STABLE FRAMES AND WEIGHTS
E108

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Abstract. Was paper 839 in the author-s list till winter 2023 when it was divided to three.

Part I: We would like to generalize imaginary elements, weight of $\mathrm{ortp}(a, M, N)$, $\mathbf{P}$-weight, $\mathbf{P}$-simple types, etc. from [She90, Ch.III,V,§4] to the context of good frames. This requires allowing the vocabulary to have predicates and function symbols of infinite arity, but it seemed that we do not suffer any real loss.

Part II: Good frames were suggested in [She09d] as the (bare bones) right parallel among a.e.c. to superstable (among elementary classes). Here we consider (μ, λ, κ)-frames as candidates for being the right parallel to the class of $|T|^+$-saturated models of a stable theory (among elementary classes). A loss as compared to the superstable case is that going up by induction on cardinals is problematic (for cardinals of small cofinality). But this arises only when we try to lift. For this context we investigate the dimension.

Part III: In the context of Part II, we consider the main gap problem for the parallel of somewhat saturated model; showing we are not worse than in the first order case.

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Anotated Content

Part I: Beautiful frames: weight and simplicity

§ 0. Introduction

We consider here the directions listed in the abstract[1]

In part I we assume $\mathfrak{s}$ is a good λ-frame with some extra properties from [She09e], e.g. as in the assumption of [She09e, §12], so we shall assume knowledge of [She09e], and the basic facts on good λ-frames from [She09c].

We can look at results from [She90] which were not regained in beautiful λ-frames. Well, of course, we are far from the main gap for the original $\mathfrak{s}$ ([She90, Ch.XIII]) and there are results which are obviously more strongly connected to elementary classes, particularly ultraproducts. This leaves us with parts of type theory: semi-regular types, weight, **P**-simple [2] types, "hereditarily orthogonal to **P**" (the last two were defined and investigated in [She78, Ch.V,§0 + Def4.4-Ex4.15], [She90, Ch.V,§0,pg.226,Def4.4-Ex4.15,pg.277-284]).

Note that "a type q is p-simple (or **P**-simple)" and "q is hereditarily orthogonal to p (or **P**)" are essentially the [3] "internal" and "foreign" in Hrushovshi's profound works.

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[1]As we have started this in 2002 and have not worked on it for long, we intend to make public what is in reasonable state.

[2]The motivation is for suitable $\mathbf{P}$ (e.g. a single regular type) that on the one hand $\text{stp}(a, A) \pm \mathbf{P} \Rightarrow \text{stp}(a/E, A)$ is **P**-simple for some equivalence relation definable over A and on the other hand if $\text{stp}(a_i, A)$ is **P**-simple for $i < \alpha$ then $\Sigma\{w(a_i, A) \cup \{a_j : j < i\}) : i < \alpha\}$ does not depend on the order in which we list the a_i's. Note that **P** here is $\mathscr{P}$ there.

[3]Note, "foreign to **P**" and "hereditarily orthogonal to **P**" are equivalent. Now ($\mathbf{P} = \{p\}$ for ease)

- (a) $q(x)$ is $p(x)$-simple when for some set A, in $\mathfrak{C}$ we have $q(\mathfrak{C}) \subseteq \text{acl}(A \cup \bigcup p_i(\mathfrak{C}))$
- (b) $q(x)$ is $p(x)$-internal when for some set A, in $\mathfrak{C}$ we have $q(\mathfrak{C}) \subseteq \text{dcl}(A \cup p(\mathfrak{C}))$.

Note

- (α) internal implies simple
- (β) if we aim at computing weights it is better to stress acl as it covers more
- (γ) but the difference is minor and
- (δ) in existence it is better to stress dcl, also it is useful that $\{F \restriction (p(\mathfrak{C}) \cup q(\mathfrak{C}) : F$ an automorphism of $\mathfrak{C}$ over $p(\mathfrak{C}) \cup \text{Dom}(p)\}$ is trivial when $q(x)$ is p-internal but not so for p-simple (though form a pro-finite group).

§ 1. I, Weight and P-weight

Recalling [She09c], [She09e]

Context 1.1. 1) $\mathfrak{s}$ is a full good$^+$ λ-frame, with primes, $K^{3,\mathrm{vq}}_{\mathfrak{s}} = K^{3,\mathrm{qr}}_{\mathfrak{s}}, \perp = \underset{\mathrm{wk}}{\perp}$ and $p \perp M \Leftrightarrow p \underset{\mathrm{a}}{\perp} M$, note that as $\mathfrak{s}$ is full, $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) = \mathscr{S}^{\mathrm{na}}_{\mathfrak{s}}(M)$; also $\mathfrak{k}_{\mathfrak{s}} = \mathfrak{k}[\mathfrak{s}] = (K^{\mathfrak{s}}, \le_{\mathfrak{k}_{\mathfrak{s}}})$ is the a.e.c.
2) $\mathfrak{C}$ is an $\mathfrak{s}$-monster so it is $K^{\mathfrak{s}}_{\lambda^+}$-saturated over λ and $M <_{\mathfrak{s}} \mathfrak{C}$ means $M \le_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ and $M \in K_{\mathfrak{s}}$. As $\mathfrak{s}$ is full, it has regulars.

Observation 1.2. $\mathfrak{s}^{\mathrm{reg}}$ *satisfies all the above except being full.*

Proof. See [She09e, 10.18=L10.p19tex] and Definition [She09e, 10.17=L10.p18tex]. $\square$

Claim 1.3. *1) If $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ then we can find b, N and a finite $\mathbf{J}$ such that:*

$\circledast$ (a) $M \le_{\mathfrak{s}} N$
(b) $\mathbf{J} \subseteq N$ *is a finite independent set in* (M, N)
(c) $c \in \mathbf{J} \Rightarrow \mathrm{ortp}(c, M, N)$ *is regular, recalling* ortp *stands for orbital type*
(d) $(M, N, \mathbf{J}) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$
(e) $b \in N$ *realizes* p.

2) We can add, if M is brimmed, that

(f) $(M, N, b) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$.

3) In (2), $|\mathbf{J}|$ depends only on (p, M).
4) If M is brimmed, then we can work in $\mathfrak{s}$(brim) and get the same $\|\mathbf{J}\|$ and N (so $N \in K_{\mathfrak{s}}$) brimmed.

Proof. 1) By induction on $\ell < \omega$, we try to choose N_ℓ, a_ℓ, q_ℓ such that:

$(*)$ (a) $N_0 = M$
(b) $N_\ell \le_{\mathfrak{s}} N_{\ell+1}$
(c) $q_\ell \in \mathscr{S}_{\mathfrak{s}}(N_\ell)$, so possibly $q_\ell \notin \mathscr{S}^{\mathrm{na}}_{\mathfrak{s}}(N_\ell)$
(d) $q_0 = p$
(e) $q_{\ell+1} \restriction N_\ell = q_\ell$
(f) $q_{\ell+1}$ forks over N_ℓ so now necessarily $q_\ell \notin \mathscr{S}^{\mathrm{na}}_{\mathfrak{s}}(N_\ell)$
(g) $(N_\ell, N_{\ell+1}, a_\ell) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$
(h) $r_\ell = \mathrm{ortp}(a_\ell, N_\ell, N_{\ell+1})$ is regular
(i) r_ℓ either is $\perp M$ or does not fork over M.

If we succeed to carry the induction for all $\ell < \omega$ let $N = \cup\{N_\ell : \ell < \omega\}$; as this is a countable chain, there is $q \in \mathscr{S}(N)$ such that $\ell < \omega \Rightarrow q \restriction N_\ell = q$ and as q is not algebraic (because each q_n is not), and $\mathfrak{s}$ is full, clearly $q \in \mathscr{S}_{\mathfrak{s}}(N)$; but q contradicts the finite character of non-forking. So for some $n \ge 0$ we are stuck, but this cannot occur if $q_n \in \mathscr{S}^{\mathrm{na}}_{\mathfrak{s}}(N_n)$. [Why? By 1.2, equivalently $\mathfrak{s}^{\mathrm{reg}}$ has enough regulars and then we can apply [She09e, 8.3=L6.1tex].] So for some $b \in N_n$ we have $q_n = \mathrm{ortp}(b, N_n, N_n)$, i.e., b realizes q_n hence it realizes p.

Let $\mathbf{J} = \{a_\ell : \text{ortp}(a_\ell, N_\ell, N_{\ell+1})$ does not fork over $N_0\}$. By [She09e, 6.2] we have $(M, N_n, \mathbf{J}) = (N_0, N_n, \mathbf{J}) \in K^{3,\text{vq}}_{\mathfrak{s}}$ hence $\in K^{3,\text{qr}}_{\mathfrak{s}}$ by [She09e] so we are done.
2) Let $N, b, \mathbf{J}$ be as in part (1) with $|\mathbf{J}|$ minimal. We can find $N' \le_{\mathfrak{s}} N$ such that $(M, N', b) \in K^{3,\text{pr}}_{\mathfrak{s}}$ and we can find $\mathbf{J}'$ such that $\mathbf{J}' \subseteq N'$ is independent regular in (M, N') and maximal under those demands. Then we can find $N'' \le_{\mathfrak{s}} N'$ such that $(M, N'', \mathbf{J}') \in K^{3,\text{qr}}_{\mathfrak{s}}$. If $\text{ortp}_{\mathfrak{s}}(b, N'', N') \in \mathscr{S}^{\text{na}}_{\mathfrak{s}}(N'')$ is not orthogonal to M we can contradict the maximality of $\mathbf{J}'$ in N' as in the proof of part (1), so $\text{ortp}_{\mathfrak{s}}(b, N'', N') \perp M$ (or $\notin \mathscr{S}^{\text{na}}_{\mathfrak{s}}(N)$). Also without loss of generality $(N'', N', b) \in K^{3,\text{pr}}_{\mathfrak{s}}$, so by [She09e] we have $(M, N', \mathbf{J}') \in K^{3,\text{qr}}_{\mathfrak{s}}$. Hence there is an isomorphism f from N' onto N'' which is the identity of $M \cup \mathbf{J}'$ (by the uniqueness for $K^{3,\text{qr}}_{\mathfrak{s}}$). So using $(N', f(b), \mathbf{J}')$ for $(N, b, \mathbf{J})$ we are done.
3) If not, we can find $N_1, N_2, \mathbf{J}_1, \mathbf{J}_2, b$ such that $M \le_{\mathfrak{s}} N_\ell \le_{\mathfrak{s}} N$ and the quadruple $(M, N_\ell, \mathbf{J}_\ell, b)$ is as in (a)-(e)+(f) of part (1)+(2) for $\ell = 1, 2$. Assume toward contradiction that $|\mathbf{J}_1| \ne |\mathbf{J}_2|$ so without loss of generality $|\mathbf{J}_1| < |\mathbf{J}_2|$.

By "$(M, N_\ell, b) \in K^{3,\text{pr}}_{\mathfrak{s}}$" without loss of generality $N_2 \le_{\mathfrak{s}} N_1$.

By [She09e, 10.15=L10b.11tex(3)] for some $c \in \mathbf{J}_2 \backslash \mathbf{J}_1, \mathbf{J}_1 \cup \{c\}$ is independent in (M, N_1), contradiction to $(M, N, \mathbf{J}_1) \in K^{3,\text{vq}}_{\mathfrak{s}}$ by [She09e, 10.15=L10b.11tex(4)].
4) Similarly. $\square_{1.3}$

Definition 1.4. 1) For $p \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M)$, let the weight of $p, w(p)$ be the unique natural number such that: if $M \le_{\mathfrak{s}} M', M'$ is brimmed, $p' \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M')$ is a non-forking extension of p <u>then</u> it is the unique $|\mathbf{J}|$ from Claim 1.3(3), it is a natural number.
2) Let $w_{\mathfrak{s}}(a, M, N) = w(\text{ortp}_{\mathfrak{s}}(a, M, N))$.

Claim 1.5. *1) If $p \in \mathscr{S}^{bs}(M)$ regular, <u>then</u> $w(p) = 1$.*
2) If $\mathbf{J}$ is independent in (M, N) and $c \in N$, <u>then</u> for some $\mathbf{J}' \subseteq \mathbf{J}$ with $\le w_{\mathfrak{s}}(c, M, N)$ elements, $\{c\} \cup (\mathbf{J} \backslash \mathbf{J}')$ is independent in (M, N).

Proof. Easy by now. $\square_{1.5}$

Note that the use of $\mathfrak{C}$ in Definition 1.6 is for transparency only and can be avoided, see 1.10 below.

Definition 1.6. 1) We say that $\mathbf{P}$ is an M^*-based family (inside $\mathfrak{C}$) <u>when</u>:

- (a) $M^* <_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ and $M^* \in K_{\mathfrak{s}}$
- (b) $\mathbf{P} \subseteq \cup\{\mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M) : M \le_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ and $M \in K_{\mathfrak{s}}\}$
- (c) $\mathbf{P}$ is preserved by automorphisms of $\mathfrak{C}$ over M^*.

2) Let $p \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M)$ where $M \le_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$

- (a) we say that p is hereditarily orthogonal to $\mathbf{P}$ (or $\mathbf{P}$-foreign) <u>when</u>:
  if $M \le_{\mathfrak{s}} N \le_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}, q \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(N), q \restriction M = p$, <u>then</u> q is orthogonal to $\mathbf{P}$
- (b) we say that p is $\mathbf{P}$-regular <u>when</u> p is regular, not orthogonal to $\mathbf{P}$ and if $q \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M'), M \le_{\mathfrak{s}} M' <_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ and q is a forking extension of p <u>then</u> q is hereditarily orthogonal to $\mathbf{P}$
- (c) p is weakly $\mathbf{P}$-regular <u>if</u> it is regular and is not orthogonal to some $\mathbf{P}$-regular p'.

3) $\mathbf{P}$ is normal <u>when</u> $\mathbf{P}$ is a set of regular types and each of them is $\mathbf{P}$-regular.
4) For $q \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M), M <_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ let $w_{\mathbf{P}}(q)$ be defined as the natural number satisfying the following

⊛ if $M \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_2 \leq_{\mathfrak{s}} \mathfrak{C}, M_\ell$ is $(\lambda, *)$-brimmed, $b \in M_2, \mathrm{ortp}_{\mathfrak{s}}(b, M_1, M_2)$ is a non-forking extension of $q, (M_1, M_2, b) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}, (M_1, M_2, \mathbf{J}) \in K^{3,\mathrm{qr}}_{\mathfrak{s}}$ and $\mathbf{J}$ is regular in (M_1, M_2), i.e. independent and $c \in \mathbf{J} \Rightarrow \mathrm{ortp}_{\mathfrak{s}}(c, M_1, M_2)$ is a regular type then $w_{\mathbf{P}}(q) = |\{c \in \mathbf{J} : \mathrm{ortp}_{\mathfrak{s}}(c, M_1, M)$ is weakly $\mathbf{P}$-regular$\}|$.

5) We replace $\mathbf{P}$ by p if $\mathbf{P} = \{p\}$, where $p \in \mathscr{S}^{\mathrm{bs}}(M^*)$ is regular (see 1.7(1)).

Claim 1.7. *1) If $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ is regular then $\{p\}$ is an M-based family and is normal.*
2) Assume $\mathbf{P}$ is an M^*-based family. If $q \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ and $M^* \leq_{\mathfrak{s}} M \leq_{\mathfrak{k}[\mathfrak{s}]} \mathfrak{C}$ then $w_{\mathbf{P}}(q)$ is well defined (and is a natural number).
3) In Definition 1.6(4) we can find $\mathbf{J}$ such that for every $c \in \mathbf{J}_1$ we have: $\mathrm{ortp}(c, M_1, M)$ *is weakly-$\mathbf{P}$-regular* $\Rightarrow \mathrm{ortp}(c, M_1, M)$ *is $\mathbf{P}$-regular.*

Proof. Should be clear. $\square_{1.7}$

Discussion 1.8. It is tempting to try to generalize the notion of $\mathbf{P}$-simple ($\mathbf{P}$-internal in Hrushovski's terminology) and semi-regular. An important property of those notions in the first order case is that: e.g. if $(\bar{a}/A) \pm p, p$ regular then for some equivalence relation E definable over $A, \mathrm{ortp}(\bar{a}/E, A)$ is $\pm p$ and is $\{p\}$-simple. So assume that $p, q \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ are not orthogonal, and we can define an equivalence relation $\mathscr{E}^{p,q}_M$ on $\{c \in \mathfrak{C} : c$ realizes $p\}$, defined by

$$c_1 \mathscr{E}^{p,q}_M c_2 \quad \underline{\text{iff}} \text{ for every } d \in \mathfrak{C} \text{ realizing } q \text{ we have } \mathrm{ortp}_{\mathfrak{s}}(c_1 d, M, \mathfrak{C}) = \mathrm{ortp}_{\mathfrak{s}}(c_2 d, M, \mathfrak{C}).$$

This may fail (the desired property) even in the first order case: suppose p, q are definable over $a^* \in M$ (on getting this, see later) and we have $\langle c_\ell : \ell \leq n\rangle, \langle M_\ell : \ell < n\rangle$ such that $\mathrm{ortp}(c_\ell, M_\ell, \mathfrak{C}) = p_\ell$ each p_ℓ is parallel to $p, c_\ell \mathscr{E}^{p,q}_{M_\ell} c_{\ell+1}$ but c_0, c_n realizes p, q respectively and $\{c_0, c_n\}$ is independent over M_0. Such a situation defeats the attempt to define a $\mathbf{P} - \{q\}$-simple type $p/\mathscr{E}$ as in [She90, Ch.V].

In first order logic we can find a saturated N and $a^* \in N$ such that $\mathrm{ortp}(M, \bigcup_\ell M_\ell \cup \{c_0, \ldots, c_n\})$ does not fork over a^* and use "average on the type with an ultrafilter c over $q(\mathfrak{C}) + a^*_t$" (for suitable a^*_t's). But see below.

Discussion 1.9. : 1) Assume ($\mathfrak{s}$ is full and) every $p \in \mathscr{S}^{\mathrm{na}}_{\mathfrak{s}}(M)$ is representable by some $a_p \in M$ (in [She90], e.g. the canonical base $\mathrm{Cb}(p)$). We can define for $\bar{a}, \bar{b} \in {}^{\omega>}\mathfrak{C}$ when $\mathrm{ortp}(\bar{a}, \bar{b}, \mathfrak{C})$ is stationary (and/or non-forking). We should check the basic properties. See §3.
2) Assume $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ is regular, definable over $\bar{a}^*$ (in the natural sense). We may wonder if the niceness of the dependence relation hold for $p \restriction \bar{a}^*$?

If you feel that the use of a monster model is not natural in our context, how do we "translate" a set of types in $\mathfrak{C}^{\mathrm{eq}}$ preserved by every automorphism of $\mathfrak{C}$ which is the identity on A? by using a "place" defined by:

Definition 1.10. 1) A local place is a pair $\mathbf{a} = (M, A)$ such that $A \subseteq M \in K_{\mathfrak{s}}$ (compare with 8.2.
2) The places $(M_1, A_1), (M_2, A_2)$ are equivalent if $A_1 = A_2$ and there are n and $N_\ell \in K_{\mathfrak{s}}$ for $\ell \leq n$ satisfying $A \subseteq N_\ell$ for $\ell = 0, \ldots, n$ such that $M_1 = N_0, M_2 = N_n$ and for each $\ell < n, N_\ell \leq_{\mathfrak{s}} N_{\ell+1}$ or $N_{\ell+1} \leq_{\mathfrak{s}} N_\ell$. We write $(M_1, A_1) \sim (M_2, A_2)$ or $M_1 \sim_{A_1} M_2$.

3) For a local place $\mathbf{a} = (M, A)$ let $K_{\mathbf{a}} = K_{(M,A)} = \{N : (N, A) \sim (M, A)\}$; so in $(M, A)/\sim$ we fix both A as a set and the type it realizes in M over $\emptyset$.
4) We call such class $K_{\mathbf{a}}$ a place.
5) We say that $\mathbf{P}$ is an invariant set[4] of types in a place $K_{(M,A)}$ when:

- (a) $\mathbf{P} \subseteq \{\mathscr{S}^{\rm bs}_{\mathfrak{s}}(N) : N \sim_A M\}$
- (b) membership in $\mathbf{P}$ is preserved by isomorphism over A
- (c) if $N_1 \le_{\mathfrak{s}} N_2$ are both in $K_{(M,A)}, p_2 \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N_2)$ does not fork over N_1 then $p_2 \in \mathbf{P} \Leftrightarrow p_2 \restriction N_1 \in \mathbf{P}$.

6) We say $M \in K_{\mathfrak{s}}$ is brimmed over A when for some N we have $A \subseteq N \le_{\mathfrak{s}} M$ and M is brimmed over N.

Claim/Definition 1.11. *1) If $A \subseteq M \in K_{\mathfrak{s}}$ and $\mathbf{P}_0 \subseteq \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ then there is at most one invariant set $\mathbf{P}'$ of types in the place $K_{(M,A)}$ such that $\mathbf{P}^+ \cap \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M) = \mathbf{P}_0$ and $M \le_{\mathfrak{s}} N \wedge p \in \mathbf{P}^+ \cap \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N) \Rightarrow$ (p does not fork over M).*
2) If in addition M is brimmed[5] over A then we can omit the last demand in part (1).
3) If $\mathbf{a} = (M_1, A), (M_2, A) \in K_{\mathbf{a}}$ then $K_{(M_2,A)} = K_{\mathbf{a}}$.

Proof. Easy. $\square_{1.11}$

Definition 1.12. 1) If in 1.11 there are such $\mathbf{P}$, we denote it by $\text{inv}(\mathbf{P}_0) = \text{inv}(\mathbf{P}_0, M)$.
2) If $\mathbf{P}_0 = \{p\}$, then let $\text{inv}(p) = \text{inv}(p, M) = \text{inv}(\{p\})$.
3) We say $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ does not split (or is definable) over A when $\text{inv}(p)$ is well defined.

[4] Really a class
[5] M is brimmed over A means that for some $M_1, A \subseteq M_1 \le_{\mathfrak{s}} M$ and M is brimmed over M_1.

§ 2. I Imaginary elements, an essential-(μ, λ)-a.e.c. and frames

§ 2(A). **Essentially a.e.c.**

We consider revising the definition of a.e.c. $\mathfrak{k}$, by allowing function symbols in $\tau_{\mathfrak{k}}$ with infinite number of places while retaining local characters, e.g., if $M_n \le M_{n+1}, M = \cup\{M_n : n < \omega\}$ is uniquely determined. Before this we introduce the relevant equivalence relations. In this context, we can give name to equivalence classes for equivalence relations on infinite sequences.

