THE HEIGHT OF THE AUTOMORPHISM TOWER OF A GROUP

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ABSTRACT. For a group G with trivial center there is a natural embedding of G into its automorphism group, so we can look at the latter as an extension of the group. So an increasing continuous sequence of groups, the automorphism tower, is defined, the height is the ordinal where this becomes fixed, arriving to a complete group. We show that for many such κ there is such a group of cardinality κ which is of height $> 2^\kappa$, so proving that the upper bound essentially cannot be improved.

§ 0. INTRODUCTION

For a group G with trivial center there is a natural embedding of G into its automorphism group $\mathrm{Aut}(G)$ where $g \in G$ is mapped to the inner automorphism $x \mapsto gxg^{-1}$ which is defined and is not the identity for $g \neq e_G$ as G has a trivial center, so we can view $\mathrm{Aut}(G)$ as a group extending G. Also the extension $\mathrm{Aut}(G)$ is a group with trivial center, so we can continue defining $G^{\langle \alpha \rangle}$ increasing with α for every ordinal α; let τ_G be when we stop, i.e., the first α such that $G^{\langle \alpha+1 \rangle} = G^{\langle \alpha \rangle}$ (or $\alpha = \infty$ but see below) hence $\beta > \alpha \Rightarrow G^{\langle \beta \rangle} = G^{\langle \alpha \rangle}$ (see Definition 0.2). How large can τ_G be?

Weilandt [Wie39] proves that for finite G, τ_G is finite. Thomas' [Tho85] celebrated work proves for infinite G that $\tau_G \leq (2^{|G|})^+$, in fact as noted by Felgner and Thomas $\tau_G < (2^{|G|})^+$. Thomas shows also that $\tau_\kappa \geq \kappa^+$. Later he ([Tho98]) showed that if $\kappa = \kappa^{<\kappa}$, $2^\kappa = \kappa^+$ (hence $\tau_\kappa \leq \kappa^{++}$ in $\mathbf{V}$) and $\lambda \geq \kappa^{++}$ and we force by $\mathbb{P}$, the forcing of adding λ Cohen subsets to κ, then in $\mathbf{V}^{\mathbb{P}}$ we still have $\tau_\kappa \leq \kappa^{++}$ though 2^κ is $\geq \lambda$ (and $\mathbf{V}, \mathbf{V}^{\mathbb{P}}$ has the same cardinals).

Just, Shelah and Thomas [JST99] proved that when $\kappa = \kappa^{<\kappa} < \lambda$, in some forcing extension (by a specially constructed κ-complete κ^+-c.c. forcing notion) we have $\tau_\kappa \geq \lambda$, so consistently $\tau_\kappa > 2^\kappa > \kappa^+$ for some κ. An important lemma there which we shall use (see 0.6 below) is that if G is the automorphism group of a structure of cardinality κ, $H \subseteq G$, and $|H| \leq \kappa$ then $\tau'_{G,H}$, the normalizer length of H in G (see Definition 0.3(2)), is $< \tau_\kappa$. Concerning groups with center, Hamkins shows that $\tau_G <$ the first strongly inaccessible cardinal above $|G|$. On the subject see the forthcoming book of Thomas.

Theorem 0.1. *If κ is strong limit singular of uncountable cofinality <u>then</u> $\tau_\kappa > 2^\kappa$.*

It would have been nice if the lower bound for τ_κ, κ^+, would (consistently) be the correct one for all κ simultaneously, but Theorem 0.1 shows that this is not so.

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Note that Theorem 0.1 shows that provably in ZFC, in general the upper bound $(2^\kappa)^+$ cannot be improved. See Conclusion 3.13 for proof of the theorem, quoting results from pcf theory. We thank Simon Thomas, the referee, Itay Kaplan and Daniel Herden for many valuable complaints detecting serious problems in earlier versions.

The program, described in a simplified way, is that for each so called "κ-parameter $\mathbf{p}$" which includes a partial order $I = I_{\mathbf{p}}$, we define a group $G_{\mathbf{p}}$ and a two element subgroup $H_{\mathbf{p}}$ such that $\langle \mathrm{nor}^\alpha_{G_{\mathbf{p}}}(H_{\mathbf{p}}) : \alpha \le \mathrm{rk}_{\mathbf{p}}\rangle$ "reflects" $\mathrm{rk}_{\mathbf{p}} = \mathrm{rk}^{<\infty}(I_{\mathbf{p}})$, the natural rank on I (see Definition 1.2), so in particular $\tau'_{G_{\mathbf{p}},H_{\mathbf{p}}} = \mathrm{rk}^{<\infty}(I_{\mathbf{p}})$. (Actually in the end we shall get only "H of cardinality $\le \kappa$").

We use an inverse system $\mathfrak{s} = \langle J, \mathbf{p}_u, \pi_{u,v} : u \le_J v\rangle$ of κ-parameters where $\pi_{u,v}$ maps $I_{\mathbf{p}_v}$ to $I_{\mathbf{p}_u}$; however, in general the $\pi_{u,v}$-s do not preserve order (but do preserve it in some weak global sense) where J is an $\aleph_1$-directed partial order. Now for each $u \in J$, we can define the group $G_{\mathbf{p}_u}$; and we can take inverse limit in two ways.

Way 1: The inverse limit $\mathbf{p}_{\mathfrak{s}}$ (with $\pi_{u,\mathfrak{s}}$ for $u \in J$ of $\mathfrak{s}$) is a κ-parameter and so the group $G_{\mathbf{p}_{\mathfrak{s}}}$ is well defined.

Way 2: The inverse system $\langle G_{\mathbf{p}_u}, \hat{\pi}_{u,v} : u \le_J v\rangle$ of groups, where $\hat{\pi}_{u,v}$ is the (partial) homomorphism from $G_{\mathbf{p}_v}$ to $G_{\mathbf{p}_u}$ induced by $\pi_{u,v}$, has an inverse limit $G_{\mathfrak{s}}$.

Now

(A) concerning $G_{\mathbf{p}_{\mathfrak{s}}}$, we normally have good control over $\mathrm{rk}_{\mathbf{p}_{\mathfrak{s}}}$ hence on the normalizer length of $H_{\mathbf{p}_{\mathfrak{s}}}$ inside $G_{\mathbf{p}_{\mathfrak{s}}}$

(B) $G_{\mathfrak{s}}$ is (more exactly can be represented good enough as) inverse limit of groups of cardinality $\le \kappa$ hence is isomorphic to $\mathrm{Aut}(\mathfrak{A})$ for some structure $\mathfrak{A}$ of cardinality $\le \kappa$

(C) in the good case $G_{\mathbf{p}_{\mathfrak{s}}} = G_{\mathfrak{s}}$ so we are done (by 0.6).

In §3 we work to get the main result.

There are obvious possible improvement of the results here, say trying to prove $\delta_\kappa \le \tau_\kappa$ (see Definition 0.5) for every κ. But more importantly, a natural conjecture, at least for me was $\tau_\kappa = \delta_\kappa$ because all the results so far on τ_κ have a parallel for δ_κ (though not inversely). In particular it seemed reasonable that for $\kappa = \aleph_0$ the lower bound was right, i.e., $\tau_\kappa = \omega_1$. See more in Kaplan-Shelah [KS09].

Definition 0.2. 1) For a group G with trivial center, define the group $G^{\langle\alpha\rangle}$ with trivial center for an ordinal α, increasing continuous with α such that $G^{\langle 0\rangle} = G$ and $G^{\langle\alpha+1\rangle}$ is the group of automorphisms of $G^{\langle\alpha\rangle}$ identifying $g \in G^{\langle\alpha\rangle}$ with the inner automorphisms it defines. We may stipulate $G^{\langle -1\rangle} = \{e_G\}$.

[We know that $G^{\langle\alpha\rangle}$ is a group with trivial center increasing continuous with α and for some $\alpha < (2^{|G|+\aleph_0})^+$ we have $\beta > \alpha \Rightarrow G^{\langle\beta\rangle} = G^{\langle\alpha\rangle}$.]

2) The automorphism tower height of the group G is

$$\tau_G = \tau^{\mathrm{atw}}_G = \min\left\{\alpha : G^{\langle\alpha\rangle} = G^{\langle\alpha+1\rangle}\right\}$$

Clearly $\beta \ge \alpha \ge \tau_G \Rightarrow G^{\langle\beta\rangle} = G^{\langle\alpha\rangle}$. (Here 'atw' stands for automorphism tower.)

3) Let $\tau_\kappa = \tau^{\mathrm{atw}}_\kappa$ be the least ordinal τ such that $\tau_G < \tau$ for every group G of cardinality $\le \kappa$; we call it the group tower ordinal of κ.

Now we define the normalizer (group theorists write $N_G(H)$, but probably for others $\mathrm{nor}_G(H)$ will be clearer: at least this is so for the author).

Definition 0.3. 1) Let H be a subgroup of G.

We define $\mathrm{nor}^\alpha_G(H)$, a subgroup of G, by induction on the ordinal α, increasing continuous with α. We may add $\mathrm{nor}^{-1}_G(H) = \{e_G\}$.

Case 1: $\alpha = 0$.
$\mathrm{nor}^0_G(H) = H$.
Case 2: $\alpha = \beta + 1$.
$\mathrm{nor}^\alpha_G(H) = \mathrm{nor}_G(\mathrm{nor}^\beta_G(H))$, see below.
Case 3: α a limit ordinal

$$\mathrm{nor}^\alpha_G(H) = \bigcup\{\mathrm{nor}^\beta_G(H) : \beta < \alpha\}$$

where $\mathrm{nor}_G(H) = \{g \in G : gHg^{-1} = H\}$. (Equivalently, $(\forall x \in H)[gxg^{-1} \in H,\ g^{-1}xg \in H]$.)

2) Let $\tau'_{G,H} = \tau^{\mathrm{nlg}}_{G,H}$, the normalizer length of H in G, be

$$\min\{\alpha : \mathrm{nor}^\alpha_G(H) = \mathrm{nor}^{\alpha+1}_G(H)\}$$

so $\beta \geq \alpha \geq \tau'_{G,H} \Rightarrow \mathrm{nor}^\beta_G(H) = \mathrm{nor}^\alpha_G(H)$. (nlg stands for 'normalizer length.')

3) Let $\tau'_\kappa = \tau^{\mathrm{nlg}}_\kappa$ be the least ordinal τ such that $\tau > \tau'_{G,H}$ whenever $G = \mathrm{Aut}(\mathfrak{A})$ for some structure $\mathfrak{A}$ on κ and $H \subseteq G$ is a subgroup satisfying $|H| \leq \kappa$.

4) $\tau''_\kappa = \tau^{\mathrm{nlf}}_\kappa$ is the least ordinal τ such that $\tau > \tau^{\mathrm{nlg}}_{G,H}$ wherever $G = \mathrm{Aut}(\mathfrak{A})$, $\mathfrak{A}$ a structure of cardinality $\leq \kappa$, H a subgroup of G of cardinality $\leq \kappa$ and

$$\mathrm{nor}^\infty_G(H) = \bigcup\{\mathrm{nor}^\alpha_G(H) : \alpha \text{ an ordinal}\} = G.$$

Definition 0.4. We say that G is a κ-automorphism group <u>if</u> G is the automorphism group of some structure of cardinality $\leq \kappa$.

Definition 0.5. Let $\delta_\kappa = \delta(\kappa)$ be the first ordinal α such that there is no sentence $\psi \in \mathbb{L}_{\kappa^+,\omega}$ satisfying:

(A) $\psi \vdash$ "$<$ is a linear order"
(B) for every $\beta < \alpha$ there is a model M of ψ such that $(|M|, <^M)$ has order type $\geq \beta$.
(C) for every model M of ψ, $(|M|, <^M)$ is a well ordering.

See on this, e.g. [She90, VII,§5].

Our proof of better lower bounds relies on the following result from [JST99].

Lemma 0.6. $\tau'_\kappa \leq \tau_\kappa$.

Question 0.7. 1) Is it consistent that for some κ, $\tau'_\kappa < \tau_\kappa$? Is this provable in ZFC? Is the negation consistent?
2) Similarly for the inequalities $\delta_\kappa < \tau'_\kappa$, (and $\delta_\kappa < \tau'_\kappa < \tau_\kappa$).

Observation 0.8. *For every* $\kappa \geq \aleph_0$ *we have* $\tau^{\mathrm{atw}}_\kappa \geq \tau^{\mathrm{nlg}}_\kappa \geq \tau^{\mathrm{nlf}}_\kappa$.

Proof. By 0.6 and checking the definitions of $\tau^{\mathrm{nlg}}_\kappa, \tau^{\mathrm{nlf}}_\kappa$. In fact we mostly work on proving that in 0.1, $\tau^{\mathrm{nlf}}_\kappa > 2^\kappa$. □

Notation: For a group G and $A \subseteq G$, let $\langle A \rangle_G$ be the subgroup of G generated by A.

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A more detailed explanation of the proof:

We would like to derive the desired group from a partial order I representing the ordinal desired as $\tau'_{G,H}$ in some way and the tower of normalizers of an appropriate subgroup of this length. It seems natural to say that if $t \in I$ represent the ordinal α then the $s <_I t$ will represent ordinals $< \alpha$ so we use the depth in I

$$\mathrm{dp}_I(t) = \bigcup\{\mathrm{dp}_I(s) + 1 : s <_I t\}.$$

For each $t \in I$ we would like to have a generator g_t of the group (we denote the group by K_I and g_t is really denoted by $g_{(\langle t \rangle, \langle\ \rangle)}$) exemplifying that the normalizer tower does not stop at $\alpha = \mathrm{dp}_I(t)$, say g_t will be in the $(\alpha+1)$-th normalizer but not in the α-th normalizer. But we need a witness for g_t not being in the earlier $(\beta+1)$-th normalizer, $\beta < \alpha$.

Now β is represented by some $s <_I t$, so we have witnesses $g_{(\langle t,s\rangle,\langle 0\rangle)}, g_{(\langle t,s\rangle,\langle 1\rangle)}$, the first in the first member of the normalizer sequence, the second in the $(\beta+1)$-th normalizer not in the β-th normalizer. So we have a long normalizer tower of the subgroup $G_I^{<0}$, the one generated by

$$\{g_{(\bar{t},\eta)} : \eta(\ell) = 0 \text{ for some } \ell < \ell g(\eta) = \ell g(\bar{t}) - 1,\ \bar{t} \in {}^{\ell g(\bar{t})}I \text{ is } <_I\text{-decreasing}\}.$$

Now §1 is dedicated to defining and investigating those groups.

However $G_I^{<0} = \langle g_{(\bar{t},\eta)} : \bar{t}$ ends with a $<_I$-minimal member$\rangle$ (which by this scheme will be the first in the normalizer tower described above) is too big. So in 2.1 we use a semi-direct product $K_I = G_I *_{\mathbf{h}} L_I$, where L_I is an abelian group with every element of order two, generated by $\{h_{gG_I^{<0}} : g \in G_I\}$ with $(\mathbf{h}(g_1))h_{gG_I^{<0}} = h_{(g_1 g)G_I^{<0}}$ and try to show that the normalizer tower of the subgroup $H_I = \{e, h_{eG_I^{<0}}\}$ of K_I has the same height.

But we have to make K_I a κ-automorphism group. We only almost have it: (and under the present description necessarily fail) we will represent it as $\mathrm{Aut}(M)/N$ for some structure M of cardinality $\le \kappa$ and normal subgroup N of it of cardinality $\le \kappa$; this suffices.

From where will M come from? We will represent I as an inverse limit of some kind of $\mathfrak{t} = \langle I_u, \pi_{u,v} : u \le_J v\rangle$ where I_u is a partial order of cardinality $\le \kappa, \pi_{u,v}$ a mapping from I_v to I_u (commuting). It seemed natural, *a priori*, to demand that $\pi_{u,v}$ is order preserving but it seemingly does not work out. It seemed natural, *a priori*, to prove that whenever $\mathfrak{t}$ is as above there is an inverse limit, etc. We find it more transparent to treat the matter axiomatically: the limit is given inside, i.e. as $\mathfrak{s}$ which is $\mathfrak{t}$ + a limit v^*; and $J^{\mathfrak{t}} = J^{\mathfrak{s}} \setminus \{v^*\}$ is directed.

Also, we demand that $J^{\mathfrak{t}}$ is $\aleph_1$-directed (otherwise in the limit of the groups we have elements represented as infinite products of limits of the generators).
We shall derive the structure M from $\mathfrak{t}$ so its automorphisms come from members of K_{I_u} (for $u \in J^{\mathfrak{t}}$). Well, not exactly by formal terms for it, to enable us to project to $u' \le_{J[\mathfrak{t}]} u$; recalling that $\pi_{u,v}$ does not necessarily preserve order. To make things smooth we demand that $J^{\mathfrak{t}}$ is a linear order (say, $\mathrm{cf}(\kappa)$) when, as in the main case, κ is singular strong limit of uncountable cofinality.

More specifically, if $s, t \in I$ then for every large enough $u \in J^{\mathfrak{t}}$,

$$s <_{I_{v^*}} t \Leftrightarrow \pi_{u,v^*}(s) <_{I_u} \pi_{u,v^*}(t)$$

(note the order of the quantifiers). Then we define a structure M derived from $\mathfrak{t}$. So the automorphism group of M is the inverse limit of groups which comes from the formal definitions of elements of K_{I_u}-s. Each depend on finitely many generators, which in different u-s give different reduced forms.

Now they are defined from some $\bar{t} \in {}^k(I_u)$ using "I_{v^*} is the inverse limit . . ." The "important" t_u-s, those which really affect, will form an inverse system. (Without loss of generality, the length k is constant on an end segment. Here we use "$J^{\mathfrak{t}}$ is $\aleph_1$-directed.") So for those ℓ-s, the sequence $\langle t_{u,\ell} : u \in J^{\mathfrak{t}}\rangle$ has limit $t_{v^*,\ell}$ (say, for $\ell < k_*$).

So $\langle t_{u_*,\ell} : \ell < k_*\rangle$ has the same quantifier type in I_u whenever $u_* \le u \le v^*$ for some $u_* < v^*$. The other t-s still has influence, so it is enough to find for them a pseudo limit: $t_{v^*,\ell}$ such that they will have the same affect on how the "important" $t_{u,\ell}$ are used (this is the essential limit).

All this gives an approximation to $\mathrm{Aut}(M) \cong K_{I_{v^*}}$. The "almost" means that we divide by the subgroup of the automorphism of M which are id_{K_u} for every

$u \in J^{\mathfrak{t}}$ large enough. This is a normal subgroup of cardinality $\leq \kappa$ so we are done except constructing such systems.

§ 1. Constructing groups from partial orders and long normalizer sequences

Discussion 1.1. Our aim is, for a partial order I, to define a group $G = G_I$ and a subgroup $H = H_I$ such that the normalizer length of H inside G reflects the depth of the well founded part of I. Eventually we would like to use I of large depth such that $|H_I| \leq \kappa$ and the normalizer length of H inside G_I is $> 2^\kappa$, even equal to the depth of I.

For clarity we first define an approximation. In particular, H appears only in §2. How do we define the group $G = G_I$ from the partial order I? For each $t \in I$ we would like to have an element associated with it (it is $g_{(\langle t\rangle,\langle\ \rangle)}$) such that it will "enter" $\text{nor}^\alpha_G(H)$ exactly for $\alpha = \text{rk}_I(t) + 1$. We intend that among the generators of the group commuting is the normal case, and we need witnesses that $g_{(\langle t\rangle,\langle\ \rangle)} \notin \text{nor}^{\beta+1}_G(H)$ wherever $\beta < \text{rk}_I(t)$ and $\beta > 0$. It is natural that if $\text{rk}_I(t_1) = \beta$ and $t_1 <_I t_0 := t$ then we use t_1 to represent β, as witness; more specifically, we construct the group such that conjugation by $g_{(\langle t\rangle,\langle\ \rangle)}$ interchanges $g_{(\langle t_0,t_1\rangle,\langle 0\rangle)}$ and $g_{(\langle t_0,t_1\rangle,\langle 1\rangle)}$ and one of them, say $g_{(\langle t_0,t_1\rangle,\langle 1\rangle)}$, belongs to $\text{nor}^{\beta+1}_G(H)\setminus \text{nor}^\beta_G(H)$ whereas the other one, $g_{(\langle t_0,t_1\rangle,\langle 0\rangle)}$, belongs to $\text{nor}^1_G(H)$. Iterating we get the elements $x \in X_I$ defined below.

To "start the induction," we add to G an element g_* of order 2 getting K_I, commuting with $g \in G$ iff g is intended to be in the low level (e.g. $g_{(\bar t,\eta)}, t_n \in I$ is minimal, see notation below). We could have in this section considered only a partial order I, and the groups G_I (and later K_I) derived from it. But as anyhow we shall use it in the context of κ-p.o.w.i.s., we do it in this frame (of course if $J^{\mathfrak{s}} = \{u\}$, then $\mathfrak{s}$ is essentially just I_u).

Note that for our main result it suffices to deal with the case $\text{rk}(I) < \infty$.

Definition 1.2. Let I be a partial order (so $\neq \varnothing$).
1) $\text{rk}_I : I \to \text{Ord} \cup \{\infty\}$ is defined by $\text{rk}_I(t) \geq \alpha$ iff $(\forall \beta < \alpha)(\exists s <_I t)[\text{rk}_I(s) \geq \beta]$.
2) $\text{rk}^{<\infty}_I(t)$ is defined as $\text{rk}_I(t)$ if $\text{rk}_I(t) < \infty$ and is defined as

$$\bigcup\{\text{rk}_I(s) + 1 : s <_I t,\ \text{rk}_I(s) < \infty\}$$

in general.
3) Let $\text{rk}(I) = \bigcup\{\text{rk}_I(t) + 1 : t \in I\}$ stipulating $\alpha < \infty = \infty + 1$.
4) $\text{rk}^{<\infty}(I) = \bigcup\{\text{rk}_I(t) + 1 : t \in I \text{ and } \text{rk}_I(t) < \infty\}$.
5) Let $I_{[\alpha]} = \{t \in I : \text{rk}_I(t) = \alpha\}$.
6) I is non-trivial <u>when</u> $\{s : s \leq_I t \text{ and } \text{rk}_I(s) \geq \beta\}$ is infinite for every $t \in I$ satisfying $\text{rk}^{<\infty}_I(t) > \beta$ (used in the proof of 1.10(1); if $\text{rk}(I) < \infty$ then it is equivalent to demand "$\text{rk}_I(s) = \beta$").
7) I is explicitly non-trivial <u>when</u> each E_I-equivalence class is infinite, where

$$E_I = \big\{(t_1,t_2) \in I \times I : (\forall s \in I)[s <_I t_1 \Leftrightarrow s <_I t_2 \wedge t_1 <_I s \Leftrightarrow t_2 <_I s]\big\}.$$

Definition 1.3. 1) $\mathfrak{s}$ is a κ-p.o.w.i.s. (partial order weak inverse system) <u>when</u>:

- (A) $\mathfrak{s} = (J, \bar I, \bar\pi)$, so $J = J^{\mathfrak{s}} = J[\mathfrak{s}]$, $\bar I = \bar I^{\mathfrak{s}}$, $\bar\pi = \bar\pi^{\mathfrak{s}}$.
- (B) J is a directed partial order of cardinality $\leq \kappa$.
- (C) $\bar I = \langle I_u : u \in J\rangle = \langle I^{\mathfrak{s}}_u : u \in J\rangle$
- (D) $I_u = I^{\mathfrak{s}}_u$ is a partial order of cardinality $\leq \kappa$.
- (E) $\bar\pi = \langle \pi_{u,v} : u \leq_J v\rangle$

(F) $\pi_{u,v}$ is a partial mapping from I_v into I_u. (No preservation of order is required!)

(G) If $u \le_J v \le_J w$ then $\pi_{u,w} = \pi_{u,v} \circ \pi_{v,w}$.

2) $\mathfrak{s}$ is a p.o.w.i.s. means κ-p.o.w.i.s. for some κ.

3) For $u \in J$ let $X_u = X_u^{\mathfrak{s}}$ be the set of x such that for some $n < \omega$:

(A) $x = (\bar{t}, \eta) = (\bar{t}^x, \eta^x)$

(B) η^x is a function from $\{0, \dots, n-1\}$ to $\{0,1\}$.

(C) $\bar{t} = \langle t_\ell : \ell \le n \rangle = \langle t^x_\ell : \ell \le n \rangle$, where $t_\ell \in I^{\mathfrak{s}}_u$ is $<_{I^{\mathfrak{s}}_u}$-decreasing: i.e. $t_n <_{I^{\mathfrak{s}}_u} t_{n-1} <_{I^{\mathfrak{s}}_u} \dots <_{I^{\mathfrak{s}}_u} t_0$.

3A) In fact, we define X_I similarly for every partial order I, so $X^{\mathfrak{s}}_u = X_{I^{\mathfrak{s}}_u}$.

4) In part (3), for $x \in X^{\mathfrak{s}}_u$, let $n(x) = \ell g(\bar{t}^x) - 1$ and so $t^x_{n(x)}$ is the last element in the sequence $\bar{t}$.

5) For $x \in X^{\mathfrak{s}}_u$ and $n \le n(x)$ let $y = x \restriction n \in X^{\mathfrak{s}}_u$ (with $n(y) = n$) be defined by:

$$\bar{t}^y := \bar{t}^x \restriction (n+1) = \langle t^x_0, \dots, t^x_n \rangle$$

$$\eta^y := \eta^x \restriction n = \eta^x \restriction \{0, \dots, n-1\}.$$

6) We define $\mathrm{rk}^2_u = \mathrm{rk}^{2,\mathfrak{s}}_u$ to be the function from X_u to $\{-1\} \cup \mathrm{Ord} \cup \{\infty\}$ as follows:

(A) If $x \in X_u$ and $\{\eta^x(\ell) : \ell < n(x)\} \subseteq \{1\}$ (e.g., $n(x) = 0$) then let $\mathrm{rk}^2_u(x) := \mathrm{rk}_{I_u}(t^x_{n(x)})$.

(B) If $x \in X_u$ and $\{\eta^x(\ell) : \ell < n(x)\} \nsubseteq \{1\}$ then let $\mathrm{rk}^{2,\mathfrak{s}}_u(x) = -1$. (Yes, -1!)

7) We say that $\mathfrak{s}$ is *nice* <u>when</u> every $I^{\mathfrak{s}}_u$ is non-trivial and $\pi_{u,v}$ is a function from I_v into I_u, i.e., the domain of $\pi^{\mathfrak{s}}_{u,v}$ is I_v.

8) $X^{<\alpha}_u := \{x \in X^{\mathfrak{s}}_u : \mathrm{rk}^2_u(x) < \alpha\}$ and $X^{\le\alpha}_u := \{x \in X^{\mathfrak{s}}_u : \mathrm{rk}^2_u(x) \le \alpha\}$. Note that $X^{\le\alpha}_u = X^{<\alpha+1}_u$ when $\alpha < \infty$. Of course, we may write $X^{<\alpha,\mathfrak{s}}_u, X^{\le\alpha,\mathfrak{s}}_u$ and note that $X^{<0}_u = \{x \in X^{\mathfrak{s}}_u : 0 \in \mathrm{Rang}(\eta^x)\}$.

Definition 1.4. Assume $\mathfrak{s}$ is a κ-p.o.w.i.s. and $u \in J^{\mathfrak{s}}$.

1) Let $G_u = G^{\mathfrak{s}}_u$ be the group generated by $\{g_x : x \in X^{\mathfrak{s}}_u\}$ freely except the equations in $\Gamma_u = \Gamma^{\mathfrak{s}}_u$ where Γ_u consists of

(A) $g_x^{-1} = g_x$; that is, g_x has order 2 for each $x \in X_u$.

(B) $g_{y_1} g_{y_2} = g_{y_2} g_{y_1}$ when $y_1, y_2 \in X_u$ and $n(y_1) = n(y_2)$.

(C) $g_x g_{y_1} g_x^{-1} = g_{y_2}$ when $\circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$ holds (see below).

1A) Let $\circledast_{x,y} = \circledast^u_{x,y} = \circledast^{u,\mathfrak{s}}_{x,y}$ mean that $\circledast_{x,y_1,y_2}$ for some y_1, y_2 such that $y \in \{y_1, y_2\}$, see below.

1B) Let $\circledast_{x,y_1,y_2} = \circledast^u_{x,y_1,y_2} = \circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$ mean that:

(A) $x, y_1, y_2 \in X_u$

(B) $n(x) < n(y_1) = n(y_2)$

(C) $y_1 \restriction n(x) = y_2 \restriction n(x)$

(D) $\bar{t}^{y_1} = \bar{t}^{y_2}$

(E) For $\ell < n(y_1)$ we have: $\eta^{y_1}(\ell) \ne \eta^{y_2}(\ell)$ <u>iff</u> $\ell = n(x) \wedge x = y_1 \restriction n(x)$.

2) Let $G^{<\alpha}_u = G^{<\alpha,\mathfrak{s}}_u$ be defined similarly to $G^{\mathfrak{s}}_u$ except that it is generated only by $\{g_x : x \in X^{<\alpha}_u\}$, freely except the equations from $\Gamma^{<\alpha}_u = \Gamma^{<\alpha,\mathfrak{s}}_u$, where $\Gamma^{<\alpha}_u$ is the set of equations from Γ_u among $\{g_x : x \in X^{<\alpha}_u\}$.

Similarly $G^{\le\alpha}_u, \Gamma^{\le\alpha}_u$; note that $G^{\le\alpha}_u = G^{<\alpha+1}_u, \Gamma^{\le\alpha}_u = \Gamma^{<\alpha+1}_u$ if $\alpha < \infty$.

3) For $X \subseteq X_u$ let $G_{u,X} = G^{\mathfrak{s}}_{u,X}$ be the group generated by $\{g_y : y \in X\}$ freely except the equations in $\Gamma_{u,X} = \Gamma^{\mathfrak{s}}_{u,X}$ which is the set of equations from Γ_u mentioning only generators among $\{g_y : y \in X\}$.

Observation 1.5. *1) The sequence $\langle X^{<\alpha}_u : \alpha \le \mathrm{rk}(I^{\mathfrak{s}}_u)\rangle$ is $\subseteq$-increasing continuous.*
2) If $x, y \in X_u$ are such that $x \ne y = x \restriction n$ then $\mathrm{rk}^2_u(y) \ge \mathrm{rk}^2_u(x)$ and if equality holds then $\mathrm{rk}^2_u(x) = \infty = \mathrm{rk}^2_u(y)$ <u>or</u> both are -1.
3) If a partial order I is explicitly non-trivial <u>then</u> I is non-trivial.

Proof. Check. $\square$

Observation 1.6. *For a κ-p.o.w.i.s. $\mathfrak{s}$.*
1) $\circledast^{u,\mathfrak{s}}_{x,y}$ holds <u>iff</u>:

- *(A) $x, y \in X_u$ and*
- *(B) $n(y) \ge n(x) + 1$.*

2) If $x \in X^{\mathfrak{s}}_u$ then $\{(y_1, y_2) : \circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$ holds$\}$ is a permutation of order two of $\{y \in X^{\mathfrak{s}}_u : n(y) > n(x)\}$.
3) Moreover, the permutation in part (2) maps each $\{y \in X^{\mathfrak{s}}_u : n(y) = k\}$ onto itself when $k \in (n(x), \omega)$ and it maps $\Gamma_{u\{y \in X^{\mathfrak{s}}_u : n(y) > k\}}$ onto itself when $n(x) \le k < \omega$.
4) $\circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$ iff $\circledast^{u,\mathfrak{s}}_{x,y_2,y_1}$.
5) For $x, y \in X^{\mathfrak{s}}_u$, in the group $G^{\mathfrak{s}}_u$ the elements g_x, g_y commute except when $x \ne y \wedge (x = y \restriction n(x) \vee y = x \restriction n(y))$. In this case, if $n(x) < n(y)$ there is $y' \ne y$ such that $\circledast_{x,y,y'}$ and $\eta^y(\ell) = \eta^{y'}(\ell) \Leftrightarrow \ell \ne n(x)$.

Proof. (details on (2),(3) see the proof of 1.7). $\square$

We first sort out how elements in $G^{\mathfrak{s}}_u$ and various subgroups can be (uniquely) represented as products of the generators.

Claim 1.7. *Assume that $\mathfrak{s}$ is a κ-p.o.w.i.s., $u \in J^{\mathfrak{s}}$ and $<^*$ is any linear order of X_u such that*

- *$\boxdot$ if $x \in X_u, y \in X_u$ and $n(x) > n(y)$ <u>then</u> $x <^* y$.*

1) Any member of G_u is equal to a product of the form $g_{x_1} \ldots g_{x_m} (x_\ell \in X_u)$ where $x_\ell <^* x_{\ell+1}$ for $\ell = 1, \ldots, m-1$. Moreover, this representation is unique.
2) Similarly for $G^{\le\alpha}_u, G^{<\alpha}_u$ (using $X^{\le\alpha}_u, X^{<\alpha}_u$ respectively instead X_u) hence $G^{\le\alpha}_u, G^{<\alpha}_u$ are subgroups of G_u.
3) In part (1) we can replace G_u and X_u by $G = G_{u,X}$ and X respectively when $X \subseteq X_u$ is such that $[\{x, y_1, y_2\} \subseteq X_u \wedge \circledast^{u,\mathfrak{s}}_{x,y_1,y_2} \wedge \{x, y_1\} \subseteq X \Rightarrow y_2 \in X]$. Hence $G_{u,X}$ is equal to $\langle g_x : x \in X\rangle_{G_u}$.
4) If $g = g_{y_1} \ldots g_{y_m}$ where $y_1, \ldots, y_m \in X_u$ and $g = g_{x_1} \ldots g_{x_n} \in G_u$ and $x_1 <^* \ldots <^* x_n$ <u>then</u> $n \le m$.
5) $\langle G^{<\alpha}_u : \alpha \le \mathrm{rk}(I^{\mathfrak{s}}_u), \alpha$ an ordinal$\rangle$ is an increasing continuous sequence of groups with last element $G^{<\infty}_u$.
6) $\{gG^{<0}_u : g \in G_u\}$ is a partition of G_u (to right cosets of G_u over $G^{<0}_u$).
7) If $<^1, <^2$ are two linear orders of X_u as in $\boxdot$ above and $G_u \models$ "$g_{x_1} \ldots g_{x_k} = g_{y_1} \ldots g_{y_m}$" and $x_1 <^1 \ldots <^1 x_k$ and $y_1 <^2 \ldots <^2 y_m$ (or just $x_1 <^1 \ldots <^1 x_k, n(y_1) \ge n(y_2) \ge \ldots \ge n(y_m)$ and $\langle y_\ell : \ell = 1, \ldots, m\rangle$ is with no repetitions), <u>then</u>:

- *(A) $k = m$*
- *(B) for every i we have $\{\ell : n(x_\ell) = i\} = \{\ell : n(y_\ell) = i\}$ and this set is a convex subset of $\{1, \ldots, m\}$.*

(So the only difference is permuting $g_{x_{\ell(1)}}, g_{x_{\ell(2)}}$ when $n(x_{\ell(1)}) = n(x_{\ell(2)})$.
8) If $I \subseteq I_u$ and $X = X_I$ then $G_{u,X} \cap G^{<0}_u$ is the subgroup of $G_{u,X}$ generated by

$$\{g_x : x \in X, \mathrm{Rang}(\eta^x) \nsubseteq \{1\}\}$$

i.e., the (naturally defined) $G^{<0}_I, (G_I := G_{u,X_I}, G^{<0}_I := G^{<0}_{u,X_I})$.
9) If $I_\ell \subseteq I^{\mathfrak{s}}_u$ for $\ell = 1, 2, 3$ (so $\le_{I_\ell} = \le_I \restriction I_\ell$) and $I_1 \cap I_2 = I_3$ <u>then</u> $G_{I_1} \cap G_{I_2} = G_{I_3}$ and $G^{<0}_{I_1} \cap G^{<0}_{I_2} = G^{<0}_{I_3}$.

Proof. 1),2),3) Recall that each generator has order two. We can use standard combinatorial group theory (but in the rewriting process below we do not assume knowledge of it); the point is that in the rewriting the number of generators in the word does not increase (so no need of $<^*$ being a well ordering).

We now give a full self-contained proof reducing everything to (3). For part of (2) we consider $G = G_u^{<\alpha}, X = X_u^{<\alpha} \subseteq X_u, \Gamma = \Gamma_u^{<\alpha}$ for α an ordinal or infinity and for part (1) and the rest of part (2) consider $G = G_u^{\leq\beta}, X = X_u^{\leq\beta} \subseteq X_u, \Gamma = \Gamma_u^{\leq\beta}$ for β an ordinal or infinity (recall that G_u, X_u is the case $\beta = \infty$). Now in parts (1),(2) for the set X, the condition from part (3) holds by 1.5(2).

[Why? So assume $\circledast^u_{x,y_1,y_2}$ and e.g. $x, y_1 \in X_u^{<\alpha}$ and we should prove that $y_2 \in X_u^{<\alpha}$. If $y_1 = y_2$ this is trivial so assume $y_1 \neq y_2$, hence necessarily $y_1 \restriction n(x) = x = y_2 \restriction n(x)$ and $n(x) < n(y_1) = n(y_2)$ and $\bar t^{y_1} = \bar t^{y_2}$ and $\eta^{y_1}(\ell) = \eta^{y_2}(\ell) \Leftrightarrow \ell \neq n(x)$. If η^x is not constantly one then also η^{y_2} is not constantly one hence $y_2 \in X_u^{<0}$ so fine. If η^x is constantly one then $\alpha > \mathrm{rk}^2_u(x) = \mathrm{rk}_{I_u}(t^x_{n(x)}) \geq \mathrm{rk}_{I_u}(t^{y_1}_{n(y_1)}) = \mathrm{rk}_{I_u}(t^{y_2}_{n(y_2)}) \geq \mathrm{rk}^2_u(y_2)$ hence $y_2 \in X_u^{\leq\alpha}$ so fine.]

So it is enough to prove part (3). Now recall that $G = G_{u,X}$ and

(A) "$\circledast_1$" every member of G can be written as a product $g_{x_1} \ldots g_{x_n}$ for some $n < \omega, x_\ell \in X$

[Why? As the set $\{g_x : x \in X\}$ generates G and $G \models$ "$g_x^{-1} = g_x$".]

(B) "$\circledast_2$" if in $g = g_{x_1} \ldots g_{x_n}$ we have $x_\ell = x_{\ell+1}$ <u>then</u> we can omit both

[Why? As $g_x g_x = e_G$ for every $x \in X$ by clause (a) of Definition 1.4(1)]

(C) "$\circledast_3$" if $1 \leq \ell < n$ and $g = g_{x_1} \ldots g_{x_n}$ and we have $x_{\ell+1} <^* x_\ell$ and $[m \in \{1, \ldots, n\} \setminus \{\ell, \ell+1\} \Rightarrow y_m = x_m]$ <u>then</u> we can find $y_\ell, y_{\ell+1} \in X$ such that $g = g_{y_1} \ldots g_{y_n}$ and $y_\ell <^* y_{\ell+1}$ and, in fact, $y_{\ell+1} = x_\ell$.

[Why does $\circledast_3$ hold? By Definition 1.4(1) and Observation 1.6(5) one of the following cases occurs. <u>Case 1</u>: $g_{x_\ell}, g_{x_{\ell+1}}$ commutes.

Let $y_\ell = x_{\ell+1}, y_{\ell+1} = x_\ell$. <u>Case 2</u>: Not Case 1 but $\circledast^{u,\mathfrak{s}}_{x_{\ell+1},x_\ell}$, see Definition 1.4(1A).

By clause (b) of Definition 1.4(1B) we have $n(x_{\ell+1}) < n(x_\ell)$. So by $\boxdot$ of the assumption of the present claim we have $x_\ell <^* x_{\ell+1}$, contradiction. <u>Case 3</u>: Not Case 1 but $\circledast^{u,\mathfrak{s}}_{x_\ell,x_{\ell+1}}$, see Definition 1.4(1A).

By 1.6(5) there is $y_\ell \in X$ such that $n(y_\ell) = n(x_{\ell+1}) > n(x_\ell), \bar t^{y_\ell} = \bar t^{x_{\ell+1}}, [i < n(x_{\ell+1}) \Rightarrow (\eta^{y_\ell}(i) = \eta^{x_{\ell+1}}(i) \Leftrightarrow i \neq n(x_\ell))]$ and $\circledast_{x_\ell,x_{\ell+1},y_\ell}$.

Let $y_{\ell+1} = x_\ell$, clearly $y_{\ell+1}, y_\ell \in X$. By Definition 1.4(1), we have $g_{x_\ell} g_{x_{\ell+1}} g_{x_\ell}^{-1} = g_{y_\ell}$ hence $g_{x_\ell} g_{x_{\ell+1}} = g_{y_\ell} g_{x_\ell} = g_{y_\ell} g_{y_{\ell+1}}$ and clearly $n(y_{\ell+1}) = n(x_\ell) < n(y_\ell)$ hence $y_\ell <^* x_\ell = y_{\ell+1}$, so we are done.

The three cases exhaust all possibilities (according to whether $n(x_\ell) = n(x_{\ell+1}), n(x_\ell) > n(x_{\ell+1})$ or $n(x_\ell) < n(x_{\ell+1})$ hence $\circledast_3$ is proved.]

$\circledast_4$ every $g \in G$ can be represented as $g_{x_1} \ldots g_{x_n}$ with $x_1 <^* x_2 <^* \ldots <^* x_n$.

[Why? Really the proofs below of $\circledast_4$ and $\circledast_5$ are incredibly detailed, but try to serve complaints about the proof being only implicit, not to mention errors in earlier versions; so a reader who "sees" those assertions (or parts) can jump ahead.

Without loss of generality g is not the unit of G. By $\circledast_1$ we can find $x_1, \ldots, x_n \in X$ such that $g = g_{x_1} \ldots g_{x_n}$ and $n \geq 1$. Choose such a representation satisfying

$\otimes$ (a) with minimal n and

(b) for this n, with minimal $m \in \{1, \ldots, n+1\}$ such that $x_m <^* \ldots <^* x_n$ and $1 \leq \ell < m \leq n \Rightarrow x_\ell \leq^* x_m$, and

(c) for this pair (n,m) if $m>2$ then with maximal ℓ where $\ell \in \{1,\dots,m-1\}$ satisfies x_ℓ is $<^*$-maximal among $\{x_1,\dots,x_{m-1}\}$ that is $k \in \{1,\dots,m-1\} \Rightarrow x_k \le^* x_\ell$.

Easily there is such a sequence $(x_1,\dots,x_n)$, noting that $m=n+1$ is O.K. for (b) and there is ℓ as in $\otimes(c)$.

By $\circledast_2$ and clause (a) of $\otimes$ we have $x_\ell \ne x_{\ell+1}$ when ℓ from $\otimes(c)$ is well defined (i.e., if $m>2$).

Now $m=2$ is impossible (as then $m=1$ can serve), if $m=1$ we are done, and if $m>2$ then ℓ is well defined and $\ell=m-1$ is impossible (as then $m-1$ can serve instead m). Lastly by $\circledast_3$ applied to this ℓ, we could have improved ℓ to $\ell+1$, contradiction.]

$\circledast_5$ the representation in $\circledast_4$ is unique.

[Why does $\circledast_5$ hold? Assume toward contradiction that $g_{x'_1}\cdots g_{x'_{n_1}} = g_{y'_1}\cdots g_{y'_{n_2}}$ where $x'_1 <^* \dots <^* x'_{n_1}$ and $y'_1 <^* \dots <^* y'_{n_2}$ and $(x'_1,\dots,x'_{n_1}) \ne (y'_1,\dots,y'_{n_2})$. For $k \le m < \omega$ let $X^{<k,m>} = \{x \in X : k \le n(x) < m\}$ and let $G^{<k,m>} = G^{\mathfrak{s}}_{u,X^{\langle k,m\rangle}}$, i.e. be the group generated by $\{g_x : x \in X^{<k,m>}\}$ freely except the equations in $\Gamma^{<k,m>}$, i.e., the equations from $\Gamma_{u,X^{<k,m>}}$, i.e., the equations from Definition 1.4(1) mentioning only its generators, i.e. generators from $\{g_x : x \in X^{<k,m>}\}$. Now clearly if $\circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$, see Definition 1.4(1B) then $n(y_1)=n(y_2) \Rightarrow [y_1 \in X^{<k,m>} \Leftrightarrow y_2 \in X^{<k,m>}]$ so the set $X^{<k,m>} \subseteq X$ satisfies the requirement in part (3) of 1.7 which we are proving; so what we have proved for X holds for $X^{<k,m>}$. In particular $\circledast_1 - \circledast_4$ above gives that for every $g \in G^{<k,m>}$ there are n and $x_1 <^* \dots <^* x_n$ from $X^{<k,m>}$ such that $G^{<k,m>} \models$ "$g = g_{x_1}\cdots g''_{x_n}$. Also it is enough to prove the uniqueness for $G^{<k,m>}$ (for every $k \le m < \omega$), i.e., we can assume $x'_1,\dots,x'_{n_1},y'_1,\dots,y'_{n_2} \in X$ as if the equality holds (though $(x'_1,\dots,x'_{n_1}) \ne (y'_1,\dots,y'_{n_2})$), finitely many equations of $\Gamma_{u,X}$ imply the undesirable equation and for some $k \le m < \omega$ they are all from $\Gamma^{<k,m>}$ and $\{x'_1,\dots,x'_{n_1},y'_1,\dots,y'_{n_2}\} \subseteq X^{<k,m>}$, hence already in $G^{\langle k,m\rangle}$ we get this undesirable equation.

Now for $k<m<\omega$ and $x \in X^{<k,k+1>}$ let $\pi^{k,m}_x$ be the following permutation of $X^{\langle k+1,m\rangle}$:

$\boxdot_0$ $\pi^{k,m}_x$ maps $y_1 \in X^{\langle k+1,m\rangle}$ to y_2 if $\circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$.

It is easy but we shall check that

$\boxdot_1$ For k,m,x as above,
- (a) $\pi^{k,m}_x$ is a permutation of order 2 of $X^{\langle k+1,m\rangle}$ which maps $\Gamma^{\langle k+1,m\rangle}$ onto itself
- (b) $\pi^{k,m}_x$ induces an automorphism $\hat\pi^{k,m}_x$ of $G^{\langle k+1,m\rangle}$: the one mapping g_{y_1} to g_{y_2} when $\pi^{k,m}_x(y_1)=y_2$
- (c) the automorphisms $\hat\pi^{k,m}_x$ of $G^{\langle k+1,m\rangle}$ for $x \in X^{<k,k+1>}$ pairwise commute
- (d) the automorphism $\hat\pi^{k,m}_x$ of $G^{\langle k+1,m\rangle}$ is of order two.

Why $\boxdot_1$? By Definition 1.4(1B) we have $\circledast_{x,y,y_1} \wedge \circledast_{x,y,y_2} \Rightarrow y_1 = y_2$ hence $\pi^{k,m}_x$ is a partial function. Next if $y \in X^{<k+1,m>}$ then $n(y) \ge k+1 > k = n(x)$ hence by 1.6(1) we have $\circledast_{x,y}$, which by Definition 1.4(1A) there is $y_1 \in X$ such that $\circledast_{x,y,y_1}$, this implies $n(y_1)=n(y)$ so as $y \in X^{<k+1,m>}$ also $y_1 \in X^{<k+1,m>}$, so $[y \in X^{<k+1,m>} \Rightarrow \pi^{k,m}_x(y) = y_1 \in X^{<k+1,m>}]$. So $\pi^{k,m}_x$ is a function from $X^{<k+1,m>}$ onto itself. By 1.6(4) we have $\pi^{k,m}_x(y_1)=y_2 \Rightarrow \pi^{k,m}_x(y_2)=y_1$ hence $\pi^{k,m}_x$ is one to one (so is a permutation) and has order two, so the first phrase of (i) holds. For the second phrase it suffices to show that every equation from $\Gamma^{<k+1,m>}$ is mapped to an equation from the same set. If the equation is from

Definition 1.4(1)(a), i.e. $g_y^{-1} = g_y$ it follows from "$\pi_x^{k,m}$ is a permutation of order 2 of $X^{<k+1,m>}$". If the equation is from Definition 1.4(1)(b), i.e. $g_{y_1}g_{y_2} = g_{y_2}g_{y_1}$ where $y_1, y_2 \in X^{<k+1,m>}$ and $n(y_1) = n(y_2)$ then it suffices to note $n(\pi_x^{k,m}(y_1)) = n(y_1) = n(y_2) = n(\pi_x^{k,m}(y_2))$.

Lastly, if the equation is from Definition 1.4(1)(c), i.e. has the form $g_y g_{y_1} g_y^{-1} = g_{y_2}$ where $y, y_1, y_2 \in X^{<k+1,m>}$ and $\circledast_{y,y_1,y_2}$ holds, let $y' = \pi_x^{k,m}(y), y_1' = \pi_x^{k,m}(y_1), y_2' = \pi_x^{k,m}(y_2)$, and it suffices to show that $y', y_1', y_2' \in X^{<k+1,m>}$ and $\circledast_{y',y_1',y_2'}$. First, $y', y_1', y_2' \in X^{<k+1,m>}$ as $\pi_x^{k,m}$ is a permutation of $X^{<k+1,m>}$.

Now, recalling $n(y) \geq k+1 > n(x)$, if $y \restriction n(x) \neq x, y_\ell \restriction n(y) = y$ then for $\ell = 1,2$, as $\circledast_{y,y_1,y_2}$ we have $n(y_\ell) > n(y) > n(x)$ and $y_\ell \restriction n(x) = y \restriction n(x) \neq x$ hence by Definition 1.4(1B), $\circledast_{x,y,y}, \circledast_{x,y_1,y_1}, \circledast_{x,y_2,y_2}$ hence $\pi_x^{k,m}$ maps y, y_1, y_2 to y, y_1, y_2 respectively, so the desired conclusion is trivial. If $(y \restriction n(x) \neq x) \wedge (y_\ell \restriction n(y) \neq y)$ or $(y \restriction n(x) = x) \wedge (y_\ell \restriction n(y) \neq y)$ we can also get the result. So we can assume $y \restriction n(x) = x$ and $y_\ell \restriction n(y) = y$ and as above $y_\ell \restriction n(x) = x$ for $\ell = 1,2$. So by Definition 1.4(1B) as $\circledast_{x,y,y'}$ we have $\bar{t}^y = \bar{t}^{y'}, \eta^y(i) = \eta^{y'}(i) \Leftrightarrow i < n(y) \wedge i \neq n(x)$ and as $\circledast_{x,y_\ell,y_\ell'}$ we have $\bar{t}^{y_\ell'} = \bar{t}^{y_\ell}, \eta^{y_\ell}(i) = \eta^{y_\ell'}(i) \Leftrightarrow i < n(y_\ell) \wedge i \neq n(x)$ for $\ell = 1,2$ and as $\circledast_{y,y_1,y_2}$ we have $\bar{t}^y = \bar{t}^{y_\ell} \restriction (n(y)+1), \eta^{y_1} \restriction n(y) = \eta^{y_2} \restriction n(y) = \eta^y, \bar{t}^{y_1} = \bar{t}^{y_2}$ and $\eta^{y_1}(i) = \eta^{y_2}(i) \Leftrightarrow i < n(y_1) \wedge i \neq n(y)$.

Hence $\bar{t}^{y'} = \bar{t}^{y_\ell'} \restriction (n(y')+1), \bar{t}^{y_1'} = \bar{t}^{y_2'}, \eta^{y_1'} \restriction n(y') = \eta^{y_2'} \restriction n(y') = \eta^{y'}$, and $\eta^{y_1'}(i) = \eta^{y_2'}(i) \Leftrightarrow i < n(y_1') \wedge i \neq n(y')$ recalling $\eta^{y_1'}(i) \neq 1 \Leftrightarrow \eta^{y_1'}(i) = 0$. So we have finished proving clause (i).

Clause (ii) of $\boxdot_1$ follows from clause (i).

As for clause (iii) note that for $x_1 \neq x_2 \in X$ such that $n(x_1) = k = n(x_2)$ and $y \in X^{<k+1,m>}$ we have $\pi_{x_1}^{k,m}(y) \neq y \Rightarrow y \restriction n(x_1) = x_1 \Rightarrow y \restriction n(x_2) = y \restriction n(x_1) = x_1 \neq x_2 \Rightarrow \pi_{x_2}^{k,m}(y) = y$, so "$\pi_{x_1}^{k,m}, \pi_{x_2}^{k,m}$ commute" follows, hence by (ii) it follows that "$\hat{\pi}_{x_1}^{k,m}, \hat{\pi}_{x_2}^{k,m}$ commute" as required.

Lastly, clause (iv) follows from "$\pi_x^{k,m}$ is a permutation <u>of order two of $X^{<k+1,m>}$</u>".

We prove this revised formulation of the uniqueness, the one on $G_{u,X^{<k,m>}}$ by induction on $m-k$.

Note that (recalling assumption $\boxdot$ of 1.7)

(∗) if $x \in X^{<k,k+1>}, y \in X^{<\ell,\ell+1>}$ and $x <^* y$ then $\ell \leq k$.

If $m-k=0$, then $G^{<k,m>}$ is the trivial group so the uniqueness is trivial.

Also the case $k = m-1$ is trivial too as in this case $G^{\langle k,m \rangle}$ is generated by $\{g_x : x \in X^{<k,m>}$, i.e. $x \in X$ and $n(x) = k\}$ freely except that they pairwise commute (i.e. clause (b) of Definition 1.4(1)) and each has order 2 (i.e. clause (a) of Definition 1.4(1)) because clause (c) there is empty in the present case.

So

$\odot$ $G^{<k,k+1>}$ is actually a vector space over $\mathbb{Z}/2\mathbb{Z}$ with basis $\{g_x : x \in X^{<k,k+1>}\}$, well in additive notation, so the uniqueness is clear.

So assume that $m-k \geq 2$, now we need

$\boxdot_{k,m}^2$ if $x_1', \ldots, x_{n_1}', y_1', \ldots, y_{n_2}'$ from $X^{\langle k,m \rangle}$ are as above in $G^{<k,m>}$ then $(x_1', \ldots, x_{n_1}') = (y_1', \ldots, y_{n_2}')$.

We can prove the induction step.

Now we define a mapping π from $\{g_x : x \in X^{<k,k+1>}\}$ to $\mathrm{Aut}(G^{<k+1,m>})$ by $x \mapsto \hat{\pi}_x^{k,m}$. Now $\odot$ above describes $G^{<k,k+1>}$ and by $\boxdot_1$ the mapping π maps $\Gamma^{<k,k+1>}$ to equations which are satisfied by $\mathrm{Aut}(G^{<k+1,m>})$, hence there is a homomorphism $\hat{\pi}$ from $G^{<k,k+1>}$ into $\mathrm{Aut}(G^{<k+1,m>})$.

Hence by 1.9 the twisted product $\hat{G} = G^{<k,k+1>} *_{\hat{\pi}} G^{<k+1,m>}$ is well defined. Let $\varkappa$ be the following mapping from $\{g_x : x \in X^{<k,m>}\}$ to $\hat{G}$: if $x \in X^{<k,k+1>}$

then $\varkappa(g_x) := (g_x, e_{G^{<k+1,m>}}) \in G^{<k,k+1>} \times G^{<k+1,m>}$ and if $x \in X^{<k+1,m>}$ then $\varkappa(g_x) := (e_{G^{<k,k+1>}}, g_x) \in G^{<k,k+1>} \times G^{<k+1,m>}$.

Now easily every equation from $\Gamma^{<k,m>}$ is mapped by $\varkappa$ to an equation satisfied in $\hat{G}$ (if it is from $\Gamma^{<k+1,m>}$ then we use the definition of $G^{<k+1,m>} = G_{u,X^{<k+1,m>}}$, if it is from $\Gamma^{<k,m>} \setminus \Gamma^{<k+1,m>}$, then we check by cases according to the clauses of Definition 1.4(1), if it is clause (a) the equation has the form $g_x^2 = e, x \in X^{<k,k+1>}$ and use $G^{<k,k+1>} \models$ "$g_x^2 = e$". If the equation is from clause (b) then it has the form $g_x g_y = g_y g_x$ where $x, y \in X^{<k,k+1>}$ and use "$G^{<k,k+1>}$ is abelian".

Lastly, if the equation is from clause (c) then the equation has the form $g_x g_{y_1} g_x^{-1} = g_{y_2}$ where $x \in X^{<k,k+1>}, y_1, y_2 \in X^{<k+1,m>}$ and $\circledast_{x,y_1,y_2}$ holds; then we use (e) of 1.9(2).

So as $G^{<k,m>}$ is generated by $\{g_x : x \in X^{<k,m>}\}$ freely except the equations from $\Gamma^{<k,m>}$ it follows that $\varkappa$ can be (uniquely) extended to a homomorphism from $G^{<k,m>}$ into $\hat{G}$. Let us return to the statment in $\circledast_5$. So assume $x'_1 <^* \ldots <^* x'_{n_1}$ and $y'_1 <^* \ldots <^* y'_{n_2}$ are from $X^{<k,m>}$ and $G^{<k,m>} \models$ "$g_{x'_1} \cdots g_{x'_{n_1}} = g_{y'_1} \cdots g_{y'_{n_2}}$".

If $\{x'_i, y'_j : i = 1, \ldots, n_1$ and $j = 1, \ldots, n_2\} \subseteq X^{<k+1,m>}$ using $\varkappa$ and recalling 1.9(2)(d) and that G_2 there stands for $G^{<k+1,m>}$ here we get a counterexample to $\circledast_5$ for $G^{<k+1,m>}$ but $m-(k+1) < m-k$ so we are done by the induction hypothesis. So by the demand on $<^*$, we have $x'_{n_1} \in X^{<k,k+1>} \vee y'_{n_2} \in X^{<k,k+1>}$. Now let $\hat{n}_1, \hat{n}_2$ be such that $g_{x_i} \in G^{<k+1,m>} \Leftrightarrow i < \hat{n}_1$ and $g_{y_j} \in G^{<k+1,m>} \Leftrightarrow j < \hat{n}_2$.

Let $\hat{\varkappa}_1 : G^{<k,m>} \to G^{<k,k+1>}$ and $\hat{\varkappa}_2 : G^{<k,m>} \to G^{<k+1,m>}$ be such that $g \in G^{<k,m>} \Rightarrow \varkappa(g) = (\hat{\varkappa}_1(g), \hat{\varkappa}_2(g))$. Applying $\hat{\varkappa}_1$ clearly $g_{x_{\hat{n}_1}} g_{x_{\hat{n}_1+1}} \cdots g_{x_{n_1}} = g_{y_{\hat{n}_2}} g_{y_{\hat{n}_2+1}} \cdots g_{y_{n_2}}$ and $(x_{\hat{n}_1}, x_{\hat{n}_1+1}, \ldots, x_{n_1}) = (y_{\hat{n}_2}, y_{\hat{n}_2+1}, \ldots, y_{n_2})$ with $\odot$, "dividing" $G^{<k,m>} \models$ "$g_{x_1} \cdots g_{x_{\hat{n}_1-1}} = g_{y_1} \cdots g_{y_{\hat{n}_2-1}}$" and we have dealt with this above. So 1),2),3) holds.

4) Included in the proof of $\circledast_4$ inside the proof of parts (1),(2),(3).

5) For $\alpha < \beta \leq \infty$, clearly $X_u^{<\alpha} \subseteq X_u^{<\beta}$ and $\Gamma_u^{<\alpha} \subseteq \Gamma_u^{<\beta}$ hence there is a homomorphism from $G_u^{<\alpha}$ into $G_u^{<\beta}$. This homomorphism is one-to-one (because of the uniqueness clause in part (2)) hence the homomorphism is the identity. So the sequence is $\subseteq$-increasing, the continuity follows by $\mathrm{rk}_u^2(x) = \alpha < \infty \Leftrightarrow g_x \in G_u^{<\alpha+1} \setminus G_u^{<\alpha}$.

6),7),8),9) Easy. $\square$

Observation 1.8. *Assume that* $\mathbf{n}$ *is a natural number* > 1, G *a group and* J *a set with:*

- *(A)* f_t *is an automorphism of* G *of order* $\mathbf{n}$ *for* $t \in J$ *(i.e.* $f_t^{\mathbf{n}} = \mathrm{id}_G$*)*
- *(B)* $f_t, f_s \in \mathrm{Aut}(G)$ *commute for any* $s, t \in J$.

