BOREL SETS WITHOUT PERFECTLY MANY OVERLAPPING TRANSLATIONS IV

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ABSTRACT. We show that, consistently, there exists a Borel set $B \subseteq {}^\omega 2$ admitting a sequence $\langle \eta_\alpha : \alpha < \lambda \rangle$ of distinct elements of ${}^\omega 2$ such that $(\eta_\alpha + B) \cap (\eta_\beta + B)$ is uncountable for all $\alpha, \beta < \lambda$ but with no perfect set P such that $|(\eta + B) \cap (\nu + B)| \geq 6$ for any distinct $\eta, \nu \in P$. This answers two questions from our previous works, [4, Problem 5.1], [6, Problem 7.6].

1. INTRODUCTION

In the series of articles [4, 5, 6] we investigated the existence of Borel sets with many, but not too many pairwise non-disjoint translations. For instance, in [5], for a countable ordinal $\varepsilon < \omega_1$ and an integer $2 \leq \iota < \omega$ we constructed a Σ^0_2 set $B \subseteq {}^\omega 2$ with the following property.

> *In some ccc forcing notion there is a sequence $\langle \rho_\alpha : \alpha < \aleph_\varepsilon \rangle$ of distinct elements of ${}^\omega 2$ such that*
>
> $$\left| (\rho_\alpha + B) \cap (\rho_\beta + B) \right| \geq 2\iota \text{ for all } \alpha, \beta < \lambda$$
>
> *but in no extension there is a perfect set of such ρ's.*

Similar resuts for the general case of perfect Abelian Polish groups were presented in [6]. However, in all those cases when discussing nonempty intersections we considered finite intersections only. It seemed that our arguments really needed a finite enumeration of "witnesses for nondisjointness". So in [4, Problem 5.1] and [6, Problem 7.6] we asked if there is a ccc forcing notion $\mathbb{P}$ adding a Σ^0_2 subset B of the Cantor space ${}^\omega 2$ such that

> *for some $H \subseteq {}^\omega 2$ of size λ, the intersections $(B+h) \cap (B+h')$ are infinite (uncountable, respectively) for all $h, h' \in H$, but for every perfect set $P \subseteq {}^\omega 2$ there are $x, x' \in P$ with the intersection $(B+x) \cap (B+x')$ finite (countable, respectively).*

In the present paper we answer the above two questions positively. Our forcing construction slightly generalizes and simplifies that of [4, 5]. This allows us to show a stronger result:

> *If $\lambda < \lambda_{\omega_1}$ then some ccc forcing notion adds a Σ_2^0 set B
> which has λ translations with pairwise uncountable intersec-
> tions, while for every perfect set $P \subseteq {}^\omega 2$ there are $x, x' \in P$
> with $|(B + x) \cap (B + x')| < 6$.*

The article is organized as follows. First, in Section 2, we recall the splitting
rank from Shelah [7]. This rank was fundamental for the question of no per-
fect squares and it is fundamental for problems of nondisjoint translations
as well. Then, in the third section we introduce nice indexed bases $\bar{\mathcal{O}}$ and we
define when translations of a Σ_2^0 set have $\bar{\mathcal{O}}$–large intersection. This allows
us to put in the same framework sets with finite, infinite and uncountable
intersections. We also analyze when a Σ_2^0 set may have a perfect set of trans-
lations with $\bar{\mathcal{O}}$–large intersections and we introduce a non-disjointness rank
on finite approximations. Our main consistency theorem is presented in the
fourth section. In the final part of the paper we summarize our results and
pose a few relevant problems.

Notation: Our notation is standard and compatible with that of classical
textbooks (like Jech [2] or Bartoszyński and Judah [1]). However, in forcing
we keep the older convention that *a stronger condition is the larger one*.

(1) For a set u we let $u^{\langle 2 \rangle} = \{(x, y) \in u \times u : x \neq y\}$.
(2) The Cantor space ${}^\omega 2$ of all infinite sequences with values 0 and 1 is
    equipped with the natural product topology and the group operation
    of coordinate-wise addition $+$ modulo 2.
(3) Ordinal numbers will be denoted be the lower case initial letters of
    the Greek alphabet $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ as well as ξ. Finite ordinals (non-
    negative integers) will be denoted by letters $a, b, c, d, i, j, k, \ell, m, n, M$
    and ι.
(4) The Greek letters κ, λ will stand for uncountable cardinals.
(5) For a forcing notion $\mathbb{P}$, all $\mathbb{P}$–names for objects in the extension via
    $\mathbb{P}$ will be denoted with a tilde below (e.g., $\underset{\sim}{\tau}$, $\underset{\sim}{X}$), and $\underset{\sim}{G}_\mathbb{P}$ will stand
    for the canonical $\mathbb{P}$–name for the generic filter in $\mathbb{P}$.

We fully utilize the algebraic properties of $({}^\omega 2, +)$, in particular the fact
that all elements of ${}^\omega 2$ are self-inverse.

2. The splitting rank

In this section we remind some basic facts from [7, Section 1] concerning
a rank (on models with countable vocabulary) which will be used in the
construction of a forcing notion in the fourth section. This rank and relevant
proofs were also presented in [4, Section 2].

Let λ be a cardinal and $\mathbb{M}$ be a model with the universe λ and a countable
vocabulary τ.

Definition 2.1.   (1) By induction on ordinals δ, for finite non-empty
        sets $w \subseteq \lambda$ we define when $\mathrm{rk}(w, \mathbb{M}) \geq \delta$. Let $w = \{\alpha_0, \dots, \alpha_n\} \subseteq \lambda$,
        $|w| = n + 1$.

(a) $\mathrm{rk}(w) \geq 0$ if and only if for every quantifier free formula $\varphi \in \mathcal{L}(\tau)$ and each $k \leq n$, if $\mathbb{M} \models \varphi[\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n]$ then the set

$$\big\{\alpha \in \lambda : \mathbb{M} \models \varphi[\alpha_0, \ldots, \alpha_{k-1}, \alpha, \alpha_{k+1}, \ldots, \alpha_n]\big\}$$

is uncountable;

(b) if δ is limit, then $\mathrm{rk}(w, \mathbb{M}) \geq \delta$ if and only if $\mathrm{rk}(w, \mathbb{M}) \geq \gamma$ for all $\gamma < \delta$;

(c) $\mathrm{rk}(w, \mathbb{M}) \geq \delta + 1$ if and only if for every quantifier free formula $\varphi \in \mathcal{L}(\tau)$ and each $k \leq n$, if $\mathbb{M} \models \varphi[\alpha_0, \ldots, \alpha_k, \ldots, \alpha_n]$ then there is $\alpha^* \in \lambda \setminus w$ such that

$$\mathrm{rk}(w \cup \{\alpha^*\}, \mathbb{M}) \geq \delta \quad \text{and} \quad \mathbb{M} \models \varphi[\alpha_0, \ldots, \alpha_{k-1}, \alpha^*, \alpha_{k+1}, \ldots, \alpha_n].$$

(2) The rank $\mathrm{rk}(w, \mathbb{M})$ of a finite non-empty set $w \subseteq \lambda$ is defined by:
  - $\mathrm{rk}(w, \mathbb{M}) = -1$ if $\neg(\mathrm{rk}(w, \mathbb{M}) \geq 0)$, and
  - $\mathrm{rk}(w, \mathbb{M}) = \infty$ if $\mathrm{rk}(w, \mathbb{M}) \geq \delta$ for all ordinals δ, and
  - for an ordinal δ: $\mathrm{rk}(w, \mathbb{M}) = \delta$ if $\mathrm{rk}(w, \mathbb{M}) \geq \delta$ but $\neg(\mathrm{rk}(w, \mathbb{M}) \geq \delta + 1)$.

Definition 2.2. For an ordinal ε and a cardinal λ let $\mathrm{NPr}_\varepsilon(\lambda)$ be the following statement: "there is a model $\mathbb{M}^*$ with the universe λ and a countable vocabulary τ^* such that $\sup\{\mathrm{rk}(w, \mathbb{M}^*) : \emptyset \neq w \in [\lambda]^{<\omega}\} < \varepsilon$."
   $\mathrm{Pr}_\varepsilon(\lambda)$ is the negation of $\mathrm{NPr}_\varepsilon(\lambda)$.

Observation 2.3. *If λ is uncountable and $\mathrm{NPr}_\varepsilon(\lambda)$, then there is a model $\mathbb{M}^*$ with the universe λ and a countable vocabulary τ^* such that*

  - $\mathrm{rk}(\{\alpha\}, \mathbb{M}^*) \geq 0$ *for all $\alpha \in \lambda$ and*
  - $\mathrm{rk}(w, \mathbb{M}^*) < \varepsilon$ *for every finite non-empty set $w \subseteq \lambda$.*

Proposition 2.4 (See [7, Claim 1.7] and/or [4, Proposition 2.6]).
   (1) $\mathrm{NPr}_1(\omega_1)$.
   (2) *If $\mathrm{NPr}_\varepsilon(\lambda)$, then $\mathrm{NPr}_{\varepsilon+1}(\lambda^+)$.*
   (3) *If $\mathrm{NPr}_\varepsilon(\mu)$ for $\mu < \lambda$ and $\mathrm{cf}(\lambda) = \omega$, then $\mathrm{NPr}_{\varepsilon+1}(\lambda)$.*

Proposition 2.5 (See [7, Conclusion 1.8] and/or [4, Proposition 2.7]). *Assume $\beta < \alpha < \omega_1$, $\mathbb{M}$ is a model with a countable vocabulary τ and the universe μ, $m, n < \omega$, $n > 0$, $A \subseteq \mu$ and $|A| \geq \beth_{\omega \cdot \alpha}$. Then there is $w \subseteq A$ with $|w| = n$ and $\mathrm{rk}(w, \mathbb{M}) \geq \omega \cdot \beta + m$ [1].*

Definition 2.6. Let λ_{ω_1} be the smallest cardinal λ such that $\mathrm{Pr}_{\omega_1}(\lambda)$.

Corollary 2.7.   (1) *If $\alpha < \omega_1$, then $\mathrm{NPr}_{\omega_1}(\aleph_\alpha)$.*
   (2) $\mathrm{Pr}_{\omega_1}(\beth_{\omega_1})$ *holds true.*
   (3) $\aleph_{\omega_1} \leq \lambda_{\omega_1} \leq \beth_{\omega_1}$.

Corollary 2.8 (See [4, Proposition 2.10 and Corollary 2.11]). *Let $\mu = \beth_{\omega_1} \leq \kappa$. If $\mathbb{P}$ is a ccc forcing notion, then $\Vdash_\mathbb{P} \mathrm{Pr}_{\omega_1}(\mu)$. In particular, if $\mathbb{C}_\kappa$ be the forcing notion adding κ Cohen reals, then $\Vdash_{\mathbb{C}_\kappa} \lambda_{\omega_1} \leq \mu \leq \mathfrak{c}$.*

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[1] " $\cdot$ " stands for the ordinal multiplication

3. SPECTRUM OF TRANSLATION NON-DISJOINTNESS

We want to analyze sets with many non-disjoint translations in more detail, restricting ourselves to Σ_2^0 subsets of $^\omega 2$. In this section we will keep the following assumptions.

Assumptions 3.1. Let $\bar{T} = \langle T_n : n < \omega \rangle$, where each $T_n \subseteq {}^{\omega>}2$ is a tree with no maximal nodes (for $n < \omega$). Let $B = \bigcup_{n<\omega} \lim(T_n)$.

Definition 3.2.   (1) Let $\mathcal{L}$ consist of all non-empty sets $u \subseteq {}^{\omega>}2$ such that $u \subseteq {}^\ell 2$ for some $\ell = \ell(u) < \omega$.
  (2) *A simple base* is a (strict) partial order $\mathcal{O} = (\mathcal{O}, \prec)$ such that $\mathcal{O} \subseteq \mathcal{L}$ and for $u, u' \in \mathcal{O}$:
     (a) if $u \prec u'$ then $\ell(u) < \ell(u')$ and $u = \{\eta{\restriction}\ell(u) : \eta \in u'\}$,
     (b) there is a $v \in \mathcal{O}$ such that $u \prec v$,
     (c) if $\rho \in {}^{\ell(u)}2$ then $u + \rho \in \mathcal{O}$, and if $\rho \in {}^{\ell(u')}2$ and $u \prec u'$ then $u + \rho{\restriction}\ell(u) \prec u' + \rho$.
  (3) Let $(\mathcal{O}, \prec)$ be a simple base. An $\mathcal{O}$–*tower* is a $\prec$–increasing sequence $\bar{u} = \langle u_n : n < \omega \rangle \subseteq \mathcal{O}$ (so $u_n \prec u_{n+1}$ for all $n < \omega$). *The cover of an $\mathcal{O}$–tower $\bar{u}$ is the set* $\mathcal{C}(\bar{u}) \overset{\text{def}}{=} \{\eta \in {}^\omega 2 : (\forall n < \omega)(\eta{\restriction}\ell(u_n) \in u_n)\}$.
  (4) *An indexed base* is a sequence $\bar{\mathcal{O}} = \langle \mathcal{O}_i : i < i^* \rangle$ where $0 < i^* \leq \omega$ and each $\mathcal{O}_i$ is a simple base.

Definition 3.3. Let $\bar{\mathcal{O}} = \langle \mathcal{O}_i : i < i^* \rangle$ be an indexed base.
  (1) We say that two translations $B + x$ and $B + y$ of the set[2] B (for $x, y \in {}^\omega 2$) have $\bar{\mathcal{O}}$–*large intersection* if for some $\langle \bar{u}_i : i < i^* \rangle$ for every $i < i^*$ we have:
     • $\bar{u}_i$ is an $\mathcal{O}_i$–tower,
     • for some $n_1, n_2 < \omega$,
     $$\mathcal{C}(\bar{u}_i) \subseteq \big(\lim(T_{n_1}) + x \big) \cap \big(\lim(T_{n_2}) + y \big),$$
     • $\mathcal{C}(\bar{u}_i) \cap \mathcal{C}(\bar{u}_j) = \emptyset$ whenever $j < i^*$, $j \neq i$.
     In the above situation we may also say that $(B + x) \cap (B + y)$ *is $\bar{\mathcal{O}}$–large*.
  (2) We say that B is *perfectly orthogonal to $\bar{\mathcal{O}}$–small* (or a $\bar{\mathcal{O}}$–**pots**–set) if there is a perfect set $P \subseteq {}^\omega 2$ such that the translations $B + x$, $B + y$ have a $\bar{\mathcal{O}}$–large intersection for all $x, y \in P$.
     The set B is *an $\bar{\mathcal{O}}$–**npots**–set* if it is not $\bar{\mathcal{O}}$–**pots**.
  (3) We say that B has λ *many pairwise $\bar{\mathcal{O}}$–nondisjoint translations* if for some set $X \subseteq {}^\omega 2$ of cardinality λ, for all $x, y \in X$ the translations $B + x$, $B + y$ have a $\bar{\mathcal{O}}$–large intersection.
  (4) We define the *spectrum of translation $\bar{\mathcal{O}}$–nondisjointness of B* as
  $$\text{stnd}_{\bar{\mathcal{O}}}(B) = \big\{(x, y) \in {}^\omega 2 \times {}^\omega 2 : \begin{array}{l} \text{the translations } B + x, B + y \\ \text{have a } \bar{\mathcal{O}}\text{–large intersection.} \end{array} \big\}.$$

Example 3.4.   (1) Let $3 \leq \iota \leq \omega$. Put $\mathcal{O}^0 = \{u \in \mathcal{L} : |u| = 1\}$ and let a relation $\prec^0$ be defined by:

_____
[2]Remember Assumptions 3.1

$u \prec^0 v$ if and only if $\ell(u) < \ell(v) \;\wedge\; u = \{\eta{\restriction}\ell(u) : \eta \in v\}$.

Then $(\mathcal{O}^0, \prec^0)$ is a simple base and $\bar{\mathcal{O}}^\iota = \langle \mathcal{O}^0 : i < \iota \rangle$ is an indexed base. Two translations $B + x$ and $B + y$ of the set B (for $x, y \in {}^\omega 2$) have $\bar{\mathcal{O}}^\iota$–large intersection if and only if $(B + x) \cap (B + y)$ has at least ι members.

(2) Let $\mathcal{O}^{\mathrm{per}} = \{u \in \mathcal{L} : |u| \geq 3\}$ and let a relation $\prec^{\mathrm{per}}$ be defined by
$\qquad u \prec^{\mathrm{per}} v$ if and only if
$\qquad u = \{\eta{\restriction}\ell(u) : \eta \in v\} \;\wedge\; (\forall \nu \in u)(|\{\eta \in v : \nu \lhd \eta\}| \geq 2)$.
Then $(\mathcal{O}^{\mathrm{per}}, \prec^{\mathrm{per}})$ is a simple base and $\bar{\mathcal{O}}^{\mathrm{per}} = \langle \mathcal{O}^{\mathrm{per}} \rangle$ is an indexed base. Two translations $B + x$ and $B + y$ of the set B (for $x, y \in {}^\omega 2$) have $\bar{\mathcal{O}}^{\mathrm{per}}$–large intersection if and only if $(B + x) \cap (B + y)$ is uncountable.