Definition 2.1. We say that $\mathfrak{k}$ is an essentially $[\lambda,\mu)$-a.e.c. or ess-$[\lambda,\mu)$-a.e.c. and we may write (μ,λ) instead of $[\lambda,\mu)$ <u>if</u> ($\lambda < \mu$ and) it is an object consisting of:

- $I(a)$ a vocabulary $\tau = \tau_{\mathfrak{k}}$ which has predicates and function symbols of possibly infinite arity but $\le \lambda$
- (b) a class of $K = K_{\mathfrak{k}}$ of τ-models
- (c) a two-place relation $\le_{\mathfrak{k}}$ on K

such that

- $II(a)$ if $M_1 \cong M_2$ <u>then</u> $M_1 \in K \Leftrightarrow M_2 \in K$
- (b) if $(N_1, M_1) \cong (N_2, M_2)$ <u>then</u> $M_1 \le_{\mathfrak{k}} N_1 \Leftrightarrow M_2 \le_{\mathfrak{k}} N_2$
- (c) every $M \in K$ has cardinality $\in [\lambda,\mu)$
- (d) $\le_{\mathfrak{k}}$ is a partial order on K
- III_1 if $\langle M_i : i < \delta\rangle$ is $\le_{\mathfrak{k}}$-increasing and $|\bigcup\limits_{i<\delta} M_i| < \mu$ <u>then</u> there is a unique $M \in K$ such that $|M| = \cup\{|M_i| : i < \delta\}$ and $i < \delta \Rightarrow M_i \le_{\mathfrak{k}} M$
- III_2 if in addition $i < \delta \Rightarrow M_i \le_{\mathfrak{k}} N$ <u>then</u> $M \le_{\mathfrak{k}} N$
- IV if $M_1 \subseteq M_2$ and $M_\ell \le_{\mathfrak{k}} N$ for $\ell = 1,2$ <u>then</u> $M_1 \le_{\mathfrak{k}} M_2$
- V if $A \subseteq N \in K$, <u>then</u> there is M satisfying $A \subseteq M \le_{\mathfrak{k}} N$ and $\|M\| \le \lambda + |A|$ (here it is enough to restrict ourselves to the case $|A| \le \lambda$).

Definition 2.2. 1) We say $\mathfrak{k}$ is an ess-λ-a.e.c. <u>if</u> it is an ess-$[\lambda,\lambda^+)$-a.e.c.
2) We say $\mathfrak{k}$ is an ess-a.e.c. <u>if</u> there is λ such that it is an ess-$[\lambda,\infty)$-a.e.c., so $\lambda = \mathrm{LST}(\mathfrak{k})$.
3) If $\mathfrak{k}$ is an ess-$[\lambda,\mu)$-a.e.c. and $\lambda \le \lambda_1 < \mu_1 \le \mu$ then let $K^{\mathfrak{k}}_{\lambda_1} = (K_{\mathfrak{k}})_{\lambda_1} = \{M \in K_{\mathfrak{k}} : \|M\| = \lambda_1\}, K^{\mathfrak{k}}_{\lambda_1,\mu_1} = \{M \in K_{\mathfrak{k}} : \lambda_1 \le \|M\| < \mu_1\}$.
4) We define $\Upsilon^{\rm or}_{\mathfrak{k}}$ as in [She09b, 0.8=L11.1.3A(2)].
5) We may omit the "essentially" <u>when</u> $\mathrm{arity}(\tau_{\mathfrak{k}}) = \aleph_0$ where $\mathrm{arity}(\mathfrak{k}) = \mathrm{arity}(\tau_{\mathfrak{k}})$ and for vocabulary $\tau, \mathrm{arity}(\tau) = \min\{\kappa$: every predicate and function symbol have arity $< \kappa\}$.

We now consider the claims on ess-a.e.c.

Claim 2.3. *Let $\mathfrak{k}$ be an ess-$[\lambda,\mu)$-a.e.c.*
1) The parallel of $\mathrm{Ax}(III)_1, (III)_2$ holds with a directed family $\langle M_t : t \in I\rangle$.
2) If $M \in K$ we can find $\langle M_{\bar a} : \bar a \in {}^{\omega>}M\rangle$ such that:

- (a) $\bar a \subseteq M_{\bar a} \le_{\mathfrak{k}} M$
- (b) $\|M_{\bar a}\| = \lambda$
- (c) *if $\bar b$ is a permutation of $\bar a$ then $M_{\bar a} = M_{\bar b}$*

(d) *if* $\bar a$ *is a subsequence of* $\bar b$ *then* $M_{\bar a} \le_{\mathfrak k} M_{\bar b}$.

3) If $N \le_{\mathfrak k} M$ *we can add in (2) that* $\bar a \in {}^{\omega>}N \Rightarrow M_{\bar a} \subseteq N$.
4) If for simplicity $\lambda_* = \lambda + \sup\{\Sigma\{|R^M| : R \in \tau_{\mathfrak k}\} + \Sigma\{|F^M| : F \in \tau_{\mathfrak k}\} : M \in K_{\mathfrak k}$ *has cardinality* $\lambda\}$ *then* $K_{\mathfrak k}$ *and* $\{(M,N) : N \le_{\mathfrak k} M\}$ *essentially are* $\mathrm{PC}_{\chi,\lambda_*}$*-classes where* $\chi = |\{M/\cong : M \in K^{\mathfrak k}_\lambda\}|$, *noting that* $\chi \le 2^{2^\theta}$. *That is,* $\langle M_{\bar a} : \bar a \in {}^{\omega>}A\rangle$ *satisfying clauses (b),(c),(d) of part (2) such that* $A = \cup\{|M_{\bar a}| : \bar a \in {}^{\omega>}A\}$ *represent a unique* $M \in K_{\mathfrak k}$ *with universe* A *and similarly for* $\le_{\mathfrak k}$, *(on the Definition of* $\mathrm{PC}_{\chi,\lambda_*}$, *see* [She09a, 1.4(3)]*). Note that if in* $\tau_{\mathfrak k}$ *there are no two distinct symbols with the same interpretation in every* $M \in K_{\mathfrak k}$ *then* $|\tau)k_*| \le 2^{2^\lambda}$.
5) The results on omitting types in [She99] *or* [She09b, 0.9=L0n.8,0.2=0n.11] *hold, i.e., if* $\alpha < (2^{\lambda_*})^+ \Rightarrow K^{\mathfrak k}_{\sqsupseteq_\alpha} \ne \emptyset$ *then* $\theta \in [\lambda,\mu) \Rightarrow K_\theta \ne \emptyset$ *and there is an* EM*-model, i.e.,* $\Phi \in \Upsilon^{\mathrm{or}}_{\mathfrak k}$ *with* $|\tau_\Phi| = |\tau_{\mathfrak k}| + \lambda$ *and* $\mathrm{EM}(I,\Phi)$ *having cardinality* $\lambda + |I|$ *for any linear order* I.
6) The lemma on the equivalence of being universal model homogeneous and of being saturated (see [She09f, 3.18=3.10] *or* [She09c, 1.14=L0.19]*) holds.*
7) We can generalize the results of [She09c, §1] *on deriving an ess-*(∞,λ)*-a.e.c. from an ess* λ*-a.e.c.*

Proof. The same proofs, on the generalization in 2.3(7), see in §5 below. The point is that, in the term of §5, our $\mathfrak k$ is a (λ,μ,κ)-a.e.c. (automatically with primes). $\square_{2.3}$

Remark 2.4. 1) In 2.3(4) we can decrease the bound on χ if we have more nice definitions of K_λ, e.g., if $\mathrm{arity}(\tau) \le \kappa$ then $\chi = 2^{(\lambda^{<\kappa}+|\tau|)}$ where $\mathrm{arity}(\tau) = \min\{\kappa$: every predicate and function symbol of τ has arity $< \kappa\}$.
2) We may use above $|\tau_{\mathfrak s}| \le \lambda, \mathrm{arity}(\tau_{\mathfrak k}) = \aleph_0$ to get that $\{(M,\bar a)/\cong M \in K^{\mathfrak k}_\lambda, \bar a \in {}^\lambda M$ list $M\}$ has cardinality $\le 2^\lambda$. See also 2.18.
3) In 2.10 below, if we omit "$\mathbb E$ is small" and $\lambda_1 = \sup\{|\mathrm{seq}(M)/\mathbb E_M| : M \in K^{\mathfrak k}_\lambda\}$ is $< \mu$ then $\mathfrak k_{[\lambda_1,\mu)}$ is an ess-$[\lambda_1,\mu)$-a.e.c.
4) In Definition 2.1, we may omit axiom V and define $\mathrm{LST}(\mathfrak k) \in [\lambda,\mu]$ naturally, and if $M \in K^{\mathfrak k}_\lambda \Rightarrow \mu > |\mathrm{seq}(M)/\mathbb E_M|$ then in 2.10(1) below we can omit "$\mathbb E$ is small".
5) Can we preserve in such "transformation" the arity finiteness? A natural candidate is trying to code $p \in \mathscr S^{\mathrm{bs}}_{\mathfrak s}(M)$ by $\{\bar a : \bar a \in {}^{\omega>}M\}$ and there are $M_0 \le_{\mathfrak s} M_1$ such that $M \le_{\mathfrak s} M_1$ and $\mathrm{ortp}(a_\ell, M_0, M_1)$ is parallel to p and $\bar a$ is independent in $(M_0,M_1)\}$. If e.g., $K_{\mathfrak s}$ is saturated this helps but still we suspect it may fail.
6) What is the meaning of ess-$[\lambda,\mu)$-a.e.c.? Can we look just at $\langle M_t : t \in I\rangle, I$ directed, $t \le_I s \Rightarrow M_t \le_{\mathfrak s} M_{\mathfrak s} \in K_\lambda$? But for isomorphism types we take a kind of completion and so make more pairs isomorphic but $\bigcup\limits_{t\in I} M_t$ does not determine $\bar M = \langle M_t : t \in I\rangle$ and the completion may depend on this representation.
7) If we like to avoid this and this number is λ', then we should change the definition of $\mathrm{seq}(N)$ (see 2.5(b)) to $\mathrm{seq}'(N) = \{\bar a : \ell g(\bar a) = \lambda$ and for some $M \le_{\mathfrak s} N$ from $K^{\mathfrak k}_\lambda, \langle a_{1+\alpha} : \alpha < \lambda\rangle$ list the members of M and $a_0 \in \{\gamma : \gamma < \mu_*\}$.

§ 2(B). **Imaginary Elements and Smooth Equivalent Relations.**

Now we return to our aim of getting canonical base for orbital types.

Definition 2.5. Let $\mathfrak{k} = (K_{\mathfrak{k}}, \le_{\mathfrak{k}})$ be a λ-a.e.c. or just ess-$[\lambda, \mu)$-a.e.c. (if $\mathfrak{k}_\lambda = \mathfrak{k}_{\mathfrak{s}}$ we may write $\mathfrak{s}$ instead of $\mathfrak{k}_\lambda$, see 2.11). We say that $\mathbb{E}$ is a smooth $\mathfrak{k}_\lambda$-equivalence relation <u>when</u>:

(a) $\mathbb{E}$ is a function with domain $K_{\mathfrak{k}}$ mapping M to $\mathbb{E}_M$
(b) for $M \in K_{\mathfrak{k}}, \mathbb{E}_M$ is an equivalence relation on a subset of $\text{seq}(M) = \{\bar{a} : \bar{a} \in {}^\lambda M$ and $M \restriction \text{Rang}(\bar{a}) \le_{\mathfrak{k}} M\}$ so $\bar{a}$ is not necessarily without repetitions; note that $\mathfrak{k}$ determines λ, pedantically when non-empty
(c) if $M_1 \le_{\mathfrak{k}} M_2$ then $\mathbb{E}_{M_2} \restriction \text{seq}(M_1) = \mathbb{E}_{M_1}$
(d) if f is an isomorphism from $M_1 \in K_{\mathfrak{s}}$ onto M_2 <u>then</u> f maps $\mathbb{E}_{M_1}$ onto $\mathbb{E}_{M_2}$
(e) if $\langle M_\alpha : \alpha \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous <u>then</u> $\{\bar{a}/\mathbb{E}_{M_\delta} : \bar{a} \in \text{seq}(M_\delta)\} = \{\bar{a}/\mathbb{E}_{M_\delta} : \bar{a} \in \bigcup\limits_{\alpha<\delta} \text{seq}(M_\alpha)\}$.

2) We say that $\mathbb{E}$ is small <u>if</u> each $\mathbb{E}_M$ has $\le \|M\|$ equivalence classes.

Remark 2.6. 1) Note that if we have $\langle \mathbb{E}_i : i < i^*\rangle$, each $\mathbb{E}_i$ is a smooth $\mathfrak{k}_\lambda$-equivalence relation and $i^* < \lambda^+$ <u>then</u> we can find a smooth $\mathfrak{k}_\lambda$-equivalence relation $\mathbb{E}$ such that essentially the $\mathbb{E}_M$-equivalence classes are the $\mathbb{E}_i$-equivalence classes for $i < i^*$; in detail: without loss of generality $i^* \le \lambda$ and $\bar{a}\mathbb{E}_M\bar{b}$ <u>iff</u> $\ell g(\bar{a}) = \ell g(\bar{b})$ and

$\circledast_1$ $i(\bar{a}) = i(\bar{b})$ and if $i(\bar{a}) < i^*$ then $\bar{a} \restriction [1 + i(\bar{a}) + 1, \lambda)\mathbb{E}_{i(\bar{a})}\bar{b} \restriction [1 + i(\bar{b}) + 1, \lambda)$ where $i(\bar{a}) = \text{Min}\{j : (j + 1 < i(*)) \wedge a_0 \ne a_{1+j}$ or $j = \lambda\}$.

2) In fact $i^* \le 2^\lambda$ is O.K., e.g. choose a function $\mathbf{e}$ from $\{e : e$ an equivalence relation on λ to i^* and for $\bar{a}, \bar{b} \in \text{seq}(M)$ we let $i(\bar{a}) = \mathbf{e}(\{(i,j) : a_{2i+1} = a_{2j+1}\}$ and

$\circledast_2$ $\bar{a} \in \mathbb{E}_M\bar{b}$ <u>iff</u> $i(\bar{a}) = i(\bar{b})$ and $\langle a_{2i} : i < \lambda\rangle\mathbb{E}_{i(\bar{a})}\langle b_{2i} : i < \lambda\rangle$.

3) We can redefine $\text{seq}(M)$ as ${}^{\lambda\ge}M$, then have to make minor changes above.

Definition 2.7. Let $\mathfrak{k}$ be a λ-a.e.c. or just ess-$[\lambda, \mu)$-a.e.c. and $\mathbb{E}$ a small smooth $\mathfrak{k}$-equivalence relation and the reader may assume for simplicity that the vocabulary $\tau_{\mathfrak{k}}$ has only predicates. Also assume $F_*, c_*, P_* \notin \tau_{\mathfrak{k}}$. We define τ_* and $\mathfrak{k}_* = \mathfrak{k}\langle\mathbb{E}\rangle = (K_{\mathfrak{k}_*}, \le_{\mathfrak{k}_*})$ as follows:

(a) $\tau^* = \tau \cup \{F_*, c_*, P_*\}$ with P_* a unary predicate, c_* an individual constant and F_* a λ-place function symbol
(b) $K_{\mathfrak{k}_*}$ is the class of $\tau^*_{\mathfrak{k}}$-models M^* such that for some model $M \in K_{\mathfrak{k}}$ we have:
  (α) $|M| = P_*^{M^*}$
  (β) if $R \in \tau$ then $R^{M^*} = R^M$
  (γ) if $F \in \tau$ has arity α then $F^{M^*} \restriction M = F^M$ and for any $\bar{a} \in {}^\alpha(M^*), \bar{a} \notin {}^\alpha M$ we have $F^{M^*}(\bar{a}) = c_*^{M^*}$ (or allow partial function or use $F^{M^*}(\bar{a}) = a_0$ when $\alpha > 0$ and $F^{M^*}(\langle\rangle)$ when $\alpha = 0$, i.e. F is an individual constant);
  (δ) F_* is a λ-place function symbol and:
    (i) if $\bar{a} \in \text{seq}(M)$ then $F_*^{M^*}(\bar{a}) \in |M^*| \backslash |M| \backslash \{c_*^{M^*}\}$
    (ii) if $\bar{a}, \bar{b} \in \text{Dom}(\mathbb{E}) \subseteq \text{seq}(M)$ then $F_*^{M^*}(\bar{a}) = F_*^{M^*}(\bar{b}) \Leftrightarrow \bar{a}\mathbb{E}_M\bar{b}$

(*iii*) if $\bar{a} \in {}^{\lambda}(M^*)$ and $\bar{a} \notin \mathrm{Dom}(\mathbb{E}) \subseteq \mathrm{seq}(M)$ then $F_*^{M^*}(\bar{a}) = c_*^{M^*}$

(ε) $c_*^{M^*} \notin |M|$ and if $b \in |M^*| \backslash |M| \backslash \{c_*^{M^*}\}$ then for some $\bar{a} \in \mathrm{Dom}(\mathbb{E}) \subseteq \mathrm{seq}(M)$ we have $F_*^{M^*}(\bar{a}) = b$

(*c*) $\leq_{\mathfrak{k}_*}$ is the two-place relation on $K_{\mathfrak{k}_*}$ defined by: $M^* \leq_{\mathfrak{k}_*} N^*$ if

(α) $M^* \subseteq N^*$ and

(β) for some $M, N \in \mathfrak{k}$ as in clause (b) we have $M \leq_{\mathfrak{k}} N$.

Definition 2.8. 1) In 2.7(1) we call $M \in \mathfrak{k}$ a witness for $M^* \in K_{\mathfrak{k}_*}$ if they are as in clause (b) above.
2) We call $M \leq_{\mathfrak{k}} N$ witness for $M^* \leq_{\mathfrak{k}^*_\lambda} N^*$ if they are as clause (c) above.

Discussion 2.9. Up to now we have restricted ourselves to vocabularies with each predicate and function symbol of finite arity, and this restriction seems very reasonable. Moreover, it seems a priori that for a parallel to superstable, it is quite undesirable to have infinite arity. Still our desire to have imaginary elements (in particular canonical basis for types) forces us to accept them. The price is that inthe class of τ-models the union of increasing chains of τ-models is not a well defined τ-model, more accurately we can show its existence, but not smoothness; however inside the class $\mathfrak{k}$ it will be.

Claim 2.10. *1) If $\mathfrak{k}$ is a $[\lambda, \mu)$-a.e.c. or just an ess-$[\lambda, \mu)$-a.e.c. and $\mathbb{E}$ a small smooth $\mathfrak{k}$-equivalence relation then $\mathfrak{k}\langle\mathbb{E}\rangle$ is an ess-$[\lambda, \mu)$-a.e.c.*
2) If $\mathfrak{k}$ has amalgamation and $\mathbb{E}$ is a small $\mathfrak{k}$-equivalence class then $\mathfrak{k}\langle\mathbb{E}\rangle$ has amalgamation property.

Proof. The same proofs. Left as an exercise to the reader. $\square_{2.10}$

§ 2(C). **Good Frames.**

Now we return to good frames.

Definition 2.11. 1) We say that $\mathfrak{s}$ is a good ess-$[\lambda, \mu)$-frame if Definition [She09c, 2.1=L1.1tex] is satisfied except that:

(*a*) in clause (A), $\mathfrak{K}_{\mathfrak{s}} = (K_{\mathfrak{s}}, \leq_{\mathfrak{s}}), \mathfrak{k}$ is an ess-$[\lambda, \mu)$-a.e.c. and $\mathfrak{K}[\mathfrak{s}]$ is an ess-(∞, λ)-a.e.c.

(*b*) $K_{\mathfrak{s}}$ has a superlimit model in χ in every $\chi \in [\lambda, \mu)$

(*c*) $K^{\mathfrak{s}}_\lambda / \cong$ has cardinality $\leq 2^\lambda$, for convenience.

Discussion 2.12. We may consider other relatives as our choice and mostly have similar results. In particular:

(*a*) we can demand less: as in [SV, §2] we may replace $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}$ by a formal version of $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}$

(*b*) we may demand goodness only for $\mathfrak{s}_\lambda$, i.e. $\mathfrak{s}$ restriction the class of models to $K^{\mathfrak{s}}_\lambda$ and have only the formal properties above so amalgamation and JEP are required only for models of cardinality λ.

Claim 2.13. *All the definitions and results in* [She09c], [She09e] *and §1 here work for good ess-$[\lambda, \mu)$-frames.*

Proof. No problem. $\square_{2.13}$

Definition 2.14. If $\mathfrak{s}$ is a $[\lambda,\mu)$-frame or just an ess-$[\lambda,\mu)$-frame and $\mathbb{E}$ a small smooth $\mathfrak{s}$-equivalence relation <u>then</u> let $\mathfrak{t} = \mathfrak{s}\langle\mathbb{E}\rangle$ be defined by:

- (a) $\mathfrak{k}_\mathfrak{t} = \mathfrak{k}_\mathfrak{s}\langle\mathbb{E}\rangle$
- (b) $\mathscr{S}^{\rm bs}_\mathfrak{t}(M^*) = \{{\rm ortp}_{\mathfrak{k}_\mathfrak{t}}(a,M^*,N^*) : M^* \le_{\mathfrak{k}_\mathfrak{t}} N^*$ and if $M \le_\mathfrak{k} N$ witness $M^*,N^* \in \mathfrak{k}_\mathfrak{t}$ then $a \in N\backslash M$ and ${\rm ortp}_\mathfrak{s}(a,M,N) \in \mathscr{S}^{\rm bs}_\mathfrak{s}(M)\}$
- (c) non-forking similarly.

Remark 2.15. We may add: if $\mathfrak{s}$ is[6] an NF-frame we define $\mathfrak{t} = \mathfrak{s}\langle\mathbb{E}\rangle$ as an NF-frame similarly, see [She09e].

Claim 2.16. *1) If $\mathfrak{s}$ is a good ess-$[\lambda,\mu)$-frame, $\mathbb{E}$ a small, smooth $\mathfrak{s}$-equivalence relation <u>then</u> $\mathfrak{s}\langle\mathbb{E}\rangle$ is a good ess-$[\lambda,\mu)$-frame.*
2) In part (1) for every κ, $\dot I(\kappa, K^{\mathfrak{s}<\mathbb{E}>}) = \dot I(\kappa, K^\mathfrak{s})$.
3) If $\mathfrak{s}$ has primes/regulars <u>then</u> $\mathfrak{s}\langle\mathbb{E}\rangle$ has.

Remark 2.17. We may add: if $\mathfrak{s}$ is an NF-frame <u>then</u> so is $\mathfrak{s}\langle\mathbb{E}\rangle$, hence $(\mathfrak{s}\langle\mathbb{E}\rangle)^{\rm full}$ is a full NF-frame; see [She09e].

Proof. Straightforward. $\square_{2.16}$

Our aim is to change $\mathfrak{s}$ inessentially such that for every $p \in \mathscr{S}^{\rm bs}_\mathfrak{s}(M)$ there is a canonical base, etc. The following claim shows that in the context we have presented this can be done.

