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BOOLEAN TYPES IN DEPENDENT THEORIES

ITAY KAPLAN, ORI SEGEL, AND SAHARON SHELAH

Abstract. The notion of a complete type can be generalized in a natural manner to allow assigning a value in an arbitrary Boolean algebra $\mathcal{B}$ to each formula. We show some basic results regarding the effect of the properties of $\mathcal{B}$ on the behavior of such types, and show they are particularity well behaved in the case of NIP theories. In particular, we generalize the third author's result about counting types, as well as the notion of a smooth type and extending a type to a smooth one. We then show that Keisler measures are tied to certain Boolean types and show that some of the results can thus be transferred to measures—in particular, giving an alternative proof of the fact that every measure in a dependent theory can be extended to a smooth one. We also study the stable case. We consider this paper as an invitation for more research into the topic of Boolean types.

§1. Introduction. A complete type over a set A in the variable x is a maximal consistent set of formulas from $L_x(A)$, the set of formulas in x with parameters from A. Of course, one can think of $L_x(A)$ as a Boolean algebra by identifying formulas which define the same set in $\mathfrak{C}$ (the monster model). Viewed this way, a type can also be defined as a homomorphism of Boolean algebras from $L_x(A)$ to the Boolean algebra 2. The idea behind this work is to generalize this definition by allowing an arbitrary Boolean algebra $\mathcal{B}$ in the range. We call these homomorphisms $\mathcal{B}$*-types over* A (see Definition 2.3). Without any assumptions, Boolean types may behave very wildly, but it turns out that if the ambient theory T is dependent (NIP) then there are some restrictions on their behavior which gives some credence to the claim that this is the right context to study such types in full generality.

Let us consider an example. Suppose that T is the theory of the random graph in the language $\{R\}$ and that $\mathcal{B}$ is any Boolean algebra. Let $M \models T$ be of size $\geq |\mathcal{B}|$ and let $h : M \to \mathcal{B}$ be a surjective map. Let $p : L_x(M) \to \mathcal{B}$ be the unique homomorphism defined by mapping formulas of the form $x\, R\, a$ to $h(a)$ and formulas of the form $x = a$ to 0 (such a homomorphism exists by quantifier elimination and [5, Proposition 5.6]). Thus for any $\mathcal{B}$ there is a type whose image has size $|\mathcal{B}|$.

However, when T is NIP (dependent) this fails. For example, suppose that $T = DLO$ (the theory of $(\mathbb{Q}, <)$), and suppose that $\mathcal{B}$ is a Boolean algebra with c.c.c (the countable chain condition: if $\langle \mathfrak{a}_i \mid i < \aleph_1 \rangle$ are non-zero elements then for some $i, j < \aleph_1$, $\mathfrak{a}_i \wedge \mathfrak{a}_j \neq 0$), for example, the algebra of measurable sets up to measure 0 in some (real) probability space. Let M be any model and let

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$p : L_x(M) \to \mathcal{B}$ be any $\mathcal{B}$-type. Suppose that the image of p has size $\left(2^{\aleph_0}\right)^+$. By quantifier elimination, the image of p is the algebra generated by elements of the form $p(x < a)$ for $a \in M$ (since $x = a$ is equivalent to $\neg(x < a \wedge a < x)$). It follows that $|\{p(x < a) \mid a \in M\}| = \left(2^{\aleph_0}\right)^+$, so we can find $\left\langle a_i \,\middle|\, i < \left(2^{\aleph_0}\right)^+ \right\rangle$ such that $p(x < a_i) \neq p(x < a_j)$ for $i \neq j$. By Erdős–Rado, we may assume that $\langle a_i \mid i < \omega_1 \rangle$ is either increasing or decreasing. Assume the former. Then $p(x < a_i) <^{\mathcal{B}} p(x < a_j)$ so $p(a_i \leq x) \cdot p(x < a_j) = p(a_i \leq x < a_j) \neq 0$ for all $i < j < \omega_1$ and thus $\{p(a_i \leq x < a_{i+1}) \mid i < \omega_1\}$ is a set of size $\aleph_1$ of nonzero mutually disjoint elements from $\mathcal{B}$, contradiction. This boundedness of the image generalizes to any NIP theory T (see Proposition 2.12 below).

Let us consider another example. In the classical settings, any type can be realized in an elementary extension. Once a type is realized, it has a unique extension to any model. Boolean types that have unique extensions are called *smooth*. Going back to the theory of the random graph, let us assume that $\mathcal{B}$ is the algebra of Borel subsets of 2^κ up to measure 0 (with the measure μ being the product measure; see [2, 254J]), and assume that $|M| \geq \kappa$. Let $h : M \to \kappa$ be surjective. Let $p : L_x(M) \to \mathcal{B}$ be defined by $p(x\,R\,a) = U_{h(a)} = \{\eta \in 2^\kappa \mid \eta(h(a)) = 1\}$ so that its measure is $1/2$ (to be precise we should put the class of this set, but we will ignore this nuance for this discussion) and $p(x = a) = 0$ for any $a \in M$ (again such a $\mathcal{B}$-type exists). Note that if $N \succ M$ and q is a $\mathcal{B}$-type extending p, then $q(x = a) = 0$ for any $a \in N$ since otherwise it has positive measure, which leads to a contradiction (since for any conjunction φ of atomic formulas or their negations over M which a realizes satisfies $\mu(q(\varphi)) \geq \mu(q(x = a))$ and the left-hand side tends to 0 as the number of distinct conditions in φ grows by the choice of p). Thus by Sikorski (see Fact 2.1), we have a lot of freedom in extending q to any $N' \succ N$. Hence no smooth extension of p exists. Again, when T is NIP and the Boolean algebra is nice enough, every type has a smooth extension (see, e.g., Corollary 2.35 below).

1.1. Structure of the paper. In Section 2 we prove all the main results on Boolean types. In Section 2.1 we give the basic definitions. In particular we define two kinds of maps between Boolean types: those induced by elementary maps (like in the classical setting) and those induced by embeddings of the Boolean algebra itself. In Section 2.2 we then give bounds to the number of Boolean types up to conjugation (in particular generalizing a result of the third author [10, Theorem 5.21]) or just image conjugation. In Section 2.3 we define and discuss smooth Boolean types as well as a stronger notion, that of a realized Boolean type. We prove that under NIP, if the algebra is complete, smooth types exist.

In Section 3 we relate Boolean types to Keisler measures, i.e., finitely additive measures on definable sets. In Section 3.2 we apply the results of Section 2.2 to count Keisler measures up to conjugation (we also give a more direct proof, using the VC-theorem). In Section 3.3 we give an alternative proof to the well-known fact that every Keisler measure extends to a smooth one [11, Proposition 7.9], using the results of Section 2.3.

In Section 4 we analyze the case where the theory (or just one formula) is stable, as well as the totally transcendental case, showing that in this case Boolean types are locally averages of types (and in the t.t. case this is true for complete types as well).

Throughout, let T be a complete first-order theory. Most of the time, we will only deal with dependent T. We use standard notations, e.g., $\mathfrak{C}$ is a monster model for T. As usual, all sets and models are subsets or elementary substructures of $\mathfrak{C}$ of cardinality $< |\mathfrak{C}|$.

§2. Boolean types.

2.1. Basic definitions. In this subsection we define Boolean types, which are the basic objects studied in this paper. We also define when two such types are conjugate to each other. We finish this subsection by briefly discussing the algebraic properties of Boolean types.

Let us start by recounting some standard notation for Boolean algebras.

Let $\mathcal{B}$ be a Boolean algebra, and denote by 0 and 1 the distinguished elements of $\mathcal{B}$ corresponding to $\bot$ and $\top$ for formulas. We denote by $\mathcal{B}^+$ the set of all nonzero elements of $\mathcal{B}$.

Let $\mathfrak{a}, \mathfrak{b} \in \mathcal{B}$. We denote by $-\mathfrak{a}$, $\mathfrak{a} + \mathfrak{b}$, and $\mathfrak{a} \cdot \mathfrak{b}$ the complement, sum, and product—corresponding to $\neg, \vee$, and $\wedge$ for formulas, respectively (for example, $-0 = 1$). We also write $\mathfrak{a} - \mathfrak{b} = \mathfrak{a} \cdot (-\mathfrak{b})$. We say $\mathfrak{a}, \mathfrak{b}$ are disjoint if $\mathfrak{a} \cdot \mathfrak{b} = 0$. We also write $(-1) \cdot \mathfrak{a}$ for $-\mathfrak{a}$ and $1 \cdot \mathfrak{a}$ for $\mathfrak{a}$.

Every Boolean algebra has a canonical order relation: $\mathfrak{a} \leq \mathfrak{b}$ iff $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{a}$ or equivalently $\mathfrak{a} + \mathfrak{b} = \mathfrak{b}$. This corresponds to $\rightarrow$ for formulas. Recall that 0 and 1 are the minimum and maximum with respect to this order.

If the supremum of a set $A \subseteq \mathcal{B}$ exists we denote it by $\sum A$; likewise, if the infimum exists we denote it by $\prod A$. An algebra is complete if both always exist.

A (κ-)complete subalgebra of $\mathcal{B}$ is some subalgebra $\mathcal{B}' \subseteq \mathcal{B}$ such that if $A \subseteq \mathcal{B}'$ (and $|A| < \kappa$), and if $\sum A$ exists in $\mathcal{B}$, then $\sum A \in \mathcal{B}'$.

If $\mathfrak{b} \in \mathcal{B}^+$, we define the relative algebra $\mathcal{B}|_{\mathfrak{b}}$—its universe is $\{\mathfrak{a} \in \mathcal{B} \mid \mathfrak{a} \leq \mathfrak{b}\}$, with $+, \cdot$ and 0 inherited from $\mathcal{B}$, and with $1_{\mathcal{B}|_{\mathfrak{b}}} = \mathfrak{b}$ and $(-\mathfrak{a})_{\mathcal{B}|_{\mathfrak{b}}}$ being $\mathfrak{b} - \mathfrak{a}$. Note that $\mathcal{B}|_{\mathfrak{b}}$ is complete if $\mathcal{B}$ is.

There is a natural homomorphism $\pi_{\mathfrak{b}} : \mathcal{B} \to \mathcal{B}|_{\mathfrak{b}}$ given by $\pi_{\mathfrak{b}}(\mathfrak{a}) = \mathfrak{a} \cdot \mathfrak{b}$.

Since we will deal with homomorphisms of Boolean algebras, we will also need the following facts (Sikorski's extension theorem):

FACT 2.1 [5, Theorems 5.5 and 5.9]. *Assume $\mathcal{B}$ is a complete Boolean algebra and $\mathcal{A}$ is any Boolean algebra. Assume $A \subseteq \mathcal{A}$ a subset and $f : A \to \mathcal{B}$ is a function.*

Then there is a homomorphism $g : \mathcal{A} \to \mathcal{B}$ extending f iff, for any $\mathfrak{a}_0, \dots, \mathfrak{a}_{n-1} \in A$ and $\varepsilon_0, \dots, \varepsilon_{n-1} \in \{\pm 1\}$ such that $\prod_{i<n} \varepsilon_i \mathfrak{a}_i = 0$, we also have $\prod_{i<n} \varepsilon_i f(\mathfrak{a}_i) = 0$.

FACT 2.2 [5, Proposition 5.8 and Theorem 5.9]. *Assume $\mathcal{B}$ is a complete Boolean algebra and $\mathcal{A}$ is any Boolean algebra. Assume $\mathcal{A}' \subseteq \mathcal{A}$ is a subalgebra, and $f : \mathcal{A}' \to \mathcal{B}$ is a homomorphism.*

Assume further that $\mathfrak{a} \in \mathcal{A}$, $\mathfrak{b} \in \mathcal{B}$. Then there exists a homomorphism $g : \mathcal{A} \to \mathcal{B}$ extending f such that $g(\mathfrak{a}) = \mathfrak{b}$ iff $\sum_{\mathfrak{a}' \in \mathcal{A}', \mathfrak{a}' \leq \mathfrak{a}} f(\mathfrak{a}') \leq \mathfrak{b} \leq \prod_{\mathfrak{a}' \in \mathcal{A}', \mathfrak{a} \leq \mathfrak{a}'} f(\mathfrak{a}')$.

DEFINITION 2.3. Suppose $\mathcal{B}$ is a Boolean algebra and $\mathfrak{C}$ is the monster model of a complete theory T.

For a set $A \subseteq \mathfrak{C}$, a *(complete)$\mathcal{B}$-type in x over A* is a Boolean algebra homomorphism from the algebra of formulas $L_x(A)$ consisting of formulas in x over A up to equivalence in $\mathfrak{C}$ to $\mathcal{B}$. By slight abuse of notation, we will use a formula to refer to its equivalence class in $L_x(A)$. The set of all complete $\mathcal{B}$-types in x over A is denoted by $S^x_{\mathcal{B}}(A)$. The set $S^n_{\mathcal{B}}(A)$ for n a natural number, or an ordinal, will be the set of all complete $\mathcal{B}$-types over A in some n fixed variables.

An elementary permutation $\pi : A \to A$ is a bijective elementary map. The group of elementary permutations $\pi : A \to A$ acts on $S^x_{\mathcal{B}}(A)$ by $(\pi * p)(\varphi(x, a)) = p\left(\varphi\left(x, \pi^{-1}(a)\right)\right)$. Say that $p_1, p_2 \in S^x_{\mathcal{B}}(A)$ are *elementarily conjugates over A* if they are in the same orbit.

We say that two $\mathcal{B}$-types are *image conjugates* if there is some partial isomorphism of Boolean algebras σ whose domain contains the image of p_2 such that $p_1 = \sigma \circ p_2$.