Then there are K *and* $\langle g_t : t \in J \rangle$ *such that*

- (α) *K is a group*
- (β) *G is a normal subgroup of K*
- (γ) *K is generated by $G \cup \{g_t : t \in J\}$*
- (δ) *if $a \in G$ and $t \in J$ then $g_t^{-1} a g_t = f_t(a)$*
- (ε) *if $<_*$ is a linear order of J then every member of K has a one and only one representation as $g_{t_1}^{b_1} g_{t_2}^{b_2} \ldots g_{t_n}^{b_n} x$ where $x \in G, n < \omega, t_1 <_* \ldots <_* t_n$ are from J and $b_1, \ldots, b_n \in \{1, \ldots, \mathbf{n}-1\}$*
- (ζ) *$g_t^{\mathbf{n}} = e_G$.*

Proof. A case of twisted product, see below. (Compare also with the proof of 1.7(3), $\boxdot_{k,m}^2$). Set $K = \bigoplus_{t \in J} \mathbb{Z}/\mathbf{n}\mathbb{Z} g_t *_\pi G$, where $\pi(g_t) = f_t \in \mathrm{Aut}(G)$. $\square$

Claim 1.9. *1) Assume G_1, G_2 are groups and π is a homomorphism from G_1 into $\mathrm{Aut}(G_2)$, we define the twisted product $G = G_1 *_\pi G_2$ as follows:*

- *(A) the set of elements is $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$*
- *(B) the product operation is $(g_1, g_2) * (h_1, h_2) = (g_1 h_1, g_2^{\pi(h_1)} h_2)$ where*
  - *(α) $g_2^{\pi(h_1)}$ is the image of g_2 by the automorphism $\pi(h_1)$ of G_2*
  - *(β) $g_1 h_1$ is a G_1-product*
  - *(γ) $g_2^{\pi(h_1)} h_2$ is a G_2-product.*

2)

- *(A) such group G exists*
- *(B) in G every member has one and only one representation as $g_1' g_2'$ where $g_1' \in G_1 \times \{e_{G_2}\}, g_2' \in \{e_{G_1}\} \times G_2$*
- *(C) the mapping $g_1 \mapsto (g_1, e)$ embeds G_1 into G*
- *(D) the mapping $g_2 \mapsto (e, g_2)$ embeds G_2 into G*
- *(E) so up to renaming, for each $h_1 \in G_1$ conjugating by it (i.e. $g \mapsto h_1^{-1} g h_1$) inside G acts on G_2 as the automorphism $\pi(h_1)$ of G_2.*

3) If H_1, H_2 are subgroups of G_1, G_2 respectively, and $g_1 \in H_1 \Rightarrow \pi(g_1)$ maps H_2 onto itself and $\pi' : H_1 \to \mathrm{Aut}(H_2)$ is $\pi'(x) = \pi(x) \restriction H_2$ <u>then</u> $\{(h_1, h_2) : h_1 \in H_1, h_2 \in H_2\}$ is a subgroup of $G_1 *_\pi G_2$ and is in fact $H_1 *_{\pi'} H_2$; we denote π' by $\pi[H_1, H_2]$.
4) If the pairs (H_1^a, H_2^a) and (H_1^b, H_2^b) are as in part (3) and $H_1^c := H_1^a \cap H_1^b, H_2^c := H_2^a \cap H_2^b$ <u>then</u> the pair (H_1^c, H_2^c) is as in part (3) and $(H_1^a *_{\pi[H_1^a, H_2^a]} H_2^a) \cap (H_1^b *_{[H_1^b, H_2^b]} H_2^b) = (H_1^c *_{\pi[H_1^c, H_2^c]} H_2^c)$.

Proof. Known and straight. $\square$

Claim 1.10. *Let $\mathfrak{s}$ be a κ-p.o.w.i.s., $u \in J^{\mathfrak{s}}$ and $I_u = I_u^{\mathfrak{s}}$ be non-trivial, see Definition 1.2(6).*
1) If $0 \le \alpha < \infty$ <u>then</u> the normalizer of $G_u^{<\alpha}$ in G_u is $G_u^{<\alpha+1}$.
2) If $\alpha = \mathrm{rk}^{<\infty}(I_u)$ <u>then</u> the normalizer of $G_u^{<\alpha}$ in G_u is $G_u^{<\infty} = G_u^{<\alpha}$.

Proof. 1) First

- $(*)_1$ if $x \in X_u$ and $\mathrm{rk}_u^2(x) = \alpha$ then conjugation by g_x in G_u maps $\{g_y : y \in X_u^{<\alpha}\} = \{g_y : y \in X_u \text{ and } \mathrm{rk}_u^2(y) < \alpha\}$ onto itself.

[Why? As $g_x = g_x^{-1}$ it is enough to prove that conjugation by g_x maps the set into itself, i.e. to prove for every $y \in X_u^{<\alpha}$ that: $g_x g_y g_x^{-1} \in \{g_z : z \in X_u^{<\alpha}\}$. As $\mathrm{rk}_u^2(x) = \alpha$ and $\alpha \ge 0$ by the assumptions of the claim it follows that $\mathrm{Rang}(\eta^x) \subseteq \{1\}$.

Now for each such y, one of the following cases occurs. <u>Case (i)</u>: g_x, g_y commutes so $g_x g_y g_x^{-1} = g_y \in \{g_z : z \in X_u^{<\alpha}\}$.

In this case the desired conclusion holds trivially. <u>Case (ii)</u>: $n(y) \le n(x)$ and not case (i).

As case (i) does not occur, necessarily $n(y) < n(x)$ and $y = x \restriction n(y)$ by 1.6(5). Also it follows that $t_{n(x)}^x <_{I_u^{\mathfrak{s}}} t_{n(y)}^y$, so as $\mathrm{rk}_{I_u}(t_{n(x)}^x) = \mathrm{rk}_u(x) = \alpha < \infty$ (recalling $\mathrm{Rang}(\eta^x) \subseteq \{1\}$) we have $\mathrm{rk}_{I_u}(t_{n(y)}^y) > \alpha$. Now $\mathrm{Rang}(\eta^y) \subseteq \mathrm{Rang}(\eta^x) \subseteq \{1\}$, so necessarily $\mathrm{rk}_u^2(y) > \alpha$, contradiction. <u>Case (iii)</u>: $n(y) > n(x)$ and not case (i).

As in case (ii) by 1.6(5) we have $x = y \restriction n(x)$.

Clearly $t^y_{n(y)} <_{I^{\mathfrak{s}}_u} t^y_{n(x)} = t^x_{n(x)}$ so as $\mathrm{rk}^2_u(x) \geq 0$ necessarily $\mathrm{rk}_{I_u}(t^x_{n(x)}) = \mathrm{rk}^2_u(x) = \alpha \in [0,\infty)$ hence $\mathrm{rk}_{I_u}(t^y_{n(y)}) < \mathrm{rk}_{I_u}(t^x_{n(x)}) = \alpha$ and so $\mathrm{rk}^2_u(y) \leq \mathrm{rk}_{I_u}(t^y_{n(y)}) < \alpha$.

Let $y_1 = y$ and by 1.6(1),(5) and Definition 1.4(1A) there is y_2 such that $\circledast^{u,\mathfrak{s}}_{x,y_1,y_2}$ hence $G_u \models$ "$g_x g_y g_x^{-1} = g_{y_2}$" and $\bar{t}^y = \bar{t}^{y_1} = \bar{t}^{y_2}$, so $\mathrm{rk}^2_u(y_2) \leq \mathrm{rk}_{I_u}(t^{y_2}_{n(y_2)}) = \mathrm{rk}_{I_u}(t^{y_1}_{n(y_1)}) < \alpha$ hence $y_2 \in X^{<\alpha}_u$ and so $g_{y_2} \in G^{<\alpha}_u$ so we are done.
So $(*)_1$ holds.] Now by $(*)_1$ it follows that g_x normalizes $G^{<\alpha}_u$ for every member g_x of $\{g_x : \mathrm{rk}^2_u(x) = \alpha\}$, hence clearly $\mathrm{nor}_{G_u}(G^{<\alpha}_u) \supseteq (G^{<\alpha}_u) \cup \{g_x : \mathrm{rk}^2_u(x) = \alpha$ and $x \in X_u\}$ but the latter generates $G^{<\alpha+1}_u$ hence

$(*)_2$ $\mathrm{nor}_{G_u}(G^{<\alpha}_u) \supseteq G^{<\alpha+1}_u$.

Second assume $g \in G_u \setminus G^{<\alpha+1}_u$, let $<^*$ be a linear ordering of X_u as in $\boxdot$ of 1.7. We can find $k < \omega$ and $x_1,\dots,x_k$ from X_u such that $g = g_{x_1} g_{x_2} \dots g_{x_k}$ and so it suffices to prove by induction on k that: if $g = g_{x_1} \dots g_{x_k} \in G_u \setminus G^{<\alpha+1}_u$ then $g \notin \mathrm{nor}_{G_u}(G^{<\alpha}_u)$. By 1.7(1),(4) without loss of generality $x_1 <^* \dots <^* x_k$. As $g \notin G^{<\alpha+1}_u$ necessarily not all the x_m-s are from $X^{<\alpha+1}_u$ hence for some $m, g_{x_m} \notin G^{<\alpha+1}_u$.

$(*)_3$ Without loss of generality $g_{x_1}, g_{x_k} \notin G^{<\alpha+1}_u$.

[Why? So assume $g_{x_k} \in G^{<\alpha+1}_u$ hence

(a) (a) $g_{x_k} \in \mathrm{nor}_{G_u}(G^{<\alpha}_u)$ (as we have already proved $G^{<\alpha+1}_u \subseteq \mathrm{nor}_{G_u}(G^{<\alpha}_u)$)

(b) (b) $\mathrm{nor}_{G_u}(G^{<\alpha}_u)$ is a subgroup of G_u hence

(c) (c) $g = g_{x_1} \dots g_{x_{k-1}} g_{x_k} \in \mathrm{nor}_{G_u}(G^{<\alpha}_u)$ iff $g_{x_1} \dots g_{x_{k-1}} \in \mathrm{nor}_{G_u}(G^{<\alpha}_u)$.

By the induction hypothesis on k we are done. Similarly if $g_{x_1} \in G^{<\alpha+1}_u$ then derive $g \in \mathrm{nor}_{G_u}(G^{<\alpha}_u)$ iff $g_{x_2} \dots g_{x_k} \in \mathrm{nor}_{G_u}(G^{<\alpha}_u)$ to finish.]

As $\mathrm{rk}^2_u(x_1) \geq \alpha+1$ and I_u is non-trivial (recall Definition 1.2(6)) we can find $t^* \in I_u$ such that

$(*)_4$ (a) $t^* <_{I_u} t^{x_1}_{n(x_1)}$

(b) $\mathrm{rk}_{I_u}(t^*) \geq \alpha$

(c) $t^* \notin \{t^x_\ell : x \in \{x_1,\dots,x_k\}$ and $\ell \in \{0,\dots,n(x)\}\}$.

Let $m(*)$ be maximal such that $1 \leq m(*) \leq k$ and $(\exists i)(x_{m(*)} = x_1 \restriction i)$.
Now we choose $y \in X^{\mathfrak{s}}_u$ as follows:

$(*)_5$ (a) $\bar{t}^y = \bar{t}^{x_{m(*)}} {}^\frown \langle t^* \rangle$

(b) $\eta^y \restriction n(x_{m(*)}) = \eta^{x_{m(*)}}$

(c) $\eta^y(n(x_{m(*)})) = 0$.

Note that

$(*)_6$ $x_{m(*)} = y \restriction n(x_{m(*)})$ and $y \in X^{<0}_u$ and $n(y) = n(x_{m(*)}) + 1$ and

$(*)_7$ $n(x_1) \geq \dots \geq n(x_{m(*)}) \geq n(x_{m(*)+1}) \geq \dots \geq n(x_k)$.

[Why? Recall that the sequence $\langle x_\ell : 1 \leq \ell \leq k\rangle$ is $<^*$-increasing hence by property $\boxdot$ of $<^*$ the sequence $\langle n(x_\ell) : 1 \leq \ell \leq k\rangle$ is non-increasing.]

We now try to define $\langle y_\ell : \ell = 1,\dots,k+1\rangle$ by induction on ℓ as follows :

$(*)_8$ $y_1 = y$ and $G_u \models$ "$g^{-1}_{x_\ell} g_{y_\ell} g_{x_\ell} = g_{y_{\ell+1}}$" if well defined.

So

$(*)_9$ $y_\ell = y$ for $\ell = 1,\dots,m(*)$ and so is well defined.

[Why? We prove it by induction on ℓ. For $\ell = 1$ this is given. So assume that this holds for ℓ and we shall prove it for $\ell+1$ when $\ell+1 \leq m(*)$. Now $\neg(\bar{t}^y = \bar{t}^{x_\ell} \restriction (n(y)+1))$, i.e. $\bar{t}^y$ is not an initial segment of $\bar{t}^{x_\ell}$ by the choice of t^* (and y) and hence $y \neq x_\ell \restriction n(y)$ hence $\neg(y = x_\ell \restriction n(y) \wedge n(y) < n(x_\ell))$

and we also have $\neg(x_\ell = y \restriction n(x_\ell) \wedge n(x_\ell) < n(y))$ as otherwise $x_\ell = x_{m(*)} \restriction n(x_\ell)$ but $n(x_\ell) \geq n(x_{m(*)})$ as $x_\ell <^* x_{m(*)}$ hence $x_\ell = x_{m(*)}$, but $\ell \neq m(*)$ hence $x_\ell \neq x_{m(*)}$, contradiction. Together by 1.6(5) the elements g_y, g_{x_ℓ} commute so as by the induction hypothesis $y_\ell = y$ it follows that $y_{\ell+1} = y$ so we are done.]

Now:

$(*)_{10}$ $y_{m(*)+1}$ is well defined and satisfies $(*)_5(a),(b)$ and also $(*)_5(c)$ when we replace 0 by 1.

[Why? By the definition of G_u in 1.4(1),(1B).]

$(*)_{11}$ $y_{m(*)+1} \notin X^{<\alpha}_u$.

[Why? By $(*)_3, x_1 \notin X^{<\alpha+1}_u$ hence η^{x_1} is constantly one; but $x_{m(*)} = x_1 \restriction n(x_{m(*)})$ hence $\eta^{x_{m(*)}}$ is constantly one. Now $\eta^{y_{m(*)+1}} = \eta^{x_{m(*)}}{}^\frown\langle 1\rangle$ by $(*)_{10}$ hence $\eta^{y_{m(*)+1}}$ is constantly one. So $\mathrm{rk}^2_u(y_{m(*)+1}) = \mathrm{rk}_{I_u}(t^{y_{m(*)+1}}_{n(y_{m(*)+1})}) = \mathrm{rk}_{I_u}(t^*) \geq \alpha$ recalling $(*)_4$, so we are done.]

$(*)_{12}$ if $\ell \in \{m(*)+1,\dots,k+1\}$ then $y_\ell = y_{m(*)+1}$ and y_ℓ is well defined.

[Why? We prove this by induction on ℓ. For $\ell = m(*)+1$ this is trivial by $(*)_{10}$. For $\ell+1 \in \{m(*)+2,\dots,k+1\}$, it is enough to prove that $y_{m(*)+1}, x_\ell$ commute. Now $\neg(\bar t^{y_{m(*)+1}} = \bar t^{x_\ell} \restriction (n(y)+1))$ because $n(y_{m(*)+1}) = n(y) = n(x_{m(*)}) + 1 \geq n(x_\ell) + 1 > n(x_\ell)$ hence $\neg\big(y_{m(*)+1} = x_\ell \restriction n(y_{m(*)+1}) \wedge n(y_{m(*)+1}) < n(x_\ell)\big)$; also $\neg\big(x_\ell = y_{m(*)+1} \restriction n(x_\ell) \wedge n(x_\ell) < n(y_{m(*)+1})\big)$ as otherwise this contradicts the choice of $m(*)$. So by 1.6(5) they commute indeed.]

$(*)_{13}$ $g^{-1} g_y g = g_{y_{k+1}}$.

[Why? We can prove by induction on $\ell = 1,\dots,k+1$ that $(g_{x_1}\cdots g_{x_{\ell-1}})^{-1} g_y (g_{x_1}\cdots g_{x_{\ell-1}}) = g_{y_\ell}$, by the definition of the y_ℓ-s, i.e., by $(*)_8$ and they are well defined by $(*)_9 + (*)_{10} + (*)_{12}$.]

$(*)_{14}$ $g^{-1} g_y g = g_{m(*)+1}$.

[Why? By $(*)_{12}$ and $(*)_{13}$.]

$(*)_{15}$ $g^{-1} g_y g \notin G^{<\alpha}_u$.

[Why? By $(*)_{14} + (*)_{11}$.]

So by $(*)_6$ we have $g_y \in G^{<0}_u \subseteq G^{<\alpha}_u$ and by $(*)_{15}$ we have $g^{-1} g_y g \notin G^{<\alpha}_u$ hence g does not normalize $G^{<\alpha}_u$, so we have carried the induction on k. As g was any member of $G_u \setminus G^{<\alpha+1}_u$ we get $\mathrm{nor}_{G_u}(G^{<\alpha}_u) \subseteq G^{<\alpha+1}_u$.

Together with $(*)_2$ we are done.

2) Follows. □

§ 2. Correcting the group

The $G^{\mathfrak{s}}_u$-s from §1 have long towers of normalizers but the "base", $G^{<0,\mathfrak{s}}_u$ is in general of large cardinality. Hence we replace below $G^{\mathfrak{s}}_u$ by $K^{\mathfrak{s}}_u$ and $G^{<0,\mathfrak{s}}_u$ by $H^{\mathfrak{s}}_u$.

Definition 2.1. Let $\mathfrak{s}$ be a κ-p.o.w.i.s.

1) For $u \in J^{\mathfrak{s}}$:

(A) recall 1.7(6): $\mathcal{A}_u = \mathcal{A}^{\mathfrak{s}}_u := \{g G^{<0}_u : g \in G_u\}$ is a partition of G (to right cosets of $G^{<0}_u$ inside G_u);

(B) for every $f \in G_u$ a permutation ∂_f of $\mathcal{A}_u$ is defined by $\partial_f(g_1 G^{<0}_u) = (f g_1) G^{<0}_u$, we may write it also as $f(g_1 G^{<0}_u)$

(C) let $L_u = L^{\mathfrak{s}}_u$ be the group generated by $\{h_{\mathbf{a}} : \mathbf{a} \in \mathcal{A}_u\}$ freely except $h_{\mathbf{a}} h_{\mathbf{b}} = h_{\mathbf{b}} h_{\mathbf{a}}$ and $h^{-1}_{\mathbf{a}} = h_{\mathbf{a}}$ for $\mathbf{a}, \mathbf{b} \in \mathcal{A}_u$; for $g \in G_u$ let $h_g = h_{g G^{<0}_u}$

(D) let $\mathbf{h}_u = \mathbf{h}^{\mathfrak{s}}_u$ be the homomorphism from G_u into the automorphism group of L_u such that $f \in G_u \wedge \mathbf{a} \in \mathcal{A}_u \Rightarrow (\mathbf{h}_u(f))(h_{\mathbf{a}}) = h_{f\mathbf{a}}$

(E) let $K_u = K^{\mathfrak{s}}_u$ be $G_u *_{\mathbf{h}_u} L_u$, the twisted product of G_u, L_u with respect to the homomorphism $\mathbf{h}_u$, see 1.9, and we identify G_u with $G_u \times \{e_{L_u}\}$ and L_u with $\{e_{G_u}\} \times L_u$

(F) let $H_u = \{(e_{G_u}, h_{e_{G_u} G^{<0}_u}), (e_{G_u}, e_{L_u})\}$ a subgroup of K_u and let $h_* := h_{e_{G_u}} = h_{e_{G_u} G^{<0}_u} \in L_u$, i.e. the pair (e_{G_u}, h_*) is the unique member of H_u which is not the unit.

2) For $\alpha \le \infty$ let $K^{<\alpha}_u = K^{<\alpha,\mathfrak{s}}_u$ be the subgroup $\{(g,h) : g \in G^{<\alpha}_u$ and $h \in L_u\}$ of K_u. Similarly $K^{\le\alpha}_u = K^{\le\alpha,\mathfrak{s}}_u$.

3) For $u \in J^{\mathfrak{s}}$ let

(A) $D_u = D^{\mathfrak{s}}_u = \{(v,g) : v \le_{J[\mathfrak{s}]} u$ and $g \in K^{\mathfrak{s}}_v\}$

(B) $Z^0_u = Z^{0,\mathfrak{s}}_u := \{(\bar t, \eta) : \bar t = \langle t_\ell : \ell \le n\rangle, n < \omega, t_\ell \in I_u$ for each $\ell \le n$ and $\eta \in {}^n 2\}$ and let $z = (\bar t^z, \eta^z) = (\langle t^z_\ell : \ell \le n\rangle, \eta^z)$ and $n(z) = n$ for $z \in Z^0_u$; this is compatible with Definition 1.3(3); note that here $\bar t$ is not necessarily decreasing

(C) $Z^1_u = Z^{1,\mathfrak{s}}_u := \{\langle x_\ell : \ell < k\rangle : k < \omega$, each x_ℓ is from $Z^0_u\}$ and let $z = (\langle x^z_\ell : \ell < k(z)\rangle)$ if $z \in Z^1_u$

(D) $Z_u := Z^0_u \cup Z^1_u$

(E) for $z \in Z_u$ we define $\mathrm{his}(z)$, a finite subset of I_u by
  (a) (α) if $z = (\langle t_\ell : \ell \le n\rangle, \eta) \in Z^0_u$ then $\mathrm{his}(z) = \{t_\ell : \ell \le n\}$
  (b) (β) if $z \in Z^1_u$ say $z = \langle(\langle t^k_\ell : \ell \le \ell_k\rangle, \eta^k) : k < k^*\rangle \in Z^1_u$ then $\mathrm{his}(z) = \{t^k_\ell : k < k^*$ and $\ell \le \ell_k\}$

(F) for $z \in Z_u$ let $n(z) = \Sigma\{\ell_k : k < k^*\}$ if $z = \langle(\langle t^k_\ell : \ell \le \ell_k\rangle, \eta^k) : k < k^*\rangle \in Z^1_u$ and $n(z)$ is already defined if $z \in Z^0_u$ in clause (b).

Observation 2.2. *In Definition 2.1:*
1) For $u \in J^{\mathfrak{s}}, K_u$ is well defined and G_u, L_u are subgroups of K_u (after the identification).
2) For $I \subseteq I^{\mathfrak{s}}_u$ let $L^{\mathfrak{s}}_{u,I}$ be the subgroup of $L^{\mathfrak{s}}_u$ generated by $\{h_{gG^{<0}_u} : g \in G^{\mathfrak{s}}_{u,X_I}\}$. If $I_1, I_2 \subseteq I^{\mathfrak{s}}_u$ then $L^{\mathfrak{s}}_{u,I_1} \cap L^{\mathfrak{s}}_{u,I_2} = L^{\mathfrak{s}}_{u,I_1\cap I_2}$. (Saharon says: The latter should be wrong!)
3) For $I \subseteq I^{\mathfrak{s}}_u$ let $K^{\mathfrak{s}}_{u,I}$ be the subgroup of $K^{\mathfrak{s}}_u$ generated by $G^{\mathfrak{s}}_{u,X_I} \cup L^{\mathfrak{s}}_{u,I}$. Then

(A) $K^{\mathfrak{s}}_{u,I}$ normalizes $L^{\mathfrak{s}}_{u,I}$ inside $K^{\mathfrak{s}}_u$

(B) $K^{\mathfrak{s}}_{u,I}$ is $G^{\mathfrak{s}}_{u,X_I} *_\pi L^{\mathfrak{s}}_{u,I}$ for the natural π, i.e. $\pi = \mathbf{h}^{\mathfrak{s}}_u \restriction G^{\mathfrak{s}}_{u,X_I}$.

Also

(A) if $I_1, I_2 \subseteq I^{\mathfrak{s}}_u$ then $K^{\mathfrak{s}}_{u,I_1} \cap K^{\mathfrak{s}}_{u,I_2} = K^{\mathfrak{s}}_{u,I_1\cap I_2}$.

Proof. Easy (recall 1.7(8),(9), 1.9(2),(3)).

We want to point out that in the proof of clause (2) the following theorem is needed:

If $I_\ell \subseteq I^{\mathfrak{s}}_u$ for $\ell = 1,2, g_\ell \in G_{u,X_{I_\ell}}$ with $h_{g_1} = h_{g_2}, I_3 = I_1 \cap I_2$ then there exists some $g_3 \in G_{u,X_{I_3}}$ with $h_{g_1} = h_{g_2} = h_{g_3}$.

Its proof is similar to 1.7(3) and is left to the reader.

SAHARON FILL! (Daniel)

Please observe: 2.2.2) "If $I_1, I_2 \subseteq I^{\mathfrak{s}}_u$ then $L^{\mathfrak{s}}_{n,I_1} \cap L^{\mathfrak{s}}_{n,I_2} = L^{\mathfrak{s}}_{n,I_1\cap I_2}$" being wrong implies that also the following is wrong:
2.2.3)(c)
2.6.3)(c)(β)
2.7.2) and (2.7.3) – (otherwise add/give proof!)

$\boxdot_9$ on p.38, proof 3.4 (uses 2.7.3)!
3.4 GAME OVER! Saharon, please break the above chain of conclusions!! $\square$

Definition 2.3. 1) If I is a partial order then ${}^k I$ is the set of $\bar t = \langle t_\ell : \ell < k\rangle$ where $t_\ell \in I$.
2) If $\bar t \in {}^k I$ then $\mathrm{tp}_{\mathrm{qf}}(\bar t, \varnothing, I) = \{(\iota, \ell_1, \ell_2) : \iota = 0$ and $I \models$ "$t_{\ell_1} < t_{\ell_2}$" or $\iota = 1$ and $t_{\ell_1} = t_{\ell_2}$ or $\iota = 2$ and $I \models$ "$t_{\ell_1} > t_{\ell_2}$" and $\iota = 3$ if none of the previous cases$\}$.
2A) Let $\mathcal{S}^k = \{\mathrm{tp}_{\mathrm{qf}}(\bar t, \varnothing, I) : \bar t \in {}^k I$ and I is a partial order$\}$.
3) We say $\bar t \in {}^k I$ realizes $p \in \mathcal{S}^k$ when $p = \mathrm{tp}_{\mathrm{qf}}(\bar t, \varnothing, I)$.
4) If $k_1 < k_2$ and $p_2 \in \mathcal{S}^{k_2}$ then $p_1 := p_2 \restriction k_1$ is the unique $p_1 \in \mathcal{S}^{k_1}$ such that if $p_2 = \mathrm{tp}_{\mathrm{qf}}(\bar t, \varnothing, I)$ then $p_1 = \mathrm{tp}_{\mathrm{qf}}(\bar t \restriction k_1, \varnothing, I)$.

Remark 2.4. Below each member of $\Lambda^0_k, \Lambda^1_k, \Lambda^2_k$ will be a description of an element of $G^{\mathfrak s}_u, \mathcal{A}^{\mathfrak s}_u, K^{\mathfrak s}_u$ respectively from a k-tuple of members of $I^{\mathfrak s}_u$. Of course, a member of $Z^{\mathfrak s}_u$ is a description of a generator of $K^{\mathfrak s}_u$.

Definition 2.5. 1) For $k < \omega$ let $\Lambda^0_k = \cup\{\Lambda^0_{k,p} : p \in \mathcal{S}^k\}$ where for $p \in \mathcal{S}^k$ we let $\Lambda^0_{k,p}$ be the set of sequences of the form $\langle(\bar\ell_j, \eta_j) : j < j(*)\rangle$ such that:

- (A) for each j for some $n = n(\bar\ell_j, \eta_j)$ we have $\bar\ell_j = \langle \ell_{j,i} : i \le n(\bar\ell_j, \eta_j)\rangle$ is a sequence of numbers $< k$ of length $n+1$ such that $p = \mathrm{tp}_{\mathrm{qf}}(\bar t, \varnothing, I) \Rightarrow \langle t_{\ell_{j,i}} : i \le n(\bar\ell_j, \eta_j)\rangle$ is $<_I$-decreasing
- (B) for each $j, \eta_j \in {}^n 2$ where $n = n(\bar\ell_j, \eta_j)$.

2) For any p.o.w.i.s. $\mathfrak s, u \in J^{\mathfrak s}, \bar t \in {}^k(I_u)$ and $\rho = \langle(\bar\ell_j, \eta_j) : j < j(*)\rangle \in \Lambda^0_k$, let $g^u_{\bar t,\rho} = g^{u,\mathfrak s}_{\bar t,\rho} = (\ldots g_{(\bar t^j,\eta_j)} \ldots)_{j<j(*)}$, the product taken in $G_u \subseteq K_u$ (so if $j(*) = 0$ it is $e_{G_u} = e_{K_u}$) where

- (A) $\bar t^j = \mathrm{seq}_{\rho,j}(\bar t) := \langle t_{\ell_{j,i}} : i \le n(\bar\ell_j, \eta_j)\rangle$
- (B) if $\bar t^j$ is decreasing (in I_u) then $g_{(\bar t^j,\eta_j)} \in G_u \subseteq K_u$ is already well defined, if not then $g_{(\bar t^j,\eta_j)} := e_{K_u}$.

2A) For a p.o.w.i.s. $\mathfrak s, u \in J^{\mathfrak s}, \bar t \in {}^k(I^{\mathfrak s}_u)$ and $\rho = \langle(\bar\ell_j, \eta_j) : j < j(*)\rangle \in \Lambda^0_k$ let $z^u_{\bar t,\rho} = z^{u,\mathfrak s}_{\bar t,\rho}$ be the following member of $Z^{1,\mathfrak s}_u$: it is $\langle x_{\bar t,\rho,j} : j < j(*)\rangle$ where $x_{\bar t,\rho,j} = x_{\bar t,(\bar\ell_j,\eta_j)} = (\langle t_{\ell_{j,i}} : i \le n(\bar\ell_j, \eta_j)\rangle, \eta_j)$. For $p \in \mathcal{S}^k$ and $\rho = \langle(\bar\ell_j, \eta_j) : j < j(*)\rangle \in \Lambda^0_{k,p}$ let $\mathrm{supp}(\rho) = \cup\{\mathrm{Rang}(\bar\ell_j) : j < j(*)\}$ and if $\bar t \in {}^k(I^{\mathfrak s}_u)$ let $\mathrm{sup}(\bar t, \rho) = \{t_\ell : \ell \in \mathrm{supp}(\rho)\}$.
2B) We say $\rho \in \Lambda^0_{k,p}$ is p-reduced when: $p \in \mathcal{S}^k$ and for every p.o.w.i.s. $\mathfrak s, u \in J^{\mathfrak s}$ and $\bar t \in {}^k(I^{\mathfrak s}_u)$ realizing p (in $I^{\mathfrak s}_u$), for no $\rho' \in \Lambda^0_{k,p}$ do we have $\mathrm{supp}(\rho') \subset \mathrm{supp}(\rho)$ and $g^{u,\mathfrak s}_{\bar t,\rho'} = g^{u,\mathfrak s}_{\bar t,\rho}$.
2C) We say that $\rho \in \Lambda^0_{k,p}$ is explicitly p-reduced when the sequence is with no repetitions and $\langle n(\bar\ell_j, \eta_j) : j < j(*)\rangle$ is non-increasing (the length can be zero).
3) For $k < \omega$ let $\Lambda^1_k = \cup\{\Lambda^1_{k,p} : p \in \mathcal{S}^k\}$ where for $p \in \mathcal{S}^k$ we let $\Lambda^1_{k,p}$ be the set of $\rho = \langle(\bar\ell_j, \eta_j) : j < j(*)\rangle \in \Lambda^0_{k,p}$ such that: for every $\mathfrak s$ and $u \in J^{\mathfrak s}$ if $\bar t \in {}^k(I^{\mathfrak s}_u)$ realizes p then there is no $\rho' \in \Lambda^0_{k,p}$ with $\mathrm{supp}(\rho') \subset \mathrm{supp}(\rho)$ and satisfying $g^{u,\mathfrak s}_{\bar t,\rho} G^{<0}_u = g^{u,\mathfrak s}_{\bar t,\rho'} G^{<0}_u$.
4) For $k < \omega$ and $p \in \mathcal{S}^k$ let $\Lambda^2_{k,p}$ be the set of finite sequences ϱ of length ≥ 1 such that $\varrho(0) \in \Lambda^0_{k,p}$ and $0 < i \Rightarrow \varrho(i) \in \Lambda^1_{k,p}$. Let $\Lambda^2_k = \cup\{\Lambda^2_{k,p} : p \in \mathcal{S}^k\}$.
5) For any $\mathfrak s$, if $u \in J^{\mathfrak s}, \bar t \in {}^k(I_u)$ and $\varrho = \langle \rho_i : i < i(*)\rangle \in \Lambda^2_k$ then $g_{\bar t,\varrho} \in K_u$ (recalling $i(*) \ge 1$) is $g_{\bar t,\rho_0} h_{g_{\bar t,\rho_1}} h_{g_{\bar t,\rho_2}} \ldots h_{g_{\bar t,\rho_{i(*)-1}}}$ (product in K_u) where $g_{\bar t,\rho_\ell}$ is

from clause (2), recalling that $h_g = h_{gG^{<0}_u}$ is from clause (c) of Definition 2.1(1).
5A) For any p.o.w.i.s. $\mathfrak{s}, u \in J^{\mathfrak{s}}, \bar{t} \in {}^k(I^{\mathfrak{s}}_u)$ and $\varrho = \langle \rho_i : i < i(*)\rangle \in \Lambda^2_k$, let $z^u_{\bar{t},\varrho} = z^{u,\mathfrak{s}}_{\bar{t},\varrho}$ be $\langle z^u_{\bar{t},\rho_i} : i < i(*)\rangle$.
5B) For $p \in \mathcal{S}^k$ and $\varrho \in \Lambda^2_{k,p}$ let $\mathrm{supp}(\varrho) = \cup\{\mathrm{supp}(\varrho(i)) : i < i(*)\}$.
5C) We say $\varrho \in \Lambda^2_{k,p}$ is p-reduced when for every p.o.w.i.s. $\mathfrak{s}, u \in J^{\mathfrak{s}}$ and $\bar{t} \in {}^k(I^{\mathfrak{s}}_u)$ realizing p, for no $\varrho' \in \Lambda^2_{k,p}$ do we have (in $K^{\mathfrak{s}}_u$) $g^{u,\mathfrak{s}}_{\bar{t},\varrho'} = g^{u,\mathfrak{s}}_{\bar{t},\varrho}$ and $\mathrm{supp}(\varrho') \subset \mathrm{supp}(\varrho)$.

Definition 2.6. 1) For $\rho_1, \rho_2 \in \Lambda^0_{k,p}$ we say $\rho_1 \mathscr{E}^0_{k,p} \rho_2$ or ρ_1, ρ_2 are 0-p-equivalent when: for every p.o.w.i.s. $\mathfrak{s}$ and $u \in J^{\mathfrak{s}}$ and $\bar{t} \in {}^k(I^{\mathfrak{s}}_u)$ realizing p the elements $g^{u,\mathfrak{s}}_{\bar{t},\rho_1}, g^{u,\mathfrak{s}}_{\bar{t},\rho_2}$ of $G^{\mathfrak{s}}_u$ are equal.
2) For $\rho_1, \rho_2 \in \Lambda^1_{k,p}$ we say $\rho_1 \mathscr{E}^1_{k,p} \rho_2$ or ρ_1, ρ_2 are 1-p-equivalent when: for every p.o.w.i.s. $\mathfrak{s}$ and $u \in J^{\mathfrak{s}}$ and $\bar{t} \in {}^k(I_u)$ realizing p we have $g^{u,\mathfrak{s}}_{\bar{t},\rho_1} G^{<0}_u = g^{u,\mathfrak{s}}_{\bar{t},\rho_2} G^{<0}_u$.
3) For $\varrho_1, \varrho_2 \in \Lambda^2_{k,p}$ we say that $\varrho_1 \mathscr{E}^2_{k,p} \varrho_2$ or ϱ_1, ϱ_2 are 2-p-equivalent, <u>when</u>: for every p.o.w.i.s. $\mathfrak{s}$ and $u \in J^{\mathfrak{s}}$ and $\bar{t} \in {}^k(I_u)$ realizing p the element $g^{u,\mathfrak{s}}_{\bar{t},\varrho_1}$ and $g^{u,\mathfrak{s}}_{\bar{t},\varrho_2}$ of $K^{\mathfrak{s}}_u$ are equal.

Claim 2.7. *Claim 1) In Definition 2.5 parts (2B),(3),(5C) saying "for every p.o.w.i.s. $\mathfrak{s}, u \in J^{\mathfrak{s}}$ and $\bar{t} \in {}^k(I_u)$ realizing p" is equivalent to saying "for some ...".*
2) In Definition 2.6, $\mathscr{E}^\iota_{k,p}$ is an equivalence relation on $\Lambda^\iota_{k,p}$ for $\iota = 0,1,2$. For $\iota = 0,2$ every $\mathscr{E}^\iota_{k,p}$-equivalence class contains a p-reduced member and for $\iota = 0$ even an explicitly p-reduced one. Explicitly p-reduced implies p-reduced.
3) For every p.o.w.i.s. $\mathfrak{s}$, <u>if</u> $u \in J^{\mathfrak{s}}$ and $\bar{t} \in {}^k(I^{\mathfrak{s}}_u)$ realizes $p \in \mathcal{S}^k$ then

- *(A) for $\rho_1, \rho_2 \in \Lambda^0_{k,p}$ we have*
  - *(α) $g^{u,\mathfrak{s}}_{\bar{t},\rho_1} = g^{u,\mathfrak{s}}_{\bar{t},\rho_2}$ iff $\rho_1 \mathscr{E}^0_{k,p} \rho_2$*
  - *(β) if $\bar{t}$ is with no repetition and ρ_1, ρ_2 are explicitly p-reduced, <u>then</u> they are $\rho_1 \mathscr{E}^0_{k,p} \rho_2$ <u>iff</u> letting $\rho_i = \langle(\bar{\ell}^i_j, \eta^i_j) : j < j_i\rangle$ for $i = 1,2$ we have*
    - *(i) $j_1 = j_2$*
    - *(ii) for some permutation π of $\{0,\dots,j_1-1\}$ we have*

      $$(\bar{\ell}^2_j, \eta^2_j) = (\bar{\ell}^1_{\pi(j)}, \eta^1_{\pi(j)})$$ *(so ρ_2 is a permutation of ρ_1, compare 1.7(7))*
- *(B) for $\rho_1, \rho_2 \in \Lambda^1_{k,p}$ we have*
  - *(α) $g^{u,\mathfrak{s}}_{\bar{t},\rho_1} G^{<0}_u = g^{u,\mathfrak{s}}_{\bar{t},\rho_2} G^{<0}_u$ iff $\rho_1 \mathscr{E}^1_{k,p} \rho_2$*
- *(C) for $\varrho_1, \varrho_2 \in \Lambda^2_{k,p}$ we have*
  - *(α) $g^{u,\mathfrak{s}}_{\bar{t},\varrho_1} = g^{u,\mathfrak{s}}_{\bar{t},\varrho_2}$ iff $\varrho_1 \mathscr{E}^2_{k,p} \varrho_2$*
  - *(β) if $\bar{t}$ is with no repetition, $\varrho_1 \mathscr{E}^2_{k,p} \varrho_2$ and ϱ_1, ϱ_2 are p-reduced then $\mathrm{supp}(\varrho_1) = \mathrm{supp}(\varrho_2)$.*

Proof. Straight, (recalling Claim 1.7(3),(7), Observation 2.2(2) and note that (3) elaborates (1)). $\square$

Claim 2.8. *Assume $k < \omega, p \in \mathcal{S}^k$, $\mathfrak{s}$ is a p.o.w.i.s., $u \in J^{\mathfrak{s}}$ and $\bar{t}_1, \bar{t}_2 \in {}^kI$ satisfy $p = \mathrm{tp}_{\mathrm{qf}}(\bar{t}_\ell, \varnothing, I^{\mathfrak{s}}_u)$ for $\ell = 1,2$.*
1) If $\rho \in \Lambda^0_{k,p}$ and ρ is p-reduced and $g_{\bar{t}_1,\rho} = g_{\bar{t}_2,\rho} \in G^{\mathfrak{s}}_u$, <u>then</u> $\bar{t}_2 \restriction \mathrm{supp}(\rho)$ is a permutation of $\bar{t}_1 \restriction \mathrm{supp}(\rho)$.
2) If $\rho \in \Lambda^1_{k,p}$ and $g^{u,\mathfrak{s}}_{\bar{t}_1,\rho} G^{<0}_u = g^{u,\mathfrak{s}}_{\bar{t}_2,\rho} G^{<0}_u$ <u>then</u> $\bar{t}_1 \restriction \mathrm{supp}(\rho)$ is a permutation of $\bar{t}_2 \restriction$

supp(ρ).
3) If $\varrho \in \Lambda^2_{k,p}$ is p-reduced and $g^{u,\mathfrak{s}}_{\bar t_1,\varrho} = g^{u,\mathfrak{s}}_{\bar t_2,\varrho}$ then similarly $\bar t_1 \restriction \mathrm{supp}(\varrho)$ is a permutation of $\bar t_2 \restriction \mathrm{supp}(\varrho)$ and both are with no repetition.
4) For every $\varrho_1 \in \Lambda^2_{k,p}$ there is a p-reduced ϱ_2 such that for every p.o.w.i.s., $u \in J^{\mathfrak{s}}$ and $\bar t \in {}^k(I^{\mathfrak{s}}_u)$ realizing p we have $g^{u,\mathfrak{s}}_{\bar t,\varrho_1} = g^{u,\mathfrak{s}}_{\bar t,\varrho_2}$. (Similarly for $\Lambda^0_{k,p}, \Lambda^1_{k,p}$).

Proof. Straight. $\square$

Definition 2.9. Let $\mathfrak{s}$ be a κ-p.o.w.i.s.
1) For $u \le_{J[\mathfrak{s}]} v$ let $\hat\pi^0_{u,v}$ be the following partial mapping from $Z^{0,\mathfrak{s}}_v$ to $Z^{0,\mathfrak{s}}_u$, recalling Definition 2.1(3)(b):

$x \in \mathrm{Dom}(\hat\pi^0_{u,v})$ iff $x \in Z^{0,\mathfrak{s}}_v$ and $\pi_{u,v}(t^x_\ell)$ is well defined for $\ell \le n(x)$ and then $\hat\pi_{u,v}(x) = (\langle \pi_{u,v}(t^x_\ell) : \ell \le n(x)\rangle, \eta^x)$.
2) For $u \le_{J[\mathfrak{s}]} v$ let $\hat\pi^1_{u,v} = \hat\pi^{1,\mathfrak{s}}_{u,v}$ be the following partial mapping from Z^1_v to Z^1_u: if $z \in Z^1_v$ so $z = \langle (\bar t^k, \eta^k) : k < k^*\rangle$ and $\bar t^k = \langle t^k_\ell : \ell \le \ell_k\rangle, t^k_\ell \in I_v$ for $k < k^*, \ell \le \ell_k$ <u>then</u> $\hat\pi^1_{u,v}(z) = \langle (\langle \pi_{u,v}(t^k_\ell) : \ell \le \ell_k\rangle, \eta^k) : k < k^*\rangle$ when each $\pi_{u,v}(t^k_\ell)$ is well defined.
3) For $u \le_{J[\mathfrak{s}]} v$ let $\hat\pi_{u,v}$ be $\hat\pi^0_{u,v} \cup \hat\pi^1_{u,v}$.
4) For $u \in J^{\mathfrak{s}}$ and $z \in Z_u$ let $\partial_{u,z}$ be the following permutation of $D_u = D^{\mathfrak{s}}_u$ where D_u is from Definition 2.1(3)(a). For each $(v,g) \in D_u$ we define $\partial_{u,z}((v,g))$ as follows: <u>Case 1</u>: $z \in \mathrm{Dom}(\hat\pi^0_{v,u}) \subseteq Z^0_u$ and $\hat\pi_{v,u}(z) \in X^{\mathfrak{s}}_v$, i.e., $\langle \pi_{v,u}(t^z_\ell) : \ell \le n(z)\rangle$ is $<_{I_u}$-decreasing.

Then let $\partial_{u,z}((v,g)) = (v, g_{\hat\pi_{v,u}(z)} g)$ noting $g_{\hat\pi_{v,u}(z)} \in G_v \subseteq K_v$. <u>Case 2</u>: $z \in \mathrm{Dom}(\hat\pi^1_{v,u}) \subseteq Z^1_u$ so $z = \langle x_\ell : \ell < k\rangle$ and $x_\ell \in \mathrm{Dom}(\hat\pi^0_{v,u})$ for $\ell < k$ and let $x'_\ell := \hat\pi^0_{v,u}(x_\ell) \in X^{\mathfrak{s}}_v$ for $\ell < k$.

Then let $\partial_{u,z}((v,g)) = (v,g')$ where $g' \in K_v$ is defined by $h_{g_{x'_0} \cdots g_{x'_{k-1}}} g$, as product in K_v noting $g_{x'_\ell} \in G_v \subseteq K_v$ for $\ell < k$. <u>Case 3</u>: Neither Case 1 nor Case 2.

Then let $\partial_{u,z}((v,g)) = (v,g)$.

Observation 2.10. *In Definitions 2.1, 2.9:*
1) If $u \le_{J[\mathfrak{s}]} v$ <u>then</u> $\hat\pi_{u,v}$ is a partial mapping from Z_v to Z_u.
2) In part (1), $\hat\pi_{u,v}$ maps Z^0_v, Z^1_v to Z^0_u, Z^1_u respectively, that is it maps $Z^\ell_v \cap \mathrm{Dom}(\hat\pi_{u,v})$ into Z^ℓ_u for $\ell = 0,1$.
3) If $u \le_{J[\mathfrak{s}]} v$ and $\mathfrak{s}$ is nice or just $\mathrm{Dom}(\pi_{u,v}) = I_v$ <u>then</u> $\mathrm{Dom}(\hat\pi_{u,v}) = Z_v$.

Proof. 1),2),3) Check. $\square$

Claim 2.11. *1) $\mathrm{nor}_{K_u}(H_u)$ is $K^{<0}_u$ where H_u is from Definition 2.1(1)(f).*
2) $\mathrm{nor}^{1+\alpha}_{K_u}(H_u)$ is $K^{<\alpha}_u$ for $\alpha \ge 0$ if I_u is non-trivial.

Proof. 1) As H_u has two elements e_{K_u} and (e_{G_u}, h_*) clearly an element of K_u normalizes H_u iff it commutes with $h_* \in L_u \subseteq K_u$. Now when does $(g,h) \in G_u *_{\mathbf{h}_u} L_u$ commute with $(e_{G_u}, h_{e_{G_u} G_u^{<0}})$. Note that

$$(g,h)(e_{G_u}, h_{e_{G_u} G_u^{<0}}) = (g, h + h_{e_{G_u} G_u^{<0}})$$

$$(e_{G_u}, h_{e_{G_u} G_u^{<0}})(g,h) = (g, (\mathbf{h}_u(g))(h_{e_{G_u} G_u^{<0}}) + h).$$

As L_u is commutative, "h_* and (g,h) commute in K_u" iff in L_u

$$(\mathbf{h}_u(g))(h_{e_{G_u} G_u^{<0}}) = h_{e_{G_u} G_u^{<0}}.$$

By the definition of $\mathbf{h}_u \in \mathrm{Hom}(G_u, \mathrm{Aut}(L_u))$ in 2.1(1)(d),(e) this means

$$(ge_{G_u})G_u^{<0} = e_{G_u} G_u^{<0}.$$

i.e.

$$g \in G_u^{<0}.$$

We can sum that: $(g,h) \in G_u *_{\mathbf{h}_u} L_u$ belongs to $\mathrm{nor}_{K_u}(H_u)$ iff (g,h) commutes with h_* iff $g \in G_u^{<0}$ iff $(g,h) \in K_u^{<0}$, as required.

2) Let $\mathbf{f}_u : K_u \to G_u$ be defined by $\mathbf{f}_u((g,h)) = g$. Clearly

$(*)_1$ $\mathbf{f}_u$ is a homomorphism from K_u onto G_u and for every ordinal $\alpha \geq 0$, it maps $K_u^{<\alpha}$ onto $G_u^{<\alpha}$ so $\mathbf{f}_u(K_u^{<\alpha}) = G_u^{<\alpha}$ and moreover $\mathbf{f}_u^{-1}(G_u^{<\alpha}) = K_u^{<\alpha}$ (see the definition of $K_u^{<\alpha}$ in 2.1(2)).

Also

$(*)_2$ $\mathrm{Ker}(\mathbf{f}_u) = \{e_{G_u}\} \times L_u \subseteq K_u^{<0}$.

Now we prove by induction on the ordinal $\alpha \geq 0$ that $\mathrm{nor}_{K_u}^{1+\alpha}(H_u) = K_u^{<\alpha}$. For $\alpha = 0$ this holds by part (1). For α limit this holds as both $\langle \mathrm{nor}_{K_u}^{\beta}(H_u) : \beta \leq \alpha \rangle$ and $\langle K_u^{<\beta} : \beta \leq \alpha \rangle$ are increasing continuous.

Lastly, for $\alpha = \beta + 1 > 0$ we have for any $f \in K_u$

$$\begin{aligned} f \in \mathrm{nor}_{K_u}^{1+\alpha}(H_u) &\Leftrightarrow f \in \ \mathrm{nor}_{K_u}(\mathrm{nor}_{K_u}^{1+\beta}(H_u)) \\ &\Leftrightarrow f \in \mathrm{nor}_{K_u}(\mathbf{f}_u^{-1}(G_u^{<\beta})) \\ &\Leftrightarrow f(\mathbf{f}_u^{-1}(G_u^{<\beta}))f^{-1} = \mathbf{f}_u^{-1}(G_u^{<\beta}) \\ &\Leftrightarrow \mathbf{f}_u(f) G_u^{<\beta} \mathbf{f}_u(f)^{-1} = G_u^{<\beta} \\ &\Leftrightarrow \mathbf{f}_u(f) \in \mathrm{nor}_{G_u}(G_u^{<\beta}) \\ &\Leftrightarrow \mathbf{f}_u(f) \in G_u^{<\alpha} \Leftrightarrow f \in K_u^{<\alpha}. \end{aligned}$$

[Why? The first $\Leftrightarrow$ by the definition of $\mathrm{nor}_{K_u}^{1+\alpha}(-)$, the second $\Leftrightarrow$ by the induction hypothesis, the third $\Leftrightarrow$ by the definition of $\mathrm{nor}_{K_u}(-)$, the fourth $\Leftrightarrow$ by $(*)_1$, the fifth $\Leftrightarrow$ by the definition of $\mathrm{nor}_{G_u}(-)$, the sixth $\Leftrightarrow$ by 1.10(1), the seventh $\Leftrightarrow$ by $(*)_1$.] $\square$

Observation 2.12. *Let $\mathfrak{s}$ be a p.o.w.i.s.*
1) For $u \in J^{\mathfrak{s}}$ and $x \in Z_u^{\mathfrak{s}}$ we have: $\partial_{u,x}$ is a well defined function and is a permutation of $D_u^{\mathfrak{s}}$.
2) If $u \leq_{J[\mathfrak{s}]} v$ then $D_u^{\mathfrak{s}} \subseteq D_v^{\mathfrak{s}}$.
3) If $u \leq_{J[\mathfrak{s}]} v$ and $y \in Z_v^{\mathfrak{s}}$ and $x = \hat{\pi}_{u,v}(y)$ then $\partial_{u,x} = \partial_{v,y} \restriction D_u$.

Proof. Straight. $\square$

Definition 2.13. Definition Let $\mathfrak{s}$ be a κ-p.o.w.i.s.
1) Let $\mathbf{S}^k = \{\mathbf{q} : \mathbf{q}$ is a function with domain $\mathcal{S}^k$ and for $p \in \mathcal{S}^k, \mathbf{q}(p) \in \Lambda^2_{k,p}\}$, on $\Lambda^2_{k,p}$, see Definition 2.5(4) above.
2) We say that $\mathbf{q} \in \mathbf{S}^k$ is disjoint when $\langle \text{supp}(\mathbf{q}(p)) : p \in \mathcal{S}^k \rangle$ is a sequence of pairwise disjoint sets. We say that $\mathbf{q}$ is reduced when $\mathbf{q}(p)$ is p-reduced for every $p \in \mathcal{S}^k$.
3) Let $Z^2_u = Z^{2,\mathfrak{s}}_u$ be $\cup\{Z^{2,k}_u : k < \omega\}$, where $Z^{2,k}_u = Z^{2,k,\mathfrak{s}}_u$ is the set of pairs $(\bar{t}, \mathbf{q})$ where $\bar{t} \in {}^k(I^{\mathfrak{s}}_u)$ and $\mathbf{q} \in \mathbf{S}^k$.
4) For $z = (\bar{t}, \mathbf{q}) \in Z^2_u$ let $\partial_{u,z} = \partial^{\mathfrak{s}}_{u,z}$ be the following permutation of D_u: if $v \leq_{J[\mathfrak{s}]} u$ and $(v, g) \in \{v\} \times K_v$ then $\partial^{\mathfrak{s}}_{u,z}((v, g)) = (v, g'g)$ where $g' = g^{v,\mathfrak{s}}_{\pi_{v,u}(\bar{t}),\mathbf{q}(p)}$ where $p = \text{tp}_{\text{qf}}(\pi_{v,u}(\bar{t}), \varnothing, I^{\mathfrak{s}}_v)$, and, of course, $\pi_{v,u}(\langle t_\ell : \ell < k\rangle) = \langle \pi_{v,u}(t_\ell) : \ell < k\rangle$. If $\pi_{v,u}(\bar{t})$ is not well-defined set $g' = 1$ trivially again.
5) For $(\bar{t}, \mathbf{q}) \in Z^2_u$ let $g_{\bar{t},\mathbf{q}} = g^u_{\bar{t},\mathbf{q}} = g^{u,\mathfrak{s}}_{\bar{t},\mathbf{q}} = g_{\bar{t},\mathbf{q}(p)}$ where $p = \text{tp}_{\text{qf}}(\bar{t}, \varnothing, I_u)$. Let $g^v_{\bar{t},\mathbf{q}} = g^{v,\mathfrak{s}}_{\bar{t},\mathbf{q}} = g^v_{\pi_{v,u}(\bar{t}),\mathbf{q}}$ when $v \leq_{J[\mathfrak{s}]} u$ and $\pi_{v,u}(\bar{t})$ is well-defined.

Remark 2.14. We can add $\{\partial^{\mathfrak{s}}_{u,z} : z \in Z^{2,\mathfrak{s}}_u\}$ to the generators of $F^{\mathfrak{s}}_u$ defined in 2.16 below.

Observation 2.15. *In Definition 2.13(4), $\partial^{\mathfrak{s}}_{u,z}$ is a well defined permutation of $D^{\mathfrak{s}}_u$.*

Proof. Easy. □

Definition 2.16. Let $\mathfrak{s}$ be a p.o.w.i.s.
1) Let $F_u = F^{\mathfrak{s}}_u$ be the subgroup of the group of permutations of $D^{\mathfrak{s}}_u$ generated by $\{\partial_{u,z} : z \in Z^{\mathfrak{s}}_u\}$.
2) For a p.o.w.i.s. $\mathfrak{s}$ let $M_{\mathfrak{s}}$ be the following model: set of elements: $\{(u, g) : u \in J^{\mathfrak{s}}$ and $g \in K^{\mathfrak{s}}_u\} \cup \{(1, u, f) : u \in J^{\mathfrak{s}}$ and $f \in F^{\mathfrak{s}}_u\}$. relations: $P^{M_{\mathfrak{s}}}_{1,u}$, a unary relation, is $\{(u, g) : g \in K_u\}$ for $u \in J^{\mathfrak{s}}$,
$P^{M_{\mathfrak{s}}}_{2,u}$, a unary relation is $\{(1, u, f) : f \in F_u\}$ for $u \in J^{\mathfrak{s}}$
$R^{M_{\mathfrak{s}}}_{u,v,h}$, a binary relation, is $\{((v, g), (1, u, f)) : f \in F_u, g \in K_v$ and $f((v, h)) = (v, g)\}$ for $u \in J^{\mathfrak{s}}$ and $v \leq_{J[\mathfrak{s}]} u$ and $h \in K_v$.

Observation 2.17. *If $\mathfrak{s}$ is a κ-p.o.w.i.s. and $v \leq_{J[\mathfrak{s}]} u$ and $f \in F_u$ then f maps $\{v\} \times K_v = P^{M_{\mathfrak{s}}}_{1,v}$ onto itself.*

Remark 2.18. Remark If $\pi \in F^{\mathfrak{s}}_u$ and $v \leq_{J[\mathfrak{s}]} u$ then $\pi \restriction (\{v\} \times K_v)$ comes directly from $K^{\mathfrak{s}}_v$, but the relation between the $\langle \pi \restriction (\{v\} \times K_v) : v \leq_{J[\mathfrak{s}]} u\rangle$ are less clear.

Claim 2.19. *Let $\mathfrak{s}$ be a p.o.w.i.s.*
1) $\varkappa$ is an automorphism of $M_{\mathfrak{s}}$ iff:

⊛ (a) *$\varkappa$ is a function with domain $M_{\mathfrak{s}}$*
  (b) *for every $u \in J^{\mathfrak{s}}$ we have:*
    (α) *$\varkappa \restriction D_u \in F^{\mathfrak{s}}_u$*

(β) *letting* $f_u = \varkappa \restriction D_u$ *we have* $(1,u,f) \in P^{M_\mathfrak{s}}_{2,u} \Rightarrow \varkappa((1,u,f))$

$= (1,u,f_u f)$ *where* $f_u f$ *is the product in* F_u.

2) If $f_u \in F_u$ *for* $u \in J^\mathfrak{s}$ *and* $f_u \subseteq f_v$ *for* $u \le_{J[\mathfrak{s}]} v$ *then there is one and only one automorphism* $\varkappa$ *of* $M_\mathfrak{s}$ *such that* $u \in J^\mathfrak{s} \Rightarrow f_u \subseteq \varkappa$.

Proof. First assume that $\bar f = \langle f_u : u \in J^\mathfrak{s}\rangle$ is as in part (2). We define $\varkappa_{\bar f}$, a function with domain $M_\mathfrak{s}$ by:

$\circledast_1$ (a) if $a = (u,g) \in P^{M_\mathfrak{s}}_{1,u}$ and $u \in J^\mathfrak{s}$ then $\varkappa_{\bar f}(a) = f_u(a)$

(b) if $a = (1,u,f) \in P^{M_\mathfrak{s}}_{2,u}$ then $\varkappa_{\bar f}(a) = (1,u,f_u f)$.

So

$\circledast_2$ (a) $\varkappa_{\bar f}$ is a well defined function

(b) $\varkappa_{\bar f}$ is one to one

(c) $\varkappa_{\bar f}$ is onto $M_\mathfrak{s}$

(d) $\varkappa_{\bar f}$ maps $P^{M_\mathfrak{s}}_{1,u}$ onto $P^{M_\mathfrak{s}}_{1,u}$ and $P^{M_\mathfrak{s}}_{2,u}$ onto $P^{M_\mathfrak{s}}_{2,u}$ for $u \in J^\mathfrak{s}$

(e) also $\bar f' = \langle f_u^{-1} : u \in J^\mathfrak{s}\rangle$ satisfies the condition of part (2) and

$\varkappa_{\bar f'}$ is the inverse of $\varkappa_{\bar f}$

(f) $\varkappa_{\bar f}$ maps $R^{M_\mathfrak{s}}_{u,v,h}$ onto itself.

[Why? The only non-trivial one is clause (f) and in it by clause (e) it is enough to prove that $\varkappa_{\bar f}$ maps $R^{M_\mathfrak{s}}_{u,v,h}$ into $R^{M_\mathfrak{s}}_{u,v,h}$. So assume $v \le_{J[\mathfrak{s}]} u, h \in K_v$ and $((v,g),(1,u,f)) \in R^{M_\mathfrak{s}}_{u,v,h}$ hence $f \in F_u, g \in K_v$ and $f((v,h)) = (v,g)$. So $\varkappa_{\bar f}((v,g)) = f_v((v,g))$ and $\varkappa_{\bar f}(1,u,f) = (1,u,f_u f)$ and we would like to show that $(f_v((v,g)),(1,u,f_u f)) \in R^{M_\mathfrak{s}}_{u,v,h}$.

This means that $(f_u f)((v,h)) = f_v((v,g))$. We know that $f((v,h)) = (v,g)$ hence $(f_u f)((v,h)) = f_u(f((v,h))) = f_u((v,g))$ so we have to show that $f_u((v,g)) = f_v((v,g))$. But $v \le_{J[\mathfrak{s}]} u$ hence (by the assumption on $\bar f$) we have $f_v \subseteq f_u$ hence $f_u((v,g)) = f_v((v,g))$ so we are done.]

So we have shown that

$\circledast_3$ if $\bar f = \langle f_u : u \in J^\mathfrak{s}\rangle$ is as in part (2) then $\varkappa_{\bar f}$ is an automorphism of $M_\mathfrak{s}$.

Next

$\circledast_4$ if $\varkappa \in \mathrm{Aut}(M_\mathfrak{s})$ and $\varkappa \restriction D_u$ is the identity for each $u \in J^\mathfrak{s}$ then $\varkappa = \mathrm{id}_{M_\mathfrak{s}}$.

[Why? By the $P^{M_\mathfrak{s}}_{2,u}$-s, $R^{M_\mathfrak{s}}_{u,v,h}$-s and $F^\mathfrak{s}_u$ being a group of permutations of D_u.]

$\circledast_5$ the mapping $\varkappa \mapsto \langle \varkappa \restriction D_u : u \in J^\mathfrak{s}\rangle$ is a homomorphism from $\mathrm{Aut}(M_\mathfrak{s})$ into $\{\bar f : \bar f \text{ as above}\}$ with coordinatewise product, with kernel $\{\varkappa \in \mathrm{Aut}(M_\mathfrak{s}) : \varkappa \restriction D_u = \mathrm{id}_{D_u}$ for every $u \in J^\mathfrak{s}\}$.

