D)$.

(2) By (1).

(3) This follows by the proof of Lemma 1.3, but as I was asked, we repeat the proof of 1.3 with the required changes. Without loss of generality, $\lambda \in \mathrm{pcf}(\mathfrak{a})$.

[Why? If $\lambda > \max\mathrm{pcf}(\mathfrak{a})$ then $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}] = \mathcal{P}(\mathrm{pcf}(\mathfrak{a}))$, so the conclusion is trivial. If not, let $\lambda' = \min(\mathrm{pcf}(\mathfrak{a}) \setminus \lambda)$, so $\lambda' \in \mathrm{pcf}(\mathfrak{a})$ and $J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}] = J^{\mathrm{pcf}}_{<\lambda'}[\mathfrak{a}]$.]

We let $J = J^{\mathrm{pcf}}_{<\lambda}[\mathfrak{a}]$. Remember that (by [She94b, VIII, 2.6]) there is

$$\big\langle \mathfrak{b}_\theta[\mathfrak{a}] : \theta \in \mathrm{pcf}(\mathfrak{a})\big\rangle,$$

a generating sequence for $\mathfrak{a}$. For $\mu \in \mathrm{pcf}(\mathfrak{a})$, let $\langle f^\mu_\alpha : \alpha < \mu\rangle$ exemplify $\mu = \mathrm{tcf}(\prod \mathfrak{b}_\mu[\mathfrak{a}], J_{<\mu}[\mathfrak{a}])$, $f^\mu_\alpha \in \prod \mathfrak{a}$. By [She94a, 3.1], without loss of generality

$(*)_0$ $(\forall f \in \prod \mathfrak{a})\Big[\bigvee_\alpha f \restriction \mathfrak{b}_\mu[\mathfrak{a}] \le f^\mu_\alpha\Big]$.

Without loss of generality, for $\theta \in \mathfrak{a}$, we can fix the following values of f^θ_α: $f^\theta_\alpha(\theta) = \alpha$ if $\alpha < \theta$ and $f^\theta_\alpha(\theta') = 0$ if $\alpha < \theta < \theta' \in \mathfrak{a}$. For any $f \in \prod \mathfrak{a}$ we define a function $f^\mathfrak{b} \in \prod \mathfrak{b}$ by:

$$f^\mathfrak{b} \restriction \mathfrak{a} = f$$

and for $\theta \in \mathfrak{b} \setminus \mathfrak{a}$:

$$f^\mathfrak{b}(\theta) = \min\big\{\beta : f \restriction \mathfrak{b}_\theta[\mathfrak{a}] \le f^\theta_\beta \mod J_{<\theta}[\mathfrak{a}]\big\}.$$

Let f vary on $\prod \mathfrak{a}$. Clearly

$(*)_1$ $f_1 \le f_2 \Rightarrow f^\mathfrak{b}_1 \le f^\mathfrak{b}_2$.

Sub-fact 1.7. If $f_1 \le f_2 \mod J_{<\lambda}[\mathfrak{a}]$ (both in $\prod \mathfrak{a}$ of course) then $f^\mathfrak{b}_1 \le f^\mathfrak{b}_2 \mod J$.

Proof of the subfact. Let $\mathfrak{c} = \{\theta \in \mathfrak{a} : f_1(\theta) \ge f_2(\theta)\}$, so $\mathfrak{c} \in J_{<\lambda}[\mathfrak{a}]$. Hence for some $n < \omega$ and $\sigma_1 < \ldots < \sigma_n$ from $\lambda \cap \mathrm{pcf}(\mathfrak{a})$, (hence $< \lambda$) we have $\mathfrak{c} \subseteq \bigcup_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}]$. So by the definition of the $f^\mathfrak{b}_i$-s we have:

$(*)_2$ If $\mu \in \mathfrak{b}$ and $\mathfrak{b}_\mu[\mathfrak{a}] \cap \bigcap_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}] \in J_{<\mu}[\mathfrak{a}]$, then $f^\mathfrak{b}_1(\mu) \le f^\mathfrak{b}_2(\mu)$.