Finally, we say that $p_1, p_2 \in S^x_{\mathcal{B}}(A)$ are *conjugates over A* if there is some $\pi : A \to A$ as above, and some partial Boolean isomorphism σ as above such that $p_1 = \sigma \circ (\pi * p_2)$.

Remark 2.4. Note that $\sigma \circ (\pi * p) = \pi * (\sigma \circ p)$. Thus, as image conjugation and elementary conjugation are clearly equivalence relations, so is conjugation.

Remark 2.5. Note that when $\mathcal{B} = 2$ (that is $\{0, 1\}$), a complete $\mathcal{B}$-type is the same as a complete type.

Also, the two notions of elementarily conjugation and conjugation identify in this case.

Remark 2.6. Note that $L_x(A)$ is isomorphic to the quotient of the Lindenbaum–Tarski algebra $\mathcal{L}_{A,x}$ (the algebra of L formulas over A in x up to logical equivalence; see [6, Chapter 26]) by the filter F generated by all sentences φ that hold in $\mathfrak{C}$ (where parameters from A are considered to be constants).

Indeed, let $f : \mathcal{L}_{A,x} \to L_x(A)$ be the canonical map sending a formula to its equivalence class in $\mathfrak{C}$. We want to show that $f^{-1}(\top) = F$.

$\psi(\mathfrak{C}) = \mathfrak{C}^x$ iff $\mathfrak{C} \vDash \forall x\,(\psi(x))$;

and since $\vdash (\forall x \psi(x)) \to \psi(x)$, we get $\forall x \psi(x) \leq \psi(x)$ in $\mathcal{L}_{A,x}$ and thus $\psi(x) \in F$.

On the other hand if $\psi(x) \in F$ then for some sentence φ such that $\mathfrak{C} \vDash \varphi$ we have $\varphi \leq \psi(x)$ in $\mathcal{L}_{A,x}$, that is $\vdash \varphi \to \psi(x)$, and certainly $\psi(\mathfrak{C}) = \mathfrak{C}^x$.

This means that for any Boolean algebra $\mathcal{B}$, $\{f \in Hom(\mathcal{L}_{A,x}, \mathcal{B}) \mid f[F] = 1\}$ are canonically isomorphic to $Hom(L_x(A), \mathcal{B})$.

Thus we can define $S^x_{\mathcal{B}}(A)$ to be $\left\{p \in Hom(\mathcal{L}_{A,x}, \mathcal{B}) \mid F \subseteq p^{-1}(1)\right\}$ without changing anything.

Example 2.7. One might wonder if there are any natural examples of Boolean types.

Let $\langle \mathcal{B}_i \rangle_{i \in I}$ be a sequence of Boolean algebras. Then the product algebra $\prod_{i \in I} \mathcal{B}_i$ is defined in the usual sense of products of algebraic structures—its elements are choice functions, and operations are preformed coordinatewise.

By the universal property of product algebras (see [5, Proposition 6.3]),

$$S^x_{\prod_{i \in I} \mathcal{B}_i}(A) \cong \prod_{i \in I} S^x_{\mathcal{B}_i}(A).$$

The correspondence is given by $\langle p_i \rangle_{i \in I} \mapsto \left(\varphi \mapsto \langle p_i(\varphi) \rangle_{i \in I}\right)$.

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In particular, for any cardinal λ, $S^x_{2^\lambda}(A) \cong \left(S^x_2(A)\right)^\lambda = \left(S^x(A)\right)^\lambda$ naturally—where the sequence $\langle p_i \rangle_{i<\lambda}$ corresponds to the Boolean type p satisfying $\varphi \in p_i \iff p(\varphi)_i = 1$.

Thus we can consider a sequence of λ complete types to be the same thing, for all intents as purposes, as a $\mathcal{B}$-type for $\mathcal{B} = 2^\lambda$.

This case will give us an idea about the behavior of general Boolean types.

2.2. Counting Boolean types. The main result of this section is Corollary 2.17 (generalizing a result by the third author [10, Theorem 5.21]), which says that when T is NIP, the number of Boolean types over a saturated model up to conjugation has a bound dependent not on $|\mathcal{B}|$ but on the chain condition satisfied by $\mathcal{B}$. The strategy will be to encode Boolean types up to conjugacy as complete types in a long variable tuple.

Throughout this subsection, A is a subset of $\mathfrak{C}$.

2.2.1. Counting Boolean types up to conjugation. We first concern ourselves with the question of the number of Boolean types up to conjugation.

Lemma 2.8. *There exists an injection $f : S^x_{2^\lambda}(A) \to S^{|x|\cdot\lambda}(A)$ such that if $f(p_1)$ and $f(p_2)$ are conjugates then p_1 and p_2 are elementary conjugates.*

Proof. Note first that $\left(S^x(A)\right)^\lambda$ can be embedded in $S^{|x|\cdot\lambda}(A)$—choose for each $p \in S^x(A)$ some $b_p \in \mathfrak{C}$ realizing it; and send each $\langle p_i \rangle_{i<\lambda}$ to $\operatorname{tp}\left(\langle b_{p_i} \rangle_{i<\lambda} / A\right)$. Obviously, this is injective.

For $p \in S^x_{2^\lambda}(A)$, let $\overline{p} \in \left(S^x(A)\right)^\lambda$ be the corresponding sequence (as in Example 2.7) and take its image $q \in S^{|x|\cdot\lambda}(A)$ under this embedding. Then we define $f(p) = q$.

Assume $p_1, p_2 \in S^x_{2^\lambda}(A)$ and let $q_1 = f(p_1), q_2 = f(p_2)$. If q_1, q_2 are elementary conjugates as witnessed by π, so are $\overline{p_1}, \overline{p_2}$ (in each coordinate) and thus also p_1, p_2.

Indeed for any $\varphi(x, a)$ and $i < \lambda$,

$$p_1\left(\varphi\left(x, \pi^{-1}(a)\right)\right)_i = 1 \iff \mathfrak{C} \models \varphi\left(b_{(p_1)_i}, \pi^{-1}(a)\right) \iff \\ \mathfrak{C} \models \varphi\left(b_{(p_2)_i}, a\right) \iff \left(p_2\left(\varphi(x, a)\right)\right)_i = 1,$$

where for any formula φ, $p(\varphi) \in 2^\lambda$, so $p(\varphi)$ is a function from λ to 2 and $p(\varphi)_i$ is the image of i.

Thus $p_1\left(\varphi\left(x, \pi^{-1}(a)\right)\right) = p_2\left(\varphi(x, a)\right) \Rightarrow p_2 = \pi * p_1$. $\dashv$

As an immediate corollary we have the following:

Corollary 2.9. *The number of types in $S^x_{2^\lambda}(A)$ up to elementary conjugation is at most the number of types in $S^{\lambda|x|}(A)$ up to conjugation.*

Definition 2.10. Fix some complete Boolean algebra $\mathcal{B}$ and regular cardinal κ. $\mathcal{B}$ is κ-c.c if there is no antichain (a set of pairwise disjoint elements from $\mathcal{B}^+$) in $\mathcal{B}$ of size κ.

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DEFINITION 2.11. Suppose $p \in S^x_{\mathcal{B}}(A)$, and $\varphi(x, y)$ is some formula. Then $p \restriction \varphi$, or $p|_\varphi$, is the restriction of p to the definable sets of the form $\varphi(x, a)$ for a some tuple (in the length of y) from A.

PROPOSITION 2.12. *Assume $\varphi(x, y)$ is NIP and $\mathcal{B}$ is κ-c.c. Then for any $p \in S^x_{\mathcal{B}}(A)$, the image of $p|_\varphi$ has cardinality $\leq 2^{<\kappa}$.*
In particular if T is NIP this holds for every φ.

PROOF. Recall that a subset X of $\mathcal{B}$ is *independent* if every nontrivial finite product from it is non-empty: for every $\mathfrak{a}_1, \dots, \mathfrak{a}_n$ and $\mathfrak{b}_1, \dots, \mathfrak{b}_m$ in X such that $\mathfrak{a}_i \neq \mathfrak{b}_j$ for all i, j, the product $\mathfrak{a}_1 \cdot \dots \cdot \mathfrak{a}_n \cdot -\mathfrak{b}_1 \cdot \dots \cdot -\mathfrak{b}_m$ is not 0. By [5, Theorem 10.1], [9], if λ is a cardinal such that λ is regular and $\mu^{<\kappa} < \lambda$ for all $\mu < \lambda$, then every subset $X \subseteq \mathcal{B}$ of cardinality λ has an independent subset $Y \subseteq X$ of cardinality λ.

Let $\lambda = (2^{<\kappa})^+$. It is easy to see that for all $\mu < \lambda$, $\mu^{<\kappa} < \lambda$—since κ is regular, for every $\beta < \kappa$ every function from $|\beta|$ to $\sup\left\{2^{|i|}\right\}_{i<\kappa} = 2^{<\kappa}$ is contained in some $2^{|\alpha|}$ for $\alpha < \kappa$.

Thus

$$\begin{aligned}(2^{<\kappa})^{|\beta|} &\leq \sum_{\alpha<\kappa}\left|\left(2^{|\alpha|}\right)^{|\beta|}\right| = \sum_{\alpha<\kappa} 2^{|\alpha||\beta|} \leq \kappa \cdot \sup\left\{2^{|\alpha||\beta|} \mid \alpha < \kappa\right\} \\ &= \kappa \cdot \sup\left\{2^{|\alpha|} \mid \alpha < \kappa\right\} = 2^{<\kappa},\end{aligned}$$

so

$$\mu^{<\kappa} = sup\left\{\mu^{|\beta|}\right\}_{\beta<\kappa} \leq sup\left\{(2^{<\kappa})^{|\beta|}\right\}_{\beta<\kappa} \leq 2^{<\kappa} < \lambda.$$

Since T is NIP, $\varphi(x, y)$ has dual VC-dimension $n < \omega$: for any $\langle a_i \rangle_{i<n+1} \in A^{|y|}$, $\{\varphi(b, y)\}_{b\in\mathfrak{C}^{|x|}}$ does not shatter $\langle a_i \rangle_{i<n+1}$.

Thus for any such sequence $\langle a_i \rangle_{i<n+1}$ exists $I \subseteq n+1$ such that

$$\bigwedge_{i\in I} \varphi(x, a_i) \wedge \bigwedge_{i\in(n+1)\setminus I} \neg\varphi(x, a_i)$$

is inconsistent.

We get

$$\begin{aligned}&\prod_{i\in I} p(\varphi(x, a_i)) \cdot \prod_{i\in(n+1)\setminus I} - p(\varphi(x, a_i)) \\ &= p\left(\bigwedge_{i\in I} \varphi(x, a_i) \wedge \bigwedge_{i\in(n+1)\setminus I} \neg\varphi(x, a_i)\right) = 0,\end{aligned}$$

so $Im\left(p|_\varphi\right)$ has no independent set of size $n+1$, let alone λ—thus $\left|Im\left(p|_\varphi\right)\right| < \lambda \Rightarrow \left|Im\left(p|_\varphi\right)\right| \leq 2^{<\kappa}$. ⊣

REMARK 2.13. If φ has IP and $\mathcal{B}$ is complete, then there is a $\mathcal{B}$-type p such that $Im\left(p|_\varphi\right) = \mathcal{B}$. Indeed by compactness there is a sequence of parameters $\langle b_\mathfrak{a} \mid \mathfrak{a} \in \mathcal{B}\rangle$ such that $\langle \varphi(x, b_\mathfrak{a}) \mid \mathfrak{a} \in \mathcal{B}\rangle$ is independent as a subset of $L_x\left(\{b_\mathfrak{a}\}_{\mathfrak{a}\in\mathcal{B}}\right)$, and $f : \{\varphi(x, b_\mathfrak{a})\}_{\mathfrak{a}\in\mathcal{B}} \to \mathcal{B}$ defined as $f(\varphi(x, b_\mathfrak{a})) = \mathfrak{a}$ can be extended to a $\mathcal{B}$-type by Fact 2.1, since there are no (non-trivial) Boolean relations on the domain.

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Now assume NIP and that $\mathcal{B}$ has κ-c.c. Let $\mathcal{A}$ be the set of subalgebras B of $\mathcal{B}$ of size at most $\mu = 2^{<\kappa} + |x| + |T|$. Proposition 2.12 says that for $p \in S^x_{\mathcal{B}}(A)$, the algebra B_p, which is the image of p, is in $\mathcal{A}$. For each $B \in \mathcal{A}$, choose some enumeration $\langle \mathfrak{a}_{B,i} \mid i < \mu \rangle$ (maybe with repetitions) of B^+. Let us say that B_1 is conjugate to B_2 if there is a (unique) isomorphism $\sigma : B_1 \to B_2$ taking $\mathfrak{a}_{B_1,i}$ to $\mathfrak{a}_{B_2,i}$.

Given $B \in \mathcal{A}$, for each $i < \mu$, choose an ultrafilter $D_{B,i}$ (of B) which contains $\mathfrak{a}_{B,i}$ (equivalently a homomorphism from B to 2 such that $D_{B,i}(\mathfrak{a}_{B,i}) = 1$; see [5, Proposition 2.15]). We ask that if B_1 and B_2 are conjugates, say via σ, then $\sigma\left(D_{B_1,i}\right) = D_{B_2,i}$—or in the language of homomorphisms, $D_{B_1,i} = D_{B_2,i} \circ \sigma$. To achieve this we choose representatives for the conjugacy classes and a sequence of ultrafilters for each representative and construct the others from it. Note that if B_1 is the chosen representative, and $\sigma_1 : B_3 \to B_2$ and $\sigma_2 : B_2 \to B_1$ witness the conjugacy then $\sigma_2 \circ \sigma_1 : B_3 \to B_1$ witnesses that B_1 and B_3 are conjugates. Thus $D_{B_3,i} = D_{B_1,i} \circ (\sigma_2 \circ \sigma_1) = \left(D_{B_1,i} \circ \sigma_2\right) \circ \sigma_1 = D_{B_2,i} \circ \sigma_1$.