[Why? Easy. Observe that $\varkappa \restriction D_u \in F_u$ for every $u \in J^\mathfrak{s}$.]

$\circledast_6$ the mapping above is onto.

[Why? Easy by $\circledast_3$.

Given $\varkappa \in \mathrm{Aut}(M_\mathfrak{s})$, let $f_u = \varkappa \restriction D_u$. Clearly $f_u \in F_u$ and $u \le_{J[\mathfrak{s}]} v \Rightarrow f_u \subseteq f_v$ so $\bar f = \langle f_u : u \in J^\mathfrak{s}\rangle$ is as above so by $\circledast_3$ we know $\varkappa_{\bar f}$ is an automorphism of $M_\mathfrak{s}$ and $\varkappa_{\bar f}\varkappa^{-1}$ is an automorphism of $M_\mathfrak{s}$ which is the identity on each D_u hence by $\circledast_4$ is $\mathrm{id}_{M_\mathfrak{s}}$. So $\varkappa = \varkappa_{\bar f}$, is as required.]

$\circledast_7$ the mapping above is one to one.

[Why? Easy by $\circledast_4$.]

Together both parts should be clear. $\square$

Definition 2.20. Definition 1) We say that $\mathbf{q}_1, \mathbf{q}_2 \in \mathbf{S}^k$ are $\mathcal{S}$-equivalent where $\mathcal{S} \subseteq \mathcal{S}^k$ when $p \in \mathcal{S} \Rightarrow \mathbf{q}_1(p)\mathscr{E}^2_{k,p}\mathbf{q}_2(p)$.
2) Omitting $\mathcal{S}$ means $\mathcal{S} = \mathcal{S}^k$.

Claim 2.21. *Claim Let $\mathfrak{s}$ be a nice κ-p.o.w.i.s. (or just* $\mathrm{Dom}(\pi_{u,v}) = I_v$ *for all* $u \leq_{J[\mathfrak{s}]}$*).*
1) If $u \in J^{\mathfrak{s}}$ and $f \in F^{\mathfrak{s}}_u$ <u>then</u> for some k and $\bar{t} = \langle \bar{t}_\ell : \ell < k\rangle \in {}^k(I^{\mathfrak{s}}_u)$ and $\mathbf{q} \in \mathbf{S}^k$ we have:

- $(*)$ *$f = \partial_{u,(\bar{t},\mathbf{q})}$ (so if $v \leq_{J[\mathfrak{s}]} u$ then $f \restriction (\{v\} \times K^{\mathfrak{s}}_v)$ is moving by multiplication by $g^v_{\pi_{v,u}(\bar{t}),\mathbf{q}}$, e.g. $g \in K_v \Rightarrow f((v,g)) = (v, g^v_{\pi_{v,u}(\bar{t}),\mathbf{q}} g)$.*

2) $\{\partial_{u,(\bar{t},\mathbf{q})} : (\bar{t},\mathbf{q}) \in Z^2_u\}$ is a group of permutations of $D^{\mathfrak{s}}_u$ which includes $F^{\mathfrak{s}}_u$.
3) For every $\mathbf{q} \in \mathbf{S}^k$ there is a reduced $\mathbf{q}' \in \mathbf{S}^k$ which is equivalent to it (see Definition 2.13(2)).

Proof. 2),3) Straight.
1) We use freely Definition 2.13. Recall that $F^{\mathfrak{s}}_u$ is the group of permutations of $D^{\mathfrak{s}}_u$ generated by $\{\partial_{u,z} : z \in Z^{\mathfrak{s}}_u\}$. Hence it is enough to prove that $f \in F^{\mathfrak{s}}_u$ satisfies the conclusion of the claim in the following cases. <u>Case 0</u>: f is the identity.

It is enough to let $k = 0$ so $\bar{t} = \varnothing$, $\mathcal{S}^k$ is a singleton $\{\varnothing\}$ and $\mathbf{q}(\varnothing)$ is the empty sequence $\langle\langle\ \rangle\rangle \in \Lambda^2_k$ of length 1, i.e. we use in Definition 2.13(3) the case $k = 0$ and in Definition 2.5(1) the case $j(*) = 0$. <u>Case 1</u>: $f = \partial_{u,z}$ where $z \in Z^0_u$.

So $z = (\bar{t}^z, \eta^z)$. We set $k = n(z) + 1, \bar{t} = \bar{t}^z \in {}^k(I^{\mathfrak{s}}_u)$ and define $\mathbf{q}$ as follows:

(A) if $p \in \mathcal{S}^k$ describes a decreasing sequence then
$$\mathbf{q}(p) = \langle(\langle 0,1,2,\dots,k-1\rangle, \eta^z)\rangle \in \Lambda^2_k$$
as sequence of length 1

(B) if not, then $\mathbf{q}(p) = \langle\langle\ \rangle\rangle$ as in Case 0.

<u>Case 2</u>: $f = \partial_{u,z}$ where $z \in Z^1_u$.
Also clear. <u>Case 3</u>: $f = f_1 f_2$ (product in $F^{\mathfrak{s}}_u$) where $f_1, f_2 \in F^{\mathfrak{s}}_u$ satisfy the conclusion of the claim.

Just combine the definitions. Here we make use of $\mathfrak{s}$ being a nice κ-p.o.w.i.s. and 2.10(3) to avoid those cases where it is impossible to choose $\bar{t} \in \mathrm{Dom}\pi_{v,u}$, meaning that $f = \partial_{u,(\bar{t},\mathbf{q})}$ always acts trivially on $\{v\} \times K^{\mathfrak{s}}_v$ while f_1, f_2 may not be trivial themselves. <u>Case 4</u>: $f = f^{-1}$ where $f \in F^{\mathfrak{s}}_u$ satisfies the conclusion of the claim.

Easy, too. $\square$

Remark 2.22. If $q \in \mathcal{S}^k$ and $\mathbf{q}_1, \mathbf{q}_2 \in \mathbf{S}^k$ and $v \leq_{J[\mathfrak{s}]} u, \bar{t} \in {}^k(I_u)$ and $q = \mathrm{tp}_{\mathrm{qf}}(\pi^{\mathfrak{s}}_{v,u}(\bar{t}), \varnothing, I_v)$ and $\mathbf{q}_1(q), \mathbf{q}_2(q)$ are not $\mathscr{E}^2_{k,q}$-equivalent, <u>then</u> $g^v_{\bar{t},\mathbf{q}_1} \neq g^v_{\bar{t},\mathbf{q}_2}$.

Proof. This is by Claim 2.7(3C). $\square$

§ 3. The main result

We can prove that every κ-p.o.w.i.s has a limit, but for our application it is more transparent to consider κ-p.o.w.i.s $\mathfrak{s}$ which is the κ-p.o.w.i.s. $\mathfrak{t}$ + its limit.

Definition 3.1. We say that $\mathfrak{s}$ is the limit of $\mathfrak{t}$ as witnessed by v_* when (both are p.o.w.i.s. and)

(A) $J^{\mathfrak{t}} \subseteq J^{\mathfrak{s}}$ and $J^{\mathfrak{s}} = J^{\mathfrak{t}} \cup \{v_*\}, v_* \notin J^{\mathfrak{t}}$ and $u \in J^{\mathfrak{s}} \Rightarrow u \leq_{J[\mathfrak{s}]} v_*$

(B) $I^{\mathfrak{s}}_u = I^{\mathfrak{t}}_u$ and $\pi^{\mathfrak{s}}_{u,v} = \pi^{\mathfrak{t}}_{u,v}$ when $u \leq_{J[\mathfrak{s}]} v <_{J[\mathfrak{s}]} v_*$

(C) if $t \in I^{\mathfrak{s}}_{v_*}$ then for some $u = u_t \in J^{\mathfrak{t}}$ we have $t \in \mathrm{Dom}(\pi^{\mathfrak{s}}_{u_t,v_*})$

(D) if $s,t \in I^{\mathfrak{s}}_{v_*}$ <u>then</u> for some $u = u_{s,t} \in J^{\mathfrak{t}}$ for every v satisfying $u \leq_{J[\mathfrak{s}]} v \leq_{J[\mathfrak{s}]} v_*$ we have $I^{\mathfrak{s}}_{v_*} \models$ "$s \leq t$" $\Leftrightarrow \pi^{\mathfrak{s}}_{v,v_*}(s) \leq_{I^{\mathfrak{s}}_v} \pi^{\mathfrak{s}}_{v,v_*}(t)$

(E) if $\langle t_u : u \in J^{\mathfrak{t}}_{\geq w}\rangle$ is a sequence satisfying $w \in J^{\mathfrak{t}}, J^{\mathfrak{t}}_{\geq w} = \{u : w \leq u \in J^{\mathfrak{t}}\}; t_u \in I^{\mathfrak{s}}_u$ and $w \leq u_1 \leq u_2 \in J^{\mathfrak{t}} \Rightarrow \pi_{u_1,u_2}(t_{u_2}) = t_{u_1}$, <u>then</u> there is a unique $t \in I^{\mathfrak{s}}_{v_*}$ such that $u \in J^{\mathfrak{t}}_{\geq w} \Rightarrow \pi_{u,v_*}(t) = t_u$.

Definition 3.2. We say that $\mathfrak{s}$ is an existential limit of $\mathfrak{t}$ <u>when</u>: clauses (a)-(e) of Definition 3.1 hold and

(A) assume that
  - (α) $u_* \in J^{\mathfrak{t}}$
  - (β) $k_1, k_2 < \omega$ and $k = k_1 + k_2$
  - (γ) $\mathscr{E}$ is an equivalence relation on $\mathcal{S}^k$
  - (δ) $\bar{e} = \langle e_u : u \in J^{\mathfrak{t}}_{\geq u_*}\rangle$, where e_u is an $\mathscr{E}$-equivalence class
  - (ε) $\bar{t} \in {}^{k_1}(I^{\mathfrak{s}}_{v_*})$
  - (ζ) for every $v \in J^{\mathfrak{t}}_{\geq u_*}$ there is $\bar{s}_v \in {}^{k_2}(I^{\mathfrak{t}}_v)$ such that:

    if $u_* \leq_{J[\mathfrak{t}]} u \leq_{J[\mathfrak{t}]} v$ then e_u is the $\mathscr{E}$-equivalence class of

    $\mathrm{tp}_{\mathrm{qf}}(\bar{t}^u{}^\frown \bar{s}^{u,v}, \varnothing, I^{\mathfrak{t}}_u)$ where $\bar{t}^u = \pi^{\mathfrak{s}}_{u,v_*}(\bar{t})$ and $\bar{s}^{u,v} = \pi^{\mathfrak{t}}_{u,v}(\bar{s}_v)$.

<u>Then</u> there are $u_* \leq u^* \in J^{\mathfrak{t}}, \bar{s} \in {}^{k_2}(I^{\mathfrak{s}}_{v_*})$ such that for every $u \in J^{\mathfrak{t}}_{\geq u^*}$, $\mathrm{tp}_{\mathrm{qf}}(\pi^{\mathfrak{s}}_{u,v_*}(\bar{t}{}^\frown\bar{s}), \varnothing, I^{\mathfrak{t}}_u)$ belongs to e_u (and is constantly p^* for some $p^* \in \mathcal{S}^k$).

Remark 3.3. We may say "$\mathfrak{s}$ is semi-limit of $\mathfrak{t}$" when in clause (d) we replace $\Leftrightarrow$ by $\Rightarrow$. We may consider using this weaker version and/or omit linearity in our main theorem, but the present version suffices.

Claim 3.4. *Main $K^{\mathfrak{s}}_{v_*}$ is an almost κ-automorphism group (see below) <u>when</u>:*

$\boxtimes$ (a) *$\mathfrak{s}, \mathfrak{t}$ are both p.o.w.i.s.*

(b) *$\mathfrak{s}$ is an existential limit of $\mathfrak{t}$ as witnessed by v_**

(c) *$J^{\mathfrak{t}}$ is $\aleph_1$-directed, linear (i.e., for every $u,v \in J^{\mathfrak{t}}$ we have*

$u \leq_{J[\mathfrak{t}]} v$ or $v \leq_{J[\mathfrak{t}]} u$) and unbounded

(d) *$\mathfrak{t}$ is a κ-p.o.w.i.s. (so $\kappa \geq |J^{\mathfrak{t}}|$ and $\kappa \geq |I^{\mathfrak{t}}_u|$ for $u \in J^{\mathfrak{t}}$)*

(e) *$\mathfrak{t}$ is nice (see Definition 1.3(7)).*

Definition 3.5. G is an almost κ-automorphism group when: there is a κ-automorphism group G^+ and a normal subgroup G^- of G^+ of cardinality $\leq \kappa$ such that G is isomorphic to G^+/G^-, i.e., there is a homomorphism from G^+ onto G with kernel G^-.

Before proving 3.4 we explain: why will being almost κ-automorphism group help us in proving our intended result? Recalling Definition 0.3 and Observation 0.8:

Claim 3.6. *For any ordinal α, if there is an almost κ-automorphism group G with a subgroup H of cardinality $\leq \kappa$ such that $\tau'_{G,H} = \alpha$ [such that $\mathrm{nor}^\alpha_G(H) = G \wedge (\forall \beta < \alpha)(\mathrm{nor}^\beta_G(H) \neq G)$] <u>then</u> there is a κ-automorphism group G' with a subgroup H' of cardinality $\leq \kappa$ such that $\tau'_{G',H'} = \alpha$ [such that $\mathrm{nor}^\alpha_{G'}(H') = G' \wedge (\forall \beta < \alpha)(\mathrm{nor}^\beta_{G'}(H') \neq G')$].*

Proof. Easy.

Let G^+, G^- be as in Definition 3.5 and h be a homomorphism from G^+ onto G with kernel G^- and let $H^+ = \{x \in G^+ : h(x) \in H\}$.

So it is easy to check each of the following statements:

$\circledast$ (a) H^+ is a subgroup of G^+

(b) $|H^+| \leq |H| \times |G^-| \leq \kappa\kappa = \kappa$

(c) G^+ is a κ-automorphism group

(d) $\mathrm{nor}^\beta_{G^+}(H^+) = \{x \in G^+ : h(x) \in \mathrm{nor}^\beta_G(H)\}$ for every $\beta \leq \infty$

(e) $\tau'_{G,H} = \tau'_{G^+,H^+}$

(f) $\mathrm{nor}^\beta_G(H) = G$ then $\mathrm{nor}^\beta_{G^+}(H^+) = G^+$ for every $\beta \leq \infty$.

Together (G^+, H^+) exemplifies the desired conclusion. $\square$

Proof. 3.4 Let G^+ be the automorphism group of $M_{\mathfrak{t}}$ and let G^- be the following subgroup of G^+

$$\{\varkappa \in G^+ : \text{for some } u \in J^{\mathfrak{t}} \text{ we have } u \leq_J v \wedge g \in K_v \Rightarrow \varkappa((v,g)) = (v,g)\}.$$

Easily

$\circledast_1$ G^- is a subgroup of G^+
[Why? As $J^{\mathfrak{t}}$ is linear.]

$\circledast_2$ for every $\varkappa \in G^+$ we can find $\bar{f}^\varkappa = \langle f^\varkappa_u : u \in J^{\mathfrak{t}} \rangle$ such that

(a) $f^\varkappa_u \in F^{\mathfrak{t}}_u$

(b) $\varkappa \restriction D^{\mathfrak{t}}_u = f^\varkappa_u$

(c) $\varkappa \restriction P^{M_{\mathfrak{t}}}_{2,u}$ is $(1,u,f) \mapsto (1,u,f^\varkappa_u,f)$.

[Why? By Claim 2.19.]

$\circledast_3$ G^- (and also $M_{\mathfrak{t}}$) has cardinality $\leq \kappa$.
[Why? As $|J^{\mathfrak{t}}| \leq \kappa$, it suffices to prove that for each $u \in J^{\mathfrak{t}}$, the subgroup $G^-_u := \{\varkappa \in G^+ : \varkappa \restriction P^{M_{\mathfrak{t}}}_{1,v}$ is the identity when $u \leq_{J[\mathfrak{t}]} v\}$ has cardinality $\leq \kappa$, but this has not more elements as $F^{\mathfrak{t}}_u$ because $\varkappa \mapsto \varkappa \restriction D^{\mathfrak{t}}_u$ is an injective function from G^-_u into $F^{\mathfrak{t}}_u$ and $J^{\mathfrak{t}}$ is linear. As $|F^{\mathfrak{t}}_u| \leq \aleph_0 + |Z^{\mathfrak{t}}_u| = \aleph_0 + |I^{\mathfrak{t}}_u| \leq \kappa$ we are done.]

$\circledast_4$ G^- is a normal subgroup of G^+.
[Why? By its definition, more elaborately

(*a*) each G^-_u is a normal subgroup of G^+.
[Why? As all members of $\mathrm{Aut}(M_{\mathfrak{t}})$ map each $\{v\} \times K_v$ onto itself.]
(*b*) $u \le_{J[\mathfrak{t}]} v \Rightarrow G^-_u \subseteq G^-_v$.
[Why? Check the definitions.]
(*c*) $G^- = \cup\{G^-_u : u \in J^{\mathfrak{t}}\}$.
[Why? Trivially.]

Together we are done proving $\circledast_4$.]

$\circledast_5$ For $x \in Z^{\mathfrak{s}}_{v_*}$ let $\varkappa_x$ be the following automorphism of $M_{\mathfrak{t}}$, it is defined as in $\circledast_2$ by $\langle f^x_u : u \in J^{\mathfrak{t}}\rangle$ where $f^x_u = \partial^{\mathfrak{s}}_{v_*,x} \restriction D^{\mathfrak{t}}_u$ is from Definition 2.9(4).

$\circledast_6$ For every $x \in Z^{\mathfrak{s}}_{v_*}$, $\varkappa_x$ is a well defined automorphism of $M_{\mathfrak{t}}$.
[Why? Look at the definitions and 2.19.]

The main point is

$\circledast_7$ G^+ is generated by $\{\varkappa_x : x \in Z^{\mathfrak{s}}_{v_*}\} \cup G^-$.

Why? Clearly the set is a set of elements of G^+. So assume $\varkappa \in G^+$ and let $\bar f^\varkappa = \langle f^\varkappa_u : u \in J^{\mathfrak{t}}\rangle$ be as in $\circledast_2$, they are fixed for awhile.

By 2.21 for each $u \in J^{\mathfrak{t}}$ there are $k = k^u$ and $\bar t = \bar t^u \in {}^{k^u}(I^{\mathfrak{t}}_u)$ and $\mathbf{q} = \mathbf{q}^u \in \mathbf{S}^{k^u}$ such that (the "disjoint" as we can replace $\bar t$ by $\bar t\char`^\bar t$ or even $\bar t\char`^\bar t\char`^ \ldots \char`^\bar t$ with $|\mathcal{S}^{k^u}|$ copies, the "reduced" by 2.21(3)):

$\boxdot_1$ $f^\varkappa_u = \partial_{u,(\bar t^u,\mathbf{q}^u)}$, i.e., if $v \le_{J[\mathfrak{t}]} u$ then $(\varkappa \equiv) f^\varkappa_u \restriction (\{v\} \times K^{\mathfrak{t}}_v)$ is a multiplication from the left (of the $K^{\mathfrak{t}}_v$-coordinate) by $g^v_{\pi^{\mathfrak{t}}_{v,u}(\bar t^u),\mathbf{q}^u}$, and $\mathbf{q}^u$ is reduced and disjoint, see Definition 2.13(2),(5).

The choices are not necessarily unique, in particular

$\boxdot_2$ if $u^1 \le_{J[\mathfrak{t}]} u^2$ then $(k^{u^2}, \pi_{u^1,u^2}(\bar t^{u^2}), \mathbf{q}^{u^2})$ can serve as $(k^{u^1}, \bar t^{u^1}, \mathbf{q}^{u^1})$.

Also

$\boxdot_3$ the set of possible $(k^u, \mathbf{q}^u)$ is countable.

As $J^{\mathfrak{t}}$ is $\aleph_1$-directed and linear

$\boxdot_4$ for some pair $(k^*, \mathbf{q}^*)$ the set $\{u \in J^{\mathfrak{t}} : k^u = k^* \text{ and } \mathbf{q}^u = \mathbf{q}^*\}$ is cofinal in $J^{\mathfrak{t}}$.

Together, without loss of generality for some $k^*, \mathbf{q}$

$\boxdot_5$ $k^u = k^*$ and $\mathbf{q}^u = \mathbf{q}$ for every $u \in J^{\mathfrak{t}}$.

Let E be an ultrafilter on $J^{\mathfrak{t}}$ such that $u \in J^{\mathfrak{t}} \Rightarrow \{v : u \le_{J[\mathfrak{t}]} v\} \in E$. Such an E exists as $J^{\mathfrak{t}}$ is linear. For each $u \in J^{\mathfrak{t}}$ there are $A_u, p_u, w(u)$ such that

$\boxdot_6$ (*a*) $A_u \in E$ and
(*b*) $p_u \in \mathcal{S}^{k^*}$
(*c*) if $v \in A_u$ then $u \le_{J[\mathfrak{t}]} v$ and $p_u = \mathrm{tp}_{\mathrm{qf}}(\pi^{\mathfrak{t}}_{u,v}(\bar t^v), \varnothing, I_u)$
(*d*) $w(u) \in A_u$.

For $p \in \mathcal{S}^{k^*}$ let

$\boxdot_7$ (*a*) $Y_p = \{u \in J^{\mathfrak{t}} : p_u = p\}$
(*b*) $\bar s^{u,v} = \pi^{\mathfrak{t}}_{u,v}(\bar t^v) \restriction \mathrm{supp}(\mathbf{q}(p_u))$ for $u \in J^{\mathfrak{t}}, v \in A_u$
(*c*) $\bar s^u = \bar s^{u,w(u)}$.

So

$\boxdot_8$ $\langle Y_p : p \in \mathcal{S}^{k^*}\rangle$ is a partition of $J^{\mathfrak{t}}$.

Fix $p \in \mathcal{S}^{k^*}$ for awhile so for each $u \in Y_p$ and $v \in A_u$ by $\boxdot_1$, $\varkappa \restriction (\{u\} \times K^{\mathfrak{t}}_u)$ is multiplication from the left by $g^u_{\pi^{\mathfrak{t}}_{u,v}(\bar t^v),\mathbf{q}}$ (it was $\mathbf{q}^v$ but we have already agreed that $\mathbf{q}^v = \mathbf{q}$). But $p = \mathrm{tp}_{\mathrm{qf}}(\pi^{\mathfrak{t}}_{u,v}(\bar t^v), \varnothing, I_u)$ as $u \in Y_p, v \in A_u$ and so by Definition 2.13(5) we know that $g^u_{\pi^{\mathfrak{t}}_{u,v}(\bar t^v),\mathbf{q}}$ is $g^u_{\pi^{\mathfrak{t}}_{u,v}(\bar t^v),\mathbf{q}(p)}$.

Now $\mathbf{q}(p) \in \Lambda^2_{k^*}$ so $\mathbf{q}(p) = \langle \rho^p_0, \rho^p_1, \dots, \rho^p_{i(p)-1}\rangle$ and recall

$$g^u_{\pi^{\mathfrak{t}}_{u,v}(\bar t^v),\mathbf{q}(p)} \text{ is } g_{\bar t,\rho^p_0} h_{g_{\bar t,\rho^p_1} G^{<0}_u} \dots \text{ with } \bar t = \pi^{\mathfrak{t}}_{u,v}(\bar t^v);$$

so it depends only on $\pi^{\mathfrak{t}}_{u,v}(\bar t^v) \restriction \operatorname{supp}(\mathbf{q}(p))$ only.

Now consider any two members v_1, v_2 of A_u (so they are above u) comparing the two expressions for $\varkappa \restriction (\{u\} \times K^{\mathfrak{t}}_u)$ one coming from v_1 the second from v_2 we conclude that $g^u_{\pi^{\mathfrak{t}}_{u,v_1}(\bar t^{v_1}),\mathbf{q}(p)} = g^u_{\pi^{\mathfrak{t}}_{u,v_2}(\bar t^{v_2}),\mathbf{q}(p)}$. As $\mathbf{q}$ is reduced also $\mathbf{q}(p)$ is p-reduced hence by 2.8(3) we conclude that

$\boxdot_9$ if ($p \in \mathcal{S}^{k^*}, u \in Y_p \subseteq J^{\mathfrak{t}}$ and) $v_1, v_2 \in A_u$ then $\pi^{\mathfrak{t}}_{u,v_1}(\bar t^{v_1}) \restriction \operatorname{supp}(\mathbf{q}(p))$ is a permutation of $\pi^{\mathfrak{t}}_{u,v_2}(\bar t^{v_2}) \restriction \operatorname{supp}(\mathbf{q}(p))$
this means

$\boxdot_{10}$ if $u \in J^{\mathfrak{t}}$ and $v_1, v_2 \in A_u$ then $\bar s^{u,v_1}$ is a permutation of $\bar s^{u,v_2}$.

Hence for each $u \in J^{\mathfrak{t}}$

$\boxdot_{11}$ if $v \in A_u$ then $\bar s^{u,v}$ is a permutation of $\bar s^u = \bar s^{u,w(u)}$.

As there are only finitely many permutations of $\bar s^u$, there are $\omega(u), A'_u$ such that

$\boxdot_{12}$ for $u \in J^{\mathfrak{t}}$:
- (a) $A'_u \in E$
- (b) $A'_u \subseteq A_u$
- (c) $\bar s^u = \bar s^{u,v}$ for every $v \in A'_u$.

Now

$\boxdot_{13}$ if $p \in \mathcal{S}^{k^*}$ and $u_1 \le_{J[\mathfrak{t}]} u_2$ are from Y_p then $\pi^{\mathfrak{t}}_{u_1,u_2}(\bar s^{u_2}) = \bar s^{u_1}$.

[Why? As E is an ultrafilter on $J^{\mathfrak{t}}$ and $A'_{u_1}, A'_{u_2} \in E$ we can find $v \in A'_{u_1} \cap A'_{u_2}$. So for $\ell = 1,2$ we have $\bar s^{u_\ell} = \pi^{\mathfrak{t}}_{u_\ell,v}(\bar t^v) \restriction \operatorname{supp}(\mathbf{q}(p)) = \pi^{\mathfrak{t}}_{u_\ell,v}(\bar t^v \restriction \operatorname{supp}(\mathbf{q}(p)))$.

As $\pi^{\mathfrak{t}}_{u_1,v} = \pi^{\mathfrak{t}}_{u_1,u_2} \circ \pi^{\mathfrak{t}}_{u_2,v}$ we conclude $\bar s^{u_1} = \pi^{\mathfrak{t}}_{u_1,u_2}(\bar s^{u_2})$ is as required.]

Let $\mathcal{S}' = \{p \in \mathcal{S}^{k^*} : Y_p$ is an unbound subset of $J^{\mathfrak{t}}\}$, so for some $u_* \in J^{\mathfrak{t}}$ we have

$\boxdot_{14}$ $J^{\mathfrak{t}}_{\ge u_*} \subseteq \cup\{Y_p : p \in \mathcal{S}'\}$.

Also without lose of generality

$\boxdot_{15}$ $k^* = k^*_1 + k^*_2$ and $\{0, \dots, k^*_1 - 1\} = \cup\{\operatorname{supp}(\mathbf{q}(p)) : p \in \mathcal{S}'\}$

$\boxdot_{16}$ for $p \in \mathcal{S}'$ and $\ell \in \operatorname{supp}(\mathbf{q}(p))$, so $s^u_\ell = (\bar s^u)_\ell$ is well defined for $u \in Y_p$, there is a unique $t_\ell \in I^{\mathfrak{s}}_{v_*}$ such that:

$$u \in Y_p \Rightarrow \pi^{\mathfrak{s}}_{u,v_*}(t_\ell) = s^u_\ell.$$

[Why? By clause (e) of Definition 3.1, $\boxdot_{13}$ and the linearity of $J^{\mathfrak{t}}$.]

Next we can find $\bar t$ such that

$\boxdot_{17}$ (a) $\bar t = \langle t_\ell : \ell < k^*_1\rangle$

(b) if $p \in \mathcal{S}'$ and $\ell \in \operatorname{supp}(\mathbf{q}(p))$ then $t_\ell \in I^{\mathfrak{s}}_{v_*}$ is as in $\boxdot_{16}$.

[Why? For $\ell \in \cup\{\operatorname{supp}(\mathbf{q}(p)) : p \in \mathcal{S}'\}$ use $\boxdot_{16}$. As $\mathbf{q}$ is disjoint (see Definition 2.13(2)) there is no case of "double definition".]

By clause (d) of Definition 3.1, possibly increasing u_*,

$\boxdot_{18}$ $p^* = \operatorname{tp}_{\rm qf}(\pi^{\mathfrak{s}}_{u,v_*}(\bar t), \varnothing, I_u)$ for every $u \in J^{\mathfrak{t}}_{\ge u_*}$.

$\boxdot_{19}$ let $\mathscr{E}$ be the following equivalence relation on $\mathcal{S}^{k^*}$, $p_1 \mathscr{E} p_2 \Leftrightarrow \mathbf{q}(p_1) \mathscr{E}^1_{k^*_1, p\restriction k^*_1} \mathbf{q}(p_2)$; note they are actually from $\mathcal{S}^{k^*_1}$ and so "$\mathscr{E}^1_{k^*_1,p\restriction k^*_1}$-equivalent" is meaningful, see Definition 2.3(4)

$\boxdot_{20}$ let $\bar e = \langle e_u : u \in J^{\mathfrak{t}}_{\ge u_*}\rangle$ be defined by $e_u = p_u/E$

$\boxdot_{21}$ $E, \bar t, \bar e, \langle \pi^{\mathfrak{t}}_{u,w(u)}(\bar t^{w(u)}) : u \in J^{\mathfrak{t}}_{\ge u_*}\rangle$ satisfies the demands $(f)(\alpha) - (\zeta)$ from Definition 3.2.

[Why? Check.]

Recall $p^* = \mathrm{tp}(\bar t, \varnothing, I^{\mathfrak s}_{v_*})$ here so let $\bar s \in {}^{(k^*_2)}(I^{\mathfrak s}_{v_*})$ be as guaranteed to exist by Definition 3.2. Let $\bar t^{v^*} := \bar t\hat{\ }\bar s$. So possibly increasing $u_* \in J^{\mathfrak t}$ for some p^* we have

$\boxdot_{22}$ if $u \in J^{\mathfrak t}_{\geq u_*}$ then $p^* = \mathrm{tp}(\pi^{\mathfrak s}_{u,v_*}(\bar t\hat{\ }\bar s), \varnothing, I^{\mathfrak s}_u) = \mathrm{tp}(\bar t\hat{\ }\bar s, \varnothing, I^{\mathfrak s}_{v_*})$.

Let

$\boxdot_{23}$ (a) $\varrho^* = \mathbf{q}(p^*)$ so $\varrho^* \in \Lambda^2_{k^*_1,p^*}$ and let $\varrho^* = \langle \rho_\ell : \ell < \ell(*)\rangle$

(b) $\bar t_u = \pi^{\mathfrak s}_{u,v_*}(\bar t)$ for $u \in J^{\mathfrak t}$

(c) let $z_u = z^{u,s}_{\bar t_u,\varrho} \in Z^{1,\mathfrak s}_u$ (see Definition 2.5(5A))

(d) let $f_u = \partial^{\mathfrak s}_{u,z_u} \in F^{\mathfrak s}_u$; (this is not the same as $f^{\varkappa}_u!$).

Now

$\boxdot_{24}$ for $u_1 \leq_{J[\mathfrak t]} u_2$ we have $f_{u_1} \subseteq f_{u_2}$.

[Why? Check.]

$\boxdot_{25}$ $\varkappa_{\bar f}$ is a finite product of members of $\{\varkappa_x : x \in Z^{\mathfrak s}_{v_*}\}$.

[Why? Recall $\varkappa_x$ for $x \in Z^{\mathfrak s}_{v^*}$ is from $\circledast_5$. Now use $\boxdot_{23}$.] Lastly

$\boxdot_{26}$ $(\varkappa^{-1}_{\bar f})\varkappa \in G^+ = \mathrm{Aut}(M_{\mathfrak t})$ is the identity on $P^{M_{\mathfrak t}}_u$ whenever $u \in J^{\mathfrak t}_{\geq u_*}$.

[Why? By $\boxdot_{24}$ and our choices.]

$\boxdot_{25}$ $(\varkappa_{\bar f}) \in (G^-_{u_*} \subseteq) G^-$.

[Why? By $\boxdot_{25}$ and the definition of $(G_{u^*}$ and$)$ G^-.]

$\boxdot_{28}$ $\varkappa$ is the product (in G^+) of $\varkappa_{\bar f} \in G^-$ and $(\varkappa^{-1}_f)\varkappa \in \langle\{\varkappa_x : x \in Z^{\mathfrak s}_{v_*}\}\rangle$.

[Why? $\boxdot_{25} + \boxdot_{27}$ this is clear.]

As $\varkappa$ was any a member of G^+ we are done proving $\circledast_7$.

$\circledast_8$ there is a homomorphism $\mathbf{h}$ from $K^{\mathfrak s}_{v_*}$ onto G^+/G^- which maps g_x to $\varkappa_x G^-$ for $x \in Z^{\mathfrak s}_{v_*}$.

[Why? By $\circledast_7$ there is at most one such homomorphism and if it exists it is onto.

So it is enough to show that for any group term, σ if $K^{\mathfrak s}_{v_*}$ satisfies $K_{v_*} \models$ "$\sigma(g_{x_1}, \dots, g_{x_{k-1}}) = e$" then $\sigma(\varkappa_{x_0}, \dots, \varkappa_{x_{k-1}}) \in G^-$. Let $\langle t_\ell : \ell < \ell^*\rangle$ list $\cup\{\mathrm{his}(x_\ell) : \ell < k\} \subseteq I^{\mathfrak s}_{v_*}$ and let $u_* \in J^{\mathfrak t}$ be such that: if $u_* \leq_{J[\mathfrak t]} u$ and $\ell(1), \ell(2) < \ell^*$ we have $I^{\mathfrak s}_{v_*} \models t_{\ell(1)} <_I t_{\ell(2)}$ iff $I^{\mathfrak t}_u \models \pi_{u,v_*}(t_{\ell(1)}) < \pi_{u,v^*}(t_{\ell(2)})$ and similarly for equality, see clause (d) of Definition 3.1.

Let $t_{u,\ell} = \pi_{u,v_*}(t_\ell), x_{u,\ell} = \hat\pi_{u,v_*}(x_\ell)$. By the definition of G^- it is enough to show that: if $u_* \leq_{J[\mathfrak t]} u$ then $K_u \models$ "$\sigma(g_{x_{u,0}}, \dots, g_{x_{u,k_1}}) = e_{K_u}$". By the analysis in 1.7 and §2 (i.e., twisted product) this should be clear.]

$\circledast_9$ $\varkappa^*$ is one to one.

[Why? By part of the analysis as for $\circledast_7$.]

By $\circledast_8 + \circledast_9$ we are done.

The problem is in verifying clause (ζ) of (f) of Definition 3.2. Now if $u \in J^{\mathfrak t}_{\geq u_*}$ we can find $w_p[u] \in \mathfrak t \geq v$ for each $p \in \mathcal S'$ such that

$\odot$ (α) $v \leq_{J[\mathfrak t]} w_p[u] \in Y_p$

(β) $\bar t^{w_p[u]} \restriction \mathrm{supp}(\mathbf{q}(p)) = \pi^{\mathfrak s}_{w_p[u]}(\bar t \restriction \mathrm{supp}(\mathbf{q}(p))$.

Let $w[p] \in \cap\{A'_{w_p[u]} : p \in \mathcal S'\}$ be a $\leq_{J[\mathfrak t]}$-common upper bound of $\{w_p[u] : p \in S'\} \cup \{u\}$.

Lastly, let $\bar s_u = (\pi^{\mathfrak t}_{u,w[u]}(\bar t^{w[u]})) \restriction [k^*_1, k^*)$. $\square$

Main Claim 3.4, p.40 <u>Once more on $\boxdot_{21}$</u>:

I do not see why the definition of $\mathscr E$ and $\bar s^{u,v}$ given on pg.40A has property 3.2(ζ). Even worse: I momentarily have some doubts that this works. <u>Try on a counter-example</u>:

Let $p_j \in \mathcal{S}', j \in \{1,2\}$ with $p_1 \neq p_2$. Thus, in particular, $\sup(\mathbf{q}(p_1)) \cap \operatorname{supp}(\mathbf{q}(p_2)) = \varnothing$. let $i(j) \in \operatorname{supp}(\mathbf{q}(p_j))$ be chosen.

There seems to be no argument preventing the following to happen: for every $p \in \mathcal{S}'$ and every $\bar{t}'$ realizing p the elements $\bar{t}'(i(1))$ and $\bar{t}'(i(2))$ are comparable, i.e. (see Definition 2.3)

$$\forall p \in \mathcal{S}' : \{(0, i(1), i(2)), (2, i(1), i(2))\} \cap p \neq \varnothing,$$

while for the constructed limit $\bar{t}$ in $\boxdot_{17}$ holds

$$(3, i(1), i(2)) \in p^*$$

(see $\boxdot_{18}$), i.e. $\bar{t}(i(1))$ and $\bar{t}(i(2))$ are incomparable.

The consequence for 3.2(ζ) is

$$\begin{aligned} \operatorname{tp}_{\mathrm{qf}}(\bar{t}^u \char`^ \bar{s}^{u,v}, \varnothing, I^{\mathfrak{t}}_u) &= \operatorname{tp}_{\mathrm{qf}}(\pi^{\mathfrak{s}}_{u,v_*}(\bar{t}) \char`^ \pi^{\mathfrak{t}}_{u,v}(\bar{s}_v, \varnothing, I^{\mathfrak{t}}_u) \\ &\Rightarrow p^* =_{\boxdot_{18}} \operatorname{tp}_{\mathrm{qf}}(\pi^{\mathfrak{s}}_{u,v^*}(\bar{t}), \varnothing, I^{\mathfrak{t}}_u) \\ &= \operatorname{tp}_{\mathrm{qf}}(\bar{t}^u \char`^ \bar{s}^{u,v}, \varnothing, I^{\mathfrak{t}}_u) \\ &\Rightarrow_{(1)<(2)} \operatorname{tp}_{\mathrm{qf}}(\bar{t}^u \char`^ \bar{s}^{u,v}, \varnothing, I^{\mathfrak{t}}_u) \notin \mathcal{S}' \end{aligned}$$

while $p_u \in_{\boxdot_{14}} \mathcal{S}'$.

In particular $\operatorname{tp}_{\mathrm{qf}}(\bar{t}^u \char`^ \bar{s}^{u,v}, \varnothing, I^{\mathfrak{t}}_u) \notin_{40A} e_u =_{\boxdot_{20}} p_u/\mathscr{E} \subseteq \mathcal{S}'$ (Contradiction!) [For me the main obstacle here seems to be $Y_{p_1} \cap Y_{p_2} =_{\boxdot_8} \varnothing$.] Saharon please: make me see and give the missing argument! Otherwise FIX! (Maybe 3.1 and 3.2 need additional properties?)

Theorem 3.7. *Assume*

- *(A)* $\aleph_0 < \operatorname{cf}(\theta) = \theta \leq \kappa$
- *(B)* $\mathcal{F}_\alpha \subseteq {}^\alpha\kappa$ *for* $\alpha < \theta$ *has cardinality* $\leq \kappa$ *(also* $\mathcal{F}_\alpha \subseteq {}^\alpha\beta$ *for some* $\beta < \kappa^+$ *is O.K.)*
- *(C)* $\mathcal{F} = \{f \in {}^\theta\kappa : f \restriction \alpha \in \mathcal{F}_\alpha$ *for every* $\alpha < \theta\}$
- *(D)* $\gamma = \operatorname{rk}(\mathcal{F}, <_{J^{\mathrm{bd}}_\theta})$, *necessarily* $< \infty$ *so* $< (\kappa^\theta)^+$
- *(E) if* $f_1, f_2 \in \mathcal{F}$, *then* $f_1 <_{J^{\mathrm{bd}}_\theta} f_2$ *or* $f_2 <_{J^{\mathrm{bd}}_\theta} f_1$ *or* $f_2 =_{J^{\mathrm{bd}}_\theta} f_1$ *(follows from (f))*
- *(F) for stationarily many* $\delta < \theta$ *we have: if* $f_1, f_2 \in \mathcal{F}_\delta$, *then for some* $\alpha < \delta$ *we have* $\beta \in (\alpha, \delta) \Rightarrow (f_1(\beta) < f_2(\beta) \Leftrightarrow f_1(\alpha) < f_2(\alpha))$.

Then $\tau^{\mathrm{atw}}_\kappa \geq \tau^{\mathrm{nlg}}_\kappa \geq \tau^{\mathrm{nlf}}_\kappa > \gamma$ *(on* $\tau^{\mathrm{nlf}}_\kappa$ *see Definition 0.3(4)).*

Theorem 3.8. *We can in Theorem 3.7 weaken clause (f) to*

- *(f)′* (α) $S \subseteq \theta$ *is a stationary set consisting of limit ordinals*
  - (β) D *is a normal filter on* θ
  - (γ) $S \in D$
  - (δ) $\bar{J} = \langle J_\delta : \delta \in S \rangle$
  - (ε) J_δ *is an ideal on* δ *extending* J^{bd}_δ *for* $\delta \in S$
  - (ζ) *if* $S' \subseteq S$ *is stationary,* $S' \in D^+$ *and* $w_\delta \in J_\delta$ *for* $\delta \in S'$, *then*

$$\cup\{\delta \setminus w_\delta : \delta \in S'\} \textit{ contains an end segment of } \theta$$

(η) *if* $\delta \in S$ *and* $f_1, f_2 \in \mathcal{F}$, *then* $f_1 \restriction \delta <_{J_\delta} f_2 \restriction \delta$ *or*

$f_2 \restriction \delta <_{J_\delta} f_1 \restriction \delta$ *or* $f_1 \restriction \delta =_{J_\delta} f_2 \restriction \delta$

Remark 3.9. 1) We can justify (f)$'$ by pcf theory quotation, see below.
2) We should prove that the p.o.w.i.s. being existential holds.

Note that in proving 3.7, 3.8 the main point is the "existential limit". This proof has affinity to the first step in the elimination of quantifiers in the theory of $(\omega, <)$. For this it is better if $I_\theta = (\mathcal{F}, <_{J^{\rm bd}_\theta})$ has many cases of existence. Toward this we "padded it" in $(*)_0$ of the proof - take care of successors ($f \in \mathcal{F} \Rightarrow f+1 \in \mathcal{F}$), have zero ($0_\theta \in \mathcal{F}$) without losing the properties we have.
2) The demand of 3.7 may seem very strong, but by pcf theory it is natural.

Observation 3.10. *1) Theorem 3.8 implies Theorem 3.7.*
2) If $(a)-(d)$ *of 3.7 holds, then* $(f) \Rightarrow (f)'$.
3) If $(a)-(d)$ *of 3.7 holds, then* $(f)' \Rightarrow (e)$.

Proof. 1) By 2).
2) Let

$$\begin{aligned} S := \{\delta < \theta : &\delta \text{ is a limit ordinal and if } f_1, f_2 \in \mathcal{F}_\delta, \\ &\text{then for some } \alpha < \delta \text{ we have } \beta \in (\alpha, \delta) \Rightarrow \\ &(f_1(\beta) < f_2(\beta) \Leftrightarrow f_1(\alpha) < f_2(\alpha))\}. \end{aligned}$$

By (f) we know that S is a stationary subset of θ. Let $\mathscr{D}_\theta$ be the club filter on θ and $D := \mathscr{D}_\theta + S$, it is a normal filter on θ and $S \in D$. So sub-clauses $(\alpha), (\beta), (\gamma)$ of $(f)'$ hold.

Let $J_\delta = J^{\rm bd}_\delta$ for $\delta \in S$ so $\bar{J} = \langle J_\delta : \delta \in S \rangle$ satisfies sub-clauses $(\delta), (\varepsilon)$ of $(f)'$. To prove (ζ) assume $S' \subseteq S$ stationary, $S' \in D^+$ and $w_\delta \in J_\delta$ for $\delta \in S'$. Then $\sup(w_\delta) < \delta$ and S' is a stationary subset of θ hence by Fodor's lemma for some $\beta(*) < \theta$ the set $S'' = \{\delta \in S' : \sup(w_\delta) = \beta(*)\}$ is a stationary subset of θ and so $[\beta(*), \theta)$ is an end segment of θ and is equal to $\cup\{[\beta(*), \delta) : \delta \in S''\}$ which is included in $\cup\{\delta \setminus w_\delta : \delta \in S'\}$, as required in (ζ) from $(f)'$, so sub-clause (ζ) really holds.

To prove sub-clause (η) of clause $(f)'$ note that what it says is what is said in (f).
3) Should be clear. Given $f_1, f_2 \in \mathcal{F}$; by sub-clause (η) of $(f)'$ for each $\delta \in S$ there are $w_\delta \in J_\delta$ and $\ell_\delta < 3$ such that $(\ell_\delta = 0 \wedge \alpha \in \delta \setminus w_\delta) \Rightarrow f_1(\alpha) < f_2(\alpha)$ and $(\ell_\delta = 1 \wedge \alpha \in \delta \setminus w_\delta) \Rightarrow f_1(\alpha) = f_2(\alpha)$ and $(\ell_\delta = 2 \wedge \alpha \in \delta \setminus w_\delta) \Rightarrow f_1(\alpha) > f_2(\alpha)$. So for some $\ell < 3$ the set $S' := \{\delta \in S : \ell_\delta = \ell\}$ is stationary ($S' \in D^+$ without loss of generality), hence $\cup\{\delta \setminus w_\delta : \delta \in S'\}$ includes an end segment of θ and we are easily done. $\square$

Proof. 3.8 Without loss of generality

$(*)_0$ (a) $(\forall f \in \mathcal{F})(\exists^\infty g \in \mathcal{F})\big(f \restriction [1, \theta) = g \restriction [1, \theta)\big)$;

moreover for $f \in \mathcal{F}$ we have

$\omega = \{g(0) : g \in \mathcal{F} \text{ and } g \restriction [1, \theta) = f \restriction [1, \theta)\}$

(b) $\alpha < \beta < \theta \Rightarrow \mathcal{F}_\alpha = \{f \restriction \alpha : f \in \mathcal{F}_\beta\}$; moreover $\alpha < \theta \Rightarrow \mathcal{F}_\alpha =$

$\{f \restriction \alpha : f \in \mathcal{F}\}$

(c) if $f \in \mathcal{F}$, <u>then</u> $f + 1 \in \mathcal{F}$

(d) the $f \in {}^\theta\{0\}$, the constantly zero function, belongs to $\mathcal{F}$.

[Why? Let $\mathcal{F}' = \{f \in {}^\theta\kappa$: for some $n < \omega$ we have $(\forall 0 < \alpha < \theta)(f(\alpha) = u) \wedge f(0) < \omega$ or for some $f' \in \mathcal{F}$ and $n < \omega$ we have $(\forall 0 < \alpha < \theta)(f(\alpha) = \omega(1 + f'(\alpha)) + n) \wedge f(0) < \omega\}$ and for $\alpha < \theta$, replace $\mathcal{F}_\alpha$ by $\mathcal{F}'_\alpha = \{f \restriction \alpha : f \in \mathcal{F}'\}$. Now check that $(a) - (e), (f)'$ of the assumption still holds.]

We define $\mathfrak{s} = (J, \bar{I}, \bar{\pi})$ as follows:

$(*)_1$ (a) $J = (\theta + 1, <)$

(b)(α) let $I_\theta = (\mathcal{F}, <_{J^{\rm bd}_\theta})$ and

(β) $I_\alpha = (\mathcal{F}_{1+\alpha+1}, <_{\alpha+1})$ for $\alpha < \theta$ where

$$f_1 <_{\alpha+1} f_2 \Leftrightarrow f_1(1 + \alpha) < f_2(1 + \alpha)$$

(c) for $\alpha \le \beta < \theta + 1$ let $\pi_{\alpha,\beta} : I_\beta \to I_\alpha$ be

$$\pi_{\alpha,\beta}(f) = f \restriction (1 + \alpha + 1).$$

Note that

$(*)_2$ I_α is explicitly non-trivial for all $\alpha \in J$ (see Definition 1.2(7)).

[Why? By $(*)_0(a)$ and the choice of $<_{I_\alpha}$ in $(*)_1(b)$.]

$(*)_3$ $\mathfrak{s} = (J, \bar{I}, \bar{\pi})$ is a p.o.w.i.s. even nice.

$(*)_4$ $\mathfrak{s}$ is a limit of $\mathfrak{t} := \mathfrak{s} \restriction \theta = ((\theta, <), \bar{I} \restriction \theta, \bar{\pi} \restriction \theta)$.

[Why? Note that clause (d) of Definition 3.1 holds by clause (e) of Theorem 3.7. Easy to check the other clauses.]

$(*)_5$ $\mathfrak{t}$ is a nice κ-p.o.w.i.s.

[Why? This follows from clause (a),(b) of Theorem 3.7.]

Now $K^{\mathfrak{s}}_\theta$ is an almost κ-automorphism group by Claim 3.4, the "existential limit" holds by $(*)_6$ below (note: J is linear). Now ${\rm rk}^{<\infty}(I^{\mathfrak{s}}_\theta) = {\rm rk}(I^{\mathfrak{s}}_\theta) = \gamma$ and $H^{\mathfrak{s}}_\theta$ is a subgroup of $K^{\mathfrak{s}}_\theta$ of cardinality $2 \le \kappa$.

Combining Claim 1.10 and Claim 2.11 we have

$$\tau^{\rm nlg}_{K^{\mathfrak{s}}_\theta, H^{\mathfrak{s}}_\theta} = {\rm rk}^{<\infty}(I^{\mathfrak{s}}_\theta) = \gamma$$

with ${\rm nor}^\infty_{K^{\mathfrak{s}}_\theta}(H^{\mathfrak{s}}_\theta) = K^{\mathfrak{s}}_\theta$ and thus $\tau^{\rm atw}_\kappa \ge \tau^{\rm nlg}_\kappa \ge \tau^{\rm nlf}_\kappa > \tau^{\rm nlg}_{K^{\mathfrak{s}}_\theta, H^{\mathfrak{s}}_\theta} = \gamma$ by 0.8 and Claim 3.6.

We still have to check

$(*)_6$ "$\mathfrak{s}$ is an existential limit of $\mathfrak{t}$", see Definition 3.2.

That is we have to prove clause (f) of 3.2, so we should prove its conclusion, assuming its assumption which means in our case

$\circledast_1$ (a) $k = k_1 + k_2, \mathscr{E}$ is an equivalence relation on $\mathcal{S}^k$

(b) $\bar{f} \in {}^{k_1}\mathcal{F}$ and $\alpha(*) < \theta$

(c) $\bar{e} = \langle e_\alpha : \alpha \in [\alpha(*), \theta)\rangle$ is such that $e_\alpha \in \mathcal{S}^k/\mathscr{E}$

(d) $\langle \bar{g}^\alpha : \alpha \in [\alpha(*), \theta)\rangle$ is such that $\bar{g}^\alpha \in {}^{k_2}(\mathcal{F}_{1+\alpha+1})$

(e) if $\alpha(*) \le \alpha \le \beta < \theta$ then:

e_α is the $\mathscr{E}$-equivalence class of ${\rm tp}_{\rm qf}(\langle f_\ell \restriction (1{+}\alpha{+}1) : \ell < k_1\rangle{}^\frown\langle g^\beta_\ell \restriction (1{+}\alpha{+}1) : \ell < k_2\rangle, \varnothing, I_\alpha)$.

Without loss of generality [recalling clause (e) of Theorem 3.7 and $(*)_0(c)$]

$\circledast_2$ (f) $\langle f_\ell : \ell < k_1 \rangle$ is $\le_{J^{\rm bd}_\theta}$-increasing

(g) f_0 is constantly zero

(h) for each $\ell < k_1 - 1$ we have: $f_{\ell+1} = f_\ell \bmod J^{\rm bd}_\theta$ or $f_{\ell+1} = f_\ell + 1 \bmod J^{\rm bd}_\theta$ or $f_\ell + \omega \le f_{\ell+1} \bmod J^{\rm bd}_\theta$

(i) $\langle f_\ell : \ell < k_1 \rangle$ is without repetition

(j) $\langle f_\ell(0) : \ell < k_1 \rangle$ is without repetition.

Possibly increasing $\alpha(*) < \theta$, without loss of generality

$\circledast_3$ if $\alpha \in [\alpha(*), \theta)$ and $\ell_1, \ell_2 < k_1$ then $f_{\ell_1}(\alpha) \le f_{\ell_2}(\alpha) \Leftrightarrow f_{\ell_1}(\alpha(*)) \le f_{\ell_2}(\alpha(*))$.

Hence by clause (f) of $\circledast_2$

$\circledast_4$ $\langle f_\ell(\alpha(*)) : \ell < k_1 \rangle$ is non-decreasing.

For notational simplicity

$\circledast_5$ (a) replace $\bar g^\delta (\delta \in [\alpha(*), \theta))$ by $\langle g^\delta_\ell : \ell < k \rangle := \langle f_\ell \restriction (1+\delta+1) : \ell < k_1 \rangle \hat{\ } \bar g^\delta$

(b) for $\ell_1, \ell_2 < k$ let $g^\delta_{\ell_1} = g^\delta_{\ell_2} \Leftrightarrow g^\delta_{\ell_1}(0) = g^\delta_{\ell_2}(0)$ without loss of generality.

Next for some p^*

$\circledast_6$ $p^* \in \mathcal{S}^k$ and for some stationary $S' \subseteq S$ from D^+, for every $\delta \in S'$ for the J_δ-majority of $\alpha < \delta$, say $\alpha \in \delta \setminus w_\delta, w_\delta \in J_\delta$, we have $p^* = {\rm tp}_{\rm qf}(\langle g^\delta_\ell \restriction (1+\alpha+1) : \ell < k_1 \rangle, \varnothing, I_\alpha)$. Without loss of generality $S' \subseteq (\alpha(*), \theta)$ and $(0, \alpha(*)) \subseteq \omega_\delta$.

[Why? By sub-clause (η) of clause $(f)'$, as $J^{\rm bd}_\delta \subseteq J_\delta$ is an ideal (applied to $(g^\delta_{\ell_1}, g^\delta_{\ell_2})$ for every $\ell_1, \ell_2 < k$) for each $\delta \in S$ ($S \subseteq (\alpha(*), \theta)$ without loss of generality) we can choose $w_\delta \in J_\delta$ and $q_\delta \in \mathcal{S}^k$ such that for every $\alpha \in \delta \setminus w_\delta$ we have ${\rm tp}_{\rm qf}(\langle g^\delta_\ell \restriction (1+\alpha+1) : \ell < k \rangle, \varnothing, I_\alpha)$ is equal to q_δ. For each $p \in \mathcal{S}^k$ let $S_p = \{\delta \in S : q_\delta = p\}$. So $S = \cup\{S_p : p \in \mathcal{S}^k\}$, hence for some p we have S_p stationary ($S_p \in D^+$ without loss of generality). So let $S' = S_p, p^* = p$.]

So considering the way $\bar g^\delta$ was defined by $\circledast_5$

$\circledast_7$ there are $\mathscr{E}^*_1, \mathscr{E}^*_2, <_*$ such that

(a) $\mathscr{E}^*_1$ is an equivalence relation on $k = \{0, \dots, k-1\}$

(b) $\mathscr{E}^*_2$ is an equivalence relation on k refining $\mathscr{E}^*_1$

(c) $<_*$ linearly orders $k/\mathscr{E}^*_1$

(d) if $\delta \in S', \alpha \in \delta \setminus w_\delta$ so $p^* = {\rm tp}_{\rm qf}(\langle g^\delta_\ell \restriction (1+\alpha+1) : \ell < k \rangle, \varnothing, I_\alpha)$ then:

(α) $\ell_1 \mathscr{E}^*_1 \ell_2$ iff $g^\delta_{\ell_1}(1+\alpha) = g^\delta_{\ell_2}(1+\alpha)$

(β) $\ell_1 \mathscr{E}^*_2 \ell_2$ iff $g^\delta_{\ell_1} \restriction (1+\alpha+1) = g^\delta_{\ell_2} \restriction (1+\alpha+1)$

(γ) $(\ell_1/\mathscr{E}^*_1) <_* (\ell_2/\mathscr{E}^*_1)$ iff $g^\delta_{\ell_1}(1+\alpha) < g^\delta_{\ell_2}(1+\alpha)$.

Let $\langle u_0, \dots, u_{m-1} \rangle$ list the $\mathscr{E}^*_1$-equivalence classes in $<_*$-increasing order. Necessary $0 \in u_0$.

Using $(f')(\zeta)$ on $\circledast_6$ let be $\alpha^* \in S'$ with $[\alpha^*, \theta) \subseteq \cup\{\delta \setminus \omega_\delta : \delta \in S'\}$. Thus in particular $p^* \in e_\alpha$ for all $\alpha \in [\alpha^*, \theta)$ by $\circledast_1(e)$. We now define $g_\ell \in {}^\theta\kappa$ for $\ell < k$ as follows: necessarily for a unique $i = i(\ell), \ell \in u_i$ and let $i_1 = i_1(\ell) \le i$ be maximal such that $u_{i_1} \cap \{0, \dots, k_1 - 1\} \ne \varnothing, j_2 = j_2(\ell) = \min(\{u_{i_1} \cap \{0, \dots, k_1 - 1\})$. It is well defined as necessary $0 \in u_0$ because f_0 is constantly zero. Now we let

$\boxdot_0$ $g_\ell = (g^{\alpha_*}_\ell \restriction \{0\}) \cup ((f_{j_2} + (i - i_1)) \restriction [1, \theta))$.

Now

$\boxdot_1$ if $\ell < k_1$ then $g_\ell = f_\ell$
[Why? Check the definition $g^{\alpha_*}_\ell(0) = f_\ell(0)$ as $g^{\alpha_*}_\ell = f_\ell$.]

$\boxdot_2$ $g_\ell \in \mathcal{F}$ for $\ell < k$
[Why? As $f_{j_2} \in \mathcal{F}$ and clauses (a)+(c) of $(*)_0$.]

$\boxdot_3$ if $\ell_1 \mathscr{E}^*_2 \ell_2$ then $g_{\ell_1} = g_{\ell_2}$
[Why? First, as $\ell_1 \mathscr{E}^*_2 \ell_2$ we have $g_{\ell_1}(0) = g^{\alpha_*}_{\ell_1}(0) = g^{\alpha_*}_{\ell_2}(0) = g_{\ell_2}(0)$. Second, clearly $i(\ell_1) = i(\ell_2), i_1(\ell_1) = i_1(\ell_2)$ and $j_2(\ell_1) = j_2(\ell_2)$ hence for $\alpha \in [1,\theta)$ we have

$$\begin{aligned} g_{\ell_1}(\alpha) = & f_{j_2(\ell_1)}(\alpha) + (i(\ell_1) - i_1(\ell_1)) = \\ & f_{j_2(\ell_2)}(\alpha) + (i(\ell_2) - i_1(\ell_2)) = g_{\ell_2}(\alpha). \end{aligned}$$

So we are done.]

$\boxdot_4$ if $\ell_1, \ell_2 < k$ but $\neg(\ell_1 \mathscr{E}^*_2 \ell_2)$ then $g_{\ell_1} \neq g_{\ell_2}$
[Why? As $\neg(\ell_1 \mathscr{E}^*_2 \ell_2)$ by $\circledast_5(b)$ we have $g^{\alpha_*}_{\ell_1}(0) \neq g^{\alpha_*}_{\ell_2}(0)$, hence $g_{\ell_1}(0) = g^{\alpha_*}_{\ell_1}(0) \neq g^{\alpha_*}_{\ell_2}(0) = g_{\ell_2}(0)$ hence $g_{\ell_1} \neq g_{\ell_2}$.

$\boxdot_5$ if $\ell_1, \ell_2 < k, \ell_1 \mathscr{E}^*_1 \ell_2$ then $\neg(g_{\ell_1} <_{J^{\rm bd}_\theta} g_{\ell_2})$
[Why? As $g_{\ell_1} \restriction [1,\theta) = g_{\ell_2} \restriction [1,\theta)$, so $g_{\ell_1} = g_{\ell_2} \mod J^{\rm bd}_\theta$, so $\neg(g_{\ell_1} <_{J^{\rm bd}_\theta} g_{\ell_2})$.]

$\boxdot_6$ if $\ell_1, \ell_2 < k$ and $(\ell_1/\mathscr{E}^*_1) <_* (\ell_2/\mathscr{E}^*_1)$ then $g_{\ell_1} <_{J^{\rm bd}_\theta} g_{\ell_2}$.
[Why? Obviously $i(\ell_1) < i(\ell_2), i_1(\ell_1) \leq i_1(\ell_2)$ and $j_2(\ell_1) \leq j_2(\ell_2)$ by $\circledast_4$. But by $\circledast_2(h)$ we have $f_{j_2(\ell_1)} + (i_1(\ell_2) - i_1(\ell_1)) \leq_{J^{\rm bd}_\theta} f_{j_2(\ell_2)}$ thus $f_{j_2(\ell_1)} + (i(\ell_1) - i_1(\ell_1)) <_{J^{\rm bd}_\theta} f_{j_2(\ell_1)} + (i(\ell_2) - i_1(\ell_1)) \leq_{J^{\rm bd}_\theta} f_{j_2(\ell_2)} + (i(\ell_2) - i_1(\ell_2))$ and $g_{\ell_1} <_{J^{\rm bd}_\theta} g_{\ell_2}$.]