Proposition 3.5. *Let $\bar{\mathcal{O}}$ be an indexed base and let $\bar{T}, B$ be as in Assumptions 3.1.*

(1) *The set B is a $\bar{\mathcal{O}}$–**pots**–set if and only if there is a perfect set $P \subseteq {}^\omega 2$ such that $P \times P \subseteq \mathrm{stnd}_{\bar{\mathcal{O}}}(B)$.*

(2) *The set $\mathrm{stnd}_{\bar{\mathcal{O}}}(B)$ is Σ^1_1.*

(3) *Let $\mathfrak{c} < \lambda \leq \mu$ and let $\mathbb{C}_\mu$ be the forcing notion adding μ Cohen reals. Then, remembering Definition 3.3(2),*

$\qquad \Vdash_{\mathbb{C}_\mu}$ *" if B has λ many pairwise $\bar{\mathcal{O}}$–nondisjoint translations, then B is a $\bar{\mathcal{O}}$–**pots**–set ".*

(4) *Assume $\mathrm{Pr}_{\omega_1}(\lambda)$. If B has λ many pairwise $\bar{\mathcal{O}}$–nondisjoint translations, then it is an $\bar{\mathcal{O}}$–**pots**–set.*

Proof. (1,2) Straightforward; in evaluation of the complexity of $\mathrm{stnd}_{\bar{\mathcal{O}}}(B)$ note that for $\mathcal{O}_i$–towers $\bar{u}_i = \langle u^i_n : n < \omega \rangle$, $x \in {}^\omega 2$ and $k < \omega$:
$\quad \mathcal{C}(\bar{u}_i) \subseteq \lim(T_k) + x$ if and only if $(\forall n < \omega)(u^i_n \subseteq T_k + x)$, and
$\quad \mathcal{C}(\bar{u}_{i_1}) \cap \mathcal{C}(\bar{u}_{i_2}) = \emptyset$ if and only if $(\exists \ell < \omega)(\forall n_1, n_2 > \ell)(u^{i_1}_{n_1}{\restriction}\ell \cap u^{i_2}_{n_2}{\restriction}\ell = \emptyset)$.

(3) This is a consequence of (1,2) above and Shelah [7, Fact 1.16].

(4) By [7, Claim 1.12(1)]. $\qquad\square$

To carry out our arguments we need to assume that our indexed base $\bar{\mathcal{O}}$ satisfies some additional properties.

Definition 3.6. An indexed base $\bar{\mathcal{O}} = \langle \mathcal{O}_i : i < i^* \rangle$ is *nice* if it satisfies the following demands (i)–(v).

(i) Either $i^* \geq 6$ or for some $i < i^*$ we have
$$\big(\forall u \in \mathcal{O}_i\big)\big(\exists v \in \mathcal{O}_i\big)(u \prec v \;\wedge\; |v| \geq 6).$$

(ii) If $i < i^*$, $u \prec_i v \prec_i v' \prec_i v''$, and $\ell(v) \leq \ell \leq \ell(v')$, then $\{\eta{\restriction}\ell : \eta \in v'\} \in \mathcal{O}_i$ and $u \prec_i \{\eta{\restriction}\ell : \eta \in v'\} \prec_i v''$.

(iii) If $i < i^*$, $u \prec_i v$, $\ell(v) < \ell$ and $v' \subseteq {}^\ell 2$ is such that for each $\nu \in v$ the set $\{\eta \in v' : \nu \lhd \eta\}$ has exactly one element, then $v' \in \mathcal{O}_i$ and $u \prec_i v'$.

(iv) Suppose $u \prec_i v$ and $u' \subseteq u$ is such that $u' \in \mathcal{O}_i$. Let $v' = \{\eta \in v : \eta{\restriction}\ell(u) \in u'\}$. Then $v' \in \mathcal{O}_i$ and $u' \prec_i v'$.

(v) If $i^* = \omega$, then for each $i < i^*$ there are infinitely many $j < i^*$ such that $\mathcal{O}_i = \mathcal{O}_j$.

Observation 3.7. *The indexed bases $\bar{\mathcal{O}}^\iota$ and $\bar{\mathcal{O}}^{\mathrm{per}}$ introduced in Example 3.4 are nice.*

Proposition 3.8. *Suppose an indexed base $\bar{\mathcal{O}} = \langle \mathcal{O}_i : i < i^* \rangle$ is nice. Then:*

(⊛) *If $2 \leq K < \omega$ and $\bar{u}^k$ (for $k < K$) is an $\mathcal{O}_{i(k)}$–tower for some $i(k) < i^*$, then there are $\mathcal{O}_{i(k)}$–towers $\bar{v}^k = \langle v_n^k : n < \omega \rangle$ (for $k < K$) such that*

- $\mathcal{C}(\bar{v}^k) = \mathcal{C}(\bar{u}^k)$, $v_0^k = u_0^k$ and
- $\bigcap\limits_{k \in K} \{\ell(v_n^k) : n < \omega\}$ *is infinite.*

Proof. Induction on K. For $K = 2$ we proceed as follows. Let $\bar{u}^0$ be an $\mathcal{O}_{i(0)}$–tower and $\bar{u}^1$ be an $\mathcal{O}_{i(1)}$–tower. Choose inductively a sequence $\langle n_k : k < \omega \rangle$ so that

- $5 < n_0 < n_1 < n_2 < n_3 < \ldots$,
- $\ell(u_5^1) < \ell(u_{n_0}^0)$,
- if $\ell(u_j^1) \leq \ell(u_{n_k}^0) < \ell(n_{j+1}^1)$, then $\ell(n_{j+5}^1) \leq \ell(n_{k+1}^0)$.

For $k < \omega$ let $j(k)$ be such that $\ell(u_{j(k)}^1) \leq \ell(u_{n_k}^0) < \ell(n_{j(k)+1}^1)$. Put $v_k = \{\eta{\restriction}\ell(u_{n_k}^0) : \eta \in u_{j(k)+1}^1\}$. By 3.6(ii), $v_k \in \mathcal{O}_{i(1)}$ and $u_{j(k)-1}^1 \prec_{i(1)} v_k \prec_{i(1)} u_{j(k)+2}^1$. The rest should be clear. $\square$

For the rest of this section we will be assuming the following.

Assumptions 3.9.   (1) $\bar{T} = \langle T_n : n < \omega \rangle$, B are as in Assumptions 3.1,

(2) $\bar{\mathcal{O}} = \langle \mathcal{O}_i : i < i^* \rangle$ is a nice indexed base. Also, $\mathcal{O}_i = (\mathcal{O}_i, \prec_i)$,

(3) there are distinct $x, y \in {}^\omega 2$ such that $(B + x) \cap (B + y)$ is $\bar{\mathcal{O}}$–large.

Definition 3.10. Let $\mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ consist of all tuples

$$\mathbf{m} = (\ell^{\mathbf{m}}, \iota^{\mathbf{m}}, u^{\mathbf{m}}, \bar{h}^{\mathbf{m}}, \bar{g}^{\mathbf{m}}) = (\ell, \iota, u, \bar{h}, \bar{g})$$

such that:

(a) $0 < \ell < \omega$, $u \subseteq {}^\ell 2$ and $2 \leq |u|$, and $\iota = i^*$ if $i^* < \omega$, and $3 \leq \iota < \omega$ otherwise;

(b) $\bar{g} = \langle g_i : i < \iota \rangle$, where[3] $g_i : u^{\langle 2 \rangle} \longrightarrow \mathcal{O}_i$ is such that $g_i(\eta, \nu) = g_i(\nu, \eta)$ and $\ell\big(g_i(\eta, \nu)\big) = \ell$ for each $(\eta, \nu) \in u^{\langle 2 \rangle}$;

(c) if $(\eta, \nu) \in u^{\langle 2 \rangle}$ and $i < i' < \iota$, then $g_i(\eta, \nu) \cap g_{i'}(\eta, \nu) = \emptyset$,

(d) $\bar{h} = \langle h_i : i < \iota \rangle$, where $h_i : u^{\langle 2 \rangle} \longrightarrow \omega$;

(e) for each $(\eta, \nu) \in u^{\langle 2 \rangle}$, if $\sigma \in g_i(\eta, \nu)$ then $\eta + \sigma \in T_{h_i(\eta, \nu)}$.

Definition 3.11. Assume $\mathbf{m} = (\ell, \iota, u, \bar{h}, \bar{g}) \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ and $\rho \in {}^\ell 2$. We define $\mathbf{m} + \rho = (\ell', \iota', u', \bar{h}', \bar{g}')$ by

- $\ell' = \ell$, $\iota' = \iota$, $u' = \{\eta + \rho : \eta \in u\}$,

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[3]remember $u^{\langle 2 \rangle} = \{(\eta, \nu) \in u \times u : \eta \neq \nu\}$

- $\bar{g}' = \langle g_i' : i < \iota' \rangle$, where $g_i' : (u')^{\langle 2 \rangle} \longrightarrow \mathcal{O}_i : (\eta + \rho, \nu + \rho) \mapsto g_i(\eta, \nu) + \rho$,
- $\bar{h}' = \langle h_i' : i < \iota' \rangle$, where $h_i' : (u')^{\langle 2 \rangle} \longrightarrow \omega$ are such that $h_i'(\eta + \rho, \nu + \rho) = h_i(\eta, \nu)$ for $(\eta, \nu) \in u^{\langle 2 \rangle}$.

Also if $\rho \in {}^\omega 2$, then we set $\mathbf{m} + \rho = \mathbf{m} + (\rho{\restriction}\ell)$.

Observation 3.12. (1) *If* $\mathbf{m} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ *and* $\rho \in {}^{\ell^{\mathbf{m}}} 2$, *then* $\mathbf{m} + \rho \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$.
(2) *For each* $\rho \in {}^\omega 2$ *the mapping* $\mathbf{M}_{\bar{T}, \bar{\mathcal{O}}} \longrightarrow \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}} : \mathbf{m} \mapsto \mathbf{m} + \rho$ *is a bijection.*

Definition 3.13. Assume $\mathbf{m}, \mathbf{n} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$. We say that $\mathbf{n}$ *strictly extends* $\mathbf{m}$ ($\mathbf{m} \sqsubset \mathbf{n}$ in short) if and only if:

- $\ell^{\mathbf{m}} < \ell^{\mathbf{n}}$, $\iota^{\mathbf{m}} \leq \iota^{\mathbf{n}}$, $u^{\mathbf{m}} = \{\eta{\restriction}\ell^{\mathbf{m}} : \eta \in u^{\mathbf{n}}\}$, and
- for every $(\eta, \nu) \in (u^{\mathbf{n}})^{\langle 2 \rangle}$ such that $\eta{\restriction}\ell^{\mathbf{m}} \neq \nu{\restriction}\ell^{\mathbf{m}}$ and each $i < \iota^{\mathbf{m}}$ we have
  - $g_i^{\mathbf{m}}(\eta{\restriction}\ell^{\mathbf{m}}, \nu{\restriction}\ell^{\mathbf{m}}) \prec g_i^{\mathbf{n}}(\eta, \nu)$, and
  - $h_i^{\mathbf{m}}(\eta{\restriction}\ell^{\mathbf{m}}, \nu{\restriction}\ell^{\mathbf{m}}) = h_i^{\mathbf{n}}(\eta, \nu)$.

Definition 3.14. (1) By induction on ordinals α we define $D^{\bar{T}}(\alpha) \subseteq \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$. We declare that:

- $D^{\bar{T}}(0) = \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$,
- if α is a limit ordinal, then $D^{\bar{T}}(\alpha) = \bigcap_{\beta < \alpha} D^{\bar{T}}(\beta)$,
- if $\alpha = \beta + 1$, then $D^{\bar{T}}(\alpha)$ consists of all $\mathbf{m} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ such that for each for each $\nu \in u^{\mathbf{m}}$ there is an $\mathbf{n} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ satisfying
  - $\mathbf{m} \sqsubset \mathbf{n}$ and $\mathbf{n} \in D^{\bar{T}}(\beta)$, and if $i^* = \omega$ then $\iota^{\mathbf{m}} < \iota^{\mathbf{n}}$, and
  - the set $\{\eta \in u^{\mathbf{n}} : \nu \lhd \eta\}$ has at least two elements

(2) We define a function[4] $\mathrm{ndrk}_{\bar{\mathcal{O}}}^{\bar{T}} = \mathrm{ndrk} : \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}} \longrightarrow \mathrm{ON} \cup \{\infty\}$ as follows.
If $\mathbf{m} \in D^{\bar{T}}(\alpha)$ for all ordinals α, then we say that $\mathrm{ndrk}(\mathbf{m}) = \infty$.
Otherwise, $\mathrm{ndrk}(\mathbf{m})$ is the first ordinal α for which $\mathbf{m} \notin D^{\bar{T}}(\alpha + 1)$.
(3) We also define

$$\mathrm{NDRK}_{\bar{\mathcal{O}}}(\bar{T}) = \mathrm{NDRK}(\bar{T}) = \sup\{\mathrm{ndrk}(\mathbf{m}) + 1 : \mathbf{m} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}\}.$$

Lemma 3.15. (1) *The relation* $\sqsubset$ *is a strict partial order on* $\mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$.
(2) *If* $\mathbf{m}, \mathbf{n} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ *and* $\mathbf{m} \sqsubset \mathbf{n}$ *and* $\mathbf{n} \in D^{\bar{T}}(\alpha)$, *then* $\mathbf{m} \in D^{\bar{T}}(\alpha)$.
(3) *If* $\alpha < \beta$ *then* $D^{\bar{T}}(\beta) \subseteq D^{\bar{T}}(\alpha)$. *Hence for* $\mathbf{m} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$, $\mathbf{m} \in D^{\bar{T}}(\alpha)$ *if and only if* $\alpha \leq \mathrm{ndrk}(\mathbf{m})$.
(4) *If* $\mathbf{m} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ *and* $\rho \in {}^\omega 2$ *then* $\mathrm{ndrk}(\mathbf{m}) = \mathrm{ndrk}(\mathbf{m} + \rho)$.
(5) *If* $\mathbf{m} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ *and* $\mathrm{ndrk}(\mathbf{m}) \geq \omega_1$, *then there is an* $\mathbf{n} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ *such that* $\mathbf{m} \sqsubset \mathbf{n}$, $|\{\eta \in u^{\mathbf{n}} : \nu \lhd \eta\}| \geq 2$ *for each* $\nu \in u^{\mathbf{m}}$, *if* $i^* = \omega$ *then* $\iota^{\mathbf{m}} < \iota^{\mathbf{n}}$, *and* $\mathrm{ndrk}(\mathbf{n}) \geq \omega_1$.
(6) *If* $\mathbf{m} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ *and* $\infty > \mathrm{ndrk}(\mathbf{m}) = \beta > \alpha$, *then there is* $\mathbf{n} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ *such that* $\mathbf{m} \sqsubset \mathbf{n}$ *and* $\mathrm{ndrk}(\mathbf{n}) = \alpha$.
(7) *If* $\mathrm{NDRK}(\bar{T}) \geq \omega_1$, *then* $\mathrm{NDRK}(\bar{T}) = \infty$.

---

[4]ndrk stands for **nondisjointness ran**k

(8) *Assume* $\mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ *and* $u' \subseteq u^{\mathbf{m}}$, $|u'| \geq 2$. *Put* $\ell' = \ell^{\mathbf{m}}$, $\iota' = \iota^{\mathbf{m}}$, *and for* $i < \iota'$ *let* $h'_i = h^{\mathbf{m}}_i {\restriction} (u')^{\langle 2 \rangle}$ *and* $g'_i = g^{\mathbf{m}}_i {\restriction} (u')^{\langle 2 \rangle}$. *Let* $\mathbf{m} {\restriction} u' = (\ell', u', i', \bar{h}', \bar{g}')$. *Then* $\mathbf{m} {\restriction} u' \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ *and* $\mathrm{ndrk}(\mathbf{m}) \leq \mathrm{ndrk}(\mathbf{m} {\restriction} u')$.

Proof. Exactly the same as for [4, Lemma 3.10]. □

Proposition 3.16. *For a nice indexed base* $\bar{\mathcal{O}}$ *the following conditions (a) – (d) are equivalent.*

(a) $\mathrm{NDRK}_{\bar{\mathcal{O}}}(\bar{T}) \geq \omega_1$.
(b) $\mathrm{NDRK}_{\bar{\mathcal{O}}}(\bar{T}) = \infty$.
(c) B *is perfectly orthogonal to* $\bar{\mathcal{O}}$*–small (see 3.3(2)).*
(d) *In some ccc forcing extension, the set* B *has* λ_{ω_1} *many pairwise* $\bar{\mathcal{O}}$*– nondisjoint translations (see 3.3(3)).*

Proof. The proof follows closely the lines of [4, Proposition 3.11].
(c) $\Rightarrow$ (d)   Assume (c) and let $P \subseteq {}^{\omega}2$ be a perfect set such that the translations $B + x$, $B + y$ have $\bar{\mathcal{O}}$–large intersection for all $x, y \in P$. Let $\kappa = \beth_{\omega_1}$. By Corollary 2.8, $\Vdash_{\mathbb{C}_\kappa} \lambda_{\omega_1} \leq \mathfrak{c}$. By Proposition 3.5(2), the formula "$P \times P \subseteq \mathrm{stnd}_{\bar{\mathcal{O}}}(B)$" is Π^1_2, so it holds in the forcing extension by $\mathbb{C}_\kappa$. Now we easily conclude (d).