Claim 2.18. <u>*The imaginary elements Claim*</u>
Assume $\mathfrak{s}$ a good λ-frame or just a good ess-$[\lambda,\mu)$-frame.
1) If $M_* \in K_\mathfrak{s}$ and $p^* \in \mathscr{S}^{\rm bs}_\mathfrak{s}(M_*)$, <u>then</u>[7] there is a small, smooth $\mathfrak{k}_\mathfrak{s}$-equivalence relation $\mathbb{E} = \mathbb{E}_{\mathfrak{s},M_*,p^*}$ and function $\mathbf{F}$ such that:

- $(*)$ *if $M_* \le_\mathfrak{s} N$ and $\bar a \in {\rm seq}(N)$ so $M =: N \restriction {\rm Rang}(\bar a) \le_\mathfrak{s} N$ and $M \cong M_*$, <u>then</u>*
  - (α) *$\mathbf{F}(N,\bar a)$ is well defined iff $\bar a \in {\rm Dom}(\mathbb{E}_N)$ and then $\mathbf{F}(N,\bar a)$ belongs to $\mathscr{S}^{\rm bs}_\mathfrak{s}(N)$*
  - (β) *$S \subseteq \{(N,\bar a,p) : N \in K_\mathfrak{s}, \bar a \in {\rm Dom}(\mathbb{E}_N)\}$ is the minimal class such that:*
    - (i) *if $\bar a \in {\rm seq}(M_*)$ and p does not fork over $M_*{\restriction}{\rm Rang}(\bar a)$ then $(M_*,\bar a,p) \in S$*
    - (ii) *S is closed under isomorphisms*
    - (iii) *if $N_1 \le_\mathfrak{s} N_2, p_2 \in \mathscr{S}^{\rm bs}_\mathfrak{s}(N_2)$ does not fork over $\bar a \in {\rm seq}(N_1)$ then $(N_2,\bar a,p_2) \in S \Leftrightarrow (N_1,\bar a,p_2{\restriction}N_1) \in S$*
    - (iv) *if $\bar a_1,\bar a_2 \in {\rm seq}(N), p \in \mathscr{S}^{\rm bs}_\mathfrak{s}(N)$ does not fork over $N{\restriction}{\rm Rang}(\bar a_\ell)$ for $\ell = 1,2$ then $(N_2,\bar a_1,p) \in S \Leftrightarrow (N_2,\bar a_2,p) \in S$*
  - (γ) *$\mathbf{F}(N,\bar a) = p$ iff $(N,\bar a,p) \in S$ hence if $\bar a,\bar b \in {\rm seq}(N)$ then: $\bar a\mathbb{E}_N\bar b$ iff $\mathbf{F}(\bar a,N) = \mathbf{F}(\bar b,N)$.*

[6] The reader may ignore this version.
[7] note that there may well be an automorphism of M^* which maps p^* to some $p^{**} \in \mathscr{S}^{\rm bs}_\mathfrak{s}(M^*)$ such that $p^{**} \ne p^*$.

2) There are unique small[8] *smooth* $\mathbb{E}$*-equivalence relation* $\mathbb{E}$ *called* $\mathbb{E}_{\mathfrak{s}}$ *and function* $\mathbf{F}$ *such that:*

$(**)(\alpha)$ $\mathbf{F}(N,\bar{a})$ *is well defined iff* $N \in K_{\mathfrak{s}}$ *and* $\bar{a} \in \mathrm{seq}(N)$

(β) $\mathbf{F}(N,\bar{a})$*, when defined, belongs to* $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$

(γ) *if* $N \in K_{\mathfrak{s}}$ *and* $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$ *<u>then</u> there is* $\bar{a} \in \mathrm{seq}(N)$ *such that* $\mathrm{Rang}(\bar{a}) = N$ *and* $\mathbf{F}(N,\bar{a}) = p$

(δ) *if* $\bar{a} \in \mathrm{seq}(M)$ *and* $M \leq_{\mathfrak{s}} N$ *<u>then</u>* $\mathbf{F}(N,\bar{a})$ *is (well defined and is) the non-forking extension of* $\mathbf{F}(M,\bar{a})$

(ε) *if* $\bar{a}_\ell \in \mathrm{seq}(N)$ *and* $\mathbf{F}(N,\bar{a}_\ell)$ *is well defined for* $\ell = 1,2$ *<u>then</u>* $\bar{a}_1 \mathbb{E}_N \bar{a}_2 \Leftrightarrow \mathbf{F}(N,\bar{a}_1) = \mathbf{F}(N,\bar{a}_2)$

(ζ) $\mathbf{F}$ *commute with isomorphisms.*

3) For $\mathfrak{t} = \mathfrak{s}\langle\mathbb{E}\rangle$ *where* $\mathbb{E}$ *as in part (2) and* $M^* \in K_{\mathfrak{t}}$ *as witnessed by* $M \in K_{\mathfrak{s}}$ *and* $p^* \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{t}}(M^*)$ *is projected to* $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ *let* $\mathrm{bas}(p^*) = \mathrm{bas}(p) = \mathbf{F}(\bar{a}, M^*)/\mathbb{E}$ *whenever* $\mathbf{F}(M,\bar{a}) = p$*. That is, assume* M_ℓ *witness that* $M^*_\ell \in K_{\mathfrak{t}}$*, for* $\ell = 1,2$ *and* $(M^*_1, M^*_2, a) \in K^{3,\mathrm{bs}}_{\mathfrak{t}}$ *then* $(M_1, M_2, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ *and* $p^* = \mathrm{ortp}_{\mathfrak{t}}(a, M^*_1, M^*_2), p = \mathrm{ortp}_{\mathfrak{s}}(a, M_1, M_2)$*; <u>then</u> in* $\mathfrak{t}$*:*

(α) *if* $M^*_\ell \leq_{\mathfrak{s}} M^*, p_\ell \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{t}}(M^*_\ell)$*, then* $p^*_1 \| p^*_2 \Leftrightarrow \mathrm{bas}(p^*_1) = \mathrm{bas}(p^*_2)$

(β) $p^* \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{t}}(M^*)$ *does not split over* $\mathrm{bas}(p^*)$*, see Definition 1.12(3) or* [She09e, §2 end].

Proof. 1) Let $M^{**} \leq_{\mathfrak{s}} M^*$ be of cardinality λ such that p^* does not fork over M^{**}. Let $\bar{a}^* = \langle a_\alpha : \alpha < \lambda\rangle$ list the element of M^{**}.

We say that $p_1 \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_1)$ is a weak copy of p^* when there is a witness (M_0, M_2, p_2, f) which means:

$\circledast_1$ (*a*) $M_0 \leq_{\mathfrak{s}} M_2$ and $M_1 \leq_{\mathfrak{s}} M_2$

(*b*) if $\|M_1\| = \lambda$ then $\|M_2\| = \lambda$

(*c*) f is an isomorphism from M^{**} onto M_0

(*d*) $p_2 \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_2)$ is a non-forking extension of p_1

(*e*) p_2 does not fork over M_0

(*f*) $f(p^* \restriction M^{**})$ is $p_2 \restriction M_0$.

For $M_1 \in K^{\mathfrak{s}}_\lambda, p_1 \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_1)$ which is a weak copy of p^*, we say that $\bar{b}$ explicate its being a weak copy <u>when</u> for some witness (M_0, M_2, p_2, f) and $\bar{c}$

$\circledast_2$ (*a*) $\bar{b} = \langle b_\alpha : \alpha < \lambda\rangle$ list the elements of M_1

(*b*) $\bar{c} = \langle c_\alpha : \alpha < \lambda\rangle$ list the element of M_2

(*c*) $\{\alpha : b_{2\alpha} = b_{2\alpha+1}\}$ code the folowing sets

(α) the isomorphic type of $(M_2, \bar{c})$

(β) $\{(\alpha,\beta) : b_\alpha = c_\beta\}$

(γ) $\{(\alpha,\beta) : f(a^*_\alpha) = c_\beta\}$

Now

[8]for small we use stability in λ

$\circledast_3$ if $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ is a weak copy of p^* then for some $\bar a \in \text{seq}(M)$, there is a $M_1 \le_{\mathfrak{s}} M$ over which p does not fork such that $\bar a$ list M_1 and explicate $p \restriction M_1$ is a weak copy of p^*

$\circledast_4$ (a) if $M \in K^{\mathfrak{s}}_\lambda$ and $\bar b$ explicate $p^* \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ is a weak copy of p^*, then from $\bar b$ and M we can reconstruct p_1

(b) call it $p_{M,\bar b}$

(c) if $M \le_{\mathfrak{s}} N$ let $p_{N,\bar b}$ be its non-forking extension in $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(N)$ we also call it $\mathbf{F}(N,\bar b)$.

Now we define $\mathbb{E}$, so for $N \in K_{\mathfrak{s}}$ we define a two-place relation $\mathbb{E}_N$

$\circledast_5$ (α) $\mathbb{E}_N$ is on $\{\bar a$: for some $M \le_{\mathfrak{s}} N$ of cardinality λ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ which is a copy of p^*, the sequence $\bar a$ explicates p being a weak copy of $p^*\}$

(β) $\bar a_1 \mathbb{E}_N \bar a_2$ iff ($\bar a_1, \bar a_2$ are as above and) $p_{N,\bar a_1} = p_{N,\bar a_2}$.

Now

$\odot_1$ for $N \in K_{\mathfrak{s}}, \mathbb{E}_N$ is an equivalence relation on $\text{Dom}(E_N) \subseteq \text{seq}(N)$

$\odot_2$ if $N_1 \le_{\mathfrak{s}} N_2$ and $\bar a \in \text{seq}(N_1)$ then $\bar a \in \text{Dom}(\mathbb{E}_{N_1}) \Leftrightarrow \bar a \in \text{Dom}(\mathbb{E}_{N_2})$

$\odot_3$ if $N_1 \le_{\mathfrak{s}} N_2$ and $\bar a_1, \bar a_2 \in \text{Dom}(\mathbb{E}_{N_1})$ then $\bar a_2 \mathbb{E}_{N_1} \bar a_2 \Leftrightarrow \bar a_1 \mathbb{E}_{N_2} \bar a_2$

$\odot_4$ if $\langle N_\alpha : \alpha \le \delta\rangle$ is $\le_{\mathfrak{s}}$-increasing continuous and $\bar a_1 \in \text{Dom}(\mathbb{E}_{N_\delta})$ then for some $\alpha < \delta$ and $\bar a_2 \in \text{Dom}(\mathbb{E}_{N_\alpha})$ we have $\bar a_1 \mathbb{E}_{N_\delta} \bar a_2$.

[Why? Let $\bar a_2$ list the elements of $M_1 \le_{\mathfrak{s}} N_\delta$ and let $p = p_{N_\delta,\bar a_1}$ so $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N_\delta)$, hence for some $\alpha < \delta, p$ does not fork over M_α hence for some $M'_1 \le_{\mathfrak{s}} M_\alpha$ of cardinality λ, the type p does not fork over M'_1. Let $\bar a_2$ list the elements of M'_1 such that it explicates $p \restriction M'_1$ being a weak copy of p^*. So clearly $\bar a_2 \in \text{Dom}(\mathbb{E}_{N_\alpha}) \subseteq \text{Dom}(\mathbb{E}_{N_\delta})$ and $\bar a_1 \mathbb{E}_{N_\delta} \bar a_2$.]

Clearly we are done.

2) Similar only we vary (M^*, p^*) but it suffices to consider 2^λ such pairs.

3) Should be clear. $\square_{2.18}$

Definition/Claim 2.19. Assume that $\mathfrak{s}$ is a good ess-$[\lambda,\mu)$-frame so without loss of generality is full. We can repeat the operations in 2.18(3) and 2.16(2), so after ω times we get $\mathfrak{t}_\omega$ which is full (that is $\mathscr{S}^{\rm bs}_{\mathfrak{t}_\omega}(M^\omega) = \mathscr{S}^{\rm na}_{\mathfrak{t}_\omega}(M^\omega)$) and $\mathfrak{t}_\omega$ has canonical type-bases as witnessed by a function $\text{bas}_{\mathfrak{t}_\omega}$, see Definition 2.20.

Proof. Should be clear. $\square_{2.17}$

Definition 2.20. We say that $\mathfrak{s}$ has type bases if there is a function bas$(-)$ such that:

(a) if $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ then bas(p) is (well defined and is) an element of M

(b) p does not split over bas(p), that is any automorphism[9] of M over bas(p) maps p to itself

(c) if $M \le_{\mathfrak{s}} N$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N)$ then: bas$(p) \in M$ iff p does not fork over M

[9]there are reasonable stronger version, but it follows that the function bas$(-)$ satisfies them

(d) if f is an isomorphism from $M_1 \in K_{\mathfrak{s}}$ onto $M_2 \in K_{\mathfrak{s}}$ and $p_1 \in \mathscr{S}^{\rm bs}(M_1)$ then $f({\rm bas}(p_1)) = {\rm bas}(f(p_1))$.

Remark 2.21. In §3 we can add:

(e) strong uniqueness: if $A \subseteq M \leq_{\mathfrak{k}(\mathfrak{s})} \mathfrak{C}, p \in \mathscr{S}(A, \mathfrak{C})$ well defined, <u>then</u> for at most one $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ do we have: q extends p and ${\rm bas}(p) \in A$. (needed for non-forking extensions).

Definition 2.22. We say that $\mathfrak{s}$ is equivalence-closed <u>when</u>:

(a) $\mathfrak{s}$ has type bases $p \mapsto {\rm bas}(p)$
(b) if $\mathbb{E}_M$ is a definition of an equivalence relation on ${}^{\omega>}M$ preserved by isomorphisms and $\leq_{\mathfrak{s}}$-extensions (i.e. $M \leq_{\mathfrak{s}} N \Rightarrow \mathbb{E}_M = \mathbb{E}_N \restriction {}^{\omega>}M$) <u>then</u> there is a definable function F from ${}^{\omega>}M$ to M such that $F^M(\bar{a}) = F^M(\bar{b})$ iff $\bar{a} E_M \bar{b}$ (or work in $\mathfrak{C}$).

To phrase the relation between $\mathfrak{k}$ and $\mathfrak{k}'$ we define.

Definition 2.23. Assume $\mathfrak{k}_1, \mathfrak{k}_2$ are ess-$[\lambda, \mu)$-a.e.c.
1) We say $\mathbf{i}$ is an interpretation in $\mathfrak{k}_2$ <u>when</u> $\mathbf{i}$ consists of

(a) a predicate $P^*_{\mathbf{i}}$
(b) a subset $\tau_{\mathbf{i}}$ of $\tau_{\mathfrak{k}_2}$.

2) In this case for $M_2 \in K_{\mathfrak{k}_2}$ le $M_2^{[\mathbf{i}]}$ be the $\tau_{\mathbf{i}}$-model $M_1 = M_2^{[\mathbf{i}]}$ with

- universe $P^{M_2}_{\mathbf{i}}$
- $R^{M_1} = R^{M_2} \restriction |M_1|$ for $R \in \tau_{\mathbf{i}}$
- F^{M_1} similarly, so F^{M_2} can be a partial function even if F^{M_2} is full.

3) We say that $\mathfrak{k}_1$ is $\mathbf{i}$-interpreted (or interpreted by $\mathbf{i}$) in $\mathfrak{k}_2$ <u>when</u>:

(a) $\mathbf{i}$ is an interpretation in $\mathfrak{k}_1$
(b) $\tau_{\mathfrak{k}_1} = \tau_{\mathbf{i}}$
(c) $K_{\mathfrak{k}_1} = \{M_2^{[\mathbf{i}]} : M_2 \in K_{\mathfrak{k}_2}\}$
(d) if $M_2 \leq_{\mathfrak{k}_2} N_2$ then $M_2^{[\mathbf{i}]} \leq_{\mathfrak{k}_1} N_2^{[\mathbf{i}]}$
(e) if $M_1 \leq_{\mathfrak{k}_1} N_1$ and $N_1 = N_2^{[\mathbf{i}]}$, so $N_2 \in K_{\mathfrak{k}_2}$ <u>then</u> for some $M_2 \leq_{\mathfrak{k}} N_2$ we have $M_1 = M_2^{[\mathbf{i}]}$
(f) if $M_1 \leq_{\mathfrak{k}_1} N_1$ and $M_1 = M_2^{[\mathbf{i}]}$, so $M_2 \in K_{\mathfrak{k}_2}$ <u>then</u> possible replacing M_2 by a model isomorphic to it over M_1, there is $N_2 \in K_{\mathfrak{k}_2}$ we have $M_2 \leq_{\mathfrak{k}_2} N_2$ and $N_1 = N_2^{[\mathbf{i}]}$.

Definition 2.24. 1) Assume $\mathfrak{k}_1$ is interpreted by $\mathbf{i}$ in $\mathfrak{k}_2$. We say strictly interpreted when: if $M_2^{[\mathbf{i}]} = N_2^{[\mathbf{i}]}$ then M_2, N_2 are isomorphic over $M_2^{[\mathbf{i}]}$.
2) We say $\mathfrak{k}_1$ is equivalent to $\mathfrak{k}_2$ if there are n and $\mathfrak{k}'_0, \ldots, \mathfrak{k}'_n$ such that $\mathfrak{k}_1 = \mathfrak{k}'_0, \mathfrak{k}_2 = \mathfrak{k}'_n$ and for each $\ell < n, \mathfrak{k}_\ell$ is strictly interpreted in $\mathfrak{k}_{\ell+1}$ or vice versa. Actually we can demand $n = 2$ and k_ℓ is strictly interpreted in $\mathfrak{k}'_1$ for $\ell = 1, 2$.

Definition 2.25. As above for (good) ess-$[\lambda, \mu)$-frame.

Claim 2.26. *Assume $\mathfrak{s}$ is a good* ess $-$ $[\lambda, \mu)$*-frame.* <u>*Then*</u> *there is $\mathfrak{C}$ (called a μ-saturated for $K_{\mathfrak{s}}$) such that:*

(a) *$\mathfrak{C}$ is a $\tau_{\mathfrak{s}}$-model of cardinality $\leq \mu$*

(b) *$\mathfrak{C}$ is a union of some $\leq_{\mathfrak{s}}$-increasing continuous sequence $\langle M_\alpha : \alpha < \mu \rangle$*

(c) *if $M \in K_{\mathfrak{s}}$ so $\lambda \leq \|M\| < \mu$* <u>*then*</u> *M is $\leq_{\mathfrak{s}}$-embeddable into some M_α from clause (b)*

(d) *$M_{\alpha+1}$ is brimmed over M_α for $\alpha < \mu$.*

§ 3. I **P**-simple types

We define the basic types over sets not necessary models. Note that in Definition 3.5 there is no real loss using C of cardinality $\in (\lambda, \mu)$, as we can replace λ by $\lambda_1 = \lambda + |C|$ and so replace $K_{\mathfrak{k}}$ to $K^{\mathfrak{k}}_{[\lambda_1,\mu)}$.

Hypothesis 3.1. 1) $\mathfrak{s}$ is a good ess-$[\lambda, \mu)$-frame, see Definition 2.11.
2) $\mathfrak{s}$ has type bases, see Definition 2.20.
3) $\mathfrak{C}$ denote some μ-saturated model for $K_{\mathfrak{s}}$ of cardinality $\leq \mu$, see 2.26.
4) But $M, A, \dots$ will be $<_{\mathfrak{k}(\mathfrak{s})} \mathfrak{C}, \subseteq \mathfrak{C}$ respectively but of cardinality $< \mu$.

Definition 3.2. Let $A \subseteq M \in K_{\mathfrak{s}}$.
1) $\mathrm{dcl}(A, M) = \{a \in M$: if $M' \leq_{\mathfrak{s}} M'', M \leq_{\mathfrak{s}} M''$ and $A \subseteq M'$ then $a \in M'$ and for every automorphism f of $M', f \restriction A = \mathrm{id}_A \Rightarrow f(a) = a\}$.
2) $\mathrm{acl}(A, M)$ is defined similarly but only with the first demand.

Definition 3.3. 1) For $A \subseteq M \in K_{\mathfrak{s}}$ let

$$\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(A, M) = \{q \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) : \mathrm{bas}(q) \in \mathrm{dcl}(A, \mathfrak{C})\}.$$

2) We call $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(A, M)$ regular <u>if</u> p as a member of $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ is regular.

Definition 3.4. 1) $\mathbb{E}_{\mathfrak{s}}$ is as in Claim 2.18(2).
2) If $A \subseteq M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$, then $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(A, M)$ <u>iff</u> p is definable over A, see 1.12(3) <u>iff</u> $\mathrm{inv}(p)$ from Definition 1.12 is $\subseteq A$ and well defined.

Definition 3.5. Let $A \subseteq \mathfrak{C}$.
1) We define a dependency relation on $\mathrm{good}(A, \mathfrak{C}) = \{c \in \mathfrak{C}$: for some $M <_{\mathfrak{k}(\mathfrak{s})} \mathfrak{C}, A \subseteq M$ and $\mathrm{ortp}(c, M, \mathfrak{C})$ is definable over some finite $\bar{a} \subseteq A\}$ as follows:

- ⊛ c depends on $\mathbf{J}$ in $(A, \mathfrak{C})$ iff there is no $M <_{\mathfrak{k}(\mathfrak{s})} \mathfrak{C}$ such that $A \cup \mathbf{J} \subseteq M$ and $\mathrm{ortp}(c, M, \mathfrak{C})$ is the non-forking extension of $\mathrm{ortp}(c, \bar{a}, \mathfrak{C})$ where $\bar{a}$ witnesses $c \in \mathrm{good}(A, \mathfrak{C})$.

2) We say that $C \in {}^{\mu>}[\mathfrak{C}]$ is good over (A, B) <u>when</u> there is a brimmed $M <_{\mathfrak{k}(\mathfrak{s})} \mathfrak{C}$ such that $B \cup A \subseteq M$ and $\mathrm{ortp}(C, M, \mathfrak{C})$ (see Definition 1.12(3)) is definable over A. (In the first order context we could say $\{c, B\}$ is independent over A but here this is problematic as $\mathrm{ortp}(B, A, \mathfrak{C})$ is not necessary basic).
3) We say $\langle A_\alpha : \alpha < \alpha^* \rangle$ is independent over A in $\mathfrak{C}$, see [She09e, L8.8,6p.5(1)] <u>if</u> we can find $M, \langle M_\alpha : \alpha < \alpha^* \rangle$ such that:

⊛(a) $A \subseteq M \leq_{\mathfrak{k}(\mathfrak{s})} M_\alpha <_{\mathfrak{s}} \mathfrak{C}$ for $\alpha < \alpha^*$
(b) M is brimmed
(c) $A_\alpha \subseteq M_\alpha$
(d) $\mathrm{ortp}(A_\alpha, M, \mathfrak{C})$ definable over A (= does not split over A)
(e) $\langle M_\alpha : \alpha < \alpha^* \rangle$ is independent over M.

3A) Similarly "over (A, B)".
4) We define locally independent naturally, that is every finite subfamily is independent.

Claim 3.6. *Assume $a \in \mathfrak{C}, A \subseteq \mathfrak{C}$.*
1) $a \in \mathrm{good}(A, \mathfrak{C})$ iff a realizes $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ for some M satisfying $A \subseteq M <_{\mathfrak{k}(\mathfrak{s})} \mathfrak{C}$.