Together $p^* = {\rm tp}_{\rm qf}(\langle g_\ell : \ell < k\rangle, \varnothing, I_\theta) \in e_\alpha$ for all $\alpha \in [\alpha^*,\theta)$ proving the conclusion of Definition 3.2, the definition of existential limit, i.e. $(*)_6$. $\square$

Theorem 3.8, p.48 Question concerning $\boxdot_1$: $\boxdot_1$ seems to be wrong! Why: Let

$\ell_1, \ell_2 < k_1$ with $f_{\ell_1} =_{J^{\rm bd}_\theta} f_{\ell_2}$ (but $f_{\ell_1} \neq f_{\ell_2}$!)
Then $\circledast_7(d)(\alpha)$ implies

$$i(\ell_1) = i(\ell_2), i_1(\ell_1) = i_1(\ell_2) \text{ and } j_2(\ell_1) = j_2(\ell_2).$$

Thus $g_{\ell_1} \restriction [1,\theta) = g_{\ell_2} \restriction [1,\theta)$ follows by $\boxdot_0$ and $\boxdot_1$ would imply

$$f_{\ell_1} \restriction [1,\theta) = f_{\ell_2} \restriction [1,\theta)!$$

That does not hold in general.

Thus only

$\boxdot'_1$ if $\ell < k_1$ then $g_\ell =_{J^{\rm bd}_\theta} f_\ell$.

A possible solution: Theorem 3.8 remains true if weakening the conclusion of Definition 3.2 to: Then there are $\bar t' \in {}^{k_1}(I^{\mathfrak s}_{v_*})$ and $\bar s \in {}^{k_2}(I^{\mathfrak s}_{v_*}$ such that for every $u \in J^{\mathfrak t}_{\geq u_*}$ large enough ${\rm tp}_{\rm qf}(\pi^{\mathfrak s}_{u,v_*}(\bar t' {}^\frown \bar s), \varnothing, I^{\mathfrak t}_u) = p_* \in e_u$ (for some constant $p_* \in \mathcal{S}^k$). Saharon, please check: is that enough to prove 3.4? Otherwise improve $(*)_0$ of

3.8,p.44. pg.43 in $(*)_0$, change (c) to:

if $f \in \mathcal{F}$ and $\alpha < \theta$ then $f' = f + 1_{[\alpha,\theta)} \in \mathcal{F}$, i.e.

$$f'(\beta) = \begin{cases} f(\beta) & \text{if } \beta < \alpha \\ f(\beta)+1 & \text{if } \beta \in [\alpha, \theta) \end{cases}$$

pg.46: Let $m_\ell < k_1$ be maximal $m < k_1$ such that $(m/\mathscr{E}_0) \leq_* (\ell/\mathscr{E}_0)$ exists as $g_0 = f_0 = 0_\theta$ by $\circledast_2(g)$; let $n_\ell = \{\iota/\mathscr{E}_1 : \iota < k_2, (m/\mathscr{E}_0) \leq_* (\iota/\mathscr{E}_0) < (\ell/\mathscr{E}_0)\}$. So $n_\ell = 0$ if $\ell < k_1$ or just $u_\ell \cap \{0, \dots, k_1\} \neq 0$

$\boxdot_1$ we define $g_\ell \in {}^\theta\kappa$ as follows:

(a) $g_\ell \restriction \delta^* = g_\ell^{\delta^*} \restriction \delta^*$

(b) $g_\ell \restriction [\delta^*, \theta) = g_{m_\ell} + n_\ell$, i.e. $g_{m_\ell} + 1_{[\delta^*,\theta)}$.

The rest should be clear (but we give details?). Main Claim 3.8, pg.48:

Dear Saharon!

In 46A you gave a revised proposal for $\boxdot_0$. It is conform to replacing $\boxdot_0$ by

$\boxdot_0'$ $g_\ell = (g_\ell^{\alpha_*} \restriction \alpha_*) \cup ((f_{j_2} + (i - i_1)) \restriction [\alpha_*, \theta))$.

This most certainly solves $\boxdot_1$, but now $\boxdot_2$ is violated. This can/must be fixed by enhancing $(*)_0$, pg.44 once more:

$(c)_1$ if $f \in \mathcal{F}$, then $f + 1 \in \mathcal{F}$

$(c)_2$ if $f_1, f_2 \in \mathcal{F}, \alpha \in \theta$, then $(f_1 \restriction \alpha) \cup (f_2 \restriction [\alpha, \theta)) \in \mathcal{F}$.

This again seems to force replacement of (b),(c) in Theorem 3.7 + 3.8 as follows:

$(b)'$ $\mathcal{F}_\alpha \subseteq \bigcup\limits_{\beta \leq \alpha} {}^{[\beta,\alpha)}\kappa$ for $\alpha < \theta$ has cardinality $\leq \kappa$

$(c)'$ $\mathcal{F} = \{f \in {}^\theta\kappa | \exists \beta \in \theta$ with $f \restriction [\beta, \alpha) \in \mathcal{F}_\alpha$ for all $\beta \leq \alpha < \theta\}$.

Question:

1) Does the pcf-argument with these changes still hold?

2) Does this (hopefully) fix all gaps around 3.7 and 3.8?

Saharon: I can do 2), but 1) needs YOU!!!

We quote

Claim 3.11. *Assume* $\mathrm{cf}(\kappa) = \theta > \aleph_0, \alpha < \kappa \Rightarrow (\alpha)^\theta < \kappa$ *and* $\lambda = \kappa^\theta$. *Then we can find* $\langle \mathcal{F}_i : i < \theta \rangle, S, D, J_\delta$ *satisfying the conditions from 3.8 with* $\gamma = \lambda$ *(and more).*

Proof. By 3.12 and [She94]. $\square$

Claim 3.12. *Assume*

$\circledast$ (a) $\bar\lambda = \langle \lambda_i : i < \theta \rangle$ *is an increasing sequence of regular cardinals with limit* κ

(b) $\lambda = \mathrm{tcf}(\prod\limits_{i<\theta} \lambda_i, <_{J_\theta^{\mathrm{bd}}})$

(c) $\max \mathrm{pcf}\{\lambda_i : i < j\} < \kappa$ *for every* $j < \theta$.

1) Then there are D, S^*, u *such that*

(α) $u \in [\theta]^\theta, S^* \subseteq \theta$ *is stationary*

(β) *there are no* $u_\varepsilon \in [u]^\theta$ *for* $\varepsilon < \theta$ *such that for some club* E *of* $\theta, \delta \in E \cap S^*$ *for at least one* $\varepsilon < \delta$ *we have* $\max \mathrm{pcf}\{\lambda_i : i \in \delta \cap u_\varepsilon\} < \max\mathrm{pcf}\{\lambda_i : i \in \delta \cap u\}$ *hence*

(γ) D *is a normal filter on* θ *where:* D *is* $\{S \subseteq \theta$*: for every sequence* $\langle u_\varepsilon : \varepsilon < \theta \rangle$ *of subsets of* u *each of cardinality* θ *and for every club* E *of* θ*, if* $\delta \in E \cap S \cap S^*$ *then for every* $\varepsilon < \delta$ *we have* $\max \mathrm{pcf}\{\lambda_i : i \in \delta \cap u_\varepsilon\} = \max\mathrm{pcf}\{\lambda_i : i \in \delta \cap u\}\}$

(δ) *for* $\delta \in S^*$ *let* $J_\delta = \{u' \subseteq \delta : \mathrm{maxpcf}\{\lambda_i : i \in \delta \setminus u' < \mathrm{maxpcf}\{\lambda_i : i < \delta\}\}$.

2) We can choose $\mathcal{F}_i \subseteq \prod\limits_{j<i} \lambda_j$ *for* $i < \theta$ *such that all the conditions in* **??** *hold.*

Proof. By [She94, II,3.5], see on this [She, §18]. □

Conclusion 3.13. *If* κ *is strong limit singular of uncountable cofinality* *<u>then</u>* $\tau^{\mathrm{atw}}_\kappa \geq \tau^{\mathrm{nlg}}_\kappa \geq \tau^{\mathrm{nlf}}_\kappa > 2^\kappa$.

Proof. By 3.8 and Claim 3.11. □

Remark 3.14. 1) If $\kappa = \kappa^{\aleph_0}$ do we have $\tau^{\mathrm{atw}}_\kappa \geq \tau^{\mathrm{nlg}}_\kappa \geq \tau^{\mathrm{nlf}}_\kappa > \kappa^+$? But if $\kappa = \kappa^{<\kappa} > \aleph_0$ then quite easily yes.
2) In 3.13 we can weaken "κ is strong limit". E.g. if κ has uncountable cofinality and $\alpha < \kappa \Rightarrow |\alpha|^{\mathrm{cf}(\kappa)} < \kappa$, then $\tau^{\mathrm{nlf}}_\kappa > \kappa^{\mathrm{cf}(\kappa)}$; see more in [She, §18].
3) We elsewhere will weaken the assumption in 3.7, 3.8 but deduce only that $\tau^{\mathrm{nlg}}_\kappa$ is large.

§ 3(A). **Private appendix.**

Definition 3.15. We say that $\mathfrak{s}$ is an almost limit of $\mathfrak{t}$ when the demands from Definition 3.1 holds except that we weaken clause (d) to

$(d)^-$ (α) if $I^{\mathfrak{s}}_{v^*} \models$ "$s < t$" then for some $u_* \in J^{\mathfrak{t}}$ we have $v \in J^{\mathfrak{t}}_{\geq u^*} \Rightarrow I^{\mathfrak{s}}_v \models$

$(\pi^{\mathfrak{s}}_{v,u^*}(s) < \pi^{\mathfrak{s}}_{v,v^*}(t)$

(β) if $n < \omega$ and $t_0, \dots, t_{n-1} \in I^{\mathfrak{s}}_{v^*}$ and $u \in J^{\mathfrak{t}}$ <u>then</u> for some v we have
  - (a) $u \leq_{J[\mathfrak{t}]} v$
  - (b) for $\ell, k < n$ we have $t^{\mathfrak{s}}_{v^*} \models t_\ell < t_k$ iff

    $I^{\mathfrak{s}}_v \models \pi^{\mathfrak{s}}_{v,v^*}(t_\ell) < \pi^{\mathfrak{s}}_{v,v^*}(t_k)$ (similarly for equality but

    this follows)
  - (c) if we use Definition 4.1 also for $\ell < n$ we have $t_\ell \in P^{I_{v^*}} \Leftrightarrow$

    $\pi^{\mathfrak{s}}_{v,v^*}(t_\ell) \in P^{I_v}$.

* * *

Claim 3.16. *Assume that* $\kappa = \kappa^{<\kappa} > \aleph_0$. *<u>Then</u>* $\tau^{\mathrm{nlf}} > \kappa^+$.

Proof. Let $\mathcal{T}$ be the set of t such that

(α) $t = (\alpha_t, <_t)$

(β) α_t is an ordinal $\leq \kappa$

(γ) $<_t$ is a well ordering on α_t.

We define a two-place relation $<_I$ on $\mathcal{T}$:

$$t_1 <_{\mathcal{T}} t_2 \ \underline{\text{iff}}\ \alpha_{t_1} < \alpha_{t_2} \wedge <_{t_1} = <_{t_2} \restriction \alpha_{t_1}.$$

Let $\mathcal{T}_\alpha = \{t \in \mathcal{T} : \alpha_t = \alpha\}$.

We define $\mathfrak{s} = (J, \bar{I}, \bar{\pi})$ as follows:

$(*)_1$ (a) $J = (\kappa + 1, <)$

(b) for $\alpha \leq \kappa$ we define I_α as follows

(α) its set of elements is $\{(t, \beta, n) : t \in \mathcal{T}_{1+\gamma}, \beta < 1 + \alpha \text{ and } n < \omega\}$

(β) I_ℓ is ordered by; $(t_1, \beta_1, n_1) < (t_2, \beta_2, n_2)$ $\underline{\text{iff}}$ $t_1 = t_2 \wedge \beta_1 <_t \beta_2$

(c) for $\alpha_1 < \alpha_2 \leq \kappa$ let $\pi_{\alpha_1,\alpha_2} : I_\beta \to I_\alpha$ be defined by: for $(t, \beta, n) \in I_{\alpha_2}, t = (\alpha_2, <_t)$ let $\pi_{\alpha_1,h_2}((t, \beta, n)) = ((\alpha_1, <_t \restriction \alpha_1, \beta, n))$ if $\beta < \alpha_1$.

So $\mathrm{Dom}(\pi_{\alpha_1,\alpha_2})$ have domain $\subset I_{\alpha_2}$ but it is onto I_2.

The rest is like the proof of 3.8 but easier. $\square$

§ 4. More cardinals

We would like to weaken the demand in Definition 3.1(d), i.e. using only $\mathfrak{s}$ is a semi-limit of $\mathfrak{t}$ and avoid using "existential limit". That is we would like to strengthen Theorem 3.7 omitting clause (f). There is a price: we weaken the conclusion from "$\tau_\kappa^{\text{nlf}} \geq \gamma$" to "$\tau_\kappa^{\text{nlg}} \geq \gamma$". We mention only the places we change (and use bold face (or gothic) versions of the latter for the new version).

Definition 4.1. (0) I denotes $(I, <_I, P^I), <_I$ a partial order on $I, P^I \subseteq \{t \in I : t$ is $<_I$-minimal$\}$ (needed? for finite??]

Definition 4.2. We define $\mathbf{X}_u^{\mathfrak{s}}$ as we have defined $X_u^{\mathfrak{s}}$ except replacing clause (c) by

$(c)'$ $\bar{t} = \langle t_\ell : \ell \leq n\rangle = \langle t_\ell^x : \ell \leq n\rangle$ where $t_\ell \in I_u^{\mathfrak{s}}$

$$\mathbf{X}_u^{<0} = \{x : t_{n(x)}^x \in P^{I_u^{\mathfrak{s}}} \text{ and } \bar{t}^x \text{ is } <_{I_u}\text{-decreasing (nec?)}\}$$

$$\mathbf{X}_u^{<1+\alpha} = X_u^{<0} \cup \{x : \mathrm{rk}(t_{n(*)}^x) < 1 + \alpha\}.$$

Definition 4.3. We define $\mathbf{G}_u^{\mathfrak{s}}$ as we have defined $G_u^{\mathfrak{s}}$ in 1.4, but it is generated by $\{g_x : x \in \mathbf{X}_u^{\mathfrak{s}}\}$ however the set of equations is the same.

Claim 4.4. *$\mathbf{G}_u^{\mathfrak{s}}$ is freely generated by $G_u^{\mathfrak{s}} \cup \{g_x : x \in \mathbf{X}_u^{\mathfrak{s}} \setminus X_u^{\mathfrak{s}}\}$ except the equations which hold in $G_u^{\mathfrak{s}}$ and*

$$g_x = y_x^{-1}$$

for

$$x \in \mathbf{X}_u^{\mathfrak{s}} \setminus X_u^{\mathfrak{s}}.$$

Claim 4.5. *Let $\mathfrak{s}$ be a nice κ-p.o.w.i.s.*

1) If $0 \leq \alpha < \infty$ $\underline{\text{then}}$ the normalizer of $G_u^{<\alpha}$ in $\mathbf{G}_u$ is $G_u^{<\alpha+1} \subseteq G_u \subseteq \mathbf{G}_u$.

2) If $\alpha = \mathrm{rk}(I_u)$ $\underline{\text{then}}$ the normalizer of $G_u^{<\alpha}$ in $\mathbf{G}_u$ is $G_u^{<\infty} = G_u^\alpha$.

Proof. By 4.4 and 1.10. $\square$

Definition 4.6. Let $\mathfrak{s}$ be a κ-p.o.w.i.s.
1) For $u \in J^{\mathfrak{s}}$ let $\mathbf{L}_u = \mathbf{L}_u^{\mathfrak{s}}$ be the group generated by $\{h_g : g \in \mathbf{G}_u\}$ freely except the equations

(A) $h_g^{-1} = h_g$
(B) $h_{g_1} h_{g_2} = h_{g_2} h_{g_1}$
(C) $h_{g_1} = h_{g_2}$ when $g_1 G_u^{<0} = g_2 G_u^{<0}$.

1A) Let $\mathfrak{h}_u = \mathfrak{h}_u^{\mathfrak{s}}$ be the homomorphism from G_u into the automoorphism group of $\mathbf{L}_u$ such that

$$f \in \mathbf{G}_u \wedge g \in G_u \Rightarrow (\mathfrak{h}_u(f))(h_g) = h_{f_g}.$$

2) Let $\mathbf{K}_u$ be $\mathbf{G}_u *_{\mathfrak{h}_u} \mathbf{L}_u$ the semi-direct product of $\mathbf{G}_u$ with $\mathbf{L}_u$ over the homorphism $\mathfrak{h}_u$.

Claim 4.7. *Main Like 3.4 but*

(b)′ $\mathfrak{s}$ *is an almost limit or at least (?) semi-limit of* $\mathfrak{t}$ *as witnessed by* v_*.

Theorem 4.8. *Like 3.7 but we omit clauses (f),(g) from the assumption and weaken the conclusion to* $\tau_\kappa^{\rm nlg} > \gamma$.

Conclusion 4.9. *Rephrase Saharon.*

Definition 4.10. In part (3) clause (b): now $g_{[\bar{t}^j, \eta_j]}$ is well defined for every j.

* * *

Claim 4.11. *[?] In the main claim 3.4 we can weaken assumption (b) to*

(b)$^-$ $\mathfrak{s}$ *is an almost limit of* $\mathfrak{t}$ *as witnessed by* v^*.

Proof. Similar to the proof of 3.4. But G^+ is not exactly. A possibility is to redo §1 (and §2) in which we have "various kinds of "*s*
eqt". Further for every n-type we have a set of partial order on it (those which in 4.11) will appear unboundedly in the reflection. $\square$

Claim 4.12. *In Claim 3.7 we can omit assumption (e).*

Proof. Without loss of generality $(*)_0$ from the proof of 3.7 holds.
However, we define $\mathfrak{s} = (J, \bar{I}, \bar{\pi})$ somewhat differently

$(*)_1$ (*a*) $J = (\theta + 1, <)$
(*b*) (α) $I_\theta = (\mathcal{F}, <_{J_\theta^{\rm bd}})$

(β) $I_\alpha = (\mathcal{F}_{1+\alpha}, <_{\alpha+1})$ for $\alpha < \theta$ where:

(i) if $1+\alpha$ is a successor ordinal, say $\beta+1$ and $f_1, f_2 \in \mathcal{F}_{1+\alpha}$ then $f_1 <_\alpha f_2 \Leftrightarrow f_1(\beta) < f_2(\beta)$

(ii) if α is a limit ordinal and $f_1, f_2 \in \mathcal{F}_{1+\alpha}$ then $f_1 <_\alpha f_2 \Leftrightarrow (\forall^* \beta < \alpha)(f_1(\beta) < f_2(\beta))$ where $(\forall^* \beta < \alpha)$ means for every large enough $\beta < \alpha$

(γ) for $\alpha < \beta < \theta + 1$ let $\pi_{\alpha,\beta} : I_\beta \to I_\alpha$ be $\pi_{\alpha,\beta}(f) = f \restriction (1+\alpha)$.

The new point is checking clause $(d)^-$ in Definition 3.15 of almost limit. Now if $n < \omega$ and $f_0, \ldots, f_{n-1} \in \mathcal{F}$ then for some club C of θ we have for $\ell, k < n$ and $\alpha \in C : f_\ell <_\theta f_k \Leftrightarrow (f_\ell \restriction \alpha) <_\alpha (f_k \restriction \alpha)$. $\square$

How to revise §1:

Best is if: if on I we have orders $<_1 \subseteq \leq_2$ then from the group for $(I, <_1)$ there is a projection for the one for $(I, <_2)$. This tends to press for a group with "all is free except some conjugations". [Alternatively] The "toward free" approach: 0)

Also non-decreasing sequences in $(\langle t_\ell : \ell \leq n\rangle, \eta)$.

1) Definition 1.4(1)(c) omit (b)

(b) $G^{u,3}_{<\alpha}$ think how to define

2) Definition 1.2(1B): add $y_1 \restriction n(x) = x = y_2 \restriction n(*)$.

3) Observation 1.6(1) and $x = y \restriction n(*)$

4) Omit 1.6(2),(3),(6).

5) Claim 1.7: replace:(a) each $g \in G^{\mathfrak{s}}_u$ we can canonically represent as $g_{x_1} \ldots g_{x_n}$ such that $g_\ell \neq g_{\ell+1}$ and $\neg\circledast_{x_\ell, x_{\ell+1}}$; (b) the order disappears.

SAHARON!

6) 1.7(4),(7) use canonical instead increase

Proof. Immediate by $G^{<k,\ell>}$ and HNN extension.

7) Claim 1.10: represent?

8) Definition 2.1: (a) we demand $\Pi_{u,v}$ maps $\{t : \mathrm{rk}^{\mathfrak{s}}_{I[v]}(t) = 0\}$ onto $\{\ell : \mathrm{rk}^{\mathfrak{s}}_{I[u]}(t) = 0\}$.

(b) and what about $x \in X_I$ with $\bar{t}^x$ non $<_I$-decreasing? $\square$

* * *

Question: $J^{\mathfrak{t}} = \omega$, the limit is too large still can we commute?

Alternative to clause (f) of the Theorem 3.7

Question: Can we replace equality on $\{u : u_* \leq_{J[\mathfrak{t}]} u\}$ by equality on $\{u : u\ eq_{J[\mathfrak{t}]} u_*\}$ for some u_*?

Moved from $3.8(f)'(\gamma)$,pg.36:

(γ) if $k < \omega$ and $\bar{f}^\delta \in {}^k(\mathcal{F}_\delta)$ for $\delta \in S$ then we can find $\alpha(*) < \delta$ and $\bar{p} = \langle p_\alpha : \alpha \in [\alpha(*), \delta)\rangle$ such that $p_\alpha \in \mathcal{S}^k$ and for every $\beta \in [\alpha(*), \delta)$ for some $\delta \in S \setminus \beta$ we have $\alpha \in [\alpha(*), \beta) \to p_\alpha = \mathrm{tp}_{\mathrm{qf}}(\langle f^\delta_\ell(\alpha) : \ell < k\rangle, \varnothing, (\theta, <))$.

Remark 4.13. Assume

(A) $\mathcal{F}$ is closed under $\min\{f,g\}, f+1, 0_\theta$.

Old proof of 3.13,pg.39: Let $\theta = \mathrm{cf}(\kappa)$ so $\aleph_0 < \theta = \mathrm{cf}(\theta) < \kappa$, and let $\mathcal{F} = {}^\theta\kappa = \{f : f \text{ a function from } \theta \text{ to } \kappa\}$ and $\mathcal{F}_\alpha = \{f \restriction \alpha : f \in \mathcal{F}\}$ for $\alpha < \theta$. Clearly the assumption of 3.7 hence its conclusion: $\tau^{\mathrm{nf}}_\kappa > \gamma$ where $\gamma = \mathrm{rk}(\mathcal{F}, <_{J^{\mathrm{bd}}_\theta})$. But $\mathrm{rk}(\mathcal{F}, <_{J^{\mathrm{bd}}_\theta}) > 2^\kappa$ as: $\kappa^\theta = 2^\kappa$ (by cardinal arithmetic) and $\mathrm{rk}(\mathcal{F}, <_{J^{\mathrm{bd}}_\theta}) = \mathrm{rk}_{J^{\mathrm{bd}}_\theta}(\langle \kappa : i < \theta\rangle)$, see [She94] (as there is a sequence $\langle f_\alpha : \alpha < 2^\kappa\rangle$ in ${}^\theta\kappa$ which is $<_{J^{\mathrm{bd}}_\theta}$-increase by [She94, §1,VII] because there is a sequence $\langle \lambda_i : i < \theta\rangle$ of cardinals $< \kappa$ with $\mathrm{tcf}(\prod\limits_{i<\theta} \lambda_i, <_{J^{\mathrm{bd}}_\theta}) = \lambda$ for any regular $\lambda \in (\kappa, 2^\kappa]$). Still we do not have clause (e) of 3.7. By a variant of [She94, II,3.5], from [She, Part C,§18=k.1tex,pg.56]; there is $\mathcal{F}$ as required (well, if 2^κ is regular, if it is singular we have to combine, see more there).

§ 5. Looks like old stuff

§ 5(A). **Old §1: The Groups.**

Discussion 5.1. How do we define the group $G = G_{\mathbf{p}}$ from the parameter $\mathbf{p}$ which is a partial order I (as the first try to be refined by additional information)? For each $t \in I$ we would like to have an element associated with it ($g_{(\langle t\rangle,\langle\ \rangle)}$) such that it will "enter" $\mathrm{nor}^\alpha_G(H)$ exactly for $\alpha = \mathrm{rk}_I(t)+1$. We intend that among the generators of the group commuting is the normal case so we need witnesses that $g_{(\langle t\rangle,\langle\ \rangle)} \notin \mathrm{nor}^{\beta+1}_G(H)$ wherever $\beta < \alpha = \mathrm{rk}_I(t), \beta > 0$. It is natural that if $\mathrm{rk}_I(t_1) = \beta$ and $t_1 <_I t_0 =: t$ then we use t_1 to represent β, as witness; more specifically, we construct the group such that conjugation by $g_{(\langle t\rangle,\langle\ \rangle)}$ interchange $g_{(\langle t_0,t_1\rangle,\langle 0\rangle)}$ and $g_{(<t_0,s_0>,<1>)}$ and one of them, say $g_{(\langle t_0,t_1\rangle,\langle 0\rangle)}$ belongs to $\mathrm{nor}^{\beta+1}_G(H) \setminus \mathrm{nor}^\beta_G(H)$ whereas the other one, $g_{(<t_0,0>,<1>)}$, belongs to $\mathrm{nor}^1_G(H)$. Iterating we get the elements $x \in X_I$ defined below. To "start the induction", some of the elements $g_{(\alpha,\ell)}(\alpha \in Z^{\mathbf{P}}, \ell < 2)$ are used to generate H and not using all of them will help to make $\mathrm{nor}^1_{G_I}(H_I)$ having the desired value. However, we have to decide for each $g_{(\bar t,\nu)}$ for $(\bar t,\nu)$ as above, for which $g_{(\alpha,\ell)}(\alpha \in Z^{\mathbf{P}}, \ell < 2)$ does conjugation by $g_{(\bar t,\nu)}$ maps $g_{(\alpha,\ell)}$ to itself and for which it does not. For this we choose subsets $A_{(\bar t,\nu)} \subseteq Z^{\mathbf{P}}$ to code our decisions when $(\bar t,\nu)$ is as above and well defined, and make the conjugation with the generators intended to generate $\mathrm{nor}^1_G(H)$ appropriately.

Note that the exact use of $\mathrm{rk}^{<\infty}_I$ (and later its role in $\mathrm{rk}^{2,<\infty}_{\mathbf{P}}$ hence $X^{<\alpha}_{\mathbf{P}}, G^{<\alpha}_{\mathbf{P}}$) is necessarily for the fine determination of $\tau^{\mathrm{nlg}}_{G,H}$, if your, e.g. mind only $|\tau^{\mathrm{nlg}}_{G,H}|$ it does not matter.

Definition 5.2. Let I be a partial order (so $\neq \emptyset$).
1a) $\mathrm{rk}_I : I \to \mathrm{Ord} \cup \{\infty\}$ is defined by $\mathrm{rk}_I(t) \geq \alpha$ iff $(\forall \beta < \alpha)(\exists s <_I t)[\mathrm{rk}_I(s) \geq \beta]$.
1b) $\mathrm{rk}^{<\infty}_I(t)$ is defined as $\mathrm{rk}_I(t)$ if $\mathrm{rk}_I(t) < \infty$ and is defined as $\cup\{\mathrm{rk}_I(s)+1 : s$ satisfies $s <_I t$ and $\mathrm{rk}_I(s) < \infty\}$ in general.
1c) Let $\mathrm{rk}(I) = \cup\{\mathrm{rk}_I(t)+1 : t \in I\}$ stipulating $\alpha < \infty = \infty + 1$.
1d) $\mathrm{rk}^{<\infty}_I = \mathrm{rk}^{<\infty}(I) = \cup\{\mathrm{rk}^{<\infty}_I(t)+1 : t \in I\}$.
1e) Let $I_{[\alpha]} = \{t \in I : \mathrm{rk}(t) = \alpha\}$.
2) Let X_I be the set of objects x satisfying:

- $(*)$ x is a pair, $x = (\bar t, \eta) = (\bar t^x, \eta^x)$ such that for some $n = n(x)$
  - (b) $\bar t = \langle t_\ell : \ell \leq n\rangle$ is a $<_I$-decreasing sequence of members of I
  - (c) $\eta \in {}^n2$.

[note that $\bar{t}$ has length $n+1$ whereas η has length n.]
2A) For $x \in X_I$ let $n = n(x), t_\ell = t_\ell(x), \eta = \eta^x, \bar{t}^x = \langle t^\ell(x) : \ell \le n(x)\rangle$ and let $t(x) = t_{n(x)}(x)$.
2B) For $x = (\bar{t}, \eta) \in X_I$ let $\mathrm{rk}_I^{<\infty}(x) = \mathrm{rk}_I^{<\infty}(t(x))$ and $\mathrm{rk}_I(x) = \mathrm{rk}_I(t(x))$.
2C) For $x \in X_I$ and $n \le n(x)$ let $x \restriction n = (\langle t_\ell^x : \ell \le n\rangle, \eta^x \restriction n)$.
3) I is non-trivial if $\{s : s \le_I t \text{ and } \mathrm{rk}_I(s) = \beta\}$ is infinite for every $t \in I$ satisfying $\mathrm{rk}_I^{<\infty}(t) > \beta$ (used in the proof of 5.16(1)).
4) I is explicitly non-trivial if each E_I-equivalence class is infinite where $E_I = \{(t_1, t_2) : t_2 \in I, t_2 \in I \text{ and } (\forall s \in I)(s <_I t_1 \equiv s <_I t_2)\}$.

Definition 5.3. 1) Let $\Lambda_m^* =^{\mathrm{df}} \{\eta, \varrho) : \eta \in {}^m 2, \varrho$ is a function from ${}^{m\ge}2$ to $\{0,1\}\}$.
2) Let $\Lambda_{<m}^* =^{\mathrm{df}} \cup\{\Lambda_k^* : k < m\}$ and $\Lambda_{\le m}^* =^{\mathrm{d}f} \Lambda_{m+1}^*$ and $\Lambda^* =^{\mathrm{d}f} \cup\{\Lambda_m^* : m < \omega\}$.
3) For any pair (η, ϱ) let $\mathbf{k}((\eta, \varrho))$ be the k such that $(\eta, \varrho) \in \Lambda_k^*$.
4) If $k < m$ and $(\eta, \varrho) \in \Lambda_m^*$ then we define $(\eta, \varrho) \restriction k =^{\mathrm{d}f} (\eta \restriction k, \varrho \restriction {}^{k\ge}2)$.
5) For $\upsilon \in \Lambda_m^*$ let $\upsilon = (\eta^\upsilon, \varrho^\upsilon)$.
6) Let $\Lambda_m^- = \mathcal{P}(\{\upsilon \in \Lambda_m^* : 0 \in \mathrm{Rang}(\eta^\upsilon)\}$.

Definition 5.4. 1) For $k, m < \omega$ and $(\eta, \varrho) \in \Lambda_k^*$ let $\varkappa = \varkappa_{m,(\eta,\varrho)}$ be the following permutation of Λ_m^*. (Note that if $k \ge m$ then this permutation is the identity). For $(\eta_1, \varrho_1) \in \Lambda_m^*$ we define $(\eta_2, \varrho_2) = \varkappa_{m,((\eta,\varrho))}(\eta_1, \varrho_1) \in \Lambda_m^*$ such that $k =^{\mathrm{d}f} \mathbf{k}((\eta_2, \varrho_2)) = \mathbf{k}(\eta_1, \varrho_1))$ as follows: Case 1: $(\eta_1, \varrho_1) \restriction k \ne (\eta, \varrho)$.

In this case we have $(\eta_2, \varrho_2) = (\eta_1, \varrho_1)$. Case 2 : Not case 1.

First $\eta_2(i) = \eta_1(i)$ iff $i \ne k$ (and $i < m$)
Second for $\rho_2 \in {}^m 2$ the value of $\varrho_2(\rho_2)$ is : $\varrho_1(\rho_1)$ when $\rho_1 \in {}^m 2$ has the same length as ρ_2 and $\rho_1(i) \ne \rho_2(i)$ iff $i = lg(\eta) \wedge \rho \triangleleft \rho_2$ for $i < m$
2) Let $\varkappa = \varkappa^2 - m, \upsilon$ be the permutation of $\mathcal{P}(\Lambda_m^*)$ induced by $\varkappa_\upsilon^1$ that is, for $\Lambda \subseteq \Lambda_m^*, \varkappa(\Lambda) = \{\varkappa((\upsilon) : \upsilon \in \Lambda\}$; we may omit the 2.

Definition 5.5. 1) Let $\mathbf{m}$ be the following function: for an ordinal $\beta = \omega\alpha + m$ we let $\mathbf{m}(\beta) = m$ (here as κ is always $> \aleph_0$ this is fine, but if it is equal we better change the values of $\mathbf{m}$ on the natural numbers such that each has $\aleph_0$ natural numbers as pre-images).
2) For a set Z of ordinals let $Z * 2 =^{\mathrm{df}} \cup\{\{\alpha\} \times \Lambda_{\mathbf{m}(\alpha)}^* : \alpha \in Z\}$.
3) For a set Z of ordinals let $Z *' 2 =^{\mathrm{d}f} \cup\{(\alpha, (\eta, \varrho)) : \alpha \in Z, (\eta, \varrho) \in \Lambda_{\mathbf{m}(\alpha)}^*, \neg[\mathrm{Rang}(\eta) \subseteq \{1\}]\}$.
4) We say that $\odot_{(\eta,\varrho),(\eta_1,\varrho_1),(\eta_2,\varrho_2)}$ when:

(A) $k =^{\mathrm{d}f} \mathbf{k}((\eta, \varrho)) < \mathbf{k}((\eta_1, \varrho_1)) = \mathbf{k}((\eta_2, \varrho_2))$
(B) $(\eta_1, \varrho_1) \restriction k = (\eta, \varrho) = (\eta_2, \varrho_2) \restriction k$
(C) $\eta_1(i) \ne \eta_2(i) \Rightarrow i = \ell g(\eta_1)$ for $i < \ell g(\eta_1) = \ell g(\eta_2)$, of course
(D) $\varrho_1(\rho) \ne \varrho_2(\rho) \Leftrightarrow \rho = \eta$ for $\rho \in {}^{m\le}2$ of course.

Definition 5.6. [USED??] 1) Let H_m be the subgroup of $\mathrm{per}(\Lambda_m^*)$ generated by $\{g_{m,(\eta,\varrho)} : (\eta, \varrho) \in \Lambda_{<m}^*$.
2) For $k \le m$ let H_m^k be the subgroup of H_m generated by $\{\varkappa_{m,(\eta,\varrho)} : (\eta, \varrho) \in \Lambda_{<m}^*$ and $0 \in \mathrm{Rang}(\eta)$ or $\ell g(\eta) \ge m - k\}$.

Observation 5.7. *1) For $(\eta, \varrho) \in \Lambda^*_{<m}$ we have: $\varkappa^1_{m,(\eta,\varrho)}$ is a permutation of Λ^*_m of order two and $\varkappa^2_{m,(\eta,\varrho)}$ is a permutation of $\mathcal{P}(\Lambda^*_m)$ of order two.*
2) If $i \in \{1,2\}, k < m\omega$ and $(\eta_1, \varrho_1), (\eta_2, \varrho_2)$ belongs to Λ^*_k and $(\eta_\ell, \varrho_\ell) \neq (\eta_{3-\ell}, \varrho_{3-\ell}) \restriction \mathbf{k}((\eta_\ell, \varrho_\ell))$ for $\ell = 1, 2$ then $\varkappa^i_{m,(\eta_1,\varrho_1)}, \varkappa^i_{m,(\eta_2,\varrho_2)}$ commute.
3) If $(\eta_\ell, \varrho_\ell) \in \Lambda^*_{<m}$ and $k_\ell =^{df} \mathbf{k}((\eta_\ell, \varrho_\ell)$ for $\ell = 0, 1$ and $(\eta_0, \varrho_0) = (\eta_1, \varrho_1) \restriction k_0$ then for one and only one $(\eta_2, \varrho_2) \in \Lambda^*_{k_1}$ we have $\odot_{(\eta_0,\varrho_0),(\eta_1,\varrho_1),(\eta_2,\varrho_2)}$ holds.
4)If $\odot^m_{(\eta,\varrho),(\eta_1,\varrho_1),(\eta_2,\varrho_2)}$ and $\mathbf{k}((\eta_1, \varrho_1)) \leq m$ then in the permutation group of Λ^*_m we have $\varkappa_{m,(\eta,\varrho)}\varkappa_{m,(\eta_1,\varrho_1)}\varkappa^{-1}_{m,(\eta,\varrho)} = \varkappa_{m,(\eta_2,\varrho_2)}$.
5) If $\upsilon \in \Lambda^*_{<m}$ then $\varkappa^1_{m,\upsilon}$ maps Λ^*_m onto (equivalently into) itself iff $\upsilon \in \Lambda^*_{<m}$; similarly $\varkappa^2_{m,\upsilon}$ maps $\mathcal{P}(\Lambda^*_m)$ onto (equivalently into) itself iff $\upsilon \in \Lambda^*_{<m}$.

Proof. Easy. □

Definition 5.8. 1) We say that $\mathbf{p}$ is a κ-parameter <u>when</u>:

(A) $\mathbf{p} = (I, \bar{A}, Z, Y) = (I^{\mathbf{p}}, \bar{A}^{\mathbf{p}}, Z^{\mathbf{p}})$ but let $I[\mathbf{p}] = I^{\mathbf{p}}$
(B) I is a partial order
(C) $\bar{A} = \langle A_x : x \in X_I \rangle$ and $A_x \subseteq Z$ so $A_x = A^{\mathbf{p}}_x$
(D) $Z \subseteq \kappa$ (and we assume that $X_I \cap (\kappa * 2) = \varnothing$, of course).

2) For a κ-parameter $\mathbf{p}$

(A) let $X_{\mathbf{p}}$ be $X_{I[\mathbf{p}]}$ and $X^+_{\mathbf{p}} = X_{\mathbf{p}} \cup (Z^{\mathbf{p}} \times 2)$ and for $x \in Z^{\mathbf{p}} \times 2 = X^+_{\mathbf{p}} \setminus X_{\mathbf{p}}$ let $n(x) = \omega$; let $X^\ell_{\mathbf{p}}$ be $X^+_{\mathbf{p}}$ if $\ell = 1, X_{\mathbf{p}}$ if $\ell = 2$
(B) let $\mathrm{rk}^1_{\mathbf{p}} : X^+_{\mathbf{p}} \to \{-1\} \cup \mathrm{Ord} \cup \{\infty\}$ be defined by $x \in X_{\mathbf{p}} \Rightarrow \mathrm{rk}^1_{\mathbf{p}}(x) = \mathrm{rk}_{I[\mathbf{p}]}(x)$ and $x \in Z^{\mathbf{p}} \times 2 \Rightarrow \mathrm{rk}^1_{\mathbf{p}}(x) = -1$
(C) let $\mathrm{rk}^2_{\mathbf{p}} : X^+_{\mathbf{p}} \to \{-1\} \cup \mathrm{Ord} \cup \{\infty\}$ and $\ell : X_{\mathbf{p}} \to \omega \cup \{\infty\}$ be defined by
  (α) if $x \in Z^{\mathbf{p}} \times 2$ then $\mathrm{rk}^2_{\mathbf{p}}(x) = -1$
  (β) if $x \in X_{\mathbf{p}}$ and $\mathrm{Rang}(\eta^x) \subseteq \{1\}$ (e.g., $n(x) = 0$) then $\mathrm{rk}^2_{\mathbf{p}}(x) = \mathrm{rk}_{I[\mathbf{p}]}(x) (= \mathrm{rk}_{I[\mathbf{p}]}(t(x)))$ and we let $\ell(x) = \infty$
  (γ) if $x \in X_{\mathbf{p}}$ and $\mathrm{Rang}(\eta^x) \nsubseteq \{1\}$ let $\ell(x) = \min\{\ell : \eta^x(\ell) = 0\}$ and $\mathrm{rk}^2_{I[\mathbf{p}]}(x) = 0$ (yes, zero)
(D) $\mathrm{rk}^{1,<\infty}_{I[\mathbf{p}]}(x)$, $\mathrm{rk}^{2,<\infty}_{\mathbf{p}}(x)$ are defined similarly using $\mathrm{rk}^{<\infty}_{I[\mathbf{p}]}(-)$ instead $\mathrm{rk}_{I[\mathbf{p}]}(-)$
(E) $\mathrm{rk}^2(\mathbf{p}) = \mathrm{rk}^2_{I[\mathbf{p}]}$, etc.
(F) for $x \in X_I$ let $\varrho^{\mathbf{p}}_{x,\alpha}$ be the function from ${}^{n(x)\geq}2$ to the set $\{0, 1\}$ defined as follows:

$$\text{for } \rho \in {}^k2, k \leq n(x) \text{ we have } \varrho^{\mathbf{p}}_{x,\alpha}(\rho) = 1 \text{ iff } \alpha \in A_{(\bar{t}^x \restriction k,\rho)}$$

(G) For $x \in X_{\mathbf{p}}$ let $\eta^{\mathbf{p}}_x$ be η^x if $\mathrm{rk}^{2,<\inf}_{\mathbf{p}}(x) > 0$ and let it be $(\eta^x) \restriction n(x)) \cong \langle 0 \rangle$ otherwise
(H) for $x \in X_{\mathbf{p}}$ let $\upsilon^{\mathbf{p}}_x = (\eta^{\mathbf{p}}_\upsilon, \varrho^{\mathbf{p}}_{x,\alpha})$ and $\varkappa^{\mathbf{p}}_{x,\alpha} = \varkappa^2_{\mathbf{m}}(\alpha), \upsilon^{\mathbf{p}}_{x,\alpha}$
(I) $\bar{\varkappa}^{\mathbf{p}}_x =^{\mathrm{df}} \langle \varkappa^{\mathbf{p}}_{x,\alpha} : \alpha \in Z^{\mathbf{p}} \rangle$.

3) We say $\mathbf{p}$ is a nice κ-parameter when:

(A) $\mathbf{p}$ is a κ-parameter
(B) if $x \in X_{\mathbf{p}}$ and $\mathrm{rk}^2_{\mathbf{p}}(x) = 0$ then $A_x \subseteq Y$, (used in the proof of 5.16(2))

(C) if $k, m < \omega$ and $x_0, x_1, \ldots, x_k \in X_{I[\mathbf{p}]}$ are with no repetitions and $\mathrm{rk}^2_{\mathbf{p}}(x_0) > 0$ then $A_{x_0} \nsubseteq \cup\{A_{x_\ell} : \ell = 1, \ldots, k\} \cup \{\alpha \in Z^{\mathbf{p}} : \mathbf{m}(\alpha) \le m\}$, (used in the proof of **??**(1))

(D) if $x \neq y \in X_{I_{\mathbf{p}}}$ then $A_x \neq A_y$.

Definition 5.9. Assume $\mathbf{p}$ is a κ-parameter. Below if we omit the superscript ℓ we mean 2.
1) Let $G^1_{\mathbf{p}} = G^1[\mathbf{p}]$ be the group generated by $\{g_x : x \in X^+_{\mathbf{p}}\}$ freely except the equations in $\Gamma^1_{\mathbf{p}}$ where $\Gamma^1_{\mathbf{p}}$ consists of

(A) $g_x^{-1} = g_x$, that is g_x has order 2, for each $x \in X^+_{\mathbf{p}}$

(B) $g_{y_1} g_{y_2} = g_{y_2} g_{y_1}$ when $y_1, y_2 \in Z^{\mathbf{p}} * 2$

(C) $g_x g_{y_1} g_x^{-1} = g_{y_2}$ when $\circledast_{x,y_1,y_2}$, see below.

1A) Let $G^2_{\mathbf{p}} = G^2[\mathbf{p}]$ be the group generated by $\{g_x : x \in X_{\mathbf{p}}\}$ freely except the equations in $\Gamma^2_{\mathbf{p}}$ where $\Gamma^2_{\mathbf{p}}$ consists of

(A) $g_x^{-1} = g_x$, that is g_x has order 2, for each $x \in X_{\mathbf{p}}$

(B) $g_{y_1} g_{y_2} = g_{y_2} g_{y_1}$, i.e., g_{y_1}, g_{y_2} commute when $\neg\circledast^2_{x,y}$ and $\neg\circledast^2_{y,x}$ see below

(C) $g_x g_{y_1} g_x^{-1} = g_{y_2}$ when $\circledast^2_{x,y_1,y_2}$, see below

this includes "x, y commute if $x \in X_{\mathbf{p}}, y = (\alpha, \ell) \in Z^{\mathbf{p}} \times 2$ and $\alpha \in Z^{\mathbf{p}} \setminus A^{\mathbf{p}}$".
1B) Let $\circledast_{x,y_1,y_2}$ means that $\circledast^1_{x,y_1,y_2}$ or $\circledast^2_{x,y_1,y_2}$, see below. Let $\circledast_{x,y}$ mean that $\circledast_{x,y_1,y_2}$ for some y_1, y_2 such that $y \in \{y_1, y_2\}$ and $\circledast^1_{x,y_1}, \circledast^2_{x,y_1}$ are defined similarly.
1C) Let $\circledast^1_{x,y_1,y_2}$ means that $x \in X_{\mathbf{p}}$ and for some $\alpha \in Z^{\mathbf{p}}$ we have $y_\ell \in \{\alpha\} \times \Lambda_{\mathbf{m}(\alpha)}$ for $\ell = 1, 2$ and $\odot_{\Upsilon[x], \Upsilon[y_1], \upsilon[y_2]}$.
1D) Let $\circledast^2_{x,y_1,y_2}$ means that:

(A) $x, y_1, y_2 \in X_{\mathbf{p}}$

(B) $n(x) < n(y_1) = n(y_2)$

(C) $y_1 \restriction n(x) = x = y_2 \restriction n(x)$

(D) $\bar{t}^{y_1} = \bar{t}^{y_2}$

(E) $\eta^{y_1}(\ell) = \eta^{y_2}(\ell)$ for every $\ell < n(y_1)$ which is $\neq n(x)$

(F) $\eta^{y_1}(n(x)) \neq \eta^{y_2}(n(x))$.

2) For $\ell \in \{1, 2\}$ let $G^{1,\le\alpha}_{\mathbf{p}}$ is defined similarly to $G^\ell_{\mathbf{p}}$ except that it is generated only by $X^{\ell,<\alpha}_{\mathbf{p}} =: \{g_x : x \in X^\ell_{\mathbf{p}} \wedge \mathrm{rk}^{2,<\infty}_{\mathbf{p}}(x) < \alpha\}$ freely except the equations from $\Gamma^{\ell,<\alpha}_{\mathbf{p}}$, where $\Gamma^{\ell,<\alpha}_{\mathbf{p}}$ is the set of equations from $\Gamma^\ell_{\mathbf{p}}$ among $\{g_x : x \in X^{\ell,<\alpha}_{\mathbf{p}}\}$.

Similarly $G^{1,\le\alpha}_{\mathbf{p}}, X^{\ell,\le\alpha}_{\mathbf{p}}$ so $X^{\ell,\le\infty}_{\mathbf{p}} = X^{\ell,<\infty}_{\mathbf{p}} = X^\ell_{\mathbf{p}}$ and $G^{\ell,\le\infty}_{\mathbf{p}} = G^{\ell,<\infty}_{\mathbf{p}} = G^\ell_{\mathbf{p}}$; note that $G^{1,\le\alpha}_{\mathbf{p}} = G^{1,<\alpha+1}_{\mathbf{p}}, X^{\ell,\le\alpha}_{\mathbf{p}} = X^{\ell,<\alpha+1}_{\mathbf{p}}$ if $\alpha < \infty$.
3) Let $H^\ell_{\mathbf{p}}$ be the subgroup of $G^\ell_{\mathbf{p}}$ generated by $\{g_y : y \in Z^{\mathbf{p}} *^\ell 2\}$.
4) For $X \subseteq X_{\mathbf{p}}, Z \subseteq Z^{\mathbf{p}}$ let $G^1_{\mathbf{p},X,Z}$ be the group generated by $\{g_y : y \in X \cup (Z * 2)\}$ freely except the equations in $\Gamma^1_{\mathbf{p},X,Z}$ which is the set of equations from $\Gamma^1_{\mathbf{p}}$ mentioning only generators among $\{g_y : y \in X \cup (Z * 2)\}$.
4A) For $X \subseteq X_{\mathbf{p}}$ we define $G^2_{\mathbf{p},X}$ similarly.

Observation 5.10. *1) The sequence $\langle X^{\ell,<\alpha}_{\mathbf{p}} : \alpha \le \mathrm{rk}^{<\infty}(\mathbf{p})\rangle$ is $\subseteq$-increasing.*
2) If $\ell \in \{1, 2\}$ and $x, y \in X_{\mathbf{p}}$ and $y = x \restriction n \neq y$ and $\ell \in \{1, 2\}$ then $\mathrm{rk}^\ell_{\mathbf{p}}(y) \le \mathrm{rk}^\ell_{\mathbf{p}}(x)$ and if equality holds then $\mathrm{rk}^1_{\mathbf{p}}(x) = \infty = \mathrm{rk}^1_{\mathbf{p}}(y)$ <u>or</u> both are zero and $\ell = 2$.
3) If a partial order I is explicitly non-trivial <u>then</u> I is non-trivial.

Proof. Check. $\square$

Observation 5.11. *For a κ-parameter $\mathbf{p}$:*
1) $\circledast^1_{x,y}$ holds iff $x \in X_{\mathbf{p}}$ and $y \in Z^{\mathbf{p}} * 2 \subseteq X^+_{\mathbf{p}} \setminus X_{\mathbf{p}}$.
2) $\circledast^2_{x,y}$ holds iff:

- (α) *$x, y \in X^+_{\mathbf{p}}$ and $n(y) \geq n(x) + 1$*
- (β) *$y \restriction n = x$.*

4) If $\circledast^2_{x,y_1,y_2}$ then $y_1 \restriction n(x) = x = y_2 \restriction n(x)$ and $n(y_1) = n(y_2)$.
5) $\circledast^\ell_{x,y_1,y_2}$ iff $\circledast^\ell_{x,y_2,y_1}$ for $\ell = 1, 2$.

Proof. Easy. $\square$

We first sort out how elements in $G_{\mathbf{p}}$ and various subgroups can be (uniquely) represented as products of the generators.

Claim 5.12. *Assume that $\mathbf{p}$ is a κ-parameter and $<^*$ is any linear order of $X_{\mathbf{p}}$ such that*

$\boxdot$ *if $x \in X_{\mathbf{p}}, y \in X_{\mathbf{p}}$ and $n(x) > n(y)$ (we could have demanded just*

$$(\exists n < n(x))[y = x \restriction n]) \text{ \underline{then} } x <^* y.$$

1) Any member of $G_{\mathbf{p}}$ is equal to a product of the form $g_{x_1} \ldots g_{x_m}$ where $x_\ell <^* x_{\ell+1}$ for $\ell = 1, \ldots, m-1$. Moreover, this representation is unique.
2) Similarly for $G^{\leq \alpha}_{\mathbf{p}}, G^{< \alpha}_{\mathbf{p}}$ (using $X^{\leq \alpha}_{\mathbf{p}}, X^{< \alpha}_{\mathbf{p}}$ respectively instead $X_{\mathbf{p}}$) hence $G^{\leq \alpha}_{\mathbf{p}}, G^{< \alpha}_{\mathbf{p}}$ are subgroups of $G_{\mathbf{p}}$.
*3) If $y <^* x$ are from $X_{\mathbf{p}}$ and g_x, g_y do not commute (in $G_{\mathbf{p}}$) then $\circledast_{x,y}$ of Definition **??**(1)(b) holds hence $(y, n(x))$ determines x uniquely, in fact, $x = y \restriction n(x)$, see 5.2(2B).*
4) If $g = g_{y_1} \ldots g_{y_m}$ where $y_1, \ldots, y_m \in X_I$ and $g = g_{x_1} \ldots g_{x_n} \in G_{\mathbf{p}}$ and $x_1 <^* \ldots <^* x_n$ then $n \leq m$.
5) $\langle G^{< \alpha}_{\mathbf{p}} : \alpha \leq \mathrm{rk}^{<\infty}(I^{\mathbf{p}}) \rangle$ is an increasing continuous sequence of groups with last element $G^2_{\mathbf{p}}$.
6) $H_{\mathbf{p}} \subseteq G^{<0}_{\mathbf{p}}$ is a subgroup of cardinality $\leq \kappa$.
7) In part (1) we can replace $G_{\mathbf{p}}, X_{\mathbf{p}}$ by $G = G_{\mathbf{p},X}, X$ when $X \subseteq X_{\mathbf{p}}$ is such that $[\{x, y_1, y_2\} \subseteq X \wedge \circledast^2_{x,y_1,y_2} \wedge \{x, y_1\} \subseteq X \Rightarrow y_2 \in X]$. Hence $G_{\mathbf{p},X}$ is equal to $\langle \{g_x : x \in X\} \rangle_{G_{\mathbf{p}}}$.

Proof. 1),2),7) Recall that each generator has order two. We can use standard combinatorial group theory (the rewriting process but below we do not assume knowledge of it); the point is that in the rewriting the number of generators in the word do not increase (so no need of $<^*$ being a well ordering).
For a full self-contained proof, for part of (2) we consider $G = G^{<\alpha}_{\mathbf{p}}, X = X^{<\alpha}_{\mathbf{p}} \cap X_{\mathbf{p}}, \Gamma = \Gamma^{<\alpha}_{\mathbf{p}}$ for α an ordinal or infinity and for part (1) and the rest of part (2) consider $G = G^{\leq \beta}_{\mathbf{p}}, X = X^{\leq \beta}_{\mathbf{p}} \cap X_{\mathbf{p}}, \Gamma = \Gamma^{\leq \beta}_{\mathbf{p}}$ for β an ordinal or infinity (recall that $G_{\mathbf{p}}, X_{\mathbf{p}}$ is the case $\beta = \infty$ CHECK!!). The condition from part (7) holds by **??**(2) so it is enough to prove part (7). Now recall that $G^2 = G^2_{\mathbf{p},X}$ and

- $\circledast_1$ every member of G can be written as a product $g_{x_1} \ldots g_{x_n}$ for some $n < \omega, x_\ell \in X$
  [Why? As the set $\{g_x ; x \in X\}$ generates G.]
- $\circledast_2$ if in $g = g_{x_1} \ldots g_{x_n}$ we have $x_\ell = x_{\ell+1}$ then we can omit both
  [Why? As $g_x g_x = e_G$ for every $x \in X$ by clause (a) of Definition **??**(1)]

$\circledast_3$ if $1 \le \ell < n$ and $g = g_{x_1} \ldots g_{x_n}$ and we have $x_{\ell+1} <^* x_\ell$ and $m \in \{1,\ldots,n\} \setminus \{\ell,\ell+1\} \Rightarrow y_m = x_m$ then we can find $y_\ell, y_{\ell+1} \in X^+$ such that $g = g_{y_1} \ldots g_{y_n}$ and $y_\ell <^* y_{\ell+1}$ and, in fact, $y_{\ell+1} = x_\ell$.

[Why does $\circledast_3$ hold? By Definition 5.9(1) one of the following cases occurs. Case 1:

$g_{x_\ell}, g_{x_{\ell+1}}$ commutes.

Let $y_\ell = x_{\ell+1}, y_{\ell+1} = x_\ell$. Case 2: $\circledast^2_{x_{\ell+1},x_\ell}$, see Definition 5.9(1B).

By clause (b) of Definition 5.9(1) we have $n(x_{\ell+1}) < n(x_\ell)$. So by $\boxdot$ of the assumption we have $x_\ell <^* x_{\ell+1}$, contradiction. Case 3: $\circledast^2_{x_\ell,x_{\ell+1}}$, see Definition 5.9(1D).

Clearly there is $y_\ell \in X$ such that $n(y_\ell) = n(x_{\ell+1}) > n(x_\ell), \bar t^{y_\ell} = \bar t^{x_{\ell+1}}$ and $i < n(x_{\ell+1}) \Rightarrow (\eta^{y_\ell}(i) = \eta^{x_{\ell+1}}(i)) \equiv (i \ne n(x_\ell))$.

Let $y_{\ell+1} = x_\ell$, clearly $y_{\ell+1}, y_\ell \in X$. By Definition 5.9(1), $g_{x_\ell} g_{x_{\ell+1}} g^{-1}_{x_\ell} = g_{y_\ell}$ hence $g_{x_\ell} g_{x_{\ell+1}} = g_{y_\ell} g_{x_\ell} = g_{y_\ell} g_{y_{\ell+1}}$ and clearly $y_\ell <^* x_\ell = y_{\ell+1}$, so we are done. The three cases exhaust all possibilities $\circledast_3$ is proved.]

$\circledast_4$ every $g \in G$ can be represented as $g_{x_1} \ldots g_{x_n}$ with $x_1 <^* x_2 <^* \ldots <^* x_n$.

[Why? Without loss of generality g is not the unit of G. By $\circledast_1$ we can find $x_1,\ldots,x_n \in X_1$ such that $g = g_{x_1} \ldots g_{x_n}$ and $n \ge 1$. Choose such representation

$\bigotimes$ (a) with minimal n and

(b) for this n, with minimal $m \in \{1,\ldots,n+1\}$ such that $x_m <^* \ldots <^* x_n$ and $1 < m \le n \Rightarrow \bigwedge\limits_{\ell=1}^{m-1} x_\ell <^* x_m$, and

(c) for this pair (n,m) if $m > 2$ then with maximal ℓ where $\ell \in \{1,\ldots,m-1\}$ satisfies x_ℓ is $<^*$-maximal among $\{x_1,\ldots,x_{m-1}\}$.

Easily there is such a sequence $(x_1,\ldots,x_n)$, noting that $m = n+1$ is O.K. for (b) and there is x_ℓ as in $\bigotimes(c)$ by $\bigotimes(a)$.

By $\circledast_2$ and clause (a) of $\otimes$ we have $x_\ell \ne x_{\ell+1}$ (when ℓ (from $\otimes(c)$) is well defined, i.e., if $m > 2$).

Now $m = 2$ is impossible (as then $m = 1$ can serve), if $m = 1$ we are done, and if $m > 2$ then $\ell = m-1$ is impossible (as then $m-1$ can serve instead m). Lastly by $\circledast_3$ applied to this ℓ, we could have improved ℓ to $\ell+1$.]

$\circledast_5$ the representation in $\circledast_4$ is unique.

[Why does $\circledast_5$ hold? Assume toward contradiction that $g_{x'_1} \ldots g_{x'_{n_1}} = g_{y'_1} \ldots g_{y'_{n_2}}$ where $x'_1 <^* \ldots <^* x'_{n_1}$ and $y'_1 <^* \ldots <^* y'_{n_2}$ and $(x'_1,\ldots,x'_{n_1}) \ne (y'_1,\ldots,y'_{n_2})$. Without loss of generality among all such examples, $(n_1+n_2+1)^2+n_1$ is minimal.

Let $Y_n =: \{x \in X : n(x) = n\}$.

So $\langle Y_n : n < \omega\rangle$ is a partition of X^+.

For $k \le m < \omega$ let $X^{<k,m>} = \{x \in X^+ : x \in \bigcup\{Y_\ell : k \le \ell < m\}\}$ and let $G^{<k,m>}$ be the group generated freely by $\{g_x : x \in X^{<k,m>}\}$ except the equations in $\Gamma^{<k,n>}$, i.e., from the equations from $\Gamma_{\mathbf{p},X^{<k,m>}}$, i.e., from Definition **??**(4) mentioning only its generators, $\{y_x : x \in X^{<k,m>}\}$. Now clearly if $\circledast^2_{x,y_1,y_2}$, see Definition **??**(1A) then $n \le \omega \Rightarrow [y_2 \in Y_n \equiv y_2 \in Y_n]$. Hence the proof of $\circledast_1 - \circledast_4$ above gives that for every $g \in G^{<k,m>}$ there are n and $x_1 <^* \ldots <^* x_n$ from $X^{<k,m>}$ such that $G^{<n,m>} \models$ "$g = g_{x_1} \ldots g_{x_n}$". Also it is enough to prove the uniqueness for $G^{<k,m>}$ (for every $k \le m < \omega$), i.e., we can assume $x'_1,\ldots,x'_{n_1}, y'_1,\ldots,y'_{n_2} \in X^{<k,m>}$ as if it fail , finitely many equations implies the undesirable equation and for some $k \ge m < \omega$ they are from $\Gamma^{\langle,k,m,\rangle}$, hence already in $G^{\langle k,m\rangle}$ we get this undesirable equation.