Let $D_B = \langle D_{B,i} \rangle_{i<\mu} : B \to 2^\mu$, the product homomorphism.

For $i < \mu$ and $p \in S^x_{\mathcal{B}}(A)$, let $\widetilde{q}_p \in S^x_{2^\mu}(A)$ be the 2^μ-type $D_{B_p} \circ p$. Finally, let $q_p = f(\widetilde{q}_p)$ in $S^{\langle x_i \mid i<\mu \rangle}(A)$ where $f : S^x_{2^\mu}(A) \to S^{\langle x_i \mid i<\mu \rangle}(A)$ is as in Lemma 2.8.

Proposition 2.14 (Assuming $\mathcal{B}$ has κ-c.c. and T has *NIP*). *Suppose* $p_1, p_2 \in S^x_{\mathcal{B}}(A)$ *and* $B = B_{p_1} = B_{p_2}$ *and that* $\widetilde{q}_{p_1} = \widetilde{q}_{p_2}$. *Then* $p_1 = p_2$.

Proof. If not, then for some $\varphi(x, a)$, $p_1(\varphi(x, a)) \neq p_2(\varphi(x, a))$.

WLOG $p_1(\varphi(x, a)) \nleq p_2(\varphi(x, a))$, and let $\mathfrak{b} = p_1(\varphi(x, a)) - p_2(\varphi(x, a))$, so $\mathfrak{b} \in B^+$. For some $i < \mu$, $\mathfrak{b} = \mathfrak{a}_{B,i}$. It follows that $p_1(\varphi(x, a)) - p_2(\varphi(x, a)) \in D_{B,i}$; hence $\widetilde{q}_{p_1}(\varphi(x, a))_i = 1$ and $\widetilde{q}_{p_2}(\varphi(x, a))_i = 0$ that is $\widetilde{q}_{p_1} \neq \widetilde{q}_{p_2}$. ⊣

Corollary 2.15 (Assuming $\mathcal{B}$ has κ-c.c. and T has *NIP*). *Suppose* $p_1, p_2 \in S^x_{\mathcal{B}}(A)$ *and* $B = B_{p_1} = B_{p_2}$. *If* q_{p_1} *and* q_{p_2} *are conjugates over* A, *then* p_1 *and* p_2 *are elementarily conjugates over* A.

Proof. Suppose that $\pi * q_{p_1} = q_{p_2}$ for an elementary permutation $\pi : A \to A$. Then also, by Lemma 2.8 (to be precise, by the choice of q_{p_1}, q_{p_2}), $\pi * \widetilde{q}_{p_1} = \widetilde{q}_{p_2}$. Note that $\pi * \widetilde{q}_{p_1} = \widetilde{q}_{\pi * p_1}$.

But $B_{\pi * p_1} = B_{p_1}$ (since π just permutes the domain); thus we can apply Proposition 2.14 to $\pi * p_1, p_2$. ⊣

Corollary 2.16 (Assuming $\mathcal{B}$ has κ-c.c. and T has *NIP*). *If* $p_1, p_2 \in S^x_{\mathcal{B}}(A)$, B_{p_1} *and* B_{p_2} *are conjugates, and* q_{p_1} *and* q_{p_2} *are conjugates over* A, *then* p_1 *and* p_2 *are conjugates over* A.

Proof. Suppose $\sigma : B_{p_1} \to B_{p_2}$ takes $\mathfrak{a}_{B_{p_1},i}$ to $\mathfrak{a}_{B_{p_2},i}$. Then $\sigma \circ p_1$ and p_2 satisfy the condition of Corollary 2.15: note that $B_{\sigma \circ p_1} = Im(\sigma \circ p_1) = \sigma(Im(p_1)) = \sigma\left(B_{p_1}\right) = B_{p_2}$.

Further, by the way we chose the D_B's, $D_{B_{p_1}} = D_{B_{p_2}} \circ \sigma$; thus

$$\widetilde{q}_{p_1} = D_{B_{p_1}} \circ p_1 = D_{B_{p_2}} \circ \sigma \circ p_1 = D_{B_{\sigma \circ p_1}} \circ \sigma \circ p_1 = \widetilde{q}_{\sigma \circ p_1},$$

and thus $q_{\sigma \circ p_1} = q_{p_1}$ and q_{p_2} are conjugates.

So for some elementary permutation $\pi : A \to A$, $\pi * (\sigma \circ p_1) = p_2$. ⊣

Corollary 2.17. *Assume* $\mathcal{B}$ *has* κ*-c.c. and* T *has NIP, and let* $\mu = 2^{<\kappa} + |T| + |x|$.

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The number of types in $S^x_{\mathcal{B}}(A)$ up to conjugation is bounded by the number of types in $S^{|x|\cdot\mu}(A)$ (equivalently $S^{\mu}(A)$) up to conjugation $+\,2^{\mu}$. Hence, if λ,κ are cardinals and α and ordinal such that $\lambda=\lambda^{<\lambda}=\aleph_\alpha=\kappa+\alpha\geq\kappa\geq\beth_\omega+\mu^+$, and if $A=M$ is a saturated model of size λ, this number is bounded by $2^{<\kappa}+|\alpha|^{\mu}+2^{\mu}=2^{<\kappa}+|\alpha|^{\mu}$.

PROOF. Given p, map it to the pair consisting of the conjugacy class of q_p and the quantifier-free type of the algebra B_p, enumerated by $\langle \mathfrak{a}_{B_p,i}\mid i<\mu\rangle$ (whose number is bounded by 2^{μ}, since the language of Boolean algebras is finite). By Corollary 2.16, the preimage of an element under this map is contained in a single conjugacy class. The second part of the statement follows immediately from Theorem 5.21 in [10], since $\mu\geq|T|$. ⊣

2.2.2. Counting types up to image conjugation. One may wonder whether we can get a meaningful bound for $\left|S^x_{\mathcal{B}}(A)\right|$ (without taking elementary conjugation into consideration). For example, the universal property of the product gives us a simple equation: $\left|S^x_{2^{\mu}}(A)\right|=\left|S^x(A)\right|^{\mu}$.

Note that if $\sigma: B\xrightarrow{\sim}B'$ is an isomorphism between two different subalgebras of $\mathcal{B}$ and $p\in S^x_{\mathcal{B}}(A)$ is such that $Im(p)=B$, then $\sigma\circ p\in S^x_{\mathcal{B}}(A)$ is different from p. This means that if $\mathcal{B}$ has many copies of small subalgebras then we necessarily have at least as many $\mathcal{B}$ types. Thus if we want to give a bound that is independent of the size of $\mathcal{B}$, we must restrict ourselves to counting up to image conjugation. Note that 2 has only a single subalgebra (itself) and a single partial isomorphism (the identity), so for 2-types (i.e., classical types) counting types up to image conjugation is the same as just counting types.

COROLLARY 2.18. *Assume $\mathcal{B}$ has κ-c.c and T has NIP, and assume $|A|\geq\aleph_0$. We have the following bounds on λ, the number of types in $S^x_{\mathcal{B}}(A)$ up to image conjugation, where $\mu=2^{<\kappa}+|T|+|x|$.*

If T is stable, $\lambda\leq|A|^{\mu}$.

If T is NIP, $\lambda\leq(ded\,|A|)^{\mu}$, where $ded\ \theta$ is the supremum on the number of cuts on a linearly ordered set of cardinality $\leq\theta$.

If T has IP and x is finite then λ could be maximal, i.e., $\sup\left\{\left|S^x_{\mathcal{B}}(A)\right|\,\middle|\,|A|\leq\kappa\right\}=|\mathcal{B}|^{\kappa}$ *for all $\kappa\geq|\mathcal{B}|+|T|$.*

PROOF. The proof of Corollary 2.16 shows also that if $p_1,p_2\in S^x_{\mathcal{B}}(A)$, B_{p_1} and B_{p_2} are conjugates, and $q_{p_1}=q_{p_2}$, then p_1 and p_2 are image conjugates.

We conclude that the number of $\mathcal{B}$ types over A up to image conjugation is at most $\left|S^{\mu}(A)\right|$.

Let $\phi(x,y)$ be some partitioned formula, and denote by $S_\phi(A)$ the set of ϕ-types, that is, a maximal consistent set of formulas of the form $\phi(x,b)$ or $\neg\phi(x,b)$ where b is a y-tuple in A.

For $\theta=\sup\{|S_\phi(A)|\mid\phi(x,y)\}$, $\left|S^{\mu}(A)\right|\leq\theta^{\mu}$, since the number of formulas in μ variables is at most $\mu+|T|=\mu$, and the map $p\mapsto\langle p|_\phi\rangle_\phi$ is injective.

According to [11, Proposition 2.69], if T has NIP then $\theta\leq ded\,|A|$.

Further, by the preceding remarks there, if T is stable then $\theta\leq|A|$ and thus $\lambda\leq|A|^{\mu}$ for stable T.

Assume that T has IP. Note that $2^{\kappa}\leq|B|^{\kappa}\leq\kappa^{\kappa}=2^{\kappa}$. If $\phi(x,y)$ has IP then there is some model $M\models T$ of size κ such that $|S_\phi(M)|=2^{|M|}$. Note that the algebra 2 is embedded in $\mathcal{B}$ so $S_2^x(M)$ embeds into $S_{\mathcal{B}}^x(M)$. Finally any two of these types are not image conjugates since there is only one embedding of 2 to $\mathcal{B}$. ⊣

2.3. Smooth Boolean types. In this section we define the notion of a smooth Boolean type analogously to a smooth Keisler measure: these are Boolean types which have a unique extension to every larger parameter set. The main result of this section is that every Boolean type in an NIP theory can be extended to a smooth one (see Corollary 2.35). This mirrors a similar result for Keisler measures (see [11, Proposition 7.9]), which we recover later in Section 3.3. We also discuss the stronger notion of a realized Boolean type.

Definition 2.19. Let $A \subseteq B$, $p \in S_{\mathcal{B}}^x(A)$, and $q \in S_{\mathcal{B}}^x(B)$; then q *extends* p if it extends it as a function, that is for any formula $\varphi(x, a)$ over A, $p(\varphi(x, a)) = q(\varphi(x, a))$ (technically, the images of the equivalence classes are the same).

We say that p is *smooth* if for every such B there exists a unique $\mathcal{B}$-type q over B extending p.

Remark 2.20. If $\mathcal{B} = 2$ and $A = M$ is a model, a type is smooth iff it is realized, that is equal to $\operatorname{tp}(a/M)$, for some tuple a in M.

Remark 2.21. As we remarked in Example 2.7, $p \in S_{2^\kappa}^x(A)$ is essentially equivalent to a sequence $\langle p_i \rangle_{i<\kappa}$ of complete types via $p(\varphi)_i = 1 \iff \varphi \in p_i$; it is obvious that q extends p iff q_i extends p_i for all i; and thus p is smooth iff p_i is smooth for each i.

Assume $|x| < \omega$ (i.e., x is a finite tuple). Then this case gives rise naturally to the following definition:

Definition 2.22. A $\mathcal{B}$-type $p \in S_{\mathcal{B}}^x(M)$ over a model M is called *realized* if $\sum_{a\in M} p(x = a)$ exists and equals 1. (Here and later, when we write $a \in M$ in the sum, we mean $a \in M^x$.)

Remark 2.23. If $\mathcal{B} = 2$, then $\sum_{a\in M} p(x = a) = 1$ iff there exists $a \in M$ such that $p(x = a) = 1$. Therefore this definition agrees with the classical one for complete types.

If $\mathcal{B} = 2^\kappa$, again by Example 2.7, $p \in S_{\mathcal{B}}^x(M)$ corresponds to a sequence of complete types and

$$\sum_{a\in M} p(x = a) = 1 \iff \forall i < \kappa \left(\sum_{a\in M} p(x = a)\right)_i = 1 \iff$$
$$\forall i < \kappa \exists a \in M\, (x = a) \in p_i.$$

That is p is realized iff every p_i is realized (in M).

Therefore, by Remark 2.21, in this case p is smooth iff every p_i is smooth iff every p_i is realized iff p is realized.

One direction of the last remark works in general:

Claim 2.24. *Assume $p \in S_{\mathcal{B}}^x(M)$ is realized. Then p is smooth.*

Proof. Let $\mathfrak{b}_a = p(x = a) \in \mathcal{B}$ for each x-tuple a in M. Assume $q \in S^x_{\mathcal{B}}(N)$ extends p. Let $\varphi(x, c) \in L_x(N)$ be some formula.

Then by [5, Lemma 1.33b], since $\sum_{a \in M} \mathfrak{b}_a$ exists and equals 1 by assumption,

$$q(\varphi(x,c)) = q(\varphi(x,c)) \cdot \sum_{a \in M} \mathfrak{b}_a = \sum_{a \in M} q(\varphi(x,c)) \cdot \mathfrak{b}_a$$
$$= \sum_{a \in M} q(\varphi(x,c)) \cdot q(x = a) = \sum_{a \in M} q(x = a \wedge \varphi(x,c)).$$

And in particular the RHS exists.

But for any $a \in M$, if $\mathfrak{C} \models \varphi(a, c)$ then $(x = a \wedge \varphi(x,c)) = (x = a)$ (as definable sets) and thus $q(x = a \wedge \varphi(x,c)) = q(x = a) = \mathfrak{b}_a$, while if $\mathfrak{C} \models \neg\varphi(a,c)$ then $(x = a \wedge \varphi(x,c)) = \bot$ and thus $q(x = a \wedge \varphi(x,c)) = 0$.