Claim 3.7. *1) If $A_\alpha \subseteq \mathfrak{C}$ is good over $(A, \bigcup_{i<\alpha} A_i)$ for $\alpha < \alpha^*, \alpha^* < \omega$ then $\langle A_\alpha : \alpha < \alpha^* \rangle$ is independent over A.*
2) Independence is preserved by reordering.
3) If $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(\bar{a}, \mathfrak{C})$ is regular then on $p(\mathfrak{C}) = \{c : c$ realizes $p\}$ the independence relation satisfies:

- *(a) like (1)*
- *(b) if b^1_ℓ depends on $\{b^0_0, \ldots, b^0_{n-1}\}$ for $\ell < k$ and b^2 depends on $\{b^1_\ell : \ell < k\}$ then b^2 depends on $\{b^0_\ell : \ell < n\}$*
- *(c) if b depends on $\mathbf{J}, \mathbf{J} \subseteq \mathbf{J}'$ then b depends on $\mathbf{J}'$.*

Remark 3.8. 1) However, we have not mentioned finite character but the local independence satisfies it trivially.

Proof. Easy. $\square_{3.7}$

Definition 3.9. 1) Assume $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(\bar{a}, \mathfrak{C})$. We say that q is explicitly (p, n)-simple when ,:

- $\circledast$ there are $b_0, \ldots, b_{n-1}, c$ such that [10]:
  - (a) b_ℓ realizes p
  - (b) c realizes q
  - (c) b_ℓ is not good [11] over $(\bar{a}, c)$ for $\ell < n$
  - (d) $\langle b_\ell : \ell < n \rangle$ is independent over $\bar{a}$
  - (e) $\langle c, b_0, \ldots, b_{n-1} \rangle$ is good over $\bar{a}$
  - (f) if [12] c' realizes q then $c = c'$ iff for every $b \in p(\mathfrak{C})$ we have: b is good over $(\bar{a}, c)$ iff b is good over $(\bar{a}, , c')$.

1A) We say that a is explicitly (p, n)-simple over A if $\mathrm{ortp}(a, A, \mathfrak{C})$ is; similarly in the other definitions replacing (p, n) by p means "for some n".
2) Assume $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(\bar{a}, \mathfrak{C})$ and $\mathbf{P}$ as in Definition 1.6. We say that q is $\mathbf{P}$-simple if we can find n and explicitly $\mathbf{P}$-regular types $p_0, \ldots, p_{n-1} \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(\bar{a}, \mathfrak{C})$ such that: each $c \in p(\mathfrak{C})$ is definable by its type over $\bar{a} \cup \bigcup_{\ell<n} p_\ell(\mathfrak{C})$,
3) In part (1) we say strongly explicitly (p, n)-simple if there are $k > a$ and $\langle \bar{a}^*_\ell : \ell < \omega \rangle$ and $r \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(\bar{a}, \mathfrak{C})$ such that [on finitely many see 3.9(3B) below]:

- (a) $\{\bar{a}^*_\ell : \ell < \omega\} \in r(\mathfrak{C})$ is independent for any $c', c'' \in p(\mathfrak{C})$, we have $c' = c''$ iff for infinitely many $m < \omega$ ($\equiv$ for all but finitely many $\models$, see claim on average) for every $b \in p(\mathfrak{C})$ we have:
  - $(*)$ b is good over $(\bar{a}, \bar{a}, a^*_m, c)$ iff b is good over $(\bar{a}, a, a^*_m, c)$, (compare with 3.13, 3.17!).

3A) In part (1) we say weakly (p, n)-simple if in $\circledast$, clause (f) is replaced by

$(f)'$ if b is good over $(\bar{a}, a^*_m)$ then c', c' realizes the same type over $\bar{a}a^*_m b$.

[10] clause (c) + (e) are replacements for c is algebraic over $\bar{a} + \{b_\ell : \ell < n\}$ and each b_ℓ is necessary

[11] not good here is a replacement to "$\mathrm{ortp}(b_\ell, \bar{a} + c, \mathfrak{C})$ does not fork over $\bar{a}$"

[12] this seems a reasonable choice here but we can take others; this is an unreasonable choice for first order

3B) In part (1) we say (p,n)-simple if for some $\bar{a}^* \in {}^{\omega>}\mathfrak{C}$ good over $\bar{a}$ for every $c \in q(\mathfrak{C})$ there are $b_0,\dots,b_{n-1} \in p(\mathfrak{C})$ such that $c \in \mathrm{dcl}(\bar{a},\bar{a}^*,b_0,\dots,b_{n-1})$ and $\bar{a}\char`^\langle b_0,\dots,b_{n-1}\rangle$ is good over $\bar{a}$ if simple.
4) Similarly in (2).
5) We define $gw_p(b,\bar{a}), p$ regular parallel to some $p' \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a})$ (gw for general weight). Similarly for $gw_p(q)$.

We first list some obvious properties.

Claim 3.10. *1) If c is $\mathbf{P}$-simple over $\bar{a}, \bar{a} \subseteq A \subset \mathfrak{C}$ then $w_p(c,A)$ is finite.*
2) The obvious implications.

Claim 3.11. *1) [Closures of the simple bs].*
2) Assume $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a},\mathfrak{C})$. If $\bar{b}_1,\bar{b}_2$ are p-simple over A then

(a) *$\bar{b}_1\char`^\bar{b}_2$ is p-simple (of course, $\mathrm{ortp}_{\mathfrak{s}}(\bar{b}_2\bar{b}_2,\bar{a},\mathfrak{C})$ is not necessary in $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a},\mathfrak{C})$ even if $\mathrm{ortp}_{\mathfrak{s}}(\bar{b}_\ell,\bar{a},\mathfrak{C}) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a},\mathfrak{C})$ for $\ell = 1,2$)*
(b) *also $\mathrm{ortp}(\bar{b}_2,\bar{a}b_1,\mathfrak{C})$ is $\mathbf{P}$-simple.*

2) If $\bar{b}_\alpha$ is p-simple over $\bar{a}$ for $\alpha < \alpha^*, \pi : \beta^* \to \alpha^*$ one to one into, then $\sum\limits_{\alpha<\alpha^*} gw_p(b_\alpha,\bar{a}_* \cup \bigcup\limits_{\ell<\alpha} b_\alpha) = \sum\limits_{\beta<\beta^*} gw(b_{\pi(\beta)})\bar{a} \cup \bigcup\limits_{i<\beta} \bar{b}_{\pi(i)})$.

Claim 3.12. *[$\mathfrak{s}$ is equivalence-closed].*
Assume that $p,q \in \mathscr{S}^{\mathrm{bs}}(M)$ are not weakly orthogonal (e.g. see 6.9). Then for some $\bar{a} \in {}^{\omega>}M$ we have: p,q are definable over $\bar{a}$ (works without being stationary) and for some $\mathfrak{k}_{\mathfrak{s}}$-definable function $\mathbf{F}$, for each $c \in q(\mathfrak{C}), \mathrm{ortp}_{\mathfrak{s}}(\mathbf{F}(c,\bar{a}),\bar{a},\mathfrak{C}) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a},\mathfrak{C})$ and is explicitly (p,n)-simple for some n, (if, e.g., M is $(\lambda,*)$-brimmed then $n = w_p(q)$.

Proof. We can find n and $c_1,b_0,\dots,b_{-1} \in \mathfrak{C}$ with c realizing q, b_ℓ realizing $p, \{b_\ell,c\}$ is not independent over M and n maximal. Choose $\bar{a} \in {}^{\omega>}M$ such that $\mathrm{ortp}_{\mathfrak{s}}(\langle c,b_0,\dots,b_{n-1}\rangle,M,\mathfrak{C}\rangle)$ is definable over $\bar{a}$. Define $E_{\bar{a}}$, an equivalence relation on $q(\mathfrak{C}) : c_1 E_{\bar{a}} c_2$ iff for every $b \in p\mathfrak{C})$, we have b is good over $(a,c_1) \Rightarrow b$ is good over $(\bar{a},c_2)$. By "$\mathfrak{s}$ is eq-closed", we are done. $\square_{3.12}$

Claim 3.13. *1) Assume $p,q \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ are weakly orthogonal (see e.g. 6.9(1)) but not orthogonal. Then we can find $\bar{a} \in {}^{\omega>}M$ over which p,q are definable and $r_1 \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a},\mathfrak{C})$ such that letting $p_1 = p \restriction \bar{a}, q_1 = q \restriction \bar{a}, n =: w_p(q) \geq 1$ we have:*

$\circledast^n_{\bar{a},p_1,q_1,r_2}$ (a) *$p_1,q_1,r_1 \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(\bar{a}), \bar{a} \in {}^{\omega>}\mathfrak{C}$*
(b) *p_1,q_1 are weakly orthogonal (see e.g. Definition 6.9(1))*
(c) *if $\{a^*_n : n < \omega\} \subseteq r_1(\mathfrak{C})$ is independent over $\bar{a}$ and c realizes q then for infinitely many $m < \omega$ there is $b \in p(\mathfrak{C})$ such that b is good over $(\bar{a},a^*_n)$ but not over $(\bar{a},a^*_n,c)$*
(d) *in (c) really there are n independent such that (but not $n+1$).*

2) If $\circledast^n_{\bar{a},p_1,q_1,r_1}$ then (see Definition 3.9(3) for some definable function $\mathbf{F}$, if c realizes $q_1, c^* = F(c,\bar{a})$ and $\mathrm{ortp}_{\mathfrak{n}}(c^*,\bar{a},\mathfrak{C})$ is (p_1,n)-simple.

See proof below.

Claim 3.14. *0) Assuming $A \subseteq \mathfrak{C}$ and $a \in \mathfrak{C}$ is called finitary when it is definable over $\{a_0, \dots, a_{n-1}\}$ where each a_ℓ is in $\mathfrak{C}$ and is good over A inside $\mathfrak{C}$.*
1) If $a \in \mathrm{dcl}(\cup\{A_i : i < \alpha\} \cup A, \mathfrak{C})$ and $\mathrm{ortp}(a, A, \mathfrak{C})$ is finitary over (A) and $\{A_i : i < \alpha\}$ is independent over A then for some finite $u \subseteq \alpha$ we have $a \in \mathrm{dcl}(\cup\{A_i : i \in u\} \cup A, \mathfrak{C})$.
2) If $\mathrm{ortp}(b, \bar{a}, \mathfrak{C})$ is $\mathbf{P}$-simple, then it is finitary.
3) If $\{A_i : i < \alpha\}$ is independent over A and a is finitary over A then for some finite $u \subseteq \alpha$ (even $|u| < \mathrm{wg}(c, A)$), $\{A_i : i \in \alpha \backslash u\}$ is independent over $A, A \cup \{c\}$) (or use $(A', A''), (A', A'' \cup \{c\})$).

Definition 3.15. 1) $\mathrm{dcl}(A) = \{a$: for every automorphism f of $\mathfrak{C}, f(a) = a\}$.
2) $\mathrm{dcl}_{\mathrm{fin}}(A) = \cup\{\mathrm{dcl}(B) : B \subseteq A$ finite$\}$.
3) a is finitary over A if there are $n < \omega$ and $c_0, \dots, c_{n-1} \in \mathrm{good}(A)$ such that $a \in \mathrm{dcl}(A \cup \{c_0, \dots, c_{n-1}\}$ (or $\mathrm{dcl}_{\mathrm{fin}}$?).
4) For such A let $\mathrm{wg}(a, A)$ be $w(\mathrm{tp}(a, A, \mathfrak{C}))$ when well defined.
5) Strongly simple implies simple.

Claim 3.16. *In Definition 3.9(3), for some $m, k < \omega$ large enough, for every $c \in q(\mathfrak{C})$ there are $b_0, \dots, b_{m-1} \in \bigcup_{\ell<n} p_\ell(\mathfrak{C})$ such that $c \in \mathrm{dcl}(\bar{\mathrm{a}} \cup \{\mathrm{a}^*_\ell : \ell < \mathrm{k}\} \cup \{\mathrm{b}_\ell : \ell < \mathrm{m}\})$.*

Proof. Let $M_1, M_2 \in K_{\mathfrak{s}(\mathrm{brim})}$ be such that $M \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_2, M_1$ $(\lambda, *)$-brimmed over $M, p_\ell \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\ell)$ a non-forking extension of $p, q_\ell \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\ell)$ is a non-forking extension of $q, c \in M_2$ realizes q_1 and $(M_1, M_2, c) \in K^{3,\mathrm{pr}}_{\mathfrak{s}(\mathrm{brim})}$. Let $b_\ell \in p_1(M_2)$ for $\ell < n^* =: w_p(q)$ be such that $\{b_\ell : \ell < n^*\}$ is independent in (M_1, M_2), let $\bar{a}^* \in {}^{\omega>}(M_1)$ be such that $\mathrm{ortp}_{\mathfrak{s}}(\langle c, b_0, \dots, b_{n-1}\rangle, M_1, M_2)$ is definable over $\bar{a}^*$ and $r = \mathrm{ortp}_{\mathfrak{s}}(\bar{a}^*, M_1, M_2), r^+ = \mathrm{ortp}(\bar{a}^* {}^\frown \langle b_0, \dots, b_{n-1}\rangle, M, M_2)$.

Let $\bar{a} \in {}^{\omega>}M$ be such that $\mathrm{ortp}_{\mathfrak{s}}(\bar{a}^*, \langle c, b_0, \dots, b_{n-1}\rangle, M, M_2)$ is definable over $\bar{a}$. As M_1 is $(\lambda, *)$-saturated over M there is $\{\bar{a}^*_f : f < \omega\} \subseteq r(\mathfrak{C})$ independent in (M, M_1) moreover letting $a^*_\omega = \bar{a}^*$, we have $\langle a^*_\alpha : \alpha \leq \omega\rangle$ is independent in (M, M_1). Clearly $\mathrm{ortp}_{\mathfrak{s}}(c\bar{a}^*_n, M, M_2)$ does not depend on n hence we can find $\langle\langle b^\alpha_\ell : \ell < n\rangle : \alpha \leq \omega\rangle$ such that $b^\alpha_\ell \in M_2, b^\omega_\ell = b_\ell$ and $\{c\bar{a}^*_\alpha, b^\alpha_0 \dots b^\alpha_{n-1} : \alpha \leq \omega\}$ (as usual as index set is independent in (M_1, M_2).

The rest should be clear. $\square_{3.13}$

Definition 3.17. Assume $\bar{a} \in {}^{\omega>}\mathfrak{C}, n < \omega, p, q, r \in \mathscr{S}^{\mathrm{bs}}(M)$ are as in the definition of p-simple$^{[-]}$ but p, q are weakly orthogonal (see e.g. Definition 6.9(1)) let p be a definable related function such that for any $\bar{a}^\nu_\ell \in r(\mathfrak{C}), \ell < k^*$ independent mapping $c \mapsto \langle\{b \in q(\mathfrak{C}) : R\mathfrak{C} \models R(b, c, \bar{a}^*_\ell)\}$ is a one-to-one function from $q(\mathfrak{C})$ into $\{\langle J_\ell : \ell < k^*\rangle : J_\ell \subseteq p(\mathfrak{C})$ is closed under dependence and has p-weight $n^*\}$
1) We can define $E = E_{p,q,r}$ a two-place relation over $r(\mathfrak{C}) : \bar{a}^*_1 E \bar{a}^*_2$ iff $\bar{a}_1, \bar{a}_2 \in r(\mathfrak{C})$ have the same projection common to $p(\mathfrak{C})$ and $q(\mathfrak{C})$.
2) Define unit-less group on r/E and its action on $q(\mathfrak{C})$.

Remark 3.18. 1) A major point is: as q is p-simple, $w_p(-)$ acts "nicely" on $p(\mathfrak{C})$ so if $c_1, c_2, c_3 \in q(\mathfrak{C})$ then $w_p(\langle c_1, c_2, c_3\rangle\bar{a}) \leq 3n^*$. This enables us to define average using a finite sequence seem quite satisfying. Alternatively, look more at averages of independent sets.
2) Silly Groups: Concerning interpreting groups note that in our present context, for every definable set P^M we can add the group of finite subsets of P^M with symmetric difference (as addition).

3) The axiomatization above has prototype $\mathfrak{s}$ where $K_{\mathfrak{s}} = \{M : M$ a κ-saturated model of $T\}, \le_{\mathfrak{s}} = \prec \restriction K_{\mathfrak{s}}, \bigcup\limits_{\mathfrak{s}}$ is non-forking, T a stable first order theory with $\kappa(T) \le \mathrm{cf}(\kappa)$. But we may prefer to formalize the pair $(\mathfrak{t}, \mathfrak{s}), \mathfrak{s}$ as above, $K_{\mathfrak{t}}$ = models of $T, \le_{\mathfrak{t}} = \prec \restriction K_{\mathfrak{t}}, \bigcup\limits_{\mathfrak{t}}$ is non-forking.

From $\mathfrak{s}$ we can reconstruct a $\mathfrak{t}$ by closing $\mathfrak{k}_{\mathfrak{s}}$ under direct limits, but in interesting cases we end up with a bigger $\mathfrak{t}$.

Part II Generalizing Stable Classes

§ 4. II Introduction

In this part we try to deal with classes like "$\aleph_1$-saturated models of a first order theory T and even a stable one" rather than of "a model of T". The parallel problem for "model of T, even superstable one" is the subject of [She09d].

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Now some construction goes well by induction on cardinality, say by dealing with $(\lambda, \mathscr{P}(n))$-system of models but not all. E.g. starting with $\aleph_0$ we may consider $\lambda > \aleph_0$, so can find $F : [\lambda]^{\aleph_0} \to \lambda$ such that there is an infinite decreasing sequence of F-closed subsets of $\lambda, u \in [\lambda]^{<\aleph_0} \Rightarrow F(u) = 0$ maybe such that $u \in [\lambda]^{\leq \aleph_0} \Rightarrow |c\ell_F(u)| \leq \aleph_0$. Let $\langle u_\alpha : \alpha < \alpha_* \rangle$ list $\{c\ell_F(u) : u \in [\lambda]^{\leq \aleph_0}\}$ such that $c\ell_F(u_\alpha) \subseteq c\ell_F(u_\beta) \Rightarrow \alpha \leq \beta$ we try to choose M_{u_α} by induction on α.

§ 5. II Axiomatizing a.e.c. without full continuity

§ 5(A). a.e.c.

Classes like "$\aleph_1$-saturated models of a first order T, which is not superstable", does not fall under a.e.c., still they are close and below we suggest a framework for them. So for increasing sequences of short length we have weaker demand.

We shall say more on primes later.

We shall lift a (μ, λ, κ)-a.e.c. to $(\infty, \lambda, \kappa)$-a.e.c. (see below), so actually k_λ suffice, but for our main objects, good frames, this is more complicated as its properties (e.g., the amalgamation property) are not necessarily preserved by the lifting.

This section generalizes [She09c, §1], in some cases the differences are minor, sometimes the whole point is the difference.

Convention 5.1. 1) In this section $\mathfrak{k}$ will denote a directed a.e.c., see Definition 5.2, may write d.a.e.c. (the d stands for directed).
2) We shall write (outside the definitions) $\mu_{\mathfrak{k}}, \lambda_{\mathfrak{k}}, \kappa_{\mathfrak{k}}$.

Definition 5.2. Assume $\lambda < \mu, \lambda^{<\kappa} = \lambda$ (for notational simplicity) and $\alpha < \mu \Rightarrow |\alpha|^{<\kappa} < \mu$ and κ is regular.

We say that $\mathfrak{k}$ is a $(\mu, \lambda, \kappa) - 1$-d.a.e.c. (we may omit or add the "(μ, λ, κ)" by Ax(0)(d) below, similarly in similar definitions; if $\kappa = \aleph_0$ we may omit it, instead $\mu = \mu_1^+$ we may write $\leq \mu_1$) <u>when</u> the axioms below hold; we write d.a.e.c. or 0-d.a.e.c. when we omit Ax(III)(b),(IV)(b) Ax(0) $\mathfrak{k}$ consists of

- (a) $\tau_{\mathfrak{k}}$, a vocabulary with each predicate and function symbol of arity $\leq \lambda$
- (b) K, a class of τ-models
- (c) a two-place relation $\leq_{\mathfrak{k}}$ on K
- (d) the cardinals $\mu = \mu_{\mathfrak{k}}, \mu(\mathfrak{k}), \lambda = \lambda_{\mathfrak{k}} = \lambda(\mathfrak{k})$ and $\kappa = \kappa_{\mathfrak{k}} = \kappa(\mathfrak{k})$ (so $\mu > \lambda = \lambda^{<\kappa} \geq \kappa = \text{cf}(\kappa)$ and $\alpha < \mu \Rightarrow |\alpha|^{<\kappa} < \mu$)

such that

- (e) if $M_1 \cong M_2$ then $M_1 \in K \Leftrightarrow M_2 \in K$
- (f) if $(N_1, M_1) \cong (N_2, M_2)$ <u>then</u> $M_1 \leq_{\mathfrak{k}} N_1 \Rightarrow M_2 \leq_{\mathfrak{k}} N_2$
- (g) every $M \in K$ has cardinality $\geq \lambda$ but $< \mu$

$(Ax(I)(a))$ $M \leq_{\mathfrak{k}} N \Rightarrow M \subseteq N$

$(Ax(II)(a))$ $\leq_{\mathfrak{k}}$ is a partial order

$Ax(III)$ assume that $\langle M_i : i < \delta \rangle$ is a $\leq_{\mathfrak{k}}$-increasing sequence and $\| \cup \{M_i : i < \delta\}\| < \mu$ <u>then</u>

- (a) (existence of unions) if $\text{cf}(\delta) \geq \kappa$ then there is $M \in K$ such that $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} M$ and $|M| = \cup\{|M_i| : i < \delta\}$ but not necessarily $M = \bigcup_{i<\delta} M_i$
- (b) (existence of limits) there is $M \in K$ such that $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} M$

$Ax(IV)(a)$ (weak uniqueness of limit = weak smoothness) for $\langle M_i : i < \delta \rangle$ as above,

- (a) if $\text{cf}(\delta) \geq \kappa$ and M is as in Ax(III)(a) and $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} N$ then $M \leq_{\mathfrak{k}} N$

(b) if $N_\ell \in K$ and $i < \delta \Rightarrow M_i \le_{\mathfrak{k}} N_\ell$ for $\ell = 1, 2$ then there are $N \in K$ and f_1, f_2 such that f_ℓ is a $\le_{\mathfrak{k}}$-embedding of N_ℓ into N for $\ell = 1, 2$ and $i < \delta \Rightarrow f_1 \restriction M_i = f_2 \restriction M_i$

$Ax(V)$ if $N_\ell \le_{\mathfrak{k}} M$ for $\ell = 1, 2$ and $N_1 \subseteq N_2$ then $N_1 \le_{\mathfrak{k}} N_2$

$Ax(VI)$ (L.S.T. property) if $A \subseteq M \in K, |A| \le \lambda$ then there is $M \le_{\mathfrak{k}} N$ of cardinality λ such that $A \subseteq M$.