Now for $k \geq m < \omega$ and $x \in Y_k$ let $\pi^{k,m}_x$ be the following permutation of $X^{\langle k+1,m\rangle}$: it maps $y_1 \in X^{\langle k+1,m\rangle}$ to y_2 if $\circledast^2_{x,y_1,y_2}$ and it maps $y \in X^{\langle k+1,m\rangle}$ to y if $\neg\circledast^2_{x,y}$.

It is easy to check that

$\boxdot_1$ For k,m,x as above,

(i) $\pi^{k,m}_x$ is a permutation of $X^{\langle k+1,m\rangle}$ which maps $\Gamma^{\langle k+1,m\rangle}$ onto itself

(ii) so $\pi^{k,m}_x$ induce an automorphism $\hat{\pi}^{k,m}_x$ of $G^{\langle k,m\rangle}$: the one mapping g_{y_1} to g_{y_2} when $\pi^{k,m}_x(y_1) = y_2$

(iii) the automorphisms $\hat{\pi}^{k,m}_x$ of $G^{\langle k,m\rangle}$ for $x \in Y_k$ pairwise commute

(iv) the automorphism $\hat{\pi}^{k,m}_x$ of $G^{\langle k,m\rangle}$ is of order two by induction on $m-k$.

Note that

$(*)$ if $x \in Y_k, y \in Y_\ell$ and $x <^* y$ then $\ell \leq k$.

If $m-k=0$, then $G^{<k,m>}$ is the trivial group so the uniqueness is trivial.

Also the case $k = m-1$ is trivial , $G^{\langle k,m\rangle}$ is actually a vector space over $\mathbb{Z}/2\mathbb{Z}$ with basis $\{g_x : x \in Y_k\}$, well in additive notation so the uniqueness is clear.

So assume that $m-k \geq 2$, now

$\boxdot^2_{k,m}$ $k \geq m < \omega$ and if $x'_1,\dots,x'_{n_1},y'_1,\dots,y'_{n_2}$ from $X^{\langle k,m\rangle}$ are as above in $G^2_{X^{\langle k,m\rangle}}$ then $\langle x'_1,\dots,x'_{n_1}\rangle = \langle y'_1,\dots,y'_{n_2}\rangle$.

We prove this.

So 1),2),7) holds.

3) Check (by (1) and the definition of $G_{\mathbf{p}}$).

4) Included in the proof of $\circledast_4$ inside the proof of parts (1),(2),(7).

5) For $\alpha < \beta \leq \infty$, as clearly $X^{<\alpha}_{\mathbf{p}} \subseteq X^{<\beta}_{\mathbf{p}}$ and $\Gamma^{<\alpha}_{\mathbf{p}} \subseteq \Gamma^{<\beta}_{\mathbf{p}}$ hence there is a homomorphism from $G^{<\alpha}_{\mathbf{p}}$ into $G^{<\beta}_{\mathbf{p}}$. This homomorphism is the one-to-one (because of the uniqueness clause in part (2)) hence the homomorphism is the identity. So the sequence is $\subseteq$-increasing, the $\subset$ follows by part (1), the uniqueness we have $\mathrm{rk}^\infty_I(t) = \alpha \Rightarrow g_{(\langle t\rangle,\langle s\rangle)} \in G^{\leq\alpha+1}_{\mathbf{p}} \setminus G^{<\alpha}_{\mathbf{p}}$.

6) $H_{\mathbf{p}}$ is generated by $\leq |Z^{\mathbf{p}} * 2| = \kappa \times 2 = \kappa$ generators. $\square$

Observation 5.13. *Assume that*

(A) G is a group

(B) f_t is an automorphism of G for $t \in J$

(C) $f_t, f_s \in \mathrm{Aut}(G)$ commute for any $s,t \in J$.

Then there are K and $\langle g_t : t \in J\rangle$ such that

(α) K is a group

(β) G is a normal subgroup of K

(γ) H is generated by $G \cup \{g_t : t \in J\}$

(δ) if $a \in G$ and $t \in G$ then $g_t a g_t^{-1} = f_t(a)$

(ε) if $<_*$ is a linear orer of J then every member of K has a one and only one representation as $x g^{b_1}_{t_1} g^{b_2}_{t_2} \dots g^{b_n}_{t_n}$ when $x \in G, n < \omega, t_1 <_* \dots <_* t_n$ are from J and $b_1,\dots,b_n \in \mathbb{Z}\setminus\{0\}$.

Proof. A case of twisted product see below. (It is also a case of repeated HNN extensions). $\square$

Definition 5.14. Definition/ 1) Assume G_1, G_2 are groups and π is a homomorphism from G_2 into $\mathrm{Aut}(G_1)$, we define the twisted product $G = G_1 *_\pi G_2$ as follows:

- (A) the set of elements is $G_1 \times G_2 = \{(g_1, g_2) : g_1 \in G_1, g_2 \in G_2\}$
- (B) the product operation is $(g_1, g_2) * (h_1, h_2) = (g_1, h_1^{\pi(g_2)}, g_2 h_2)$ where
  - (α) $h_1^{\pi(g_2)}$ is the image of h_1 by the automorphism $\pi(g_2)$ of G_1
  - (β) $g_1 h_1^{\pi(g_2)}$ is a G_1-product
  - (γ) $g_2 h_2$ is a G_2-product.

2)

- (A) such group G exists
- (B) in G every member has one and only one representation as g'_1, g'_2 when $g'_1 \in G_1 \times \{e_{G_2}\}, g'_2 \in \{e_{G_1}\} \times G_2$
- (C) the mapping $g_1 \mapsto (g_1, e)$ embeds G_1 into G
- (D) the mapping $g_2 \mapsto (e, g_2)$ embeds G_2 into G
- (E) so up to renaming, each $g_2 \in G_2$ conjugating by it inside G acts on G_1 as the automorphism $\pi(g_2)$ of G_1.

Observation 5.15. *[?] Let* $\mathbf{p}$ *be a nice* κ*-parameter.*
1) If $a \in Z^{\mathbf{p}}, m < 2$ *and* $g \in G_{\mathbf{p}}$ *then* $g g_{(a,v)} g^{-1} \in \{g_{(a,v)} : \upsilon \in \Lambda^*_{\mathbf{m}(a)}\}$.
2) If $G_{\mathbf{p}} \models$ "$g_{x_1} \ldots g_{x_n} = g_{y_1} \ldots g_{y_m}$" *where* $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_m\} \in X^+_I$ *and* $Z \subseteq Z^{\mathbf{p}}$ *and we omit* g_{x_ℓ} *if* $x_\ell \in Z * 2$ *and we omit* g_y *if* $y \in Z * 2$ *then the equation still holds.*

Proof. By 5.12 and its proof. $\square$

Claim 5.16. *Let* $\mathbf{p}$ *be a nice* κ*-parameter and* $I = I^{\mathbf{p}}$ *be non-trivial.*
1) If $0 < \alpha < \mathrm{rk}^{<\infty}_{I[\mathbf{p}]}$ *<u>then</u> the normalizer of* $G^{<\alpha}_{\mathbf{p}}$ *in* $G_{\mathbf{p}}$ *is* $G^{<\alpha+1}_{\mathbf{p}}$.
2) If $\alpha = \mathrm{rk}^{<\infty}_{I[\mathbf{p}]}$ *<u>then</u> the normalizers of* $G^{<\alpha}_{\mathbf{p}}$ *in* $G_{\mathbf{p}}$ *is* $G^{<\infty}_{\mathbf{p}} = G^{<\alpha}_{\mathbf{p}}$.

Proof. 1) First if $x \in X_{\mathbf{p}}$ and $\mathrm{rk}^{2,<\infty}_{\mathbf{p}}(x) = \alpha$ then conjugation by g_x in $G^2_{\mathbf{p}}$ maps $X^{<\alpha}_{\mathbf{p}} = \{g_y : y \in X_{\mathbf{p}}$ and $\mathrm{rk}^{2,<\infty}_{\mathbf{p}}(y) < \alpha\}$ onto itself.

[Why? It is enough to prove for every $y \in X^{<\alpha}_{\mathbf{p}}$ that: if $y \in X_{\mathbf{p}}$, $\mathrm{rk}^{2,<\infty}_{\mathbf{p}}(y) < \alpha$ then $g_x g_y g_x^{-1} \in X^{<\alpha}_{\mathbf{p}}$. Now for each such g_y, one of the following two cases occurs:

(*iii*)

- (A) g_x, g_y commutes so $g_x g_y g_x^{-1} = g_y \in X^{2,<\alpha}_{\mathbf{p}}$
- (*ii*) (*i*) fails.

In case (i) the desired statement trivially holds, so assume that (*ii*) holds.

As $z \in \{x \restriction n : n \le n(x)\} \Rightarrow \mathrm{rk}^{2,<\infty}_{\mathbf{p}}(z) \ge \mathrm{rk}^{2,<\infty}_{\mathbf{p}}(x) \ge \alpha \Rightarrow g_z \notin X^{<\alpha}_{\mathbf{p}}$ and g_x, g_y does not commute, by 5.12(3) we get that $x = y \restriction n(x), n(x) < n(y)$. (As $\mathrm{rk}^{2,<\infty}_{\mathbf{p}}(x) \ge \alpha > 0$ by Definition **??**(2)(c)(γ) necessarily η^x is constantly 1, but not used.) Hence $g_x g_y g_x^{-1} = g_{y'}$ where $\bar{t}^{y'} = \bar{t}^y$ (and $\eta^{y'}(\ell) = \eta^{y'}(\ell)) \equiv (\ell = n(x))$, hence $g_{y'} \in X^{<\alpha}_{\mathbf{p}}$ as required.]

So really g_x normalize $G^{<\alpha}_{\mathbf{p}}$.

As this holds for every member of $\{g_x : \mathrm{rk}^{2,<\infty}_{\mathbf{p}}(x) = \alpha\}$, clearly $\mathrm{nor}_{G_{\mathbf{p}}}(G^{<\alpha}_{\mathbf{p}}) \supseteq (G^{<\alpha}_{\mathbf{p}}) \cup \{g_x : \mathrm{rk}^{2,<\infty}_{\mathbf{p}}(x) = \alpha$ and $x \in X_{\mathbf{p}}\}$ but the latter generates $G^{<\alpha+1}_{\mathbf{p}}$ hence $\mathrm{nor}_{G_{\mathbf{p}}}(G^{<\alpha}_{\mathbf{p}}) \supseteq G^{<\alpha+1}_{\mathbf{p}}$.

Second assume $g \in G_{\mathbf{p}} \setminus G_{\mathbf{p}}^{<\alpha+1}$, let $<^*$ be a linear ordering of $X_{\mathbf{p}}$ as in $\boxdot$ of 5.12; so by 5.12 we can find $k < \omega$ and $x_1 <^* \ldots <^* x_k$ from $X_{\mathbf{p}}$ such that $g = g_{x_1} g_{x_2} \ldots g_{x_k}$. As $g \notin G_{\mathbf{p}}^{<\alpha+1}$ necessarily not all the g_{x_m} are from $X_{\mathbf{p}}^{<\alpha+1}$ hence for some $m, g_{x_m} \notin G_{\mathbf{p}}^{<\alpha+1}$; and by the definition of $G_{\mathbf{p}}^{<\alpha+1}$, $\mathrm{rk}_{\mathbf{p}}^{2,<\infty}(x_m) \geq \alpha + 1$ hence η^{x_m} is constantly 1 and without loss of generality m is the minimal such m.

Let $m(*) \in [m, k]$ be such that

(A) $x_{m(*)} = x_m \restriction n(x_{m(*)})$

(ii) under (i), $n(x_{m(*)})$ is minimal;

there is such $m(*)$ as m satisfied the condition in clause (i). Of course, $\mathrm{rk}_{\mathbf{p}}^{2,<\infty}(x_{m(*)}) \geq \mathrm{rk}_{\mathbf{p}}^{2,<\infty}(x_m) \geq \alpha + 1$. Hence we can find t^* such that (recalling I is non-trivial, see Definition 5.2(3)):

(A) $t^* <_I t(x_{m(*)})$

(B) $\mathrm{rk}_I(t^*) = \alpha$

(C) $t^* \notin \{t_\ell(x) : x \in \{x_1, \ldots, x_k\}$ and $\ell \in \{0, \ldots, n(x)\}\}$.

We can let $n = n(x_{m(*)})$ and choose

$$y_1 = (\langle t_0(x_{m(*)}), \ldots, t_n(x_{m(*)}), t^* \rangle, \eta^{x_{m(*)}} {}^\frown \langle 0 \rangle)$$

$$y_2 = (\langle t_0(x_{m(*)}), \ldots, t_n(x_{m(*)}), t^* \rangle, \eta^{x_{m(*)}} {}^\frown \langle 1 \rangle).$$

So $y_1, y_2 \in X_{\mathbf{p}}$, $\mathrm{rk}_{\mathbf{p}}^2(y_1) = 0$, $\mathrm{rk}_{\mathbf{p}}^{2,<\infty}(y_2) = \alpha$ but $0 < \alpha$ by the assumption of part (1) hence $g_{y_1} \in G_{\mathbf{p}}^{<1} \subseteq G_{\mathbf{p}}^{<\alpha}$ and by 5.12 $g_{y_2} \in G_{\mathbf{p}}^{<\alpha+1} \wedge g_{y_2} \notin G_{\mathbf{p}}^{<\alpha}$. Now

(A) conjugating by $g_{x_{m(*)}}$ maps g_{y_1} to g_{y_2}.
Moreover,

(B) y_1, y_2 commutes with $g_{x_m}, \ldots, g_{x_k}$ except $g_{x_{m(*)}}$.
[Why? Assume toward contradiction that this fails for $\ell \in \{m, \ldots, k\} \setminus \{m(*)\}$ and $y_i, i \in \{1, 2\}$; clearly by 5.12(3) we get $y_i = x_\ell \restriction n(y_i) \neq x_\ell$ or $x_\ell = y_i \restriction n(x_\ell) \neq y_i$. By the choice of t^* (i.e., see clause (c) above) the first case does not occur hence the second one occurs. As $\ell \in [m, k]$ by the choice of $m(*)$ the second case implies that $n(x_\ell) \leq n(y_i) - 1 = n(x_{m(*)})$ and it also implies $x_\ell = y_i \restriction n(x_\ell) = x_{m(*)} \restriction n(x_\ell) = x_m \restriction n(x_\ell)$. As $\ell \in [m, k]$ by the choice of $m(*)$ we necessarily have $n(x_\ell) = n(x_{m(*)})$ hence by the previous equality $x_\ell = x_{m(*)}$, but $\ell \neq m(*) \Rightarrow (x_\ell <^* x_{m(*)}) \vee (x_{m(*)} <^* x_\ell) \Rightarrow x_{m(*)} \neq x_\ell$ hence $\ell = m(*)$, contradiction.]

By clauses (d) + (e) we have $g g_{y_1} g^{-1} = g_1 \ldots g_{m-1}((g_m \ldots g_k) g_{y_1} (g_k^{-1} \ldots g_m^{-1})) g_{m-1}^{-1} \cdots g_1^{-1} = (g_1 \ldots g_{m-1}) g_{y_2} (g_{m-1} \ldots g_1)^{-1}$. But $g_1, \ldots, g_{m-1} \in G_{\mathbf{p}}^{<\alpha+1}$ by the choice of m and $G_{\mathbf{p}}^{<\alpha}$ is a normal subgroup of $G_{\mathbf{p}}^{<\alpha+1}$ (as we have proved that $G_{\mathbf{p}}^{<\alpha+1} \subseteq \mathrm{nor}_{G_{\mathbf{p}}}(G_{\mathbf{p}}^{<\alpha})$). So conjugation by $(g_1 \ldots g_{m-1})$ maps $G_{\mathbf{p}}^{<\alpha}$ onto $G_{\mathbf{p}}^{<\alpha}$ and so necessarily it maps $G_{\mathbf{p}}^{<\alpha+1} \setminus G_{\mathbf{p}}^{<\alpha}$ onto $G_{\mathbf{p}}^{<\alpha+1} \setminus G_{\mathbf{p}}^{<\alpha}$ but $g_{y_2} \in G_{\mathbf{p}}^{<\alpha+1} \setminus G_{\mathbf{p}}^{<\alpha}$. Hence together $g g_{y_1} g^{-1} = (g_1 \ldots g_{m-1}) g_{y_2} (g_{m-1} \ldots g_1)^{-1} \in G_{\mathbf{p}}^{<\alpha+1} \setminus G_I^{<\alpha}$. But as said above $g_{y_1} \in G_{\mathbf{p}}^{<\alpha}$, so $g \notin \mathrm{nor}_{G_{\mathbf{p}}}(G_{\mathbf{p}}^{<\alpha})$.

As g was any member of $G_{\mathbf{p}} \setminus G_{\mathbf{p}}^{1,<\alpha+1}$ we deduce that $\mathrm{nor}_{G_{\mathbf{p}}}(G_{\mathbf{p}}^{1,<\alpha}) \subseteq G_{\mathbf{p}}^{<\alpha+1}$. As we have shown the other inclusion earlier we are done.

2) Similar (and is not really needed). $\square$

Definition 5.17. 1) Let $H'_{\mathbf{p}}$ be the abelian group generated freely by $\{g_y : y \in Z^{\mathbf{p}} * 2\}$ freely except that each generator has order two.

1) The mapping $\varkappa^{\mathbf{p}}$ from $\{g_x : x \in X_{\mathbf{p}}\}$ into the group $\mathrm{per}(Z^{\mathbf{p}} * 2)$ of permutations of $Z^{\mathbf{p}} * 2$ is defined by: $\varkappa^{\mathbf{p}}(g_x)((\alpha, \upsilon')) = (\alpha, \varkappa_{\mathbf{m}(\alpha), \upsilon^{\mathbf{p}}_{x,\alpha}}(\upsilon')$.

2) We can above replace $Z^{\mathbf{p}} * 2$ by $H'_{\mathbf{p}}$ and we call it $\varkappa^{\mathbf{p}}_*$, so $\varkappa^{\mathbf{p}}_*(g_y) = g_{\varkappa^{\mathbf{p}}(y)}$.

3) We call $\hat{\varkappa}^{\mathbf{p}}$ the extension of the the mapping $\varkappa^{\mathbf{p}}_*$ to a homomorphism from the group $G^2_{\mathbf{p}}$ into the group of automorphism of $H'_{\mathbf{p}}$.
4) Let $\pi^1_{\mathbf{p}}$ is the homomorphism from $G^1_{\mathbf{p}}$ into the twisted product $H'_{\mathbf{p}} * G^2_{\mathbf{p}}$ defined by:

(A) for $x \in X_{\mathbf{p}}$, we let $\pi^1_{\mathbf{p}}(g_x) = g_x$, i.e., (e, g_x)

(ii) for $y \in Z^{\mathbf{p}} * 2$ we let $\pi^1_{\mathbf{p}}(g_y) = g_y$, i.e, (g_y, e).

Claim 5.18. *1) The mapping in Definition 5.17 is well defined, i.e,* $\varkappa^{\mathbf{p}}(g_x)$ *is really a permutation of* $Z^{\mathbf{p}} * 2$.
2) $\varkappa^{\mathbf{p}}_*$ *is well defined and the images are automorphisms of* $H'_{\mathbf{p}}$.
3) Moreover, this mapping respect the equations from $\Gamma^1_{\mathbf{p}}$ *hence* $\hat{\varkappa}^{\mathbf{p}}$ *is a homomorphism from* $G^2_{\mathbf{p}}$ *into the group of automorphism of* $H'_{\mathbf{p}}$.
4) In Definition 5.17(4), the mapping π^1 *is a well defined homomorphism from* $G^1_{\mathbf{p}}$ *into the twisted product.*

Proof. Check. $\square$

Claim 5.19. *1) The normalizer of* $H^1_{\mathbf{p}}$ *in* $G^1_{\mathbf{p}}$ *is* $G^{1,<1}_{\mathbf{p}}$.
2) If $1 \leq \alpha \leq \mathrm{rk}^{<\infty}(\mathbf{p})$ *<u>then</u> the* α*-th normalizer of* $H^1_{\mathbf{p}}$ *in* $G^1_{\mathbf{p}}$ *is* $G^{1,<\alpha}_{\mathbf{p}}$.
3) $\tau^{\mathrm{nlg}}_{G_{\mathbf{p}},H_{\mathbf{p}}} = \mathrm{rk}^{<\infty}_{\mathbf{p}}$.
4) First, $G^{1,<0}_{\mathbf{p}}$ *is abelian (as it is generated by* $\langle g_y : y \in Z^{\mathbf{p}} * 2\rangle$ *which pairwise commutes); as* $H^1_{\mathbf{p}} \subseteq G^{1,<0}_{\mathbf{p}}$ *it follows that* $G^{1,<0}_{\mathbf{p}} \subseteq \mathrm{nor}_{G^1_{\mathbf{p}}}(H^1_{\mathbf{p}}))$.

Second, if $x \in X_{\mathbf{p}}$, $rk^2_{\mathbf{p}}(x) = 0$ *then* $\alpha \in Z^{\mathbf{p}} \Rightarrow u^{\mathbf{p}}_{\kappa,\alpha} \in \Gamma^-$ *(as* $\mathbf{p}$ *is a nice* κ*-parameter, see clause (b) of Definition* **??***(3) + ADD). Now for any* $g_{(\alpha,\upsilon)} \in H^1_{\mathbf{p}}$ *(i.e.,* $(\alpha, \upsilon) \in (Z^{\mathbf{p}} * 2)$ *conjugation by* g_x *inside* $G_{\mathbf{p}}$ *maps* $g_{(\alpha,\upsilon)}$ *to* $g_{(\alpha,\upsilon')}$ *with* $u' \in \Lambda'$ *iff* $u' \in \Lambda^-$ *such that*

(A) if $\mathbf{m}(\alpha) > n(x)$ *then* $\upsilon' \in \Lambda^-_{n(x)}$

(B) if $\mathbf{m}(\alpha) \leq n(x)$ *then* $\upsilon' = u$ *hence* $\in \Lambda^-_{n(x)}$ *so in both cases to a member of* $H_{\mathbf{p}}$. *Together* $G^{<1}_{\mathbf{p}} \subseteq \mathrm{nor}_{G_{\mathbf{p}}}(H_{\mathbf{p}})$.

Third, if $g \in G_{\mathbf{p}} \setminus G^{<1}_{\mathbf{p}}$ *then let* $<^*$ *be as in 5.12(1) and* $g = g_{x_1} \cdots g_{x_k}$ *for some* $x_1 <^* \ldots <^* x_k$ *from* $X^+_{\mathbf{p}}$ *and necessarily for some* $m \in \{1, \ldots, k\}$ *we have* $rk^{2,<\infty}_{\mathbf{p}}(x_m) \geq 1$. *As* $\mathbf{p}$ *is a nice* κ*-parameter (see Definition* **??***(3), clause (c)) there in* $\alpha \in A_{x_m} \setminus \cup \bigcup \{A_{x_\ell} : \ell \in \{1, \ldots, k\} \setminus \{m\}$ *and* $x_\ell \in X_{\mathbf{p}}\}$ *such that* $\mathbf{m}(\alpha) > n(x_m)$. *So* g_{x_ℓ} *commute with* $g_{(\alpha,0)}$ *and* $g_{(\alpha,1)}$ *if* $\ell \in \{1, \ldots, k\} \setminus \{m\}$ *and* $G_{\mathbf{p}} \models g_{x_m}(g_{(\alpha,0)})g^{-1}_{x_m} = g_{(\alpha,1)}$.

So if $\ell \in \{1, \ldots, k\} \cup \{m\}$, *conjugation by* g_{x_ℓ} *maps the sets* $\{g_{(\alpha,\upsilon)} : \upsilon \in \Lambda^-_{\mathbf{m}(\alpha)}\}$ *and* $\{g_{(\alpha,\upsilon)} : \upsilon \in \Lambda^*_m \setminus \Lambda^-_m\}$ *onto themselves. By conjugation* g_{x_m} *maps their union onto itself by mix then. As* $\Lambda^-_m \subseteq \Lambda_n, \{g_{(\alpha,\upsilon)} : \upsilon \in \Lambda^-_m\} = H \cap \{g_{(\alpha,\upsilon)} : \upsilon \in \Lambda^*_n\}$ *clearly for some* $\upsilon_1 \in \Lambda^-_m, u_2 \in \Lambda^*_m \setminus \Lambda^-_m$ *we have* $G^1_{\mathbf{p}} \models$ *"*$(g_1, \ldots, g_k)^{-1} g_{(\alpha,u_1)}(g_i, \cdots, g_k) = g_{(\alpha,u_2)}$*" but* $g_{(\alpha,\upsilon_1)} \in H_{\mathbf{p}}, g_{(\alpha,\upsilon_2)} \notin H_{\mathbf{p}}$ *so* $g \notin \mathrm{nor}_{G_{\mathbf{p}}}(H_{\mathbf{p}})$.

As this holds for every $g \in G_I \setminus G^{<1}_{\mathbf{p}}$, *clearly* $\mathrm{nor}_{G_{\mathbf{p}}}(H) \subseteq G^{<1}_{\mathbf{p}}$. *As we have proved above the other inclusion, together we get equality.*
5) It follows by 5.16(1) + part (2), as $\langle G^{<\alpha}_{\mathbf{p}} : \alpha \leq \infty\rangle$ *is an increasing continuous sequence.*
6) Follows by part (2) and the definitions (0.4(2)) and the non-triviality of $I^{\mathbf{p}}$ *implies the rank is* ≥ 1.

§ 5(B). **Private Appendix**
§2 **Easier group.**

Definition 5.20. For a κ-parameter $\mathbf{p}$.
1) Let $F_{\mathbf{p}} = F[\mathbf{p}]$ be the group generated by $\{g_x : x \in X^+_{\mathbf{p}}\}$ freely except the equations in $\Gamma^*_{\mathbf{p}}$ which are

- (A) $g_x = g_x^{-1}$ for $x \in Z^{\mathbf{p}} * 2$
- (B) $g_x g_y = g_y g_x$ for $x, y \in Z^{\mathbf{p}} * 2$
- (C) $g_x g_{y_1} g_x^{-1} = g_{y_2}$ when $\circledast^1_{x,y_1,y_2}$, see Definition 5.9(1c).

2) We define $F^{<0}_{\mathbf{p}} =: \langle \{g_x : x \in Z^{\mathbf{p}} * 2\}\rangle_{F[\mathbf{p}]}$ (identify it with $G^{1,<0}_{\mathbf{p}}$) and $H^{\mathbf{p}} = \langle \{g_x : x \in Z^{\mathbf{p}} * 2\}\rangle_{F[\mathbf{p}]}$ (and identify it with $H^1_{\mathbf{p}} \Rightarrow H^2_{\mathbf{p}}$), justification by 5.21(1) below).
3) Let $\pi^2_{\mathbf{p}}$ be the unique homomorphism from $F^{\mathbf{p}}$ onto $G^1_{\mathbf{p}}$ satisfying

$$\pi_{\mathbf{p}}(g_x) = g_x \text{ for } x \in X^+_{\mathbf{p}}.$$

3A) Let $\pi_{\mathbf{p}}$ be $\pi^2_{\mathbf{p}} \circ \pi^1_{\mathbf{p}} \in \mathrm{Hom}(F_{\mathbf{p}}, F^2_{\mathbf{p}})$.

4) Let $\mathbf{F}^1_{\mathbf{p}} = F_1[\mathbf{p}]$ be the subgroup of $F_{\mathbf{p}}$ generated by $\{g_x : x \in X_{\mathbf{p}}\}$.
5) Let $\mathbf{F}^{<\alpha}_{\mathbf{p}}$ be $\{g \in F_{\mathbf{p}} : \pi^{\mathbf{p}}(g) \in G^{<\alpha}_{\mathbf{p}}\}$.
6) For $X \subseteq X_{\mathbf{p}}$ and $Z \subseteq Z^{\mathbf{p}}$ let $F_{\mathbf{p},X,Z}$ be the group generated by $\{g_x : x \in X \cup (Z * 2)\}$ freely except $\Gamma^*_{\mathbf{p},X,Z} =$ the equations of Γ^*_p mentioning only the generators we have listed.

Claim 5.21. *0) The identification of $H^1_{\mathbf{p}}, H^2_{\mathbf{p}}$ and $H'_{\mathbf{p}}$ (from 5.17, 5.18) is justified.*
1) $\pi^2_{\mathbf{p}}$ is really a homomorphism from $F_{\mathbf{p}}$ onto $G^1_{\mathbf{p}}$ which is the identity on $H_{\mathbf{p}}$.
2) The subgroup of $F_{\mathbf{p}}$ generated by $\{g_y : y \in Z^{\mathbf{p}} * 2\}$ satisfies:

- *(A) it is abelian*
- *(B) every element has order 2*
- *(C) it can be considered as a vector space over $\mathbb{Z}/2\mathbb{Z}$ with basis $\{g_y : y \in Z^{\mathbf{p}} \times 2\}$.*

3) $F^1_{\mathbf{p}}$ is a gree group generated freely by $\{g_x : x \in X_{\mathbf{p}}\}$.
4) $F^{<0}_{\mathbf{p}}$ is a normal subgroup of $\mathbf{F}_{\mathbf{p}}$ and for $x \in X_{\mathbf{p}}$, conjugation by g_x in $F^{\mathbf{p}}$ acts on $H_{\mathbf{p}}$ as the following permutation $\mathrm{per}(g_x)$ of $\{g_y : y \in Z^{\mathbf{p}} * 2\}$ (its basis as a vector space): $g_x g_{(\alpha,\upsilon)} g_x^{-1}$ is $(\alpha, \varkappa^{\mathbf{p}}_x(\upsilon))$. The permutations $\langle \mathrm{per}(g_x) : x \in X_{\mathbf{p}}\rangle$ pairwise commute.
5) $F_{\mathbf{p}}$ is the twisted product of $H_{\mathbf{p}}$ and $F^1_{\mathbf{p}}$.
6) For $\alpha \in Z^{\mathbf{p}}, H^{\mathbf{p}} \cap \{g_y : y \in \{\alpha\} * \Lambda^*_{\mathbf{m}(\alpha)}\}$ is equal to $\{g_y : y \in \{\alpha\} * \Lambda^*_{\mathbf{m}(\alpha)}\}$.
7) If $X \subseteq X_{\mathbf{p}}$ and $Z \subseteq Z^{\mathbf{p}}$ then $F^*_{\mathbf{p},X,Z}$ is essentially $\langle \{g_x : x \in X \cup (Z * 2)\}\rangle_{F_{\mathbf{p}}}$.

Proof. Straight. □

Claim 5.22. *1) $F^{<1}_{\mathbf{p}} = \mathrm{nor}_{\mathbf{F}[\mathbf{p}]}(H_{\mathbf{p}})$ and $\pi_{\mathbf{p}}$ maps $\mathrm{nor}_{F_{\mathbf{p}}}(H_{\mathbf{p}})$ onto $G^{2,<1}_{\mathbf{p}}$ and $\mathrm{Ker}(\pi) \subseteq \mathrm{nor}_{F[\mathbf{p}]}(H_{\mathbf{p}})$.*
2) $\pi_{\mathbf{p}}$ maps $\mathrm{nor}^{1+\alpha}_{F_{\mathbf{p}}}(H_{\mathbf{p}})$ onto $\mathrm{nor}^{\alpha}_{G_{\mathbf{p}}}(G^{2,<1}_{\mathbf{p}})$ for $\alpha < \infty$ so $F^{<\alpha}_{\mathbf{p}} = \mathrm{nor}^{\alpha}_{F_{\mathbf{p}}}(H_{\mathbf{p}})$.
3) $\tau^{\mathrm{nlg}}_{F_{\mathbf{p}},H_{\mathbf{p}}}$ is equal to $\tau^{\mathrm{nlg}}_{G_{\mathbf{p}},H_{\mathbf{p}}}$ and $\mathrm{nor}^{\infty}_{F_{\mathbf{p}}}(H_{\mathbf{p}}) = F^{\mathbf{p}}$ iff $\mathrm{nor}^{\infty}_{G_{\mathbf{p}}}(H_{\mathbf{p}}) = G_{\mathbf{p}}$.

Proof. 1) For every $x \in X^+_p$ and $\alpha < \kappa$ clearly in $F_{\mathbf{p}}$ conjugating by g_x maps $\{\alpha\} \times \Lambda^*_{\mathbf{m}(\alpha)}$ onto itself.
(Why? If $x \in Z^{\mathbf{p}} * 2, g_x$ commutes with them and if $x \in X_{\mathbf{p}}$ also.)

Hence this holds for every $g \in F_{\mathbf{p}}$ hence it follows that $gg_{(\alpha,v)}g^{-1} = g^{-1}g_{(\alpha,v)}g$ and so by the choice of $H_{\mathbf{p}}$:

$(*)_1$ for $g \in F_{\mathbf{p}}$ we have $g \in \mathrm{nor}_{F_{\mathbf{p}}}(H_{\mathbf{p}})$ iff for every $\alpha \in Z^{\mathbf{p}}$, we have conjugation by g maps $\{g_y : y \in \{\alpha\} * \Lambda^*_{\mathbf{m}(\alpha)}\}$ onto itself.

Similarly

$(*)_2$ for $g \in G_{\mathbf{p}}$ we have $g \in \mathrm{nor}_{G_{\mathbf{p}}}(H_{\mathbf{p}})$ iff for every $\alpha \in Z^{\mathbf{p}}$ we have conjugation by g maps $\{g_y : y \in \{\alpha\} * \Lambda^*_{\mathbf{m}(\alpha)}\}$ onto itself.

As $\pi_{\mathbf{p}}$ maps $F_{\mathbf{p}}$ onto $G_{\mathbf{p}}$ and is the identity on $H_{\mathbf{p}}$ (which includes $\{g_{(\alpha,v)} : \alpha \in Z^{\mathbf{p}}\}$, clearly $\pi_{\mathbf{p}}$ maps $\mathrm{nor}_{F_{\mathbf{p}}}(H_{\mathbf{p}})$ onto $\mathrm{nor}_{G_{\mathbf{p}}}(H_{\mathbf{p}})$ and $\mathrm{Ker}(\pi_{\mathbf{p}}) \subseteq \mathrm{nor}_{F[\mathbf{p}]}(H_{\mathbf{p}})$.
2) So by 5.16(1) we have $\mathrm{nor}_{G_{\mathbf{p}}}(H_{\mathbf{p}}) = G^{<1}_{\mathbf{p}}$ but (by Definition 5.20), $F^{<1} = \{g \in F_{\mathbf{p}}, \pi_{\mathbf{p}}(g) \in G^{<1}_{\mathbf{p}}\}$ so together we get $\mathrm{nor}_{F_{\mathbf{p}}}(H_{\mathbf{p}}) = F^{<1}_{\mathbf{p}}$.
3) We prove this by induction on α.
For $\alpha = 0$ we have $\mathrm{nor}^0_{F_{\mathbf{p}}}(H_{\mathbf{p}}) = H_{\mathbf{p}}$, $\mathrm{nor}^0_{G_{\mathbf{p}}}(H_{\mathbf{p}})$ and $\pi_{\mathbf{p}}$ is the identity on $H_{\mathbf{p}}$.

For $\alpha = 1$ use part (1).

For α limit this is trivial.

For $\alpha = \beta + 1$ note that $\mathrm{Ker}(\pi_{\mathbf{p}})$ is $\subseteq \mathrm{nor}^1_{F_{\mathbf{p}}}(H_{\mathbf{p}}) \subseteq \mathrm{nor}^\beta_{F_{\mathbf{p}}}(H_{\mathbf{p}}) = F^{<\beta}_{\mathbf{p}}$ and $\pi_{\mathbf{p}}$ maps $F^{<\beta}_{\mathbf{p}}$ onto $G^{2,<\beta}_{\mathbf{p}}$ hence it follows that $\mathrm{nor}_{G_{\mathbf{p}}}(\pi_{\mathbf{p}}(F^{<\beta}_{\mathbf{p}})) = \pi_{\mathbf{p}}(\mathrm{nor}_{F_{\mathbf{p}}}(G^{2,<\beta}_\beta))$.

Hence

$$\begin{aligned}
\mathrm{nor}^\alpha_{G_{\mathbf{p}}}(H_{\mathbf{p}}) &= \mathrm{nor}_{G_{\mathbf{p}}}(\mathrm{nor}^\beta_{G^2_{\mathbf{p}}}(H_{\mathbf{p}}) \\
&= \mathrm{nor}_{G_{\mathbf{p}}}(G^{2,<\beta}_{\mathbf{p}}) = \mathrm{nor}_{G_{\mathbf{p}}}(\pi_{\mathbf{p}}(F^{<\beta}_{\mathbf{p}})) \\
&= \pi_{\mathbf{p}}(\mathrm{nor}_{F_{\mathbf{p}}}(F^{<\beta}_{\mathbf{p}})) = \pi_{\mathbf{p}}(\mathrm{nor}_{F_{\mathbf{p}}}(\mathrm{nor}^\beta_{F_{\mathbf{p}}}(H_{\mathbf{p}})) \\
&= \pi_{\mathbf{p}}(\mathrm{nor}^\alpha_{F_{\mathbf{p}}}(H_{\mathbf{p}})).
\end{aligned}$$

So we are done. $\square$

We can below use simplified κ-parameters, does not matter.

Definition 5.23. 1) $\mathfrak{s}$ is a κ-p.o.w.i.s. (partial order weak inverse system) when:

(A) $\mathfrak{s} = (J, \bar{\mathbf{p}}, \bar{\pi})$ so $J = J^{\mathfrak{s}} = J[\mathfrak{s}], \bar{p} = \bar{p}^{\mathfrak{s}}, \bar{\pi} = \bar{\pi}^{\mathfrak{s}}$

(B) J is a directed partial order of cardinality $\leq \kappa$

(C) $\bar{\mathbf{p}} = \langle \mathbf{p}_u : u \in J\rangle$

(D) $\mathbf{p}_u$ is a κ-parameter, $I_u = I^{\mathbf{p}}_u$ is a partial order of cardinality $\leq \kappa$ and let $I^{\mathfrak{s}}_u = I^{\mathbf{p}^{\mathfrak{s}}_u}$, $X^{\mathfrak{s}}_u = X_{\mathbf{p}^{\mathfrak{s}}_u}$, $Z^{\mathfrak{s}}_u = Z^{\mathbf{p}^{\mathfrak{s}}_u}$, $A^{\mathfrak{s}}_{u,x} = A^{p^{\mathfrak{s}}_u}_x$ when the latter is defined

(E) $\bar{\pi} = \langle \pi_{u,v} : u \leq_J v\rangle$

(F) $\pi_{u,v}$ is a partial mapping from I_v into I_u

(G) if $u \leq_J v \leq_J w$ then $\pi_{u,w} = \pi_{u,v} \circ \pi_{v,w}$ (may use $\subseteq$)

(H) $u \leq_J v \Rightarrow Z^{\mathbf{p}_u} \subseteq Z^{\mathbf{p}_v}$ and use $\mathrm{id}_{Z^{\mathbf{p}_u}} \cup \pi_{u,v})$ hence $\varrho^{\mathbf{p}}_x = \varrho^{\mathbf{p}}_{\pi_{u,v}}(x), v^{\mathbf{p}}_{\pi(u,v(x))}$

(I) if $x \in \mathrm{Dom}(\pi_{u,v})$ then $A^{\mathfrak{s}}_{v,x} \cap Z^{\mathfrak{s}}_u = A^{\mathfrak{s}}_{u,\pi_{u,v}(x)}$.

2) We define $\pi^+_{u,v} = \pi^{+,\mathfrak{s}}_{u,v}$ when $u \leq_{J[\mathfrak{s}]} v$ as follows:

(A) $\pi^+_{u,v}$ is a partial mapping from $X^+_{\mathbf{p}_v}$ into $X^+_{\mathbf{p}_u}$

(B) for $x \in X_{\mathbf{p}_v}$,

(α) $x \in \mathrm{Dom}(\pi^+_{u,v})$ iff: for every w satisfying $u \le_{J[\mathfrak{s}]} w \le_{J[\mathfrak{s}]} v$ and $\ell < n(x)$ we have
$[\pi_{w,v}(t_{\ell+1}(x)) <_{I_w} \pi_{w,v}(t_\ell(x))]$
(β) $\pi^+_{u,v}(x) = (\langle \pi_{u,v}(t_0(x), \ldots, \pi_{u,v}(t_{n(x)}(x)) \rangle, \eta^x)$

(C) for $y \in X^+_{\mathbf{p}_v} \setminus X_{\mathbf{p}_v} = Z^{\mathbf{p}_v} * 2$ we have:
(α) $y \in \mathrm{Dom}(\pi^+_{u,v})$ iff $y \in Z^{\mathbf{p}_u} * 2$
(β) $\pi^+_{u,v}(y) = y$ for $y \in Z^{\mathbf{p}_u} * 2$.

3) If $u \le_{J[\mathfrak{s}]} v$, then $\breve{\pi}_{u,v} = \breve{\pi}^{\mathfrak{s}}_{u,v}$ is the partial homomorphism from $F_{\mathbf{p}_2}$ into $F_{\mathbf{p}_1}$ with domain the subgroup of $F^+_{\mathbf{p}_2}$ generated by $\{g_x : x \in \mathrm{Dom}(\pi^+_{u,v})\}$ mapping g_x to $g_{\pi^+_{u,v}(x)} \in F_{\mathbf{p}_1}$; see justification below.
4)[?] We say $\mathfrak{s}$ is linear if $J^{\mathfrak{s}}$ is a linear (= total) order. [USED?]
5) We say $\mathfrak{s}$ is nice when every $p^{\mathfrak{s}}_u$ is nice.[?]

Claim 5.24. *If $\mathfrak{s}$ is a κ-p.o.w.i.s and $J^{\mathfrak{s}} \models$ "$v \le u \le w$" then*
(A) $\breve{\pi}^{\mathfrak{s}}_{u,v}$ are well defined (homomorphisms)
(B) $\pi^+_{w,v} \subseteq \pi^+_{w,u} \circ \pi^+_{u,v}$ and $\breve{\pi}_{w,v} \subseteq \breve{\pi}_{w,u} \circ \breve{\pi}_{u,v}$
(C) if $J^{\mathfrak{s}}$ is a linear order then in clause (b) we get equalities.

Proof. Clause (a): It is enough to prove that (when $u \le_{J[\mathfrak{s}]} v$): $\pi^{+,\mathfrak{s}}_{u,s}$ maps the set of equations $\Gamma_{\mathbf{p},\mathrm{Dom}(\pi^+_{u,v}),Z^{\mathfrak{s}}_u}$ onto the set of equations $\Gamma_{\mathbf{p},\mathrm{Rang}(\pi^+_{u,v}),Z^{\mathfrak{s}}_u}$.

Looking as the definitions this is obvious.
Clause (b): Easy.
Clause (c): Easy, in fact we have chosen Definition 6.10(2)(b) such that those equalities will hold. $\square$

§ 5(C). old §4.

Definition 5.25. 1) We say that $\mathbf{k}$ is a simplified κ-parameter when
(A) $\mathbf{k} = (S, \bar{A}, Z) = (S^{\mathbf{k}}, \bar{\upsilon}^{\mathbf{k}}, Z^{\mathbf{k}})$
(B) S a set
(C) $Z \subseteq \kappa$
(D) $\bar{\kappa} = \langle \varkappa_{x,\alpha} : x \in S, \alpha \in Z \rangle$ and $\varkappa_{x,\alpha} \in \mathrm{per}(\mathcal{P}(\Lambda^*_{\mathbf{m}(\alpha)}))$.

2) $S^+_{\mathbf{k}} = S^{\mathbf{k}} \cup (Z^{\mathbf{k}} * 2)$ and we always assume that this is a disjoint union.
3) For a simplified κ-parameter let $F_{\mathbf{k}}$ be the group generated by $\{g_x : x \in S^+_{\mathbf{k}}\}$ freely except the equations in $\Gamma_{\mathbf{k}}$ which are
(A) $g_x = g_x^{-1}$ for $x \in Z^{\mathbf{k}} * 2$
(B) $g_x g_y = g_y g_x$ for $x, y \in Z^{\mathbf{k}} * 2$
(C) $g_x g_{y_1} g_x^{-1} = g_{y_2}$ when for some $\alpha \in Z^{\mathbf{k}}$ we have $x \in S^{\mathbf{k}}, \{y_1, y_2\} \subseteq \{\alpha\} * 2$ and $\varkappa_{x,u}(\upsilon^{y_1}) = \upsilon^{y_2}$.

4) Let $H_{\mathbf{k}}$ be the subgroup of $F_{\mathbf{k}}$ generated by $\{g_x : x \in S_{\mathbf{k}}$ or for some $\alpha \in Z^{\mathbf{k}}$ we have $x \in \{\alpha\} * \Lambda^-_m$.
5) For a κ-parameter $\mathbf{p}$ let $\mathbf{k}(\mathbf{p})$ be $(X_{I[\mathbf{p}]}, \bar{\varkappa}^{\mathbf{p}}, Z^{\mathbf{p}})$ where $\langle \varkappa^{\mathbf{p}}_{x,\alpha} : x \in X_{\mathbf{p}}, \alpha \in Z^{\mathbf{p}} \rangle$.
6) We say $\mathbf{k}$ is one to one if $\bar{A}^{\mathbf{k}}$ is with no repetitions.

Claim 5.26. *Assume $\mathbf{p}$ is a κ-parameter.*
1) $\mathbf{k}(\mathbf{p})$ is a simplified κ-parameter.
2) If $\mathbf{p}$ is nice then $\mathbf{k}(\mathbf{p})$ is one to one.
3) The mapping $g_x \mapsto g_x (x \in X^+_{\mathbf{p}})$ induces an isomorphism from $F_{\mathbf{p}}$ onto $F_{\mathbf{k}(\mathbf{p})}$.

Proof. Easy. □

Claim 5.27. *For* $\mathbf{k}$ *a simplified* κ*-parameter, the parallel to 5.21 holds.*

Proof. Easy. □

* * *

We can below use simplified κ-parameters, does not matter.

Definition 5.28. 1) $\mathfrak{s}$ is a κ-p.o.w.i.s. (partial order weak inverse system) <u>when</u>:

(A) $\mathfrak{s} = (J, \bar{\mathbf{p}}, \bar{\pi})$ so $J = J^{\mathfrak{s}} = J[\mathfrak{s}], \bar{p} = \bar{p}^{\mathfrak{s}}, \bar{\pi} = \bar{\pi}^{\mathfrak{s}}$

(B) J is a directed partial order of cardinality $\leq \kappa$

(C) $\bar{\mathbf{p}} = \langle \mathbf{p}_u : u \in J \rangle$

(D) $\mathbf{p}_u$ is a κ-parameter, $I_u = I_u^{\mathbf{p}}$ is a partial order of cardinality $\leq \kappa$ and let $I_u^{\mathfrak{s}} = I^{\mathbf{p}_u^{\mathfrak{s}}}$, $X_u^{\mathfrak{s}} = X_{\mathbf{p}_u^{\mathfrak{s}}}$, $Z_u^{\mathfrak{s}} = Z^{\mathbf{p}_u^{\mathfrak{s}}}$, $A_{u,x}^{\mathfrak{s}} = A_x^{p_u^{\mathfrak{s}}}$ when the latter is defined

(E) $\bar{\pi} = \langle \pi_{u,v} : u \leq_J v \rangle$

(F) $\pi_{u,v}$ is a partial mapping from I_v into I_u

(G) if $u \leq_J v \leq_J w$ then $\pi_{u,w} = \pi_{u,v} \circ \pi_{v,w}$ (may use $\subseteq$)

(H) $u \leq_J v \Rightarrow Z^{\mathbf{p}_u} \subseteq Z^{\mathbf{p}_v}$ and use $\mathrm{id}_{Z^{\mathbf{p}_u}} \cup \pi_{u,v})$ hence $\varrho_x^{\mathbf{p}} = \varrho_{\pi_{u,v}}^{\mathbf{p}}(x), \upsilon_{\pi(u,v(x))}^{\mathbf{p}}$

(I) if $x \in \mathrm{Dom}(\pi_{u,v})$ then $A_{v,x}^{\mathfrak{s}} \cap Z_u^{\mathfrak{s}} = A_{u,\pi_{u,v}(x)}^{\mathfrak{s}}$.

2) We define $\pi_{u,v}^+ = \pi_{u,v}^{+;\mathfrak{s}}$ when $u \leq_{J[\mathfrak{s}]} v$ as follows:

(A) $\pi_{u,v}^+$ is a partial mapping from $X_{\mathbf{p}_v}^+$ into $X_{\mathbf{p}_u}^+$

(B) for $x \in X_{\mathbf{p}_v}$,

(α) $x \in \mathrm{Dom}(\pi_{u,v}^+)$ <u>iff</u>: for every w satisfying $u \leq_{J[\mathfrak{s}]} w \leq_{J[\mathfrak{s}]} v$ and $\ell < n(x)$ we have $[\pi_{w,v}(t_{\ell+1}(x)) <_{I_w} \pi_{w,v}(t_\ell(x))]$

(β) $\pi_{u,v}^+(x) = (\langle \pi_{u,v}(t_0(x), \ldots, \pi_{u,v}(t_{n(x)}(x)) \rangle, \eta^x)$

(C) for $y \in X_{\mathbf{p}_v}^+ \setminus X_{\mathbf{p}_v} = Z^{\mathbf{p}_v} * 2$ we have:

(α) $y \in \mathrm{Dom}(\pi_{u,v}^+)$ <u>iff</u> $y \in Z^{\mathbf{p}_u} * 2$

(β) $\pi_{u,v}^+(y) = y$ for $y \in Z^{\mathbf{p}_u} * 2$.

3) If $u \leq_{J[\mathfrak{s}]} v$, <u>then</u> $\check{\pi}_{u,v} = \check{\pi}_{u,v}^{\mathfrak{s}}$ is the partial homomorphism from $F_{\mathbf{p}_2}$ into $F_{\mathbf{p}_1}$ with domain the subgroup of $F_{\mathbf{p}_2}^+$ generated by $\{g_x : x \in \mathrm{Dom}(\pi_{u,v}^+)\}$ mapping g_x to $g_{\pi_{u,v}^+(x)} \in F_{\mathbf{p}_1}$; see justification below.

4)[?] We say $\mathfrak{s}$ is linear <u>if</u> $J^{\mathfrak{s}}$ is a linear (= total) order. [USED?]

5) We say $\mathfrak{s}$ is nice when every $p_u^{\mathfrak{s}}$ is nice.[?]

Claim 5.29. *If* $\mathfrak{s}$ *is a* κ-p.o.w.i.s *and* $J^{\mathfrak{s}} \models$ "$v \leq u \leq w$" <u>*then*</u>

(A) $\check{\pi}_{u,v}^{\mathfrak{s}}$ *are well defined (homomorphisms)*

(B) $\pi_{w,v}^+ \subseteq \pi_{w,u}^+ \circ \pi_{u,v}^+$ *and* $\check{\pi}_{w,v} \subseteq \check{\pi}_{w,u} \circ \check{\pi}_{u,v}$

(C) if $J^{\mathfrak{s}}$ *is a linear order then in clause (b) we get equalities.*

Proof. Clause (a): It is enough to prove that (when $u \le_{J[\mathfrak{s}]} v$): $\pi^{+,\mathfrak{s}}_{u,s}$ maps the set of equations $\Gamma_{\mathbf{p},\mathrm{Dom}(\pi^+_{u,v}),Z^{\mathfrak{s}}_u}$ onto the set of equations $\Gamma_{\mathbf{p},\mathrm{Rang}(\pi^+_{u,v}),Z^{\mathfrak{s}}_u}$.

Looking as the definitions this is obvious.

Clause (b): Easy.

Clause (c): Easy, in fact we have chosen Definition 6.10(2)(b) such that those equalities will hold. $\square$

§ 5(D). **old? §3.**

Definition 5.30. We say that $\mathfrak{s}$ is the limit of $\mathfrak{t}$, both κ-p.o.w.i.s. as witnessed by v_* when

(A) $J^{\mathfrak{t}} \subseteq J^{\mathfrak{s}} = J^{\mathfrak{s}} = J^{\mathfrak{t}} \cup \{v_*\}, v_* \notin J^{\mathfrak{t}}$ and $u \in J^{\mathfrak{s}} \Rightarrow u \le_{J[\mathfrak{s}]} v_*$

(B) $\mathbf{p}^{\mathfrak{s}}_u = \mathbf{p}^{\mathfrak{t}}_u, \pi^{\mathfrak{s}}_{u,v} = \pi^{\mathfrak{t}}_{u,v}$ when $u \le_{J[\mathfrak{s}]} v <_{J[\mathfrak{s}]} v_*$

(C) $J^{\mathfrak{t}}$ is directed

(D) if $t \in I^{\mathfrak{s}}_{v_*}$ then for some $u = u_t \in J$ we have $t \in \mathrm{Dom}(\pi^{\mathfrak{s}}_{u_t,v_*})$, moreover $J^{\mathfrak{s}} \models$ "$u_t \le v < v_*$" $\Rightarrow t \in \mathrm{Dom}(\pi^{\mathfrak{s}}_{v,v_*})$

(E) if $I^{\mathfrak{s}}_{v_*} \models$ "$s < t$" then for some $u = u_{s,t} \in J^{\mathfrak{t}}$ we have $u \le_{J[\mathfrak{s}]} v <_{J[\mathfrak{s}]} v_* \Rightarrow \pi^{\mathfrak{s}}_{v,v_*}(s) <_{I^{\mathfrak{s}}_v} \pi^{\mathfrak{s}}_{v,v_*}(t)$

(F) if $s,t \in I^{\mathfrak{s}}_{v_*}$ and the conclusion of clause (e) holds then $I^{\mathfrak{s}}_{v_*} \models s <_{\mathfrak{t}} t$

(G) if $\langle t_u : u \in J_{\ge w}\rangle$ is a sequence satisfying $w \in J, J_{\ge w} = \{u : w \le u \in J\}; t_u \in I^{\mathfrak{s}}_u$ and $w \le u_1 \le u_2 \in J$ we have $\pi_{u_1,u_2}(t_{u_2}) = t_{u_1}$, then there is a unique $t \in I^{\mathfrak{s}}_{v_*}$ such that $u \in J_{\ge w} \Rightarrow \pi_{u,v_*}(t) = t_u$.

Claim 5.31. *$G^{\mathfrak{s}}_{v_*}$ is a κ-automorphism group when:*

$\boxtimes$ *(a) $\mathfrak{s},\mathfrak{t}$ are both nice κ-p.o.w.i.s*

(b) $\mathfrak{s}$ is the limit of $\mathfrak{t}$ as witnessed by v_*

(c) $J^{\mathfrak{t}}$ is $\aleph_1$-directed

(d) $\kappa \ge |J^{\mathfrak{t}}|$ and $\kappa \ge |I^{\mathfrak{t}}_u|$ for $u \in J^{\mathfrak{t}}$.

Proof. Let $\mathbf{p}_u = \mathbf{p}^{u]} = \mathbf{p}^{\mathfrak{s}}_u$ for $u \in J^{\mathfrak{s}}$, etc. First Presentation:

For $u \in J^{\mathfrak{t}}$ let

(A) $S_u = X_{\mathbf{p}_u} \cup \{(2,v,x)$: we have $u \le_{J[\mathfrak{t}]} v, x \in X^+_{\mathbf{p}[v]}, x \notin \mathrm{Dom}(\pi^{+,\mathfrak{s}}_{u,v})$

(B) for $s \in S_u$ let $\bar{\varkappa}^u_s = \langle \varkappa^u_{s,\alpha} : \alpha \in Z^{\mathbf{p}[u]}\rangle$ satisfying $\varkappa^u_{s,\alpha} \in \mathrm{per}(\mathcal{P}(\Lambda^*_{\mathbf{m}(\alpha)}))$ be defined as follows:

  (α) if $s \in X_{\mathbf{p}_u}$ then $\varkappa^u_{s,\alpha} = \kappa^{\mathbf{p}[u]}_{s,\alpha}$

  (β) if $s = (2,v,x)$ then $\varkappa_{s,\alpha} = \kappa^{\mathbf{p}[v]}_{x,\alpha}$

  (γ) $s \in Z^{\mathbf{p}[u]} * 2$ then $\kappa^u_{s,\alpha}$ is the identity on $\mathcal{P}(\Lambda^*_{\mathbf{m}(\alpha)})$ or any $\{\varkappa_{\mathbf{m}(\alpha),u} : \upsilon \in \Lambda_{\mathbf{m}(\alpha)}\}$

(C) K_u is the group generated by $\{g_x : x \in S_u\}$ freely except

  (α) $g_x = g^{-1}_x$ when $x \in Z^{\mathbf{p}[u]} * 2$

  (β) $g_{y_1} g_{y_2} = g_{y_2} g_{y_1}$ when $y_1, y_2 \in Z^{\mathbf{p}[u]} * 2$

  (γ) $g_x g_{y_1} g^{-1}_x = g_{y_2}$, if for some $\alpha \in Z^{\mathbf{p}[u]}, \{y_1,y_2\} \subseteq \{\alpha\} * 2$ and $\varkappa^u_{x,\alpha}(\upsilon^{y_1}) = \upsilon^{y_2}$.

 Note

$(*)_1$ in K_u:

  (α) g_{y_1}, g_{y_2} commute if $y_1, y_2 \in Z^{\mathbf{p}[u]} * 2$

(β) $g_x g_{y_1} g_x^{-1} = g_{y_2}$ if $\circledast^1_{x,y_1,y_2}[\mathbf{p}_u]$

(γ) conjugating by g_x maps H onto itself when $x \in S_u \setminus X^+_{\mathbf{p}_u}$ so $x = (2, v, x), u \le_J v, x \in I_v^{\mathbf{p}[v]} \setminus \mathrm{Dom}(\pi^+_{u,v})$

$(*)_2$ (a) $\langle \{g_y : y \in Z^{\mathbf{p}[u]} * 2\rangle_{K_u}$ is, essentially, $G^{<0}_{\mathbf{p}[u]}$

(b) the subgroup of K_u which $\langle g_y : y \in S_u \setminus Z^{\mathbf{p}[u]} * 2\rangle$ generates,

it generates it freely call it K^1_u

(c) K_u is the twisted product of K^1_u and $G^{<0}_{\mathbf{p}[u]}$.

So as in the proof of 5.21

$(*)_3$ $F_{\mathbf{p}_u} \subseteq K_u$.

Now for $u <_{J[\mathfrak{t}]} v$ let $\pi^*_{u,v}$ be the following mapping from S_v to S_u: for $x \in S$. Case 1: If $x \in \mathrm{Dom}(\pi^{\mathfrak{t},+}_{u,v})$ then $\pi^*_{u,v}(x) = \pi^{\mathfrak{t},+}_{u,v}(x)$. Case 2: $x \in X^+_{\mathbf{p}_v} \setminus \mathrm{Dom}(\pi^{\mathfrak{t},+}_{u,v})$

then $\pi^{\mathfrak{b}}_{u,v}(x) = (2, v, \bar{\varkappa}^v_x \restriction Z^{\mathbf{p}[u]})$. Case 3: $x \in S_v \setminus X^+_{\mathbf{p}_v}$.

So $x = (r, v, x)$ and let $\pi^*_{u,v}(x) = (2, v, x)$.

Now

$(*)_4$ (a) for $u <_{J[\mathfrak{t}]} v, \pi^*_{u,v}$ is a function from S_v into S_u (could have arranged

onto, if $J^{\mathfrak{t}}$ is linear this holds)

(b) for $u_0 <_{J[\mathfrak{t}]} u_1 <_{J_{[\mathfrak{t}]}} u_2$ we have $\pi^*_{u_0,u_2} = \pi^*_{u_0,u_1} \circ \pi_{u_1,u_2}$

(c) for $u_1 <_{J[\mathfrak{t}]} u_2, \pi^*_{u_0,u_2}$ induce a mappng $\pi^{+,\mathfrak{b}}_{u_1,u_2}$ from $\{g_x : x \in S_{u_2}\}$

into $\{g_x : x \in S_{u_1}\}$ which has one and only one extension

$\hat{\pi}^{\mathfrak{t}}_{u_1,u_2}$ which is a homomorphism from K_{u_2} into K_{u_1}

(d) $F_{\mathbf{p}_{v_*}}$ is the inverse limit of $\langle K_u, \pi^{\mathfrak{b}}_{u_1,u_2} : u \in J^{\mathfrak{t}}, u_1 \le_{J[\mathfrak{t}]} u_2\rangle$.

Why? Check.

Now it follows that $F_{\mathbf{p}_{v_*}}$ is a κ-automorphism group. Now we can improve the conclusion. Can we waive the $\aleph_1$-directed? See in the continuation. $\square$

Alternative presentation:

For each $u \in J^{\mathfrak{t}}$ we define $\mathbf{k}_u = \mathbf{k}[u] = (S_u, \bar{\varkappa}^u, Z^u)$ by

$(*)_0$ (a) S_u as in (a) above

(b) $\bar{\varkappa}^u = \langle \bar{\varkappa}^u_s : s \in S^u\rangle, \bar{\varkappa}^u_s$ for $u \in J^{\mathfrak{t}}, s \in X_{\mathbf{p}_u}$ as in (b) above

(c) $Z^u = Z^{\mathbf{p}[u]}$.