(d) $\Rightarrow$ (a)   Assume (d) and let $\mathbb{P}$ be the ccc forcing notion witnessing this assumption, $G \subseteq \mathbb{P}$ be generic over $\mathbf{V}$. Let us work in $\mathbf{V}[G]$.
   Let $\langle \eta_\alpha : \alpha < \lambda_{\omega_1} \rangle$ be a sequence of distinct elements of ${}^{\omega}2$ such that

$$\left(\forall \alpha < \beta < \lambda_{\omega_1} \right) \left((B + \eta_\alpha) \cap (B + \eta_\beta) \text{ is } \bar{\mathcal{O}}\text{–large} \right).$$

Remember Definition 3.2(3): an $\mathcal{O}_i$–tower is an $\prec_i$–increasing sequence $\bar{u} = \langle u_n : n < \omega \rangle$ and its cover $\mathcal{C}(\bar{u})$ is the set $\{ \eta \in {}^{\omega}2 : (\forall n < \omega)(\eta {\restriction} \ell(u_n) \in u_n) \}$.
   Let $\tau = \{ R_{\mathbf{m}} : \mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}} \}$ be a vocabulary where each $R_{\mathbf{m}}$ is a $|u^{\mathbf{m}}|$–ary relational symbol. Let $\mathbb{M} = \left(\lambda_{\omega_1}, \{ R^{\mathbb{M}}_{\mathbf{m}} \}_{\mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}} \right)$ be the model in the vocabulary τ, where for $\mathbf{m} = (\ell, \iota, u, h, g) \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ the relation $R^{\mathbb{M}}_{\mathbf{m}}$ is defined by

$$
\begin{aligned}
R^{\mathbb{M}}_{\mathbf{m}} = \ \ & \Big\{ (\alpha_0, \ldots, \alpha_{|u|-1}) \in (\lambda_{\omega_1})^{|u|} : \{ \eta_{\alpha_0} {\restriction} \ell, \ldots, \eta_{|u|-1} {\restriction} \ell \} = u \text{ and} \\
& \text{for each distinct } j_1, j_2 < |u| \text{ and every } i < \iota \\
& \text{there is an } \mathcal{O}_i\text{–tower } \bar{u}^i(j_1, j_2) = \langle u^i_n(j_1, j_2) : n < \omega \rangle \text{ such that} \\
& g_i(\eta_{\alpha_{j_1}} {\restriction} \ell, \eta_{\alpha_{j_2}} {\restriction} \ell) = u^i_0(j_1, j_2) \text{ and } \mathcal{C}\left(\bar{u}^i(j_1, j_2) \right) \text{ is included in} \\
& [\lim(T_{h_i(\eta_{\alpha_{j_1}} {\restriction} \ell, \eta_{\alpha_{j_2}} {\restriction} \ell)}) + \eta_{\alpha_{j_1}}] \cap [\lim(T_{h_i(\eta_{\alpha_{j_2}}, \eta_{\alpha_{j_1}})}) + \eta_{\alpha_{j_2}}] \Big\}.
\end{aligned}
$$

Claim 3.16.1.   (1) *If* $\alpha_0, \alpha_1, \ldots, \alpha_{j-1} < \lambda_{\omega_1}$ *are distinct,* $j \geq 2$, *then for infinitely many* $k < \omega$ *there is* $\mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$ *such that*

$$\ell^{\mathbf{m}} = k, \quad u^{\mathbf{m}} = \{ \eta_{\alpha_0} {\restriction} k, \ldots, \eta_{\alpha_{j-1}} {\restriction} k \} \quad \text{and} \quad \mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}].$$

(2) *Assume that* $\mathbf{m} \in \mathbf{M}_{\bar{T},\bar{\mathcal{O}}}$, $j < |u^{\mathbf{m}}|$, $\alpha_0, \alpha_1, \ldots, \alpha_{|u^{\mathbf{m}}|-1} < \lambda_{\omega_1}$ *and* $\alpha^* < \lambda_{\omega_1}$ *are all pairwise distinct and such that*
   • $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_j, \ldots, \alpha_{|u^{\mathbf{m}}|-1}]$ *and*
   • $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}, \alpha^*, \alpha_{j+1}, \ldots \alpha_{|u^{\mathbf{m}}|-1}]$.

Then for infinitely many $k < \omega$ there is an $\mathbf{n} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ such that $\mathbf{m} \sqsubset \mathbf{n}$ and $\ell^{\mathbf{n}} = k$, $u^{\mathbf{n}} = \{\eta_{\alpha_0} \restriction k, \ldots, \eta_{\alpha_{|u^{\mathbf{m}}|-1}} \restriction k, \eta_{\alpha^*} \restriction k\}$ and $\mathbb{M} \models R_{\mathbf{n}}[\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}, \alpha^*]$, and if $i^* = \omega$ then also $\iota^{\mathbf{m}} < \iota^{\mathbf{n}}$.

(3) *If $\mathbf{m} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ and $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}]$, then*

$$\mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}\}, \mathbb{M}) \leq \mathrm{ndrk}_{\bar{\mathcal{O}}}^{\bar{T}}(\mathbf{m}).$$

Proof of the Claim. (1) It is a simpler version of the proof below.

(2) By the definition of $R_{\mathbf{m}}^{\mathbb{M}}$, since $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}, \alpha^*, \alpha_{j+1}, \ldots \alpha_{|u^{\mathbf{m}}|-1}]$ and $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_j, \ldots, \alpha_{|u^{\mathbf{m}}|-1}]$, we may choose a sequence

$$\langle \bar{u}^i(j_1, j_2) : (j_1, j_2) \in (|u^{\mathbf{m}}| + 1)^{\langle 2 \rangle} \wedge i < \iota^{\mathbf{m}} \rangle$$

satisfying the following demands. Letting $\alpha_{|u^{\mathbf{m}}|} = \alpha^*$, for $(j_1, j_2) \in (|u^{\mathbf{m}}| + 1)^{\langle 2 \rangle}$ and $i < \iota^{\mathbf{m}}$:

- $\bar{u}^i(j_1, j_2) = \bar{u}^i(j_2, j_1)$ is a $\mathcal{O}_i$–tower,
- if $\{j_1, j_2\} \neq \{j, |u^{\mathbf{m}}|\}$, then $u_0^i(j_1, j_2) = g_{\mathbf{m}}^{\mathbf{m}}(\eta_{\alpha_{j_1}} \restriction \ell^{\mathbf{m}}, \eta_{\alpha_{j_2}} \restriction \ell^{\mathbf{m}})$,
- if $i_1 < i_2 < \iota^{\mathbf{m}}$, then $\mathcal{C}(\bar{u}^{i_1}(j_1, j_2)) \cap \mathcal{C}(\bar{u}^{i_2}(j_1, j_2)) = \emptyset$,
- if $\{j_1, j_2\} \neq \{j, |u^{\mathbf{m}}|\}$, then $\mathcal{C}(\bar{u}^i(j_1, j_2))$ is included in

$$[\lim(T_{h_i^{\mathbf{m}}(\eta_{\alpha_{j_1}} \restriction \ell^{\mathbf{m}}, \eta_{\alpha_{j_2}} \restriction \ell^{\mathbf{m}})}) + \eta_{\alpha_{j_1}}] \cap [\lim(T_{h_i^{\mathbf{m}}(\eta_{\alpha_{j_2}} \restriction \ell^{\mathbf{m}}, \eta_{\alpha_{j_1}} \restriction \ell^{\mathbf{m}})}) + \eta_{\alpha_{j_2}}],$$

- for some N_i', N_i'' we have

$$\mathcal{C}(\bar{u}^i(j, |u^{\mathbf{m}}|)) \subseteq [\lim(T_{N_i'}) + \eta_{\alpha_j}] \cap [\lim(T_{N_i''}) + \eta_{\alpha^*}].$$

Since $\bar{\mathcal{O}}$ is nice (and $\iota^{\mathbf{m}}$ and $u^{\mathbf{m}}$ are finite), we may use 3.8(⊛) and modify $\bar{u}^i(j_1, j_2)$ (without changing $u_0^i(j_1, j_2)$) and demand that the set

$$A = \bigcap_{i < \iota^{\mathbf{m}}} \bigcap_{j_1 < j_2 \leq |u^{\mathbf{m}}|} \{\ell(u_n^i(j_1, j_2)) : n \in \omega\}$$

is infinite. Let $\ell_0 \in A \setminus (\ell^{\mathbf{m}} + 1)$ be bigger than the second element of $A \setminus (\ell^{\mathbf{m}} + 1)$ and such that $\eta_{\alpha_{|u^{\mathbf{m}}|}} \restriction \ell_0 \neq \eta_{\alpha_j} \restriction \ell_0$, and $x \restriction \ell_0 \neq y \restriction \ell_0$ whenever $x \in \mathcal{C}(\bar{u}^{i_1}(j_1, j_2))$, $y \in \mathcal{C}(\bar{u}^{i_2}(j_1, j_2))$, $(j_1, j_2) \in (|u^{\mathbf{m}}| + 1)^{\langle 2 \rangle}$ and $i_1 < i_2 < \iota^{\mathbf{m}}$.

Let $\iota = \iota^{\mathbf{m}} = i^*$ if $i^* < \omega$ and let $\iota = \iota^{\mathbf{m}} + 1$ otherwise. In the latter case we also have to choose $\mathcal{O}_{\iota^{\mathbf{m}}}$–towers $\bar{u}^{\iota^{\mathbf{m}}}(j_1, j_2)$, but to ensure the demand 3.10(c) we will have to modify the already chosen towers $\bar{u}^i(j_1, j_2)$ (for $i < \iota^{\mathbf{m}}$). Fix $(j_1, j_2) \in (|u^{\mathbf{m}}| + 1)^{\langle 2 \rangle}$ for a moment. Let

$$K = \sum \{|u_n^i(j_1, j_2)| : \ell(u_n^i(j_1, j_2)) = \ell_0 \wedge i < \iota^{\mathbf{m}} \wedge n < \omega\}.$$

By 3.6(v) and the assumptions on $\langle \eta_\alpha : \alpha < \lambda_{\omega_1} \rangle$, there are infinitely many $\mathcal{O}_{\iota^{\mathbf{m}}}$–towers $\bar{v}^k$ such that their covers are pairwise disjoint and included in $(\lim(T_{k_1}) + \eta_{\alpha_{j_1}}) \cap (\lim(T_{k_2}) + \eta_{\alpha_{j_2}})$ for some k_1, k_2.

Choose $\ell(j_1, j_2) \in A \setminus (\ell_0 + 1)$ so large, that there are more than $K + 1$ many k's for which the sets $\{\eta \restriction \ell(j_1, j_2) : \eta \in v_n^k\}$ are pairwise disjoint (for large n) and $\ell(v_5^k) < \ell(j_1, j_2)$ for all those k's. For $i < i^{\mathbf{m}}$ let $n(i), m(i)$ be such that $\ell(u_{n(i)}^i(j_1, j_2)) = \ell_0$ and $\ell(u_{m(i)}^i(j_1, j_2)) = \ell(j_1, j_2)$, and let $v^i \subseteq$

$u^i_{m(i)}(j_1, j_2)$ be such that for each $\nu \in u^i_{n(i)}(j_1, j_2)$ the set $\{\eta \in v^i : \nu \lhd \eta\}$ has exactly one element. By 3.6(iii) we have

$$v^i \in \mathcal{O}_i \quad \text{and} \quad u^i_{n(i)-1}(j_1, j_2) \prec_i v^i.$$

Using repeatedly 3.6(iv) we may modify the towers $\bar{u}^i(j_1, j_2)$ (for $i < i^{\mathbf{m}}$) and demand that

- for each $i < \iota^{\mathbf{m}}$, for some $n^*(i)$,

$$\ell(u^i_{n^*(i)}(j_1, j_2)) = \ell(j_1, j_2) \text{ and } \left|u^i_{n^*(i)}(j_1, j_2)\right| = \left|\{\eta{\upharpoonright}\ell_0 : \eta \in u^i_{n^*(i)}(j_1, j_2)\}\right|.$$

Looking back at the towers $\bar{v}^k$, we may choose one, $\bar{v}^{k^*} = \bar{v}(j_1, j_2)$, which has the property that for all large n

$$\{\eta{\upharpoonright}\ell(j_1, j_2) : \eta \in v_n(j_1, j_2)\} \cap \bigcup\{u^i_{n^*(i)}(j_1, j_2) : i < \iota^{\mathbf{m}}\} = \emptyset.$$

Now unfix (j_1, j_2) and set $\ell = \max\{\ell(j_1, j_2) : (j_1, j_2) \in (|u^{\mathbf{m}}| + 1)^{\langle 2 \rangle}\}$.

Suppose $j_1 < j_2 \leq |u^{\mathbf{m}}|$ and let n be such that $\ell(v_{n-1}(j_1, j_2)) < \ell \leq \ell(v_n(j_1, j_2))$. By 3.6(ii), we may let

- $u^{\iota^{\mathbf{m}}}_0(j_1, j_2) = u^{\iota^{\mathbf{m}}}_0(j_2, j_1) = \{\eta{\upharpoonright}\ell : \eta \in v_n(j_1, j_2)\}$,
- $u^{\iota^{\mathbf{m}}}_m(j_1, j_2) = u^{\iota^{\mathbf{m}}}_m(j_2, j_1) = v_{n+m}(j_1, j_2)$ for $m > 0$,

getting a $\mathcal{O}_{\iota^{\mathbf{m}}}$–tower $\bar{u}^{\iota^{\mathbf{m}}}(j_1, j_2)$. We also fix $k(j_1, j_2), k(j_2, j_1)$ such that

$$\mathcal{C}(\bar{u}^{\iota^{\mathbf{m}}}(j_1, j_2)) \subseteq \left(\lim(T_{k(j_1, j_2)} + \eta_{\alpha_{j_1}}\right) \cap \left(\lim(T_{k(j_2, j_1)} + \eta_{\alpha_{j_2}}\right).$$

If $i^* = \iota^{\mathbf{m}} < \omega$, then the procedure leading to the choice of $\bar{u}^{\iota^{\mathbf{m}}}(j_1, j_2)$ is not present and we just let $\ell = \min(A \setminus (\ell_0 + 1))$.

Let $u = \{\eta_{\alpha_0}{\upharpoonright}\ell, \ldots, \eta_{\alpha_{|u^{\mathbf{m}}|-1}}{\upharpoonright}\ell, \eta_{\alpha^*}{\upharpoonright}\ell\}$.

For each $i < \iota$ and $(j_1, j_2) \in (|u^{\mathbf{m}}| + 1)^{\langle 2 \rangle}$ put $g_i(\eta_{\alpha_{j_1}}{\upharpoonright}\ell, \eta_{\alpha_{j_2}}{\upharpoonright}\ell) = u^i_n(j_1, j_2)$, where n is such that $\ell(u^i_n(j_1, j_2)) = \ell$. This defines $g_i : u^{\langle 2 \rangle} \longrightarrow \mathcal{O}_i$ for $i < \iota$. For $(\nu_1, \nu_2) \in u^{\langle 2 \rangle}$ we also set

$$h_i(\nu_1, \nu_2) = \begin{cases} h^{\mathbf{m}}_i(\nu_1{\upharpoonright}\ell^{\mathbf{m}}, \nu_2{\upharpoonright}\ell^{\mathbf{m}}) & \text{if } \nu_1{\upharpoonright}\ell^{\mathbf{m}} \neq \nu_2{\upharpoonright}\ell^{\mathbf{m}}, \ i < \iota^{\mathbf{m}}, \\ N'_i & \text{if } \nu_1 \lhd \eta_{\alpha_j}, \ \nu_2 \lhd \eta_{\alpha^*}, \ i < \iota^{\mathbf{m}}, \\ N''_i & \text{if } \nu_1 \lhd \eta_{\alpha^*}, \ \nu_2 \lhd \eta_{\alpha_j}, \ i < \iota^{\mathbf{m}}, \\ k(j_1, j_2) & \text{if } \nu_1 \lhd \eta_{\alpha_{j_1}}, \ \nu_2 \lhd \eta_{\alpha_{j_2}}, \ i = \iota^{\mathbf{m}} < \iota. \end{cases}$$

It should be clear that $\mathbf{n} = (\ell, \iota, u, g, h) \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ is as required.

(3) By induction on β we show that

> *for every* $\mathbf{m} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ *and all* $\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1} < \lambda_{\omega_1}$ *such that* $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}]$:
> $\beta \leq \text{rk}(\{\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}\}, \mathbb{M})$ *implies* $\beta \leq \text{ndrk}(\mathbf{m})$.

STEPS $\beta = 0$ AND β IS LIMIT: Straightforward.

STEP $\beta = \gamma + 1$: Suppose $\mathbf{m} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ and $\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1} < \lambda_{\omega_1}$ are such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}]$ and $\gamma + 1 \leq \text{rk}(\{\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}\}, \mathbb{M})$. Let $\nu \in u^{\mathbf{m}}$, so $\nu = \eta_{\alpha_j}{\upharpoonright}\ell^{\mathbf{m}}$ for some $j < |u^{\mathbf{m}}|$. Since $\gamma + 1 \leq \text{rk}(\{\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}\}, \mathbb{M})$ we may find $\alpha^* \in \lambda_{\omega_1} \setminus \{\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}\}$ such that

$$\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}, \alpha^*, \alpha_{j+1}, \ldots, \alpha_{|u^{\mathbf{m}}|-1}]$$

and $\mathrm{rk}(\{\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}, \alpha^*\}, \mathbb{M}) \geq \gamma$. By clause (2) we may find $\mathbf{n} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ such that $\mathbf{m} \sqsubset \mathbf{n}$ and $u^{\mathbf{n}} = \{\eta_{\alpha_0} \restriction \ell^{\mathbf{n}}, \ldots, \eta_{\alpha_{|u^{\mathbf{m}}|-1}} \restriction \ell^{\mathbf{n}}, \eta_{\alpha^*} \restriction \ell^{\mathbf{n}}\}$, and if $i^* = \omega$ then $\iota^{\mathbf{m}} < \iota^{\mathbf{n}}$, and $\mathbb{M} \models R_{\mathbf{n}}[\alpha_0, \ldots, \alpha_{|u^{\mathbf{m}}|-1}, \alpha^*]$. Then also $|\{\eta \in u^{\mathbf{n}} : \nu \lhd \eta\}| \geq 2$. By the inductive hypothesis we have also $\gamma \leq \mathrm{ndrk}(\mathbf{n})$. Now we may easily conclude that $\gamma + 1 \leq \mathrm{ndrk}(\mathbf{m})$. $\qquad\square$

By the definition of λ_{ω_1},

($\odot$) $\sup\{\mathrm{rk}(w, \mathbb{M}) : \emptyset \neq w \in [\lambda_{\omega_1}]^{<\omega}\} \geq \omega_1$

Now, suppose that $\beta < \omega_1$. By ($\odot$), there are distinct $\alpha_0, \ldots, \alpha_{j-1} < \lambda_{\omega_1}$, $j \geq 2$, such that $\mathrm{rk}(\{\alpha_0, \ldots, \alpha_{j-1}\}, \mathbb{M}) \geq \beta$. By Claim 3.16.1(1) we may find $\mathbf{m} \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}}$ such that $\mathbb{M} \models R_{\mathbf{m}}[\alpha_0, \ldots, \alpha_{j-1}]$. Then by Claim 3.16.1(3) we also have $\mathrm{ndrk}_{\bar{\mathcal{O}}}^{\bar{T}}(\mathbf{m}) \geq \beta$. Consequently, $\mathrm{NDRK}(\bar{T}) \geq \omega_1$.

