HIGHER DIMENSIONAL UNIVERSAL FUNCTIONS FROM LOWER DIMENSIONAL ONES

SAHARON SHELAH AND JURIS STEPRĀNS

Abstract. It is shown that the existence of a universal function on pairs implies the existence of a universal function for triples.

1. Introduction

Following Sierpiński, in [1] a very general definition of higher dimensional universal functionss is given which, in the case of triples on ω_1 would say that $U : \omega_1^3 \to \lambda$ is universal if for every $H : \omega_1^3 \to \lambda$ there are are embedding $e_i : \omega_1 \to \omega_1$ for $i \in 3$ such that

$$H(\alpha_0, \alpha_1, \alpha_2) = U(e_0(\alpha_0), e_1(\alpha_1), e_2(\alpha_2))$$

for each $(\alpha_0, \alpha_1, \alpha_2) \in \omega_1^3$. However, in this brief note the stronger and, perhaps, more natural definition of universal function will be based on the definition of a universal graph introduced by Rado in [3].

Definition 1.1 (Rado)**.** For $k \in \omega$ define $U \subseteq [\omega_1]^k$ to be a universal hypergraph if for every hypergraph $H \subseteq [\omega_1]^k$ there is a one-to-one function $e : \omega_1 \to \omega_1$ such that

$$(\forall a \in [\omega_1]^k)\ a \in H \text{ if and only if } e[a] \in U$$

where $e[a]$ denotes the image of a under e.

Definition 1.2. For $k \in \omega$ define a function $U \subseteq [\omega_1]^k \to \lambda$ to be universal if for every function $G : [\omega_1]^k$ there is a single $e : \omega_1 \to \omega_1$ such that

$$(\forall a \in [\omega_1]^k)\ H(a) = U(e[a]).$$

In Proposition 6.2 of [1] it is shown that if there is a universal function from ω_1^2 to ω_1 in the sense of Sierpiński then there is also a universal function from ω_1^3 to ω_1, also in the sense of Sierpiński. In this note, the companion result for universality in the sense of Rado will be established.

Theorem 1.1. *If there is a universal function from $[\omega_1]^2$ to ω as in Definition 1.2 then there is a universal function from $[\omega_1]^3$ to ω as well.*

However, the results in thois note do not answer rhe following question.

Question 1.1. If there is a universal graph $U \subseteq [\omega_1]^2$ — in other words, U can be considered as a universal function from $[\omega_1]^2$ to 2 — then is there a subset of $[\omega_1]^3$ that is a universal hypergraph for triples?

The interest in universal functions is based largely on the result of [4] that shows that universality can hold for graphs even in the absence of saturated graphs. While [4] did not address the question of universal functions from $[\omega_1]^2$ to ω explicitly, the methods will also yield such universal functions. However, this does follow explictly from the results of [2] which modified the methods of [4] to apply to a larger class of structures of cardinality $\aleph_1$, including functions from $[\omega_1]^2$ to ω. It should also be

The first author's research for this paper was partially supported by the United States-Israel Binational Science Foundation (grant no. 2006108), and by the National Science Foundation (grant no. NSF-DMS 1833363). This is article has been assigned number 1088 in the first author's list of publications that can be found at http://shelah.logic.at. The second author's research for this paper was partially supported by NSERC of Canada.

noted that it is shown in [5] that it is consistent that there is a universal graph on ω_1 in the sense of Rado, yet no universal function from $[\omega_1]^2$ to ω. Hence, the hypothesis used in this note is significant and one cannot expect the results here to directly solve Question 1.1.

2. Universal 3-dimensional functions from 2-dimensional ones

Definition 2.1. Suppose that $U : [\omega_1]^2 \to H(\aleph_0)$ is a universal function. Let $U^* : [\omega_1]^3 \to \omega$ be defined by letting $U^*(\{\alpha,\beta,\gamma\}) = U(\{\beta,\gamma\})(U(\{\alpha,\gamma\}))$ if $\{\alpha,\beta,\gamma\} \in [\omega_1]^3$ and

- $\max(\{\alpha,\beta\}) = \gamma$
- $|U(\{\alpha,\gamma\})| \lneq |U(\{\beta,\gamma\})|$
- $U(\{\beta,\gamma\}$ is a function with $U(\{\alpha,\gamma\}$ in its domain

and letting $U^*(\{\alpha,\beta,\gamma\}) = 0$ otherwise.