Thus we get necessarily (since the supremum never changes when adding or removing 0's)

$$q(\varphi(x,c)) = \sum_{a \in \varphi(M,c)} \mathfrak{b}_a.$$

That is, we get q is uniquely determined. ⊣

One may naturally ask if every smooth type is realized. We start with the following result:

Claim 2.25. *Assume $p \in S^x_{\mathcal{B}}(M)$ is smooth for $\mathcal{B}$ a complete Boolean algebra. Then $\sum_{a \in M} p(x = a)$ is maximal: for any extension $q \in S^x_{\mathcal{B}}(N)$ of p,*

$$\sum_{a \in N} q(x = a) = \sum_{a \in M} p(x = a).$$

Proof. Assume otherwise. Since for any $a \in M$, $q(x = a) = p(x = a)$, we have in particular some $a \in N \setminus M$ be such that $q(x = a) > 0$.

However, there is always a type $q' \in S^x_{\mathcal{B}}(N)$ extending p such that $q'(x = a) = 0$. Indeed, for any consistent $\varphi(x, b) \in L_x(M)$ it cannot be $\varphi(x,b) \to x = a$, as that would imply that a is definable over M; but $M \prec N$ thus its definable closure is itself.

Thus by Fact 2.2, there exists q' as required.

This means that if $q(x = a) > 0$, q and q' are distinct extensions of p; thus p is not smooth. ⊣

The property in Claim 2.25 has an alternative formulation which is somewhat easier to reason about:

Claim 2.26. *Assume $\mathcal{B}$ is complete. A type $p \in S^x_{\mathcal{B}}(M)$ has the property that $\sum_{a \in M} p(x = a)$ is not maximal iff there exists a subalgebra B' of $\mathcal{B}$ such that $Im(p) \subseteq B'$ and an atom $\mathfrak{a} \in B'$ such that $\mathfrak{a} \leq -\sum_{a \in M} p(x = a)$.*

Proof. Assume q extends p and $\sum_{a \in N} q(x = a) > \sum_{a \in M} p(x = a)$ and let $c \in N \setminus M$ such that $q(x = c) > 0$. Let $B' = Im(q)$ and $\mathfrak{a} = q(x = c)$.

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Then $\mathfrak{a}$ must be an atom in B', since if $\mathfrak{b} \leq \mathfrak{a}$ and $\mathfrak{b} \in B'$, let $\varphi(x, b)$ such that $q(\varphi(x, b)) = \mathfrak{b}$. If $\mathfrak{C} \vDash \varphi(c, b)$ then $\varphi(x, b) \wedge x = c$ is the same as $x = c$ and thus $\mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a} = q(\varphi(x, b) \wedge x = c) = q(x = c) = \mathfrak{a}$; similarly if $\mathfrak{C} \nvDash \varphi(c, b)$ then $\varphi(x, b) \wedge x = c$ is $\perp$ and thus we likewise get $\mathfrak{b} = 0$. Finally since $\mathfrak{a}$ is disjoint from $p(x = a)$ for any $a \in M$, $\mathfrak{a} \leq -\sum_{a \in M} p(x = a)$.

On the other hand, let B' and $\mathfrak{a} \in B'$ be an atom as in the claim. Let $D_{\mathfrak{a}} : B' \to 2$ be the principal ultrafilter generated by $\mathfrak{a}$ represented as a homomorphism (i.e., $D_{\mathfrak{a}}(\mathfrak{b}) = 1 \iff \mathfrak{a} \leq \mathfrak{b}$) and let $p' = D_{\mathfrak{a}} \circ p$ which is a complete type over M. p' is not realized in M: as $\mathfrak{a} \leq -\sum_{a \in M} p(x = a)$, $D_{\mathfrak{a}}\left(\sum_{a \in M} p(x = a)\right) = 0$; thus $p'(x = a) = D_{\mathfrak{a}}(p(x = a)) \leq D_{\mathfrak{a}}\left(\sum_{a \in M} p(x = a)\right) = 0$ for any a in M.

Let c realize p' outside of M, and let N containing c and M. Then for any $\varphi(x, b) \in L_x(M)$, if $\varphi(x, b) \to x = c$ then $\varphi(x, b) = \perp$ (since c cannot be definable over M) and thus

$$p(\varphi(x, b)) = 0 \leq \mathfrak{a};$$

and if $x = c \to \varphi(x, b)$ then $\mathfrak{C} \vDash \varphi(c, b)$ and thus

$$D_{\mathfrak{a}}(p(\varphi(x, b))) = p'(\varphi(x, b)) = 1 \Rightarrow \mathfrak{a} \leq p(\varphi(x, b)).$$

Thus by Fact 2.2, p can be extended to a type q over N which satisfies $q(x = c) = \mathfrak{a}$ and thus

$$\sum_{a \in N} q(x = a) > \sum_{a \in M} p(x = a),$$

as required. ⊣

As a corollary of Claim 2.26 we get:

REMARK 2.27. If $p \in S^x_{\mathcal{B}}(M)$ is onto and $\mathcal{B}$ atomless and complete, then $\sum_{a \in M} p(x = a)$ is maximal and in fact $p(x = a) = 0$ for all a.

On the other hand if $\mathcal{B}$ is atomic (that is there is an atom under every positive element) then $\sum_{a \in M} p(x = a)$ is maximal iff $\sum_{a \in M} p(x = a) = 1$ (that is iff p is realized).

EXAMPLE 2.28. Let $L = \{E_B\}_{B \in \mathcal{B}(\mathbb{R})}$ (one unary predicate E_B for every Borel set in $\mathbb{R}$), $T = Th_L(\mathbb{R})$ (with the obvious interpretations), and $\mathcal{B}$ be the algebra $\mathcal{B}(\mathbb{R})/I$ where $I = \{B \in \mathcal{B}(\mathbb{R}) \mid \mu(B) = 0\}$ where μ is the Lebesgue measure; $\mathcal{B}$ is a σ-complete and c.c.c.—thus complete—as well as atomless.

Then T proves that every Boolean combination of the unary predicates is equivalent to single unary predicate, and by a standard argument eliminates quantifiers. Thus for any $M \subseteq \mathbb{R}$, $L_x(M)$ is isomorphic to $\mathcal{B}(\mathbb{R})$ with $E_B(x) \mapsto B$ ($x = a$ is equivalent to $E_{\{a\}}(x)$). Let $p : L_x(M) \to \mathcal{B}$ be the projection.

We get that for any $q : L_x(N) \to \mathcal{B}$ extending p, q is a surjection to an atomless Boolean algebra; therefore it sends atoms to 0, that is $q(x = a) = 0$ for all $a \in N$.

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Thus since $x = y$ is the only atomic formula involving both x and a parameter, and since T eliminates quantifiers, q is uniquely determined.

Thus p is smooth, but not realized by Remark 2.27.

DEFINITION 2.29. Let $p \in S^x_{\mathcal{B}}(M)$ be a Boolean type and let $\varphi(x, y)$ be a formula. We say that the image of p with respect to φ is maximal, or that $Im\left(p|_{\varphi(x,y)}\right)$ is maximal, if for any $N \supseteq M$ and for any $q \in S^x_{\mathcal{B}}(N)$ extending p we have $\{q(x, b) \mid b \in N\} = \{p(x, a) \mid a \in M\}$.

If the image of p is maximal with respect to every φ we say that the image of p is maximal.

The following proposition gives us a way to extend types in a way that maximizes their images, in the following precise sense:

PROPOSITION 2.30. *Assume $\mathcal{B}$ is a Boolean algebra. Let $p \in S^x_{\mathcal{B}}(M)$ be a $\mathcal{B}$-type over M. Then there exists $N \supseteq M$ and a type q over N extending p such that the image of q is maximal.*

PROOF. Let $\langle \varphi_i(x, y) \mid i < |T| \rangle$ be an enumeration of all partitioned formulas (recall that we are assuming that x is a finite tuple) and let $\langle \mathfrak{b}_\alpha \mid \alpha < |\mathcal{B}| \rangle$ be an enumeration of the elements of $\mathcal{B}$.

We construct recursively two increasing sequences with respect to the lexicographic order on $(|T| + 1) \times (|\mathcal{B}| + 1)$:

1. An increasing sequence of models $\langle M_{i,\alpha} \mid \alpha \leq |\mathcal{B}|, i \leq |T| \rangle$.
2. An increasing (with respect to extension) sequence of $\mathcal{B}$-types $\langle p_{i,\alpha} \mid \alpha \leq |\mathcal{B}|, i \leq |T| \rangle$ such that $p_{i,\alpha} \in S^x_{\mathcal{B}}(M_{i,\alpha})$.

The construction is as follows:

For $(0, 0)$, $M_{0,0} = M$, $p_{0,0} = p$.

Fix i and assume we have $M_{i,\alpha}, p_{i,\alpha}$ for $\alpha < |\mathcal{B}|$; if there exist M' and p' over M' such that $M_{i,\alpha} \subseteq M'$, p' extends p and $\mathfrak{b}_\alpha \in Im\left(p'|_{\varphi_i}\right)$, let $M_{i,\alpha+1} = M'$, $p_{i,\alpha+1} = p'$; otherwise let $M_{i,\alpha+1} = M_{i,\alpha}$, $p_{i,\alpha+1} = p_{i,\alpha}$. Assume we have $M_{i,\alpha}, p_{i,\alpha}$ for all $\alpha < \delta \leq |\mathcal{B}|$ a limit ordinal; then define $M_{i,\delta} = \bigcup_{\alpha<\delta} M_{i,\alpha}$ and $p_{i,\delta} = \bigcup_{\alpha<\delta} p_{i,\alpha}$. Note that since $(p_{i,\alpha})_{\alpha<\delta}$ is a chain of homomorphism this is a well-defined homomorphism.

Assume we have $M_{i,|\mathcal{B}|}, p_{i,|\mathcal{B}|}$ for $i < |T|$ and let $M_{i+1,0}, p_{i+1,0} = M_{i,|\mathcal{B}|}, p_{i,|\mathcal{B}|}$. Finally assume we have $M_{i,|\mathcal{B}|}, p_{i,|\mathcal{B}|}$ for all $i < j \leq |T|$ a limit ordinal. Then define $M_{j,0} = \bigcup_{i<j} M_{i,|\mathcal{B}|}$ and $p_{j,0} = \bigcup_{i<j} p_{i,|\mathcal{B}|}$.

Now let $N = M_{|T|,|\mathcal{B}|}$, $q = p_{|T|,|\mathcal{B}|}$. Then if for any $\mathfrak{b}_\alpha$ and φ_i we have that $\mathfrak{b}_\alpha \in Im\left(q'|_{\varphi_i}\right)$ for some extension q' of q then the same holds for q: any extension of q is also an extension of $p_{i,\alpha}$ and thus by construction we have $\mathfrak{b}_\alpha \in Im\left(p_{i,\alpha+1}|_{\varphi_i}\right) \subseteq Im\left(q|_{\varphi_i}\right)$. ⊣

REMARK 2.31. If $\mathcal{B}$ has κ-c.c. and T is dependent, then by Propositions 2.12 and 2.30 we can take N to be of cardinality $\leq |M| + 2^{<\kappa} + |T|$, by choosing a preimage $\varphi(x, b)$ for every element in $Im(q)$ and restricting q to a structure containing M and each of the b's (which, from Löwenheim–Skolem, we can take to be no larger than $|M| + 2^{<\kappa} + |T|$).

COROLLARY 2.32. *Suppose $\mathcal{B}$ is complete. Then every $\mathcal{B}$-type $p \in S^x_{\mathcal{B}}(M)$ can be extended to a $\mathcal{B}$-type $q \in S^x_{\mathcal{B}}(N)$ such that $\sum_{a \in N} q(x = a)$ is maximal.*

PROOF. If the image of p with respect to $x = y$ is maximal then $\sum_{a \in M} p(x = a)$ is maximal: indeed, assume q over N extends p. Then $\sum_{a \in M} p(x = a) < \sum_{a \in N} q(x = a)$ implies in particular that $Im(p|_{x=y}) \neq Im(q|_{x=y})$.

We conclude from Proposition 2.30 that every $\mathcal{B}$-type p over M has an extension q over N such that $\sum_{a \in N} q(x = a)$ is maximal. ⊣

Corollary 2.32 essentially reproves the fact that if $\mathcal{B} = 2^\lambda$ then every type can be extended to a smooth type—since in this case by Remark 2.27, $\sum_a q(x = a)$ is maximal iff q is realized and by Remark 2.23 this happens iff q is smooth.

A similar approach can still be useful for any algebra. In the following discussion we no longer need the notion of realized Boolean types, and thus no longer assume that x is a finite tuple. We start with a useful claim:

CLAIM 2.33. *Assume $\mathcal{A}, \mathcal{B}$ are Boolean algebras where $\mathcal{B}$ is complete, $\mathcal{A}' \subseteq \mathcal{A}$ a subalgebra, and $p : \mathcal{A}' \to \mathcal{B}$ and $p_1, p_2 : \mathcal{A} \to \mathcal{B}$ homomorphisms such that $p \subseteq p_1, p_2$ and $p_1 \neq p_2$. Then there exist distinct extensions $q_1, q_2 : \mathcal{A} \to \mathcal{B}$ of p and $\mathfrak{a} \in \mathcal{A}$ such that $q_1(\mathfrak{a}) < q_2(\mathfrak{a})$ and $\sigma \circ q_1 \neq q_2$ for any automorphism σ of $\mathcal{B}$.*

PROOF. Let $\mathfrak{a} \in \mathcal{A} \setminus \mathcal{A}'$ be such that $p_1(\mathfrak{a}) \neq p_2(\mathfrak{a})$.