All the considerations above where carried out in $\mathbf{V}[G]$. However, the rank function $\mathrm{ndrk}_{\bar{\mathcal{O}}}^{\bar{T}}$ is absolute, so we may also claim that in $\mathbf{V}$ we have $\mathrm{NDRK}_{\bar{\mathcal{O}}}(\bar{T}) \geq \omega_1$. $\qquad\square$

4. THE MAIN RESULT

In this section we construct a forcing notion adding a sequence $\bar{T}$ of subtrees of $^{\omega >}2$ such that $\mathrm{NDRK}_{\bar{\mathcal{O}}^6}(\bar{T}) < \omega_1$ and yet with many $\bar{\mathcal{O}}$–nondisjoint translations (for a nice $\bar{\mathcal{O}}$). The sequence $\bar{T}$ will be added by finite approximations, so we will need a finite version of Definition 3.10.

Definition 4.1. Assume that

(a) $0 < n, M < \omega$, $\bar{t} = \langle t_m : m < M\rangle$, and each t_m is a subtree of $^{n \geq}2$ in which all terminal branches are of length n,

(b) $T_j \subseteq {}^{\omega >}2$ (for $j < \omega$) are trees with no maximal nodes, $\bar{T} = \langle T_j : j < \omega\rangle$ and $t_m = T_m \cap {}^{n \geq}2$ for $m < M$,

(c) $\mathbf{M}_{\bar{T}, \bar{\mathcal{O}}^6}$ is defined as in Definition 3.10 for $\bar{\mathcal{O}}^6$ introduced in Example 3.4(1).

We let $\mathbf{M}_{\bar{t}, \bar{\mathcal{O}}^6}^n$ consist of all tuples $\mathbf{m} = (\ell^{\mathbf{m}}, 6, u^{\mathbf{m}}, \bar{h}^{\mathbf{m}}, \bar{g}^{\mathbf{m}}) \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}^6}$ such that $\ell^{\mathbf{m}} \leq n$ and $\mathrm{rng}(h_i^{\mathbf{m}}) \subseteq M$ for each $i < 6$.

The extension relation $\sqsubset$ on $\mathbf{M}_{\bar{t}, \bar{\mathcal{O}}^6}^n$ is inherited from $\mathbf{M}_{\bar{T}, \bar{\mathcal{O}}^6}$ (see Definition 3.13).

Observation 4.2.   (1) *The Definition of $\mathbf{M}_{\bar{t}, \bar{\mathcal{O}}^6}^n$ does not depend on the choice of $\bar{T}$, as long as the clause 4.1(c) is satisfied.*

(2) *If $\mathbf{m} \in \mathbf{M}_{\bar{t}, \bar{\mathcal{O}}^6}^n$ and $\rho \in {}^{\ell^{\mathbf{m}}}2$, then $\mathbf{m} + \rho \in \mathbf{M}_{\bar{t}, \bar{\mathcal{O}}^6}^n$ (remember Definition 3.11).*

Lemma 4.3 (See [3, Lemma 2.3].). *Let $0 < \ell < \omega$ and let $\mathcal{B} \subseteq {}^{\ell}2$ be a linearly independent set of vectors (in $({}^{\ell}2, +)$ over $\mathbb{Z}_2$). If $\mathcal{A} \subseteq {}^{\ell}2$, $|\mathcal{A}| \geq 5$ and $\mathcal{A} + \mathcal{A} \subseteq \mathcal{B} + \mathcal{B}$, then for a unique $x \in {}^{\ell}2$ we have $\mathcal{A} + x \subseteq \mathcal{B}$.*

Theorem 4.4. *Assume that an uncountable cardinal λ satisfies $\mathrm{NPr}_{\omega_1}(\lambda)$ and suppose that $\bar{\mathcal{O}} = \langle \mathcal{O}_i : i < i^*\rangle$ is a nice indexed base. Then there is a*

ccc forcing notion $\mathbb{P}$ *of size* λ *such that*

$\Vdash_{\mathbb{P}}$ " *for some* Σ_2^0 $\bar{\mathcal{O}}^6$–**npots**–*set* $B = \bigcup\limits_{n<\omega} \lim(T_n) \subseteq {}^\omega 2$ *there is*

a sequence $\langle \eta_\alpha : \alpha < \lambda \rangle$ *of distinct elements of* ${}^\omega 2$ *such that all intersections* $(\eta_\alpha + B) \cap (\eta_\beta + B)$ *are* $\bar{\mathcal{O}}$–*large for* $\alpha, \beta < \lambda$ ".

Proof. Fix a countable vocabulary $\tau = \{R_{n,\zeta} : n, \zeta < \omega\}$, where $R_{n,\zeta}$ is an n–ary relational symbol (for $n, \zeta < \omega$). By the assumption on λ, we may fix a model $\mathbb{M} = (\lambda, \{R_{n,\zeta}^{\mathbb{M}}\}_{n,\zeta<\omega})$ in the vocabulary τ with the universe λ and an ordinal $\alpha^* < \omega_1$ such that:

($\circledast$)$_{\mathrm{a}}$ for every n and a quantifier free formula $\varphi(x_0, \ldots, x_{n-1}) \in \mathcal{L}(\tau)$ there is $\zeta < \omega$ such that for all $a_0, \ldots, a_{n-1} \in \lambda$,

$$\mathbb{M} \models \varphi[a_0, \ldots, a_{n-1}] \Leftrightarrow R_{n,\zeta}[a_0, \ldots, a_{n-1}],$$

($\circledast$)$_{\mathrm{b}}$ $\sup\{\mathrm{rk}(v, \mathbb{M}) : \emptyset \neq v \in [\lambda]^{<\omega}\} < \alpha^*$,

($\circledast$)$_{\mathrm{c}}$ the rank of every singleton is at least 0.

For a nonempty finite set $v \subseteq \lambda$ let $\mathrm{rk}(v) = \mathrm{rk}(v, \mathbb{M})$, and let $\zeta(v) < \omega$ and $k(v) < |v|$ be such that $R_{|v|,\zeta(v)}, k(v)$ witness the rank of v. Thus letting $\{a_0, \ldots, a_k, \ldots a_{n-1}\}$ be the increasing enumeration of v and $k = k(v)$ and $\zeta = \zeta(v)$, we have

($\circledast$)$_{\mathrm{d}}$ if $\mathrm{rk}(v) \geq 0$, then $\mathbb{M} \models R_{n,\zeta}[a_0, \ldots, a_k, \ldots, a_{n-1}]$ but there is no $a \in \lambda \setminus v$ such that

$$\mathrm{rk}(v \cup \{a\}) \geq \mathrm{rk}(v) \quad \text{and} \quad \mathbb{M} \models R_{n,\zeta}[a_0, \ldots, a_{k-1}, a, a_{k+1}, \ldots, a_{n-1}],$$

($\circledast$)$_{\mathrm{e}}$ if $\mathrm{rk}(v) = -1$, then $\mathbb{M} \models R_{n,\zeta}[a_0, \ldots, a_k, \ldots, a_{n-1}]$ but the set

$$\{a \in \lambda : \mathbb{M} \models R_{n,\zeta}[a_0, \ldots, a_{k-1}, a, a_{k+1}, \ldots, a_{n-1}]\}$$

is countable.

Without loss of generality we may also require that (for $\zeta = \zeta(v)$, $n = |v|$)

($\circledast$)$_{\mathrm{f}}$ for every $b_0, \ldots, b_{n-1} < \lambda$

$$\text{if } \mathbb{M} \models R_{n,\zeta}[b_0, \ldots, b_{n-1}] \text{ then } b_0 < \ldots < b_{n-1}.$$

Now we will define a forcing notion $\mathbb{P}$. *A condition* p in $\mathbb{P}$ is a tuple

$$\left(w^p, n^p, \iota^p, M^p, \bar{\eta}^p, \bar{t}^p, \bar{r}^p, \bar{h}^p, \bar{g}^p, \mathcal{M}^p\right) = \left(w, n, \iota, M, \bar{\eta}, \bar{t}, \bar{r}, \bar{h}, \bar{g}, \mathcal{M}\right)$$

such that the following demands $(*)_1$–$(*)_{11}$ are satisfied.

$(*)_1$ $w \in [\lambda]^{<\omega}$, $|w| \geq 5$, $5 \leq n$, $M < \omega$, $\iota < \omega$ and if $i^* < \omega$ then $\iota = i^*$.

$(*)_2$ $\bar{\eta} = \langle \eta_\alpha : \alpha \in w \rangle \subseteq {}^n 2$.

$(*)_3$ $\bar{t} = \langle t_m : m < M \rangle$, where $\emptyset \neq t_m \subseteq {}^{n \geq} 2$ for $m < M$ is a tree in which all terminal branches are of length n and $t_m \cap t_{m'} \cap {}^n 2 = \emptyset$ for $m < m' < M$.

$(*)_4$ $\bar{r} = \langle r_m : m < M \rangle$, where $0 < r_m \leq n$ for $m < M$.

$(*)_5$ $\bar{h} = \langle h_i : i < \iota \rangle$, where $h_i : w^{\langle 2 \rangle} \longrightarrow M$ are such that $h_i(\alpha, \beta) = h_i(\beta, \alpha)$.

$(*)_6$ $\bar{g} = \langle g_i : i < \iota \rangle$, where $g_i : w^{\langle 2 \rangle} \longrightarrow \mathcal{O}_i$ are such that $\ell(g_i(\alpha, \beta)) = n$, $g_i(\alpha, \beta) = g_i(\beta, \alpha)$ and, for each $(\alpha, \beta) \in w^{\langle 2 \rangle}$, $\left|\bigcup\limits_{i<\iota} g_i(\alpha, \beta)\right| \geq 6$.

$(*)_7$ For each $m < M$,

$$t_m \cap {}^n 2 = \bigcup \left\{ \eta_\alpha + g_i(\alpha, \beta) : (\alpha, \beta) \in w^{\langle 2 \rangle} \text{ and } i < \iota \text{ and } h_i(\alpha, \beta) = m \right\}.$$

$(*)_8$ The family

$$\left\{ \eta_\alpha : \alpha \in w \right\} \cup \bigcup \left\{ g_i(\alpha, \beta) : (\alpha, \beta) \in w^{\langle 2 \rangle} \wedge i < \iota \right\}$$

is a linearly independent set of vectors in ${}^n 2$ (over the field $\mathbb{Z}_2$); in particular there are no repetitions in the representation above and all elements are non-zero vectors.

$(*)_9$ $\mathcal{M}$ consists of all triples $\mathfrak{d} = (\ell^\mathfrak{d}, v^\mathfrak{d}, \mathbf{m}^\mathfrak{d}) = (\ell, v, \mathbf{m})$ such that
   $(*)_9^a$ $0 < \ell \le n$, $v \subseteq w$, $5 \le |v|$, and $\eta_\alpha \restriction \ell \ne \eta_\beta \restriction \ell$ for distinct $\alpha, \beta \in v$,
   $(*)_9^b$ $\mathbf{m} \in \mathbf{M}^n_{\bar{t}, \bar{\mathcal{O}}^6}$, $\ell^{\mathbf{m}} = \ell$, $u^{\mathbf{m}} = \{ \eta_\alpha \restriction \ell : \alpha \in v \}$,
   $(*)_9^c$ for each $(\alpha, \beta) \in (v)^{\langle 2 \rangle}$ and $i < 6$ we have $r_{h_i^{\mathbf{m}}(\eta_\alpha \restriction \ell, \eta_\beta \restriction \ell)} \le \ell^\mathfrak{d}$,
   $(*)_9^d$ $\left(\forall (\alpha, \beta) \in v^{\langle 2 \rangle} \right) \left(\forall i < 6 \right) \left(\exists j < \iota \right) \left(h_i^{\mathbf{m}}(\eta_\alpha \restriction \ell, \eta_\beta \restriction \ell) = h_j(\alpha, \beta) \right)$.

$(*)_{10}$ If $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}$, $\ell^{\mathfrak{d}_0} = \ell^{\mathfrak{d}_1} = \ell$, $\rho \in {}^\ell 2$, and $\mathbf{m}^{\mathfrak{d}_1} = \mathbf{m}^{\mathfrak{d}_0} + \rho$, then $\operatorname{rk}(v^{\mathfrak{d}_0}) = \operatorname{rk}(v^{\mathfrak{d}_1})$, $\zeta(v^{\mathfrak{d}_0}) = \zeta(v^{\mathfrak{d}_1})$, $k(v^{\mathfrak{d}_0}) = k(v^{\mathfrak{d}_0})$ and if $\alpha \in v^{\mathfrak{d}_0}$, $\beta \in v^{\mathfrak{d}_1}$ are such that $|\alpha \cap v^{\mathfrak{d}_0}| = k(v^{\mathfrak{d}_0}) = k(v^{\mathfrak{d}_1}) = |\beta \cap v^{\mathfrak{d}_1}|$, then $(\eta_\alpha \restriction \ell) + \rho = \eta_\beta \restriction \ell$.

$(*)_{11}$ Suppose that
   - $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}$, $\mathbf{m}^{\mathfrak{d}_0} \sqsubset \mathbf{m}^{\mathfrak{d}_1}$ and $v^{\mathfrak{d}_0} \subseteq v^{\mathfrak{d}_1}$, and
   - $\alpha_0 \in v^{\mathfrak{d}_0}$, $|\alpha_0 \cap v^{\mathfrak{d}_0}| = k(v^{\mathfrak{d}_0})$, $\operatorname{rk}(v^{\mathfrak{d}_0}) = -1$.

   Then $\left| \{ \nu \in u^{\mathbf{m}^{\mathfrak{d}_1}} : (\eta_{\alpha_0} \restriction \ell^{\mathfrak{d}_0}) \trianglelefteq \nu \} \right| = 1$.

To define *the order $\le$ of $\mathbb{P}$* we declare for $p, q \in \mathbb{P}$ that $p \le q$ if and only if
   - $w^p \subseteq w^q$, $n^p \le n^q$, $M^p \le M^q$, $\iota^p \le \iota^q$ and
   - $t_m^p = t_m^q \cap {}^{n^p \ge} 2$ and $r_m^p = r_m^q$ for all $m < M^p$, and
   - $\eta_\alpha^p \trianglelefteq \eta_\alpha^q$ for all $\alpha \in w^p$, and
   - $h_i^q \restriction (w^p)^{\langle 2 \rangle} = h_i^p$ and $g_i^p(\alpha, \beta) \preceq_i g_i^q(\alpha, \beta)$ for $i < \iota^p$ and $(\alpha, \beta) \in (w^p)^{\langle 2 \rangle}$.

Claim 4.4.1. (1) $(\mathbb{P}, \le)$ *is a partial order of size* λ.
   (2) *For each* $\beta < \lambda$ *and* $n_0, M_0 < \omega$ *the set*

$$D_\beta^{n_0, M_0} = \left\{ p \in \mathbb{P} : n^p > n_0 \wedge M^p > M_0 \wedge \beta \in w^p \right\}$$

   is open dense in $\mathbb{P}$.
   (3) *If* $i^* = \omega$, *then for each* $\iota < \omega$ *the set* $D_\iota = \{ p \in \mathbb{P} : \iota^p \ge \iota \}$ *is open dense in* $\mathbb{P}$.

Proof of the Claim. (1) First let us argue that $\mathbb{P} \ne \emptyset$. Let $\iota = i^*$ if it is finite, and $\iota = 6$ if $i^* = \omega$. Let $w = \{ \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \}$ be any 5 element subset of λ. Using 3.2(2c)+3.6(ii) we may find $v(i, b)$ for $i < \iota$ and $b < 2$ such that for some $\ell < \omega$ for all $i < \iota$ and $b < 2$ we have

$$v(i, b) \in \mathcal{O}_i, \quad v(i, 0) \prec_i v(i, 1), \quad \text{and} \quad \ell(v(i, 1)) = \ell.$$

By 3.6(i), we may also require that if $i^* < 6$ then for some $i < \iota$ we have $|v(i, 1)| \ge 6$. Fix an enumeration

$$\left\{ (\sigma_a, i_a, j_a, k_a) : a < A \right\} = \left\{ (\sigma, i, j, k) : j < k < 5 \wedge i < \iota \wedge \sigma \in v(i, 1) \right\}.$$

Choose $n > \ell + 5$ and a sequence $\langle \rho_a : a < A + 5 \rangle \subseteq {}^n 2$ so that

- $\langle \rho_a {\upharpoonright} [\ell, n) : a < A + 5 \rangle$ is linearly independent in ${}^{[\ell,n)} 2$ over $\mathbb{Z}_2$, and
- $\sigma_a \lhd \rho_a$ for each $a < A$.

Put

- $\eta_{\alpha_b} = \rho_{A+b}$ (for $b < 5$) and $\bar\eta = \langle \eta_{\alpha_b} : b < 5 \rangle$,
- $g_i(\alpha_j, \alpha_k) = g_i(\alpha_k, \alpha_j) = \{ \rho_a : a < A \ \wedge \ j = j_a \ \wedge \ k = k_a \ \wedge \ i_a = i \}$ (for $i < \iota$ and $j < k < 5$) and $\bar g = \langle g_i : i < \iota \rangle$.

It follows from Definition 3.6(iii) that $g_i(\alpha_j, \alpha_k) \in \mathcal{O}_i$.