However

$(*)_3$ $\mathfrak{d} := \Big\{\mu \in \mathrm{pcf}(\mathfrak{a}) : \mathfrak{b}_\mu[\mathfrak{a}] \cap \bigcup_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}] \ne \varnothing \mod J_{<\mu}[\mathfrak{a}]\Big\}$
(for our fixed $\sigma_1, \ldots, \sigma_n \in \lambda \cap \mathrm{pcf}(\mathfrak{a})$) belongs to J.

[As $\mu \in \mathfrak{d}$ implies $\mu \in \mathrm{pcf}(\bigcup_{\ell=1}^{n} \mathfrak{b}_{\sigma_\ell}[\mathfrak{a}]) = \bigcup_{\ell=1}^{n} \mathrm{pcf}(\mathfrak{b}_{\sigma_\ell}[\mathfrak{a}])$, which is in J.]

Together we get subfact 1.7. □

Sub-fact 1.8. For any $g \in \prod \mathfrak{b}$, for some $f \in \prod \mathfrak{a}$, we have $g \le f^{\lambda,\mathfrak{b}}_\alpha$.

Proof of the subfact. The family J_1 of sets $\mathfrak{c} \subseteq \mathfrak{b}$ for which this holds (i.e., for each $g \in \prod \mathfrak{c}$ there is $f \in \prod \mathfrak{a}$ such that $g \le f \restriction \mathfrak{c}$) satisfies the same three properties as in 1.5:

(A) $\{\theta\} \in J_1$ for $\theta \in \lambda \cap \mathrm{pcf}(\mathfrak{a})$.
(B) J_1 is an ideal of subsets of $\mathrm{pcf}(\mathfrak{a})$.
(C) If $\{\mathfrak{c}_i : i < \kappa\} \subseteq J_1$ and $\min(\mathfrak{c}_i) > \kappa^+$ for all $i < \kappa$, then $\bigcup_{i<\kappa} \mathfrak{c}_i$ is in J_1.

We shall show their satisfaction below.

Why do (A)+(B)+(C) suffice for 1.8?
As $\mathfrak{b} \in J_*[\mathrm{pcf}(\mathfrak{a})]$.

Why? Just prove that

$(*)_4$ $\mathfrak{c} \subseteq \mathfrak{b}$ & $\mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})] \ \Rightarrow\ \mathfrak{c} \in J_1$

by induction on $\sup\{\mu^+ : \mu \in \mathfrak{c}\}$. For successor use (A)+(B). For singular, let $\langle \mu_i : i < \kappa\rangle$ be such that μ_i is strictly increasing continuous with limit $\sup(\mathfrak{c}) = \sup\{\mu^+ : \mu \in \mathfrak{c}\}$, and $\kappa^+ < \mu_0$; by the induction hypothesis $\mathfrak{c} \cap \mu_0$ and $\mathfrak{c} \cap [\mu_i, \mu_{i+1}]$ are in the ideal, by (C) we know that

$$\bigcup_{i<\kappa} \big(\mathfrak{c} \cap [\mu_i, \mu_{i+1})\big) = \mathfrak{c} \cap [\mu_0, \sup \mathfrak{c})$$

is in the ideal and, as said above, $\mathfrak{c} \cap \mu_0 \in J_1$ so by (B)

$$\mathfrak{c} = \big(\mathfrak{c} \cap \mu_0\big) \cup \big(\mathfrak{c} \cap [\mu_0, \sup \mathfrak{c})\big)$$

is in J_1; note $\sup(\mathfrak{c}) \notin \mathfrak{c}$ as $\sup(\mathfrak{c})$ is singular. As $\mathfrak{c} \in J_*[\mathrm{pcf}(\mathfrak{a})]$ implies $\mathfrak{c}$ has no inaccessible accumulation point, we have covered all cases in the induction, so $(*)_4$ holds. Now note that $\mathfrak{b} \in J_*[\mathrm{pcf}(\mathfrak{a})]$, so from $(*)_4$ we get $\mathfrak{b} \in J_1$ and by the definition of J_1 we are done.

Next, why do (A), (B), (C) hold?
We shall use $(*)_1$ from above freely.