Lemma 2.1. *If $U : [\omega_1]^2 \to \omega$ is a universal function and $F : [\omega_1]^2 \to \omega$ then there is a strictly increasing function $e : \omega_1 \to \omega_1$ such that $F(\{\alpha,\beta\}) = U(\{e(\alpha), e(\beta)\})$ for each $\{\alpha,\beta\} \in [\omega_1]^2$.*

Proof. Let $\{Z_\xi\}_{\xi\in\omega_1}$ partition ω_1 into uncountable sets and define $F^* : [\omega_1]^2 \to \omega$ by

$$F^*(\{\alpha,\beta\}) = \begin{cases} F(\{\xi,\eta\}) & \text{if } \alpha \in Z_\xi \ \& \ \beta \in Z_\eta \\ 0 & \text{otherwise.} \end{cases}$$

There is then a one-to-one function $e : \omega_1 \to \omega_1$ such that $F^*(\{\alpha,\beta\}) = U(\{e(\alpha), e(\beta)\})$ for each $\{\alpha,\beta\} \in [\omega_1]^2$. Since each Z_ξ is uncountable it is possible to select $\zeta_\xi \in Z_\xi$ such that $e(\zeta_\xi) < e(\zeta_\mu)$ whenever $\xi < \mu$. Now define $e^*(\alpha) = e(\zeta_\alpha)$ and note that e^* is strictly increasing and, moreover, $F(\{\alpha,\beta\}) = F^*(\{\zeta_\alpha,\zeta_\beta\})$ because $\zeta_\alpha \in Z_\alpha$ and $\zeta_\beta \in Z_\beta$. Then

$$U(\{e^*(\alpha), e^*(\beta)\}) = U(\{e(\zeta_\alpha), e(\zeta_\beta)\}) = F^*(\{\zeta_\alpha,\zeta_\beta\}) = F(\{\alpha,\beta\})$$

as required. □

Lemma 2.2. *Suppose that for every $D : [\omega_1]^3 \to \omega$ is there are $\{C_a\}_{a\in[\omega_1]^2}$ such that:*

1. *$C_a \in H(\aleph_0)$ is a function for each $a \in [\omega_1]^2$*
2. *$D(\{\alpha,\beta,\gamma\}) = C_{\{\beta,\gamma\}}(C_{\{\alpha,\gamma\}})$ for each $\{\alpha,\beta,\gamma\} \in [\omega_1]^3$ provided that $\gamma > \max(\{\alpha,\beta\})$ and $|C_{\{\alpha,\gamma\}}| \lneq |C_{\{\beta,\gamma\}}|$.*

If $U : [\omega_1]^2 \to H(\aleph_0)$ is a universal function then $U^* : [\omega_1]^3 \to \omega$ is also a universal function.

Proof. Given $D : [\omega_1]^3 \to \omega$ let $\{C_a\}_{a\in[\omega_1]^2}$ satisfy Requirements (1) and (2). Using that U is universal and Lemma 2.1 it is possible to find a strictly increasing function $e : \omega_1 \to \omega_1$ such that $U(\{e(\alpha), e(\beta)\}) = C_{\{\alpha,\beta\}}$ for all $\{\alpha,\beta\} \in [\omega_1]^2$. It suffices to show that

$$U^*(\{e(\alpha), e(\beta), e(\gamma)\}) = D(\{\alpha,\beta,\gamma\})$$

for all $\{\alpha,\beta,\gamma\} \in [\omega_1]^3$.

To see that this is so let $\{\alpha,\beta,\gamma\} \in [\omega_1]^3$ and suppose that $\gamma > \max(\{\alpha,\beta\})$ and $|C_{\{\alpha,\gamma\}}| \lneq |C_{\{\beta,\gamma\}}|$. It follows that

$$|U(\{e(\alpha), e(\gamma)\})| = |C_{\{\alpha,\gamma\}}| \lneq |C_{\{\beta,\gamma\}}| = |U(\{e(\beta), e(\gamma)\})|$$

and, hence, that

$$U^*(\{e(\alpha), e(\beta), e(\gamma)\}) = U(\{e(\beta), e(\gamma)\})(U(\{e(\alpha), e(\gamma)\}) = C_{\{\beta,\gamma\}}(C_{\{\alpha,\gamma\}}) = D(\{\alpha,\beta,\gamma\}).$$

□

Lemma 2.3. *Given $D : [\omega_1]^3 \to \omega$ there are $\{C_a\}_{a\in[\omega_1]^2}$ such that:*

1. *$C_a \in H(\aleph_0)$ is a function for each $a \in [\omega_1]^2$*

(2) $D(\{\alpha,\beta,\gamma\}) = C_{\{\beta,\gamma\}}(C_{\{\alpha,\gamma\}})$ *for each* $\{\alpha,\beta,\gamma\} \in [\omega_1]^3$ *provided that* $\gamma > \max(\{\alpha,\beta\})$ *and* $|C_{\{\alpha,\gamma\}}| \lneq |C_{\{\beta,\gamma\}}|$.