We also let $M = 10 \cdot \iota$ and we fix a bijection $\varphi : [w]^2 \times \iota \longrightarrow M$. Then for $j < k < 5$ and $i < \iota$ we set $h_i(\alpha_j, \alpha_k) = h_i(\alpha_k, \alpha_j) = \varphi(\{\alpha_j, \alpha_k\}, i)$. This way we defined $\bar h = \langle h_i : i < \iota \rangle$.

We put $r_m = n$ for $m < M$ and we let $t_m \subseteq {}^{n \geq} 2$ be trees in which all terminal branches are of length n and such that

$$t_m \cap {}^n 2 = \bigcup \{ \eta_\alpha + g_i(\alpha, \beta) : (\alpha, \beta) \in w^{\langle 2 \rangle} \text{ and } i < \iota \text{ and } h_i(\alpha, \beta) = m \}.$$

Finally, $\mathcal{M}$ is defined by clause $(*)_9$.

One easily verifies that $(w, n, \iota, M, \bar\eta, \bar t, \bar r, \bar h, \bar g, \mathcal{M}) \in \mathbb{P}$.

We see from the arguments above that $|\mathbb{P}| \geq \lambda$ and since there are only countably many elements p of $\mathbb{P}$ with $w^p = w$, we get $|\mathbb{P}| = \lambda$.

Clearly, $\leq$ is a partial order on $\mathbb{P}$.

(2) Let $p \in \mathbb{P}$, $\beta \in \lambda \setminus w^p$.

We will define a condition q in a manner similar to the construction in (1) above. Let $\alpha^- = \min(w^p)$ and $\alpha^+ = \max(w^p)$.

Set $w^q = w^p \cup \{\beta\}$, $\iota^q = \iota^p$.

For $(\alpha_0, \alpha_1) \in (w^q)^{\langle 2 \rangle}$ and $i < \iota^q$ pick $v(i, \alpha_0, \alpha_1) \in \mathcal{O}_i$ so that: for some ℓ, for all $i < \iota^q$ and $(\alpha_0, \alpha_1) \in (w^q)^{\langle 2 \rangle}$ we have

- $\ell(v(i, \alpha_0, \alpha_1)) = \ell$,
- if $\alpha_0, \alpha_1 \in w^p$ then $g_i^p(\alpha_0, \alpha_1) \prec_i v(i, \alpha_0, \alpha_1) = v(i, \alpha_1, \alpha_0)$,
- if $\alpha_0 \in w^p$ then $g_i^p(\alpha^+, \alpha^-) \prec_i v(i, \alpha_0, \beta) = v(i, \beta, \alpha_0)$.

Fix an enumeration

$$\{ (\sigma^a, i^a, \alpha_0^a, \alpha_1^a) : a < A \} = \{ (\sigma, i, \alpha_0, \alpha_1) : \alpha_0 < \alpha_1 \text{ are from } w^q \text{ and } i < \iota^q \ \wedge \ \sigma \in v(i, \alpha_0, \alpha_1) \}.$$

Choose $n > \ell + |w^p| + 1$ and a sequence $\langle \rho_a : a \leq A + |w^p| \rangle \subseteq {}^n 2$ so that

- $\langle \rho_a {\upharpoonright} [\ell, n) : a \leq A + |w^p| \rangle$ is linearly independent in ${}^{[\ell,n)} 2$ over $\mathbb{Z}_2$,
- $\sigma^a \lhd \rho_a$ for each $a < A$, and
- if $\alpha \in w^p$ is such that $|w^p \cap \alpha| = k$ then $\eta_\alpha^p \lhd \rho_{A+k}$.

Put

- $\eta_\beta^q = \rho_{A+|w^p|}$, and if $\alpha \in w^p$ is such that $|w^p \cap \alpha| = k$ then $\eta_\alpha^q = \rho_{A+k}$ and $\bar\eta^q = \langle \eta_\alpha^q : \alpha \in w^q \rangle$,
- $g_i^q(\alpha_0, \alpha_1) = g_i^q(\alpha_1, \alpha_0) = \{ \rho_a : a < A \wedge i = i^a \wedge \alpha_0 = \alpha_0^a \wedge \alpha_1 = \alpha_1^a \}$ (for $i < \iota^q$ and $\alpha_0 < \alpha_1$ from w^q) and $\bar g^q = \langle g_i^q : i < \iota^q \rangle$.

It follows from Definition 3.6(iii) that $g_i^q(\alpha_0, \alpha_1) \in \mathcal{O}_i$ and if $(\alpha_0, \alpha_1) \in (w^p)^{\langle 2 \rangle}$ then $g_i^p(\alpha_0, \alpha_1) \prec_i g_i^q(\alpha_0, \alpha_1)$.

We also let $M^q = M^p + \iota^q \cdot |w^p|$ and we define mappings $h_i^q : (w^q)^{\langle 2 \rangle} \longrightarrow M^q$ so that:

- if $(\alpha_0, \alpha_1) \in (w^p)^{\langle 2 \rangle}$ and $i < \iota^q$, then $h_i^q(\alpha_0, \alpha_1) = h_i^p(\alpha_0, \alpha_1)$,
- if $\alpha \in w^p$ and $i < \iota^q$, then $h_i^q(\alpha, \beta) = h_i^q(\beta, \alpha) = M^p + |\alpha \cap w^p| \cdot \iota + i$.

This way we defined $\bar{h}^q = \langle h_i^q : i < \iota^q \rangle$.

We put $r_m^q = r_m^p$ for $m < M^p$ and $r_m^q = n$ for $M^p \le m < M^q$. We let $t_m^q \subseteq {}^{n \ge 2}$ be trees in which all terminal branches are of length n and such that

$$t_m^q \cap {}^n 2 = \bigcup \left\{ \eta_\alpha^q + g_i^q(\alpha, \beta) : (\alpha, \beta) \in (w^q)^{\langle 2 \rangle} \text{ and } i < \iota^q \text{ and } h_i^q(\alpha, \beta) = m \right\}.$$

[Note that by our definitions above and by clause $(*)_7$ for p we have $t_m^p \cap {}^{n^p} 2 = t_m^q \cap {}^{n^p} 2$ for all $m < M^p$.] Naturally we also set $n^q = n$ and we define $\mathcal{M}^q$ by clause $(*)_9$.

We claim that $q = \left(w^q, n^q, \iota^q, M^q, \bar{\eta}^q, \bar{t}^q, \bar{r}^q, \bar{h}^q, \bar{g}^q, \mathcal{M}^q \right) \in \mathbb{P}$. Demands $(*)_1$–$(*)_9$ are pretty straightforward.

RE $(*)_{10}$: To justify clause $(*)_{10}$, suppose that $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}^q$, $\ell^{\mathfrak{d}_0} = \ell^{\mathfrak{d}_1} = \ell$, $\rho \in {}^\ell 2$ and $\mathbf{m} = \mathbf{m}^{\mathfrak{d}_0} = \mathbf{m}^{\mathfrak{d}_1} + \rho$, and consider the following two cases.

CASE 1: $\beta \notin v^{\mathfrak{d}_0} \cup v^{\mathfrak{d}_1}$
If $\ell \le n^p$ then $r_{h_i^{\mathbf{m}}(\eta_{\alpha_0} \restriction \ell, \eta_{\alpha_1} \restriction \ell)} \le n^p$, so $h_i^{\mathbf{m}}(\eta_{\alpha_0} \restriction \ell, \eta_{\alpha_1} \restriction \ell) < M^p$ for all $(\alpha_0, \alpha_1) \in (v^{\mathfrak{d}_0})^{\langle 2 \rangle}$. Hence also $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}^p$ and clause $(*)_{10}$ for p applies. If $\ell > n^p$ then the sequence $\langle \eta_\alpha^q \restriction \ell : \alpha \in v^{\mathfrak{d}_0} \cup v^{\mathfrak{d}_1} \rangle$ is linearly independent and

$$\{ (\eta_\alpha^q \restriction \ell) + \rho : \alpha \in v^{\mathfrak{d}_0} \} = \{ \eta_\alpha^q \restriction \ell : \alpha \in v^{\mathfrak{d}_1} \}.$$

Since $|v^{\mathfrak{d}_0}| \ge 5$ we immediately conclude $\rho = \mathbf{0}$, and therefore also $v^{\mathfrak{d}_0} = v^{\mathfrak{d}_1}$ (remember $\ell > n^p$).

CASE 2: $\beta \in v^{\mathfrak{d}_0} \cup v^{\mathfrak{d}_1}$
Say, $\beta \in v^{\mathfrak{d}_0}$. If $\alpha \in v^{\mathfrak{d}_0} \setminus \{\beta\}$, then $h_j^q(\alpha, \beta) \ge M^p$ for all $j < \iota$, and hence $r_{h_i^{\mathbf{m}}(\eta_\alpha \restriction \ell, \eta_\beta \restriction \ell)}^q = n^q$ (remember $(*)_9^d$). Consequently, $\ell = n^q$. Since the sequence $\langle \eta_\alpha^q : \alpha \in v^{\mathfrak{d}_0} \cup v^{\mathfrak{d}_1} \rangle$ is linearly independent, like before we get $\rho = \mathbf{0}$ and $v^{\mathfrak{d}_0} = v^{\mathfrak{d}_1}$.

RE $(*)_{11}$: Assume towards contradiction that for some $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}^q$ we have:

- $v_0^{\mathfrak{d}} \subseteq v_1^{\mathfrak{d}}$ and without loss of generality $|v^{\mathfrak{d}_1}| = |v^{\mathfrak{d}_0}| + 1$,
- $\alpha_0 \in v^{\mathfrak{d}_0}$, $|\alpha_0 \cap v^{\mathfrak{d}_0}| = k(v^{\mathfrak{d}_0})$, $\text{rk}(v^{\mathfrak{d}_0}) = -1$, and $\mathbf{m}^{\mathfrak{d}_0} \sqsubset \mathbf{m}^{\mathfrak{d}_1}$, and
- there is $\alpha_1 \in v^{\mathfrak{d}_1}$ such that $\eta_{\alpha_0}^q \restriction \ell^{\mathfrak{d}_0} = \eta_{\alpha_1}^q \restriction \ell^{\mathfrak{d}_0}$ but $\eta_{\alpha_0}^q \restriction \ell^{\mathfrak{d}_1} \ne \eta_{\alpha_1}^q \restriction \ell^{\mathfrak{d}_1}$.

Let $\ell_0 = \ell^{\mathfrak{d}_0}$, $\ell_1 = \ell^{\mathfrak{d}_1}$.

Suppose $\beta \in v^{\mathfrak{d}_0}$ and take $\beta' \in v^{\mathfrak{d}_0} \setminus \{\beta\}$. Then $h_j^q(\beta, \beta') \ge M^p$ for all $j < \iota$. Hence, for some $j < \iota$,

$$r_{h_0^{\mathbf{m}^{\mathfrak{d}_0}}(\eta_\beta \restriction \ell_0, \eta_{\beta'} \restriction \ell_0)}^q = r_{h_j^p(\beta, \beta')}^q = n^q = \ell_0 = \ell_1,$$

contradicting the last item in our assumptions.

If we had $v^{\mathfrak{d}_1} = v^{\mathfrak{d}_0} \cup \{\beta\}$, then considering a $\beta' \in v^{\mathfrak{d}_0} \setminus \{\alpha_0\}$ we will immediately arrive to

$$M^p > h_0^{\mathbf{m}^{\mathfrak{d}_0}}(\eta_\alpha \restriction \ell_0, \eta_{\beta'} \restriction \ell_0) = h_0^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\beta \restriction \ell_1, \eta_{\beta'} \restriction \ell_1) \ge M^p,$$

a contradiction.

Therefore the only remaining possibility is that $\beta \notin v^{\mathfrak{d}_1}$.

If $\ell_1 \leq n^p$, then $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}^p$ and clause $(*)_{11}$ for p gives us a contradiction. So assume $\ell_1 > n^p$. Since $\{\eta_\gamma^q \restriction n^p : \gamma \in v^{\mathfrak{d}_1}\}$ are all pairwise distinct, we conclude $\ell_0 < n^p$ and $\mathbf{m}^{\mathfrak{d}_0} \in \mathcal{M}^p$. We define $\mathbf{n} \in \mathbf{M}^n_{\bar{t},\bar{\mathcal{O}}^6}$ by setting:

- $\ell^{\mathbf{n}} = n^p$, $u^{\mathbf{n}} = \{\eta_\gamma^q \restriction n^p : \gamma \in v^{\mathfrak{d}_1}\} = \{\eta_\gamma^p : \gamma \in v^{\mathfrak{d}_1}\}$, $\iota^{\mathbf{n}} = 6$,
  and for $(\gamma, \gamma') \in (v^{\mathfrak{d}_1})^{\langle 2 \rangle}$ and $i < 6$:
- if $\{\gamma, \gamma'\} \neq \{\alpha_0, \alpha_1\}$, then

$$g_i^{\mathbf{n}}(\eta_\gamma^p, \eta_{\gamma'}^p) = \{\sigma \restriction n^p : \sigma \in g_i^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\gamma^q \restriction \ell_1, \eta_{\gamma'}^q \restriction \ell_1)\}$$

  and $h_i^{\mathbf{n}}(\eta_\gamma^p, \eta_{\gamma'}^p) = h_i^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\gamma^q \restriction \ell_1, \eta_{\gamma'}^q \restriction \ell_1)$,
- if $\{\gamma, \gamma'\} = \{\alpha_0, \alpha_1\}$, then we fix distinct $\sigma_0, \ldots, \sigma_5 \in \bigcup_{j < \iota^q} g_j^p(\alpha_0, \alpha_1)$

  (remember $(*)_6$ for p), and we let $g_i^{\mathbf{n}}(\eta_{\alpha_0}^p, \eta_{\alpha_1}^p) = g_i^{\mathbf{n}}(\eta_{\alpha_1}^p, \eta_{\alpha_0}^p) = \{\sigma_i\}$
  and $h_i^{\mathbf{n}}(\eta_{\alpha_0}^p, \eta_{\alpha_1}^p) = h_i^{\mathbf{n}}(\eta_{\alpha_1}^p, \eta_{\alpha_0}^p) = m$ where $\eta_{\alpha_0}^p + \sigma_i, \eta_{\alpha_1}^p + \sigma_i \in t_m^p$
  (for $i < 6$).

Since $\mathbf{m}^{\mathfrak{d}_0} \sqsubset \mathbf{m}^{\mathfrak{d}_1}$, in the case when $\{\gamma, \gamma'\} \neq \{\alpha_0, \alpha_1\}$ we have

$$g^{\mathbf{m}^{\mathfrak{d}_0}}(\eta_\gamma^p \restriction \ell_0, \eta_{\gamma'}^p \restriction \ell_0) \prec_{\mathcal{O}^0} g_i^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\gamma^p \restriction \ell_1, \eta_{\gamma'}^p \restriction \ell_1)$$

and hence $g_i^{\mathbf{n}}(\eta_\gamma^p, \eta_{\gamma'}^p) \cap g_j^{\mathbf{n}}(\eta_\gamma^p, \eta_{\gamma'}^p) = \emptyset$ whenever $i < j < 6$. Hence 3.10(c) is satisfied. Other cases and other conditions of 3.10 follow immediately by our choices, and hence

$$\mathbf{n} = (n^p, 6, u^{\mathbf{n}}, \bar{h}^{\mathbf{n}}, \bar{g}^{\mathbf{n}}) \in \mathbf{M}^n_{\bar{t},\bar{\mathcal{O}}^6}.$$

Moreover, $\mathbf{m}^{\mathfrak{d}_0} \sqsubset \mathbf{n}$ and $\mathfrak{d}_* = (n^p, v^{\mathfrak{d}_1}, \mathbf{n}) \in \mathcal{M}^p$. However, then $\mathfrak{d}_0, \mathfrak{d}_*$ contradict clause $(*)_{11}$ for p.

(3)   Let $p \in \mathbb{P}$. Set $w^q = w^p$ and $\iota^q = \iota^p + 1$. For $(\alpha_0, \alpha_1) \in (w^q)^{\langle 2 \rangle}$ and $i < \iota^q$ we use Proposition 3.8 to pick $v(i, \alpha_0, \alpha_1) \in \mathcal{O}_i$ so that: for some ℓ, for all $i < \iota^q$ and $(\alpha_0, \alpha_1) \in (w^q)^{\langle 2 \rangle}$ we have

- $\ell(v(i, \alpha_0, \alpha_1)) = \ell$,
- if $i < \iota^p$ then $g_i^p(\alpha_0, \alpha_1) \prec_i v(i, \alpha_0, \alpha_1) = v(i, \alpha_1, \alpha_0)$,
- for some $v \in \mathcal{O}_{\iota^p}$, $v \prec_{\iota^p} v(\iota^p, \alpha_0, \alpha_1) = v(\iota^p, \alpha_1, \alpha_0)$.

Fix an enumeration

$$\{(\sigma^a, i^a, \alpha_0^a, \alpha_1^a) : a < A\} = \{(\sigma, i, \alpha_0, \alpha_1) : \alpha_0 < \alpha_1 \text{ are from } w \text{ and}$$
$$i < \iota^q \wedge \sigma \in v(i, \alpha_0, \alpha_1)\}.$$

Choose $n = n^q > \ell$ and a sequence $\langle \rho_a : a < A + |w^p| \rangle \subseteq {}^n 2$ so that

- $\langle \rho_a \restriction [\ell, n) : a < A + |w^q| \rangle$ is linearly independent in $^{[\ell,n)}2$ over $\mathbb{Z}_2$, and
- $\sigma^a \lhd \rho_a$ for each $a < A$, and
- if $\alpha \in w^q$ is such that $|w^q \cap \alpha| = k$ then $\eta_\alpha^p \lhd \rho_{A+k}$.