Remark 5.3. There are some more axioms listed in 5.4(5), but we shall mention them in any claim in which they are used so no need to memorize, so 5.4(1)-(4) omit them?

Definition 5.4. 1) We say $\mathfrak{k}$ is a 4-d.a.e.c. or d.a.e.c$^+$ when it is a $(\lambda, \mu, \kappa) - 1$-d.a.e.c. and satisfies Ax(III)(d),Ax(IV)(e) below.
2) We say $\mathfrak{k}$ is a 2-d.a.e.c. or d.a.e.c.$^\pm$ when is a $(\lambda, \mu, \kappa) - 0$-d.a.e.c. and Ax(III)(d), Ax(IV)(d) below holds.
3) We say $\mathfrak{k}$ is 5-d.a.e.c. when it is 1-d.a.e.c. and Ax(III)(f) holds.
4) We say $\mathfrak{k}$ is 6-d.a.e.c. when it is a 1-d.a.e.c. and Ax(III)(f) + Ax(IV)(f).
5) Concerning Definition 5.2, we consider the following axioms:

Ax(III)(c) if I is κ-directed and $\bar{M} = \langle M_s : s \in I \rangle$ is $\le_{\mathfrak{k}}$-increasing $s \le_I t \Rightarrow M_s \subseteq M_s$ and $\Sigma\{\|M_s\| : s \in I\} < \mu$ then $\bar{M}$ has a $\le_{\mathfrak{s}}$-upper bound, M, i.e. $s \in I \Rightarrow M_s \le_{\mathfrak{k}} M$.

Ax(III)(d) (union of directed system) if I is κ-directed, $|I| < \mu, \langle M_t : t \in I \rangle$ is $\le_{\mathfrak{k}}$-increasing and $\| \cup \{M_t : t \in I\}\| < \mu$ then there is one and only one M with universe $\cup \{|M_t| : t \in I\}$ such that $M_s \le_{\mathfrak{k}} M$ for every $s \in I$ we call it the $\le_{\mathfrak{k}}$-union of $\langle M_t : t \in I \rangle$.

Ax(III)(e) like Ax(III)(c) but I is just directed

Ax(III)(f) If $\bar{M} = \langle M_i : i < \delta \rangle$ is $\le_{\mathfrak{k}}$-increasing, cf$(\delta) < \kappa$ and $| \cup \{M_i : i < \delta\}| < \mu$ then there is M which is $\le_{\mathfrak{k}}$-prime over $\bar{M}$, i.e.

(∗) if $N \in K_{\mathfrak{k}}$ and $i < \delta \Rightarrow M_i \le_{\mathfrak{k}} N$ then there is a $\le_{\mathfrak{k}}$-embedding of M into M over $\cup \{|M_i| : i < \delta\}$.

Ax(IV)(c) If I is κ-directed and $\bar{M} = \langle M_s : s \in I \rangle$ is $\le_{\mathfrak{k}}$-increasing and N_1, N_2 are $\le_{\mathfrak{s}}$-upper bounds of $\bar{M}$ then for some (N_2', f) we have $N_2 \le_{\mathfrak{k}} N_2'$ and f is a $\le_{\mathfrak{k}}$-embedding of N_1 into N_2 which is the identity on M_s for every $s \in I$ (this is a weak form of uniqueness)

Ax(IV)(d) If I is a κ-directed partial order, $\bar{M} = \langle M_s : s \in I \rangle$ is $\le_{\mathfrak{k}}$-increasing, $s \in I \Rightarrow M_s \le_{\mathfrak{k}} M$ and $|M| = \cup |M_s| : s \in I\}$, then $\bigwedge_s M_s \le_{\mathfrak{k}} N \Rightarrow M \le_{\mathfrak{k}} N$.

Ax(IV)(e) Like Ax(IV)(c) but I is just directed.

Ax(IV)(f) If I is directed and $\bar{M} = \langle M_s : s \in I \rangle$ is $\le_{\mathfrak{k}}$-increasing then there is M which is a $\le_{\mathfrak{k}}$-prime over $\bar{M}$, defined as in Ax(III)(f).

Claim 5.5. *Assume*[13] $\mathfrak{k}$ *is a d.a.e.c.*

[13]By 5.1 no need to say this

1) Ax(III)(d) implies Ax(III)(c).
2) Ax(III)(e) implies Ax(III)(c) and it implies Ax(III)(b).
3) Ax(IV)(d) implies Ax(IV)(c).
4) Ax(IV)(e) implies Ax(IV)(c) and implies Ax(III)(b).
5) In all the axioms in Definition 5.4 it is necessary that $|\cup\{M_s : s \in I\}| < \mu_{\mathfrak{k}}$.
6) Ax(IV)(b) implies that $\mathfrak{k}$ *has amalgamation.*

Definition 5.6. We say $\langle M_i : i < \alpha\rangle$ is $\le_{\mathfrak{k}}$-increasing $(\ge \kappa)$-continuous when it is $\le_{\mathfrak{k}}$-increasing and $\delta < \alpha$ and $\mathrm{cf}(\delta) \ge \kappa \Rightarrow |M_i| = \cup\{|M_j| : j < \delta\}$.

As an exercise we consider directed systems with mappings.

Definition 5.7. 1) We say that $\bar M = \langle M_t, h_{s,t} : s \le_I t\rangle$ is a $\le_{\mathfrak{k}}$-directed system when I is a directed partial order and if $t_0 \le_I t_2 \le_I t_2$ then $h_{t_2,t_0} = h_{t_2,t_1} \circ h_{t_1,t_0}$.
1A) We say that $\bar M = \langle M_t, h_{s,t} : s \le_I t\rangle$ is a $\le_{\mathfrak{k}} -\theta$-directed system when in addition I is θ-directed.
2) We omit $h_{s,t}$ when $s \le_I t \Rightarrow h_{s,t} = \mathrm{id}_{M_s}$ and write $\bar M = \langle M_t : t \in I\rangle$.
3) We say $(M, \bar h)$ is a $\le_{\mathfrak{k}}$-limit of $\bar M$ when $\bar h = \langle h_s : s \in I\rangle, h_s$ is a $\le_{\mathfrak{k}}$-embedding of M_s into M_s and $s \le_I t \Rightarrow h_s = h_t \circ h_{t,s}$.
4) We say $\bar M = \langle M_\alpha : \alpha < \alpha^*\rangle$ is $\le_{\mathfrak{k}}$-semi-continuous when: (see Ax(III)(f) in 5.4)

- (a) $\bar M$ is $\le_{\mathfrak{k}}$-increasing
- (b) if $\alpha < \alpha^*$ has cofinality $\ge \kappa$ then $M_\alpha = \cup\{M_\beta : \beta < \alpha\}$
- (c) if $\alpha < \alpha^*$ has cofinality $< \kappa$ then M_δ is $\le_{\mathfrak{k}}$-prime over $\bar M \restriction \alpha$.

Observation 5.8. *[$\mathfrak{k}$ is an d.a.e.c.]*
1) If $\bar M = \langle M_t, h_{s,t} : s \le_I t\rangle$ *is a* $\le_{\mathfrak{k}}$*-directed system, then we can find a* $\le_{\mathfrak{k}}$*-directed system* $\langle M'_t : t \in I\rangle$ *(so* $s \le_I t \Rightarrow M'_s \le_{\mathfrak{k}} M'_t$*) and* $\bar g = \langle g_t : t \in I\rangle$ *such that:*

- (a) g_t *is an isomorphism from* M_t *onto* M'_t
- (b) *if* $s \le_I t$ *then* $g_s = g_t \circ h_{s,t}$.

2) So in the axioms (III)(a),(b)(IV)(a) from Definition 5.2 as well as those of 5.4 we can use $\le_{\mathfrak{k}}$*-directed system* $\langle M_s, h_{s,t} : s \le_I t\rangle$ *with* I *as there.*
3) If $\mathfrak{k}$ *is an ess-*(μ,λ)*-a.e.c., see §1 then* $\mathfrak{k}$ *is a* $(\mu,\lambda,\aleph_0)$*-d.a.e.c. and satisfies all the axioms from 5.4.*
4) If $(M, \bar h)$ *is prime over* $\bar M = \langle M_t, h_{s,t} : s \le_I t\rangle$ *and* $\chi = \Sigma\{\|M_t\| : t \in I\}$ *then* $\|M\| \le \chi^{<\kappa}$.

Proof. Straightforward, e.g. we can use "$\mathfrak{k}$ has $(\chi^{<\kappa})$-LST", i.e. Observation 5.9. $\square_{5.8}$

More serious is proving the LST theorem in our content (recall that in the axioms, see Ax(VI), we demand it only down to λ).

Claim 5.9. *[$\mathfrak{k}$ is a* $(\mu,\lambda,\kappa) - 2$*-d.a.e.c., see Definition 5.4.]*
If $\lambda_{\mathfrak{k}} \le \chi = \chi^{<\kappa} < \mu_{\mathfrak{k}}, A \subseteq N \in \mathfrak{k}$ *and* $|A| \le \chi \le \|N\|$ *then there is* $M \le_{\mathfrak{k}} N$ *of cardinality* χ *such that* $\|M\| = \chi$.

Proof. Let $\langle u_\alpha : \alpha < \alpha(*)\rangle$ list $[A]^{<\kappa(\mathfrak{k})}$, let I be the following partial order:

$(*)_1$ (α) set of elements is $\{\alpha < \chi$: for no $\beta < \alpha$ do we have $u_\alpha \subseteq u_\beta\}$
(β) $\alpha \le_I \beta$ iff $u_\alpha \subseteq u_\beta$ (hence $\alpha \le \beta$).

Easily

$(*)_2$ (a) I is κ-directed
(b) for every $\alpha < \alpha(*)$ for some $\beta < \alpha(*)$ we have $u_\alpha \subseteq u_\beta \wedge \beta \in I$
(c) $\cup\{u_\alpha : \alpha \in I\} = A$.

Now we choose M_α by induction on $\alpha < \chi$ such that

$(*)_3$ (a) $M_\alpha \leq_{\mathfrak{k}} N$
(b) $\|M_\alpha\| = \lambda_{\mathfrak{k}}$
(c) M_α include $\cup\{M_\beta : \beta <_I \alpha\} \cup u_\alpha$.

Note that $|\{\beta \in I : \beta <_I \alpha\}| \leq |\{u : u \subseteq u_\alpha\}| = 2^{|u_\alpha|} \leq 2^{<\kappa(\mathfrak{k})} \leq \lambda_{\mathfrak{k}}$ and by the induction hypothesis $\beta < \alpha \Rightarrow \|M_\beta\| \leq \lambda_{\mathfrak{k}}$ and recall $|u_\alpha| < \kappa(\mathfrak{k}) \leq \lambda_{\mathfrak{k}}$ hence the set $\cup\{M_\beta : \beta < \alpha\} \cup u_\alpha$ is a subset of N of cardinality $\leq \lambda$ hence by Ax(VI) there is M_α as required.

Having chosen $\langle M_\alpha : \alpha \in I\rangle$ clearly by Ax(V) it is a $\leq_{\mathfrak{k}}$-directed system hence by Ax(III)(d), $M = \cup\{M_\alpha : \alpha \in I\}$ is well defined with universe $\cup\{|M_\alpha| : \alpha \in I\}$ and by Ax(IV)(d) we have $M \leq_{\mathfrak{k}} N$.

Clearly $\|M\| \leq \Sigma\{\|M_\alpha\| : \alpha \in I\} \leq |I| \cdot \lambda_{\mathfrak{k}} = \chi$, and by $(*)_2(c) + (*)_3(c)$ we have $A \subseteq \cup\{u_\alpha : \alpha < \chi\} = \cup\{u_\alpha : \alpha \in I\} \subseteq \cup\{|M_\alpha| : \alpha \in I\} = M$ and so M is as required. $\square_{5.9}$

Notation 5.10. 1) For $\chi \in [\lambda_{\mathfrak{k}}, \mu_{\mathfrak{k}})$ let $K_\chi = K^{\mathfrak{k}}_\chi = \{M \in K : \|M\| = \chi\}$ and $K_{<\chi} = \bigcup_{\mu<\chi} K_\mu$.
2) $\mathfrak{k}_\chi = (K_\chi, \leq_{\mathfrak{k}} \restriction K_\chi)$.
3) If $\lambda_{\mathfrak{k}} \leq \lambda_1 < \mu_1 \leq \mu_{\mathfrak{k}}, \lambda_1 = \lambda_1^{<\kappa}$ and $(\forall \alpha < \mu_1)(|\alpha|^{<\kappa} < \mu_1)$, <u>then</u> we define $K_{[\lambda_1,\mu_1]} = K^{\mathfrak{s}}_{[\lambda_1,\mu_1)}$ and $\mathfrak{k}_1 = \mathfrak{k}_{[\lambda_1,\mu_1]}$ similarly, i.e. $K_{\mathfrak{k}} = \{M \in K_{\mathfrak{k}} : \|M\| \in [\lambda_1, \mu_1)\}$ and $\leq_{\mathfrak{k}_1} = \leq_{\mathfrak{k}} \restriction K_{\mathfrak{k}_1}$ and $\lambda_{\mathfrak{k}_1} = \lambda_1, \mu_{\mathfrak{k}_1} = \mu_1, \kappa_{\mathfrak{k}_1} = \kappa_{\mathfrak{k}}$.

Definition 5.11. The embedding $f : N \to M$ is a $\mathfrak{k}$-embedding or a $\leq_{\mathfrak{k}}$-embedding if its range is the universe of a model $N' \leq_{\mathfrak{k}} M$, (so $f : N \to N'$ is an isomorphism onto).

Claim 5.12. *[$\mathfrak{k}$ is a 2-d.a.e.c.]*
1) For every $N \in K$ there is a $\kappa_{\mathfrak{k}}$-directed partial order I of cardinality $\leq \|N\|$ and $\bar{M} = \langle M_t : t \in I\rangle$ such that $t \in I \Rightarrow M_t \leq_{\mathfrak{k}} N, \|M_t\| \leq \mathrm{LST}(\mathfrak{k}), I \models s < t \Rightarrow M_s \leq_{\mathfrak{k}} M_t$ and $N = \bigcup_{t\in I} M_t$.
2) For every $N_1 \leq_{\mathfrak{k}} N_2$ we can find $\langle M^\ell_t : t \in I^*\rangle$ as in part (1) for N_ℓ such that $I_1 \subseteq I_2$ and $t \in I_1 \Rightarrow M^2_t = M^1_t$.

Proof. 1) As in the proof of 5.9.
2) Similarly. $\square_{5.12}$

Claim 5.13. *Assume $\lambda_{\mathfrak{k}} \leq \lambda_1 = \lambda_1^{<\kappa} < \mu_1 \leq \mu_{\mathfrak{k}}$ and $(\forall \alpha < \mu_1)(|\alpha|^{<\kappa} < \mu_1)$.*
1) <u>Then</u> $\mathfrak{k}^*_1 := \mathfrak{k}_{[\lambda_1,\mu_1)}$ as defined in 5.10(3) is a $(\lambda_1, \mu_1, \kappa_{\mathfrak{k}})$-d.a.e.c.
2) For each of the folllowing axioms if $\mathfrak{k}$ satisfies it then so does $\mathfrak{k}_1$: Ax(III)(d),(IV)(b),(IV)(c),(IV)(d).
3) If if addition $\mathfrak{k}$ satisfies Ax(III)(d),(IV)(d), this part (2) apply to all the axioms in 5.2, 5.4.

Claim 5.14. *1) If $\mathfrak{k}$ satisfies Ax(IV)(e) then $\mathfrak{k}$ satisfies Ax(III)(e) provided that $\mu_{\mathfrak{k}}$ is regular or at least the relevant I has cardinality $< \mathrm{cf}(\mu_{\mathfrak{k}})$.*
2) If Ax(III)(d),(IV)(d) we can waive $\mu_{\mathfrak{k}}$ is regular.

Proof. We prove this by induction on $|I|$.

Case 1: I is finite.
So there is $t^* \in I$ such that $t \in I \Rightarrow t \leq_I t^*$, so this is trivial.

Case 2: I is countable.
So we can find a sequence $\langle t_n : n < \omega\rangle$ such that $t_n \in I, t_n \leq_I t_{n+1}$ and $s \in I \Rightarrow \bigvee\limits_{n<\omega} s \leq_I t_n$. Now we can apply the axiom to $\langle M_{t_n}, h_{t_n,t_m} : m < n < \omega\rangle$.

Case 3: I uncountable.
First, we can find an increasing continuous sequence $\langle I_\alpha : \alpha < |I|\rangle$ such that $I_\alpha \subseteq I$ is directed of cardinality $\leq |\alpha| + \aleph_0$ and let $I_{|I|} = I = \cup\{I_\alpha : \alpha < |I|\}$.
Second, by the induction hypothesis for each $\alpha < |I|$ we choose $N_\alpha, \bar h^\alpha = \langle h_{\alpha,t} : t \in I_\alpha\rangle$ such that:

- (a) $N_\alpha \in \mathfrak{k}^{\mathfrak{s}}_{\leq\chi}$
- (b) $h_{\alpha,t}$ is a $\leq_{\mathfrak{k}}$-embedding of M_t into N_α
- (c) if $s <_I t$ are in I_α then $h_{\alpha,s} = h_{\alpha,t} \circ h_{t,s}$
- (d) if $\beta < \alpha$ then $N_\beta \leq_{\mathfrak{k}} N_\alpha$ and $t \in I_\beta \Rightarrow h_{\alpha,t} = h_{\beta,t}$.

For $\alpha = 0$ use the induction hypothesis.
For α a limit ordinal by Ax(III)(a) there is N_α s required as $I_\alpha = \cup\{I_\beta : \beta < \alpha\}$ there are no new h_t's; well we have to check $\Sigma\{\|N_\beta\| : \beta < \alpha\} < \mu_{\mathfrak{k}}$ but as we assume $\mu_{\mathfrak{k}}$ is regular this holds.
For $\alpha = \beta + 1$, by the induction hypothesis there is $(N'_\alpha, \bar g^\alpha)$ which is a limit of $\langle M_s, h_{s,t} : s \leq_{I_\alpha} t\rangle$. Now apply Ax(IV)(e); well is the directed system version with $\langle M_s, h_{s,t} : s \leq_{I_\beta} t\rangle, (N'_\alpha, \bar g_\alpha), (N_\beta, \langle h_s : s \in I_\beta\rangle$ here standing for $\bar M, N_1, N_2$ there.
So there are $N_\alpha, f^\alpha_s (s \in I_\beta)$ such that $N_\beta \leq_{\mathfrak{k}} N_\alpha$ and $s \in I_\beta \Rightarrow f^\alpha_s \cdot g_s = h_s$. Lastly, for $s \in I_\alpha \backslash I_\beta$ we choose $h_s = f^\alpha_s \circ g_s$, so we are clearly done.
2) Easy by 5.9 or 5.13. $\square_{5.14}$

§ 5(B). **Basic Notions.**

As in [She09c, §1], we now recall the definition of orbital types (note that it is natural to look at types only over models which are amalgamation bases recalling Ax(IV)(b) implies every $M \in K_{\mathfrak{k}}$ is).

Definition 5.15. 1) For $\chi \in [\lambda_{\mathfrak{k}}, \mu_{\mathfrak{k}})$ and $M \in K_\chi$ we define $\mathscr{S}(M)$ as $\mathrm{ortp}(a, M, N) : M \leq_{\mathfrak{k}} N \in K_\chi$ and $a \in N\}$ where $\mathrm{ortp}(a, M, N) = (M, N, a)/\mathscr{E}_M$ where $\mathscr{E}_M$ is the transitive closure of $\mathscr{E}^{\mathrm{at}}_M$, and the two-place relation $\mathscr{E}^{\mathrm{at}}_M$ is defined by:

$$\begin{aligned}(M, N_1, a_1)\mathscr{E}^{\mathrm{at}}_M \quad (M, N_2, a_2) &\underline{\text{iff}}\ M \leq_{\mathfrak{k}} N_\ell, a_\ell \in N_\ell, \|M\| \leq \|N_\ell\| = \|M\|^{<\kappa} \\ &\text{for } \ell = 1,2 \text{ and there is } N \in K_\chi \text{ and } \leq_{\mathfrak{k}}\text{-embeddings} \\ &f_\ell : N_\ell \to N \text{ for } \ell = 1,2 \text{ such that:} \\ &f_1 \restriction M = \mathrm{id}_M = f_2 \restriction M \text{ and } f_1(a_1) = f_2(a_2).\end{aligned}$$

(of course $M \leq_{\mathfrak{k}} N_1, M \leq_{\mathfrak{k}} N_2$ and $a_1 \in N_1, a_2 \in N_2$)

2) We say "a realizes p in N" when $a \in N, p \in \mathscr{S}(M)$ and letting $\chi = \|M\|^{<\kappa}$ for some $N' \in K_\chi$ we have $M \leq_{\mathfrak{k}} N' \leq_{\mathfrak{k}} N$ and $a \in N'$ and $p = \text{ortp}(a, M, N')$; so $M, N' \in K_\chi$ but possibly $N \notin K_\chi$.
3) We say "a_2 strongly [14] realizes $(M, N^1, a_1)/\mathscr{E}^{\text{at}}_M$ in N" when for some N^2 we have $M \leq_{\mathfrak{k}} N^2 \leq_{\mathfrak{k}} N$ and $a_2 \in N^2$ and $(M, N^1, a_1)\, \mathscr{E}^{\text{at}}_M\, (M, N^2, a_2)$.
4) We say M_0 is a $\leq_{\mathfrak{k}[\chi_0,\chi_1)}$-amalgamation base if this holds in $\mathfrak{k}_{[\chi_0,\chi_1)}$, see below.
4A) We say $M_0 \in \mathfrak{k}$ is an amalgamation base or $\leq_{\mathfrak{k}}$-amalgamation base when: for every $M_1, M_2 \in \mathfrak{k}$ and $\leq_{\mathfrak{k}}$-embeddings $f_\ell : M_0 \to M_\ell$ (for $\ell = 1, 2$) there is $M_3 \in \mathfrak{k}_\lambda$ and $\leq_{\mathfrak{k}}$-embeddings $g_\ell : M_\ell \to M_3$ (for $\ell = 1, 2$) such that $g_1 \circ f_1 = g_2 \circ f_2$.
5) We say $\mathfrak{k}$ is stable in χ when:

- (a) $\lambda_{\mathfrak{k}} \leq \chi < \mu_{\mathfrak{k}}$
- (b) $M \in K_\chi \Rightarrow |\mathscr{S}(M)| \leq \chi$
- (c) $\chi \in \text{Car}_{\mathfrak{k}}$ which means $\chi = \chi^{<\kappa}$ or the conclusion of 5.9 holds
- (d) $\mathfrak{k}_\chi$ has amalgamation.