$(*)_1$ $\mathbf{k}_u$ is a simplified *kappa*-parameter

[Why? Just check.]

[So $\mathbf{k}_u$ is in general not one to one; this helps to make the inverse limit right]

$(*)_2$ let $F_u = F_{\mathbf{k}_u}$

$(*)_3$ if $u \le_{J[\mathfrak{t}]} v$ then we define a mapping $\pi^*_{u,v}$ from S_v to S_u as follows:

(a) if $x \in \mathrm{Dom}(\pi^{\mathfrak{t},+}_{u,v}) \subseteq X_{\mathbf{p}[v]}$ then $\pi^*_{u,v}(x) = \pi^{+,\mathfrak{t}}_{u,v}(x)$

(b) if $x \in X_{\mathbf{p}[v]} \setminus \mathrm{Dom}(\pi^{\mathfrak{t},+}_{u,v})$ then $\pi^*_{u,v}(x) = (2, v, x)$

(c) assume $x = (2, v_1, x_1) \in S_v \setminus X_{\mathbf{p}[v]}$

(hence $v \le_{J[\mathfrak{t}]} v_1$ and $x_1 \in X_{\mathbf{p}[v_1]} \setminus \mathrm{Dom}(\pi^{+,\mathfrak{t}}_{v,v_1})$);

(α) $x_1 \notin \mathrm{Dom}(\pi^{+,\mathfrak{t}}_{u,v_1})$ then $\pi^*_{u,v}(x) = (2, v_1, x_1)$

(β) if $x_1 \in \mathrm{Dom}(\pi^{+,\mathfrak{t}}_{u,v_1})$ then $\pi^*_{u,v}(x) = \pi_{u,v_1}(x_1)$

$(*)_4$ for $u \le_{J[\mathfrak{t}]} v$ we have

(a) $\pi^*_{u,v}$ is a well defined function

(b) $\pi^*_{u,v}$ extend $\pi^{\mathfrak{t},+}_{u,v}$

(c) $\mathrm{Dom}(\pi^*_{u,v}) = S^{\mathbf{k}[v]} = S_v$ and $\mathrm{Rang}(\pi^*_{u,v}) \subseteq S^{\mathbf{k}[v]} = S_v$

(d) if $u_1 \le_{J[\mathfrak{t}]} u_2 \le_{J[\mathfrak{t}]} u_2$ then $\pi^*_{u_1,u_3} = \pi^*_{u_2,u_2} \circ \pi^*_{u_2,u_3}$

(e) if $x \in S_v$ then $\bar{\varkappa}^v_x \cap Z^u = \bar{\varkappa}^u_{\pi^*_{u,v}(x)}$.

[Why? Check.]

$(*)_5$ for $u \le_{J[\mathfrak{t}]} v$ let $\check{\pi}^*_{u,v}$ be the homomorphism from $F_{\mathbf{k}[v]}$ into $F_{\mathbf{k}[u]}$ such that

(a) it maps g_x to $g_{\pi^*_{u,v}(x)}$ for $x \in S_v$

(b) it maps g_x to g_x for $x \in Z^u * 2$

(c) it maps g_x to $e_{F_{\mathbf{k}[u]}}$ for $x \in (Z^v \setminus Z^u) \times 2$

$(*)_6$ $\check{\pi}^*_{u,v}$ is a well defined homomorphism from $F_{\mathbf{k}[v]}$ into $F_{\mathbf{k}[u]}$

[Why? As $F_{\mathbf{k}[u]}, F_{\mathbf{k}[v]}$ are twisted products]

$(*)_7$ $\check{\pi}^*_{u,v}$ extends $\hat{\pi}^{\mathfrak{s}}_{u,v}$

[why? check]

$(*)_8$ $F_{\mathbf{p}[v_*]}$ is the inverse limit of $\langle F_{\mathbf{k}[u]}, \check{\pi}^*_{u,v} : u \le_{J[\mathfrak{t}]} v\rangle$.

Claim 5.32. *Assume*

(A) $\aleph_0 < \theta = \mathrm{cf}(\theta) \le \kappa$

(B) $\mathcal{T}_\alpha \subseteq {}^\alpha\kappa$ *for* $\alpha < \theta$ *has cardinality* $\le \kappa$

(C) $\mathcal{F} = \{f \in {}^\theta\kappa : f \restriction \alpha \in \mathcal{T}_\alpha$ *for* $\alpha < \theta\}$

(D) $\gamma = \mathrm{rk}(\mathcal{F}, <_{J^{\mathrm{bd}}_\theta})$, *necessarily* $< \infty$ *so* $< (\kappa^\theta)^+$

(E) for every $n - \alpha \le \theta$ *and* n, *the function from* $\alpha + 1$ *to* $\{n\}$ *belongs to* $\mathcal{T}_\alpha$.

Then $\tau^{\mathrm{atw}}_\kappa \ge \tau^{\mathrm{nlg}}_\kappa \ge \tau^{\mathrm{nlf}}_\kappa > \gamma$ *(on* $\tau^{\mathrm{nlf}}_\kappa$ *see below).*

Definition 5.33. $\tau^{\mathrm{nlf}}_\kappa$ is the least ordinal τ such that $\tau > \tau^{\mathrm{nlf}}_{G,H}$ wherever $G = \mathrm{Aut}(\mathfrak{A}), \mathfrak{A}$ a structure of cardinality $\le \kappa, H$ a subgroup of G of cardinality $\le \kappa$ and $\mathrm{nor}^{<\infty}_G(H) = G$.

Proof. We define $\mathfrak{s} = (J, \bar{\mathbf{p}}, \bar{\pi})$ as follows:

(A) $J = (\theta + 1; <)$

(B) $I_\alpha = (\mathcal{T}_{\alpha+1}, <_{\alpha+1})$ for $\alpha < \theta + 1$ where

$$f_1 <_{\alpha+1} f_2 \Leftrightarrow f_1(\alpha) < f_2(\alpha)$$

(C) for $\alpha < \beta < \theta + 1$ let $\pi_{\alpha,\beta} : I_\beta \to I_\alpha$ be

$$\pi_{\alpha,\beta}(f) = f \restriction (\alpha + 1).$$

(D) let $\langle U_\alpha : \alpha < \theta\rangle$ be a partition of κ to sets, each of cardinality κ

(E) for $\alpha < \theta, \ell < 2$ let $\langle A^{\alpha,\ell}_x : x \in X_{I_\alpha}\rangle$ be an independent sequence of subsets of $U_{2\alpha+\ell}$

(F) for $\alpha < \theta$ and $x \in X_{I_\alpha}$ let
$A_x = \cup\{A^{\beta,\ell}_{\pi^+_{\beta,\alpha}(x)} : \beta \le \alpha, x \in \mathrm{Dom}(\pi^+_{\beta,\alpha}(x)$, see Definition xxx and $\mathrm{rk}^{2,<\infty}_{I_\beta}(x) = 0 \Rightarrow \ell = 0\}$

(G) for $\alpha \le \theta$ let $Z^\alpha = \cup\{A_{2\beta+\ell} : \beta \le \alpha, \beta < \theta, \ell < 2\}$.

Lastly, for $\alpha \le \theta$ let $\mathbf{p}_\alpha = (I_\alpha, \langle A_x : x \in X_{I_\alpha}, Z^\alpha)$

$(*)_2$ $\mathbf{p}_\alpha$ is a nice κ-parameter

$(*)_3$ $\mathfrak{s} = (J, \bar{\mathbf{p}}, \bar{\pi})$ is a κ-p.o.w.i.s.

$(*)_4$ $\mathfrak{s}$ is a limit of $\mathfrak{t} =: \mathfrak{s} \restriction \theta = ((\theta, <), \bar{\mathbf{p}} \restriction \theta, \bar{\pi} \restriction \theta)$.

Easy to finish. $\square$

Now we can conclude 6.16

Conclusion 5.34. *If $\kappa = \kappa^{\aleph_0}$ then $\tau_\kappa^{\rm atw} \ge \tau_\kappa^{\rm nlg} > \tau_\kappa^{\rm nlf} > \kappa^+$.*

Proof. θ is regular $\mathcal{F}_\alpha = {}^\alpha\kappa$ for $\alpha \le \theta$.

Let

(A) $J = ([\kappa]^{\aleph_0}, \subseteq)$

(B) for $u \in J, I_u = \{R : R \text{ a well ordering of } u\}$

(C) for $u \le_J v$ let $\pi_{u,v}(R) = R \restriction U$ for $f \in I_v$.

We continue on as above (imitating §3). FILL $\square$

Definition 5.35. 1) Assume that I_1, I_2 are partial orders; we say that $\pi : I_1 \to I_2$ is a homomorphism, if it is a function from I_1 into I_2 such that $s <_{I_1} t \Rightarrow \pi(s) <_{I_2} \pi(t)$.
1A) For $x = (\langle t_0, \dots, t_n\rangle, \eta) \in X_{I_1}$ and a function π from I_1 to I_2 define

$$\pi^+(x) = (\langle \pi(t_0), \dots, \pi(t_n)\rangle, \eta\rangle).$$

2) For κ-parameter that $\mathbf{p}, \mathbf{q}$ we say π is a partial homomorphism from $\mathbf{p}$ to $\mathbf{q}$ if

(A) π is a function, $\mathrm{Dom}(\pi) \subseteq I^{\mathbf{p}} \cup Z^{\mathbf{p}}$,

(B) $\pi \restriction I^{\mathbf{p}}$ is a homomorphism from $I^{\mathbf{p}} \restriction \mathrm{Dom}(\pi)$ into $I^{\mathbf{q}}$ and

(C) $\pi \restriction Z^{\mathbf{p}}$ is a partial one-to-one function from $Z^{\mathbf{p}}$ into $Z^{\mathbf{q}}$ and $x \in X_{\mathrm{Dom}(\pi) \cap I[\mathbf{p}]} \Rightarrow A^{\mathbf{q}}_{\pi^+(x)} \cap \pi(Z^{\mathbf{p}}) = \{\pi(y) : y \in A^{\mathbf{p}}_x\}$

(D) π maps $Y^{\mathbf{p}} \cap \mathrm{Dom}(\pi)$ onto $Y^{\mathbf{q}} \cap \pi(Z^{\mathbf{p}} \cap \mathrm{Dom}(\pi))$.

2A) We define $\pi^+ : X^+_{\mathbf{p}} \to X^+_{\mathbf{q}}$ by: if $x \in X_{\mathbf{p}}, \pi^+(x)$ is defined as in part (1A) and if $y = (\alpha, \ell)$ if $\alpha \in \mathrm{Dom}(\pi)$ and $\ell < 2$ then $\pi^+(y) = (\pi(\alpha), \ell)$.
2B) We may omit "partial" when $I^{\mathbf{p}} = \mathrm{Dom}(\pi)$.
3) We say that π is a partial isomorphism from I_1 to I_2 <u>when</u> π is a one-to-one function from some $A'_1 \subseteq I_1$ onto $A'_2 \subseteq I_2$ such that π is an isomorphism from $I_1 \restriction A'_1$ onto $I_2 \restriction A_2$.
4) Similarly "π is a partial isomorphism from $\mathbf{p}_1$ to $\mathbf{p}_2$" if it is a partial homomorphism from $\mathbf{p}_1$ to $\mathbf{p}_2, \pi \restriction I_1$ is a partial isomorphism from I_1 to I_2 (so $\pi \restriction Z^{\mathbf{p}_1}$ is one to one).
5) Let $\mathbf{p} \subseteq \mathbf{q}$ for κ-parameters mean that $\mathrm{id}_{I[\mathbf{p}]} \cup \mathrm{id}_{Z[\mathbf{p}]}$ is a partial isomorphism from $\mathbf{p}$ to $\mathbf{q}$.
6) If $Z^{\mathbf{q}} \subseteq Z^{\mathbf{p}}$ then when we treat $\pi : I^{\mathbf{p}} \to I^{\mathbf{q}}$ as $\pi : \mathbf{p} \to \mathbf{q}$ we mean $\pi \cup \mathrm{id}_{Z[\mathbf{q}]}$.

Of course

Claim 5.36. *In Definition 5.35, if π is a partial homomorphism from $\mathbf{p}_1$ to $\mathbf{p}_2$ <u>then</u>:*

(A) π^+ is a partial mapping from $X^+_{\mathbf{p}_1}$ into $X^+_{\mathbf{p}_2}$, see Definition 5.35(1A)

(B) if $x, y_1, y_2 \in X^+_{\mathbf{p}_1}$ and $x \in \mathrm{Dom}(\pi^+)$ and $\circledast^{\{0,1\}}_{x,y_1,y_2}$ then $y_1 \in \mathrm{Dom}(\pi^+) \Leftrightarrow y_2 \in \mathrm{Dom}(\pi^+)$

(C) $(\mathbf{p}_1, X_{\mathbf{p}_1} \cap \mathrm{Dom}(\pi^+), Z^{\mathbf{p}_1} \cap \mathrm{Dom}(\pi))$ is as in Definition **??***(4) and Claim 5.12(7)*

(D) $(\mathbf{p}_2, X_{\mathbf{p}_2} \cap \mathrm{Rang}(\pi^+) = \mathrm{Rang}(\pi^+ \restriction X_{\mathbf{p}_1}), Z^{\mathbf{p}_2} \cap \mathrm{Rang}(\pi))$ *is as in Definition* **??***(4) and Claim 5.12(7)*

(E) π^+ *maps* $\Gamma^*_{\mathbf{p}_1, X_{\mathbf{p}_1} \cap \mathrm{Dom}(\pi^+, Z^{\mathbf{p}_1} \cap \mathrm{Dom}(\pi)}$ *onto* $\Gamma^*_{\mathbf{p}_2, X_{\mathbf{p}_2} \cap \mathrm{Rang}(\pi^+), Z^{\mathbf{p}_2} \cap \mathrm{Rang}(\pi)}$ *(see Definition 5.20(6))*

(F) there is a unique homomorphism $\hat{\pi}$ *from the subgroup* $\langle \{g_x : x \in \mathrm{Dom}(\pi^+)\}\rangle_{F[\mathbf{p}_1]}$ *of* $F_{\mathbf{p}_1}$ *onto the subgroup* $\langle \{g_x : x \in \mathrm{Rang}(\pi^+)\rangle_{F[\mathbf{p}_2]}$ *of* $F_{\mathbf{p}_2}$ *mapping* g_x *to* $g_{\pi^+(x)}$ *for* $x \in \mathrm{Dom}(\pi^+)$*.*

Proof. Check (or see the proof of 5.42(2); see 6.6). $\square$

Claim 5.37. *If* $\mathbf{p}_1 \subseteq \mathbf{p}_2$ *are* κ*-parameters, then* $X_{\mathbf{p}_1} \subseteq X_{\mathbf{p}_2}, Z^{\mathbf{p}_1} \times 2 \subseteq Z^{\mathbf{p}_2} \times 2, X^+_{\mathbf{p}_1} \subseteq X^+_{\mathbf{p}_2}, \Gamma_{\mathbf{p}_1} \subseteq \Gamma_{\mathbf{p}_2}$ *and* $G_{\mathbf{p}_1}$ *is a subgroup of* $G_{\mathbf{p}_2}$ *and* $F_{\mathbf{p}_1}$ *is a subgroup of* $F_{\mathbf{p}_2}$*.*

Proof. The only non-trivial part are $G_{\mathbf{p}_1}$ is a subgroup of $G_{\mathbf{p}_2}$ which holds by 5.12(7) and "$F_{\mathbf{p}_1}$ is a subgroup of $F_{\mathbf{p}_2}$q" which holds by the properties of twisted products (see Claim 5.22(3) and Definition 5.14. $\square$

Claim 5.38. *1) If* π *is a partial homomorphism from* $\mathbf{p}_1$ *to* $\mathbf{p}_2$ *(see Definition 5.35(2)), then* $\hat{\pi}$ *from clause (f) of 5.35 is well defined and* $\hat{\pi} \restriction F^{<0}_{\mathbf{p}_1}$ *is a partial isomorphism from* $\langle \{g_y : y \in Z^{\mathbf{p}_1} \times 2\}\rangle_{F[\mathbf{p}_1]}$ *into* $\langle \{y_y : y \in Z^{\mathbf{p}_2} \times 2\rangle_{F[\mathbf{p}_2]}$ *preserving "*$g \in H$*", "*$g \notin H$*"; if* π *is onto* $Z^{\mathbf{p}_2}$ *then* $\hat{\pi}$ *is onto* $F^{<0}_{\mathbf{p}_2}$*.*
2) In Definition 5.35 and Clause (f) of 5.36, if π *is one to one then* π^+ *is one to one and also* $\hat{\pi}$ *is one to one.*

Proof. Follows from clause (d) of 5.35(2) and 5.37. $\square$

Claim 5.39. *Assume that* $\mathbf{p}_\ell$ *is a* κ*-parameter for* $\ell < 3$ *and* $\pi_\ell : \mathbf{p}_\ell \to \mathbf{p}_{\ell+1}$ *is a partial homomorphism for* $\ell = 0, 1$ *and* $\pi = \pi_1 \circ \pi_0 : \mathbf{p}_0 \to \mathbf{p}_2$*. Then* π *is a partial homomorphism from* $\mathbf{p}_0$ *into* $\mathbf{p}_2$ *and* $\hat{\pi} = \hat{\pi}_1 \circ \hat{\pi}_0$ *(and* $\pi^+ = \pi^+_1 \circ \pi^+_0$*).*

Proof. Easy. $\square$

§ 5(E). §5 **Inverse limits.**

Definition 5.40. 1) We say $\mathfrak{s}$ is a κ-p.o.i.s. (partial order inverse system, and p.o.i.s. means κ-p.o.i.s. for some κ) when:

(A) $\mathfrak{s} = (J, \bar{\mathbf{p}}, \bar{\pi})$

(B) J is a directed partial order of cardinality $\leq \kappa$

(C) $\bar{\mathbf{p}} = \langle \mathbf{p}_u : u \in J \rangle$

(D) $\mathbf{p}_u$ is a κ-parameter, $I_u = I^{\mathbf{p}_u}$ is of cardinality $\leq \kappa$ and $u \leq_J v \Rightarrow Y^{\mathbf{p}_u} \subseteq Y^{\mathbf{p}_v} \wedge (Z^{\mathbf{p}_u} \setminus Y^{\mathbf{p}_u}) \subseteq (Z^{\mathbf{p}_v} \setminus Y^{\mathbf{p}_v})$

(E) $\bar{\pi} = \langle \pi_{u,v} : u \leq_J v \rangle$

(F) each $\pi_{u,v}$ is a homomorphism from I_v into I_u (so $Z^{\mathbf{p}_u} \subseteq Z^{\mathbf{p}_u}$ and we pretend that $\pi_{u,v} \restriction Z^{\mathbf{p}_v} = \mathrm{id}_{Z[\mathbf{p}_u]}$; see 5.35(6)) and $\pi_{u,u} = \mathrm{id}_{I_u}$ (so $\mathrm{Dom}(\pi_{u,v})$ may be a proper subset of I_v) and $x \in \mathrm{Dom}(\pi_{u,v}) \Rightarrow A^{\mathbf{p}_u}_{\pi(x)} \cap Z^{\mathbf{p}_v} = A^{\mathbf{p}_v}_x$

(G) if $u_0 \leq_J u_1 \leq_J u_2$ then $\pi_{u_0,u_2} = \pi_{u_0,u_1} \circ \pi_{u_1,u_2}$
(in particular the domains of the two sides are equal).

It follows that

(A) $u \leq_J v$ implies that $H_{\mathbf{p}_u} \subseteq H_{\mathbf{p}_v}$ and $F^{<0}_{\mathbf{p}_u} \subseteq F^{<0}_{\mathbf{p}_v}$ and $H_{\mathbf{p}_u} = H_{\mathbf{p}_v} \cap G^{<0}_{\mathbf{p}_u}$.

1A) Let $\mathfrak{s} = (J^{\mathfrak{s}}, \bar{\mathbf{p}}^{\mathfrak{s}}, \bar{\pi}^{\mathfrak{s}}), \bar{\mathbf{p}}^{\mathfrak{s}} = \langle \mathbf{p}^{\mathfrak{s}}_u : u \in J^{\mathfrak{s}} \rangle, \mathbf{p}^{\mathfrak{s}}_u = (I^{\mathfrak{s}}_u, \bar{A}^{\mathfrak{s}}_u, Z^{\mathfrak{s}}_u, Y^{\mathfrak{s}}_u), \bar{A}^{\mathfrak{s}}_u = \langle A^{\mathfrak{s}}_{u,x} : x \in X^+_{\mathbf{p}^{\mathfrak{s}}_u} \rangle, \bar{\pi}^{\mathfrak{s}} = \langle \pi^{\mathfrak{s}}_{u,v} : u \leq_J v \rangle, J^{\mathfrak{s}} = J[\mathfrak{s}], \mathbf{p}_u[\mathfrak{s}] = \mathbf{p}^{\mathfrak{s}}_u, I^{\mathfrak{s}}_u = I_u[\mathfrak{s}]$ and $F^{\mathfrak{s}}_u = F_{\mathbf{p}_u[\mathfrak{s}]}$ and, of course, $\hat{\pi}^{\mathfrak{s}}_{u,v} = \hat{\pi}_{u,v}$ (see Definition 5.36).

2) We define $I^+ = I^+[\mathfrak{s}] = \text{Inv-lim}_{\text{or}}(\mathfrak{s})$, a partial order (easy to check) as follows:

(A) $\bar{t} \in \text{inv} - \lim_{\text{or}}(\mathfrak{s})$ <u>iff</u>

  (α) $\bar{t}$ has the form $\langle t_u : u \in J_{\geq w} \rangle$ for some $w \in J$ where $J_{\geq w} = \{v \in J : w \leq_J v\}$ and $u \in J_{\geq w} \Rightarrow t_u \in I_u$, and let $w[\bar{t}] = w$, we may use $J_{\geq \varnothing} = J_{\min(J)} = J$ even when J has no minimal member

  (β) if $u_1 \leq_J u_2$ are in $J_{\geq w}$ then $\pi_{u_1,u_2}(t_{u_2}) = t_{u_2}$

(B) for $\bar{s}, \bar{t} \in \text{inv} - \lim_{\text{or}}(\mathfrak{s})$ let $\bar{s} <_{I^+} \bar{t}$ <u>iff</u> there is $w \in J$ such that $w[\bar{s}] \leq_J w \wedge w[\bar{t}] \leq_J w \wedge (\forall u)(w \leq_J u \Rightarrow s_u <_{I_u} t_u)$

(C) For $\bar{s}, \bar{t} \in \text{inv} - \lim_{\text{or}}(\mathfrak{s})$ let $\bar{s} \leq_{I^+} \bar{t}$ be defined similarly

3) Let $I_{\mathfrak{s}} = I[\mathfrak{s}] = \text{inv} - \lim_{\text{or}}(\mathfrak{s})$ be the partial order $I^+/\approx$ where $\approx$ is the following two place relation:
$\bar{s} \approx \bar{t}$ iff for some $w \in J$ we have

$$w[\bar{s}] \leq_I w \wedge w[\bar{t}] \leq_J w \wedge (\forall u)(u \leq_J u \Rightarrow s_u = t_u)$$

clearly

(A) $\approx$ is an equivalence relation on I^+ and

(B) $\bar{s} \approx \bar{s}' \wedge \bar{t} \approx t' \Rightarrow (\bar{s} <_{I^+} \bar{t} \Leftrightarrow \bar{s}' <_{I^+} \bar{t}')$ and

(C) $\bar{s} \leq_{I^+} \bar{t}$ and $\neg(\bar{s} \approx \bar{t}) \Rightarrow \bar{s} <_{I^+} \bar{t}$.

3A) We define $\mathbf{p} = \mathbf{p}_{\mathfrak{s}} = \mathbf{p}[\mathfrak{s}] = \text{inv} - \lim_{\text{sy}}(\mathfrak{s})$ as $(I, \bar{A}, Z, Y)$ where

(A) $I = \text{inv} - \lim_{\text{or}}(\mathfrak{s})$

(B) $\bar{A} = \langle A_{\bar{s}/\approx} : (\bar{s}/\approx) \in \text{inv} - \lim_{\text{or}}(\mathfrak{s}) \rangle$ and $A_{\bar{s}/\approx} = \cup \{A_{s_u} : u \in J_{\geq w[\bar{s}]}\}$

(C) $Z = \cup\{Z^{\mathbf{p}_u} : u \in J\}$ and $Y = \cup\{Y^{\mathbf{p}_u} : u \in J\}$.

4) We define $\pi^{\mathfrak{s}}_u$ for $u \in I$, a partial map from $I = \text{inv} - \lim_u(\mathfrak{s})$ to I_u by $\pi^{\mathfrak{s}}_u(\bar{t}/\approx) = s$ <u>iff</u> $\bar{t} \in I^+, u \in J$ and $(\exists \bar{s})(\bar{s} \approx \bar{t} \wedge s_u = s)$.

5) We define $F^+_{\mathfrak{s}}$, a set and $F_{\mathfrak{s}}$, a group, (where $F^{\mathfrak{s}}_u = F_{\mathbf{p}_u[\mathfrak{s}]}$ is as defined in Definition 5.9(1))

(A) $F^+_{\mathfrak{s}} = \text{inv} - \lim_{\text{gr}}(\mathfrak{s}) = \text{inv} - \lim_{\text{gr}} \langle F_{\mathbf{p}_u}, \hat{\pi}_{u,v} : u \leq_J v \rangle$

that is, $G^+_{\mathfrak{s}}$ is (just) the set of $\bar g$ of the form $\langle g_u : u \in J_{\geq w}\rangle$ where $w \in J, g_u \in G_u$ and $\hat\pi_{u,v}(g_v) = g_u$ when $w \leq_J u \leq_J v$

(A) $\approx$ is defined on $F^+_{\mathfrak{s}}$ as in part (3).

5A)

(A) the group $F_{\mathfrak{s}} = \text{inv} - \lim_{\text{gr}}\langle F_{I_u}, \hat\pi_{u,v} : u \leq_J v\rangle$ is defined parallely to part (3), with co-ordinatewise multiplication

(B) $\pi^{\mathfrak{s}}_u$ is the partial homomorphism from the group $F_{\mathfrak{s}}$ (i.e., from a subgroup) into $F^{\mathfrak{s}}_u$ defined by $\pi^{\mathfrak{s}}_u(\bar g) = g'_u$ when $\bar g \approx \bar g' \wedge u \in J_{\geq w[\bar g']}$.

So for $\bar g \in F^+_{\mathfrak{s}}$ we have $\bar g = \langle g_u : u \in J_{\geq w[\bar g]}\rangle$.

6) Let $H^+_{\mathfrak{s}}$ be $\cup\{H_{\mathbf{p}_u} : u \in J\}$.

7) We naturally define $\mathbf{j} = \mathbf{j}_{\mathfrak{s}} = \mathbf{j}[\mathfrak{s}]$, an embedding of $F_{\mathbf{p}[\mathfrak{s}]}$ into $F_{\mathfrak{s}}$ as follows:

(A) $\mathbf{j}(g_y) = \langle g_{y_u} : u \in J_{\geq v}\rangle/\approx$ if $v \in J, y \in X^+_{\mathbf{p}_u} \setminus X_{\mathbf{p}_v}$ so it is the identity on $H_{\mathbf{p}[\mathfrak{s}]}$ and even $G^{<0}_{\mathbf{p}[\mathfrak{s}]}$

(B) if $x \in X_{\mathbf{p}[\mathfrak{s}]}$ let $t_\ell(x) = \langle t_{\ell,u} : u \in J_{\geq w_{1,\ell}}\rangle/\approx$ for $\ell = 0,\ldots,n(x)$ where $t_{\ell,u} \in I_u$ and let $w \in J$ be a common upper bound of $\{w_{1,0},\ldots,w_{1,n(x)}\}$ and we let $x_u = (\langle t_{\ell,u} : \ell \leq n(x)\rangle, \eta^x_u)$ for $u \in J_{\geq w}$
<u>then</u>

$$\mathbf{j}(g_x) = \langle g_{x_u} : u \in J_{\geq w}\rangle/\approx .$$

8) We say that $\mathfrak{s}$ is locally nice when for each $u \in J^{\mathfrak{s}}, \mathbf{p}^{\mathfrak{s}}_u$ is nice and $I[\mathbf{p}^{\mathfrak{s}}_u]$ is non-trivial.

9) We say that $\mathfrak{s}$ is nice if $\mathbf{p}_{\mathfrak{s}} = \text{inv} - \lim_{\text{sy}}(\mathfrak{s})$ is nice and $I[\mathbf{p}_{\mathfrak{s}}]$ is non-trivial.

Claim 5.41. *1) The inverse limits in 5.40 are well defined in particular:*

(A) if $\bar s^1 \approx \bar s^2$ *where* $\bar s^\ell = \langle s^\ell_u : u \in J_{\geq w_\ell}\rangle$ *for* $\ell = 1,2$ *then* $u \in J_{\geq w_1} \cap J_{\geq w_2} \Rightarrow s^1_u = s^2_u$

(B) if we define $\mathfrak{t}$ *by* $J_{\mathfrak{t}} = J \cup \{\mathfrak{s}\}$, *so*

$$\mathbf{p}^{\mathfrak{t}}_u = \mathbf{p}^{\mathfrak{s}}_u \text{ if } u \in J \text{ and is } \mathbf{p}_{\mathfrak{s}} \text{ if } u = \mathfrak{s}, I^{\mathfrak{t}}_u = I_{\mathbf{p}[u]}$$

$$\begin{aligned}\pi^{\mathfrak{t}}_{u,v} &\text{ is } \pi^{\mathfrak{s}}_{u,v} \text{ if } v \in J\\ &\text{is } \pi^{\mathfrak{s}}_u \text{ if } u \in J^{\mathfrak{s}} \text{ and}\\ &\text{is } \text{id}_{\mathbf{p}_u} \text{ if } u = v \in J^{\mathfrak{t}} \setminus J^{\mathfrak{s}}\end{aligned}$$

<u>*then*</u>

(α) $\mathfrak{t}$ *is a* κ*-p.o.i.s.*

(β) $H_{\mathbf{p}[\mathfrak{s}]} = \cup\{H_{\mathbf{p}^s_u[\mathfrak{s}]} : u \in J\}$.

2) The mapping $\mathbf{j}_{\mathfrak{s}}$ *from Definition 5.40(7) is really a well defined embedding of the group* $G_{\mathbf{p}[\mathfrak{s}]}$ *into the group* $G_{\mathfrak{s}}$.

3) In part (2) if $J^{\mathfrak{s}}$ *is* $\aleph_1$*-directed* <u>*then*</u>

(A) equality holds, that is $\mathbf{j}_{\mathfrak{s}}$ *maps* $F_{\mathbf{p}[\mathfrak{s}]}$ *onto* $F_{\mathfrak{s}}$

(B) $\bigwedge_{u\in J[\mathfrak{s}]} \text{rk}^2(\mathbf{p}_u) < \infty \Rightarrow \text{rk}^2(\mathbf{p}_{\mathfrak{s}}) < \infty$.

Proof. 1),2) Easy.

3) We leave clause (b) to the reader and prove clause (a). Let $\bar g \in \text{inv} - \lim_{\text{gr}}(\mathfrak{s})$ so $\bar g = \langle g_u : u \in J_{\geq w[\bar g]}\rangle$. Now for each $u \in J_{\geq w[\bar g]}, g_u \in F^{\mathfrak{s}}_u$ and let $n_u = \min\{n : g_u$ is the product of n of the generators $\{g_x : x \in X^+_{\mathbf{p}_u}\}\}$ and let $n^1_u = \min\{n : g$ is

the product of n_u of the generators from $\{g_x : x \in X^+_{\mathbf{p}_u,n}\}\}$ where $X^+_{\mathbf{p}_u,n} = \{x \in X^+_{\mathbf{p}_u,n} : x \in X_{\mathbf{p}_u} \Rightarrow |n(x)| \le n\}$. Clearly:

$(*)_1$ if $u \le v$ are in $J_{\ge w[t]}$ then $n_u \le n_v$ and $n_u = n_v \Rightarrow n^1_u \le n^1_v$.

Case 1: for every $n < \omega$ there is $u \in J_{\ge w[\bar t]}$ such that $n_u \ge n$.

Let $u(n)$ exemplify this. As J is $\aleph_1$-directed there is $u \in J$ such that $n < \omega \Rightarrow u(n) \le_J u$, so $u \in J_{\ge w[\bar g]}$ and $\ell < \omega \Rightarrow u(\ell) \le_J u \Rightarrow \ell \le n_{u[t]} \le n_u < \omega$, contradiction.

So assume that not case (1) hence for some $u^*, n(*)$

$(*)_2$ $u^* \in J$ and $u \in J_{\ge u^*} \Rightarrow u \in J_{\ge w[\bar g]} \wedge n_u = n(*)$.

Case 2: For every $n < \omega$, some $v, u^* \le v \in J$ hence $n_v = n(*)$ satisfies $n^1_v \ge n$. We get a contradiction similar to Case 1. Case 3: Neither Case 1 nor Case 2.

Hence for some $n(*) < \omega$ and $n^1(*) < \omega$ and $u^* \in J$ we have $u \in J_{\ge u^*} \Rightarrow u \in J_{\ge w[\bar g]} \wedge n_u = n(*)$ and $n^1_u = n^1(*)$. So for some v we have $w[\bar g] \le_J v$ and $(\forall u)(v \le_J u \Rightarrow n_u = n(*)$ and $n^1_u = n^1(*) \wedge w[\bar g] \le u)$. For each $u \in J_{\ge v}$ let $g_u = g_{x_{u,1}} \cdots g_{x_{u,n}}$ where $n = n(u) = n(*)$ and $x_{u,\ell} \in X^+_{\mathbf{p},n^1(*)}$ be as in 5.12 for some appropriate linear order $<^*_u$ of $X_{\mathbf{p}_u}$, recalling 5.12(4) and the generators having order 2. We now define a set $B^\ell_u \subseteq X^+_{\mathbf{p}^5_u}$ by induction on $\ell \le n(*) \times n(*)$. Let B^0_u be $\{x_{u,1}, \dots, x_{u,n(*)}\}$. Let $B^{\ell+1}_u$ be $B^\ell_u \cup \{y_1 : x, y_2 \in B^\ell_u$, and $g_x g_{y_2} g^{-1}_x = g_{y_1}$ is one of the equations in $\Gamma_{\mathbf{p}_u}\}$.

So $|B^\ell_u| \le n(*)^{2^\ell}$ and [1]

⊛ if $v \le_J u_1 \le_{J_2} u_2$ then $\{\pi_{u_1,u_2}(g_{x_{u_2,\ell}}) : \ell = 1, \dots, n(*)\} \subseteq B^{n(*)\times n(*)}_{u_1}$
[Why? By the proof of 5.12(1),(7) applied to $\langle g_{\pi^+_{u_1,u_2}(x_{u_2,\ell})} : \ell = 1, \dots, n(*)\rangle$; by the uniqueness from there, in the end of the process we necessarily get $\langle g_{x_{u_1,\ell}} : \ell = 1, \dots, n(*)\rangle$ in $\le n(*) \times n(*)$ steps, each step being exchanging two generators and in the ℓ-th step before the end all the generators appearing are from $B^\ell_{u_1}$.]

Let $m(*) = n(*)^{2^{n(x)\times n(*)}}$. Let D be an ultrafilter on J such that $u_* \in J \Rightarrow \{u \in J : u_* \le_J u\} \in D$ so we have $\{u : n_u = n(*), n^1_u = n^1(*)\} \in D$. For $u \in J_{\ge v}$ let $\langle x^u_\ell : \ell < m(*)\rangle$ list $B^{n(*)\times n(*)}_u$ possibly with repetitions (we could have avoided this). Without loss of generality $x_{u,\ell} = x^u_\ell$ for $\ell = 1, \dots, n(*)$. For each $u \in J_{\ge v}$ we can find $\eta = \eta_u$, a function from $\{1, \dots, n(*)\}$ into $\{0, 1, \dots, m(*) - 1\}$ such that the set

$$A_{u,\eta} = \{u' \in J : u \le_J u' \text{ and } 1 \le \ell \le n(*) \Rightarrow \pi^+(x_{u',\ell}) = x^u_{\eta(\ell)}\}$$

belong to D. So for some η^* and $A \in D$ we have $u \in A \Rightarrow v \le_J u$ and $\eta_u = \eta^*$ and moreover, for some set S we have

$$u \in A \Rightarrow S = \{(\ell_1, \ell_2, \ell_3) : g_{x^u,\ell_1} g_{x^u,\ell_2} g^{-1}_{x^u,\ell_1} = g_{x^u,\ell_3} \in \Gamma_{\mathbf{p}_u} \text{ and } \ell_1, \ell_2, \ell_3 < m(*)\}.$$

Let $u_1 \le_J u_2$ be from A so we can find $u_3 \in A_{u_1} \cap A_{u_2}$. We know that $\ell \in \{1, \dots, n(*)\} \Rightarrow \pi^+_{u_1,u_2}\pi^+_{u_2,u_3}(g_{x_{u_3,\ell}}) = \pi^+_{u_1,u_3}(g_{x_{u_3,\ell}})$ so $\pi^+_{u_1,u_2}(x^{u_2}_{\eta(\ell)}) = x^{u_1}_{\eta(\ell)}$. Now let $\bar t_\ell = \langle t^u_\ell : u \in J_{\ge v}\rangle$ be: $t^u_\ell = \pi_{u,u_1}(t_{u_1,\eta^*(\ell)})$ for the D-majority of $u_1 \in J$. So we are done. □

[1] alternatively let $B_u = \{y \in X^+_{\mathbf{p}_u}$: for some $(\alpha, m) \in (Z \times 2) \cap \{x_{u,1}, \dots, x_{u,m}\}$ we have $y \in \{(\alpha, 0), (\alpha, 1\}$ or $g \in X_{\mathbf{p}_u}$ and $\bar t^y \in \{\bar t^{x_{u,1}}, \dots, \bar t^{x_{u,n}}\}$.

To connect the κ-p.o.i.s. to τ'_κ we need to know that $G_{\mathbf{p}[\mathfrak{s}]}$ is a κ-automorphic group.

Claim 5.42. *Let $\mathfrak{s}$ be a κ-p.o.i.s. with* $\mathrm{Dom}(\pi^{\mathfrak{s}}_{u,v}) = F^{\mathfrak{s}}_v$ *for $v \in J^{\mathfrak{s}}$. The group $F = F_{\mathfrak{s}}$ is isomorphic to a κ-automorphism group, i.e., the automorphism group of some structure $\mathfrak{A}$ of cardinality κ.*

Proof. So in Definition 5.40(5)

- $\circledast$ for every $\bar{g} \in F^+_{\mathfrak{s}}$ there is $\bar{g}' = \langle g'_u : u \in J\rangle \in F^+_{\mathfrak{s}}$ such that $J_{\geq w[\bar{g}']} = J = J^{\mathfrak{s}}$ and $\bar{g}' \approx \bar{g}$.

Hence

- $(*)$ $J = J^{\mathfrak{s}}$ is a directed p.o. of cardinality $\leq \kappa, F_u$ a group of cardinality $\leq \kappa, \langle G_u, \hat{\pi}_{u,v} : u \leq_J v\rangle$ is an inversely directed system of groups with inverse limit $G_{\mathfrak{s}}$.

As is well known there is $\mathfrak{A}$ as required:

- (A) the universe of $\mathfrak{A}$ be $\cup\{A_u : u \in J\}$ where
$$A_u = G_u \times \{u\}$$
- (B) the relations of $\mathfrak{A}$ are
  - (α) for $u_1 \leq_I u_2$,
$$R^{\mathfrak{A}}_{u_1,u_2} = \{((g_1,u_1),(g_2,u_2)) : \hat{\pi}_{u_1,u_2}(g_2) = g_1 \in F_{u_1},\ g_2 \in F_{u_2}\}$$
  - (β) for $u \in J, g \in G_u$
$$R^{\mathfrak{A}}_{u,g} = \{((g_1,u_1),(g_2,u_1)) : g_1,g_2 \in F_u,\ F_u \models \text{“}g_2 = gg_1\text{”}\}$$

$\square$

We would like to relax the assumptions in 5.42.

Definition 5.43. 1) A partial inverse system of groups $\mathfrak{g} = \langle G_u, \pi_{u,v} : u \leq_J v \text{ from } J\rangle$ means:

- (A) J is a directed partial order
- (B) G_u a group
- (C) $\pi_{u,v}$ is a partial homomorphism from G_v to G_u, i.e. from a subgroup of G_v onto G_u
- (D) if $u_0 \leq_J u_1 \leq_J u_2$ then $\pi_{u_0,u_2} = \pi_{u_0,u_1} \circ \pi_{u_1,u_2}$ (including the domain).

2) We say that $\mathfrak{g}$ is smooth which means:

- (A) for every $v \in J$ and $x \in X^{\mathfrak{s}}_{\mathbf{p}_v}$ there is $u = \mathbf{u}^{\mathfrak{s}}_v(x)$ (necessarily unique) such that:
  - (α) $u \leq_J v$
  - (β) if $w \leq_J v$ then $g \in \mathrm{Dom}(\pi^{\mathfrak{s}}_{w,v}) \Leftrightarrow u \leq w$.

3) We say that $\mathfrak{g}$ is good <u>when</u>:

- (α) $\mathrm{Rang}(\pi^{\mathfrak{g}}_{u,v}) = G_u$
- (β) the normal subgroup of G_v which $\{g \in \mathrm{Dom}(\pi_{u,v}) : \pi_{u,v}(g) = e_{G_u}\}$ generates is disjoint to $\{g \in \mathrm{Dom}(\pi_{u,v}) : \pi_{u,v}(g) \neq e_{G_u}\}$ whenever $u \leq_J v$.

4) Let inv-lim$(\mathfrak{g})$ be the usual inverse limit (i.e., using only members of the form $\langle g_u : u \in J\rangle$) and let $\text{Inv-lim}_{\text{gr}}(\mathfrak{g})$ and $\text{inv-lim}_{\text{gr}}(\mathfrak{g})$ be defined as in Definition 5.40(5),(5A) respectively, and $\pi^{\mathfrak{g}}_u$ are the mapping from it into $G^{\mathfrak{g}}_u$.

5) We say that a κ-p.o.i.s. is smooth [or good] when the partial inverse system $\langle G^{\mathfrak{s}}_u, \pi^{\mathfrak{s}}_{u,w} : u \leq^*_{J[\mathfrak{s}]} v\rangle$ is smooth [or good].

Observation 5.44. *1) If $\mathfrak{g}$ is a partial inverse system of groups* <u>*then*</u> $u_0 \le_J u_1 \le_J u_2 \Rightarrow$ *Dom*$(\pi_{u_0,u_2}) \subseteq \mathrm{Dom}(\pi_{u_1,u_2})$.
2) If $w \in J$ and $\langle (v_u, g_u) : u \in J_{\ge w}\rangle$ satisfy the statement $(*)$ below <u>*then*</u> *for some $u_* \in J_{\ge w}$ we have $u \in J_{\ge u_*} \Rightarrow g_u \in \mathrm{Dom}(\pi_{u,v_u})$, where*

$(*)$ (i) $u \in J_{\ge w} \Rightarrow u \le_J v_u \wedge g_u \in G^{\mathfrak{g}}_{v_u}$

(ii) *if $w \le_J u_1 \le_J u_2$ then $\pi_{v,v_{u_1}}(g_{u_1}) = \pi_{v,v_{u_2}}(g_{u_2})$ are well defined for*

$$\textit{some } v \textit{ which satisfies } u_1 \le_J v \le_J v_{u_1} \wedge v \le_J v_{u_2}$$

Proof. 1) So let $u_0 <_J u_1 <_J u_2$ hence $\pi_{u_0,u_2} = \pi_{u_0,u_1} \circ \pi_{u_1,u_2}$ by clause (d) of Definition **??**(2), hence $\mathrm{Dom}(\pi_{u_0,u_2}) \subseteq \mathrm{Dom}(\pi_{u_1,u_2})$.
2) Let $u_1 \in J_{\ge w}$ and if v_{u_1} fails the demand on u_* then there is u_2 such that $v_{u_1} \le_J u_2 \wedge g_{u_2} \notin \mathrm{Dom}(\pi_{u_2,v_{u_2}})$. Let v be as guaranteed in cluase (ii) of $(*)$ so $u_1 \le_J v \le_J v_{u_1} \wedge v \le_J v_{u_2}$ and $\pi_{v,v_{u_1}}(g_{u_2}) = \pi_{v,v_{u_1}}(g_{u_1})$. Hence $g_{u_2} \in \mathrm{Dom}(\pi_{v,v_{u_2}})$ and $u_1 \le_J v \le_J v_{u_1} \le_J u_2 \le_J v_{u_2}$, so by part (1) we have $\mathrm{Dom}(\pi_{v,v_{u_2}}) \subseteq \mathrm{Dom}(\pi_{u_2,v_{u_2}})$ hence $g_{u_2} \in \mathrm{Dom}(\pi_{u_2,v_{u_2}})$, contradiction to the choice of u_2. $\square$

Claim 5.45. *If J is an $\aleph_1$-directed partial order, $\mathfrak{g} = \langle G_u, \pi_{u,v} : u \le_J v\rangle$ is a smooth good partial inverse system of groups and $\sum\limits_{u \in J} |G_u| \le \kappa$* <u>*then*</u> inv-Lim$(\mathfrak{g})$ *is a κ-automorphism group.*

Proof. For $u \in J$ let S_u be $\{(v,g) : u \le_J v$ and $g \in G^{\mathfrak{g}}_v\}$ so $u \le_J v \Rightarrow S_v \subseteq S_u$.
We define an inverse group system $\mathfrak{h} = \langle G^{\mathfrak{h}}_u, \pi^{\mathfrak{h}}_{u,v} : u \le_J v\rangle$ as follows:

(A) $G^{\mathfrak{h}}_u = G_u[\mathfrak{h}]$ is the group generated by $S^{\mathfrak{g}}_u := \{z_{(v,g)} : (v,g) \in S^{\mathfrak{g}}_u\}$ freely (as different members can become equal we should pedantically denote by $z'_{(v,g)} \in G^{\mathfrak{h}}_u$ its image but we are not so careful) except the equations in $\Gamma_u = \Gamma^{\mathfrak{g}}_u$, which consists of:
- (α) $z_{(v_1,g_1)} = z_{(v_2,g_2)}$ if for some $v, u \le_J v, v \le_J v_1, v \le_J v_2$ and $\pi^{\mathfrak{g}}_{v,v_1}(g_1) = \pi^{\mathfrak{g}}_{v,v_2}(g_2)$
- (β) $z_{(u_1,g_1)} \dots z_{(u_n,g_n)} = e$ if $n < \omega, (u_\ell, g_\ell) \in S^{\mathfrak{g}}_u$ for $\ell = 1, \dots, n$ and for some $v, (\forall \ell)[u \le v \le u_\ell]$ and letting $g'_\ell = \pi^{\mathfrak{g}}_{v,u_\ell}(g_\ell)$ we have $G^{\mathfrak{g}}_v \models$ "$g_1 \dots g_n = e_{G_u[\mathfrak{g}]}$"

(B) if $u \le_J v$ then $\pi^{\mathfrak{h}}_{u,v} : S^{\mathfrak{h}}_v \to S^{\mathfrak{g}}_u$ is defined as follows:
- (γ) if $(w,g) \in S^{\mathfrak{g}}_v$ (hence $(w,g) \in S^{\mathfrak{g}}_u$) then $\pi^{\mathfrak{h}}_{u,v}(z_{(w,g)}) = z_{(w,g)}$.

Now we investigate this object

(A) if $u \le_J v$ then $\pi^{\mathfrak{h}}_{u,v}$ can be extended to one and only one homomorphism called $\hat{\pi}_{u,v}$ from $G^{\mathfrak{h}}_v$ into $G^{\mathfrak{h}}_u$.

[Why? As $\{z_{(w,g)} : (w,g) \in S_v\}$ generates G_v and $S^{\mathfrak{g}}_v \subseteq S^{\mathfrak{g}}_u$ clearly there is at most one such mapping $\hat{\pi}^{\mathfrak{h}}_{u,v}$, but to show that it is a well defined homomorphism from the group $G^{\mathfrak{h}}_v$ into the group $G^{\mathfrak{h}}_u$ it suffices to show clause (d) below]

(A) $\pi^{\mathfrak{h}}_{u,v}$ maps every equation in $\Gamma^{\mathfrak{h}}_v$ to an equation from $\Gamma^{\mathfrak{h}}_u$.

[Why clause (d) holds? First we deal with "$z_{(w_1,g_1)} = z_{(w_2,g_2)}$" $\in \Gamma_v$ as in (α) of clause (a), so the same w which witnesses the membership in Γ_v witnesses its membership in Γ_u. Second we deal with "$z_{(u_1,g_1)} \dots z_{(u_n,g_n)} = e$", as in clause ($\beta$) so by clause ($\alpha$), without loss of generality $u_\ell = w$ for $\ell = 1, \dots, n$ so $(w, g_\ell) \in S^{\mathfrak{g}}_u$ and $G_w \models$ "$g_1 \dots g_n = e$", again the same equation appears in (β) for u.]

(A) for $u_0 \leq_J u_1 \leq_J u_2$ we have $\pi^{\mathfrak{h}}_{u_0,u_2} = \pi^{\mathfrak{h}}_{u_0,u_1} \circ \pi^{\mathfrak{h}}_{u_1,u_2}$ hence $\hat{\pi}^{\mathfrak{h}}_{u_0,u_2} = \hat{\pi}^{\mathfrak{h}}_{u_0,u_1} \circ \pi^{\mathfrak{h}}_{u_1,u_2}$.

[Why? Check the definition of $G^{\mathfrak{h}}_u$ and $\pi^{\mathfrak{h}}_{u,v}$.]

(A) $\aleph_0 + \sum\limits_{u\in J} |G^{\mathfrak{h}}_u| = \aleph_0 + \sum\limits_{u\in J} |G^{\mathfrak{g}}_u|$

[Why? Check by (h) below the $\geq$ holds and directly $\leq$ holds.]

(A) inv-lim$(\mathfrak{h}) = \mathrm{inv}-\lim_{\mathrm{gr}}(\mathfrak{h})$, see Definition **??**(4).

[Why? As $\mathrm{Dom}(\pi^{\mathfrak{h}}_{u,v}) = G^{\mathfrak{h}}_v$ when $u \leq_J v$.]

(A) the mapping $\mathbf{j}_u : G^{\mathfrak{g}}_u \to G^{\mathfrak{h}}_u$ defined by $g \mapsto z_{(u,g)}$ from $G^{\mathfrak{g}}_u$ to $G^{\mathfrak{h}}_u$ is an embedding

[Why? It is a homomorphism as $G^{\mathfrak{g}}_u \models$ "$g_1g_2g_3 = e_{G_u[\mathfrak{g}]}$" implies that "$z_{(u,g_1)}z_{(u,g_2)}z_{(u,g_3)} = e$" $\in \Gamma^{\mathfrak{g}}_u$. For proving it an embedding, by the local character it is enough to consider the case J is finite, in this case "directed" means "having a member which is $\leq_J$-above any other". Call it v_* - now by clause (α) of Definition **??**(3), as $\mathfrak{g}$ is assumed to be good the set $\{z_{(v_*,g)} : g \in G^{\mathfrak{g}}_{v_*}\}$ generates $G^{\mathfrak{y}}_u$ and without loss of generality we can replace $S^{\mathfrak{g}}_v$ by $S^{\mathfrak{g}}_{v^*}$ for $v \in J$, i.e., $S^{\mathfrak{g}}_{v^*} \subseteq S^{\mathfrak{g}}_v$ and $(\forall s_1 \in S^{\mathfrak{g}}_v)(\exists s_2 \in S^{\mathfrak{g}}_{v^*})[g_{s_1} = g_{s_2} \in \Gamma_v]$. In addition the v mentioned in clause (α) and the v mentioned in clause (β) of the definition of Γ_u (see clause (a)) can be chosen as u, hence without loss of generality $J = \{u, v_*\}$. By clause (β) of the definition of "good" and the theory of free amalgamation of two groups, (that is after we divide $G^{\mathfrak{g}}_{v_*}$ by the normal subgroup which $\mathrm{Ker}(\pi_{u,v_*})$ generates) extending a third one we are done.]

(A) $\bar{\mathbf{j}} = \langle \mathbf{j}_u : u \in J\rangle$ embed $\mathfrak{g}$ into $\mathfrak{h}$, i.e.,
  - (α) $\mathbf{j}_u \in \mathrm{Hom}(G^{\mathfrak{g}}_u, G^{\mathfrak{h}}_u)$
  - (β) $u \leq_J v \Rightarrow \mathbf{j}_u \circ \pi^{\mathfrak{g}}_{u,v} = \pi^{\mathfrak{h}}_{u,v} \circ \mathbf{j}_v$

[Why? Check.]

So

(A) $\bar{\mathbf{j}}$ induces an embedding $\mathbf{j}$ of inv-lim$_{\mathrm{gr}}(\mathfrak{g})$ into inv-lim$_{\mathrm{gr}}(\mathfrak{y}) = \mathrm{inv}-\lim(\mathfrak{h})$ so $u \in J \Rightarrow \mathbf{j}_u \circ \pi^{\mathfrak{g}}_u = \pi^{\mathfrak{h}}_u \circ \mathbf{j}$

[note that if $g \in \mathrm{inv}-\lim_{g_1}(\mathfrak{g})$ then $a = \langle g_u : u \in J_{\geq w}\rangle/\approx$ where $\langle g_u : u \in J_{\geq w}\rangle$ for each $v \in J$ we can choose $u_v \in J_{\geq w}$ such that $v \leq_J u_v$ and let $g'_u = z_{(v_u,g_{u_v})} \in G^{\mathfrak{h}}_u$ (really $z'_{(u_v,g_{u_v})}$) and $\mathbf{j}(a) = \langle g'_u : u \in J\rangle/\approx$]

(B) for every $y \in G^{\mathfrak{h}}_u$ there is (v,g) such that
  - (α) $(v,g) \in S^{\mathfrak{g}}_u$
  - (β) $G^{\mathfrak{h}}_u \models$ "$y = z_{(v,g)}$"
  - (γ) if $v \neq u$ then $g \notin \mathrm{Dom}(\pi_{u,v})$.

[Why? y can be presented as a product $z_{(v_1,y_1)}, \ldots, z_{(v_n,y_n)}$ where $(v_\ell, y_\ell) \in S^{\mathfrak{g}}_u$ (for $\ell = 1,\ldots,n$) noting that $G^{\mathfrak{y}}_u \models$ "$z^{-1}_{(v,y_\ell)} = z_{(v,y^{-1}_\ell)}$". Let $w \in J$ be such that $u \leq_J w$ and $\ell \in \{1,\ldots,n\} \Rightarrow v_\ell \leq_J w$. By clause (α) of the definition **??**(3) of "$\mathfrak{g}$ is good" there are $g'_\ell \in G_w$ such that $\pi_{u_\ell,w}(g'_\ell) = g_\ell$ for $\ell = 1,\ldots,n$. So $G^{\mathfrak{y}}_u \models$ "$z_{(v_\ell,g_\ell)} = z_{(w,g'_\ell)}$" so y is the product of $\langle z_{(w,g_\ell)} : \ell = 1,\ldots,n\rangle$ hence is $z_{(w,g)}$ where $G^{\mathfrak{g}}_w \models$ "$g = g'_1 \ldots g'_n$". If $g \in \mathrm{Dom}(\pi_{u,v})$ then use $(u, \pi_{u,v}(g))$ and if $g \notin \mathrm{Dom}(\pi_{u,v})$ uses (w,g).]

(l) $\mathbf{j}$ is onto inv-lim$(\mathfrak{h})$.

[Why? Now let $\bar{y} = \langle y_u : u \in J\rangle \in \mathrm{inv}-\lim(\mathfrak{h})$, and we should prove that $\bar{y} \in \mathrm{Rang}(\mathbf{j})$, by clauses (h) + (i) above equivalently we should prove that for some $u_* \in J$ we have: $u_* \leq_J, u \in J \Rightarrow y_u \in \mathrm{Rang}(\mathbf{j}_u)$. For each $u \in J$ we can find a pair $(v_u, g_u) \in S^{\mathfrak{g}}_{u_\alpha}$ and $G^{\mathfrak{h}}_u \models$ "$z_{(v_u,g_u)} = y_u$" and it is as in clause (k).

Let $w \in J$, so $\langle (v_u, g_u) : u \in J_{\geq w}\rangle$ is as in $(*)$ of 5.44.

[Why? Clause (i) of $(*)$ there holds, as $u \le_J v_u$ and $(v_u, g_u) \in S^{\mathfrak{g}}_{v_u}$ (by (α) of (k)) and the definition of $S^{\mathfrak{g}}_u$. The main point is, assuming $w \le_J u_1 \le_J u_2$ to prove that there is v such that $u \le_J v \le_J v_{u_1} \wedge v \le_J v_{u_2} \wedge (\pi_{v,v_{u_1}}(g_{u_1}), \pi_{v,v_{u_2}}(g_{u_2})$ are well defined and equal). By clause (b) of Definition 5.40, J is directed hence there is $v^* \in J$ such that $v_{u_1} \le_J v^* \cap v_{u_2} \le_J v^*$. By clause (α) of Definition 5.43(3), there are g^*_1, g^*_2 from G_{v^*} such that $\pi_{v_{u_1},v^*}(g^*_1) = g_{u_1}$ and $\pi_{v_{u_1},v^*}(g^*_2) = g_{u_2}$. So $G^{\mathfrak{h}}_{u_2} \models$ "$z_{(v^*,g^*_2)} = z_{(v_{u_2},g_2)}$ hence as $\bar{y} \in$ inv-lim$(\mathfrak{h})$ we get $G^{\mathfrak{h}}_{u_1} \models$ "$z_{(v^*,g^*_2)} = z_{(v_{u_1},g_1)}$". As g is smooth (see Definition **??**(2)) there is $v \in J$ as required there for (v^*, g^*_2). It is also as required in **??**(3)$(\beta)(*)(ii)$.]

Hence by the conclusion of 5.44(2) when applied to $\langle (v_u, g_u) : u \in J_{\ge w} \rangle$ we get that for some $u_* \in J_{\ge w}$ we have $u \in J_{\ge u_*} \Rightarrow g_u \in \mathrm{Dom}(\pi^{\mathfrak{g}}_{v_u})$ hence by (γ) of clause (k) we have $v_u = u$. This is enough for clause (ℓ)

(m) lim-inv$(\mathfrak{g})$ is a κ-automorphism group.

[Why? By clauses $(f) + (j) + (l)$ this group is inv-lim$(\mathfrak{y})$ which (see the proof of 5.42) is a κ-automorphism.] $\square$

Claim 5.46. *If $\mathfrak{s}$ is a smooth κ-p.o.i.s. (see Definition* **??***(2),(5)) and $\mathfrak{s}$ is good (see Definition* **??***(3),(5)), <u>then</u> $G_{\mathfrak{s}}$ is isomorphic to a κ-automorphism group.*

Proof. By **??**. $\square$

Conclusion 5.47. *If $\mathfrak{s}$ is a smooth good nice κ-p.o.i.s with $\aleph_1$-directed $J^{\mathfrak{s}}$ recalling $G_{\mathfrak{s}} = \mathrm{inv} - \lim\langle G^{\mathfrak{s}}_u, \pi_{u,v} : u \le_{J^{\mathfrak{s}}} v\rangle$ we have*

- *(A) there is a structure $\mathfrak{A}$ of cardinality κ and a $(\le \kappa)$-element subgroup $H_{\mathfrak{s}}$ of the automorphism group $\mathrm{Aut}(\mathfrak{A}) \cong G_{\mathfrak{s}}$ such that $\tau'_{G,H}$, the normalizer-depth of H in $G_{\mathfrak{s}}$ is $\mathrm{rk}^{<\infty}(I^{\mathfrak{s}})$*
- *(B) there is a group G' of cardinality κ such that its automorphism tower height, $\tau_{G'}$ is $\mathrm{rk}^{\infty}(I_{\mathfrak{s}})$*
- *(C) $\tau^{\mathrm{atw}}_\kappa \ge \tau^{\mathrm{nlg}}_\kappa > \mathrm{rk}^{<\infty}(I^{\mathfrak{s}})$*

Proof. By 5.41(1), $\mathbf{j}_{\mathfrak{s}}$ is an embedding of $G_{\mathbf{p}[\mathfrak{s}]}$ into $G_{\mathfrak{s}}$, and by 5.41(3) it is onto. By **??** there is a structure $\mathfrak{A}$ of cardinality κ such that $\mathrm{Aut}(\mathfrak{A})$, the automorphism group of $\mathfrak{A}$, is isomorphic to $G_{\mathfrak{s}}$ hence by the previous sentence to $G_{\mathbf{p}[\mathfrak{s}]}$. By 5.16(4) we have $\tau^{\mathrm{nlg}}_{G_{\mathbf{p}[\mathfrak{s}]},H_{\mathbf{p}[\mathfrak{s}]}} = \mathrm{rk}^{<\infty}(I_{\mathfrak{s}})$ and $H_{\mathbf{p}[\mathfrak{s}]}$ is a subgroup of $G_{\mathbf{p}[\mathfrak{s}]}$ with $\le \kappa$ elements hence we have $\tau^{\mathrm{nlg}}_\kappa > \mathrm{rk}^{<\infty}(I_{\mathfrak{s}})$ (recalling Definition 0.4(3)) hence by 0.6 we get also $\tau^{\mathrm{atw}}_\kappa \ge \tau^{\mathrm{nlg}}_\kappa \ge \mathrm{rk}^{<\infty}(I_{\mathfrak{s}})$.