Put

- if $\alpha \in w^q$ is such that $|w^q \cap \alpha| = k$ then $\eta_\alpha^q = \rho_{A+k}$ and $\bar{\eta}^q = \langle \eta_\alpha^q : \alpha \in w^q \rangle$,

- $g_i^q(\alpha_0, \alpha_1) = g_i^q(\alpha_1, \alpha_0) = \{\rho_a : a < A \wedge i = i^a \wedge \alpha_0 = \alpha_0^a \wedge \alpha_1 = \alpha_1^a\}$
  (for $i < \iota^q$ and $\alpha_0 < \alpha_1$ from w^q) and $\bar{g}^q = \langle g_i^q : i < \iota^q \rangle$.

It follows from Definition 3.6(iii) that $g_i^q(\alpha_0, \alpha_1) \in \mathcal{O}_i$ and if $(\alpha_0, \alpha_1) \in (w^p)^{\langle 2 \rangle}$ then $g_i^p(\alpha_0, \alpha_1) \prec_i q_i^q(\alpha_0, \alpha_1)$.

We also let $M^q = M^p + |[w^q]^2|$ and we fix a bijection $\psi : [w^q]^2 \longrightarrow [M^p, M^q)$. Then we define mappings $h_i^q : (w^q)^{\langle 2 \rangle} \longrightarrow M^q$ so that for $\alpha_0 < \alpha_1$ from w^q we have

- if $i < \iota^q$, then $h_i^q(\alpha_0, \alpha_1) = h_i^q(\alpha_1, \alpha_0) = h_i^p(\alpha_0, \alpha_1)$,
- $h_{\iota^p}^q(\alpha_0, \alpha_1) = h_{\iota^p}^q(\alpha_1, \alpha_0) = \psi(\{\alpha_0, \alpha_1\})$.

This way we defined $\bar{h}^q = \langle h_i^q : i < \iota^q \rangle$.

We put $r_m^q = r_m^p$ for $m < M^p$ and $r_m^q = n$ for $M^p \leq m < M^q$. We let $t_m^q \subseteq {}^{n \geq} 2$ be trees in which all terminal branches are of length n and such that

$$t_m^q \cap {}^n 2 = \bigcup \{\eta_\alpha^q + g_i^q(\alpha, \beta) : (\alpha, \beta) \in (w^q)^{\langle 2 \rangle} \text{ and } i < \iota^q \text{ and } h_i^q(\alpha, \beta) = m\}.$$

[Note that by our definitions above and by clause $(*)_7$ for p we have $t_m^p \cap {}^{n^p} 2 = t_m^q \cap {}^{n^p} 2$ for all $m < M^p$.] We define $\mathcal{M}^q$ by clause $(*)_9$. Like previously, one easily verifies that $q = (w^q, n^q, \iota^q, M^q, \bar{\eta}^q, \bar{t}^q, \bar{r}^q, \bar{h}^q, \bar{g}^q, \mathcal{M}^q) \in \mathbb{P}$. [The crucial point is that if $\mathfrak{d} \in \mathcal{M}^q$, $\eta, \nu \in u^{\mathbf{m}^{\mathfrak{d}}}$ and $h_i^{\mathbf{m}^{\mathfrak{d}}}(\eta, \nu) \geq M^p$, then $\ell^{\mathfrak{d}} = n^q$.] $\qquad \square$

Claim 4.4.2. *The forcing notion $\mathbb{P}$ has the Knaster property.*

Proof of the Claim. Suppose that $\langle p_\xi : \xi < \omega_1 \rangle$ is a sequence of pairwise distinct conditions from $\mathbb{P}$ and let

$$p_\xi = (w_\xi, n_\xi, \iota_\xi, M_\xi, \bar{\eta}_\xi, \bar{t}_\xi, \bar{r}_\xi, \bar{h}_\xi, \bar{g}_\xi, \mathcal{M}_\xi)$$

where $\bar{\eta}_\xi = \langle \eta_\alpha^\xi : \alpha \in w_\xi \rangle$, $\bar{t}_\xi = \langle t_m^\xi : m < M_\xi \rangle$, $\bar{r}_\xi = \langle r_m^\xi : m < M_\xi \rangle$, and $\bar{h}_\xi = \langle h_i^\xi : i < \iota_\xi \rangle$, $\bar{g}_\xi = \langle g_i^\xi : i_\xi < \iota \rangle$. By a standard Δ–system cleaning procedure we may find an uncountable set $A \subseteq \omega_1$ such that the following demands $(*)_{12}$–$(*)_{15}$ are satisfied.

$(*)_{12}$ $\{w_\xi : \xi \in A\}$ forms a Δ–system with the kernel w^*.

$(*)_{13}$ If $\xi, \varsigma \in A$, then $|w_\xi| = |w_\varsigma|$, $n_\xi = n_\varsigma$, $\iota_\xi = \iota_\varsigma$, $M_\xi = M_\varsigma$, and $t_m^\xi = t_m^\varsigma$ and $r_m^\xi = r_m^\varsigma$ (for $m < M_\xi$).

$(*)_{14}$ If $\xi < \varsigma$ are from A and $\pi : w_\xi \longrightarrow w_\varsigma$ is the order isomorphism, then

  (a) $\pi(\alpha) = \alpha$ for $\alpha \in w^* = w_\xi \cap w_\varsigma$,
  (b) if $\emptyset \neq v \subseteq w_\xi$, then $\mathrm{rk}(v) = \mathrm{rk}(\pi[v])$, $\zeta(v) = \zeta(\pi[v])$ and $k(v) = k(\pi[v])$,
  (c) $\eta_\alpha^\xi = \eta_{\pi(\alpha)}^\varsigma$ (for $\alpha \in w_\xi$),
  (d) $g_i^\xi(\alpha, \beta) = g_i^\varsigma(\pi(\alpha), \pi(\beta))$ and $h_i^\xi(\alpha, \beta) = h_i^\varsigma(\pi(\alpha), \pi(\beta))$ for $(\alpha, \beta) \in (w_\xi)^{\langle 2 \rangle}$ and $i < \iota_\xi$, and

$(*)_{15}$ $\mathcal{M}_\xi = \mathcal{M}_\varsigma$ (this actually follows from the previous demands).

Note that then also

$(*)_{16}$ if $\xi \in A$, $v \subseteq w^*$ and $\delta \in w_\xi \setminus w^*$ are such that $\mathrm{rk}(v \cup \{\delta\}) = -1$, then $k(v \cup \{\delta\}) \neq |\delta \cap v|$.

[Why? Suppose $\mathrm{rk}\big(v\cup\{\delta\}\big) = -1$ and $k = k\big(v\cup\{\delta\}\big) = |\delta\cap v|$, $j = j\big(v\cup\{\delta\}\big)$. For $\varsigma \in A$ let $\pi_\varsigma : w_\xi \longrightarrow w_\varsigma$ be the order isomorphism and let $\delta_\varsigma = \pi_\varsigma(\delta)$. By $(*)_{14}$ we know that $k = k\big(v \cup \{\delta_\varsigma\}\big) = |\delta_\varsigma \cap v|$ and $j = j\big(v \cup \{\delta_\varsigma\}\big)$. Therefore, letting $v \cup \{\delta\} = \{a_0, \ldots, a_{n-1}\}$ be the increasing enumeration, for every $\varsigma \in A$ we have $\mathbb{M} \models R_{n,j}[a_0, \ldots, a_{k-1}, \delta_\varsigma, a_{k+1}, \ldots, a_{n-1}]$. Hence the set

$$\{b < \lambda : \mathbb{M} \models R_{n,j}[a_0, \ldots, a_{k_1}, b, a_{k+1}, \ldots, a_{n-1}]\}$$

is uncountable, contradicting $(\circledast)_e$ from the beginning of the proof of the theorem.]

We will show that for distinct ξ, ς from A the conditions p_ξ, p_ς are compatible. So let $\xi, \varsigma \in A$, $\xi < \varsigma$ and let $\pi : w_\xi \longrightarrow w_\varsigma$ be the order isomorphism. We will define $q = \big(w, n, \iota, M, \bar\eta, \bar t, \bar r, \bar h, \bar g, \mathcal{M}\big)$ where $\bar\eta = \langle \eta_\alpha : \alpha \in w \rangle$, $\bar t = \langle t_m : m < M \rangle$, $\bar r = \langle r_m : m < M \rangle$, and $\bar h = \langle h_i : i < \iota \rangle$, $\bar g = \langle g_i : i < \iota \rangle$.

We set

$(*)_{17}$ $\iota = \iota_\xi$ and $w = w_\xi \cup w_\varsigma$.

Similarly to the arguments in previous claims, we first pick

$$\langle v(i, \alpha_0, \alpha_1) : (\alpha_0, \alpha_1) \in w^{\langle 2 \rangle} \ \wedge \ i < \iota \rangle$$

and an ℓ such that for all $i < \iota$ and $(\alpha_0, \alpha_1) \in w^{\langle 2 \rangle}$ we have

- $v(i, \alpha_0, \alpha_1) = v(i, \alpha_1, \alpha_0) \in \mathcal{O}_i$, $\ell\big(v(i, \alpha_0, \alpha_1)\big) = \ell$,
- if $\alpha_0, \alpha_1 \in w_\xi$ then $g_i^\xi(\alpha_0, \alpha_1) \prec_i v(i, \alpha_0, \alpha_1)$, and
- if $\alpha_0, \alpha_1 \in w_\varsigma$ then $g_i^\varsigma(\alpha_0, \alpha_1) \prec_i v(i, \alpha_0, \alpha_1)$.

(No other demands on $v(i, \alpha_0, \alpha_1)$ but symmetry if $\alpha_0 \in w_\xi \setminus w_\varsigma$ and $\alpha_1 \in w_\varsigma \setminus w_\xi$.) Then we fix an enumeration

$$\big\{(\sigma^a, i^a, \alpha_0^a, \alpha_1^a) : a < A\big\} = \big\{(\sigma, i, \alpha_0, \alpha_1) : \alpha_0 < \alpha_1 \text{ are from } w \text{ and}$$
$$i < \iota \ \wedge \ \sigma \in v(i, \alpha_0, \alpha_1)\big\}$$

and we choose $n > \ell$ and $\langle \rho_a : a < A + |w| \rangle \subseteq {}^n 2$ so that

- $\langle \rho_a{\upharpoonright}[\ell, n) : a < A + |w| \rangle$ is linearly independent in ${}^{[\ell,n)}2$ over $\mathbb{Z}_2$, and
- $\sigma^a \lhd \rho_a$ for each $a < A$, and
- if $\alpha \in w_\xi$ is such that $|w_\xi \cap \alpha| = k$ then $\eta_\alpha^\xi \lhd \rho_{A+k}$,
- if $\alpha \in w_\varsigma \setminus w_\xi$ is such that $|(w_\varsigma \setminus w_\xi) \cap \alpha| = k$ then $\eta_\alpha^\varsigma \lhd \rho_{A+|w_\xi|+k}$.

Put

$(*)_{18}$ n is the one chosen right above,

$(*)_{19}$ $\bar\eta = \langle \eta_\alpha : \alpha \in w \rangle$, where

- if $\alpha \in w_\xi$ is such that $|w_\xi \cap \alpha| = k$ then $\eta_\alpha = \rho_{A+k}$,
- if $\alpha \in w_\varsigma \setminus w_\xi$ is such that $|(w_\varsigma \setminus w_\xi) \cap \alpha| = k$ then $\eta_\alpha = \rho_{A+|w_\xi|+k}$,

$(*)_{20}$ $\bar g = \langle g_i : i < \iota \rangle$, where for $i < \iota$ and $\alpha_0 < \alpha_1$ from w we put

$$g_i(\alpha_0, \alpha_1) = g_i(\alpha_1, \alpha_0) = \big\{\rho_a : a < A \ \wedge \ i = i^a \ \wedge \ \alpha_0 = \alpha_0^a \ \wedge \ \alpha_1 = \alpha_1^a\big\}.$$

As before, by 3.6(iii), we know that $g_i(\alpha_0, \alpha_1) \in \mathcal{O}_i$ and if $(\alpha_0, \alpha_1) \in (w_\xi)^{\langle 2 \rangle}$ then $g_i^\xi(\alpha_0, \alpha_1) \prec_i g_i(\alpha_0, \alpha_1)$ and similarly for ς in place of ξ.

Let

$(*)_{21}$ $M = M_\xi + |w_\xi \setminus w_\varsigma|^2$

and let $\psi : (w_\xi \setminus w_\varsigma) \times (w_\varsigma \setminus w_\xi) \longrightarrow [M_\xi, M)$ be a bijection. Then we define

$(*)_{22}$ $\bar{h} = \langle h_i : i < \iota \rangle$, where mappings $h_i : w^{\langle 2 \rangle} \longrightarrow M$ are such that for distinct $\alpha_0, \alpha_1 \in w$ and $i < \iota$ we have
- $h_i(\alpha_0, \alpha_1) = h_i(\alpha_1, \alpha_0)$,
- if $\alpha_0, \alpha_1 \in w_\xi$, then $h_i(\alpha_1, \alpha_0) = h_i^\xi(\alpha_1, \alpha_0)$,
- if $\alpha_0, \alpha_1 \in w_\varsigma$, then $h_i(\alpha_1, \alpha_0) = h_i^\varsigma(\alpha_1, \alpha_0)$,
- if $\alpha_0 \in w_\xi \setminus w_\varsigma$ and $\alpha_1 \in w_\varsigma \setminus w_\xi$, then $h_i(\alpha_1, \alpha_0) = \psi(\alpha_0, \alpha_1)$.

$(*)_{23}$ $\bar{t} = \langle t_m : m < M \rangle$, where $t_m \subseteq {}^{n \geq 2}$ are trees in which all terminal branches are of length n and such that

$$t_m \cap {}^n 2 = \bigcup \left\{ \eta_\alpha + g_i(\alpha, \beta) : (\alpha, \beta) \in w^{\langle 2 \rangle} \text{ and } i < \iota \text{ and } h_i(\alpha, \beta) = m \right\},$$

$(*)_{24}$ $\bar{r} = \langle r_m : m < M \rangle$, where $r_m = r_m^\xi$ for $m < M_\xi$, $r_m = n$ if $M_\xi \leq m < M$.

$(*)_{25}$ $\mathcal{M}$ is defined by $(*)_9$ (for the objects introduced in $(*)_{17}$–$(*)_{24}$).

In clauses $(*)_{17}$–$(*)_{25}$ we defined all the ingredients of

$$q = \left(w, n, M, \bar{\eta}, \bar{t}, \bar{r}, \bar{h}, \bar{g}, \mathcal{M} \right).$$

We still need to argue that $q \in \mathbb{P}$ (after this it will be obvious that it is a condition stronger than both p_ξ and p_ς).

It is pretty straightforward that q satisfies demands $(*)_1$–$(*)_9$.

RE $(*)_{10}$ **:** To justify clause $(*)_{10}$, suppose that $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}$, $\ell^{\mathfrak{d}_0} = \ell^{\mathfrak{d}_1} = \ell$ and $\rho \in {}^\ell 2$ and $\mathbf{m}^{\mathfrak{d}_1} = \mathbf{m}^{\mathfrak{d}_0} + \rho$, and consider the following three cases.

CASE 1: $v^{\mathfrak{d}_0} \subseteq w_\xi$
Then for each $(\delta, \varepsilon) \in (v^{\mathfrak{d}_0})^{\langle 2 \rangle}$ and $i < \iota$ we have $h_i(\delta, \varepsilon) < M_\xi$, and consequently $\mathrm{rng}(h_j^{\mathbf{m}^{\mathfrak{d}_0}}) \subseteq M_\xi$ (for $j < 6$). Hence also $\mathrm{rng}(h_j^{\mathbf{m}^{\mathfrak{d}_1}}) \subseteq M_\xi$ (for $j < 6$). But looking at $(*)_{22}$ (and remembering $(*)_9^{\mathrm{d}}$) we now conclude $h_i(\delta, \varepsilon) < M_\xi$ for $(\delta, \varepsilon) \in (v^{\mathfrak{d}_1})^{\langle 2 \rangle}$ and $i < \iota$. Consequently, either $v^{\mathfrak{d}_1} \subseteq w_\xi$ or $v^{\mathfrak{d}_1} \subseteq w_\varsigma$.

If $v^{\mathfrak{d}_1} \subseteq w_\xi$ and $\ell \leq n_\xi$, then $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}_\xi$ and clause $(*)_{10}$ for p_ξ can be used to get the desired conclusion.

If $v^{\mathfrak{d}_1} \subseteq w_\xi$ and $\ell > n_\xi$, then $\{\eta_\alpha \restriction \ell : \alpha \in v^{\mathfrak{d}_0} \cup v^{\mathfrak{d}_1}\}$ is linearly independent and hence $\rho = \mathbf{0}$ and $v^{\mathfrak{d}_0} = v^{\mathfrak{d}_1}$.

If $v^{\mathfrak{d}_1} \subseteq w_\varsigma$ and $\ell \leq n_\xi$, then consider $v = \pi^{-1}[v^{\mathfrak{d}_1}] \subseteq w_\xi$ and $\mathfrak{d} = (\ell, v, \mathbf{m}^{\mathfrak{d}_1})$. Clearly, $\mathfrak{d} \in \mathcal{M}_\xi$ and we may use $(*)_{10}$ for p_ξ to conclude that $\mathrm{rk}(v) = \mathrm{rk}(v^{\mathfrak{d}_0})$, $\zeta(v) = \zeta(v^{\mathfrak{d}_0})$, $k(v) = k(v^{\mathfrak{d}_0})$, and if $\alpha \in v^{\mathfrak{d}_0}$, $\beta \in v$ are such that $|\alpha \cap v^{\mathfrak{d}_0}| = k(v^{\mathfrak{d}_0}) = k(v) = |\beta \cap v|$, then $(\eta_\alpha \restriction \ell) + \rho = \eta_\beta \restriction \ell$. Now we use the properties $(*)_{14}$(b,c) of π to get a similar assertions with $v^{\mathfrak{d}_1}$ in place of v.