Let $\mathfrak{b}_1 = \sum_{\mathfrak{a}' \leq \mathfrak{a}, \mathfrak{a}' \in \mathcal{A}'} p(\mathfrak{a}')$ and $\mathfrak{b}_2 = \prod_{\mathfrak{a}' \geq \mathfrak{a}, \mathfrak{a}' \in \mathcal{A}'} p(\mathfrak{a}')$. Then $\mathfrak{b}_1 \leq p_1(\mathfrak{a}) \neq p_2(\mathfrak{a}) \leq \mathfrak{b}_2$ and thus $\mathfrak{b}_1 < \mathfrak{b}_2$.

By Fact 2.2 there are extensions q_i of p such that $q_i(\mathfrak{a}) = \mathfrak{b}_i$ for $0 \leq i < 2$. Assume there is a homomorphism $\sigma : \mathcal{B} \to \mathcal{B}$ such that $\sigma \circ q_1 = q_2$.

Then for any $\mathfrak{b} \in Im(p)$, take $\mathfrak{a}' \in \mathcal{A}'$ such that $p(\mathfrak{a}') = \mathfrak{b}$. Then $\sigma(\mathfrak{b}) = \sigma(p(\mathfrak{a}')) = \sigma(q_1(\mathfrak{a}')) = q_2(\mathfrak{a}') = p(\mathfrak{a}') = \mathfrak{b}$; thus $\sigma|_{Im(p)} = \mathrm{id}_{Im(p)}$.

Further $\sigma(\mathfrak{b}_1) = \sigma(q_1(\mathfrak{a})) = q_2(\mathfrak{a}) = \mathfrak{b}_2$. We conclude that $\mathfrak{b}_2 = \sigma(\mathfrak{b}_1) \leq \sigma(\mathfrak{b}_2)$. On the other hand, for any $\mathfrak{a}' \in \mathcal{A}'$ such that $\mathfrak{a}' \geq \mathfrak{a}$, since by assumption $\mathfrak{b}_2 \leq p(\mathfrak{a}')$ we get $\sigma(\mathfrak{b}_2) \leq \sigma(p(\mathfrak{a}')) = p(\mathfrak{a}')$; thus $\sigma(\mathfrak{b}_2) \leq \prod_{a' \geq a, a' \in A'} p(\mathfrak{a}') = \mathfrak{b}_2$.

We get $\sigma(\mathfrak{b}_2) = \mathfrak{b}_2 = \sigma(\mathfrak{b}_1)$, and thus σ is not injective and in particular, not an automorphism. ⊣

PROPOSITION 2.34. *Assume that $\mathcal{B}$ is complete, and assume that N is a model of T and q is a $\mathcal{B}$-type over N that has the property in the conclusion of Proposition 2.30: it has maximal image with respect to every formula. Then for any formula $\varphi(x, y)$, if there exist $N' \supseteq N$ and extensions $q_1, q_2 \in S^x_{\mathcal{B}}(N')$ of q such that $q_1|_\varphi \neq q_2|_\varphi$, then φ is independent.*

In particular, if T has NIP, such a q is smooth.

PROOF. Assume otherwise, and let n be such that n is greater than the dual VC-dimension of $\varphi(x, y)$ (see, e.g., [11, Lemma 6.3]).

Let N' be an elementary extension of N, $q_i \in S^x_{\mathcal{B}}(N')$ extending q, and $a \in N'$ such that $q_i(\varphi(x, a)) = \mathfrak{a}_i$. By the previous claim we may assume $\mathfrak{a}_1 < \mathfrak{a}_2$ and let $\mathfrak{b} = \mathfrak{a}_2 - \mathfrak{a}_1 > 0$. Note that by Fact 2.2, for any $\mathfrak{b}' \leq \mathfrak{b}$ there exists some $q' \in S^x_{\mathcal{B}}(N')$

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extending q such that $q'(\varphi(x,a)) = \mathfrak{a}_1 + \mathfrak{b}'$; thus by assumption (on q) there exists $a' \in N$ such that $q(\varphi(x,a')) = \mathfrak{a}_1 + \mathfrak{b}'$.

Assume first $\mathfrak{b}_* \leq \mathfrak{b} \leq \mathfrak{a}_2, -\mathfrak{a}_1$ for some atom $\mathfrak{b}_*$ of $\mathcal{B}$. Let $x' \subseteq x$ be a finite tuple containing all variables from x appearing in $\varphi(x,y)$. Since $Im(q|_{x'=z})$ is maximal, we get by Claim 2.26 and the proof of Corollary 2.32 that $\mathfrak{b}_* \nleq -\sum_{c \in N^{|x'|}} q(x'=c)$; but if $\mathfrak{b}_* \nleq q(x'=c)$ for all $c \in N^{|x'|}$ then since $\mathfrak{b}_*$ is an atom we get $\mathfrak{b}_* \leq -q(x'=c)$ for all $c \in N^{|x'|}$ that is

$$\mathfrak{b}_* \leq \prod_{c \in N^{|x'|}} -q(x'=c) = -\sum_{c \in N^{|x'|}} q(x'=c).$$

Let then $c \in N^{|x'|}$ be such that $q(x'=c) \geq \mathfrak{b}_*$. Let $D_{\mathfrak{b}_*}$ be the ultrafilter corresponding to $\mathfrak{b}_*$ represented as a homomorphism to 2. Then $D_{\mathfrak{b}_*} \circ q|_{x'} \in S(A)$ is realized in N (since $D_{\mathfrak{b}_*}(q(x'=c)) = 1$). But $D_{\mathfrak{b}_*} \circ q_i|_{x'}$ are extensions of $D_{\mathfrak{b}_*} \circ q|_{x'}$ and we have $D_{\mathfrak{b}_*}(q_1(\varphi(x,a))) = D_{\mathfrak{b}_*}(\mathfrak{a}_1) = 0$ and $D_{\mathfrak{b}_*}(q_2(\varphi(x,a))) = D_{\mathfrak{b}_*}(\mathfrak{a}_2) = 1$ so $D_{\mathfrak{b}_*} \circ q$ is not smooth—contradiction.

Thus there is no atom of $\mathcal{B}$ under $\mathfrak{b}$; therefore by trivial induction there exist disjoint $\mathfrak{b}_\eta > 0$ for $\eta \in \{\pm 1\}^n$ such that $\sum_{\eta \in \{\pm 1\}^n} \mathfrak{b}_\eta = \mathfrak{b}$. Let $\mathfrak{b}_i = \sum_{\eta \in \{\pm 1\}^n, \eta(i)=1} \mathfrak{b}_\eta \leq \mathfrak{b}$. Then $\mathfrak{b} - \mathfrak{b}_i = \sum_{\eta \in \{\pm 1\}^n, \eta(i)=-1} \mathfrak{b}_\eta$; thus for any such η, $\mathfrak{b} \prod_{i<n} \eta(i)\, \mathfrak{b}_i = \mathfrak{b}_\eta > 0$.

By assumption on q, for any i there exists $a_i \in N^{|y|}$ such that $q(\varphi(x,a_i)) = \mathfrak{a}_1 + \mathfrak{b}_i$.

Let $\varphi_\eta(x, y_0, \dots, y_{n-1}) = \bigwedge_{i<n} \varphi(x,y_i)^{\eta(i)}$. Then

$$\begin{aligned} q(\varphi_\eta(x, a_0, \dots, a_{n-1})) &= \prod_{i<n} \eta(i)(\mathfrak{a}_1 + \mathfrak{b}_i) \\ &\geq \mathfrak{b} \prod_{i<n} \eta(i)(\mathfrak{a}_1 + \mathfrak{b}_i). \end{aligned}$$

Since for every $i < n$, $\mathfrak{b} \cdot (-\mathfrak{b}_i) = \mathfrak{b} - \mathfrak{b}_i \leq \mathfrak{b} \leq -\mathfrak{a}_1$, we have $\mathfrak{b} \cdot -(\mathfrak{a}_1 + \mathfrak{b}_i) = (-\mathfrak{a}_1) \cdot \mathfrak{b} \cdot (-\mathfrak{b}_i) = \mathfrak{b} - \mathfrak{b}_i$; therefore $\mathfrak{b} \cdot \eta(i)(\mathfrak{a}_1 + \mathfrak{b}_i) \geq \mathfrak{b} \cdot \eta(i)\, \mathfrak{b}_i$.

Thus

$$\begin{aligned} q(\varphi_\eta(x, a_0, \dots, a_{n-1})) &\geq \mathfrak{b} \prod_{i<n} \eta(i)(\mathfrak{a}_1 + \mathfrak{b}_i) \\ &\geq \mathfrak{b} \prod_{i<n} \eta(i)\, \mathfrak{b}_i = \mathfrak{b}_\eta > 0; \end{aligned}$$

thus $\varphi_\eta(x, a_0, \dots, a_{n-1}) \neq \bot$ and thus $N \vDash \exists x \varphi_\eta(x, a_0, \dots, a_{n-1})$ contradicting our choice of n. ⊣

By Proposition 2.30 and remark 2.31 we get:

Corollary 2.35. *If $\mathcal{B}$ is a complete Boolean algebra, every $\mathcal{B}$-type in an NIP theory can be extended to a smooth $\mathcal{B}$-type. In fact, if $\mathcal{B}$ is κ-c.c., then given a $\mathcal{B}$-*

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type $p \in S^x_{\mathcal{B}}(M)$ *we can find some* $N \succ M$ *and smooth* $q \in S^x_{\mathcal{B}}(N)$ *where* $|N| \leq |M| + |x| + 2^{<\kappa} + |T|$.

§3. Relation to Keisler measures.

3.1. Connecting Keisler measures and Boolean types. In this subsection we recall the notion of a Keisler measure, and attach to a Keisler measure a Boolean type in a canonical probability algebra in a way that preserves the measure. This preserves many of the measure's properties and will be used later to transfer results from Boolean types to Keisler measures.

Definition 3.1. A *Keisler measure* in x over a set A is a finitely additive probability measure on $L_x(A)$.

Two Keisler measures λ, λ' in x over A are conjugates if there exists an elementary map $\pi : A \to A$ such that $\lambda\left(\varphi\left(x, \pi^{-1}(a)\right)\right) = \lambda'(\varphi(x, a))$ for any formula $\varphi(x, a)$.

Definition 3.2. A *measure algebra* is a σ-complete Boolean algebra $\mathcal{B}$ (not necessarily an algebra of sets) equipped with a σ-additive measure that is positive on every element other than $0_{\mathcal{B}}$.

A *probability algebra*is a measure algebra that assigns measure 1 to $1_{\mathcal{B}}$.

Example 3.3. For every probability space, the algebra of measurable subsets up to measure 0 is a probability algebra.

Definition 3.4. Let κ be an infinite cardinal. Then $(\mathcal{U}_\kappa, \nu_\kappa)$ is the probability algebra of Borel subsets of 2^κ up to measure 0, with ν_κ the usual product measure (see [2, 254J]).

Remark 3.5. Since every probability algebra has c.c.c., every supremum or infimum is effectively countable. Thus in particular the σ-complete subalgebra generated by some subset is the same as the complete subalgebra generated by the same set.

We will show that we can attach to a Keisler measure a $\mathcal{B}$-type for a measure algebra $\mathcal{B}$.

Remark 3.6. Given a Keisler measure λ over A, we can consider it as a measure on clopen sets of $S^x(A)$ and then extend it uniquely to a regular σ-additive measure on the Borel sets of $S^x(A)$ (see [4, 416Qa]).

Let $(\mathcal{B}, \lambda)$ be the probability algebra of Borel subsets of $S^x(A)$ up to λ measure 0 and let ψ be the projection from the algebra of Borel subsets onto $\mathcal{B}$. Since the clopen sets are a basis for the topology, the complete subalgebra of $\mathcal{B}$ generated by the clopen sets is $\mathcal{B}$ itself. Note that there are at most $|T| + |A| + |x|$ clopen sets.

Fact 3.7. *Let* κ *be an infinite cardinal. If* $\mathcal{B}$ *is a probability algebra, and there is* $B \subseteq \mathcal{B}$ *such that* $|B| \leq \kappa$ *and the smallest* (σ-) *complete subalgebra of* $\mathcal{B}$ *containing* B *is* $\mathcal{B}$ *itself then there is a measure preserving homomorphism* f *from* $\mathcal{B}$ *to* $(\mathcal{U}_\kappa, \nu_\kappa)$ (*see* [3, *Lemma* 332*N*], *and see also Propositions* 331*G and* 331*F there*).

Further, every measure algebra homomorphism is an embedding.

Proposition 3.8. *Assume* $\mathcal{A} \subseteq \mathcal{U}_\kappa$ *is a complete subalgebra that can be completely generated by a set* S *such that* $|S| < \kappa$. *Assume further that* $f : \mathcal{A} \to \mathcal{U}_\kappa$ *is measure preserving* (*thus an embedding*).

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Then exists a measure preserving automorphism σ of $\mathcal{U}_\kappa$ extending f.

Proof. By following the proof of [3, Theorem 331I] ($\mathcal{U}_\kappa$ satisfies the requirements for the theorem by [3, Theorem 331K]), we find that a recursive construction of an automorphism can start from any partial isomorphism, as long as the domain of said partial isomorphism is a complete subalgebra (in Fremlin's terminology, closed subalgebra) that can be completely generated by less than κ elements. ⊣

Remark 3.9. When considering Boolean types to a measure algebra $(\mathcal{B}, \lambda)$, we will adapt the definitions of conjugate types (Definition 2.3) and conjugate subalgebras to this context, which means that partial isomorphisms are now required to keep the extra structure (that is, to preserve the measure).