6) We say $p = q \restriction M$ if $p \in \mathscr{S}(M), q \in \mathscr{S}(N), M \leq_{\mathfrak{k}} N$ and for some $N^+, N \leq_{\mathfrak{k}} N^+$ and $a \in N^+$ we have $p = \text{ortp}(a, M, N^+), q = \text{ortp}(a, N, N^+)$; note that $p \restriction M$ is well defined if $M \leq_{\mathfrak{k}} N, p \in \mathscr{S}(N)$.
7) For finite m, for $M \leq_{\mathfrak{k}} N, \bar{a} \in {}^m N$ we can define $\text{ortp}(\bar{a}, N, N)$ and $\mathscr{S}^m(M)$ similarly and $\mathscr{S}^{<\omega}(M) = \bigcup\limits_{m<\omega} \mathscr{S}^m(M)$, (but we shall not use this in any essential way, hence we choose $\mathscr{S}(M) = \mathscr{S}^1(M)$.)

Definition 5.16. 1) We say N is λ-universal above or over M if for every $M', M \leq_{\mathfrak{k}} M' \in K^{\mathfrak{k}}_\lambda$, there is a $\leq_{\mathfrak{k}}$-embedding of M' into N over M. If we omit λ we mean $\text{rnd}_{\mathfrak{k}}(\lambda) = \min(\text{Car})_{\mathfrak{k}} \backslash \lambda)$ so $\leq \|N\|^{<\kappa(\mathfrak{k})}$; clearly this implies that M is a $\leq_{\mathfrak{k}_{[\chi_0,\chi_1]}}$-amalgamation base where $\chi_0 = \|M\|, \chi_1 = (\|N\|^{<\kappa})^+$.
2) $K^3_{\mathfrak{k}} = \{(M, N, a) : M \leq_{\mathfrak{k}} N, a \in N \backslash M$ and $M, N \in K_{\mathfrak{k}}\}$, with the partial order $\leq = \leq_{\mathfrak{k}}$ defined by $(M, N, a) \leq (M', N', a')$ iff $a = a', M \leq_{\mathfrak{k}} M'$ and $N \leq_{\mathfrak{k}} N'$. We say (M, N, a) is minimal if $(M, N, a) \leq (M', N_\ell, a) \in K^3_{\mathfrak{k}}$ for $\ell = 1, 2$ implies $\text{ortp}(a, M', N_1) = \text{ortp}(a, M', N_2)$ moreover, $(M', N_1, a)\mathscr{E}^{\text{at}}_\lambda(M', N_2, a)$, (not needed if every $M' \in K_\lambda$ is an amalgamation basis).
2A) $K^{3,\mathfrak{k}}_\lambda$ is defined similarly using $\mathfrak{k}_{[\lambda,\text{rnd}(\lambda)]}$.

Generalizing superlimit, we have more than one reasonable choice.

Definition 5.17. 1) For $\ell = 1, 2$ we say $M^* \in K^{\mathfrak{k}}_\lambda$ is superlimit$_\ell$ or $(\lambda, \geq \kappa)$-superlimit$_\ell$ when clause (c) of 5.15(5) and:

- (a) it is universal, (i.e., every $M \in K^{\mathfrak{k}}_\lambda$ can be properly $\leq_{\mathfrak{k}}$-embedded into M^*), and
- (b) Case 1: $\ell = 1$ if $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing, $\text{cf}(\delta) \geq \kappa, \delta < \lambda^+$ and $i < \delta \Rightarrow M_i \cong M^*$ then $M_\delta \cong M^*$

  Case 2: $\ell = 2$ if I is a $(< \kappa)$-directed partial order of cardinality $\leq \chi$, $\langle M_t : t \in I \rangle$ is $\leq_{\mathfrak{k}}$-increasing and $t \in I \Rightarrow M_t \cong M^*$ then $\cup\{M_t : t \in I\} \cong M^*$.

[14] note that $\mathscr{E}^{\text{at}}_M$ is not an equivalence relation and certainly in general is not $\mathscr{E}_M$

2) M is λ-saturated above μ <u>if</u> $\|M\| \geq \lambda > \mu \geq \mathrm{LST}(\mathfrak{k})$ and: $N \leq_{\mathfrak{k}} M, \mu \leq \|N\| < \lambda, p \in \mathscr{S}_{\mathfrak{k}}(N)$ implies p is strongly realized in M. Let "M is λ^+-saturated" mean that "M is λ^+-saturated above λ" and $K(\lambda^+\text{-saturated}) = \{M \in K : M$ is λ^+-saturated$\}$ and "M is saturated ..." mean "M is $\|M\|$-saturated ...".

Definition 5.18. 1) We say <u>N is (λ, σ)-brimmed over M</u> if we can find a sequence $\langle M_i : i < \sigma\rangle$ which is $\leq_{\mathfrak{k}}$-increasing semi-continuous, $M_i \in K_\lambda, M_0 = M, M_{i+1}$ is $\leq_{\mathfrak{k}}$-universal over M_i and $\bigcup_{i<\sigma} M_i = N$. We say N is (λ, σ)-brimmed over A if $A \subseteq N \in K_\lambda$ and we can find $\langle M_i : i < \sigma\rangle$ as in part (1) such that $A \subseteq M_0$; if $A = \emptyset$ we may omit "over A".
2) We say N is $(\lambda, *)$-brimmed over M if for some $\sigma \in [\kappa, \lambda), N$ is (λ, σ)-brimmed over M. We say N is $(\lambda, *)$-brimmed if for some M, N is $(\lambda, *)$-brimmed over M.
3) If $\alpha < \lambda^+$ let "N is (λ, α)-brimmed over M" mean $M \leq_{\mathfrak{k}} N$ are from K_λ and $\mathrm{cf}(\alpha) \geq \kappa \Rightarrow N$ is $(\lambda, \mathrm{cf}(\alpha))$-brimmed over M.

Recall

Claim 5.19. *1) If $\mathfrak{k}$ is a (μ, λ, κ)-d.a.e.c. with amalgamation, stable in χ and $\sigma = \mathrm{cf}(\sigma)$ so $\chi \in [\lambda, \mu)$, <u>then</u> for every $M \in K^{\mathfrak{k}}_\chi$ there is $N \in K^{\mathfrak{k}}_\lambda$ universal over M which is (χ, σ)-brimmed over M (hence is S^χ_σ-limit, see* [She09a], *not used).*
2) If N_ℓ is (χ, θ)-brimmed over M for $\ell = 1, 2$, and $\kappa \leq \theta = \mathrm{cf}(\theta) \leq \chi^+$ <u>then</u> N_1, N_2 are isomorphic over M.
3) If M_2 is (χ, θ)-brimmed over M and $M_0 \leq_{\mathfrak{s}} M_1$ then M_2 is (χ, θ)-brimmed over M_0.

Proof. Straightforward for part (1); recall clause (c) of Definition 5.19(5).
2),3) As in [She09c]. $\square_{5.19}$

* * *

§ 5(C). Liftings.

Here we deal with lifting, there are two aspects. First, if $\mathfrak{k}^1, \mathfrak{k}^1$ agree in λ they agree in every higher cardinal. Second, given $\mathfrak{k}$ we can find $\mathfrak{k}_1$ with $\mu_{\mathfrak{k}_1} = \infty, (\mathfrak{k}_1)_\lambda = \mathfrak{k}_\lambda$.

Theorem 5.20. *1) If $\mathfrak{k}^\ell$ is a (μ, λ, κ)-a.e.c. for $\ell = 1, 2$ and $\mathfrak{k}^1_\lambda = \mathfrak{k}^2_\lambda$ then $\mathfrak{k}^1 = \mathfrak{k}^2$.*
2) If $\mathfrak{k}_\ell$ is a $(\mu_\ell, \lambda, \kappa)$-d.a.e.c. for $\ell = 1, 2$ and $\mathfrak{k}^1$ satisfies Ax(IV)(d) and $\mu_1 \leq \mu_2$ and $\mathfrak{k}^1_\lambda = \mathfrak{k}^2_\lambda$ <u>then</u> $\mathfrak{k}_1 = \mathfrak{k}_2[\lambda, \mu_1)$.

Proof. By 5.12. $\square_{5.20}$

Theorem 5.21. <u>*The lifting-up Theorem*</u>
1) If $\mathfrak{k}_\lambda$ is a $(\lambda^+, \lambda, \kappa)$-d.a.e.c.$^\pm$ <u>then</u> the pair $(K', \leq_{\mathfrak{k}'})$ defined below is an $(\infty, \lambda, \kappa)$-d.a.e.c.$^+$ where we define

- *(A) K' is the class of M such that M is a $\tau_{\mathfrak{k}_\lambda}$-model, and for some I and $\bar{M}$ we have*
  - *(a) I is a κ-directed partial order*
  - *(b) $\bar{M} = \langle M_s : s \in I\rangle$*

(c) $M_s \in K_\lambda$

(d) $I \models s < t \Rightarrow M_s \le_{\mathfrak{k}_\lambda} M_t$

(e) *if* $J \subseteq I$ *has cardinality* $\le \lambda$ *and is* κ*-directed and* M_J *is the* $\mathfrak{k}$*-union of* $\langle M_t : t \in J\rangle$*, see Definition 5.4, then* M_J *is a submodel of* M

(f) $M = \cup\{M_J : J \subseteq I$ *is* κ*-directed of cardinality* $\le \lambda\}$*, i.e. both for the universe and for the relations and functions.*

$(A)'$ *we call such* $\langle M_s : s \in I\rangle$ *a witness for* $M \in K'$*, we call it reasonable if* $|I| \le \|M\|^{<\kappa}$

(B) $M \le_{\mathfrak{k}'} N$ *iff for some* $I, J, \bar{M}$ *we have*

(a) J *is a* κ*-directed partial order*

(b) $I \subseteq J$ *is* κ*-directed*

(c) $\bar{M} = \langle M_s : s \in J\rangle$ *and is a* $\le_{\mathfrak{k}_\lambda}$*-increasing*

(d) $\langle M_s : s \in J\rangle$ *is a witness for* $N \in K'$

(e) $\langle M_s : s \in I\rangle$ *is a witness for* $M \in K'$.

$(B)'$ *We call such* $I, \langle M_s : s \in J\rangle$ *witnesses for* $M \le_{\mathfrak{k}'} N$ *or say* $(I, J, \langle M_s : s \in J\rangle)$ *witness* $M \le_{\mathfrak{k}'} N$.

2) For the other axioms we have implications.

Proof. The proof of part (2) is straightforward so we concentrate on part (1). So let us check the axioms one by one.

AxO(a),(b),(c) and (d): K' is a class of τ-models, $\le_{\mathfrak{k}'}$ a two-place relation on K, K' and $\le_{\mathfrak{k}'}$ are closed under isomorphisms and $M \in K' \Rightarrow \|M\| \ge \lambda$, etc.
[Why? trivially.]

AxI(a): If $M \le_{\mathfrak{k}'} N$ then $M \subseteq N$.
[Why? We use smoothness for κ-directed unions, i.e. Ax(IV)(x).]

AxII(a),(b),(c):
We prove the first, the others are easier.

Ax II(a): $M_0 \le_{\mathfrak{k}'} M_1 \le_{\mathfrak{k}'} M_2$ implies $M_0 \le_{\mathfrak{k}'} M_2$ and $M \in K' \Rightarrow M \le_{\mathfrak{k}'} M$.
[Why? The second phrase is trivial. For the first phrase let for $\ell \in \{1,2\}$ the κ-directed partial orders $I_\ell \subseteq J_\ell$ and $\bar{M}^\ell = \langle M^\ell_s : s \in J_\ell\rangle$ witness $M_{\ell-1} \le_{\mathfrak{k}'} M_\ell$.

We first observe

$\boxdot$ if I is a κ-directed partial order, $\langle M^\ell_t : t \in I\rangle$ is a $\le_{\mathfrak{k}_\lambda}$-system witnessing $M_\ell \in K'$ for $\ell = 1,2$ and $t \in I \Rightarrow M^1_t \le_{\mathfrak{k}_\lambda} M^2_t$ then $M_1 \le_{\mathfrak{k}} M_2$.

[Why? Let I_1 be the partial order with set of elements $I \times \{1\}$ ordered by $(s,1) \le_{I_1} (t,1) \Leftrightarrow s \le_I t$. Let I_2 be the partial order with set of elements $I \times \{1,2\}$ ordered by $(s_1,\ell_1) \le_{I_2} (s_2,\ell_2) \Leftrightarrow s_1 \le_I s_2 \wedge \ell_1 \le \ell_2$. Clearly $I_1 \subseteq I_2$ are both κ-directed.

Let $M_{(s,1)} = M^1_s, M_{(s,2)} = M^2_s$, so clearly $\bar{M} = \langle M_t : t \in I_\ell\rangle$ is a $\le_{\mathfrak{k}} -\kappa$-directed system witnessing $M_\ell \in K'$ for $\ell = 1,2$ and $(I_1, I_2, \bar{M})$ witness $M_1 \le_{\mathfrak{k}'} M_2$, so we are done.]

Without loss of generality J_1, J_2 are pairwise disjoint. Let $\chi = (|J_1| + |J_2|)^{<\kappa}$ so $\lambda \le \chi < \mu$ and let

$$\mathscr{U} := \{u : \begin{array}{l} u \subseteq J_1 \cup J_2 \text{ has cardinality } \leq \lambda \text{ and } u \cap I_\ell \\ \text{is } \kappa\text{-directed under } \leq_{I_\ell} \text{ for } \ell = 1,2 \text{ and } u \cap J_\ell \\ \text{is } \kappa\text{-directed under } \leq_{J_\ell} \text{ for } \ell = 1,2 \text{ and} \\ \cup\{|M^2_t| : t \in I_2\} = \cup\{|M^1_t : t \in J_1\}\}. \end{array}$$

Let $\langle u_\alpha : \alpha < \alpha^* \rangle$ list $\mathscr{U}$, and we define a partial order I:

- $(a)'$ its set of elements is $\{\alpha < \alpha^*$: for no $\beta < \alpha$ do we have $u_\alpha \subseteq u_\alpha\}$
- $(b)'$ $\alpha \leq_I \beta$ iff $u_\alpha \subseteq u_\beta \wedge \alpha \in I \wedge \beta \in I$.

Note that the set I may have card$(\sum\limits_{i<\delta} \|M_i\|)^\lambda$ which may be $> \mu_{\mathfrak{k}}$.

As in the proof of 5.9, I is κ-directed.

For $\ell = 0,1,2$ and $\alpha \in I$ let $M_{\ell,\alpha}$ be

- (a) $\leq_{\mathfrak{k}}$-union of $\langle M^\ell_t : t \in u_\alpha \cap I_1 \rangle$ if $\ell = 0$
- (b) the $\leq_{\mathfrak{k}}$-union of the $\leq_{\mathfrak{k}_\lambda}$-directed system $\langle M^1_t : t \in J_1 \rangle$, equivalently the $\leq_{\mathfrak{k}_\lambda}$-directed system of $\langle M^2_t : t \in I_2 \rangle$, when $\ell = 1$
- (c) the $\leq_{\mathfrak{k}}$-union of the $\leq_{\mathfrak{k}_\lambda}$-directed system $\langle M^2_t : t \in J_2 \rangle$ when $\ell = 2$.

Now

- $(*)_1$ if $\ell = 0,1,2$ and $\alpha \leq_I \beta$ then $M^\ell_\alpha \leq_{\mathfrak{k}_\lambda} M^\ell_\beta$
- $(*)_2$ if $\alpha \in I$ then $M^0_\alpha \leq_{\mathfrak{k}_\lambda} M^1_\alpha \leq_{\mathfrak{k}_\lambda} M^1_\alpha$
- $(*)_3$ $\langle M_{\ell,\alpha} : \alpha \in I \rangle$ is a witness for $M_\ell \in K'$
- $(*)_4$ $M_{0,\alpha} \leq_{\mathfrak{k}_\lambda} M_{2,\alpha}$ for $\alpha \in I$.

Together by $\boxdot$ we get that $M_0 \leq_{\mathfrak{k}'} M_2$ as required.

Ax III(a): In general.

Let $(I_{i,j}, J_{i,j}, \bar{M}^{i,j})$ witness $M_i \leq_{\mathfrak{k}'} M_j$ when $i \leq j < \delta$ and without loss of generality $\langle J_{i,j} : i < j < \delta \rangle$ are pairwise disjoint. Let $\mathscr{U}$ be the family of u such that for some $v \in [\delta]^{\leq \lambda}$,

- (a) $v \subseteq \delta$ has cardinality $\leq \lambda$ and has order type of cofinality $\geq \kappa$
- (b) $u \subseteq \cup\{J_{i,j} : i < j$ are from $v\}$ has cardinality $\leq \lambda$ and
- (b) for $i < j$ from v the set $u \cap J_{i,j}$ is κ-directed under $\leq_{J_{i,j}}$ and $u \cap I_{i,j}$ is κ-directed under $\leq_{I_{i,j}}$
- (c) if $i_0 \leq i_1 \leq i_2$ then $\cup\{M^{i(0),i(1)}_{t,s} : s \in u \cap J_{i(0),i(1)}\} = \cup\{M^{i(1),i(2)}_s : s \in u \cap I_{i(1),i(2)}\}$
- (d) if $i(0) \leq i(1) \leq i(2)$ are from v then $\cup\{M^{i(0),i(1)}_{t,s} : s \in u \cap J_{i(0),i(1)}\} = \cup\{M^{i(1),i(2)}_s : s \in u \cap I_{i(1),i(2)}\}$
- (e) if $i(0) \leq k(0) \leq j(1)$ and $i(1) \leq j(1)$ are from u then $\cup\{M^{i(0),j(0)}_s : s \in u \cap J^{i(0),j(0)}_s\} = \cup\{M^{i(1),j(1)}_s : s \in u \cap J_{i(1),j(1)}\}$.

Let the rest of the proof be as before.

Ax(IV)(a):
Similar, but $\mathscr{U} = \{u \subseteq I : u$ has cardinality $\le \lambda$ and is κ-directed$\}$.

Ax(III)(d):
Assuming $\mathfrak{k}$ satisfies Ax(III)(d). Similar.

Ax(IV)(d):
Assuming $\mathfrak{k}$ satisfiefs Ax(IV)(d). Similar.

Axiom V: Assume $N_0 \le_{\mathfrak{k}'} M$ and $N_1 \le_{\mathfrak{k}'} M$.
If $N_0 \subseteq N_1$, then $N_0 \le_{\mathfrak{k}'} N_1$.
[Why? Let $(I_\ell, J_\ell, \langle M^\ell_s : s \in J_\ell\rangle)$ witness $N_\ell \le_{\mathfrak{k}} M$ for $\ell = 0,1$; without loss of generality J_0, J_1 are disjoint.
Let

$$\mathscr{U} := \{u \subseteq J_0 \cup J_1 : \begin{array}{l} |u| \le \lambda \text{ and } u \cap J_\ell \text{ is } \kappa\text{-directed} \\ \text{and } u \cap I_\ell \text{ is } \kappa\text{-directed for } \ell = 0,1 \text{ and} \\ \cup\{|M^0_s| : s \in u \cap J_0\} = \cup\{|M^0_s| : s \in u \in J_1\}\}. \end{array}$$

For $u \in \mathscr{U}$ let

- $M_u = M{\restriction}\cup\{(M^\ell_s : s \in u \cap J_\ell\}$ for $i = 0,1$
- $N_{\ell,u} = N_\ell{\restriction}\{(M^\ell_s) : s \in u \cap I_\ell\}$.

Let

$(*)$ (a) $(\mathscr{U}, \subseteq)$ is κ-directed
(b) $N_{\ell,u} \le_{\mathfrak{k}} M$
(c) $M_{\ell,u} \le_{\mathfrak{k}} M_{\ell,v}$ when $u \subseteq v$ are from $\mathscr{U}$ and $\ell = 0,1$
(d) $M_{0,u} \le_{\mathfrak{k}} M_{1,u}$
(e) $N_| ell = \cup\{N_{\ell,u} : u \in \mathscr{U}\}$; as in ?

By $\boxdot$ above we are done.

Axiom VI: $\mathrm{LST}(\mathfrak{k}') = \lambda$.
[Why? Let $M \in K', A \subseteq M, |A| + \lambda \le \chi < \|M\|$ and let $\langle M_s : s \in I\rangle$ witness $M \in K'$; without loss of generality $|A| = \chi^{<\kappa}$. Now choose a directed $I \subseteq J$ of cardinality $\le |A| = \chi^{<\kappa}$ such that $A \subseteq M' =: \bigcup_{s \in I} M_s$ and so $(I, J, \langle M_s : s \in J\rangle)$ witnesses $M' \le_{\mathfrak{k}'} M$, so as $A \subseteq M'$ and $\|M'\| \le |A| + \mu$ we are done.] $\square_{5.21}$

Also if two such d.a.e.c.'s have some cardinal in common then we can put them together.

Claim 5.22. *Let $\iota \in \{1,2,3\}$ and assume $\lambda_1 < \lambda_2 < \lambda_3$ and*

(a) *$\mathfrak{k}^1$ is an $(\lambda_2^+, \lambda, \kappa) - 2$-d.a.e.c., $K^1 = K^1_{\ge\lambda}$*
(b) *$\mathfrak{k}^2$ is a $(\lambda_3, \lambda_2, \kappa) - \iota$-d.a.e.c.*
(c) *$K^{\mathfrak{k}^1}_{\lambda_2} = K^{\mathfrak{k}^2}_{\lambda_2}$ and $\le_{\mathfrak{k}^2} {\restriction} K^{\mathfrak{k}^2}_{\lambda_2} = \le_{\mathfrak{k}^1} {\restriction} K^{\mathfrak{k}^1}_{\lambda_2}$*

(d) *we define* $\mathfrak{k}$ *as follows:* $K_{\mathfrak{k}} = K_{\mathfrak{k}} \cup K_{\mathfrak{k}_2}, M \le_{\mathfrak{k}} N$ *iff* $M \le_{\mathfrak{k}^1} N$ *or* $M \le_{\mathfrak{k}^2} N$ *or for some* $M', M \le_{\mathfrak{k}^1} M' \le_{\mathfrak{k}^2} N$.

Then $\mathfrak{k}$ *is an* $(\lambda_3, \lambda_1, \kappa) - \iota$*-d.a.e.c.*

Proof. Straightforward. E.g.

Ax(III)(d): So $\langle M_s : s \in I\rangle$ is $\le_{\mathfrak{s}} -\kappa$-directed system.

If $\|M_s\| \ge \lambda_2$ for some λ, use $\langle M_s : s \le t \in I\rangle$ and clause (b) of the assumption. If $\cup\{M_s : s \in I\}$ has cardinality $\le \lambda_2$ use clause (a) in the assumption. If neither one of them holds, recall $\lambda_2 = \lambda_2^{<\kappa}$ by clause (b) of the assumption, and let

$$cU = \{u \subseteq I : |u|\ le\lambda_2, u \text{ is } \kappa\text{-directed (in I), and } \cup\{M_s : s \in u\} \text{ has cardinality } \lambda\}.$$

Easily $(\mathscr{U}, \subseteq)$ is λ_2-directed, for $u \in J$ let M_u be the $\le_{\mathfrak{s}}$-union of $\langle M_s : s \in u\rangle$. Now by clause (a) of the assumption

$(*)_1$ $M_u \in K^{\mathfrak{k}^1}_{\lambda_2} = K^{k^*}_{\lambda_2}$

$(*)_2$ if $u_1 \subseteq v$ are from $\mathscr{U}$ then $M_u \le_{\mathfrak{k}^1} M_v, M_u \le_{\mathfrak{k}^2} M_v$.