$\square_{5.47}$ $\square$

Claim 5.48. *If $\kappa = \kappa^{\aleph_0}$ <u>then</u>*

- *(A) there is a good smooth nice κ-p.o.i.s. $\mathfrak{s}$ with $\mathrm{rk}^{<\infty}(I_{\mathfrak{s}}) \ge \kappa^+$ hence*
- *(B) $\tau^{\mathrm{atw}}_\kappa \ge \tau^{\mathrm{nlg}}_\kappa > \kappa^+$.*

Proof. Clause (b) follows from clause (a) by 5.47. For proving clause (a), for each $u \in [\kappa]^{\le\aleph_0}$ we define the partial order $(I_u, <_{I_u})$ as follows (the n is to enable us to quote §1, i.e., to simplify §1)

$\circledast_1$ for $u \in [\kappa]^{\le\kappa}$ we define I_u by

(i) $I_u = \{t : t$ is a triple $(<_t, \alpha^t, \varepsilon^t)$ such that $<_t$ is a well ordering of u and $\alpha^t \in u$ and $\varepsilon^t < \kappa\}$

(ii) $t_1 <_{I_u} t_2$ iff $<_{t_1} = <_{t_2}$ and $\alpha^{t_1} <_{t_1} \alpha^{t_2}$.

For $u \subseteq v \in [\kappa]^{\le \aleph_0}$ let $\pi_{u,v} : I_v \to I_u$ be defined as follows: $\pi_{u,v}(t_2) = t_1$ iff $(<_{t_1} = <_{t_2} \restriction u)$ and $\alpha^{t_1} = \alpha^{t_2}$ and $\varepsilon^{t_1} = \varepsilon^{t_2}$. For $u \subseteq \kappa$ let $W^+_u = \{(\varrho, \nu, \bar\varepsilon) : \varrho \in {}^{[0,n]}u$ and $\nu \in {}^{[1,n]}2$ for some n and $\bar\varepsilon$ is a finite sequence of ordinals $\in u$ of length $\ell g(\varrho)$[[second nec.22]]$\}$ and $\mathscr{H}^*_u = \{h : h$ is a function from some finite subset of W^+_u into $\{0,1\}\}$. Clearly $|W^+_u| = |\mathscr{H}^*_u| = |u| + \aleph_0$ when $u \ne \varnothing$. Let $\mathbf{c}$ be a one-to-one function from $\mathscr{H}^*_\kappa$ onto κ.

Let $\langle f_\varepsilon : \varepsilon < \kappa\rangle$ be a sequence of functions from κ to $\{0,1\}$ such that for every finite function f from κ to $\{0,1\}$, for κ ordinals ε the function f_ε extends f.

Let $\langle h_\varepsilon : \varepsilon < \kappa\rangle$ be such that h_ε is the unique member of $\mathscr{H}^*_\kappa$ such that $\mathbf{c}(h) = \varepsilon$.

Let J be $\{u : u \subseteq \kappa$ is countable infinite such that $w \subseteq u$ and u is closed under $\mathbf{c}$, i.e., if $h \in \mathscr{H}^*_u$ then $\mathbf{c}(h) \in u\}$ ordered by $\subseteq$; clearly J is a cofinal subset of $([\kappa]^{\le\aleph_0}, \subseteq)$.

For $u \in J$ we define $\mathbf{p}_u = ((I_u, <_u), \bar A_u, Z_u, Y_u)$ as follows

$\circledast_3$ (a) $(I_u, <_u)$ is as defined above

(b) $\bar A_u = \langle A^u_x : x \in I_u\rangle$

where

(c) $A^u_x = \{\zeta \in u$: some y is a witness for $(x, h)\} \cup \{\zeta < \kappa$: no y witness (x, ζ) and $f_{\varepsilon^{t(x)}}(\zeta) = 1\}$, [[second, necessary?]]

where: y witness (x, ζ) means that $y = (\varrho, \nu, \zeta) \in \mathrm{Dom}(h)$ and $1 = h_\zeta(y)$ and x satisfies y which means that $\ell g(\bar\zeta) = n(x) + 1$, $\nu = \eta^x, \ell g(\varrho) = n(x) + 1$ and for each $\ell \le n(*)$ we have $\varrho(\ell) = \alpha^{t_\ell(x)}$ and $\zeta_\ell = \varepsilon^{t_\ell(x)}$

(d) $Z_u = u$

(e) $Y_u = \{\mathbf{c}(h) : h \in \mathrm{Dom}(\mathbf{c})$ and for some $y \in \mathrm{Dom}(h)$ we have $1 = h(y)$ and $y = (\varrho, \nu, \bar m) \in \mathrm{Dom}(h)$ satisfying $\mathrm{Rang}(\nu) \not\subseteq \{1\} \vee \varrho(\ell g(\varrho) - 1) = 0\}$

$\circledast_4$ for $u \le_J v$ we define $\pi_{u,v}$ as follows: $\pi_{u,v}(t_2) = t_2$ iff ($t_1 \in I_u, t_2 \in I_v$ and) $<_{t_1} = <_{t_1} \restriction u$ and $\varepsilon^{t_1} = \varepsilon^{t_2}$ so $\mathrm{Dom}(\pi_{u,v}) = \{t_2 \in I_v : \alpha^{t_2} \in v\}$.

Now

$\boxtimes_1$ $\mathfrak{s} =: (J, \langle \mathbf{p}_u : u \in J\rangle, \langle \pi_{u,v} : u \le_J v\rangle)$ is a κ-p.o.i.s.
[Why? Check.]

$\boxtimes_2$ if $u \le_J v$ then $\pi_{u,v}$ is a strict homomorphism from $\mathrm{Dom}(\pi_{u,v}) \subseteq I_v$ onto I_u
[Why? Check.]

$\boxtimes_3$ $\mathfrak{s}$ is smooth
[Why? See Definition **??**(2), so let $v \in I$ and $g \in G_v$ be given so let $G_v \models$ "$g = g_{x_1} \dots g_{x_n}$" where $x_1, \dots, x_n \in X^+_{I[v]}$. Let $T = \{t^{x_\ell}_m : \ell \in \{1, \dots, n\}$ and $m \le n(x_\ell)\}$, this is a finite subset of I_v and let $w =$ the closure under $\mathbf{c}$ of $\{0\} \cup \{\alpha^t : t \in T\}$. It is as required.]

$\boxtimes_4$ $\mathfrak{s}$ is nice (see Definition ??(3) + 5.40(8))

[Why? We should check Definition ??(3), clause (a)-(c), for $\mathbf{p}^* = \mathbf{p}_{\mathfrak{s}}$.

The partial order $I_{\mathfrak{s}}$ is non-trivial:

This is because by ??(3) as it is explicitly non-trivial (by the third coordinate in members of I_u). Clause (a) of Definition ??(3): $\mathbf{p}_{\mathfrak{s}}$ is a κ-parameter:

Why? By $\boxtimes_1$. Clause (b) of Definition ??(3):

Assume $x \in X_{\mathbf{p}}$ and $\mathrm{rk}^2_{\mathbf{p}}(x) = 0$ and we have to prove that $A^{\mathbf{p}[\mathfrak{s}]}_x \subseteq Y$. As $\mathrm{rk}^2_{\mathbf{p}}(x) = 0$, one of the following cases occurs: $0 \in \mathrm{Rang}(\eta^x)$ or $\mathrm{rk}_{I[\mathbf{p}]}(t(x)) = 0$ which means that $\neg(\exists s)(s <_{I[\mathbf{p}]} t(x))$. In the first case the inclusion holds by the definition of Y. In the second case we use our demand $u \in J \Rightarrow 0 \in u$, to show: letting $t_\ell(x) = \langle t^\ell_n : u \in J_{\geq w_\ell}\rangle / \approx$, for $\ell \leq n(x)$, without loss of generality $w_\ell = w$ and so $w \leq_J v_1 \leq_J u \Rightarrow ((\alpha^{(t^{n(x)})} = 0) \equiv (\alpha^{(t^{n(x)}_u)} = 0)) \Rightarrow ((\mathrm{rk}_{I[v]}(t^{n(x)}_v) = 0) \equiv ((\mathrm{rk}_{I[u]}(t^{n(x)}_u) = 0)$ hence $\mathrm{rk}_{I[\mathbf{p}]}(t(x)) = 0 \Leftrightarrow (\forall u)(w \leq_J u \Rightarrow \mathrm{rk}_{I[u]}(t^{n(x)}_u) = 0)$ hence $\mathrm{rk}_{\mathbf{p}}(x) = 0 \Rightarrow (\forall u)(w \leq_J u \Rightarrow A^{\mathbf{p}_u}_{\pi^+_{u,\mathfrak{s}(x)}} \subseteq Y^{\mathbf{p}_u}) \Rightarrow A^{\mathbf{p}}_x \subseteq Y$.

Clause (c) of Definition ??(3):

If $k < \omega$ and $x_0, \dots, x_k \in X_{\mathbf{p}}$ are with no repetitions and $\mathrm{rk}^2_{\mathbf{p}}(x_0) > 0$ then $A_{x_0} \nsubseteq \cup\{A_{x_\ell} : \ell = 1, \dots, k\} \cup Y$.

Easy by our choices.]

$\boxtimes_5$ $\mathfrak{s}$ is very nice.

[Why? In Definition ??(4) we have to check clauses (d),(e). We use here the freedom in choosing $\varepsilon^{t(x)}$ and ε^s for (d),(e) respectively. DETAILS?]

$\boxtimes_6$ $\mathfrak{s}$ is good (see Definition ??(3),(5))

[Why? Clause (α) of the definition of good holds by $\boxtimes_2$.

Why Clause (β) of the Definition of good holds? Assume that $\langle (v_u, g_u) : u \in J_{\geq w}\rangle$ is as there. We choose $u_n \in J_{\geq w}$ by induction on n such that $v_{u_n} \subset u_{n+1}$ (hence $u_n \subset u_{n+1}$) and $g_{u_{n+1}} \notin \mathrm{Dom}(\pi^{\mathfrak{s}}_{u_{n+1}, v_{u_{n+1}}})$ and let $u_\omega = \bigcup\{u_n : n < \omega\}$. Now for each n as $\pi^{\mathfrak{s}}_{u_n, v_{u_\omega}}(g_{u_\omega}) = \pi^{\mathfrak{s}}_{u_n, u_{n+1}}(g_{u_n})$ we have $\pi^{\mathfrak{s}}_{u_n, v_{u_\omega}}(g_{u_\omega})$ is well defined. So $g_{u_\omega} \in \bigcap\{\mathrm{Dom}(\pi^{\mathfrak{s}}_{u_n, v_{u_\omega}}) : n < \omega\}$ but easily this is equal to $\mathrm{Dom}(\pi^{\mathfrak{s}}_{u_\omega, v_{u_\omega}})$ but this implies that for some $n < \omega, m \in [n, \omega) \Rightarrow v_{u_m} = u_m$, contradiction.

Lastly, Clause (β) of the Definition of good holds by ??.]

How does the partial order $I_{\mathfrak{s}} = \mathrm{inv} - \lim(\mathfrak{s})$ look like? essentially as the disjoint sum of the well orders of κ. So any ordinal $\alpha \in [\kappa, \kappa^+)$ occurs as order type so $\mathrm{rk}^{<\infty}(I) = \kappa^+$. □

Remark 5.49. Of course, we can replace κ^+ by some higher ordinals $< \kappa^{++}$; the family of such ordinals is closed, e.g., under products and under sums of $\leq \kappa$ ordinals; but of doubtful interest.

§ 6. Less good inverse limits

We may think of the partial order $(\prod_{\alpha<\theta} f(\alpha), <_{J^{\mathrm{bd}}_\theta})$, where $f : \theta \to \mathrm{Ord}$.

It is close to being the inverse limit of the $\langle I_\alpha = (f(\alpha), <) : \alpha < \theta\rangle$ but only "in the long run (in θ)". To deal with this we deal with the case the $\pi_{u,v}$-s are not strict homomorphisms (i.e., preserving $<$), but the order on the inverse limit is determined by what occurs "late enough". We use this to prove that $\tau_\kappa > 2^\kappa$ for many cardinals κ (e.g. any strong limit singular κ).

We get here a better lower bound to τ'_κ.

For this we have to redo §1 + §2 with various changes, in particular slightly changing the definition of $X_I, X_{\mathbf{p}}$. We formulate our main result and then state the changes in the earlier definitions, claims and proofs.

Claim 6.1. *Assume*

- *(A)* $\aleph_0 < \theta = \mathrm{cf}(\theta) \le \kappa$
- *(B)* $\mathcal{T}_\alpha \subseteq {}^\alpha\kappa$ *for* $\alpha < \theta$ *has cardinality* $\le \kappa$
- *(C)* $\mathcal{F} = \mathcal{T}_\kappa = \{f \in {}^\theta\kappa : f \restriction \alpha \in \mathcal{T}_\alpha$ *for every* $\alpha < \theta\}$
- *(D)* $\gamma = \mathrm{rk}(\mathcal{F}, <_{J^{\mathrm{bd}}_\theta})$, *necessarily* $< \infty$ *so* $< (\kappa^\theta)^+$.

Then

- (α) *there is a good smooth very nice* κ-p.o.w.i.s. $\mathfrak{s}$ *(see Definition 6.10 below) with* $\mathrm{rk}(I_{\mathfrak{s}}) = \gamma$ *(see Definition 6.10(1))*
- (β) *in* $(\alpha), G_{\mathbf{p}[\mathfrak{s}]}$ *is a* κ*-automorphism group with the subgroup* $H_{\mathbf{p}[\mathfrak{s}]}$, *a* κ*-element subgroup, satisfying* $\tau'_{G_{\mathbf{p}[\mathfrak{s}]},H_{\mathbf{p}[\mathfrak{s}]}} = \gamma$ *and* $\mathrm{nor}^{<\infty}_{G_{\mathbf{p}[\mathfrak{s}]}}(H_{\mathbf{p}[\mathfrak{s}]}) = G_{\mathbf{p}[\mathfrak{s}]}$
- (γ) $\tau_\kappa \ge \tau'_\kappa > \gamma$.

Below we redefine $\mathbf{p} \le \mathbf{q}$

Definition 6.2. 1) π is a partial function from the κ-parameter $\mathbf{p}_2$ to the κ-parameter $\mathbf{p}_1$ if:

- (A) π is a function
- (B) $\mathrm{Dom}(\pi) \subseteq I^{\mathbf{p}_2} \cup Z^{\mathbf{p}_2}$
- (C) π maps $I^{\mathbf{p}_2} \cap \mathrm{Dom}(\pi)$ into $I^{\mathbf{p}_1}$
- (D) π maps $Y^{\mathbf{p}_2} \cap \mathrm{Dom}(\pi)$ into $Y^{\mathbf{p}_1}$
- (E) π maps $Z^{\mathbf{p}_2} \setminus Y^{\mathbf{p}_2}$ into $Z^{\mathbf{p}_1} \setminus Y^{\mathbf{p}_1}$.

2) For κ-parameters $\mathbf{p}, \mathbf{q}$ let $\mathbf{p} \le \mathbf{q}$ mean that $\mathrm{id}_{X_{\mathbf{p}}} \cup \mathrm{id}_{Z^{\mathbf{p}}}$ is a partial mapping from $\mathbf{p}$ to $\mathbf{q}$.

3) If $\mathrm{Dom}(\pi) \subseteq I^{\mathbf{p}_2}$ we use π for $\pi \cup \mathrm{id}_{Z^{\mathbf{p}_1}}$ (so we assume $Z^{\mathbf{p}_1} \subseteq Z^{\mathbf{p}_2}$ and $Y^{\mathbf{p}_1} = Y^{\mathbf{p}_2} \cap Z^{\mathbf{p}_1}$).

Remark 6.3. 1) We are mainly interested in cases then in that $\mathrm{rk}_{I_{[\mathbf{p}_1]}}(t) = n < \omega \Rightarrow \mathrm{rk}_{I[\mathbf{p}_2]}(\pi(t)) = n$.

Definition 6.4. [Replacing Definition 5.35] 1) If π is a partial function from a partial order I_2 into a partial order I_1, we define the mapping π^+ (really $\pi^+_{I_1,I_2}$) as follows:

- (A) π^+ is a partial mapping from X_{I_2} into X_{I_1} (note that even if $\mathrm{Dom}(\pi) = I_2$, still $\mathrm{Dom}(\pi^+)$ may be a proper subset of X_{I_2})
- (B) for $x \in X_{I_2}$
  - (α) $x \in \mathrm{Dom}(\pi^+)$ <u>iff</u> $x \in X_{I_2}, \{t_0(x), \dots, t_{n(x)}(x)\} \subseteq \mathrm{Dom}(\pi)$

    and $(\langle \pi(t_0(x), \dots, \pi(t_{n(x)}(x)) \rangle, \eta^x)$ belongs to X_{I_1}
  - (β) $\pi^+(x) = (\langle \pi(t_0(x), \dots, \pi(t_{n(x)}(x)) \rangle, \eta^x)$

1A) We say that π is a partial mapping from $\mathbf{p}_2$ into $\mathbf{p}$, if

$(a) - (e)$ is as in Definition **??**

- (A) if $x \in \mathrm{Dom}(\pi^+)$ then $\alpha \in \mathrm{Dom}(\pi) \cap Z^{\mathbf{p}_2} \Rightarrow \alpha \in A^{\mathbf{p}_2}_x \Leftrightarrow \pi(\alpha) \in A^{\mathbf{p}_1}_{\pi^+(x)}$.

2) For π a partial mapping from $\mathbf{p}_2$ to $\mathbf{p}_1$ (both are κ-parameters) we define

(A) π^+ or really $\pi^+_{\mathbf{p}_1,\mathbf{p}_2}$ is the following function

(a) π^+ is a partial mapping from $X^+_{\mathbf{p}_2}$ to $X^+_{\mathbf{p}_1}$

(b) for $x \in X_{\mathbf{p}_2}$ we behave as in (b) of part (1) (so $x \in X_{\mathbf{p}_1}, \pi^+(x) \in X_{\mathbf{p}_2}$)

(c) if $a \in Z^{\mathbf{p}_2}, m < 2$ <u>then</u>: $\pi^+((a,m))$ is well defined iff $a \in \mathrm{Dom}(\pi)$ and then its value is $(\pi(a), m)$

(B) $\hat{\pi}$ or really $\hat{\pi}_{\mathbf{p}_1,\mathbf{p}_2}$ is the partial homomorphism from $F^+_{\mathbf{p}_2}$ into $F^+_{\mathbf{p}_1}$ with domain the subgroup of $F^+_{\mathbf{p}_2}$ generated by $\{g_x : x \in \mathrm{Dom}(\pi^+)\}$ mapping g_x to $g_{\pi^+(x)} \in F_{\mathbf{p}_1}$; see justification below.

Remark 6.5. Note that the parts of Definition 6.20 (and claim 6.7) while not actually used, they serve as a warm-up for their variants which will be used. The difference is in 6.10(2), the motivation is, at least, in the case J is linear to have commutations.

Claim 6.6. *In Definition 6.20(2), if π is a partial mapping from $\mathbf{p}_1$ to $\mathbf{p}_2$ <u>then</u>:*

(A) π^+ is a well defined partial mapping from $X^+_{\mathbf{p}_1}$ into $X^+_{\mathbf{p}_2}$

(B) if $\pi^+(x_1) = x_2$ then $(x_1 \in X_{\mathbf{p}_1} \Leftrightarrow x_2 \in X_{\mathbf{p}_2})$ and $(x_1 \in X^+_{\mathbf{p}_1} \setminus X_{\mathbf{p}_1}) \Leftrightarrow (x_2 \in X^+_{\mathbf{p}_2} \setminus X_{\mathbf{p}_2})$.

Proof. Check. $\square$

Claim 6.7. *1) In Definition 6.20(1), $\pi^+ = \pi^+_{I_1,I_2}$ and in Definition 6.20(2), $\pi^+_{\mathbf{p}_1,\mathbf{p}_2}$ and $\hat{\pi}_{\mathbf{p}_1,\mathbf{p}_2}$ are well defined, in particular, $\hat{\pi}$ is really a partial homomorphism from $F^+_{\mathbf{p}_2}$ into $F^+_{\mathbf{p}_1}$. (Compare with **??**)*
2) If I_1, I_2, I_3 are partial orders and π_ℓ is a partial mapping from $I_{\ell+1}$ into I_ℓ for $\ell = 1,2$ and $\pi = \pi_2 \circ \pi_1$ <u>then</u> $\pi^+ \supseteq \pi^+_1 \circ \pi^+_2$.
3) If $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3$ are parameters and π_ℓ is a partial isomorphism from $\mathbf{p}_{\ell+1}$ into $\mathbf{p}_\ell$ for $\ell = 1,2$ and $\pi = \pi_2 \circ \pi_1$ <u>then</u> $\pi^+_{\mathbf{p}_1,\mathbf{p}_3} \supseteq \pi^+_{\mathbf{p}_1,\mathbf{p}_2} \circ \pi^+_{\mathbf{p}_2,\mathbf{p}_3}$ and $\hat{\pi}_{\mathbf{p}_1,\mathbf{p}_3} = \hat{\pi}_{\mathbf{p}_1,\mathbf{p}_2} \circ \hat{\pi}_{\mathbf{p}_2,\mathbf{p}_3}$.

Remark 6.8. 1) In 6.7(2) possibly $\pi^+ \supset \pi^+_2 \circ \pi^+_1$ and $\pi = \pi_2 \circ \pi_1$ even when π_ℓ is one to one from I_ℓ onto $I_{\ell+1}$ for $\ell = 1,2$.
2) If $\pi'_{\ell,m} : \mathbf{p}_\ell \to \mathbf{p}_m$ for $(\ell,m) \in \{(1,2),(1,3),(2,4),(3,4)\}$ and the diagram commute, this does not necessarily hold for the $\hat{\pi}_{(\ell,m)}$-s.

Proof. 1) The main point is why $\pi_{\mathbf{p}_1,\mathbf{p}_2}$ is a homomorphism.

Let $Z_1 = \mathrm{Dom}(\pi \restriction Z^{\mathbf{p}_1}), Z_2 = \mathrm{Rang}(\pi \restriction Z^{\mathbf{p}_1}), X_1 = \mathrm{Dom}(\pi^+ \restriction X_{\mathbf{p}_2})$ and $X_2 = \mathrm{Rang}(\pi^+ \restriction X_{\mathbf{p}_2})$, so clearly for $\ell = 1,2$

$(*)_\ell$ (a) if $x, y \in X_{I_\ell}, \bar{t}^x = \bar{t}^y$ and $x \in X^\ell$ then $y \in X_\ell$

(b) if $x \in X_\ell$ and $n < n(x)$ then $x \restriction n \in X_\ell$

hence

$\boxdot_\ell$ (a) if $g_x \bar{g}_{y_1} g_x^{-1} = g_{y_2}$ is one of the equations of $\Gamma^*_{\mathbf{p}_\ell}$ then

$$y_1 \in X_\ell \Leftrightarrow y_2 \in X_\ell$$

(b) $G_{\mathbf{p}_\ell,X_\ell,Z_\ell}$ is a subgroup of $F_{\mathbf{p}_\ell}$ generated by

$$\{g_y : y \in X_\ell \cup (Z_\ell \times 2)\} \text{ freely except the equations from } \Gamma^*_{\mathbf{p},X_\ell,Z_\ell}.$$

$\otimes$ (a) $\pi \restriction X_1$ is a mapping from X_1 onto X_2

(b) $\pi \restriction Z_1$ is a mapping from Z_1 onto Z_2

(c) π maps the set of equations $\Gamma^*_{\mathbf{p}_1,X_1,Z_1}$ onto the set of equations $\Gamma^*_{\mathbf{p}_2,X_2,Z_2}$.

Hence $\hat{\pi}_{\mathbf{p}_1,\mathbf{p}_2}$ is a well defined homomorphism from $F_{\mathbf{p}_1,X_1,Z_1}$ onto $F_{\mathbf{p}_2,X_2,Z_2}$ as required.

2),3) Check. $\square_{6.7}$ $\square$

Remark 6.9. Below we are mainly interested in the case J is linear.

Definition 6.10. 1) $\mathfrak{s}$ is a κ-p.o.w.i.s. (partial order weak inverse system) <u>when</u>: in Definition 5.40 we replace (f) by $(f)'$ below, i.e., (retaining clauses (a)-(e) and (g))

(A) $\mathfrak{s} = (J, \bar{\mathbf{p}}, \bar{\pi})$ so $J = J^{\mathfrak{s}} = J[\mathfrak{s}], \bar{p} = \bar{p}^{\mathfrak{s}}, \bar{\pi} = \bar{\pi}^{\mathfrak{s}}$

(B) J is a directed partial order of cardinality $\leq \kappa$

(C) $\bar{\mathbf{p}} = \langle \mathbf{p}_u : u \in J \rangle$

(D) $\mathbf{p}_u$ is a κ-parameter, $I_u = I^{\mathbf{p}}_u$ is a partial order of cardinality $\leq \kappa$ and let $I^{\mathfrak{s}}_u = I^{\mathbf{p}^{\mathfrak{s}}_u}, X^{\mathfrak{s}}_u = X_{\mathbf{p}^{\mathfrak{s}}_u}, Z^{\mathfrak{s}}_u = Z^{\mathbf{p}^{\mathfrak{s}}_u}$

(E) $\bar{\pi} = \langle \pi_{u,v} : u \leq_J v \rangle$

$(f)'$ $\pi_{u,v}$ is a partial mapping from I_v into I_u (so we assume $u \leq_J v \Rightarrow Z^{\mathbf{p}_u} \subseteq Z^{\mathbf{p}_v}$ and use $\mathrm{id}_{Z^{\mathbf{p}_u}} \cup \pi_{u,v}$)

(F) if $u \leq_J v \leq_J w$ then $\pi_{u,w} = \pi_{u,v} \circ \pi_{v,w}$ (may use $\subseteq$).

2) In Definition 6.10(1) we define $\pi^+_{u,v} = \pi^{+,\mathfrak{s}}_{u,v}$ (when $u \leq_{J[\mathfrak{s}]} v$) <u>not</u> by the general definition of 6.20 but as follows:

(A) $\pi^+_{u,v}$ is a partial mapping from $X^+_{\mathbf{p}_v}$ into $X^+_{\mathbf{p}_u}$

(B) for $x \in X^+_{\mathbf{p}_v}$,

(α) $x \in \mathrm{Dom}(\pi^+_{u,v})$ <u>iff</u>: for every w satisfying $u \leq_{J[\mathfrak{s}]} w \leq_{J[\mathfrak{s}]} v$ and $\ell < n(x)$ we have $[\pi_{w,v}(t_{\ell+1}(x)) <_{I_w} \pi_{w,v}(t_\ell(x))]$

(β) $\pi^+_{u,v}(x) = (\langle \pi_{u,v}(t_0(x), \ldots, \pi_{u,v}(t_{n(x)}(x)) \rangle, \eta^x)$

3) If $u \leq_{J[\mathfrak{s}]} v$, <u>then</u> $\check{\pi}_{u,v} = \check{\pi}^{\mathfrak{s}}_{u,v}$ is the partial homomorphism from $F_{\mathbf{p}_2}$ into $F_{\mathbf{p}_1}$ with domain the subgroup of $F^+_{\mathbf{p}_2}$ generated by $\{g_x : x \in \mathrm{Dom}(\pi^+_{u,v})\}$ mapping g_x to $g_{\pi^+_{u,v}(x)} \in F_{\mathbf{p}_1}$; see justification below.

4) We say $\mathfrak{s}$ is linear <u>if</u> $J^{\mathfrak{s}}$ is a linear (= total) order.

5) We say $\mathfrak{s}$ is nice when every $p^{\mathfrak{s}}_u$ is nice.

Claim 6.11. *If $\mathfrak{s}$ is a κ-p.o.w.i.s and $J^{\mathfrak{s}} \models$ "$v \leq u \leq w$" <u>then</u>*

(A) $\check{\pi}^{\mathfrak{s}}_{u,v}$ are well defined (homomorphisms)

(B) $\pi^+_{w,v} \subseteq \pi^+_{w,u} \circ \pi^+_{u,v}$ and $\check{\pi}_{w,v} \subseteq \check{\pi}_{w,u} \circ \check{\pi}_{u,v}$

(C) if $J^{\mathfrak{s}}$ is a linear order then in clause (b) we get equalities.

Proof. Clause (a): Similar to 5.42(2); it is enough to prove this for $\breve{\pi}$, for this it suffices to show that $\breve{\pi}$ maps the equations in $\Gamma^+_{I_1}$ into $\Gamma^+_{I_2}$ and this is proved as in the proof of clause (A) in the proof of 5.42(2).
Clause (b): Easy.
Clause (c): Easy, in fact we have chosen Definition 6.10(2)(b) such that those equalities will hold. $\square$

We now repeat Definition 5.40(1A)-(7).

Definition 6.12. Let $\mathfrak{s}$ be a κ-p.o.w.i.s.
1) Let $\mathfrak{s} = (J^{\mathfrak{s}}, \bar{\mathbf{p}}^{\mathfrak{s}}, \bar{\pi}^{\mathfrak{s}}), \mathbf{p}^{\mathfrak{s}} = \langle \mathbf{p}^{\mathfrak{s}}_u : u \in J^{\mathfrak{s}} \rangle, \bar{\pi}^{\mathfrak{s}} = \langle \pi^{\mathfrak{s}}_{u,v} : u \leq_J v \rangle, J^{\mathfrak{s}} = \bar{J}[\mathfrak{s}], I^{\mathfrak{s}}_u = I[\mathbf{p}^{\mathfrak{s}}_u]$ and $F^{\mathfrak{s}}_u = F_{\mathbf{p}^{\mathfrak{s}}_u}$.
2) We define $I^+ = I^+[\mathfrak{s}] = \text{inv} - \lim_{\text{or}}(\mathfrak{s})$, a partial order (easy to check) as follows:

(A) $\bar{t} \in \text{inv} - \lim_{\text{or}}(\mathfrak{s})$ iff

  (α) $\bar{t}$ has the form $\langle t_u : u \in J_{\geq w} \rangle$ for some $w \in J$ where $J_{\geq w} = \{v \in J : w \leq_J v\}$ and $u \in J_{\geq w} \Rightarrow t_u \in I_u$ and let $w[\bar{t}] = w$

  (β) if $u_1 \leq_J u_2$ are in $J_{\geq w}$ then $\pi_{u_1,u_2}(t_{u_2}) = t_{u_1}$

(B) for $\bar{t}, \bar{s} \in \text{inv} - \lim_{\text{or}}(\mathfrak{s})$ let $\bar{s} <_{I^+} \bar{t}$ iff there is $w \in J$ such that $w[\bar{s}] \leq_J w \wedge w[\bar{t}] \leq_J w \wedge (\forall u)(w \leq_J u \Rightarrow s_u <_{I_u} t_u)$.

3) Let $I = I_{\mathfrak{s}} = I[\mathfrak{s}] = \text{inv} - \lim_{\text{or}}(\mathfrak{s})$ be $I^+/\approx$ where $\approx$ is the following two place relation on $I^+ : \bar{s} \approx \bar{t}$ iff for some $w \in J$ we have

$$w[\bar{s}] \leq_I w \wedge w[\bar{t}] \leq_J w \wedge (\forall u)(u \leq_J u \Rightarrow s_u = t_u)$$

clearly $\bar{s} \approx \bar{s}' \wedge \bar{t} \cong t' \Rightarrow (\bar{s} <_{I^+} \bar{t} \Leftrightarrow \bar{s}' <_{I^+} \bar{t}')$ and $\bar{s} \leq_{I^+} \bar{t}$ and $\neg(\bar{s} \approx \bar{t}) \Rightarrow \bar{s} <_{I^+} \bar{t}$.

3A) We define $\mathbf{p} = \mathbf{p}[\mathfrak{s}] = \text{inv} - \lim(\mathfrak{s})$ in $(\mathbf{p}, \bar{A}, Z, Y)$ where

(A) $I = \text{inv} - \lim_{\text{or}}(\mathfrak{s})$

(B) $\bar{A} = \langle A_{\bar{s}/\approx} : \bar{s}/\approx \rangle$ belongs to inv-$\lim_{\text{or}}(\mathfrak{s})\rangle$ and $A_{\bar{s}/\approx} = \cup\{A_{s_\kappa} : u \in w[\bar{s}]\}$

(C) $Z = \cup\{Z^{\mathbf{p}_u} : u \in J\}$ and $Y = \cup\{Y^{\mathbf{p}_u} : u \in J\}$.

4) We define $\pi^{\mathfrak{s}}_u$ for $u \in I$, a partial map from $I = \text{inv} - \lim_{\text{or}}(\mathfrak{s})$ to I_u by $\pi_{u,\mathfrak{s}}(\bar{t}/\approx) = \bar{s}/\approx$ iff $\bar{t} \in I^+, u \in J$ and $(\exists \bar{s})(\bar{s} \approx \bar{t} \wedge s_u = s)$; it is well defined.
5) We define $F^+_{\mathfrak{s}}$, a set and $F_{\mathfrak{s}}$, a group, (where $F^{\mathfrak{s}}_u = F_{\mathbf{p}_u}[\mathfrak{s}]$ is as defined in Definition 5.9(1)) [2]

(A) $F^+_{\mathfrak{s}} = \text{inv} - \lim_{\text{gr}} \langle F_{\mathbf{p}_u}, \breve{\pi}_{u,v} : u \leq_J v \text{ in } J \rangle$

that is, $F^+_{\mathfrak{s}}$ is (just) the set of $\bar{g}$ of the form $\langle g_u : u \in J_{\geq w} \rangle$ such that $w \in J, g_u \in F_u$ and $\breve{\pi}_{u,v}(g_v) = g_u$ when $w \leq_J u \leq_J v$

(A) $\approx$ is defined on $F^+_{\mathfrak{s}}$ as in part (3)

(B) $F_{\mathfrak{s}} = \text{inv} - \lim_{\text{gr}} \langle F_{\mathbf{p}_u}, \breve{\pi}_{u,v} : u \leq_J v \text{ in } J \rangle$ is the inverse limit of the groups defined similarly,

(C) $\breve{\pi}^{\mathfrak{s}}_u$ is the partial homomorphism from the group $F_{\mathfrak{s}}$ (i.e., from a subgroup) into $F^{\mathfrak{s}}_u$ defined by $\pi^{\mathfrak{s}}_u(\bar{g}) = g'_u$ if $\bar{g} \approx \bar{g}' \wedge u \in J_{\geq w[\bar{g}']}$.

So for $\bar{g} \in F^+_{\mathfrak{s}}$ we have $\bar{g} = \langle g_u : u \in J_{\geq w[\bar{g}]} \rangle$.
6) Let $H^+_{\mathfrak{s}}$ be $\cup\{H_{\mathbf{p}_u} : u \in I\}$.
7) We naturally define $\mathbf{j} = \mathbf{j}_{\mathfrak{s}} = \mathbf{j}[\mathfrak{s}]$, an embedding of $F_{\mathbf{p}[\mathfrak{s}]}$ into $F_{\mathfrak{s}}$ as follows:

(A) $\mathbf{j}(g_y) = \langle g_y : u \in J \geq v \rangle/\approx$ if $v \in J, y \in X^+_{\mathbf{p}_v} \setminus X_{\mathbf{p}_v}$

[2] by this definition there may be no maximal member in $\bar{t}/\approx$, but any two members are compatible functions, so if we replace $J_{\geq w}$ by upward closed non-empty sets we have a maximal member

(B) if $x \in Z^{\mathbf{p}[\mathfrak{s}]}$ let $t_\ell(x) = \langle t_{\ell,u} : u \in J_{\geq w_{1,\ell}}\rangle/\approx$ for $\ell = 0,\dots,n(x)$ where $t_{\ell,u} \in I_u$ and let $w \in J$ be a common upper bound of $\{w_{1,0},\dots,w_{1,n(x)}\} \cup \{w_{2,1},\dots,w_{2,n(*)}\}$ and we let $x_u = (\langle t_{\ell,u} : \ell \leq n(*)\rangle, \eta^x)$ for $u \in J_{\geq w}$ <u>then</u>

$$\mathbf{j}(g_x) = \langle g_{x_u} : u \in J_{\geq w}\rangle/\approx .$$

Claim 6.13. *1) Those inverse limits are well defined, in particular: if we define* $\mathfrak{t}$ *by* $J_\mathfrak{t} = J \cup \{\mathfrak{s}\}$, *(so* $I^\mathfrak{t}_u = I^\mathfrak{s}_u$ *if* $u \in J$ *and is* $I_\mathfrak{s}$ *if* $u = \mathfrak{s}$; $\pi^\mathfrak{t}_{u,v}$ *is* $\pi^\mathfrak{s}_{u,v}$ *if* $v \in J$ *and is* $\pi_{u,\mathfrak{s}}$ *if* $u \in J^\mathfrak{s}$ *and is* id_{I_u} *if* $u = v \in J^\mathfrak{t} \setminus J^\mathfrak{s}$*)* <u>*then*</u>

(α) $\mathfrak{t}$ *is a* p.o.w.i.s

(β) $H_{\mathbf{p}[\mathfrak{s}]} = \cup\{H_{\mathbf{p}_u[\mathfrak{s}]} : u \in J\}$.

2) The mapping $\mathbf{j}_\mathfrak{s}$ *from Definition 6.12(7) is really a well defined embedding of the group* $F_{I[\mathfrak{s}]}$ *into the group* $F_\mathfrak{s}$.

3) In part (2) if $J^\mathfrak{s}_1$ *is* $\aleph_1$*-directed* <u>*then*</u>

(A) equality holds, that is $\mathbf{j}_\mathfrak{s}$ *maps* $G_{I[\mathfrak{s}]}$ *onto* $G_\mathfrak{s}$

(B) $\bigwedge\limits_{u \in J} \mathrm{rk}(I_u) < \infty \Rightarrow \mathrm{rk}(I_\mathfrak{s}) < \infty$, *etc.*

Proof. 1),2) Easy.

3) As in 5.41(3).

[Saharon: recheck for the non-linear case!!] $\square_{6.13}$ $\square$

Claim 6.14. *Assume*

(A) $\aleph_0 < \theta = \mathrm{cf}(\theta) \leq \kappa$

(B) $\mathcal{T}_\alpha \subseteq {}^\alpha\kappa$ *for* $\alpha < \theta$ *has cardinality* $\leq \kappa$

(C) $\mathcal{F} = \{f \in {}^\theta\kappa : f \restriction \alpha \in \mathcal{T}_\alpha$ *for* $\alpha < \theta\}$

(D) $\gamma = \mathrm{rk}(\mathcal{F}, <_{J^{\mathrm{bd}}_\theta})$, *necessarily* $< \infty$ *so* $< (\kappa^\theta)^+$.

<u>*Then*</u> $\tau^{\mathrm{atw}}_\kappa \geq \tau^{\mathrm{nlg}}_\kappa \geq \tau^{\mathrm{nlf}}_\kappa > \gamma$ *(on* $\tau^{\mathrm{nlf}}_\kappa$ *see below).*

Where

Definition 6.15. $\tau^{\mathrm{nlf}}_\kappa$ is the least ordinal τ such that $\tau > \tau^{\mathrm{nlf}}_{G,H}$ wherever $G = \mathrm{Aut}(\mathfrak{A}), \mathfrak{A}$ a structure of cardinality $\leq \kappa, H$ a subgroup of G of cardinality $\leq \kappa$ and $\mathrm{nor}^{<\infty}_G(H) = G$.

Proof. We define $\mathfrak{s} = (J, \bar{\mathbf{p}}, \bar{\pi})$ as follows:

(A) $J = (\theta; <)$

(B) $I_\alpha = (\mathcal{T}_{\alpha+1}, <_{\alpha+1})$ for $\alpha < \theta$ where

$$f_1 <_{\alpha+1} f_2 \Leftrightarrow f_1(\alpha) < f_2(\alpha)$$

(C) for $\alpha < \beta < \theta$ let $\pi_{\alpha,\beta} : I_\beta \to I_\alpha$ be

$$\pi_{\alpha,\beta}(f) = f \restriction (\alpha + 1).$$

Now

$(*)_1$ $\mathfrak{s}$ is a κ-p.o.w.i.s.

$(*)_2$ $\mathfrak{s}$ is linear and very nice

[Why? As in the proof of 5.48.]

$(*)_3$ $\mathfrak{s}$ is good

[Why? Assume $\alpha < \theta$ and $x \in Y_{I_\alpha}$. Let $\beta \le \alpha$ be minimal such that $\langle t_\ell(x) \restriction \beta : \ell \le n(x)\rangle$ are pairwise distinct $s_\ell(x) \restriction \beta \notin \{t_m(x) \restriction \beta : m \le \ell\}$ for $\ell \in \{1, \dots, n(x)\}$. Now β is well defined (as $\beta = \alpha$ is O.K. by the definition of $x \in Y$ and J is well ordered. Also $\beta \ne 0$ (as $\{f \restriction 0 : f \in T_{\alpha+1}\}$ is a singleton (as $n(x) > 0$ is assumed). Lastly, β cannot be a limit ordinal so $\beta' = \beta - 1, y = (\langle t_\ell(x) \restriction \beta : \ell \le n(x)\rangle, \langle s_\ell(x) \restriction \beta : 1 \le \ell \le n(x)\rangle)$ are as required.]

$(*)_4$ $I_{\mathfrak{s}}$ is (isomorphic to) $(\mathcal{F}, <_{J^{\rm bd}_x})$

[Why? This is how we define $I_{\mathfrak{s}}$ (note the difference compared to §1.]

$(*)_5$ $F_{\mathfrak{s}}$ is isomorphic to $F_{I[\mathfrak{s}]}$

[Why? By 6.13(3).]

$(*)_6$ $F_{I[\mathfrak{s}]}$ is a κ-automorphism group

[Why? By 5.45.]

Recalling §1, together clauses $(\alpha), (\beta), (\gamma)$ of 6.1 holds so we are done. $\square$

Conclusion 6.16. *1) If κ is strong limit singular of uncountable cofinality, <u>then</u> $\tau^{\rm atw}_\kappa \ge \tau^{\rm nlg}_\kappa \ge \tau^{\rm nlf}_\kappa > 2^\kappa$ (on $\tau^{\rm nlf}_\kappa$, see Definition 6.15).*
2) If $2^{\aleph_0} < 2^\theta < \kappa = \kappa^{<\theta} < \kappa^\theta$ then $\tau^{\rm nlf}_\kappa > \kappa^\theta$.

Proof. 6.16 Let $\theta = {\rm cf}(\kappa)$.

By [Shear, II,5.4,VIII,§1] for every regular $\lambda \le 2^\kappa$ there is an increasing sequence $\langle \lambda_i : i < \theta\rangle$ of regular cardinals $< \kappa$ with $(\prod\limits_{i<\kappa} \lambda_i, <_{J^{\rm bd}_\theta})$ having true cofinality λ. Clearly for any such $\langle \lambda_i : i < \theta\rangle$ we can find $f_\alpha \in \prod\limits_{i<\theta} \lambda_i$ for $\alpha < 2^\kappa$ such that $\alpha < \beta \Rightarrow f_\alpha <_{J^{\rm bd}_\theta} f_\beta$. Now we can prove by induction on α then ${\rm rk}_I(f_\alpha) \ge \alpha$ where $I = (\mathcal{F}, <_{J^{\rm bd}_\theta})$. Now $\mathcal{F}$ as in 6.1 and we know $f \in \mathcal{F} \Rightarrow {\rm rk}_I(f) < \infty$, so we are done. $\square_{6.16}$ $\square$

§ 6(A). **§5 Alternative presentation of §3,§4.** We try to give the shortest way: from §2,§3,§4 we use only 5.20, 5.21,5.22. (5.1) Definition **??**,pg.35 [natural but not

used: Definition 6.20, Claim 6.6, Claim 6.7 (5.2) Definition 6.10 κ-p.o.w.i.s.,pg.37 (5.3) Claim **??**,pg.38 [clause (a) fill proof]

§ 6(B). **Private Appendix**
§5. A different way to represent §1 is

Definition 6.17. 1) We say $Y \subseteq Y_I$ is closed when: if $x \in Y$ and $m \leq n(x)$ <u>then</u> $(\langle t_0(x), t_1(x), \dots, t_{m-1}(x), t_m(x)\rangle, \langle s_1(x), \dots, s_{m-1}(x), s_m(x)\rangle$ and $(\langle t_0(x), t_1(x), \dots, t_{m-1}(x), s_m(x)\rangle, \langle s_1(x), \dots, s_{m-1}(x), t_m(x)\rangle$ belongs to Y.
2) G^*_Y is the subgroup of X_I generated by $\{g_x : x \in Y\}$.

Claim 6.18. *1) If $y \subseteq X_I$ is finite <u>then</u> there is a finite closed $y^+ \supseteq Y$.*
2) If $y \subseteq X_I$ is closed and $<^*$ is a linear order of X^+_I as in 5.12 and $g \in G^*_Y$ <u>then</u> we can find n and $x_1 <^* \dots <^* x_n$ from Y such that $g = g_{x_1} \dots g_{x_n}$ (hence G^*_Y has $\leq 2^{|Y|}$ elements).

<u>Moved from §1, Feb 2004</u>:

Definition 6.19. 1) Let $Y_I = X_I \cup K^+_I$, we are assuming $X_I \cap K_I = \varnothing = 0$.
1A) $Y^{\leq\alpha}_I = X^{\leq\alpha}_I \cup K_I, Y^{<\alpha}_I = X^{<\alpha}_I \cup (X_I \times K_I)$.
2) G_I is the group generated by $\{g_x : x \in Y_{\mathbf{p}}\}$ freely except the equations in $\Delta_{\mathbf{p}}$, where Δ_I is defined below.
2A) $G^{\leq\alpha}_{\mathbf{p}}$ is the group generated by $\{g_x : x \in Y^{\leq\alpha}_{\mathbf{p}}\}$ freely except the equations in $\Delta^{\leq\alpha}_I$ where $\Delta^{\leq\alpha}_I$ is defined below; similarly $G^{<\alpha}_I, \Delta^{<\alpha}_I$.
3) $\Delta_I = \Delta^{\leq\infty}_I$ where $\Delta^{\leq\alpha}_I$ is the set of the following equations

- (A) $f^{-1}_x = g_x$ for $x \in Y_I$
- (B) $g_x g_y = g_y g_x$ for
  - (i) $x, y \in X_I, \neg\circledast_{x,y}, \neg\circledast_{y,x}$ or
  - (ii) $x, y \in Y_I \setminus X_I$
- (C) $g_x g_{y_1} g^{-1}_x = g_{y_2}$ if $x, y_1 y_2 \in X_I$ are as in (c) of 5.8(2)
- (D) $g_x g_{y_1} g^{-1}_x = g_{y_2}$ if $x \in X_I$ and $y_1, y_2 \in K_I$ and $K_I \models$ "$y_1 g_x = g_2$".

Now we first analyze the group K_I.
4) For $y \in Y_I$ define $\mathrm{rk}_*(y)$ as in Definition ?? if $y \in X_{\mathbf{p}}$ and as -1 otherwise (e.g., $y \in K_I$).

§ 6(C). **Private Appendix.** What about $\mathfrak{s}$ with $J^{\mathfrak{s}}$ not $\aleph_1$-directed? Even if every $I^{\mathfrak{s}}$ is well founded the inverse limit to be well founded. Still we can have large $\mathrm{rk}(I'_{\mathfrak{s}})$, but the group we get $G_{\mathfrak{s}}$ may be "bigger" than $G_{I[\mathfrak{s}]}$, see 5.40(7). However, we shall show that they are similar enough.

Claim 6.20. *[?] Assume $\mathfrak{s}$ is a κ-p.o.i.s so $(G_{I[\mathfrak{s}]}, H_{I[\mathfrak{s}]}), (G_{\mathfrak{s}}, H_{\mathfrak{s}})$ are well defined as well as the natural embedding $\mathbf{j}^{\mathfrak{s}} = \mathbf{j}[\mathfrak{s}]$ from $G_{\mathfrak{s}}$ into $G_{I[\mathfrak{s}]}$ mapping $H_{\mathfrak{s}}$ onto $H_{I(\mathfrak{s})}$ (see* **??***)*

(A) *for every ordinal $\alpha, j^{\mathfrak{s}}$ maps $\mathrm{nor}^{\alpha}_{G_{\mathfrak{s}}}(H_{\mathfrak{s}})$ onto $\mathrm{nor}^{\alpha}_{G_{I[\mathfrak{s}]}}(H_{I[\mathfrak{s}]})$*

(B) *the normatizer length of $H_{\mathfrak{s}}$ in $G_{\mathfrak{s}}$ is equal to $\mathrm{rk}^{<\infty}(I_{\mathfrak{s}})$.*

Proof. FILL! $\square$

Conclusion 6.21. *1) For every κ there is a structure $\mathfrak{A}$ of cardinality κ such that for some two element subgroups H of $\mathrm{Aut}(\mathfrak{A}')$ has normalizer length $\geq \kappa^+$ in $\mathrm{Aut}(\mathfrak{A})$. 2?*

Remark 6.22. Of course, we can get length somewhat $> \kappa^+$.

<u>Moved 2003/7 from the proof of 5.41:</u> Let

$B^1_u = X_I \cap \{x_{u,1}, \ldots, x_{u,n(u)}\}$, a finite subset of X_I

$$B^2_u = \big\{y \text{ :for some } x \in B^1_u \text{ and } m \leq n(x) \text{we have } \\ y = (\langle t_0(x), t_1(x), \ldots, t_m(x)\rangle, \langle s_1(x), \ldots, s_m(x)\rangle) \text{ or } \\ y = (\langle t_0(x), t_1(x), \ldots, t_{m-1}(x), s_m(x)\rangle, \langle s_1(x), \ldots, s_{m-1}(x), t_m(x)\rangle)\big\}$$

again a finite subset of X_I.

Let $B^3_u = \{y \in X_I : \text{for some } \ell \in \{1, \ldots, n(u)\} \text{ we have } y \in x_{u,\ell} \in X^+_I \setminus X_I\} \cup B^2_u$.

§ 6(D). **Private Appendix 2.** Assignments: 1) (2002/9/15) - get $\mathfrak{A}$ such that $\mathrm{Aug}(\mathfrak{A}) = G_{\mathfrak{s}}$ for §3 for κ-p.o.w.i.s. $\mathfrak{s}$, [seem O.K.]
2) Complete 6.7 [details]
3) Try $\mathrm{Con}(\tau'_{\aleph_0} > 2^{\aleph_0} > \aleph_1)$ start with Σ'_1-pre-relation with high length.
4) (2002/9/16)- about the normalizer problem for, e.g., $\kappa = \beth_\omega$, try to say determinacy by clubs of $X \in [\mathcal{P}(\kappa)]^{\aleph_0}$, "$X \cap W_1 \in S$" is undetermined. So is there a phrase modulo then?
5) (2002/9/28) - can we directly get $G = \mathrm{inv} - \lim\langle G_{I_u}, \hat{\pi}_{u,v} : u \le_J v\rangle$ is κ-aut? Moved from §0,p.2: [Fill! we can show $\delta(\kappa) \le \tau^*_\kappa$? But $\tau'_\kappa \le \delta(\kappa)$? Question: A connection between $\tau'_{G,H}$ when $\mathrm{nor}^\infty_G(H) = G$ and auto tower. Moved from Definition 6.34(3), clause (B):

(A) if $u \le_J v$ we define $f^{\mathfrak{s}}_{u,v} : \mathbf{M}_u \to \mathbf{M}_v$ as follows
$f_{u,v}(z) = z'$ if:
- (α) $\pi^+_{u,v}(x^z) = x^{z'}$
- (β) one of the following occurs
- (i) $n(z') = n(z)$ and $\ell < n(z) \Rightarrow (u^z_\ell, x^z_\ell) = (u^{z'}_\ell, x^{z'}_\ell)$ and

$$\pi^+_{u,v}(x^z_{n(z)}) = x^{z'}_{n(z')} \text{ and } (\forall w)(u \le_J w \le_J v \Rightarrow$$

$$\pi^+_{w,v}(x^{z'}_{n(z')}) \in X_{I_w}$$

- (ii) not (i) and $n(z') = n(z) + 1, \pi^+_{u,v}(x^{z'}) = x^z$ and

$$u <_J u^{z'}_{n(z)} \le_J u^z_{n(z)}, x^{z'}_{n(z)} = \pi^+_{u^{z'}_{n(z)}, u^z_{n(z)}}(x^z_{n(z)})$$

Moved from Claim 6.35: Definition 6.31(4), clause $(g)(\alpha)(ii)$ we add

- and for no $v_1 <_J$ do we have $\mathbf{r}(x) \le v_1$ and $(\forall w)[v_1 \le_J v \to \pi_{w,v}(x) \in X_I)$.

Moved from pg.17:

Claim 6.23. Assume $\mathfrak{s}$ is a κ-p.o.w.i.s. Then $\langle {}_*G_u, {}_*\pi_{u,v} : u \le_J v\rangle$ is an inverse system of groups (so with ${}_*\pi_{u,v}$ a homomorphism from ${}_*G_v$ into ${}_*G_u$) with inverse limit ${}_*G_{\mathfrak{s}}$ very similar to $G_{\mathfrak{s}}$, in particular for some 2-element subgroup ${}_*H$ such that $\tau'_{{}_*G_{\mathfrak{s}}, {}_*H_{\mathfrak{s}}} = \tau'_{G_{\mathfrak{s}}, H_{\mathfrak{s}}}$ and $\mathrm{nor}^\infty_{{}_*G_{\mathfrak{s}}}({}_*H_{\mathfrak{s}}) = {}_*G_{\mathfrak{s}} \Leftrightarrow \mathrm{nor}^\infty_{G_{\mathfrak{s}}}(H_{\mathfrak{s}}) = G_{\mathfrak{s}}$.

§ 6(E). **§k.** Juris note: 1) Some points as in Tuesday notes:

(A) lim sup norm
(B) cs product of forks and though they look just for the future not a necessity

(C) no bigness (which is a remnant) of ideal
(D) we use in the i-th norm that we use first ε_{i-1} then ε_{i-2}, etc., and use they decrease
(E) i still think that finding a line not covered by $\langle \mathscr{I}_0, \dots, \mathscr{I}_n \rangle$ as witnessed by $p' \geq p$ with norms dropping down by ≤ 1 its O.K., i.e., the closure inherent is using Koning is O.K. by inflating the $\mathscr{I}_\ell$-s, see below.

A try of lines/point: Choice: $n_i^\ell < \omega$, $i < w$, $\ell \leq (0)$, $n_i^\ell << n_i^{\ell+1} <<< n_{i+1}^0$, $\varepsilon_i = \frac{1}{n_i^1}$,$0 < \zeta^* << 1$ constant. Notation: 1) $SQ_n = \{(\frac{i}{2^n}, \frac{j}{2^n}) : i,j \in \{0,\dots,2^n\}, x \in SQ_n \Rightarrow x = (x^{[1]}x^{[2]})$.
2) $LN_n =$ a line $\mathscr{L}$ determined by two distinct points among $CR_n = \{(\frac{\ell}{2^n}, \frac{k}{2^n}) : \ell, k \in \{0, 2^n\}$.
3) If $\mathscr{L} \in LN_n$ let

$$nb(\mathscr{L}) = \{x \in SQ_n : \text{ the distance of } x \text{ from } \mathscr{L} \text{ is } \leq \frac{\sqrt{2}}{2^n}\}.$$

4) A SQ_n-square is a square of the form

$$\{(x,y) : \frac{\ell}{2^n} \leq x \leq \frac{\ell+1}{2^n}, \frac{k}{2^n} < y \leq \frac{k+1}{2^n}$$

denoted by s, t.

Definition 6.24. (A) Let $\Sigma(\mathfrak{c}_i) = \mathcal{P}(\{A : A \subseteq {}^{n_i^8}2, |A|/2^{n_i^8} = 2^{-i-1}\})$
$\Sigma(\mathfrak{c}_i) = \mathcal{P}(\mathfrak{c}_i) \setminus \{0\}$

(B) for $d \in \Sigma(\mathfrak{c}_i)$ we define by induction on $j \leq i$ when $\text{nor}_i(d) \geq j$; this defines the norm which is always $\leq i$
$\underline{j=0}$: always (i.e., non empty) $\underline{j=1}$: $\text{nor}(d) \geq 1$ iff ${}^{n_i^8}2 = \cup\{A : A \in d\}$

$\underline{j+1>1}$: if

- (α) $m > n_i^0$
- (β) F is a function with domain d into

  $\{S : S \text{ in the union of } \leq \zeta^* \cdot (2_i^m)^2\ SQ_m\text{-squares}\}$
- (γ) g: is a function from the set of $SQ_{n_j^8}$-squares to numbers no! $\in \{\frac{0}{2^{n_y^g}}, \frac{1}{2^{n_j^g}}\}$
- (δ) $\Sigma\{g(t) : t \in \text{Dom} g\} \leq \zeta^*$
- (ε) F obeys g which means: for every $SQ_{n_j^8}$-square t and $A \in D$.

(C) $\text{Leb}(t \cap F(A)) < \frac{1}{2^{n_j^{10}}}$ or $g(t) \geq \text{Leb}(t \cap F(A))$ (equivalently $g(t) > 0$) then we can find an SQ_m-line $\mathscr{L}$ such that $j \leq \text{nor}_i\{A \in d : \mathscr{L} \nsubseteq F(A)\}$ or even an $SQ_{m'}$-line $\mathscr{L}$ for some $m' \geq m$ - no real difference (certainly after $m' = 2^{2^m}$).

Claim 6.25. $\text{nor}_i(\mathfrak{c}_i) \geq i$ *for* $i > 2$.

Proof. Note that in the definition 6(E) we can always increase m and it is not really used. We need some m such that the range of F is appropriate. Will we have more lines? But we make the difference to make this null.

Assume toward contradiction that $\text{nor}_i(c_i) < i$.
So as before we can choose by downward induction on $j < i$ funtions F_j, g_j such that:

$\circledast(\alpha)$ F_j is $\langle F_{j,\mathscr{L}} : \mathscr{L} \in \mathbf{L}_j\rangle$ where
(β) $\mathbf{L}_j = \{j+1,\dots,i-1\}(LN_m)$, i.e., a sequence of lines

(γ) $g_j = \langle g_{j,\bar{\mathscr{L}}} : \mathscr{L} \in \mathbf{L}_j \rangle$

(δ) for each $\bar{\mathscr{L}} \in \mathbf{L}_j, (F_{j,\bar{\mathscr{L}}}, g_{j,\bar{\mathscr{L}}})$ are as in Definition 6.24

(ε) $j > \mathrm{nor}_i(\{A \in C_i$: for every appropriate $j' = j, j+1, i-1, \mathscr{L}_{j'} \nsubseteq F_{\bar{\mathscr{L}} \restriction (j',i)}(A)\})$ wherever $\bar{\mathscr{L}} = \langle \mathscr{L}_{j'} : j'' = j, j+1, \ldots, i-1 \rangle \in \mathbf{L}_j$.

For $j = i$ the demand on $A \in d$ is empty so

ε says $\alpha > \mathrm{nor}_i(\mathfrak{c}_i)$ which holds.

The induction hypothesis is by the definition.

So we have $(F_{i-1}, g_{i_1}, \ldots, F_1, g_1)$. So for each $\bar{\mathscr{L}} \in \mathbf{L}_1$ we have $1 > \mathrm{nor}_1\{A \in \mathfrak{c}_i :$ as above$\}$ hence there is $f(\bar{\mathscr{L}}) \in {}^{n_i^8} Z$ such that

$\circledast$ $f(\bar{\mathscr{L}}) \notin A$ if

$\boxdot$ $A \in \mathfrak{c}_i$ and for $j = 1, \ldots, i-1$ we have $\mathscr{L}_{j'} \nsubseteq F_{\mathscr{L} \restriction \{j+1, \ldots, i-1\}}(A)$.

NOW COMES the main point.

We have two many points.

We choose by downward induction on $j \leq i$, $\mathbf{L}_j^-$ and $\bar{S}_j =< S$. $\square$

Alternative 1

Definition 6.26. 1) For a partial order I, let

(a) $Y_I = \{(\langle t_0, \ldots, t_n \rangle, \langle s_1, \ldots, s_n \rangle) :$ (a) $t_\ell \in I$ and $s_\ell \in I$ and (b) $t_0, \ldots, t_n$ is without repetitions and (c) $s_\ell \notin \{t_0, \ldots, t_\ell\}$ for $\ell \in \{1, \ldots, n\}\}$

(b) $Y_I^+ = Y_I \cup [Y_I]^{<\aleph_0}$. 2) G_I^+ is defined as in Definition 5.2(4) using Y_I^+ instead of X_I^+.