Proposition 3.10. *Let $\kappa \geq |A| + |T| + |x|$ be some cardinal. There is an injection from the set of Keisler measures over a set A to the set of $(\mathcal{U}_\kappa, \nu_\kappa)$-types over A. Further, this injection respects conjugation; that is, if the images of λ, λ' are conjugate with $\pi : A \to A$, then so are λ, λ'.*

Proof. Let f and ψ be as in Fact 3.7, using Remark 3.6.

Let $p_\lambda : L_x(A) \to \mathcal{U}_\kappa$ be $f \circ \psi|_{L_x(A)}$ (where $L_x(A)$ is thought of as the algebra of clopen subsets of $S^x(A)$). Then, $\lambda \mapsto p_\lambda$ would be our injection.

By choice of f, $\lambda(\varphi) = \nu_\kappa(p_\lambda(\varphi))$, which means that this map is indeed injective. It follows that if p_{λ_1} and p_{λ_2} are conjugate (as in Remark 3.9, i.e., as $(\mathcal{U}_\kappa, \nu_\kappa)$-types) then λ_1 and λ_2 are conjugate:

Suppose that $\pi : A \to A$ is an elementary map and that $\sigma : B_{p_{\lambda_1}} \to B_{p_{\lambda_2}}$ is a measure preserving isomorphism such that $\sigma \circ (\pi * p_{\lambda_1}) = p_{\lambda_2}$. Then

$$\begin{aligned}\pi * \lambda_1(\varphi(x,a)) &= \lambda_1\left(\varphi\left(x, \pi^{-1}(a)\right)\right) = \nu_\kappa\left(p_{\lambda_1}\left(\varphi\left(x,\pi^{-1}(a)\right)\right)\right)\\ &= \nu_\kappa\left(\sigma\left(p_{\lambda_1}\left(\varphi\left(x,\pi^{-1}(a)\right)\right)\right)\right) = \nu_\kappa\left(p_{\lambda_2}(\varphi(x,a))\right)\\ &= \lambda_2(\varphi(x,a)).\end{aligned}$$ ⊣

3.2. Counting Keisler measures. In this subsection we count the number of Keisler measures up to elementary conjugation similarly to Section 2.2.

Remark 3.11. Corollary 2.16 and all proofs leading to it still work when we add structure to the algebra, but now we have to take into account the number of isomorphism types of the new structure, i.e., the measure. Recall that in Section 2.2 we defined $\mu = 2^{<\kappa} + |T| + |x|$ (for $\mathcal{B}\kappa$-c.c.), the maximal cardinality of the image of any single $\mathcal{B}$-type when T is NIP. In the case of a measure algebra, the number of possible isomorphism types is thus $\leq 2^\mu + \left(2^{\aleph_0}\right)^\mu = 2^\mu$ (the isomorphism type of an measure algebra B consists of the quantifier-free type of B as in the proof of Corollary 2.17 and the list of values $\{\lambda(\mathfrak{a})\}_{\mathfrak{a}\in B}$). Finally, note that every probability algebra is c.c.c (i.e., ω_1-c.c.), so in this case we can take $\kappa = \omega_1$ and then $\mu = 2^{\aleph_0} + |T| + |x|$.

By applying Proposition 3.10, Remark 3.11, and Corollary 2.17 we get that:

Corollary 3.12. *Assume T has NIP. The number of Keisler measures in x up to conjugation over a set A is bounded by the number of types in $S^{|x|\cdot\mu}(A)$ up to conjugation $+\, 2^\mu$ where $\mu = 2^{\aleph_0} + |T| + |x|$.*

Hence, if again λ, κ *are cardinals and* α *an ordinal such that* $\lambda = \lambda^{<\lambda} = \aleph_\alpha = \kappa + \alpha \geq \kappa \geq \beth_\omega + \mu^+$, *and* $A = M$ *is a saturated model of size* λ, *this number is bounded by* $2^{<\kappa} + |\alpha|^{\mu}$.

However, Keisler measures have been studied extensively in the context of NIP (see for example [11, Chapter 7]), and there are results which give a better bound. Indeed:

Proposition 3.13. *Assume* T *has NIP. Then there is an injection from the set of Keisler measures in* x *over a set* A *to the set of* 2^{μ}*-types over* A *where* $\mu = |T| + |x|$.

Further, if the images of λ, λ' *are elementarily conjugates, then* λ *and* λ' *are conjugates.*

Proof. Recall that given complete types $p_0, \ldots, p_{n-1}$, and a formula $\phi(x,b)$, $Av(p_0, .., p_{n-1}; \phi(x,b))$ is $\frac{|\{i<n | \varphi(x,b) \in p_i\}|}{n}$.

Fix a Keisler measure λ over A. By [11, Proposition 7.11], (for $X_1 = \top$) for any $m < \omega$ and partitioned formula $\phi(x,y)$ there exist $n_{m,\phi} < \omega$ and types $\left\langle p_i^{m,\phi}(x)\right\rangle_{i<n_{m,\phi}} \in S^x(A)$ such that for any y-tuple b,

$$\left|\lambda(\phi(x;b)) - Av\left(p_0^{m,\phi}, \ldots, p_{n_{m,\phi}-1}^{m,\phi}; \phi(x,b)\right)\right| < \frac{1}{m}.$$

Note that by [11, Exercise 7.12], $n_{m,\phi}$ can be chosen independently of λ.

Let $I = \{m, \phi, i \mid m < \omega, \phi \in L_x(\emptyset), i < n_{\phi,m}\}$ (note that $|S| = \mu$). Choose $p_\lambda \in S^x_{2^I}(A)$ such that $p_\lambda(\phi(x,b))_{m,\phi,i} = 1$ iff $\phi(x,b) \in p_i^{m,\phi}$.

Then

$$\lim_{m\to\infty} Av\left((p_\lambda)_{m,\phi,0}, \ldots, (p_\lambda)_{m,\phi,n_{m,\phi}-1}; \phi(x,b)\right) = \lambda(\phi(x,b)).$$

And if $\pi * p_{\lambda_1} = p_{\lambda_2}$ we find that for any $\phi(x,y)$ and b,

$$\begin{aligned} \pi * \lambda_1(\phi(x,b)) &= \lambda_1\left(\phi\left(x,\pi^{-1}(b)\right)\right) \\ &= \lim_{m\to\infty} Av\left(\left(p_{\lambda_1}\right)_{m,\phi,0}, \ldots, \left(p_{\lambda_1}\right)_{m,\phi,n_{m,\phi}-1}; \phi\left(x,\pi^{-1}(b)\right)\right), \\ &\quad \lim_{m\to\infty} Av\left(\left(p_{\lambda_2}\right)_{m,\phi,0}, \ldots, \left(p_{\lambda_2}\right)_{m,\phi,n_{m,\phi}-1}; \phi\left(x,\pi^{-1}(b)\right)\right) \\ &= \lambda_2(\phi(x,b)). \end{aligned}$$

⊣

Thus with Corollary 2.9 we get

Corollary 3.14. *Assume* T *has NIP. The number of Keisler measures up to conjugation over a set* A *is bounded by the number of types in* $S^{|x|\cdot\mu}(A)$ *up to conjugation where* $\mu = |T| + |x|$ *(with the same explicit bound as in Corollary* 3.12*, but replacing* $\mu = 2^{\aleph_0} + |T| + |x|$ *with* $\mu = |T| + |x|$*).*

Remark 3.15. This bound improves upon the one in Corollary 3.12, since μ is now potentially smaller (for small $|T|$ and $|x|$).

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3.3. Smooth Keisler measures. Recall that a Keisler measure λ is *smooth* if it has a unique extension to any set containing its domain. In this subsection we show that Proposition 3.10 preserves smoothness, that is the Boolean type is smooth iff the measure is. We use this to recover the fact that in an NIP theory measures can be extended to smooth ones.

LEMMA 3.16. *Assume λ_i is a Keisler measure over A_i for $i \in \{1,2\}$ such that $A_1 \subseteq A_2$ and λ_2 extends λ_1; let $\kappa_i = |A_i| + |T| + |x|$ and let $f_i : \mathcal{B}_i \to \mathcal{U}_{\kappa_i^+}$ be an embedding of measure algebras, where $\mathcal{B}_i$ is the algebra of Borel subsets of $S^x(A_i)$ up to λ_i-measure* 0(*see Remark* 3.6).

Fix an embedding $\imath$ of $\mathcal{U}_{\kappa_1^+}$ in $\mathcal{U}_{\kappa_2^+}$ (one exists by Fact 3.7).

Then $\mathcal{B}_1$ embeds canonically into $\mathcal{B}_2$ and there is a measure algebra homomorphism $g : \mathcal{B}_2 \to \mathcal{U}_{\kappa_2^+}$ that extends $\imath \circ f_1$ and such that $\nu_{\kappa_2^+} \circ f_2 = \nu_{\kappa_2^+} \circ g$.

PROOF. Note first that since λ_2 extends λ_1, $\mathcal{B}_1$ can be embedded into $\mathcal{B}_2$ naturally with the preimage of the projection map from $S^x(A_2)$ to $S^x(A_1)$.

Note that $(\imath \circ f_1)(\mathcal{B}_1)$ is generated as a (σ-)complete algebra by κ_1 elements (as the image of such an algebra) and that it is the complete subalgebra of $\mathcal{U}_{\kappa_2^+}$, generated as a complete subalgebra by the images of the clopen sets over A_1 (see [3, Proposition 324L]).

By Proposition 3.8 and as λ_2 extends λ_1, there is an automorphism σ of $\left(\mathcal{U}_{\kappa_2^+}, \nu_{\kappa_2^+}\right)$ extending $f_2 \circ (\imath \circ f_1)^{-1}$.

But now we get $\sigma^{-1} \circ f_2|_{\mathcal{B}_1} = \imath \circ f_1|_{\mathcal{B}_1}$; thus $g = \sigma^{-1} \circ f_2$ is as required. $\dashv$

COROLLARY 3.17. *If $p \in S^x_{\mathcal{U}_{\kappa^+}}(M)$ is smooth, then so is $\nu_{\kappa^+} \circ p$ for $\kappa \geq |M| + |T| + |x|$.*

PROOF. Assume $\lambda = \nu_{\kappa^+} \circ p$ is not smooth. So there is $N \supseteq M$ and distinct λ_1, λ_2 over N extending λ, and hence there is one of cardinality at most $|M|$ by Löwenheim–Skolem (restrict λ_1, λ_2 to a smaller model containing M and some b, for which there exists $\varphi(x,b)$ such that $\lambda_1(\varphi(x,b)) \neq \lambda_2(\varphi(x,b))$).

Let $\mathcal{B}_0$ be the measure algebra of Borel sets in $S^x(M)$ up to λ-measure 0 and I the ideal of sets of λ-measure 0 in the algebra of Borel subsets of $S^x(M)$ (see Remark 3.6). Then since $\nu_{\kappa^+}(\mathfrak{a}) = 0 \iff \mathfrak{a} = 0$, $I \cap L_x(M) = \ker(p)$; thus $L_x(M)/\ker(p)$ is naturally embedded in $\mathcal{B}_0$.

Write $p = \widetilde{p} \circ \pi$ for $\pi : L_x(M) \to L_x(M)/\ker(p)$ the projection.

Then $\widetilde{p} : L_x(M)/\ker(p) \to \mathcal{U}_{\kappa^+}$ is a measure preserving function from a subalgebra of $\mathcal{B}_0$, and the complete (in Fremlin's terminology, order-closed) subalgebra of $\mathcal{B}_0$ generated by $L_x(M)/\ker(p)$ is $\mathcal{B}_0$ itself. Thus by [3, Propositions 324O and 323J] $\widetilde{p}$ has a unique extension to a measure preserving $\overline{p} : \mathcal{B}_0 \to \mathcal{U}_{\kappa^+}$, that is $\lambda = \nu_{\kappa^+} \circ \overline{p}$ (here we treat λ as a measure on $S^x(M)$).

For each $i \in \{1,2\}$, we do the following. Let $\mathcal{B}_i$ be the measure algebra of Borel subsets of $S^x(N)$ up to λ_i-measure 0, which naturally embeds $\mathcal{B}_0$ as a subalgebra via the preimage map of the projections $S^x(N) \to S^x(M)$ (this uses the fact that λ_i extends λ_0). Take an embedding $f_i : \mathcal{B}_i \to \mathcal{U}_{\kappa^+}$ for λ_i guaranteed by Fact 3.7, and using Lemma 3.16 we can find a measure preserving $g_i : \mathcal{B}_i \to \mathcal{U}_{\kappa^+}$ extending $\overline{p}$ such that $\lambda_i = \nu_{\kappa^+} \circ g_i$.

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Let $\pi_i : L_x(N) \to \mathcal{B}_i$ be the projection, note that it extends π. Then for $p_i = g_i \circ \pi_i \in S^x_{\mathcal{U}_{\kappa^+}}(N)$ we find $\lambda_i = \nu_{\kappa^+} \circ p_i$ (when we consider λ_i as Keisler measures), and since π_i extends π and g_i extend $\overline{p}$ (thus $\widetilde{p}$) we find p_1, p_2 extend p, but they are distinct since λ_1 and λ_2 are distinct.

We conclude that p is not smooth. $\dashv$

On the other hand:

Proposition 3.18. *Assume λ is a smooth Keisler measure in x over M. Let $p \in S^x_{\mathcal{U}_{\kappa^+}}(M)$ such that $\lambda = \nu_{\kappa^+} \circ p$ for $\kappa \geq |M| + |T| + |x|$. Then p is smooth.*

Proof. Let $p_1, p_2 \in S^x_{\mathcal{U}_{\kappa^+}}(N)$ be distinct types extending p (for $N \supseteq M$). Again, without loss of generality $|N| \leq \kappa$. By Claim 2.33 we can choose p_1 and p_2 such that for no automorphism σ of $\mathcal{U}_{\kappa^+}$, $\sigma \circ p_1 = p_2$.