If $v^{\mathfrak{d}_1} \subseteq w_\varsigma$ and $\ell > n_\xi$, then we consider $v = \pi^{-1}[v^{\mathfrak{d}_1}] \subseteq w_\xi$ and use the linear independence of $\{\eta_\alpha \restriction \ell : \alpha \in v^{\mathfrak{d}_0} \cup v\}$ to conclude that $\rho = \mathbf{0}$ and $v^{\mathfrak{d}_0} = v = \pi^{-1}[v^{\mathfrak{d}_1}]$. Finally we use the properties $(*)_{14}$(b,c) of π to get the desired assertions.

CASE 2: $v^{\mathfrak{d}_0} \subseteq w_\varsigma$
Same as the previous case, just interchanging ξ and ς.

CASE 3: $v^{\mathfrak{d}_0} \setminus w_\xi \neq \emptyset \neq v^{\mathfrak{d}_0} \setminus w_\varsigma$
Then for some $(\delta, \varepsilon) \in (v^{\mathfrak{d}_0})^{\langle 2 \rangle}$ we have $h_i(\delta, \varepsilon) \geq M_\xi$ for all $i < \iota$, so necessarily $\ell = n$. Now, the linear independence of $\bar{\eta}$ implies $\rho = \mathbf{0}$ and $v^{\mathfrak{d}_0} = v^{\mathfrak{d}_1}$ and the desired conclusion follows.

RE $(*)_{11}$: Let us prove clause $(*)_{11}$ now. Suppose that $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}$, $\delta \in v^{\mathfrak{d}_0}$, $|\delta \cap v^{\mathfrak{d}_0}| = k(v^{\mathfrak{d}_0})$, $\mathrm{rk}(v^{\mathfrak{d}_0}) = -1$, and $v^{\mathfrak{d}_0} \subseteq v^{\mathfrak{d}_1}$ and $\mathbf{m}^{\mathfrak{d}_0} \sqsubseteq \mathbf{m}^{\mathfrak{d}_1}$. Assume towards contradiction that there is an $\varepsilon \in v^{\mathfrak{d}_1}$ such that

$(*)_{26}$ $\eta_\varepsilon \restriction \ell^{\mathfrak{d}_1} \neq \eta_\delta \restriction \ell^{\mathfrak{d}_1}$ but $\eta_\varepsilon \restriction \ell^{\mathfrak{d}_0} = \eta_\delta \restriction \ell^{\mathfrak{d}_0}$.

Without loss of generality $v^{\mathfrak{d}_1} = v^{\mathfrak{d}_0} \cup \{\varepsilon\}$. Since we must have $\ell^{\mathfrak{d}_0} < n$, for no $\alpha, \beta \in v^{\mathfrak{d}_0}$ we can have $(\forall i < \iota)(h_i(\alpha, \beta) \geq M_\xi)$. Therefore either $v^{\mathfrak{d}_0} \subseteq w_\xi$ or $v^{\mathfrak{d}_0} \subseteq w_\varsigma$. By the symmetry, we may assume $v^{\mathfrak{d}_0} \subseteq w_\xi$. Note that

$(*)_{27}$ if $(\alpha, \beta) \in (v^{\mathfrak{d}_1})^{\langle 2 \rangle} \setminus \{(\varepsilon, \delta), (\delta, \varepsilon)\}$ then $h_i(\alpha, \beta) < M_\xi$ for all $i < \iota$.

Now, if $v^{\mathfrak{d}_1} \subseteq w_\xi$ and $\ell^{\mathfrak{d}_1} \leq n_\xi$, then $\mathfrak{d}_0, \mathfrak{d}_1 \in \mathcal{M}_\xi$ and they contradict clause $(*)_{11}$ for p_ξ. Let us consider the possibility that $v^{\mathfrak{d}_1} \subseteq w_\xi$ but $\ell^{\mathfrak{d}_1} > n_\xi$. Define $\mathbf{n} \in \mathbf{M}^n_{\bar{t}, \bar{\mathcal{O}}^6}$ by:

- $\ell^{\mathbf{n}} = n_\xi$, $u^{\mathbf{n}} = \{\eta_\gamma \restriction n_\xi : \gamma \in v^{\mathfrak{d}_1}\}$ (note $\eta_\varepsilon \restriction n_\xi \neq \eta_\delta \restriction n_\xi$), $\iota^{\mathbf{n}} = 6$, and for $(\gamma, \gamma') \in (v^{\mathfrak{d}_1})^{\langle 2 \rangle}$ and $i < 6$:
- if $\{\gamma, \gamma'\} \neq \{\varepsilon, \delta\}$, then

$$g_i^{\mathbf{n}}(\eta_\gamma \restriction n_\xi, \eta_{\gamma'} \restriction n_\xi) = \{\sigma \restriction n_\xi : \sigma \in g_i^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\gamma \restriction \ell^{\mathfrak{d}_1}, \eta_{\gamma'} \restriction \ell^{\mathfrak{d}_1})\}$$

and $h_i^{\mathbf{n}}(\eta_\gamma \restriction n_\xi, \eta_{\gamma'} \restriction n_\xi) = h_i^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\gamma \restriction \ell^{\mathfrak{d}_1}, \eta_{\gamma'} \restriction \ell^{\mathfrak{d}_1})$, and
- for $\{\gamma, \gamma'\} = \{\delta, \varepsilon\}$ we fix any distinct $\sigma_0, \ldots, \sigma_5 \in \bigcup_{j < \iota} g_j^\xi(\delta, \varepsilon)$ and we

let $g_i^{\mathbf{n}}(\eta_\delta \restriction n_\xi, \eta_\varepsilon \restriction n_\xi) = g_i^{\mathbf{n}}(\eta_\varepsilon \restriction n_\xi, \eta_\delta \restriction n_\xi) = \{\sigma_i\}$ and $h_i^{\mathbf{n}}(\eta_\delta \restriction n_\xi, \eta_\varepsilon \restriction n_\xi) = h_i^{\mathbf{n}}(\eta_\varepsilon \restriction n_\xi, \eta_\delta \restriction n_\xi) = m$ where $(\eta_\delta \restriction n_\xi) + \sigma_i, (\eta_\varepsilon \restriction n_\xi) + \sigma_i \in t_m^\xi$ (for $i < 6$).

Since $\mathbf{m}^{\mathfrak{d}_0} \sqsubseteq \mathbf{m}^{\mathfrak{d}_1}$, in the case when $\{\gamma, \gamma'\} \neq \{\delta, \varepsilon\}$ we have

$$g_i^{\mathbf{m}^{\mathfrak{d}_0}}(\eta_\gamma \restriction \ell^{\mathfrak{d}_0}, \eta_{\gamma'} \restriction \ell^{\mathfrak{d}_0}) \prec_{\mathcal{O}^0} g_i^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\gamma \restriction \ell^{\mathfrak{d}_1}, \eta_{\gamma'} \restriction \ell^{\mathfrak{d}_1}),$$

and hence $g_i^{\mathbf{n}}(\eta_\gamma^p, \eta_{\gamma'}^p) \cap g_j^{\mathbf{n}}(\eta_\gamma^p, \eta_{\gamma'}^p) = \emptyset$ whenever $i < j < 6$. Hence 3.10(c) is satisfied. Other cases and other conditions of 3.10 follow immediately by our choices, and hence

$$\mathbf{n} = (\ell^{\mathbf{n}}, 6, u^{\mathbf{n}}, \bar{h}^{\mathbf{n}}, \bar{g}^{\mathbf{n}}) \in \mathbf{M}^n_{\bar{t}, \bar{\mathcal{O}}^6}.$$

Moreover, $\mathbf{m}^{\mathfrak{d}_0} \sqsubseteq \mathbf{n}$ and $\mathfrak{d}_* = (n_\xi, v^{\mathfrak{d}_1}, \mathbf{n}) \in \mathcal{M}_\xi$. However, then $\mathfrak{d}_0, \mathfrak{d}_*$ contradict clause $(*)_{11}$ for p_ξ.

Consequently, $v^{\mathfrak{d}_1} \setminus w_\xi \neq \emptyset$, so necessarily $\varepsilon \notin w^*$.

Suppose $|v^{\mathfrak{d}_0} \setminus w^*| \geq 2$, say $\alpha_0, \alpha_1 \in v^{\mathfrak{d}_0} \setminus w^*$. Then $h_i(\varepsilon, \alpha_0), h_i(\varepsilon, \alpha_1) \geq M_\xi$ for all $i < \iota$. But $\mathbf{m}^{\mathfrak{d}_0} \sqsubseteq \mathbf{m}^{\mathfrak{d}_1}$ implies that for $\alpha \in v^{\mathfrak{d}_0} \setminus \{\delta\}$ we have

$$h_0^{\mathbf{m}^{\mathfrak{d}_1}}(\eta_\varepsilon \restriction \ell^{\mathfrak{d}_1}, \eta_\alpha \restriction \ell^{\mathfrak{d}_1}) = h_0^{\mathbf{m}^{\mathfrak{d}_0}}(\eta_\delta \restriction \ell^{\mathfrak{d}_0}, \eta_\alpha \restriction \ell^{\mathfrak{d}_0}) < M_\xi,$$

so we arrive to a contradiction.

If we had $v^{\partial_0} \subseteq w^*$, then $v^{\partial_1} \subseteq w_\varsigma$ and we may repeat the earlier arguments with ς in place of ξ to get a contradiction. Thus the only possibility left is that $|v^{\partial_0} \setminus w^*| = 1$. Let $\{\alpha\} = v^{\partial_0} \setminus w^*$. If $\alpha \neq \delta$, then $h_0^{\mathbf{m}^{\partial_1}}(\eta_\alpha{\upharpoonright}\ell^{\partial_1}, \eta_\varepsilon{\upharpoonright}\ell^{\partial_1}) = h_0^{\mathbf{m}^{\partial_0}}(\eta_\alpha{\upharpoonright}\ell^{\partial_0}, \eta_\varepsilon{\upharpoonright}\ell^{\partial_0}) < M_\xi$ gives a contradiction like before. Therefore, $v^{\partial_0} = (v^{\partial_0} \cap w^*) \cup \{\delta\}$. But now our assumptions on v^{∂_0}, δ contradict $(*)_{16}$. $\qquad\square$

Claim 4.4.3. *Assume* $p = (w, n, \iota, M, \bar{\eta}, \bar{t}, \bar{h}, \bar{g}, \mathcal{M}) \in \mathbb{P}$. *If* $\mathbf{m} \in \mathbf{M}_{\bar{t}, \bar{\mathcal{O}}^6}^n$ *is such that* $\ell^{\mathbf{m}} = n$ *and* $|u^{\mathbf{m}}| \geq 5$, *then for some* $\rho \in {}^n 2$ *and* $v \subseteq w$ *we have* $(n, v, (\mathbf{m} + \rho)) \in \mathcal{M}$.

Proof of the Claim. Let $\mathbf{m} \in \mathbf{M}_{\bar{t}, \bar{\mathcal{O}}^6}^n$ be such that $\ell^{\mathbf{m}} = n$. Suppose $(\eta, \nu) \in (u^{\mathbf{m}})^{\langle 2 \rangle}$.

Let $g_j^{\mathbf{m}}(\eta, \nu) = \{\sigma_j\}$ for $j < 6$. Then σ_js are pairwise distinct, and if $\eta + \sigma_i = \nu + \sigma_j$ then

$$\eta + \sigma_k, \nu + \sigma_k \notin \{\eta + \sigma_i, \nu + \sigma_i\} = \{\eta + \sigma_j, \nu + \sigma_j\}$$

whenever $k \notin \{i, j\}$. Hence we may pick $j_0 < j_1 < j_2 < 6$ such that

$$\eta + \sigma_{j_0}, \nu + \sigma_{j_0}, \eta + \sigma_{j_1}, \nu + \sigma_{j_1}, \eta + \sigma_{j_2}, \nu + \sigma_{j_2}$$

are all pairwise distinct. Just to simplify notation let us assume that $j_0 = 0$, $j_1 = 1$ ad $j_2 = 2$.

For each $j < 3$ we have $\eta + \sigma_j, \nu + \sigma_j \in \bigcup_{m < M} t_m$. By clause $(*)_7$ there are $(\alpha_j, \beta_j), (\alpha_j', \beta_j') \in w^{\langle 2 \rangle}$ and $\rho_j \in \bigcup_{i < \iota} g_i(\alpha_j, \beta_j)$ and $\rho_j' \in \bigcup_{i < \iota} g_i(\alpha_j', \beta_j')$ such that $\eta + \sigma_j = \eta_{\alpha_j} + \rho_j$ and $\nu + \sigma_j = \eta_{\alpha_j'} + \rho_j'$ for $j < 3$. Then $\eta + \nu = \eta_{\alpha_j} + \eta_{\alpha_j'} + \rho_j + \rho_j'$ for all $j < 3$. We will consider 3 cases, the first two of them will be shown to be impossible.

CASE 1: $\eta_{\alpha_j} = \eta_{\alpha_j'}$ for some $j < 3$.
Then, by the linear independence demanded in $(*)_7$, $\eta_{\alpha_j} = \eta_{\alpha_j'}$ for all $j < 3$ and $\{\rho_0, \rho_0'\} = \{\rho_1, \rho_1'\} = \{\rho_2, \rho_2'\}$. But $g_i(\alpha, \beta)$'s are disjoint, so each $\rho \in \bigcup \{g_i(\alpha, \beta) : (\alpha, \beta) \in w^{\langle 2 \rangle} \wedge i < \iota\}$ uniquely determines α, β such that $\eta_\alpha + \rho, \eta_\beta + \rho \in \bigcup_{m < M} t_m$. Therefore, $|\{\alpha_0, \alpha_1, \alpha_2\}| \leq 2$ in the current case. Since $\eta + \sigma_j, \nu + \sigma_j$ are all pairwise distinct (for $j < 3$), this gives an immediate contradiction.

CASE 2: $\eta_{\alpha_j} \neq \eta_{\alpha_j'}$ and $\rho_j \neq \rho_j'$ for some (equivalently: all) $j < 3$.
Then $\{\eta_{\alpha_0}, \eta_{\alpha_0'}\} = \{\eta_{\alpha_1}, \eta_{\alpha_1'}\} = \{\eta_{\alpha_2}, \eta_{\alpha_2'}\}$ and $\{\rho_0, \rho_0'\} = \{\rho_1, \rho_1'\} = \{\rho_2, \rho_2'\}$. However, this again contradicts $\eta + \sigma_j, \nu + \sigma_j$ being pairwise distinct.

Thus the only possible case is the following:
CASE 3: $\eta_{\alpha_j} \neq \eta_{\alpha_j'}$ and $\rho_j = \rho_j'$ for all $j < 3$.
Then $\eta + \nu = \eta_{\alpha_0} + \eta_{\alpha_0'}$.

Consequently we have shown that

$$u^{\mathbf{m}} + u^{\mathbf{m}} \subseteq \{\eta_\alpha + \eta_\beta : \alpha, \beta \in w\}.$$

By Lemma 4.3 for some ρ we have $u^{\mathbf{m}} + \rho \subseteq \{\eta_\alpha : \alpha \in w\}$. Let $v = \{\alpha \in w : \eta_\alpha \in u^{\mathbf{m}} + \rho\}$. Let us argue that $(n, v, (\mathbf{m} + \rho)) \in \mathcal{M}$: demands $(*)_9^{\mathrm{a}}$–$(*)_9^{\mathrm{c}}$ are immediate consequences of our choices above. Let us verify $(*)_9^{\mathrm{d}}$.

Suppose that $(\alpha, \beta) \in v^{\langle 2 \rangle}$ and $i < 6$. Let $\eta = \eta_\alpha + \rho, \nu = \eta_\beta + \rho$ (so they are in $u^{\mathbf{m}}$) and let $\{\sigma_i\} = g_i^{\mathbf{m}}(\eta, \nu)$. Then $\eta + \sigma_i, \nu + \sigma_i \in \bigcup_{m < M} t_m$, so we may choose $(\alpha', \beta'), (\alpha'', \beta'') \in w^{\langle 2 \rangle}$ and $j', j'' < \iota$ and $\rho' \in g_{j'}(\alpha', \beta')$ and $\rho'' \in g_{j''}(\alpha'', \beta'')$ such that $\eta + \sigma_i = \eta_{\alpha'} + \rho'$ and $\nu + \sigma_i = \eta_{\alpha''} + \rho''$. Then

$$\eta_\alpha + \eta_\beta = \eta + \nu = \eta_{\alpha'} + \eta_{\alpha''} + \rho' + \rho''.$$

By the linear independence stated in $(*)_8$ we get $\rho' = \rho''$ and $\{\eta_{\alpha'}, \eta_{\alpha''}\} = \{\eta_\alpha, \eta_\beta\}$. Consequently also $\{\alpha, \beta\} = \{\alpha', \alpha''\}$ and $\{\alpha', \beta'\} = \{\alpha'', \beta''\}$ and $j' = j''$. Since $\alpha \neq \beta$ we get $\alpha' \neq \alpha''$ and thus $\alpha' = \beta''$, $\alpha'' = \beta'$. Consequently, $\{\alpha'', \beta''\} = \{\alpha', \beta'\} = \{\alpha', \alpha''\} = \{\alpha, \beta\}$. Hence $\eta + \sigma_i = \eta_{\alpha'} + \rho' \in t_{h_{j'}(\alpha, \beta)} = t_{h_{j'}(\beta, \alpha)}$ and $\nu + \sigma_i = \eta_{\alpha''} + \rho' \in t_{h_{j'}(\alpha, \beta)} = t_{h_{j'}(\beta, \alpha)}$. Therefore, $h_i^{\mathbf{m} + \rho}(\eta_\alpha, \eta_\beta) = h_i^{\mathbf{m}}(\eta, \nu) = h_{j'}(\alpha, \beta) = h_{j'}(\beta, \alpha)$. □

Define $\mathbb{P}$–names $\underset{\sim}{T}_m$ and $\underset{\sim}{\eta}_\alpha$ (for $m < \omega$ and $\alpha < \lambda$) by
$$\Vdash_\mathbb{P} \text{`` } \underset{\sim}{T}_m = \bigcup\{t_m^p : p \in G_\mathbb{P} \wedge m < M^p\} \text{ '', and}$$
$$\Vdash_\mathbb{P} \text{`` } \underset{\sim}{\eta}_\alpha = \bigcup\{\eta_\alpha^p : p \in G_\mathbb{P} \wedge \alpha \in w^p\} \text{ ''.}$$

Claim 4.4.4. (1) *For each $m < \omega$ and $\alpha < \lambda$,*
$$\Vdash_\mathbb{P} \text{`` } \underset{\sim}{\eta}_\alpha \in {}^\omega 2 \text{ and } \underset{\sim}{T}_m \subseteq {}^{\omega >} 2 \text{ is a tree without terminal nodes ''.}$$
(2) *For all $\alpha < \beta < \lambda$ we have*
$$\Vdash_\mathbb{P} \text{`` } \big(\underset{\sim}{\eta}_\alpha + \bigcup_{m < \omega} \lim(\underset{\sim}{T}_m)\big) \cap \big(\underset{\sim}{\eta}_\beta + \bigcup_{m < \omega} \lim(\underset{\sim}{T}_m)\big) \text{ is } \bar{\mathcal{O}}\text{–large ''.}$$
(3) $\Vdash_\mathbb{P} \text{`` } \bigcup_{m < \omega} \lim(\underset{\sim}{T}_m) \text{ is a } \bar{\mathcal{O}}^6\text{–\textbf{npots} set ''.}$

Proof of the Claim. (1, 2) By Claim 4.4.1 (and the definition of the order in $\mathbb{P}$).