Definition 6.27. If π is a partial function from a p.o. I_2 into a p.o. I_2 we define the mapping $\pi^+, \hat{\pi}, \breve{\pi}$ (really $\pi^+_{I_1,I_2}, \hat{\pi}_{I_1,I_2}, \breve{\pi}_{I_1,I_2}$, the $\breve{\pi}$ is not connected to the $\breve{\pi}$ from Claim 5.42 and $\pi^+, \hat{\pi}$ are not as in Definition 5.35, 6.44) as follows:

(A)(a) π^+ is a partial mapping from $Y_{I_2}^+$ into $Y_{I_1}^+$ (every $\mathrm{Dom}(\pi) = I_2$, $\mathrm{Dom}(\pi^+)$ is a proper subset of $Y_{I_2}^+$)

(A) for $x \in Y_{I_2}$

(α) $x \in \mathrm{Dom}(\pi^+)$ <u>iff</u> $\{t_0(x), \ldots, t_{n(x)}(x), s_1(x), \ldots, s_{n(x)}(x)\} \subseteq \mathrm{Dom}(\pi)$ and $\ell < n(x) \Rightarrow (t_{\ell+1}(x) <_{I_2} s_{\ell+1}(x) <_{I_2} t_\ell(x)) \vee (s_{\ell+1}(x) <_{I_2} t_{\ell+1}(x) <_{I_2} t_\ell(x))$

(β) $\pi^+(x) = (\langle \pi(t_0(x), \ldots, \pi(t_{n(x)}(x)) \rangle, \langle \pi(s_1(x)), \ldots, \pi(s_{n(x)}(x)) \rangle$

(B) for any finite $y \in Y_{I_2}$ we have $y \in \mathrm{Dom}(\pi^+)$ and $\pi^+(y) = \{\pi(x) : x \in y \wedge x \in \mathrm{Dom}(\pi^+)\}$

(C) in particular $\langle\ \rangle \in \mathrm{Dom}(\pi^+), \pi^+(\langle\ \rangle) = \langle\ \rangle$

(B) $\breve{\pi}$ is the partial homomorphism from $G_{I_2}^+$ into $G_{I_1}^+$ with domain the subgroup of $G_{I_2}^+$ generated by $\{g_x : x \in \mathrm{Dom}(\pi^+)\}$ mapping g_x to $g_{\pi^+(x)} \in G_{I_1}$; see justification below

(C) $\hat{\pi}$ is the homomorphism from $G_{I_2}^+$ into $G_{I_1}^+$ mapping for $x \in Y_{I_2}^+, g_x$ to $g_{\pi^+(x)}$ if $x \in \mathrm{Dom}(\pi^+)$ and to $e_{G_{I_2}}$ if $x \in Y_{I_2}^+ \setminus \mathrm{Dom}(\pi^+)$.

Claim 6.28. *1) In Definition 6.27, $\hat{\pi}, \check{\pi}$ are well defined, in particular, $\hat{\pi}$ is really a homomorphism from $G^+_{I_2}$ into $G^+_{I_1}$.*
2) If I_1, I_2, I_3 are partial orders and π_ℓ is a partial mapping from $I_{\ell+1}$ into I_ℓ for $\ell = 1,2$ and $\pi_3 = \pi_2 \circ \pi_1$ <u>then</u> $\pi^+ \supseteq \pi^+_2 \circ \pi^+_1$.

Remark 6.29. But the $\check{\pi}_{u,v}$ may fail to commute and possibly $\pi^+ \supset \pi^+_2 \circ \pi^+_1$.

Proof. As in 5.42(2). □

Definition 6.30. 1) We define "$\mathfrak{s}$ is a κ-p.o.w.i.s. (w for weakly) similarly to κ-p.o.i.s. in 5.40(1) except that we replace clauses $(f), (g)$ there by:

- $(f)'$ for $u \le_J v, \pi_{u,v}$ is a partial function from I_v to $I_u, \pi_{u,u} = \mathrm{id}_{I_u}$
- $(g)'$ $u \le_J \le v \le_J w$ implies only $\pi^{\mathfrak{s}}_{u,w} \supseteq \pi^{\mathfrak{s}}_{u,v} \circ \pi^{\mathfrak{s}}_{v,w}$.

<u>discussion</u>: Let $\mathfrak{s}$ be a κ-p.o.w.i.s, with $\aleph_1$-directed $J^{\mathfrak{s}}$, we would like to show that $G_{\mathfrak{s}}$ is a κ-automorphism group. We are thinking on the case $J^{\mathfrak{s}}$ is a (linear) well ordering. E.g., $\theta = \mathrm{cf}(\theta) > \aleph_0, f_\alpha \in {}^\theta\kappa$ for $\alpha < \alpha^*$ form a tree, i.e., $f_\alpha(i) = f_\beta(j) \Rightarrow i = j$ and $f_\alpha \restriction i = f_\beta \restriction j$ and f_α is $<_{J^{\mathrm{bd}}_\theta}$-increasing with α where $\mathcal{J}^{\mathrm{bd}}_\theta$ is the ideal of bounded subsets of θ.

So we choose $J = \theta, I_i = \{f_\alpha(i) : \alpha < \alpha^*\}$ where J ordered by the order of the ordinals and for $i < j$ we have $\pi_{j,i}(f_\alpha(j)) = f_\alpha(i)$ for $\alpha < \alpha^*$. Now in Definition 6.35, 6.36, 6.37 below we replace the $G^{\mathfrak{s}}_u$-s by bigger groups and extend the homomorphism $\hat{\pi}_{u,v}$ such that their cardinalities are still $\le \kappa$ but the inverse limit is, essentially, the same, by adding many copies forming a tree with few branches. So instead we have many $g_{\mathbf{m}}, \mathbf{m} = \langle x, z_0, \ldots, z_{n(*)}\rangle, z_\ell \in Z_{\mathrm{proj}_\ell(x)}$.

Definition 6.31. Let $\mathfrak{s}$ be a κ-p.o.w.i.s and $J = J^{\mathfrak{s}}$, etc.
0) $\mathfrak{s}$ is linear if $J^{\mathfrak{s}}$ is a linear order.
1) We say $\mathfrak{s}$ is smooth <u>if</u>

- (A) $\mathrm{Rang}(\pi^{\mathfrak{s}}_{u,v}) = I_u$
- (B) if $w \in J$ and $x \in Y_{I_w}$ (see below), $n(x) > 0$ <u>then</u> for some y, u (we may write $u = \mathbf{u}(x), y = \mathbf{y}(x)$) we have: $u \le_J w$ and $x \in Y_{I_u}$ and ${}_*\pi^+_{u,w}(x) = y \in Y_{I_u}$ and
  - $(*)_{u,y}$ if $u \le_J v, x' \in Y_{I_v}$ and $u' \le_J v, {}_*\pi^+_{u,v}(x') = y$ then ${}_*\pi^+_{u',v}(x')$ is well defined iff $u \le_J u'$.

we define $n(x), t_\ell(x), s_\ell(x)$ for $x \in Y_I$ as we have defined in 5.40 for $x \in X_I$ and let

$$Y^+_I = \{x : x \in Y_I\} \cup [Y_I]^{<\aleph_0}.$$

3) For $u \le_J v$ let ${}_*\pi^+_{u,v} : Y^+_{I_w} \to Y^+_{I_v}$ be defined as in clause (A) of 6.27. For $u \in J$ let ${}_*Y_u = \{y \in Y_v : u, y$ are as in clause (b) of part (1),i.e., $(*)_{u,y}$ holds$\}$.

So our aim in this section is (proved in 6.36)

Claim 6.32. *If a κ-p.o.w.i.s. $\mathfrak{s}$ is linear (see Definition 6.31 below) <u>then</u> $G_{\mathfrak{s}}$ is a κ-automorphism group.*

discussion: To define the bigger groups we shall have to define various things. The idea is that we allow ourselves to use $Y^+_{I_u}$ instead of $X^+_{I_u}$ but we add "freely" many copies to make the mapping having full domain and range, but add no more than necessary so that no more branches are added. The ones we really need are $\bar{x} = \langle x_v : v \in J_{\ge w}\rangle$ such that ${}_*\pi^+_{v_1,v_2}(x_{v_2}) = x_{v_2}$, in the interesting cases to allow it comes $Z_{o,u,2}$ in clause (g) of 6.33 but there is no need for the parallel statement in clause (h) of Definition 6.33. The set $\{(\ell,m,i) : \ell < m \le n(x)$ and $i = 1$ and $t_\ell(x_v) <_{I_v} t_m(x_v)$ or $1 \le \ell < m \le n(x)$ and $i = 2$ and $s_\ell(x_v) <_{I_u} s_m(x_v)\}$ is essentially constant.

Definition 6.33. Let $\mathfrak{s}$ be a linear κ-p.o.w.i.s. and $J = J^{\mathfrak{s}}$.
1) By induction on n we define $\mathbf{x}_n, \bar{Z}_n = \langle Z_{n,u} : u \in J\rangle$ and $\bar{f}_n = \langle f_{n,u,v} : u \le_J v\rangle$ such that

(A) $\bar{Z}_n$ is a sequence of pairwise disjoint sets each of cardinality $\le \kappa$

(B) $\mathbf{x}_n$ is a function from $\{Z_{n,u} : u \in J\}$ onto $\cup\{Y_{I_u} : u \in J\}$ mapping $Z_{n,u}$ onto Y_{I_u} for each $u \in J$ and let $Z_{n,u,x} = \{z \in Z_{n,u} : \mathbf{x}_n(z) = x\}$ and for $r \le_J u$ let ${}_rZ_{n,u} = \{z \in Z_{n,u} : \mathbf{r}(\mathbf{x}_n(z)) \le_J r\}$ and ${}_rZ_{n,u,x} = \{z \in {}_rZ_{n,u} : \mathbf{x}_n(z) = x\}$

(C) $f_{n,u,v}$ is a function from ${}_uZ_{n,v}$ into $Z_{n,u}$ and the $f_{n,u,v}$-s commute and $f_{n,u,v}(y) = x \Rightarrow \mathbf{r}(\mathbf{x}_n(y)) = \mathbf{r}(\mathbf{x}_n(x))$

(D) if $u \le_J v$ and $f_{n,u,v}(z_2) = z_1$ then ${}_*\pi^+_{u,v}(\mathbf{x}_n(z_2)) = \mathbf{x}_n(z_1)$

(E) if $m < n$ then $Z_{m,u,x} \subseteq Z_{n,u,x}, Z_{m,u} \subseteq Z_{n,u}, f_{m,v,u} \subseteq f_{n,v,m}$

(F) if $n = m+1$ and $u \le_J v$ then $Z_{0,n,u} \subseteq \mathrm{Rang}(f_{n,u,v})$

(G) for $n = 0$ and $u \in J$ we have

(α) $Z_{n,u} = Z_{n,u,1} \cup Z_{n,u,2}$ where

(i) $Z_{n,u,1} = \{\langle n,u,v,x,y,1\rangle : x \in Y_{I_v}, y \in Y_{I_u}, {}_*\pi^+_{u,v}(x) = y,$

$\mathbf{r}(x) \le_J u \le_J v\}$

(ii) $Z_{n,u,2} = \cup\{\langle n,u,v,x,y,2\rangle : x \in X_{I_v}, y \in X_{I_u}, \mathbf{r}(x) \le_J v \le_J u$

and ${}_*\pi^+_{u,v}(y) = x$ and

$(\forall w)(u \le_J w \le_J v \Rightarrow \pi^+_{w,v}(y) \in X_{I_w})\}$

(β) $\mathbf{x}_0(\langle n,u,v,x,y,1\rangle) = y$ and $\mathbf{x}_0(\langle n,u,v,x,y,2\rangle) = y$ <u>if</u> $\langle n,u,v,x,1\rangle, \langle n,u,v,x,y,2\rangle$ are as above

(γ) $f_{n,u_1,u_2}(z_2) = z_1$ <u>iff</u> $z_2 \in Z_{n,u_1}, z_1 \in Z_{n,u_2}$ and one of the following occurs

(i) for some v for $x \in X_{I_u}$ we have $z_\ell = \langle n,u_\ell,v,x,y,1\rangle \in Z_{n,u_\ell,1}$

for $\ell = 1,2$

(ii) for some v, y_1, y_2 we have $z_\ell = \langle n,u_\ell,v,x,y_\ell,2\rangle \in Z_{n,u_\ell,2}$ for

$\ell = 1,2$ and $\pi^+_{u_1,u_2}(y_2) = y_1$

(iii) for some v, y_1 we have $z_1 = \langle n,u,v,x,y_1,1\rangle \in Z_{n,u_1,1}$,

$z_2 = \langle n,u_2,v,x,y_2,2\rangle$ and $\pi^+_{v,u_2}(y_2) = x$

(H) for $n = m+1$

(α) $Z_{n,u} = Z^1_{n,u,1} \cup Z^2_{n,u,2}$ where

(i) $Z_{n,u,1} = Z_{m,u}$

(ii) $Z_{n,u,2} = \{(n,u,v,w,z,y,y') : y \in Y_{I_w}, z \in Z_{m,v},$

$\pi^+_{v,w}(y) = \mathbf{x}_m(z), \pi^+_{u,w}(y) = y', \mathbf{r}(y) \le_J v, \mathbf{r}(y) \le_J u,$

$\neg(u \le_J v), v \le_J w, u \le_J w\}$

(β) $\mathbf{x}_n(z') = x'$ iff $z' \in Z_{n,u}, \mathbf{x}_m(z') = x'$ or $z' = (n,u,v,w,z,y,y')$ and $\mathbf{x}_m(z) = x'$

(γ) for $u_1 \le_J u_2, f_{n,u_1,u_2}(z_2) = z_1$ iff one of the following cases occurs

(i) $f_{m,u_1,u_2}(z_2) = z_1$

(ii) for some v,w,z,y we have $z_\ell = (n,u_\ell,v,w,z,y) \in Z_{n,u_\ell,2}$ for

$i = 1,2$

(iii) for some v,w,z,y we have $z_2 = (n,u_2,v,w,z,y) \in Z_{n,u_2,2}$

and $\mathbf{r}(y) \le_J u_1 \le_J v$ and $z_1 = f_{m,u_1,v}(z)$.

Definition 6.34. Let $\mathfrak{s}$ be a κ-p.o.i.s.
1) Let $Z_{\omega,u} = \cup\{Z_{n,n} : n < \omega\}, f_{\omega,u,v} = \cup\{f_{n,u,v} : n < \omega\}$ and $\mathbf{x}_\omega = \cup\{\mathbf{x}_n : n < \omega\}$.
2) Let $\mathbf{M}_u = \{\mathbf{m} : \mathbf{m}$ has the form $(x, z_0, \dots, z_{n(x)})$ such that $x \in Y_u, z_\ell \in Z_{u,pr_\ell(x)}$ where $pr_\ell(x) = (\langle t_i(x) : i \le \ell\rangle, \langle s_\ell(x) : i = 1,\dots,\ell\rangle)\}$,
$\mathbf{M}^+_u = \{\mathbf{m} : \mathbf{m} \in \mathbf{M}_u\} \cup [M_u]^{<\aleph_0}$. For $\mathbf{m} \in \mathbf{M}_u$ let $\mathbf{m} = (x^{\mathbf{m}}, z^{\mathbf{m}}_0, \dots, z^{\mathbf{m}}_{n(\mathbf{m})})$ where $n(\mathbf{m}) = n(x^{\mathbf{m}})$.
3) We define $\langle {}_*G_u, {}_*\hat\pi_{u,v} : u \le_J v\rangle$ as follows

(A) ${}_*G_u$ is generated by $\{g_{\mathbf{m}} : \mathbf{m} \in \mathbf{M}_u\} \cup \{g_{\mathbf{n}} : \mathbf{n} \in [\mathbf{M}_u]^{<\aleph_0}\}$ freely except the equations
  (a) $g^{-1}_{\mathbf{m}} = g_{\mathbf{m}}, g^{-1}_{<\mathbf{m}>} = g^{-1}_{\mathbf{n}} = g_{\mathbf{n}}$
  (b) $g_{\mathbf{m}} g_{\mathbf{n}} g^{-1}_{\mathbf{m}} = g_{\mathbf{n} \triangleleft \{\mathbf{m}\}}$
  (c) if $n+1 = n(\mathbf{m}_1) = n(\mathbf{m}_2), \mathbf{m} = \mathrm{pr}_n(\mathbf{m}_1) = \mathrm{pr}_n(\mathbf{m}_2), z^{\mathbf{m}_1}_{n+1} = z^{\mathbf{m}_2}_{n+1}, (s^{\mathbf{m}_1}_{n+1}, t^{\mathbf{m}_1}_{n+1}) = (t^{\mathbf{m}_2}_{n+1}, s^{\mathbf{m}_2}_{n+1})$ then $g_{\mathbf{m}} g_{\mathbf{m}_1} g^{-1}_{\mathbf{m}} = g_{\mathbf{m}_2}$
  (d) in all other cases the generators commute

(B) if $u \le_J v$ we define ${}_*\pi^+_{u,v} : \mathbf{M}_u \to \mathbf{M}_v$ by: ${}_*\pi^+_{u,v}(\mathbf{m}_2) = \mathbf{m}_1$ iff $\mathbf{m}_2 \in Z_{w,u}, \mathbf{m}_1 \in Z_{w,u}$ and $\ell \le n(\mathbf{m}_2) \Rightarrow f_{w,u,v}(z^{\mathbf{m}_2}_\ell) = z^{\mathbf{m}_1}_\ell$ and ${}_*\pi_{u,v}(x^{\mathbf{m}_2}) = x^{\mathbf{m}_1}$

(C) if $u \le_J v$ then we define a homomorphism ${}_*\hat\pi_{u,v}$ from ${}_*G_v$ to ${}_*G_v$ as follows: it maps $g_{\langle\ \rangle}$ to $g_{\langle\ \rangle}$

it maps $g_{\mathbf{m}_2}$ to $g_{\mathbf{m}_1}$ [and $g_{\mathbf{n}_1}$ to $g_{\mathbf{m}_1}$] if
  $\circledast(a)$ $\mathbf{m}_1 \in \mathbf{M}_u, \mathbf{m}_2 \in \mathbf{M}_v$
  (b) ${}_*\pi^+_{u,v}(\mathbf{m}_2) = \mathbf{m}_1$
  (c) $\mathbf{n}_1 \in [\mathbf{M}_u]^{<\aleph_0}, \mathbf{n}_2 \in [\mathbf{M}_v]^{<\aleph_0}$
  (d) $\mathbf{n}_1 = \{{}_*\pi^+_{u,v}(\mathbf{m}_2) : \mathbf{m}_2 \in \mathbf{n}_2 \wedge \mathbf{m}_2 \in \mathrm{Dom}({}_*\pi^+_{u,v})\}$.

Claim 6.35. *Assume $\mathfrak{s}$ is a κ-p.o.w.i.s. with $J^{\mathfrak{s}}$ a linear ordering of uncountable cofinality.*
Then

(A) $\langle {}_*G_u, {}_*\hat{\pi}_{u,v} : u \leq_J v\rangle$ *is an inverse system of groups (so with* ${}_*\pi_{u,v}$ *a homomorphism from* ${}_*G_v$ *into* ${}_*G_u$*) with inverse limit isomorphic to* $G_{\mathfrak{s}}$

(B) hence $G_{\mathfrak{s}}$ *is a* κ*-automorphism group.*

Claim 6.36. *In 5.47 we can replace* κ*-p.o.i.s by* κ*-p.o.w.i.s., that is ?*

Proof. The same proof replacing FILL. $\square$

The following claim works, e.g. for strongly inaccessible cardinals, but is most interesting for κ strong limit singular of uncountable cofinality.

Conclusion 6.37. *Assume*

(A) $\aleph_0 < \theta = \mathrm{cf}(\theta) \leq \kappa$

(B) $T_\alpha \subseteq {}^\alpha\kappa$ *for* $\alpha < \theta$ *has cardinality* $\leq \kappa$

(C) $\mathcal{F} = \{f \in {}^\theta\kappa : f \restriction \alpha \in T_\alpha$ *for* $\alpha < \theta\}$

(D) $\gamma = \mathrm{rk}(\mathcal{F}, <_{J^{\mathrm{bd}}_\theta})$*, necessarily* $< \infty$ *so* $< (\kappa^\theta)^+$.

Then

(α) there is a κ*-p.o.w.i.s.* $\mathfrak{s}$ *with* $\mathrm{rk}(I_{\mathfrak{s}}) = \gamma$

(β) in (α), $G_{I[\mathfrak{s}]}$ *is a* κ*-automorphism group with* $H_{I[\mathfrak{s}]}$*, a two element subgroup,* $\tau'_{G_{I[\mathfrak{s}]}, H_{I[\mathfrak{s}]}} = \gamma$ *and* $\mathrm{nor}^{<\infty}_{G_{I[\mathfrak{s}]}}(H_{I[\mathfrak{s}]}) = G_{I[\mathfrak{s}]}$

(γ) $\tau_\kappa \geq \tau'_\kappa > \gamma$.

Proof. We define $\mathfrak{s} = (J, \bar{I}, \bar{\pi})$ as follows:

(A) $J = (\theta; <)$

(B) $I_\alpha = (T_{\alpha+1}, <_{\alpha+1})$ for $\alpha < \theta$ where

$$f_1 <_{\alpha+1} f_2 \Leftrightarrow f_1(\alpha_0) < f_2(\alpha)$$

(C) for $\alpha < \beta < \theta$ let $\pi_{\alpha,\beta} : I_\beta \to I_\alpha$ be

$$\pi_{\alpha,\beta}(f) = f \restriction (\alpha + 1).$$

Now

$(*)_1$ $\mathfrak{s}$ is a κ-p.o.w.i.s.

$(*)_2$ $\mathfrak{s}$ is linear

$(*)_3$ $\mathfrak{s}$ is smooth

[Why? Assume $\alpha < \theta$ and $x \in Y_{I_\alpha}$. Let $\beta \leq \alpha$ be minimal such that $\langle t_\ell(x) \restriction \beta : \ell \leq n(x)\rangle$ are pairwise distinct $s_\ell(x) \restriction \beta \notin \{t_m(x) \restriction \beta : m \leq \ell\}$ for $\ell \in \{1, \ldots, n(x)\}$. Now β is well defined (as $\beta = \alpha$ is O.K. by the definition of $x \in Y$ and J is well ordered. Also $\beta \neq 0$ (as $\{f \restriction 0 : f \in T_{\alpha+1}\}$ is a singleton (as $n(x) > 0$ is assumed). Lastly, β cannot be a limit ordinal so $\beta' = \beta - 1, y = (\langle t_\ell(x) \restriction \beta : \ell \leq n(x)\rangle, \langle s_\ell(x) \restriction \beta : 1 \leq \ell \leq n(x)\rangle)$ are as required.]

$(*)_4$ $I_{\mathfrak{s}}$ is (isomorphic to) $(\mathcal{F}, <_{J^{\mathrm{bd}}_x})$

[Why? This is how we define $I_{\mathfrak{s}}$ (note the difference compared to §1.]

$(*)_5$ $G_{\mathfrak{s}}$ is isomorphic to $G_{I[\mathfrak{s}]}$

[Why? By 6.13(3).]

$(*)_6$ $G_{I[\mathfrak{s}]}$ is a κ-automorphic group
[Why? By 5.45.]

Recalling §1, together clauses $(\alpha),(\beta),(\gamma)$ of 6.37 holds so we are done. $\square$

Conclusion 6.38. *1) If κ is strong limit singular of uncountable cofinality, then $\tau_\kappa \geq \tau'_\kappa \geq \tau''_\kappa > 2^\kappa$.*
2) If $\kappa = \kappa^{<\kappa} > \aleph_0$ then $\tau_\kappa \geq \delta(\kappa)$.

Proof. 1) Let $\theta = \mathrm{cf}(\kappa)$.

By [Shear, II,5.4,VIII,§1] for every regular $\lambda \leq 2^\kappa$ there is an increasing sequence $\langle \lambda_i : i < \theta \rangle$ of regular cardinals $< \kappa$ with $(\prod\limits_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\theta})$ having true cofinality λ, hence for some such $\langle \lambda_i : i < \theta \rangle$ we can find $f_\alpha \in \prod\limits_{i<\theta} \lambda_i$ for $\alpha < 2^\kappa$ such that $\alpha < \beta \Rightarrow f_\alpha <_{J^{\mathrm{bd}}_\theta} f_\beta$. Now we can prove by induction on α then $\mathrm{rk}_I(f_\alpha) \geq \alpha$ where $I = (\mathcal{F}, <_{J^{\mathrm{bd}}_\theta})$. $\mathcal{F}$ as in clause (c) of 6.37 and we know $f \in \mathcal{F} \Rightarrow \mathrm{rk}_I(f) < \infty$, so we are done. $\square$

§ 6(F). **§4 Reconstructing §3.** discussion: We may try to make §2 more similar to §2. We still have to use Y^+_I and π not order preserving: but we demand

$(g)''$ $\pi_{u,v} \circ \pi_{v,w} = \pi_{u,v}$ for $u \leq_J v \leq_J \leq_J w$.

0) 6.26, 6.28 as before but
1) In clause (A), Definition 6.27(b)(α):

$\circledast$ and $\ell < n(x) \Rightarrow [\pi(t_{\ell+1}(x)) <_{I_1} \pi(s_{\ell+1}(x)) <_{I_2} \pi(t_\ell(x))] \vee [\pi(s_{\ell+1}(x)) <_{I_1} \pi(s_\ell(x)) <_{I_2} \pi(t_\ell(x))]$.

2) Instead 6.30.
3) In Definition 6.31 we define $\hat{\pi}_{u,v}$ a partial homomorphism from G_v into G_u by:
$\hat{\pi}_{u,v}(g_x) = g_{\pi_{u,v}(x)}$ when $x \in Y_{I_v}$ and $x \in \mathrm{Dom}(\pi_{u,v})$
$\hat{\pi}_{u,v}(g_x) = e_{G_u}$ if $x \in Y_{I_v} \setminus \mathrm{Dom}(\pi_{u,v})$
$\hat{\pi}_{u,v}(g_y) = g_{\{\pi_{u,v}(x) : x \in y \cap \mathrm{Dom}(\pi_{u,v})\}}$.
4) 6.32 disappears but we need 5.40(2) with G_{I_u} replaced by $G^+_{I_u}$ notational problem: $+G^+_{\mathfrak{s}}$ of 5.40(5).
5) Repeat 5.41.
6) Repeat 5.42, now easy.
7) Repeat 5.47. SAHARON: From here on: copied part. Old proof of **??**, moved 3/2004, pgs.16-17:

Proof. **??** So toward contradiction assume

$(*)_1$ $h_1 \in G, h_2 \in G \setminus K, h_3 \in G_{\mathbf{p}}$ and $h_3 h_1 h_3^{-1} = h_2$ but for no $h \in G$ do we have $h h_1 h^{-1} = h_2$.

Let $\leq^*$ be as in $\boxdot$ of Claim 5.12. By 5.12(1) we can find $\langle x_{\ell,k} : k = 1,\dots,k^*_\ell \rangle$ for $\ell = 1,2,3$ such that

$(*)_2$ $x_{1,k} \in X^+_{\mathbf{p}}, x_{2,k} \in X_I, x_{3,k} \in X_{I[\mathbf{p}]}$
$(*)_3$ $x_{\ell,1} <^* x_{1,2} <^* \ldots <^* x_{\ell,k^*_\ell}$
$(*)_4$ $h_\ell = g_{x_{\ell,1}} \ldots g_{x_{\ell,k^*_\ell}}$.

Without loss of generality

$(*)'_4$ $x_{\ell,k} \notin (Z^{\mathbf{p}} \setminus Z) \times 2$

[why? By 5.15(1) if we define $h'_\ell = (\ldots g_{x_{\ell,k}} \ldots)_{k \in w(\ell)}$ where $w(\ell) = \{k : x_{\ell,k} \notin (Z^{\mathbf{p}} \setminus Z^{\mathbf{q}}) \times 2\}$ then $G_{\mathbf{p}} \models h'_3 h'_1 (h'_3)^{-1} = h'_2$, but $h'_1 = h_1, h'_2 = h_2$ (as they belong to $G_{\mathbf{q}}$), so without lose of generality $(*)_4$ holds.]

Without loss of generality

$(*)_5$ $G^{<0}_{\mathbf{q}} h_1 = G^{<0}_{\mathbf{q}} h_2$; moreover for some k^{**}_1, k^{**}_2 we have $\langle x_{1,k} : k \in (k^{**}, k^*_2] \rangle = \langle x_{2,k} : k \in (k^{**}_2, k^*_2] \rangle$ and $x_{1,k}$ $(k = 1, k^{**}_1) x_{2,k} (k = 1, \ldots, k^{**}_2) \in Z^{\mathbf{q}} \times 2$

[Why? By 5.15(1) if we define $h'_\ell = (\ldots, g_{\ell,k}, \ldots)_{k \in w(\ell)}$ where $w(\ell) = \{k : x_{\ell,k} \notin Z^{\mathbf{p}}\}$ then we have $h'_3 h'_1 (h'_3)^{-1} = h''_2$ hence by 5.15(2) $h'_3 \in G_{\mathbf{q}}$. So $h'_1, h'_2 \in G\mathbf{q}$ are conjugate in $G_{\mathbf{q}}$ and letting $h''_2 = h_2, h''_1 = h'_3 h_1 (h'_3)^{-1}$, the pair (h''_1, h''_2) satisfies the demands in $(*)_2$ and in $(*)_5$ (and $(*)_4$).]

$(*)_6$ if $a \in Z^{\mathbf{q}}$ then the number $|\{k : x_{1,k} \in \{a\} \times 2\}|, |\{k : x_{1,k} \in \{a\} \times 2\}|$ (both are $\in \{0,1,2\}$) are equal or one is zero and the other is 2 and $|\{k : k^{**}_{1,3} < k \leq k^*_3$ and $a \in A^{\mathbf{q}}_a\}|$ is odd.

[Why? By the proof of 5.12(1).]

Now easily h_1, h_2 are conjugate in $G_{\mathbf{q}}$ so we are done. $\square$

<u>Moved from pg.34,2004/2 from Def. **??**</u>: we define $n(x), t_\ell(x), s_\ell(x)$ for $x \in Y_I$ as we have defined in 5.40 for $x \in X_I$ and let

$$Y^+_I = \{x : x \in Y_I\} \cup [Y_I]^{<\aleph_0}.$$

3) For $u \leq_J v$ let ${}_*\pi^+_{u,v} : Y^+_{I_w} \to Y^+_{I_v}$ be defined as in clause (A) of **??**. For $u \in J$ let ${}_*Y_u = \{y \in Y_v : u, y$ are as in clause (b) of part (1),i.e., $(*)_{u,y}$ holds$\}$. <u>Moved from pg.36,2004/2 from Conclusion 6.16</u>: 2) If $\kappa = \kappa^{<\kappa} > \aleph_0$ then $\tau_\kappa \geq$

$\delta(\kappa)$. [?] <u>Moved from Definition 6.10, part (4)</u>: [?] If $u \leq_{J[\mathfrak{s}]} v$ then $\hat{\pi}_{u,v} = \hat{\pi}^{\mathfrak{s}}_{u,v}$ is

the homomorphism from $G_{\mathbf{p}_2}$ into $G_{\mathbf{p}_1}$ mapping g_x to $g_{\pi^+(x)}$ if $x \in \mathrm{Dom}(\pi^+)$ and to $e_{G_{I_2}}$ if $x \in Y^+_{I_2} \setminus \mathrm{Dom}(\pi^+)$ (hence $X \in Y_{I_2}$).

<u>Moved from Definition **??**</u>: 1) [used?] We say $\mathfrak{s}$ is smooth <u>if</u>: $J' \subseteq J$ is finite

then we can find a directed system $\langle G'_u, \pi'_{u,v} : u \leq_J v$ are from $J' \rangle$ such that $G_u \subseteq G'_u, \pi_{u,v} \subseteq \pi'_{u,v}$.

2) [used?] We say $\mathfrak{s}$ is strongly smooth if

(A) $\mathrm{Rang}(\pi^{\mathfrak{s}}_{u,v}) = I_u$

(B) if $w \in J$ and $x \in Y_{I_w}$ (see below), $n(x) > 0$ <u>then</u> for some y, u (we may write $u = \mathbf{u}(x) = \mathbf{u}^{\mathfrak{s}}_w(x), y = \mathbf{y}_w(x) = y^{\mathfrak{s}}_w(x)$) we have: $u \leq_J w$ and $x \in Y_{I_u}$ and ${}_*\pi^+_{u,w}(x) = y \in Y_{I_u}$ and

$(*)_{u,y}$ if $u \leq_J v, x' \in Y_{I_v}$ and $u' \leq_J v, {}_*\pi^+_{u,v}(x') = y$ then ${}_*\pi^+_{u',v}(x')$ is well defined iff $u \leq_J u'$.

*** <u>Moved from pg.37</u>: (and

??) The following was circumvented by using the linear case (and Definition 6.10(2). The main missing point for §3 is the parallel of **??** replacing "good".

Claim 6.39. *The group* inv-lim($\mathfrak{g}$) *is a κ-automorphism group such that*

(A) J is $\aleph_1$-directed partial order

(B) $\mathfrak{g} = \langle G_u, \pi_{u,v} : u \leq_J v \rangle$ is a weak inverse limit of groups, i.e.

(α) $\pi_{u,v} \in \mathrm{Hom}(G_v, G_u)$

(β) *if* $u \leq_J v \leq_J w$ *then* $\pi_{u,w} \supseteq \pi_{u,v} \circ \pi_{v,w}$

(C) $\kappa \geq |J| + \Sigma\{\|G_u\| : u \in J\}$.

Proof. Fill. [used?] $\square$

Claim 6.40. *[used?] 1) If* [3] $\mathfrak{s}$ *is a smooth* κ-p.o.w.i.s. *then* $\langle G^{\mathfrak{s}}_u, \hat{\pi}^{\mathfrak{s}}_{u,v} : u \leq_{J[\mathfrak{s}]} v\rangle$ *is a smooth inverse system of groups.*
2) If $\mathfrak{s}$ *is linear* κ*-p.o.w.i.s* *then* $\mathfrak{s}$ *is a strongly smooth* κ*-p.o.w.i.s.*
3) If $\mathfrak{s}$ *is strongly smooth then* $\mathfrak{s}$ *is smooth.*

Proof. 1) Easy.
2),3) Not used and implicit in the proof of **??**. $\square$

Moved from the proof of **??**,pg.16,17: We define a function π from $\{g_x : x \in X_{\mathbf{p}}\} \subseteq G_{\mathbf{p}}$ into $G_{\mathbf{p}}$ as follows:

⊛ (a) if $x \in X$ then $\pi(g_x) = g_x$

(b) if $x \in X^+_{\mathbf{p}} \setminus X$ then $\pi(g_x) = e_{G_{\mathbf{p}}}$.

Part A: This mapping π respects the equations from $\Gamma_{\mathbf{p}}$ hence can be extended to a homomorphism $\breve{\pi}$ from $G_{\mathbf{p}}$ into $G_{\mathbf{p}}$, in fact into G which is a subgroup of $G_{\mathbf{p}}$.

Now towards contradiction suppose $h \in G \setminus K$ belongs to $N_{\ell-1}$ to the normal subgroup of $G_{\mathbf{p}}$ which K generates. Clearly h is equal to a product of conjugates of members of K, i.e., for some $n < \omega$ and $h_\ell \in K, g_\ell \in G_{\mathbf{p}}$, for $\ell < n$ we have $G_{\mathbf{p}} \models h = (g_0 h g_0^{-1})(g_1 h_1 g_1^{-1}) \ldots (g_{n-1} h_{n-1} g_{n-1}^{-1})$. This implies that

$$\begin{aligned}\breve{\pi}(h) =&(\breve{\pi}(g_0)\breve{\pi}(h_0)(\breve{\pi}(g_0))^{-1})((\breve{\pi}(g_1)\\ &(\breve{\pi}(h_1)(\breve{\pi}(g_1))^{-1})\ldots(\breve{\pi}(g_{n-1})(\breve{\pi}(h_{n-1})(\breve{\pi}(g_{n-1})^{-1}).\end{aligned}$$

Now there are $g'_\ell \in G$ such that $\pi(g_\ell) = \pi(g'_\ell)$ for $\ell < n$.

[Why? We apply 5.12. Let $<^*$ be as there, we can find $x_{\ell,1} <^* \ldots <^* x_{\ell,k(\ell)}$ from $X^+_{\mathbf{p}}$ such that $G_{\mathbf{p}} \models g_\ell = g_{x_{\ell,1}} \cdots g_{x_{\ell,k(\ell)}}$, so by the choice of $<^*$ for some $k_1(\ell) \leq k(\ell)$ we have $x_{\ell,i} \in Z^{\mathbf{p}} \times 2 \Leftrightarrow i \leq k_1(\ell)$, let $w_\ell =: \{i : i \in \{1, \ldots, k(\ell)\}$ and $x_{\ell,i} \in X \cup (Z \times 2)$ and $g'_\ell = (\ldots g_{x_{\ell,i}} \ldots)_{i \in w_\ell}$. Now check.]

Hence $\breve{\pi}(h)$ is equal to $(\breve{\pi}(g'_0)\breve{\pi}(h_0)(\breve{\pi}(g'_0))^{-1})(\breve{\pi}(g'_1)\breve{\pi}(h_1)(\breve{\pi}(g'_1))^{-1}) \ldots (\breve{\pi}(g'_{n-2})(\breve{\pi}h_{n-1})\breve{\pi}(g_{n-1})^{-1}) = \breve{\pi}((g'_0 h_0 (g'_0)^{-1})(g'_1 h_1 (g'_1)^{-1}) \ldots (g'_{n-1} h_{n-1} (g'_{n-1})^{-1})$. As $h_\ell \in K, g'_\ell \in G$ and $K \triangleleft G$ clearly $(g'_0 h_0 (g'_0)^{-1})(g'_1 h_1 (g'_1)^{-1}) \ldots (g'_{n-1} h_{n-1} (g'_{n-1})^{-1})$ belongs to K call it h', so $\breve{\pi}(h) = \pi(h'), h' \in K$. So $h^* = h(h')^{-1} \in \mathrm{Ker}(\breve{\pi})$, now $h \in G \setminus K, h' \in K$ hence also $h^* \in G \setminus K$ and $\breve{\pi}(h^*) = \breve{\pi}(h)(\pi(h'))^{-1} = e_G$ and h^* is the product of conjugates of members of members K in $G_{\mathbf{p}}$.

As $h^* \in G$, by 5.12(1) apply to G let $h^* = g_{x_1} \ldots g_{x_m}$ where $x_1, \ldots, x_m$ from $X \cup (Z \times 2), x_1 <^* \ldots <^* x_m$ where $<^*$ is as in 5.12. By the demands on $<^*$ there for some $m(*) \leq m$ we have $x_1, \ldots, x_{m(*)} \in Z \times 2$ while $x_{m(*)+1}, \ldots, x_m \in X$, hence $e_{G_{\mathbf{p}}} = \breve{\pi}(h^*) = g_{x_{m(*)+1}} \cdots g_{x_m}$, so $h^* = g_{x_1} \ldots g_{x_{m(*)}}$ and let $x_\ell = (\alpha_\ell, i_\ell)$, so $\alpha_\ell \in Z, m_\ell < 2$. Clearly

$(*)_1$ for every $g \in G_{\mathbf{p}}$ the element $g h^* g^{-1}$ belongs to $\langle \{g_{(\alpha_\ell, i_0} : i < 2$ and $\ell = m(*), \ldots, m\}\rangle_{G_{\mathbf{p}}}$.

[3] in fact, $\hat{\pi}^{\mathfrak{s}}_{u,v}$ was defined such that this holds

So we shall be done if we prove $(*)_2$. Part B: Moved from 6.1,pg.30-31:

Proof. We define $\mathfrak{s} = (J, \bar{\mathbf{p}}, \bar{\pi})$ as follows:

(A) $J = (\theta; <)$

(B) $I_\alpha = (\mathcal{T}_{\alpha+1}, <_{\alpha+1})$ for $\alpha < \theta$ where

$$f_1 <_{\alpha+1} f_2 \Leftrightarrow f_1(\alpha) < f_2(\alpha)$$

(C) for $\alpha < \beta < \theta$ let $\pi_{\alpha,\beta} : I_\beta \to I_\alpha$ be

$$\pi_{\alpha,\beta}(f) = f \restriction (\alpha + 1).$$

(compare with 6.14)! □

Moved from proof of 6.7,pg.32: Hence, letting $Z^{\mathbf{p}_2} \cap \mathrm{Dom}(\pi)$

$(**)$ $G = \langle \{g_x : x \in X \cup (Z \times 2)\} \rangle_{G[\mathbf{p}_2]}$ is generated by $\{g_x : x \in X \cup (Z \times 2)\}$ freely except the equation is $\Gamma = \{\varphi \in \Gamma_{\mathbf{p}_2} : \varphi$ mention only g_x with $x \in X \cup (Z \times 2)\}$.

Moved from before 6.13,pg.33: Now the proof is similar to 5.42(2); it is enough to prove this for $\hat{\pi}$, for this it suffices to show that $\hat{\pi}$ maps the equations in Γ into $\Gamma^+_{I_2}$ and this is proved as in the proof of clause (A) in the proof of 5.42(2).

Claim 6.41. *If* $\mathbf{p}_1 \le \mathbf{p}_2$ *are* κ*-parameters,* then $X_{\mathbf{p}_1} \subseteq X_{\mathbf{p}_2}, X^+_{\mathbf{p}_1} \subseteq X^+_{\mathbf{p}_2}, \Gamma_{\mathbf{p}_1} \subseteq \Gamma_{\mathbf{p}_2}$ *and* $G_{\mathbf{p}_1}$ *is a subgroup of* $G_{\mathbf{p}_2}$. *[???]*

Moved from pg.36:

Definition 6.42. Let $\mathfrak{s}$ be a κ-p.o.w.i.s and $J = J^{\mathfrak{s}}$, etc.
1) $\mathfrak{s}$ is linear if $J^{\mathfrak{s}}$ is a linear order.

§ 6(G). **alternative 2.** Moved from Definition **??**,pg.7: 4) We say that $\mathbf{p}$ is a very nice parameter if in addition

(A) if $x_1, \ldots, x_k \in X_{\mathbf{p}}$ and $s \in Z^{\mathbf{p}}$ then there is $x \in X_{\mathbf{p}}$ such that $s \in A^{\mathbf{p}}_x$ and $\ell \in \{1, \ldots, k\} \wedge n < \omega \Rightarrow x \neq x_\ell \restriction n \wedge x_\ell \neq x \restriction n$; note even $t(x_\ell \restriction 0)$ is a well defined member of I (not used presently)[used? so we shall ignore]

(B) if $x \in X \cup \{\langle\ \rangle\}$, $\mathrm{rk}_{I[\mathbf{p}]}(t^x_{m(x)}) > 0, \ell < \omega, m < \omega, z_0, \ldots, z_{m-1} \in Z_{\mathbf{p}}$ are pairwise distinct, $u_\nu \subseteq [0, m)$ for $\nu \in {}^{n(x)+1}2$ then there are infinitely many $s \in I^{\mathbf{p}}$ such that

  $(*)$ (α) if $\mathrm{rk}_{I[\mathbf{p}]}(t^x_{n(*)}) \ge \omega$ or $x = \langle\ \rangle$ then $\mathrm{rk}_{I[\mathbf{p}]}(t) \ge \ell$

  (β) if $0 < \mathrm{rk}_{I[\mathbf{p}]}(t^x_{n(*)}) < \omega$ then $\mathrm{rk}_{I[\mathbf{p}]}(t) = \mathrm{rk}_{I[\mathbf{p}]}(t^x_{n(*)}) - 1$

  (γ) for some $y_\nu \in X_I(\nu \in {}^{n(x)+1}2)$ we have: $\eta^{y_\nu} = \nu, [x = \langle\ \rangle \Rightarrow n(y_\nu) = 1], [x \neq \langle\ \rangle \Rightarrow \bar{t}^{y_\nu} \restriction n(x) = \bar{t}^x], t^{y_\nu}_{n(x)} = s$ and $k < m \wedge (\nu \in {}^{n(x)+1}2) \wedge (\mathrm{rk}^2_{\mathbf{p}}(y_\nu) = 0 \Rightarrow z_k \in Y) \Rightarrow [(z_k \in A^{\mathbf{p}}_{y_\nu}) \equiv (k \in u_\nu)]$ (this will serve us in the proof of **??**).

Moved from Definition 5.36,p.16:

(A) if $\pi^+(x_1) = x_2$ and π is a strict homomorphism and $x_1 \in Z^{\mathbf{p}_1} \times Z$ or $x_1 = (\langle t_\ell : \ell \le n\rangle, \eta), t_\ell \in \mathrm{Dom}(\pi)$ and $\eta \in {}^n Z$ then $(x_1 \in X_{\mathbf{p}_1} \Leftrightarrow x_2 \in X_{\mathbf{p}_2})$ and $(x_1 \in X^+_{\mathbf{p}_1} \setminus X_{\mathbf{p}_1}) \Leftrightarrow (x_2 \in X^+_{\mathbf{p}_2} \setminus X_{\mathbf{p}_2})$

(B) if $\pi^+(x_1) = x_2$ and $\pi^+(y_1) = y_2$ and π is a partial isomorphism or strict homomorphism then $\circledast_{x_1,y_1}$ in Definition 5.2(4)(b) holds (for $\mathbf{p}_1$) iff $\circledast_{x_2,y_2}$ in Definition 5.2(4)(b) holds (for $\mathbf{p}_2$); in fact, this holds for each of clauses (i), (ii) there separately; for the only if part (i.e., the $\Rightarrow$ implication) we do not need the "π a partial isomorphism"

(C) if π is a strict homomorphism then the mapping $g_x \mapsto g_{\pi^+(x)}$ for $x \in X^+_{\mathbf{p}_1}$ maps every equation from $\Gamma_{\mathbf{p}_1}$ to an equation from $\Gamma_{\mathbf{p}_2}$ is not used.

2) If π is a partial isomorphism from $\mathbf{p}_1$ to $\mathbf{p}_2$ then the mapping $g_x \mapsto g_{\pi^+(x)}$ maps $\Gamma_{\mathbf{p}_1, X_{I_1} \restriction \mathrm{Dom}(\pi), Z \cap \mathrm{Dom}(\pi)}$ onto $\Gamma_{p_2, X_{I_2} \restriction \mathrm{Rang}(\pi), Z \cap \mathrm{Rang}(\pi)}$ [not used]. Moved from pgs.17-19:

Claim 6.43. [4] *1) The normal subgroup N of $G_{\mathbf{p}}$ which K generates satisfies $N \cap G = K$ when*

$\circledast$ (a) $\mathbf{p}$ *is a κ-parameter*

(b) $\mathbf{q}$ *is a very nice κ-parameter, $\mathbf{q} \le \mathbf{p}$ and $t \in I^{\mathbf{q}} \Rightarrow \min\{\omega, \mathrm{rk}_{I[\mathbf{q}]}(t)\} = \min\{\omega, \mathrm{rk}_{I[\mathbf{p}]}(t)\}$ and $G \equiv G_{\mathbf{q}}$*

(c) *K is a normal subgroup of G.*

2) We can replace clause (b) by

$(b)'$ (α) *$G = G_{\mathbf{p},X,Z}$, see Definition* **??***(4)) where $Z \subseteq Z^{\mathbf{p}}$ and $X \subseteq X_I$ is*

closed under restriction (i.e., $y = x \restriction n, x \in X \Rightarrow y \in X$) and:

if $x \in X, y \in X_{\mathbf{p}}$ and $\bar t^y = \bar t^x$ then $y \in X$

(β) *if $x \in X \cup \{\langle\ \rangle\}, n < \omega, z_0, \dots, z_{m-1} \in Z$ are pairwise distinct and*

$u_\eta \subseteq [0, m)$ for $\eta \in {}^{n(x)+1}2$ then there are infinitely many

$s \in I^{\mathbf{p}}$ such that $(\exists y \in X_p)(\bar t^y = \bar t^{x\,\frown} < s > inX)$ and $(*)$ from **??***(4)(e) holds.*

Proof. 1) Let $X = X_{\mathbf{q}}, Z = Z^{\mathbf{q}}, X^+ = X \cup (Z \times 2)$ and apply (2), possible: by clause (e) of **??**(4); (note that in (β) we even get $t \in I^{\mathbf{q}}$.

2) By 5.12(7), G is a subgroup of $G_{\mathbf{p}}$ and is generated by $g_x : x \in X \cup (Z \times 2)$ freely except the equations $\Gamma_{\mathbf{p},X,Z}$.

Assume $h \in G$ is a product of conjugates of members of K in $G_{\mathbf{p}}$. Let $<^*$ be as in $\boxdot$ of 5.12. So assume $G_{\mathbf{p}} \models$ "$h = (g_0 h_0 g_0^{-1}) \dots (g_{n-1} h_{n-1} g_{n-1}^{-1})$" where $h_0, \dots, h_{n-1} \in K, g_0, \dots, g_{n-1} \in G_{\mathbf{p}}$. By 5.12(7) we can find a sequence $\langle x_{\ell,k} : k = 1, \dots, k(\ell)\rangle$ which is $<^*$-increasing in $X^+_{\mathbf{p}}$ such that $g_\ell = g_{x_{\ell,1}} \cdots g_{x_{\ell,k(\ell)}}$. We can find $\langle y_{\ell,m} : m = 1, \dots, m(\ell)\rangle$ in $X \cup (Z \times 2)$ such that $G \models$ "$h_\ell = g_{y_{\ell,1}} \cdots g_{y_{\ell,m(\ell)}}$" (exists as $h_\ell \in K \subseteq G$), let $h = g_{z_0} \dots g_{z_{i-1}}$ where $j < i \Rightarrow z_j \in X \cup (Z \times 2)$; exists as $h \in G$.

Now we apply clause (β) of $(b)'$ in the assumption of part (2) of the claim (use it inductively to choose replacements). In detail let $Z^* \subseteq Z$ be the set of $s \in Z$ such that for some $m < 2$ we have $(s, m) \in \{x_{\ell,k} : \ell < n, k = 1, \dots, k(\ell)\} \cup \{y_{\ell,m} : \ell < n, m = 1, \dots, m(\ell)\} \cup \{z_j : j < i\}$. Let $X^{**} \subseteq X^{\mathbf{p}}$ be the minimal set such that

[4] used in the end of 5.48

$\circledast$ (a) each $x_{\ell,k}, z_i$ and $y_{\ell,m}$ belongs to it or to $Z^* \times 2$,

(b) $[x \in X^{**} \wedge n \le n(x) \Rightarrow x \restriction n \in X^{**}]$

(c) $[y', y'' \in X_{\mathbf{p}} \wedge \bar t^{y'} = \bar t^{y''} \Rightarrow y' \in X^{**} \equiv y'' \in X^{**}]$.

Let $X^* = X^{**} \cap X$; clearly X^{**} too is finite; clearly $x \in X^* \wedge m \le n(x) \Rightarrow x \restriction n \in X^*$, see 9α) of $(b)'$. Let $\langle \bar t^i : i < i(*)\rangle$ list $\{\bar t^y : y \in X^{**}\}$ with no repetitions such that $\bar t^i \triangleleft \bar t^j \Rightarrow i < j$. Let $n_i = \ell g(\bar t_i) - 1$ so $\bar t^i = \langle t^i_\ell : \ell \le n_i\rangle$. Now we choose $\bar s^i$ by induction on $i < i(*)$ such that

$\boxdot$ (i) $\ell g(\bar s^i) = n_i + 1 (= \ell g(\bar t^i))$ so $\bar s^i = \langle s^i_\ell : \ell \le n_i\rangle$

(ii) if $x \in X^*$ then $\bar t^x \trianglelefteq \bar t^i \equiv \bar s^x \trianglelefteq \bar s^j$ and $\bar t^i \trianglelefteq \bar t^x \equiv \bar s^j \trianglelefteq \bar s^x$

(iii) $A^{\mathbf{p}}_{(\bar t^i,\nu)} \cap (Z^* \times 2) = A^{\mathbf{p}}_{(\bar s^i,\nu)} \cap (Z^* \cap 2)$ for $\nu \in {}^{n_i-1}2$

(iv) $\mathrm{Min}\{\mathrm{rk}_{\mathbf{p}}(s^i_{n_i}), i(*) + |X^{**}| - i\} = \min\{\mathrm{rk}_{\mathbf{p}}(t^i_{n_i}), i(*) + |X^{**}| - i\}$

(v) $\bar s^i \in \{\bar t^x : x \in X\}$.

The induction step is possible by assumption $(b)'(\beta)$ for $\ell < n$ let $g'_\ell = g'_{\ell,1} \ldots g'_{\ell,k(\ell)}$ where:

$\circledast$ (a) $g'_{\ell,k}$ is $g_{y'_{\ell,k}}$ if $y_{\ell,k} \in X_{\mathbf{p}}$ and $y'_{\ell,k}$ is defined by: $\bar t^{y_{\ell,k}} = \bar t_i \Rightarrow y'_{\ell,k} = (\bar s_i, \eta^{y_{\ell,k}})$,

(b) $g'_{\ell,k} = e_{G_{\mathbf{p}}}$ if $y_{\ell,k} \in (Z^{\mathbf{p}} \setminus Z^*) \times 2$

(c) $g'_{\ell,k} = g_{\ell,k}$ if $y_{\ell,k} \in Z^* \times 2$.

We define h' by

$(*)$ $G \models$ "$h' = (g'_0 h_0 (g'_0)^{-1}) \ldots (g'_{n-1} h_{n-1} (g'_{n-1})^{-1})$".

So

$\odot$ $g'_{\ell,k} \in G, g'_\ell \in G$ and $h' \in K$.

[Why? First, $g'_{\ell,k} \in G$ as can be checked by cases in $\boxdot$.
Second, $g'_\ell \in G$ as the product of $\langle g'_{\ell,k} : k = 1, \ldots, k(\ell)\rangle$.
Third, h' is a product of conjugates by members of G of the members of K but K is a normal subgroup of G hence $h' \in K$.]
So it is enough to prove that $h' = h$.

Let $\mathbf{f}$ be the function with domain X^{**} such that $\mathbf{f}(\bar t^i, \eta) = (\bar s^i, \eta)$ when $i < i(*), \eta \in {}^{\ell g(\bar t^i)-1}2$.

Clearly

$(*)_1$ $\mathbf{f}$ is a one to one function from X^{**} into X

$(*)_2$ the subgroup $G_{\mathbf{p},X^{**},Z^*} = \langle \{g_y : y \in X^{**} \cup (Z^* \times 2)\rangle_{G_{\mathbf{p}}}$ is generated freely by $\{g_y : y \in X^{**} \cup (Z^* \times 2)\}$ except the equations in $\Gamma_{\mathbf{p},X^{**},Z^*}$.
[Why? As the demand in 5.12(7) holds.]

$(*)_3$ the subgroup $G_{\mathbf{p},\mathbf{f}(X^{**}),Z^*} = \langle \{g_y : y \in \mathbf{f}(X^{**}) \cup (Z^* \times 2)\}\rangle_{G_{\mathbf{p}}}$ is generated freely by $\{g_y : y \in \mathbf{f}(X^{**}) \cup (Z^* \times 2)\}$ except the equations in $\Gamma_{\mathbf{p},\mathbf{f}(X^{**}),Z^*}$.
[Why? Similarly.]

$(*)_4$ $\mathbf{f}$ maps $\Gamma_{\mathbf{p},X^{**},Z^*}$ onto $\Gamma_{\mathbf{p},\mathbf{f}(X^{**}),Z^*}$,
[Why? By the choice of $\bar s^i$-s.]
hence

$(*)_5$ $\mathbf{f}$ induces an isomorphism $\hat{\mathbf{f}}$ from $G_{\mathbf{p},X^{**},Z^*}$ onto $G_{\mathbf{p},\mathbf{f}(x^{**}),Z^*}$

$(*)_6$ $\mathbf{f}$ is the identity on X^* hence $\hat{\mathbf{f}}$ is the identity on $G_{\mathbf{p},X^*,Z^*}$
but

$(*)_7$ $h, h_0, \ldots, h_{n-1}, g_0, \ldots, g_{n-1}$ belongs to $\langle \{g_y : y \in X^* \cup (Z^* \times 2)\}\rangle_{G_{\mathbf{p}}}$ hence $\hat{\mathbf{f}}$ maps each of them to itself and

$(*)_8$ $\hat{\mathbf{f}}$ maps g_ℓ to g'_ℓ, hence recalling $G \models$ "$h = (g_0 h_0 g_0^{-1}) \dots (g_{n-1}, h_{n-1}, g_{n-1}^{-1})$ we deduce by $(*)$, $\hat{\mathbf{f}}(h) = h'$.

But $h \in G_{\mathbf{p},X^*,Z^*}$ so by $(*)_6$ $\mathbf{f}(h) = h'$ implies $h = h'$ and $h' \in G$ so we are done. $\square_{??}$ $\square$

Moved from Claim 6.6, pg.35:

(A) if $\pi^+(x_1) = x_2$ and $\pi^+(y_1) = y_2$ then $\circledast_{x_1,y_1}$ in Definition 5.2(4)(b) holds (for $\mathbf{p}_1$) iff $\circledast_{x_2,y_2}$ in Definition 5.2(4)(b) holds (for $\mathbf{p}_2$); in fact, this holds for each of clauses (i), (ii) there separately.

Moved from pg.19:

Definition 6.44. [Here?] 1) For π a homomorphism from $\mathbf{p}_1$ to $\mathbf{p}_2$, let $\hat{\pi}$ be the partial homomorphism from $F_{\mathbf{p}_1}$ into $F_{\mathbf{p}_2}$ induced by the mapping $g_x \mapsto g_{\pi^+(x)}$ for $x \in X^+_{\mathbf{p}_1}$, if there is one.
2) Similarly for π a partial homomorphism.

Moved 2005/8 from the proof of 3.6,pgs.31,32: For $p \in \mathcal{S}^{k^*}$ let $\langle k_\ell(p) : \ell < \ell(p)\rangle$ list $\mathrm{supp}(p)$, see Definition 2.5(9),(10), in increasing order, let $\bar{s} = \langle t_{\bar{k}_\ell(p_u)} : \ell < \ell(p_u)\rangle$, and let $B_p = \{k_\ell(p) : \ell < \ell(p)\}$ and let $B := \{B_p : p \in \mathcal{S}^*\}$

$\boxdot_7$ if $p \in \mathcal{S}^{k^*}$ and $u_1 \le_{J[\mathfrak{t}]} u_2$ are from Y_p then $\pi^{\mathfrak{s}}_{u_1,u_2}(\bar{s}^{u_2})$ is a permutation of $\bar{s}^{u_2}$

$\boxdot_8$ for $p \in \mathcal{S}^{k^*}$ and $u_1 \le_{J[\mathfrak{t}]} u_2$ from Y_p, let $h = \{(\ell_1,\ell_2) : \ell_1,\ell_2 < k^*$ and $\pi^{\mathfrak{s}}_{u_1,u_2}(t^{u_2}_{\ell_2}) = t^{u_2}_{\ell_1}$.

[Why? By a claim 2.7.]

Let E_p be an ultrafilter on $J^{\mathfrak{t}}$ such that $Y_p \in E_p$ and $u \in J^{\mathfrak{t}} = \{v : u \le_{J[\mathfrak{t}]} v\} \in E_p$, exists as $J^{\mathfrak{t}}$ is directed (actually one E suffice). So for each $p \in \mathcal{S}^*$ and $u \in Y_p$ thee are $A_{p,u} \in E_p$ and $h^*_{p,u}$ such that $v \in A_{p,u} \Rightarrow u \le_{J[\mathfrak{t}]} v$ and $h_{u,v} = h^*_{p,u}$. So without lose of generality

$\boxdot_9$ if $p \in \mathcal{S}^*$ and $u_1 \le_{J[\mathfrak{t}]} u_2$ are from Y_p then $k \in B_p \Rightarrow \pi^{\mathfrak{s}}_{u_1,u_2}(t^{u_2}_k) = t^{u_1}_k$.

[Why? We replace $\bar{t}^u \restriction B_p$ by $\langle \pi_{u,v}(t^v_k) : k \in A_p\rangle$ for the E_p-majority of v-s.]

Let $\mathcal{S}' = \{p \in \mathcal{S}^k : u \in Y_p$ for some $u \in J^{\mathfrak{t}}\}$. Without loss of generality

$\boxdot_{10}$ $k^* = \bigcup\{A_p : p \in \mathcal{S}'\}$.

By clause (f) of Definition 3.1 for each $\ell \in B$ there is a $t^{v^*}_\ell$ such that

$\boxdot_{11}$ $t^{v^*}_\ell \in J^{\mathfrak{s}}_{v^*}$ and $u^* \le_{J[\mathfrak{t}]} u \in J^{\mathfrak{t}} \Rightarrow \pi^{\mathfrak{s}}_{u,v^*}(t^{v^*}_\ell) = t^v_\ell$.

By clause (d) of Definition 2.1(1) for some $u_* \in J$

$\boxdot_{12}$ $u^* \le_{J[\mathfrak{t}]} u_*$ and if $u_* \le_{J[\mathfrak{t}]} u$ then $p_u = p^* := \mathrm{tp}_{\mathrm{qf}}(\langle t^{v^*}_\ell : \ell < k^*\rangle, \varnothing, I^{\mathfrak{s}}_{v^*})$.

Let $\bar{f} = \langle f_u : u \in J^{\mathfrak{t}}\rangle$ be defined as $f^*_u = g^u_{\bar{t}^{u'},\mathbf{q}^*(p_*)}$, see Definition 2.5. So

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