Let λ' be the unique extension of λ to N; in particular $\lambda' = \nu_{\kappa^+} \circ p_i$ for $i = 1, 2$. Since $\ker(p_i) = \{\varphi(x, b) \mid \lambda'(\varphi(x, b)) = 0\}$, $\mathcal{B} = L_x(M) / \ker(p_i)$ is independent of i. Let $\mathcal{B}'$ be the probability algebra of Borel sets in $S^x(M)$ up to measure 0.

We conclude that both p_i's can be written as $f_i \circ \pi$ where π is the projection from $L_x(N)$ to the algebra $\mathcal{B}$, and $f_i : \mathcal{B} \to Im(f_i)$ are measure preserving embeddings (note $Im(f_i)$ is a complete subalgebra; see [3, Proposition 324L]), and we can extend them uniquely to $\mathcal{B}'$ by [3, Propositions 324O and 323J], like in the proof of Corollary 3.17.

We get $f_2 \circ f_1^{-1}$ is a partial measure preserving isomorphism from $Im(f_1)$ into $\mathcal{U}_{\kappa^+}$ (which can, by Proposition 3.8, be extended to an automorphism) and $f_2 \circ f_1^{-1} \circ p_1 = f_2 \circ \pi = p_2$, contradiction. $\dashv$

By [3, Theorem 322Ca–c], every probability algebra is complete as a Boolean algebra (Dedekind complete, in his terminology).

Thus by choosing a sufficiently large κ (at least $\left(2^{\aleph_0}\right)^+$) in Proposition 3.10 and using Corollary 2.35 we get, recovering [11, Proposition 7.9]:

Corollary 3.19. *Every Keisler measure over an NIP theory can be extended to a smooth measure.*

Proof. Take a Keisler measure λ over a model M in x. Let $\kappa = \left(2^{\aleph_0} + |T| + |x| + |M|\right)^+$. Then by Proposition 3.10 there exists a type $p \in S^x_{\mathcal{U}_\kappa}(M)$ such that $\lambda = \nu_\kappa \circ p$.

By Corollary 2.35 there exists a model $N \supseteq M$ of cardinality at most $2^{\aleph_0} + |T| + |x| + |M|$ and a smooth extension $q \in S^x_{\mathcal{U}_\kappa}(N)$ of p.

Thus letting $\lambda' = \nu_\kappa \circ q$, we find λ' extends λ; thus by Corollary 3.17 we are done. $\dashv$

Remark 3.20. By [11, Lemma 7.8] and the remark following it, if $N \supseteq M$ is an extension and λ over N is a smooth extension of μ over M, then there is some $M \subseteq N' \subseteq N$ such that $\lambda|_{N'}$ is still smooth and $|N'| \leq |M| + |T|$.

Indeed the lemma and the remark state that λ is smooth iff for every $\varepsilon > 0$ and $\phi(x, y)$ exist $\psi_i(y), \theta_i^1(x), \theta_i^2(x)$ $(i = 1, \dots, n)$ over N such that:

1. $\left(\bigvee_{i=1}^{n} \psi_i(y)\right) = \top$.
2. If b is a y-tuple in $\mathfrak{C}$ and $\psi_i(b)$ holds then $\theta_i^1(x) \to \phi(x, b) \to \theta_i^2(x)$.
3. $\lambda\left(\theta_i^2(x)\right) - \lambda\left(\theta_i^1(x)\right) < \varepsilon$.

Thus we can take an elementary substructure N' of N containing M and all parameters in the formulas θ_i^j, ψ_i mentioned in the lemma for each $\varepsilon = \frac{1}{n}$ and $\phi(x; y)$; then all the formulas are over N' and retain the required properties, ensuring that $\lambda|_{N'}$ is still smooth.

The proof of [11, Lemma 7.8] relies on the following fact:

FACT 3.21 [11, Lemma 7.4]. *Let $\phi(x,b)$ be a formula ($b \in \mathfrak{C}$), and let λ be a Keisler measure in x over M. Let*:

$$r_1 = \sup\{\lambda(\varphi(x,a)) \mid \varphi(x,a) \in L_x(M), \mathfrak{C} \vDash \varphi(x,a) \to \phi(x,b)\} \text{ and}$$
$$r_2 = \inf\{\lambda(\varphi(x,a)) \mid \varphi(x,a) \in L_x(M), \mathfrak{C} \vDash \phi(x,b) \to \varphi(x,a)\}.$$

Then there exists an extension λ' of λ to $\mathfrak{C}$ such that $\lambda(\phi(x,b)) = r$ iff $r_1 \leq r \leq r_2$.

REMARK 3.22. For any measure algebra $\mathcal{B}$ and $F \subseteq \mathcal{B}$ there exists a countable $A \subseteq F$ such that A and F have the same set of lower bounds; hence $\prod F$ and $\prod A$ exist and are equal (since $\mathcal{B}$ has c.c.c. and is complete; see for example [3, Proposition 316E]); and if F is closed under finite products we can choose A to be a chain. Thus we conclude $\nu_\kappa(\prod F) = \inf\{\nu_\kappa(a)\}_{a\in F}$. A similar argument shows $\nu_\kappa(\sum F) = \sup\{\nu_\kappa(a)\}_{a\in F}$ if F is closed under finite sums.

Let $p \in S^x_{\mathcal{U}_\kappa}(M)$, $\phi(x,y)$ be a partitioned formula and $b \in \mathfrak{C}$ a y-tuple. Then

$$\{p(\varphi(x,a)) \mid \varphi(x,a) \in L_x(M), \phi(x,b) \to \varphi(x,a)\}$$

and

$$\{p(\varphi_2(x,a_2)) - p(\varphi_1(x,a_1)) \mid \varphi_i(x,a_i) \in L_x(M),$$
$$\varphi_1(x,a_1) \to \phi(x,b) \to \varphi_2(x,a_2)\}$$

are closed under finite products, while

$$\{p(\varphi(x,a)) \mid \varphi(x,a) \in L_x(M), \varphi(x,a) \to \phi(x,b)\}$$

is closed under finite sums.

REMARK 3.23. We can also get [11, Lemma 7.4] by combining Proposition 3.10, Remark 3.22, and Fact 2.2.

Indeed we need to only find for any $r_1 \leq r \leq r_2$ some

$$\mathfrak{b} \leq \left(\prod_{\psi(x,b):\varphi(x,a)\to\psi(x,b)} p(\psi(x,b))\right) - \left(\sum_{\psi(x,b):\psi(x,b)\to\varphi(x,a)} p(\psi(x,b))\right)$$

(for p some type corresponding to λ) such that $\nu_\kappa(\mathfrak{b}) = r - r_1$; and it is easy to see that when $\kappa \geq \aleph_0$, for every $\mathfrak{a} \in \mathcal{U}_\kappa^+$ and every $0 \leq s \leq \nu_\kappa(\mathfrak{a})$ there exists $\mathfrak{b} \leq \mathfrak{a}$ such that $\nu_\kappa(\mathfrak{b}) = s$ (indeed the same holds for any non-atomic measure algebra, but here we can see it directly).

§4. Analysis of the stable case. In this section, we analyze the stable and totally transcendental (t.t.) cases. We show that in the stable case, local Boolean types are essentially averages of classical φ-types, and that in the t.t. case the same is true for

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complete Boolean types. We start with the t.t. case, since the general stable case is similar (and easier, since local ranks are bounded by ω).

4.1. The t.t. case.

Definition 4.1. For $q \in S^x_{\mathcal{B}}(A)$, let $\operatorname{supp}(q) = \{p(x) \in S^x(A) \mid p \vdash \theta \Rightarrow q(\theta) > 0\}$.

Note that when $q \in S^x_{\mathcal{B}}(A)$, $\operatorname{supp}(q)$ is a closed subset of $S^x(A)$. Also note that if Γ is a collection of formulas, closed under finite conjunctions, such that $q(\theta) > 0$ for all $\theta \in \Gamma$ then there is a type $p \in \operatorname{supp}(q)$ such that $p \vdash \Gamma$ (because $\Gamma \cup \{\neg\psi \mid q(\psi) = 0\}$ is consistent).

We will use some basic facts from stability theory, namely:

Fact 4.2 [7, Section 3]. *For a topological space X, let $X' = X \setminus \{x \in X \mid x$ is isolated$\}$. Assume that T is t.t. and let $X = S^x(A)$ for some $A \subseteq M \models T$. For $\alpha \in$ **ord**, let $X^{(\alpha)}$ be the Cantor–Bendixon analysis of X $(X^{(\alpha+1)} = (X^{(\alpha)})'$ and $X^{(\alpha)} = \bigcap_{\beta<\alpha} X^{(\beta)}$ when α is a limit ordinal$)$. Then for some (successor)$\alpha < |T|^+$, $X^{(\alpha)} = \emptyset$. (Note that this definition is a bit different than the one in [7, Section 3], where one considers global types.)*

Remark 4.3. Note that for any topological space X, if $Y \subseteq X$ then $Y' \subseteq X'$.

Lemma 4.4. *Suppose that T is t.t. Suppose that $\mathcal{B}$ is any complete Boolean algebra. Let $A \subseteq M \models T$ and let $q \in S^x_{\mathcal{B}}(A)$. Let $X = \operatorname{supp}(q)$. Let U be the set of all isolated types $r \in X$. For each $r \in U$, let $\theta_r(x)$ be an isolating formula for r. Then:*

1. *$\{q(\theta_r) \mid r \in U\}$ is an antichain.*
2. *For any $\psi(x) \in L_x(A)$, $q(\psi) \geq \sum_{r\in U} q(\theta_r) \cdot r(\psi)$ (where we treat r as a 2-type).*

Proof. (1) Note that $0 < q(\theta_r)$ for all $r \in U$ since $r \in \operatorname{supp}(q)$. Suppose that $r_1 \neq r_2 \in U$ and $0 < \mathfrak{b} \leq q(\theta_{r_1}), q(\theta_{r_2})$. Hence $0 < \mathfrak{b} \leq q(\theta_{r_1} \wedge \theta_{r_2})$ and thus for some $r \in \operatorname{supp}(q)$, $r \vdash \theta_{r_1} \wedge \theta_{r_2}$; hence $r_1 = r = r_2$ by assumption.

(2) It is enough to show that if $\psi(x) \in r$ for some $r \in U$ then $q(\psi) \geq q(\theta_r)$. If $\psi(x) \in r$, it follows that $q(\theta_r \wedge \neg\psi) = 0$ so $q(\theta_r) = q(\psi \wedge \theta_r) \leq q(\psi)$. ⊣

Theorem 4.5. *Suppose that T is totally transcendental, A is some set, $\mathcal{B}$ is a complete Boolean algebra, and that $q \in S^x_{\mathcal{B}}(A)$ is a $\mathcal{B}$-type. Then there is a maximal antichain $\langle \mathfrak{b}_r \mid r \in U \rangle$ where $U \subseteq \operatorname{supp}(q)$ such that for all $\psi(x) \in L_x(A)$, $q(\psi) = \sum_{r\in U} \mathfrak{b}_r \cdot r(\psi)$.*

Proof. For any $0 < \mathfrak{b} \in \mathcal{B}$, let $\mathcal{B}|_{\mathfrak{b}}$ be the relative algebra. Letting $X = S^x(A)$, we try to construct a sequence $\langle \mathfrak{b}_\alpha, q_\alpha, U_\alpha, \bar{\mathfrak{c}}_\alpha \mid \alpha < \alpha^* \rangle$ for some $\alpha^* \leq |T|^+$ such that:

- $0 < \mathfrak{b}_\alpha \leq \mathfrak{b}_\beta$ for $\beta < \alpha$; $\mathfrak{b}_0 = 1$ and more generally $\mathfrak{b}_\alpha = -\sum_{\beta<\alpha, r\in U_\beta} \mathfrak{c}_{\beta,r}$; $q_\alpha \in S^x_{\mathcal{B}_\alpha}(A)$ where $\mathcal{B}_\alpha = \mathcal{B}|_{\mathfrak{b}_\alpha}$; $q_\alpha(\psi) = q(\psi) \cdot \mathfrak{b}_\alpha$ for any $\psi \in L_x(A)$; $U_\alpha \subseteq \operatorname{supp}(q_\alpha) \subseteq X^{(\alpha)}$; $\bar{\mathfrak{c}}_\alpha = \langle \mathfrak{c}_{\alpha,r} \mid r \in U_\alpha \rangle$ is an antichain contained in $\mathcal{B}_\alpha$; $q_\alpha(\psi) \geq \sum_{r\in U_\alpha} \mathfrak{c}_{\alpha,r} \cdot r(\psi)$ for any $\psi \in L_x(A)$.

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Given $\langle \mathfrak{b}_\beta, q_\beta, U_\beta, \bar{\mathfrak{c}}_\beta \mid \beta < \alpha \rangle$, if $\sum_{\beta<\alpha, r \in U_\beta} \mathfrak{c}_{\beta,r} = 1$ we stop and let $\alpha^* = \alpha$. Otherwise, let $\mathfrak{b}_\alpha$, q_α as above and let $U_\alpha \subseteq \operatorname{supp}(q_\alpha)$ be the set of isolated types in $\operatorname{supp}(q_\alpha)$ (so for $\alpha = 0$, $q_0 = q$ and $\mathfrak{b}_0 = 1$). For $r \in U_\alpha$, let $\mathfrak{c}_{\alpha,r} = q_\alpha(\theta_r)$ where θ_r isolates r. By Lemma 4.4(1), $\{\mathfrak{c}_{\alpha,r} \mid r \in U_\alpha\}$ is an antichain in $\mathcal{B}_\alpha$. Note that $q_\alpha(\psi) \geq \sum_{r \in U_\alpha} \mathfrak{c}_{\alpha,r} \cdot r(\psi)$ by Lemma 4.4(2). Now prove by induction on α that $\operatorname{supp}(q_\alpha) \subseteq X^{(\alpha)}$ (this follows from Remark 4.3 and the fact that $\operatorname{supp}(q_{\alpha+1}) \subseteq \operatorname{supp}(q_\alpha)'$) and that $\left\{\mathfrak{c}_{\beta,r} \,\middle|\, r \in U_\beta, \beta < \alpha\right\}$ is an antichain in $\mathcal{B}$.