Proof. Let $B_\xi : \omega \to \xi$ be a bijection if ξ is infinite and let B_k be the identity if $k \in \omega$. It will be shown that for each infinite $\xi \in \omega_1$ there are $C_{\{\xi,B_\xi(k)\}} \in H(\aleph_0)$ for $k \in \omega$ such that:

(1) each $C_{\{\xi,B_\xi(k)\}}$ is a function
(2) **domain**$(C_{\{\xi,B_\xi(k)\}}) = \{C_{\{\xi,B_\xi(j)\}}\}_{j\in k}$
(3) $C_{\{\xi,B_\xi(k)\}}(C_{\{\xi,B_\xi(j)\}}) = D(\{B_\xi(j), B_\xi(k), \xi\})$

and for each $k \in \omega$ there are $C_{\{k,j\}} \in H(\aleph_0)$ for $j \in k$ such that:

(4) each $C_{\{k,j\}}$ is a function
(5) **domain**$(C_{\{k,j\}}) = \{C_{\{k,\ell\}}\}_{\ell\in j}$
(6) $C_{\{k,j\}}(C_{k,\ell\}}) = D(\{\ell, j, k\})$.

To see that this suffices note that Conclusion (1) is immediate. To see that Conclusion (2) holds suppose that $\{\alpha,\beta,\gamma\} \in [\omega_1]^3$ and $\gamma > \max(\{\alpha,\beta\})$ and $|C_{\{\alpha,\gamma\}}| \lneq |C_{\{\beta,\gamma\}}|$. If γ is infinite then let $j = |C_{\{\alpha,\gamma\}}| \lneq |C_{\{\beta,\gamma\}}| = k$. It follows from Condition (2) that $B_\gamma(j) = \alpha$ and $B_\gamma(k) = \beta$. It is therefore possible to use Condition (3) to conclude that

$$C_{\{\gamma,\beta\}}(C_{\{\gamma,\alpha\}}) = C_{\{\gamma,B_\gamma(k)\}}(C_{\{\gamma,B_\gamma(j)\}}) = D(\{B_\gamma(j), B_\gamma(k), \gamma\}) = D(\alpha,\beta,\gamma).$$

On the other hand, if $\gamma = k \in \omega$ then then let $\ell = |C_{\{\alpha,k\}}| \lneq |C_{\{\beta,k\}}| = j$. It follows from Condition (5) that $\ell = \alpha$ and $j = \beta$. It is therefore possible to use Condition (6) to conclude that

$$C_{\{\gamma,\beta\}}(C_{\{\gamma,\alpha\}}) = C_{\{k,j\}}(C_{\{k,\ell\}}) = D(\{\ell, j, k\}) = D(\alpha,\beta,\gamma)$$

as required.

For each $\xi \in \omega_1$ define the functions $C_{\{\xi,B_\xi(k)\}}$ by induction on $k \in \omega \cap \xi$. Start by letting $C_{\{\xi,B_\xi(0)\}} = \varnothing$. Given $C_{\{\xi,B_\xi(j)\}}$ for $j \in k$ satisfying Condition (2) or Condition (5) note that the $C_{\{\xi,B_\xi(j)\}}$ are all distinct since they have different cardinalities. It is therefore easy to define a function $C_{\{\xi,B_\xi(k)\}} \in H(\aleph_0)$ satisfying that $C_{\{\xi,B_\xi(k)\}}(C_{\{\xi,B_\xi(j)\}}) = D(\{B_\xi(j), B_\xi(k), \xi\})$ for all $j \in k$ as required.

□

Theorem 1.1 now follows immediately from Lemma 2.2 and Lemma 2.3.

References

[1] Paul B. Larson, Arnold W. Miller, Juris Steprāns, and William A. R. Weiss. Universal functions. *Fund. Math.*, 227(3):197–246, 2014.
[2] Alan H. Mekler. Universal structures in power $\aleph_1$. *J. Symbolic Logic*, 55(2):466–477, 1990.
[3] R. Rado. Universal graphs and universal functions. *Acta Arith.*, 9:331–340, 1964.
[4] Saharon Shelah. On universal graphs without instances of CH. *Ann. Pure Appl. Logic*, 26(1):75–87, 1984.
[5] Saharon Shelah and Juris Steprāns. Universal graphs and functions on ω_1. *Ann. Pure Appl. Logic*, 172(8):Paper No. 102986, 43, 2021.

Department of Mathematics, Rutgers University, Hill Center, Piscataway, New Jersey, U.S.A. 08854-8019, and, Institute of Mathematics, Hebrew University, Givat Ram, Jerusalem 91904, Israel
Email address: shelah@math.rutgers.edu

Department of Mathematics, York University, 4700 Keele Street, Toronto, Ontario, Canada M3J 1P3
Email address: steprans@yorku.ca