(3) Let $G \subseteq \mathbb{P}$ be a generic filter over $\mathbf{V}$ and let us work in $\mathbf{V}[G]$. Let $\bar{T} = \langle (\underset{\sim}{T}_m)^G : m < \omega \rangle$.

Suppose towards contradiction that $B = \bigcup_{m < \omega} \lim\big((\underset{\sim}{T}_m)^G\big)$ is an $\bar{\mathcal{O}}^6$–**pots** set. Then, by Proposition 3.16, $\mathrm{NDRK}_{\bar{\mathcal{O}}^6}(\bar{T}) = \infty$. Using Lemma 3.15(5), by induction on $j < \omega$ we choose $\mathbf{m}_j, \mathbf{m}_j^* \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}^6}$ and $p_j \in G$ such that

(i) $\mathrm{ndrk}_{\bar{\mathcal{O}}^6}(\mathbf{m}_j) \geq \omega_1$, $|u^{\mathbf{m}_j}| > 5$ and $\mathbf{m}_j \sqsubset \mathbf{m}_j^* \sqsubset \mathbf{m}_{j+1}$,

(ii) for each $\nu \in u^{\mathbf{m}_j}$ the set $\{\eta \in u^{\mathbf{m}_{j+1}} : \nu \lhd \eta\}$ has at least two elements, and

(iii) $p_j \leq p_{j+1}$, $\ell^{\mathbf{m}_j} < \ell^{\mathbf{m}_j^*} = n^{p_j} < \ell^{\mathbf{m}_{j+1}}$ and $\mathrm{rng}(h_i^{\mathbf{m}_j}) \subseteq M^{p_j}$ for all $i < 6$, and

(iv) $|\{\eta {\restriction} n^{p_j} : \eta \in u^{\mathbf{m}_{j+1}}\}| = |u^{\mathbf{m}_j}| = |u^{\mathbf{m}_j^*}|$.

To carry out the construction we proceed as follows. Suppose we have determined $\mathbf{m}_j$ so that $\mathrm{ndrk}_{\bar{\mathcal{O}}^6}(\mathbf{m}_j) \geq \omega_1$. Using densities given in Claim 4.4.1, we find $p_j \in G$ with $n^{p_j} > \ell^{\mathbf{m}_j}$ and $\mathrm{rng}(h_i^{\mathbf{m}_j}) \subseteq M^{p_j}$ (for $i < 6$). Next we choose $\mathbf{n}$ such that $\mathbf{m}_j \sqsubset \mathbf{n}$, $\mathrm{ndrk}_{\bar{\mathcal{O}}^6}(\mathbf{n}) \geq \omega_1$, and $\ell^{\mathbf{n}} > n^{p_j}$. Using Lemma

3.15(8) (for a $u' \subseteq u^{\mathbf{n}}$ such that $\{\eta {\restriction} \ell^{\mathbf{m}_j} : \eta \in u'\} = u^{\mathbf{m}_j}$, $|u'| = |u^{\mathbf{m}_j}|$) we may also demand that $|u^{\mathbf{n}}| = |u^{\mathbf{m}_j}|$. Now we let

- $\ell = n^{p_j}$, $u = \{\eta {\restriction} \ell : \eta \in u^{\mathbf{n}}\}$,
- $\bar{h} = \langle h_i : i < 6 \rangle$, where for $i < 6$ and $(\eta, \nu) \in (u^{\mathbf{n}})^{\langle 2 \rangle}$
  $h_i(\eta {\restriction} \ell, \nu {\restriction} \ell) = h_i^{\mathbf{n}}(\eta, \nu) = h_i^{\mathbf{m}_j}(\eta {\restriction} \ell^{\mathbf{m}_j}, \nu {\restriction} \ell^{\mathbf{m}_j})$,
- $\bar{g} = \langle g_i : i < 6 \rangle$, where for $i < 6$ and $(\eta, \nu) \in (u^{\mathbf{n}})^{\langle 2 \rangle}$
  $g_i(\eta {\restriction} \ell, \nu {\restriction} \ell) = \{\rho {\restriction} \ell : \rho \in g_i^{\mathbf{n}}(\eta, \nu)\}$.

Clearly, $\mathbf{m}_j^* = (\ell, 6, u, \bar{h}, \bar{g}) \in \mathbf{M}_{\bar{T}, \bar{\mathcal{O}}^6}$ and $\mathbf{m}_j \sqsubset \mathbf{m}_j^*$. Finally use Lemma 3.15(5) to pick $\mathbf{m}_{j+1} \sqsupseteq \mathbf{n}$ such that $\mathrm{ndrk}(\mathbf{m}_{j+1}) \geq \omega_1$ and condition (ii) is satisfied. Note that $\mathbf{m}_j^* \sqsubset \mathbf{m}_{j+1}$.

Then, by (iii)+(iv), $\mathbf{m}_j, \mathbf{m}_j^* \in \mathbf{M}_{\bar{t}^{p_j}, \bar{\mathcal{O}}^6}^{n^{p_j}}$. It follows from Claim 4.4.3 that for some $w_j \subseteq w^{p_j}$ and $\rho_j \in {}^{n^{p_j}}2$ we have $(n^{p_j}, w_j, \mathbf{m}_j^* + \rho_j) \in \mathcal{M}^{p_j}$.

Fix j for a moment and consider $(n^{p_j}, w_j, \mathbf{m}_j^* + \rho_j) \in \mathcal{M}^{p_j} \subseteq \mathcal{M}^{p_{j+1}}$ and $(n^{p_{j+1}}, w_{j+1}, \mathbf{m}_{j+1}^* + \rho_{j+1}) \in \mathcal{M}^{p_{j+1}}$. (Note that since $(n^{p_j}, w_j, \mathbf{m}_j^* + \rho_j) \in \mathcal{M}^{p_j}$, we know that $r_{h_i^{\mathbf{m}_j^*}(\eta, \nu)} \leq n^{p_j}$ for all $i < 6$, $(\eta, \nu) \in u^{\mathbf{m}_j^*}$.) Since $(\mathbf{m}_j^* + (\rho_{j+1} {\restriction} n^{p_j})) \sqsubset (\mathbf{m}_{j+1}^* + \rho_{j+1})$, we may choose $w_j^* \subseteq w_{j+1}$ such that $(n^{p_j}, w_j^*, \mathbf{m}_j^* + (\rho_{j+1} {\restriction} n^{p_j})) \in \mathcal{M}^{p_{j+1}}$. Since $(\mathbf{m}_j^* + \rho_j) + (\rho_j + \rho_{j+1} {\restriction} n^{p_j}) = \mathbf{m}_j^* + (\rho_{j+1} {\restriction} n^{p_j})$, we may use clause $(*)_{10}$ for p_{j+1} to conclude that $\mathrm{rk}(w_j^*) = \mathrm{rk}(w_j)$.

Condition (ii) of the choice of $\mathbf{m}_{j+1}$ implies that

$$(\forall \gamma \in w_j^*)(\exists \delta \in w_{j+1} \setminus w_j^*)(\eta_\gamma^{p_{j+1}} {\restriction} n^{p_j} = \eta_\delta^{p_{j+1}} {\restriction} n^{p_j}).$$

Let $\delta(\gamma)$ be the smallest $\delta \in w_{j+1} \setminus w_j^*$ with the above property and let $w_j^*(\gamma) = (w_j^* \setminus \{\gamma\}) \cup \{\delta(\gamma)\}$. Then, for $\gamma \in w_j^*$, $(n^{p_j}, w_j^*(\gamma), \mathbf{m}_j^* + (\rho_{j+1} {\restriction} n^{p_j})) \in \mathcal{M}^{p_{j+1}}$ and therefore, by clause $(*)_{10}$ for p_{j+1}, we get that for each $\gamma \in w_j$:

$$\mathrm{rk}(w_j^*(\gamma)) = \mathrm{rk}(w_j^*), \quad \zeta(w_j^*(\gamma)) = \zeta(w_j^*), \quad \text{and} \quad k(w_j^*(\gamma)) = k(w_j^*).$$

Let $n = |w_j^*|$, $\zeta = \zeta(w_j^*)$, $k = k(w_j^*)$, and let $w_j^* = \{\alpha_0, \ldots, \alpha_k, \ldots, \alpha_{n-1}\}$ be the increasing enumeration. Let $\alpha_k^* = \delta(\alpha_k)$. Then clause $(*)_{10}$ also gives that $w_j^*(\alpha_k) = \{\alpha_0, \ldots, \alpha_{k-1}, \alpha_k^*, \alpha_{k+1}, \ldots, \alpha_{n-1}\}$ is the increasing enumeration. Now,

$$\mathbb{M} \models R_{n,\zeta}[\alpha_0, \ldots, \alpha_{k-1}, \alpha_k, \alpha_{k+1}, \ldots, \alpha_{n-1}] \quad \text{and}$$
$$\mathbb{M} \models R_{n,\zeta}[\alpha_0, \ldots, \alpha_{k-1}, \alpha_k^*, \alpha_{k+1}, \ldots, \alpha_{n-1}],$$

and consequently if $\mathrm{rk}(w_j^*) \geq 0$, then

$$\mathrm{rk}(w_{j+1}) \leq \mathrm{rk}(w_j^* \cup \{\alpha_k^*\}) < \mathrm{rk}(w_j^*) = \mathrm{rk}(w_j)$$

(remember $(\circledast)_d$ from the very beginning of the proof of the Theorem).

Now, unfixing j, it follows from the above considerations that for some $j_0 < \omega$ we must have:

(a) $\mathrm{rk}(w_{j_0}^*) = -1$, and
(b) $(n^{p_{j_0}}, w_{j_0}^*, \mathbf{m}_{j_0}^* + (\rho_{j_0+1} {\restriction} n^{p_{j_0}})), (n^{p_{j_0+1}}, w_{j_0+1}, \mathbf{m}_{j_0+1}^* + \rho_{j_0+1}) \in \mathcal{M}^{p_{j_0+1}}$,
(c) for each $\nu \in u^{\mathbf{m}_{j_0}^*}$ the set $\{\eta \in u^{\mathbf{m}_{j_0+1}^*} : \nu \lhd \eta\}$ has at least two elements.

However, this contradicts clause $(*)_{11}$ (for p_{j_0+1}). ☐

☐

5. CONCLUSIONS AND QUESTIONS

Corollary 5.1. *Assume* $\mathrm{NPr}_{\omega_1}(\lambda)$ *and* $\lambda = \lambda^{\aleph_0} < \mu = \mu^{\aleph_0}$.

(1) *Let* $\bar{\mathcal{O}}$ *be a nice indexed base. Then there is a ccc forcing notion* $\mathbb{Q}$ *of size* μ *forcing that:*
  - $2^{\aleph_0} = \mu$ *and there is a* Σ_2^0 *set* $B \subseteq {}^{\omega}2$ *which has* λ *many pairwise* $\bar{\mathcal{O}}$*–nondisjoint translates but does not have* λ^+ *many pairwise* $\bar{\mathcal{O}}^6$*–nondisjoint translates.*

(2) *In particular, there is a ccc forcing notion* $\mathbb{Q}'$ *of size* μ *forcing that:*
  - $2^{\aleph_0} = \mu$ *and for some* Σ_2^0 *set* $B \subseteq {}^{\omega}2$ *there are pairwise distinct* $\langle \eta_\xi : \xi < \lambda \rangle$ *such that* $(B + \eta_\xi) \cap (B + \eta_\zeta)$ *is uncountable for each* $\xi, \zeta < \lambda$*, but*
  - *for any set* $A \subseteq {}^{\omega}2$ *of size* λ^+ *there are* $x, y \in A$ *such that* $|(B + x) \cap (B + y)| < 6$.

Proof. (1) Let $\mathbb{P}$ be the forcing notion given by Theorem 4.4 and let $\mathbb{Q} = \mathbb{P} * \mathbb{C}_\mu$. The set B added by $\mathbb{P}$ is a $\bar{\mathcal{O}}^6$–**npots**–set in $\mathbf{V}^{\mathbb{P}}$, so by Proposition 3.16 we got $\mathrm{NDRK}_{\bar{\mathcal{O}}^6}(\bar{T}) = \infty$. The rank $\mathrm{ndrk}_{\bar{\mathcal{O}}^6}^{\bar{T}}$ is absolute, so in $\mathbf{V}^{\mathbb{Q}}$ we still have $\mathrm{NDRK}_{\bar{\mathcal{O}}^6}(\bar{T}) = \infty$ and thus B is a $\bar{\mathcal{O}}^6$–**npots**–set in $\mathbf{V}^{\mathbb{Q}}$. By 3.5(3) this set cannot have λ^+ pairwise $\bar{\mathcal{O}}^6$–nondisjoint translates, but it does have λ many pairwise $\bar{\mathcal{O}}$–nondisjoint translates (by absoluteness). ☐

Corollary 5.2. *Assume* **MA** *and* $\aleph_\alpha < \mathfrak{c}$, $\alpha < \omega_1$.

(1) *Let* $\bar{\mathcal{O}}$ *be a nice indexed base. Then there exists a* Σ_2^0 $\bar{\mathcal{O}}^6$*–**npots**–set* $B \subseteq {}^{\omega}2$ *which has* $\aleph_\alpha$ *many pairwise* $\bar{\mathcal{O}}$*–nondisjoint translations.*

(2) *In particular, there exists a* Σ_2^0 *set* $B \subseteq {}^{\omega}2$ *such that*
  - *for some pairwise distinct* $\langle \eta_\xi : \xi < \aleph_\alpha \rangle \subseteq {}^{\omega}2$ *the intersections* $(B + \eta_\xi) \cap (B + \eta_\zeta)$ *are uncountable for each* $\xi, \zeta < \aleph_\alpha$*, but*
  - *for every perfect set* $P \subseteq {}^{\omega}2$ *there are* $x, y \in P$ *such that* $|(B + x) \cap (B + y)| < 6$.

Proof. Standard consequence of the proof of Theorem 4.4, using the fact that "B is a $\bar{\mathcal{O}}^6$–**npots**–set" is sufficiently absolute by Proposition 3.16. ☐

Problem 5.3.   (1) Can one differentiate between various nice $\bar{\mathcal{O}}$ in the context of our results? In particular:

(2) Is it consistent that for some nice $\bar{\mathcal{O}}$ there is an Σ_2^0 $\bar{\mathcal{O}}$–**npots**–set which has $\aleph_\alpha$ many pairwise $\bar{\mathcal{O}}$–nondisjoint translations, but for some other nice $\bar{\mathcal{O}}^*$ every Σ_2^0 set with $\aleph_\alpha$ many pairwise $\bar{\mathcal{O}}^*$–nondisjoint translations is automatically $\bar{\mathcal{O}}^*$–**pots** ?

(3) Is it consistent that there is an Σ_2^0 set $B \subseteq {}^{\omega}2$ which is has $\aleph_\alpha$ many pairwise $\bar{\mathcal{O}}^{\mathrm{per}}$–nondisjoint translations, is $\bar{\mathcal{O}}^{\mathrm{per}}$–**npots**, but is also $\bar{\mathcal{O}}^6$–**pots**?

Problem 5.4.   (1) Consider the forcing notion $\mathbb{P}$ given by Theorem 4.4 for $\bar{\mathcal{O}}^{\mathrm{per}}$. In the forcing extension by $\mathbb{P}$, the ranks $\mathrm{NDRK}_{\bar{\mathcal{O}}^6}(\bar{T})$ and

$\text{NDRK}_{\bar{\mathcal{O}}^{\text{per}}}(\bar{T})$ are both countable. Are they equal? What are their values?

(2) Does there exist a sequence of trees $\bar{T}^*$ (as in Assumptions 3.1) for which the ranks $\text{NDRK}_{\bar{\mathcal{O}}^{\text{per}}}(\bar{T})$ and $\text{NDRK}_{\bar{\mathcal{O}}^{\iota}}(\bar{T})$ are different (for some/all ι)?

(3) Generalize the construction of [5] to arbitrary nice $\bar{\mathcal{O}}$.

(4) Generalize the result of the present paper to the context of arbitrary perfect Abelian Polish groups.

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