Now use clause (b) of the assumption.

Axiom V: We shall use freely

$(*)$ $\mathfrak{k}^2_\lambda = \mathfrak{k}^2$ and $\mathfrak{k}^1_\lambda = \mathfrak{k}^1$.

So assume $N_0 \le_{\mathfrak{k}} M, N_1 \le_{\mathfrak{k}} M, N_0 \subseteq N_1$.

Now if $\|N_0\| \ge \lambda_1$ use assumption (b), so we can assume $\|N_0\| < \lambda_1$. If $\|M\| \le \lambda_1$ we can use assumption (a) so we can assume $\|M\| > \lambda_1$ and by the definition of $\le_{\mathfrak{k}}$ there is $M_0' \in K^{\mathfrak{k}^1}_{\lambda_1} = K^{\mathfrak{k}^2}_{\lambda_1}$ such that $N_0 \le_{\mathfrak{k}^1} M_0' \le_{\mathfrak{k}^2} M$. First assume $\|N_1\| \le \lambda_1$, so we can find $M_1' \in K^{\mathfrak{k}^1}_{\lambda_1}$ such that $N_1 \le_{\mathfrak{k}^1} M_1' \le_{\mathfrak{k}^2} M$ (why? if $N_1 \in K^{\mathfrak{k}^1}_{<\lambda_1}$, by the definition of $\le_{\mathfrak{k}}$ and if $N_1 \in K^{\mathfrak{k}^1}_{\lambda_1}$ just choose $M_1' = N_1$). Now we can by assumption (b) find $M'' \in K^{\mathfrak{k}^1}_{\lambda_1}$ such that $M_0' \cup M_1' \subseteq M'' \le_{\mathfrak{k}} M$, hence by assumption (b) (i.e. AxV for $\mathfrak{k}^2$) we have $M_0' \le_{\mathfrak{k}} M'', M_1' \le_{\mathfrak{k}} M''$. As $N_0 \le_{\mathfrak{k}} M_0' \le_{\mathfrak{k}} M'' \in K^{\mathfrak{k}}_{\le\lambda_1}$ by assumption (a) we have $N_0 \le_{\mathfrak{k}} M''$, and similarly we have $N_1 \le_{\mathfrak{k}} M''$. So $N_0 \subseteq N_1, N_0 \le_{\mathfrak{k}} M'', N_1 \le_{\mathfrak{k}} M'$ so by assumption (b) we have $N_0 \le_{\mathfrak{k}} N_1$.

We are left with the case $\|N_1\| > \lambda$, by assumption (b) there is $N_1' \in K_{\lambda_1}$ such that $N_0 \subseteq N_1' \le_{\mathfrak{k}^2} N_2$. By assumption (b) we have $N_1' \le_{\mathfrak{k}} M$, so by the previous paragraph we get $N_0 \le_{\mathfrak{k}} N_1'$, together with the previous sentence we have $N_0 \le_{\mathfrak{k}^1} N_1' \le_{\mathfrak{k}^2} N_1$ so by the definition of $\le_{\mathfrak{k}}$ we are done. $\square_{5.22}$

Definition 5.23. If $M \in K_\chi$ is $(\chi, \ge \kappa)$-superlimit$_1$ let $K^{[M]}_\chi = \{N \in K_\chi : N \cong M\}, \mathfrak{K}^{[M]}_\chi = (K^{[M]}_\chi, \le_{\mathfrak{k}} \restriction K^{[M]}_\chi)$ and $\mathfrak{k}^{[M]}$ is the $\mathfrak{k}'$ we get in 5.21(1) for $\mathfrak{k}' = \mathfrak{k}^{[M]}_\chi$.

Claim 5.24. *1) If* $\mathfrak{k}$ *is an* (μ, λ, κ)-a.e.c., $\lambda \le \chi < \mu, M \in K_\chi$ *is* $(\chi, \ge \kappa)$*-superlimit*$_1$ then $\mathfrak{k}^{[M]}_\chi$ *is a* (χ^+, χ, κ)-d.a.e.c.

2) If in addition $\mathfrak{k}$ *is a* (μ, λ, κ)-d.a.e.c.$^\pm$ then $\mathfrak{k}^{[M]}_\chi$ *is a* (χ^+, χ, κ)*-d.a.e.c.*$^\pm$.

§ 6. II pr frames

Definition 6.1. For $\iota = 1, 2, 3, 4$. We say that $\mathfrak{s}$ is a good $(\mu, \lambda, \kappa) - \iota$-frame when $\mathfrak{s}$ consists of the following objects satisfying the following condition: μ, λ, κ (so we may write $\mu_{\mathfrak{s}}, \lambda_{\mathfrak{s}}, \kappa_{\mathfrak{s}}$ but we usually ignore them defining $\mathfrak{s}$) and

(A) $\mathfrak{k} = \mathfrak{k}_{\mathfrak{s}}$ is a (μ, λ, κ) − 6-d.a.e.c., so we may write $\mathfrak{s}$ instead of $\mathfrak{k}$, e.g. $\le_{\mathfrak{s}}$-increasing, etc. and $\chi \in [\lambda, \mu) \Rightarrow \mathrm{LST}(\chi^{<\kappa})$

(B) $\mathfrak{k}$ has a $(\lambda, \ge \kappa)$-superlimit model M^* which [15] is not $<_{\mathfrak{k}}$-maximal, i.e.,
   - (a) $M^* \in K^{\mathfrak{s}}_\lambda$
   - (b) if $M_1 \in K^{\mathfrak{s}}_\lambda$ then for some $M_2, M_1 <_{\mathfrak{s}} M_2 \in K^{\mathfrak{s}}_\lambda$ and M_2 is isomorphic to M^*
   - (c) if $\langle M_i : i < \delta \rangle$ is $\le_{\mathfrak{s}}$-increasing, $i < \delta \Rightarrow M_i \cong M$ and $\mathrm{cf}(\delta) \ge \kappa, \delta < \lambda^+$ then $\cup\{M_i : i < \delta\}$ is isomorphic to M^*

(C) $\mathfrak{k}$ has the amalgamation property, the JEP (joint embedding property), and has no $\le_{\mathfrak{k}}$-maximal member; if of $\iota \ge 2, \mathfrak{k}$ has primes$^-$ and if $\iota \ge 4, \mathfrak{k}$ has primes$^+$

(D) (a) $\mathscr{S}^{\mathrm{bs}} = \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}$ (the class of basic types for $\mathfrak{k}_{\mathfrak{s}}$) is included in $\bigcup\{\mathscr{S}(M) : M \in K_{\mathfrak{s}}\}$ and is closed under isomorphisms including automorphisms; for $M \in K_\lambda$ let $\mathscr{S}^{\mathrm{bs}}(M) = \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}} \cap \mathscr{S}(M)$; no harm in allowing types of finite sequences.
   - (b) if $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$, then p is non-algebraic (i.e., not realized by any $a \in M$).
   - (c) (density)
     if $M \le_{\mathfrak{k}} N$ are from $K_{\mathfrak{s}}$ and $M \ne N$, then for some $a \in N \backslash M$ we have $\mathrm{ortp}(a, M, N) \in \mathscr{S}^{\mathrm{bs}}$
     [intention: examples are: minimal types in [She01], regular types for superstable theories]
   - (d) bs-stability
     $\mathscr{S}^{\mathrm{bs}}(M)$ has cardinality $\le \|M\|^{<\kappa}$ for $M \in K_{\mathfrak{s}}$.

(E) (a) $\bigcup = \bigcup_{\mathfrak{s}}$ is a four place relation called nonforking with $\bigcup(M_0, M_1, a, M_3)$ implying $M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_3$ are from $K_{\mathfrak{s}}, a \in M_3 \backslash M_1$ and $\mathrm{ortp}(a, M_0, M_3) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_0)$ and $\mathrm{ortp}(a, M_1, M_3) \in \mathscr{S}^{\mathrm{bs}}(M_1)$. Also $\bigcup$ is preserved under isomorphisms.

We also write $M_1 \underset{M_0}{\overset{M_3}{\bigcup}} a$ and demand: if $M_0 = M_1 \le_{\mathfrak{k}} M_3$ both in K_λ then: $\bigcup(M_0, M_1, a, M_3)$ is equivalent to "$\mathrm{ortp}(a, M_0, M_3) \in \mathscr{S}^{\mathrm{bs}}(M_0)$".

Also we may state $M_1 \underset{M_0}{\overset{M_3}{\bigcup}} a$ "$\mathrm{ortp}(a, M_1, M_3)$ does not fork over M_0 (inside M_3)" (this is justified by clause (b) below).

[Explanation: The intention is to axiomatize non-forking of types, but we allow dealing with basic types only. Note that in [She01] we know something on minimal types but other types are something else.]

[15] follows by (C) in fact

(b) (monotonicity):

if $M_0 \le_{\mathfrak{k}} M'_0 \le_{\mathfrak{k}} M'_1 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_3 \le_{\mathfrak{k}} M'_3, M_1 \cup \{a\} \subseteq M''_3 \le_{\mathfrak{k}} M'_3$ all of them in K_λ, then $\bigcup(M_0, M_1, a, M_3) \Rightarrow \bigcup(M'_0, M'_1, a, M'_3) \Leftrightarrow \bigcup(M'_0, M'_1, a, M''_3)$, so it is legitimate to just say "ortp(a, M_1, M_3) does not fork over M_0".

[Explanation: non-forking is preserved by decreasing the type, increasing the basis (= the set over which it does not fork) and increasing or decreasing the model inside which all this occurs. The same holds for stable theories only here we restrict ourselves to "legitimate" types.]

(c) (local character):

Case 1: $\iota = 1, 2, 3$.

If $\langle M_i : i \le \delta\rangle$ is $\le_{\mathfrak{s}}$-semi-continuous and $p \in \mathscr{S}^{\rm bs}(M_\delta)$ and $\mathrm{cf}(\delta) \ge \kappa$ then for every $\alpha < \delta$ large enough, p does not fork over M_α.

Case 2: $\iota = 4$.

If I is a κ-directed partial order and $\bar{M} = \langle M_t : t \in I\rangle$ is a $\le_{\mathfrak{s}}$-directed system and M is its $\le_{\mathfrak{k}}$-union and $M \le_{\mathfrak{s}} N$ and ortp$(a, M, N) \in \mathscr{S}^{\rm bs}(M_\delta)$ then for every $s \in I$ large enough ortp(a, M, N) does not fork over M_s.

[Explanation: This is a replacement for $\kappa \ge \kappa_r(T)$; if $p \in \mathscr{S}(A)$ then there is a $B \subseteq A$ of cardinality $< \kappa$ such that p does not fork over A. The case $\iota = 2$? is a very strong demand even for stable first order theories.] It means dimensional continuity, i.e. M_δ is minimal over $\cup\{M_\alpha : \alpha < \delta\}$ and κ-saturated models.]

(d) (transitivity):

if $M_0 \le_{\mathfrak{k}} M'_0 \le_{\mathfrak{k}} M''_0 \le_{\mathfrak{k}} M_3$ and $a \in M_3$ and ortp(a, M''_0, M_3) does not fork over M'_0 and ortp(a, M'_0, M_3) does not fork over M_0 (all models are in K_λ, of course, and necessarily the three relevant types are in $\mathscr{S}^{\rm bs}$), then ortp(a, M''_0, M_3) does not fork over M_0

(e) uniqueness:

if $p, q \in \mathscr{S}^{\rm bs}(M_1)$ do not fork over $M_0 \le_{\mathfrak{k}} M_1$ (all in $K_{\mathfrak{s}}$) and $p \restriction M_0 = q \restriction M_0$ then $p = q$

(f) symmetry:

Case 1: $\ell \ge 3$.

If $M_0 \le_{\mathfrak{s}} M_\ell \le_{\mathfrak{s}} M_3$ and $(M_0, M_\ell, a_\ell) \in K^{3,\rm pr}_{\mathfrak{s}}$, see clause (j) below for $\ell = 1, 2$ then ortp$_{\mathfrak{s}}(a_2, M_1, M_3)$ does not fork over M_0 iff ortp$_{\mathfrak{s}}(a_1, M_2, M_3)$ does not fork over M_0.

Case 2: $\iota = 1, 2$.

If $M_0 \le_{\mathfrak{k}} M_3$ are in $\mathfrak{k}_\lambda$ and for $\ell = 1, 2$ we have $a_\ell \in M_3$ and ortp$(a_\ell, M_0, M_3) \in \mathscr{S}^{\rm bs}(M_0)$, then the following are equivalent:

- (α) there are M_1, M'_3 in $K_{\mathfrak{s}}$ such that $M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{K}} M'_3$, $a_1 \in M_1, M_3 \le_{\mathfrak{k}} M'_3$ and ortp(a_2, M_1, M'_3) does not fork over M_0

- (β) there are M_2, M'_3 in K_λ such that $M_0 \le_{\mathfrak{k}} M_2 \le_{\mathfrak{k}} M'_3$, $a_2 \in M_2, M_3 \le_{\mathfrak{k}} M'_3$ and ortp(a_1, M_2, M'_3) does not fork over M_0.

[Explanation: this is a replacement to "ortp$(a_1, M_0 \cup \{a_2\}, M_3)$ forks over M_0 iff ortp$(a_2, M_0 \cup \{a_1\}, M_3)$ forks over M_0"; which is not well defined in out context]

(g) [existence] if $M \leq_{\mathfrak{s}} N, p \in \mathscr{S}^{\rm bs}(M)$ then there is $q \in \mathscr{S}^{\rm bs}(N)$ a non-forking extension of p

(h) [continuity] Case 1: $\iota \geq 1$.
If $\langle M_\alpha : \alpha \leq \delta \rangle$ is $\leq_{\mathfrak{s}}$-increasing, $\leq_{\mathfrak{s}}$-semi-continuity, $M_\delta = \bigcup_{\alpha<\delta} M_\alpha$ which holds if $\text{cf}(\delta) \geq \kappa$ and $p \in \mathscr{S}(M_\delta)$ and $p \restriction M_\alpha$ does not fork over M_0 for $\alpha < \delta$ then $p \in \mathscr{S}^{\rm bs}(M_\delta)$ and it does not fork over M_0

Case 2: $\iota = 4$.
Similarly for $\bar{M} = \langle M_t : t \in I \rangle, I$ directed, $M = \cup\{M_t : t \in I\}$ is a $\leq_{\mathfrak{s}}$-upper bound of $\bar{M}$

(j) $\mathfrak{s}$ has $K^{3,\rm pr}_{\mathfrak{s}}$-primes, see 6.8

(k) Case 1: $\iota \geq 1$.
If $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N)$ then p does not fork over M for some $M \leq_{\mathfrak{s}} N$ from K_λ

Case 2: $\iota = 3, 4$.
If $M_\ell(\ell \leq 3), a_\ell, p_\ell(\ell = 1, 2)$ are as in (E)(f).

Discussion 6.2. Consider using: semi-continuous + $\text{cf}(\delta) \geq \kappa$ for $(E)(c), (E)(x)$: $\text{cf}(\delta) \geq \kappa$ stable only if $\chi = \chi^{<\kappa}$.

Claim 6.3. *1) If $\langle M_i : i < \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing, $(\Sigma\{\|M_i\| : i < \delta\}) < \mu$ and $p_i \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_i)$ does not fork over M_0 for $i < \delta$ and $[i < j \Rightarrow p_j \restriction M_i = p_i]$ then:*

- (a) *we can find M_δ such that $i < \delta \Rightarrow M_i \leq_{\mathfrak{k}} M_\delta$*
- (b) *for any such M_δ, we can find $p \in \mathscr{S}_{\mathfrak{s}}(M_\delta)$ such that $\bigwedge_{i<\delta} p \restriction M_i = p_i$ and p does not fork over M_0*
- (c) *p_δ is unique in clause (b)*
- (d) *if $\ell \geq \kappa \wedge \text{cf}(\delta) \geq \kappa$ we can add $M = \cup\{M_\alpha : \alpha < \delta\}$.*

2) Similarly for $\bar{M} = \langle M_t : t \in I \rangle, I$ directed.

Proof. 1) First choose M_δ by 6.1, Clause (A). Second choose $p_\delta \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\delta)$, a non-forking extension of p_0, exist by Ax(g) of (E) of 6.1. Now $p_\delta \restriction M_i \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_i)$ does not fork over M_0 by (b) of (E) of 6.1 and extend p_0 so is equal to p_i by (e) of (E). Third, p_δ is unique by (E)(e).
2) Should be clear, too. $\square_{6.3}$

Definition 6.4. 0) We say $\mathfrak{s}$ is full when $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M) = \{p : p \in \mathscr{S}^{\varepsilon}_{\mathfrak{k}[\mathfrak{s}]}(M)$ is not algebraic for some $\varepsilon < \kappa_{\mathfrak{s}}\}$ for every $M \in K_{\mathfrak{s}}$ [compare with (2)].
1) Assume $M_\ell \leq_{\mathfrak{s}} N$ for $\ell = 1, 2$ and $p_\ell \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\ell)$ for $\ell = 1, 2$. We say that p_1, p_2 are parallel when some $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N)$ is a non-forking extension of p_ℓ for $\ell = 1, 2$.
2) We say $\mathfrak{s}$ is type-full when $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M) = \mathscr{S}^{\rm na}_{\mathfrak{k}_{\mathfrak{s}}}(M)$ for $M \in K_{\mathfrak{s}}$.
3) We say $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ is based on $\bar{\mathbf{a}}$ when:

- (a) $\bar{\mathbf{a}}$ is a sequence from M
- (b) if $M \leq_{\mathfrak{s}} N$ and $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N)$ is a non-forking extension of p and π is an automorphism of N over $\bar{\mathbf{a}}$ then $\pi(q) = q$ (by Ax(E)(k) there is such $\bar{\mathbf{a}} \in {}^{\lambda}(M)$).

4) We say $\mathfrak{s}$ is $(< \theta)$-based when (3) there is such $\bar{\mathbf{a}} \in {}^{\theta>}M$.

Definition 6.5. 1) We say that NF is a non-forking relation on a $(\mu, \lambda, \kappa) - 1$-d.a.e.c. $\mathfrak{k}$ when in addition to 6.1(A)-(C)

(F) (a) NF is a four-place relation on $\mathfrak{k}_\mathfrak{s}, \mathrm{NF}_\mathfrak{s}(M_0, M_1, M_2, M_3)$ implies $M_0 \le_\mathfrak{k} M_\ell \le_\mathfrak{k} M_1$ and $\mathrm{NF}_\mathfrak{s}$ is preserved by isomorphisms

$(b)_1$ monotonicity: if $\mathrm{NF}_\mathfrak{s}(M_0, M_1, M_2, M_3), M_0 \le_\mathfrak{s} M'_\ell \le_\mathfrak{s} M_\ell$ for $\ell = 1, 2, M'_1 \cup M'_2 \subseteq M'_3 \le_\mathfrak{s} M, M'_3 \le_\mathfrak{s} M$ then $\mathrm{NF}_\mathfrak{s}(M_0, M', M'_2, M'_3)$

(c) symmetry: $\mathrm{NF}_\mathfrak{s}(M_0, M_1, M_2, M_3)$ implies $\mathrm{NF}_\mathfrak{s}(M_0, M_2, M_1, M_3)$

$(d)_1$ transitivity: if $\mathrm{NF}_\mathfrak{s}(M_{2\ell}, M_{2\ell+1}, M_{2\ell+3}, M_{2\ell+4})$ for $\ell = 0, 1$ then $\mathrm{NF}_\mathfrak{s}(M_0, M_1, M_4, M_5)$

$(d)_2$ long transitvity: if $\langle (N_i, M_i) : i < \delta \rangle$ is an $\mathrm{NF}_\mathfrak{s}$-sequence (i.e., M_i is $\le_\mathfrak{s}$-increasing, N_i is $\le_\mathfrak{s}$-increasing, $M_i \le_\mathfrak{s} M_i, i < j < \delta \Rightarrow$ $\mathrm{NF}_\mathfrak{s}(M_i, N_i, M_j, N_j)$ and $\Sigma\{\|N_i\| : i < \delta\} < \mu$ then we can find (N_δ, M_δ) such that
$\langle (M_i, N_i) : i \le \delta \rangle$ is an NF-sequence. But what about pr-continuity?

$(d)_2^+$ like $(d)_2$ for directed systems

(e) continuity

Definition 6.6. 1) Let $\mathfrak{s}$ be a good λ-frame and NF a non-forking relation on $\mathfrak{k}$. We say NF respects $\mathfrak{s}$ when: if $\mathrm{NF}_\mathfrak{s}(M_0, M_1, M_2, M_3)$ and $a \in M_2, \mathrm{ortp}_\mathfrak{s}(a, M_0, M_3) \in \mathscr{S}^{\mathrm{bs}}_\mathfrak{s}(M_0)$ then $\mathrm{ortp}_\mathfrak{s}(a, M_1, M_3)$ is a non-forking extension of $\mathrm{ortp}_\mathfrak{s}(a, M_0, M_2)$.
2) We say $\mathfrak{s}$ is a good $(\lambda, \mu, \kappa) -$ NF-frame when it is a good (λ, μ, χ)-frame and $\mathrm{NF}_\mathfrak{s}$ is a non-forking relation on $\mathfrak{k}_\mathfrak{s}$ which respects $\mathfrak{s}$.

Definition 6.7. We say that $\mathfrak{s}$ is a very good $(\mu, \lambda, \kappa) -$ NF-frame if it is a good $(\mu, \lambda, \kappa) -$ NF-frame and

(G) (a) $\mathfrak{k}_\mathfrak{s}$ has primes for chains and even directed systems, see Definition 5.4(4)

(b) if $\mathrm{NF}_\mathfrak{s}(M_0, M_1, M_2, M_3)$ then there is $M^*_3 \le_\mathfrak{s} M_3$ which is prime over $M_1 \cup M_2$ that is:

$(*)$ if $\mathrm{NF}_\mathfrak{s}(M'_0, M'_1, M'_2, M'_3)$ and f_ℓ is an isomorphism from M_ℓ onto M'_ℓ for $\ell = 0, 1, 2$ such that $f_0 \subseteq f_1, f_0 \subseteq f_2$ then there is a $\le_\mathfrak{s}$-embedding f_3 of M^*_3 into M'_3 extending $f_1 \cup f_2$

(c) $\mathfrak{k}_\mathfrak{s}$ has primes (see 6.8(2) below).