For (1): Let $g \in \prod \mathfrak{b}$. If $\theta \in \mathfrak{a}$, it follows from $f^{\mathfrak{b}} \restriction \mathfrak{a} = f$ and $(*)_0$. If $\theta \in \mathfrak{b} \setminus \mathfrak{a}$ ($\subseteq \mathrm{pcf}(\mathfrak{a})$), then $g(\theta) < \theta$. Let $f = f^\theta_{g(\theta)+1}$, hence

$$\big(\forall \gamma \leq g(0)\big)\big[f \not\leq f^\theta_\gamma \mod J_{<\theta}[\mathfrak{a}]\big].$$

Hence $g(\theta) < f^{\mathfrak{b}}(\theta)$; this shows $\{\theta\} \in J_1$.

For (2): Trivially, $\mathfrak{c} \subseteq \mathfrak{c}' \in J_1 \ \Rightarrow\ \mathfrak{c} \in J_1$. If $\mathfrak{c}_1, \mathfrak{c}_2 \in J_1$, $\mathfrak{c} = \mathfrak{c}_1 \cup \mathfrak{c}_2$, and $g \in \prod \mathfrak{c}$ then choose (for $\ell = 1, 2$) $f_\ell \in \prod \mathfrak{a}$ such that $g \restriction \mathfrak{c}_\ell \leq f^{\mathfrak{b}}_\ell$. Now let $f \in \prod \mathfrak{a}$ be defined by $f(\theta) = \max\{f_1(\theta), f_2(\theta)\}$, so $f \in \prod \mathfrak{a}$ and $g \restriction \mathfrak{c}_1 \leq f^{\mathfrak{b}}_1 \leq f^{\mathfrak{b}}$ and $g \restriction \mathfrak{c}_2 \leq f^{\mathfrak{b}}_2 \leq f^{\mathfrak{b}}$ hence $g \restriction (\mathfrak{c}_1 \cup \mathfrak{c}_2) \leq f^{\mathfrak{b}}$.

For (3): Let $g \in \prod \mathfrak{c}$. By assumption, for each $i < \kappa$, for some $f_i \in \prod \mathfrak{a}$, $g \restriction \mathfrak{c}_i \leq f^{\mathfrak{b}}_i$. Now $\big(\prod \mathfrak{a}, <_{J_{\leq\kappa}[\mathfrak{a}]}\big)$ is κ^+-directed, hence for some $f \in \prod \mathfrak{a}$,

$$\bigwedge_{i<\kappa} f_i <_{J_{\leq\kappa}[\mathfrak{a}]} f.$$

Without loss of generality, $g \restriction \mathfrak{a} \leq f \restriction \mathfrak{c}$. Let $\theta \in \bigcup_{i<\kappa} \mathfrak{c}_i$. If $\theta \in \mathfrak{a}$ trivially $g(\theta) \leq f(\theta)$, so assume $\theta \notin \mathfrak{a}$. Now, for some i, $\theta \in \mathfrak{c}_i$, so $\theta > \kappa$ and $f_i <_{J_{\leq\kappa}[\mathfrak{a}]} f$, hence $f_i <_{J_{<\theta}[\mathfrak{a}]} f$. Hence by their definitions $f^{\mathfrak{b}}_i(\theta) \leq f^{\mathfrak{b}}(\theta)$.

So (A), (B), (C) hold and hence $\mathfrak{b}$ is as required, i.e., we have proved subfact 1.8. $\square$

We finish by

Sub-fact 1.9. $\prod \mathfrak{b}/J$ is λ-directed.

Proof of the subfact. Assume $g_i \in \prod \mathfrak{b}$ for $i < i^* < \lambda$. By 1.8, for each $i < i^*$, for some $f_i \in \prod \mathfrak{a}$, we have $g_i \leq f_i^{\mathfrak{b}}$. But $\prod \mathfrak{a}/J_{<\lambda}[\mathfrak{a}]$ is λ-directed, hence for some $f \in \prod \mathfrak{a}$ we have

$$\bigwedge_{i<i^*} f_i < f \mod J_{<\lambda}[\mathfrak{a}].$$

By 1.7 we have

$$\bigwedge_{i<i^*} f_i^{\mathfrak{b}} \leq f^{\mathfrak{b}} \mod J,$$

hence by the previous sentence $i < i^* \Rightarrow g_i \leq f_i^{\mathfrak{b}} \leq_J f^{\mathfrak{b}}$, so $f^{\mathfrak{b}} + 1$ is a $<_J$-upper bound of $\{g_i : i < i^*\}$, as required. □

□

References

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