Finally, since for some $\beta < |T|^+$, $X^{(\beta)} = \emptyset$, it follows that $\alpha^* \leq \beta$ (otherwise, $\operatorname{supp}(q_\beta) = \emptyset$ and so $\mathfrak{b}_\beta = q_\beta(x = x) = 0$, contradiction). Hence for all $\psi \in L_x(A)$, $q(\psi) \geq \sum_{\alpha<\alpha^*} q_\alpha(\psi) \geq \sum_{r \in U_\alpha, \alpha < \alpha^*} \mathfrak{c}_{\alpha,r} \cdot r(\psi)$ and $\{\mathfrak{c}_{\alpha,r} \mid \alpha < \alpha^*, r \in U_\alpha\}$ is a maximal antichain in $\mathcal{B}$. Since this is also true for $\neg\psi$, we have equality and we are done. ⊣

4.2. The stable case. Fix a partitioned formula $\varphi(x, y)$ in some theory T, and let $A \subseteq \mathfrak{C}$. As in [7, Section 2], by a *φ-formula over A*, we will mean a formula $\psi(x) \in L_x(A)$ which is equivalent to a Boolean combination of instances of φ over A (over a model M, a φ-formula is just a Boolean combination of instances of φ over M). Let $L_{\varphi,x}(A)$ be the Boolean algebra of φ-formulas over A up to equivalence in $\mathfrak{C}$. Let $S^x_\varphi(A)$ be the set of all complete φ-types over A in x, i.e., maximal consistent sets of φ-formulas over A.

Definition 4.6 (Local Boolean type). Suppose that $\mathcal{B}$ is a Boolean algebra and $\varphi(x, y)$ is a partitioned formula. A *$\mathcal{B},\varphi$-type* over a set A is a homomorphism from $L_{\varphi,x}(A)$ to $\mathcal{B}$. Denote the set of $\mathcal{B},\varphi$-types by $S^x_{\mathcal{B},\varphi}(A)$. For $q \in S^x_{\mathcal{B},\varphi}$ let $\operatorname{supp}_\varphi(q) = \left\{p(x) \in S^x_\varphi(A) \mid p \vdash \theta \Rightarrow q(\theta) > 0\right\}$.

Similarly to the previous section, we have:

Fact 4.7 [7, Section 3]. *Assume that $\varphi(x, y)$ is stable in some complete theory T, $A \subseteq M \models T$ and let $X = S^x_\varphi(A)$. Then for some $n < \omega$, $X^{(n+1)} = \emptyset$.*

Theorem 4.8. *Suppose that $\varphi(x, y)$ is stable, A is some set, $\mathcal{B}$ is a complete Boolean algebra, and that $q \in S^x_{\mathcal{B},\varphi}(A)$ is a $\mathcal{B},\varphi$-type. Then there is a maximal antichain $\langle \mathfrak{b}_r \mid r \in U \rangle$ where $U \subseteq \operatorname{supp}_\varphi(q)$ such that for all $\psi(x) \in L_{\varphi,x}(A)$, $q(\psi) = \sum_{r \in U} \mathfrak{b}_r \cdot r(\psi)$.*

Proof. The proof is exactly as the proof of Theorem 4.5, working with $X = S^x_\varphi(A)$ and with local Boolean types, replacing Fact 4.2 with Fact 4.7. We leave the details to the reader. ⊣

Remark 4.9. When T is stable and $q \in S^x_{\mathcal{B}}(A)$, this essentially means that $q|_\varphi$ factors through $2^{|U|} \hookrightarrow \prod_{r \in U} \mathcal{B}|_{\mathfrak{b}_r} \cong \mathcal{B}$ (see [5, Proposition 6.4]). In particular we get again, more directly, that for $\mathcal{B}$ which is κ-c.c., $\left|Im\left(q|_\varphi\right)\right| \leq 2^{<\kappa}$ (see Proposition 2.12). When T is t.t., we get similarly that $|Im(q)| \leq 2^{<\kappa}$.

4.3. Non-forking. Using (the proof of) Theorem 4.8, one can recover the theory of forking in stable theories.

Definition 4.10. Let $\mathcal{B}$ be any Boolean algebra and let T be any theory. Let $A \subseteq B$ be any sets. Say that a $\mathcal{B}$-type or a $\mathcal{B},\varphi$-type q *forks over A* if for some

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formula $\theta(x)$ over B which forks over A, $q(\theta) > 0$. (For the definition of forking, see, e.g., [12, Definition 7.1.7].)

FACT 4.11 (E.g., [7, Section 2]). *If $M \prec N$, $\varphi(x, y)$ is stable, then any φ-type $p \in S^x_\varphi(M)$ has a unique non-forking extension $p|_N$ to $S_\varphi(N)$. The same is true assuming elimination of imaginaries when M is replaced by an algebraically closed set A.*

REMARK 4.12. Suppose that $p \in S^x_{\mathcal{B}}(B)$ does not fork over $A \subseteq B$. Then there is a global non-forking (over A) extension $q \in S^x_{\mathcal{B}}(\mathfrak{C})$ (and the same is true for local Boolean types). This follows from the fact that the set of forking formulas over A forms an ideal, and Fact 2.1. When B is a model M then p does not fork over M and if T is stable (or even simple) then this is true in general.

Suppose that $\varphi(x, y)$ is stable and $p \in S_{\mathcal{B},\varphi}(M)$ for some model $M \models T$. Then we can find an explicit extension: by Theorem 4.8, we can write p as the sum $\sum_{r\in U} \mathfrak{b}_r \cdot r$ for some maximal antichain $U \subseteq \operatorname{supp}_\varphi(p)$ and we let $q = \sum_{r\in U} \mathfrak{b}_r \cdot r|_{\mathfrak{C}}$ (where $r|_{\mathfrak{C}}$ is the unique global non-forking extension of r). A similar statement holds in the t.t. case.

Next we would like to prove that there is a unique non-forking extension.

LEMMA 4.13. *Suppose that $\varphi(x, y)$ is stable and $\mathcal{B}$ is any complete Boolean algebra. Let $M \prec N \models T$ and let $q \in S_{\mathcal{B},\varphi}(N)$ be non-forking over M. Let $X = \operatorname{supp}_\varphi(q|_M)$ ($q|_M$ is the restriction of q to $L_{\varphi,x}(M)$). Let U be the set of all isolated φ-types $r \in X$. For each $r \in U$, let $\theta_r(x)$ be an isolating formula for r(so it is a φ-formula). Then*

- *For any $\psi(x) \in L_{\varphi,x}(N)$, $q(\psi) \geq \sum_{r\in U} q(\theta_r) \cdot r|_N(\psi)$ (where $r|_N$ is the unique non-forking extension of r to N).*

PROOF. It is enough to show that $q(\psi) \geq q(\theta_r)$ when $\psi \in r|_N$. Suppose that $\psi(x) \in r|_N$ but $q(\theta_r \wedge \neg\psi) > 0$. Then there is some $r' \in \operatorname{supp}_\varphi(q)$ such that r' contains $\theta_r \wedge \neg\psi$. But then r' does not fork over M (because q does not fork over M) and $r'|_M$ contains θ_r and is in X and thus $r'|_M = r|_M$ and by Fact 4.11, $r' = r$. ⊣

THEOREM 4.14. *Suppose that $\varphi(x, y)$ is stable. Suppose that $n < \omega$. Then, whenever $\mathcal{B}$ is a complete Boolean algebra, $M \prec N$, and $q \in S_{\mathcal{B},\varphi}(N)$ does not fork over M, there is a maximal antichain $\langle \mathfrak{b}_r \mid r \in U \rangle$ where $U \subseteq \operatorname{supp}_\varphi(q|_M)$ which depends only on $q|_M$ such that for all $\psi(x) \in L_{\varphi,x}(N)$, $q(\psi) = \sum_{r\in U} \mathfrak{b}_r \cdot r|_N(\psi)$. In particular, q is the unique non-forking extension of $q|_M$.*

PROOF. The proof follows the same lines as in the proof of Theorem 4.8 (and Theorem 4.5), using Lemma 4.13 instead of Lemma 4.4. ⊣

REMARK 4.15. As in the classical case, we can extend these results (existence and uniqueness of non-forking extensions) for an arbitrary algebraically closed set A, assuming elimination of imaginaries.

4.4. Connection to Keisler measures. Using the general results on Boolean types we can recover and prove some results on Keisler measures. The following result appeared in [1, Fact 2.2], [8, Fact 1.1] for models.

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Corollary 4.16. *Suppose that $\varphi(x,y)$ is stable and that μ is a Keisler measure on $L_{\varphi,x}(A)$ for some set A. Then there is a countable family $\langle p_i \mid i<\omega\rangle$ of complete φ-types over A and positive real numbers $\langle \alpha_i \mid i<\omega\rangle$ such that $\sum_{i<\omega}\alpha_i = 1$ and for any $\psi(x)\in L_{\varphi,x}(A)$, $\mu(\psi)=\sum\alpha_i p_i(\psi)$.*

Similarly, if T is t.t. and μ is a Keisler measure on $L_x(A)$ then there is a countable family $\langle p_i \mid i<\omega\rangle$ of complete types over A and positive real numbers $\langle \alpha_i \mid i<\omega\rangle$ such that $\sum_{i<\omega}\alpha_i=1$ and for any $\psi(x)\in L_x(A)$, $\mu(\psi)=\sum\alpha_i p_i(\psi)$.

Proof. Given μ, let $\mathcal{B}$ be the Boolean algebra of Borel subsets of $S^x_\varphi(A)$ up to μ-measure 0 (recall Remark 3.6) and let $q\in S^x_{\mathcal{B},\varphi}(A)$ be the natural homomorphism from φ-formulas over A (up to equivalence over $\mathfrak{C}$) to $\mathcal{B}$. Now apply Theorem 4.8 to q and $\mathcal{B}$ to obtain a maximal antichain $\{\mathfrak{b}_r \mid r\in U\}$ where $U\subseteq \operatorname{supp}_\varphi(q)$ such that for all $\psi(x)\in L_x(A)$, $q(\psi)=\sum_{r\in U}\mathfrak{b}_r\cdot r(\psi)$. Note that U must be countable as $\mathcal{B}$ is c.c.c. Letting $\alpha_r=\mu(\mathfrak{b}_r)$ we are done. The second statement follows similarly from Theorem 4.5. ⊣

Definition 4.17. A Keisler measure μ on $L_x(N)$ does not fork over $M\prec N$ if whenever $\mu(\theta)>0$, θ does not for over M.

Corollary 4.18. *Suppose that $\varphi(x,y)$ is stable and that μ is a Keisler measure on $L_\varphi(M)$ for some model M. Then μ has a unique global non-forking extension to $\mathfrak{C}$. More generally, this holds when replacing M by any algebraically closed set A, assuming elimination of imaginaries.*

Proof. We use a local version of Proposition 3.10: given μ, we can find $p\in S^x_{\varphi,\mathcal{U}_\kappa}(M)$ such that $\mu=\nu_\kappa\circ p$ for $\kappa=|T|+|M|$. By Remark 4.12 there is a non-forking extension $q\in S^x_{\varphi,\mathcal{U}_\kappa}(N)$ and then we can define $\mu'=\mu\circ q$. For uniqueness, suppose that λ_1,λ_2 are two non-forking measures over $N\succ M$ extending μ. We may assume $|N|=|M|$ and let κ be as above. Let $q_1,q_2\in S^x_{\varphi,\mathcal{U}_{\kappa^+}}(N)$ be corresponding $\varphi,\mathcal{U}_{\kappa^+}$-types. By a local version of Lemma 3.16 we may assume that both q_1,q_2 extend p. Thus we are done by Theorem 4.14. The more general statement follows similarly by Remark 4.15. ⊣

Remark 4.19. Note that in the context of the first part of Corollary 4.16 where A is a model and $N\supseteq A$, the unique non-forking extension of μ to N is the weighted sum $\sum\alpha_i p_i|_N$ where $p_i|_N$ is the unique non-forking extension of p_i to N. This follows immediate from the fact that the sum does not fork. The analogous result holds in the t.t. case.

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THE HEBREW UNIVERSITY OF JERUSALEM
EINSTEIN INSTITUTE OF MATHEMATICS
EDMOND J. SAFRA CAMPUS, GIVAT RAM
JERUSALEM 91904, ISRAEL
E-mail: kaplan@math.huji.ac.il
URL: math.huji.ac.il/˜kaplan

E-mail: ori.segel@mail.huji.ac.il

THE HEBREW UNIVERSITY OF JERUSALEM
EINSTEIN INSTITUTE OF MATHEMATICS
EDMOND J. SAFRA CAMPUS, GIVAT RAM
JERUSALEM 91904, ISRAEL
and
DEPARTMENT OF MATHEMATICS, HILL CENTER-BUSCH CAMPUS, RUTGERS
THE STATE UNIVERSITY OF NEW JERSEY
110 FRELINGHUYSEN ROAD
PISCATAWAY, NJ 08854-8019, USA
E-mail: shelah@math.huji.ac.il
URL: shelah.logic.at