Definition 6.8. 0) $K^{3,\mathrm{bs}}_\mathfrak{s} = \{(M, N, a) : M \le_\mathfrak{s} N$ and $a \in N$ and $\mathrm{ortp}_\mathfrak{s}(a, M, N) \in \mathscr{S}^{\mathrm{bs}}_\mathfrak{s}(M)\}$.
1) $K^{3,\mathrm{pr}}_\mathfrak{s} = \{(M, N, a) \in K^{3,\mathrm{bs}}_\mathfrak{s}$: if $M \le N', a' \in N', \mathrm{ortp}_\mathfrak{s}(a', M, N') = \mathrm{ortp}(a, M, N)$ then there is a $\le_\mathfrak{k}$-embedding of N into N' extending id_M and mapping a to a'.
2) $\mathfrak{k}_\mathfrak{s}$ has $K^{3,\mathrm{pr}}_\mathfrak{s}$-primes if for every $M \in K_\mathfrak{s}$ and $p \in \mathscr{S}^{\mathrm{bs}}_\mathfrak{s}(M)$ there are (N, a) such that $(M, N, a) \in K^{3,\mathrm{pr}}_\mathfrak{s}$ and $\mathrm{ortp}_\mathfrak{s}(a, M, N) = p$.

Definition 6.9. 1) $[\iota \ge 3]$
Assume $p_1, p_2 \in \mathscr{S}^{\mathrm{bs}}(M)$. We say p_1, p_2 are weakly orthogonal, $p_1 \underset{\mathrm{wk}}{\perp} p_2$ when: if $M_0 \le_\mathfrak{s} M_\ell \le_\mathfrak{s} M_3, (M_0, M_\ell, a_\ell) \in K^{3,\mathrm{pr}}_\mathfrak{s}$ and $\mathrm{ortp}_\mathfrak{s}(a_\ell, M_0, M_\ell) = p_\ell$ for $\ell = 1, 2$ then $\mathrm{ortp}_\mathfrak{s}(a_2, M_1, M_3)$ does not fork over M_0 (symmetric by Ax(E)(f)).

2) We say p_1, p_2 are orthogonal, $p_1 \perp p_2$ when: if $M \leq_{\mathfrak{s}} M_2, M_1 \leq_{\mathfrak{s}} M_2$ and $q_\ell \in \mathscr{S}^{\rm bs}(M_2)$ is a non-forking extension of p_ℓ and q_ℓ does not fork over M_1 <u>then</u> $q_1 \underset{\rm wk}{\perp} q_2$.
3) We say that $\{a_t : t \in I\}$ is independent in (M_0, M_1, M_2) when

- (a) $a_t \in M_2 \backslash M_1$
- (b) $\mathrm{ortp}_{\mathfrak{s}}(a_t, M_1, M_2)$ does not fork over M_0
- (c) there is a list $\langle t(\alpha) : \alpha < \alpha(*)\rangle$ with no repetitions of I and is a $\leq_{\mathfrak{s}}$-increasing sequence $\langle M_{1,\alpha} : \alpha \leq \alpha(*)+1\rangle$ such that $M_1 \leq_{\mathfrak{s}} M_{1,0}, M_2 \leq M_{1,\alpha(*)+1}$ such that $a_{t(\alpha)} \in M_{1,\alpha+1}$ and $\mathrm{ortp}_{\mathfrak{s}}(a_{t(\alpha)}, M_{1,\alpha}, M_{1,\alpha+1})$ does not fork over M_0.

4) Let $(M, N, \mathbf{J}) \in K^{3,\rm bs}_{\mathfrak{s}}$ <u>if</u> $M \leq_{\mathfrak{s}} N$ and $\mathbf{J}$ is independent in (M, N).
5) Let $(M, N, \mathbf{J}) \in K^{3,\rm qr}_{\mathfrak{s}}$ <u>if</u>:

- (a) $M \leq_{\mathfrak{s}} N$
- (b) $\mathbf{J}$ is independent in (M, N)
- (c) if $M \leq_{\mathfrak{s}} N', h$ is a one-to-one function from $\mathbf{J}$ into N' such that $(M, N', h''(\mathbf{J})) \in K^{3,\rm bs}_{\mathfrak{s}}$ <u>then</u> there is a $\leq_{\mathfrak{s}}$-embedding g of N into N' over M extending h.

Remark 6.10. $\bar{M} = \langle M_i : i < \alpha\rangle$ is increasing semi-continuous when it is $\leq_{\mathfrak{s}}$-increasing, M_δ is prime over $\bar{M} \restriction \delta$ for every limit $\delta < \alpha$.

Remark 6.11. We now can imitate relations of the axioms (as in [She09c, §2]), and basic properties of the notions introduced in 6.9.

Definition 6.12. 1) We say p is strongly dominated by $\{p_t : t \in I\}$, possibly with repetitions, so pedantically we should use a sequence and write $p \leq^{\rm dm}_{\rm st} \{p_t : t \in I\}$); <u>when</u>:

- (a) $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N), p_t \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N_t), N_t \leq_{\mathfrak{s}} N^+ \in K_{\mathfrak{s}}, N \leq_{\mathfrak{s}} N^+$ and
- (b) if $N^+ \leq_{\mathfrak{s}} N^*$ and $a_t \in N^*$ and $\mathrm{ortp}(a_t, N^+, N^*) \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N^+)$ is parallel to p_t and $p' \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N^+)$ is parallel to p, see Definition 6.4 and $\{a_t : t \in I\}$ is independent in (N^+, N^*) <u>then</u> some $a \in N^*$ realizes p'.

2) We say p is weakly dominated by $\{p_t : t \in I\}$ and write $p \leq_{\rm wk} \{p_t : t \in I\}$ <u>when</u> for some set J and function h from J onto I we have $p \leq^{\rm dm}_{\rm st} \{p_{h(t)} : t \in J\}$.
3) Let dominated mean strongly dominated.

Claim 6.13. *1) If p is strongly dominated by $\{p_t : t \in I\}$ <u>then</u> p is weakly dominated by $\{p_t : t \in I\}$.*
2) If p is strongly dominated by $\{p_t : t \in I\}$ <u>then</u> for some $J \subseteq I$ of cardinality $< \kappa_{\mathfrak{s}}, p$ is strongly dominated by $\{p_t : t \in I\}$.
3) p is weakly dominated by $\{p_t : t \in I\}$ <u>iff</u> for some $\langle i_t : t \in I\rangle, p$ is strongly dominated by $\{p'_s : s \in \{(t,i) : t \in I, i < i_t\}\}$ where $p'_{(t,i)} = p_t, i_t < \kappa_{\mathfrak{s}}$ for each $t \in I$.
4) In Definition 6.12(2) without loss of generality $(\forall s \in I)(\exists^{<\kappa} t \in J)(h(t) = s)$.
5) [Preservation by parallelism]

Proof. Proof should be clear. $\square_{6.13}$

The following should be included in very good for 8.16, see[16] more 8.4

[16] for the case there is $M \in K_\lambda$ brimmed over $N \in K_\lambda$ for every such N this (8.4(1)(c)) work

Claim 6.14. *1) If $p \le^{\rm dm}_{\rm wk} \{p_i : i < i^*\}$ and $i < i^* \Rightarrow q \perp p_i$ then $q \perp p$ (see Definition 6.8(3)).*
1A) If $p \le^{\rm dm}_{\rm wk} \{p_i : i < i^*\}, (p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M))$ then $p \pm p_i$ for some $i < i^*$.
2) If $M \le_{\mathfrak{s}} N, q \in N, p_i \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ for $i < i^* < \kappa$, then there is $q' \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ such that for some N^+, f we have:

- (a) *$N \le_{\mathfrak{s}} N^+, f$ is an automorphism of N^+*
- (b) *f maps the non-forking extension of q in $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(N^+)$ to the non-forking extension of q' in $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(N^+)$*
- (c) *f maps the non-forking extension of p_i in $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(N^+)$ to itself.*

3) If $p \le^{\rm dm}_{\rm st} \{p_i : i < \alpha\}$ then $p \le^{\rm dm}_{\rm st} \{p_i : i < \alpha, p_i \pm p\}$ [see Def 6.12.]
4) Assume

- (a) *$p, p_i \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ for $i < i^*$*
- (b) *p_i is weakly dominated by p*
- (c) *no $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ is weakly dominated by p and orthogonal to p_i for $i < i^*$.*

Then $p \le^{\rm dim}_{\rm wk} \{p_i : i < i^*\}$.

Proof. Let $p \in \mathscr{S}^{\rm bs}(N), N \in K_A$. If $p \in \mathbf{P}^\perp$ we are done so assume $N \le N_1 \in K_{\mathfrak{s}}, q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N_1) \cap \mathbf{P}$ and $p \pm q$. Let $\bar{\mathbf{a}} \in {}^{\kappa >}N_1$ be such that q is definable over $\bar{\mathbf{a}}$, so we can find $\langle \bar{\mathbf{a}}_i : i < \kappa \rangle, \bar{a}_i \in {}^{\ell g(\bar{\mathbf{a}})}M$ such that $\langle \bar{\mathbf{a}}_i : i < \kappa \rangle \hat{\ } \langle \bar{\mathbf{a}} \rangle$ is indiscernible over $A \cup \bar{b}$ where p be definable over $\bar{\mathbf{b}} \subseteq M$.

Let $q_i \in \mathscr{S}^{\rm bs}(N_1)$ be defined over $\bar{\mathbf{a}}_i$ as q was defined over $\bar{\mathbf{a}}$ so easily $q_i \in \mathscr{P}, p \pm q_i$, so we are done. $\square_{6.14}$

Claim 6.15. *1) If $\chi = \chi^{<\kappa} \in [\lambda, \mu)$, the following is impossible:*

- (a) *$\langle M_i : i < \chi^+ \rangle$ is $\le_{\mathfrak{s}}$-increasing $\le_{\mathfrak{s}}$-semi-continuous,*
- (b) *$\langle N_i : i < \chi^+ \rangle$ is $\le_{\mathfrak{s}}$-increasing, $\le_{\mathfrak{s}}$-semi-continuous,*
- (c) *$M_i \le_{\mathfrak{s}} N_i \in K_{\le \chi}$,*
- (d) *for some stationary $S \subseteq \{\delta < \chi^+ : {\rm cf}(\gamma) \ge \kappa\}$ for every $i \in S$*
  - *there is $a_i \in M_{i+1} \backslash M_i$ such that ${\rm ortp}(a_i, N_i, N_{i+1})$ is not the non-forking extension of ${\rm ortp}(a_i, M_i, M_{i+1}) \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_i)$.*

2) Like (1) replacing (d) by:

- (d)′ *like (d) replacing • by*
  - •′ *$b_i \in N_i \backslash M_i, {\rm ortp}_{\mathfrak{s}}(b_i, M_i, N_i) \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_i)$ and ${\rm ortp}_{\mathfrak{s}}(b_i, M_{i+1}, N_{i+1})$ forks over M_i.*

Proof. Should be clear. $\square_{6.15}$

Claim 6.16. *If $p, p_i \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ for $i < \kappa_{\mathfrak{s}}$ and $i < j \Rightarrow p_2 \perp p_j$ then $p \perp p_i$ for every $i < \kappa$ large enough.*

Proof. Similar to the proof of 8.6. $\square$

Definition 6.17. 1) We say that (a good (frame$_{\mathfrak{s}}$), $\mathfrak{s}$ is θ-based$_1$ when:

- (a) if $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ then for some $\bar{\mathbf{a}} \in {}^{\theta >}M, p$ is based on $\bar{\mathbf{a}}$ (see Definition 6.4(4)).

2) We say that $\mathfrak{s}$ is θ-based$_2$ <u>when</u>:

- (a) as in part (1)
- (b) $\mathfrak{s}$ is full
- (c) if $M_1 \le_{\mathfrak{s}} M_2$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_2)$ <u>then</u> for some $\bar a_\ell \in {}^{\theta>}(M_\ell)$ the types $p, {\rm ortp}_{\mathfrak{s}}(\bar a_2, M_1, M_2)$ are based on $\bar{\mathbf{a}}_2, \bar{\mathbf{a}}_1$ respectively, (or axiomatically!).

Part III

§ 7. Introduction/Thoughts on the main gap

We address here two problems: type theory (i.e. dimension, orthogonality, etc.) for strictly stable class and the main gap concerning somewhat saturated models, the hope always was that advance in the first will help the second.

Concerning the first order case work started in [She90, Ch.V], particularly [She90, Ch.V,§5] and [She91] and was much advanced in Hernandes [Her92]; but this was not enough for the main gap for somewhat saturated models.

We deal here with the type dimension for a general framework.

* * *

The main gap for $\aleph_1 - |T|^+$-saturated model of a countable first order theory is open. A priori it has looked easier than the one for models (which was preferred as "the original question") because of the existence of prime models over any, but is still open (and for uncountable first order, $|T|^+$-saturated model as well).

Why the proof in [She90, Ch.XII] does not work? What is missing is (in $\mathfrak{C}^{\rm eq}$!)

⊛ if $M_0 \prec M_1 \prec M_2$ are $\aleph_1$-saturated, $a \in M_2 \backslash M_1$ and $(a/M_1) \pm M_0$ <u>then</u> for some $b \in M_2 \backslash M_1$ we have $b \bigcup\limits_{M_0} M_1$.

The central case is a/M_1 is orthogonal to q if $q \perp M_0$.

<u>Possible Approach 1</u>: We use T being first order countable, stable NDOP (even shallow) to understand types. See [LS06].

<u>Possible Approach 2</u>: We use the context of part II. We are poorer in knowledge <u>but</u> we have a richer $\mathfrak{C}^{\rm eq}$ so we may prove ⊛ even if its fails for T in the elementary case (this is a connection between Part I and Part II of this work).

<u>Possible Approach 3</u>: We use the context of part II. If things are not O.K. we define a derived such d.a.e.c. (as in [She09g] or [She09c]), it may have non-structure properties, if not we arrive to the same place. Similarly in limit. If we succeed enough times we shall prove that all is O.K.

<u>Possible Approach 4</u>: Now we have a maximal non-forking tree $\langle M_\eta, a_\eta : \eta \in \mathscr{T}\rangle$ inside a somewhat saturated model; for [She90], e.g. $\|M_\eta\| \le \lambda$, the models are λ^+-saturated but we use models from [She, Part II]. If M is prime over $\cup\{M_\eta : \eta \in \mathscr{T}\}$ we are done, but maybe there is a residue. This appears as: for $\eta \in \mathscr{T}$, and $p \in \mathscr{S}^{\rm bs}(M_\eta)$ the dimension of p is not exhausted by $\langle a_{\eta\hat{}<\alpha>} : \eta\hat{}\langle\alpha\rangle \in \mathscr{T}$ and $(a_{\eta\hat{}<\alpha>}/M_\eta) \pm p\}$ but the lost part is not infinite! This imposes $\le \lambda$ unary functions from $\mathscr{T}$ to $\mathscr{T}$. Now it seems to me that the question of this possible non-exhaustion arise (essentially: there is a non-algebraic $p \in (M^\perp)^\perp$ which do not 1-dominate any $q \in \mathscr{S}(M)$) is not a good dividing line, as though its negation is informative, it is not clear whether it has any consequence. However, there are two candidates for dividing lines (actually their disjunction seems so)

(A) $(*)$ we can find $M, \langle M_\eta, a_\eta : \eta \in \mathscr{T}\rangle$ as above and $\eta_* \in \mathscr{T}, \ell g(\eta) = 2, \nu_* \in \mathscr{T}, \ell g(\nu_*) = 1, \eta_* \restriction 1 \neq \nu_*$ and $p \in \mathscr{S}^{\rm bs}(M_{\eta_*}), p \perp M_{\eta \restriction 1}$ with a residue as above such that we need M_{ν_*} to explicate it.

More explicitly

$(*)'$ if $M' \le_{\mathfrak{s}} M$ is prime over $\cup\{M_\eta : \eta \in \mathscr{T}\}$ and we can find $a_{\eta_*,\nu_*} \in M \backslash M'$ such that $\text{ortp}(\mathscr{C}(a_{\eta^*,\nu^*}, M'), \cup\{M_\eta : \eta \in \mathscr{T}\})$ mark (M_{η_*}, M_{ν_*}).

Even in $(*)'$ we have to say more in order to succeed in using it.

From $(*)'$ we can prove a non-structure result: on $\mathscr{T}$ we can code any two-place relation R on $\{\eta \in \mathscr{T} : \ell g(\eta) = 1, M_\eta, M_{\eta_* \restriction 1}$ isomorphic over $M_{<>}\}$ which is $\nu_1 R \nu_2 \Leftrightarrow (\exists \nu) \bigwedge\limits_{\ell}$ [there is $\eta', \eta_\ell \triangleleft \eta' \in \mathscr{T}, \ell g(\eta') = 2$ and $\nu \in T, \ell g(\nu) = 1$ and there is $a_{\eta'\nu}$ as above].

More complicated is the case

(B) $(**)$ we can fix $(M, \langle M_\eta, a_\eta : \eta \in \mathscr{T}\rangle$ as above), $\eta_* \in T, \nu_*, \nu_{**} \in \mathscr{T}, \ell g(\eta_*) = \ell g(\nu_*) = 1 = \ell g(\nu_{**})$ such that $(\eta_*, \nu_*), (\eta_*, \nu_{**})$ are as above.

But whereas for (A) we have to make both η_* and ν_* not redundant in (B), in order to get non-structure we have to use a case of (B) which is not "a faking", e.g. cannot replace (M_{η_*}, a_{η_*}) by two such pairs.

That is, the "faker" is a case where we can find $M'_{\eta_*}, M''_{\eta_*}$ such that:

(a) $\text{NF}(M_{<>}, M'_{\eta_*}, M''_{\eta_*}, M_{\eta_*})$
(b) M_{η_*} is prime over $M'_{\eta_*} \cup M''_{\eta_*}$
(c) only (M'_{η_*}, M_{ν_*}) and $(M''_{\eta_*}, M_{\nu_{**}})$ relate.

[Possibly we have $\langle \nu_t : t \in I\rangle$ we can "divide" but not totally, probably a problem]
(C) if both (A) and (B) in the right formulation does not appear then

(α) good possibility; we can prove a structure theory: for $M, \langle M_\eta, a_\eta : \eta \in \mathscr{T}\rangle$ as above that is on each $\text{suc}_{\mathscr{T}}(\eta)$ we have a two-place relation but it is very simple, you have to glue some together or at most at tree structure.

If this fails we may fall back to approach (3).

Question 7.1. 1) For an a.e.c. $\mathfrak{k}$ when does the theory of a model in the logic $\mathscr{L} = \mathbb{L}_{\infty,\kappa}[\mathfrak{k}]$ enriched by dimension quantifiers, characterize up to isomorphism models of $\mathfrak{k}$? Similarly enriching also by game quantifiers of length $\le \kappa$.
2) Prove the main gap theorem in the version: if $\mathfrak{s}$ is n-beautiful (or $n+1$?) <u>then</u> for $K_{\lambda^{+n}}$ the main gap holds, if in particular: if $\mathfrak{s}$ has NDOP, every $M \in K_{\lambda^{+n}}$ is prime over some non-forking tree of $\le_{\mathfrak{K}[\mathfrak{s}]}$-submodels $\langle M_\eta : \eta \in \mathscr{T}\rangle$, each M_η of cardinality $\le \lambda, \mathscr{T} \subseteq {}^{\omega>}(\lambda^{+n})$. If $\mathfrak{s}$ is shallow then the tree has depth $\le \text{Depth}(\mathfrak{s}) < \lambda^+$ and we can draw conclusion on the number of models.

Discussion 7.2. [Assume stability in $\lambda_{\mathfrak{s}}$].

Let $M_0 \in K_{\mathfrak{s}}, \lambda^+_{\mathfrak{s}}$-saturated at least for the time being.
1) Assume

$\boxplus_1$ $N_0 \le_{\mathfrak{s}} N_1 \le_{\mathfrak{s}} M, N_\ell \in K^{\mathfrak{s}}_\lambda, a \in N_0$ and $(N_0, N_1, a) \in K^{3,\rm pr}_{\mathfrak{s}}$.

We choose $(N^+_{1,i}, N_{1,i}, \mathbf{I}_i)$ and if possible also (M_1, a_i) by induction on $i \le \lambda^+_{\mathfrak{s}}$ such that

$(*)(a)$ (α) $N_{0,i} \le_{\mathfrak{s}} N_{1,i} \le_{\mathfrak{s}} N^+_{1,i} \le_{\mathfrak{s}} M$

- (b) $\mathbf{I}_i \subseteq \{c \in M : \mathrm{ortp}(c, N_{1,i}, M_0) \perp N_0\}$ is independent in $(N_{1,i}, N^+_{1,i}, M)$ and minimal
- (c) $\langle N_j : j \le i\rangle$ is $\le_{gs}$-semi-continuous also $\langle N^+_j : j \le i\rangle$
- (d) if $i = j+1$ then $N^+_{1,i}$ is $\le_{\mathfrak{s}}$-universal over $N^+_{1,j}$ and $(N_0, N_{1,i}, a) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$
- (e) if $j < i$ then $\mathbf{I}_j \backslash (N_i \cap \mathbf{I}_j) \subseteq \mathbf{I}_i$
- (f) if possible
  - (α) $N_i \le_{\mathfrak{s}} M^+_i \le_{\mathfrak{s}} M$
  - (β) $(\mathbf{I}_i \backslash M_i)$ is independent in (M_i, M)
  - (γ) $a_i \in M \backslash (\mathbf{I}_i)$
  - (δ) $\mathrm{ortp}(a_i, M^*_1, M) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(N^+_i)$ is $\perp N_i$
  - (ε) $N^*_i \le N_{1,i+1}$
- (g) if $i = j+1$ thee are (b, N^+_*, N_{**}) such that $b \in N^+_{1,j} \backslash N_{1,j}, N_{1,i} \le_{\mathfrak{s}} N_* \le_{\mathfrak{s}}$ $, N_{**} \in K^{\mathfrak{s}}_{\lambda_{\mathfrak{s}}}, N^+_{1,i} \le_{\mathfrak{s}} N_{**}$ and $\mathrm{ortp}_{\mathfrak{s}}(b, N_*, N_{**})$ forks over $N_{1,j}$ then for some $b \in N^+_{1,j} \backslash N_{1,j}$ the type $\mathrm{ortp}_{\mathfrak{s}}(N_{1,i}, N^+_{1,i})$ forks over $N_{1,j}$.

There is no problem to carry the induction.

$\boxplus_2$ the following subset of $\lambda^+_{\mathfrak{s}}$ is not stationary; say disjoint to the club C:
- $S = \{i < \lambda^+_{\mathfrak{s}} : \mathrm{cf}(i) \ge \kappa_{\mathfrak{s}}$ and (M_i, a_i) is well defined$\}$
- $S_2 = \{i : \mathrm{cf}(i) \ge \kappa_{\mathfrak{s}}$ and for some $b \in N^+_{1,i}, \mathrm{tp}(b, N_{1,i}, N^+_{1,i}) = N_0$.

2) Similarly without (N_0, a) hence without $\perp N_0$; just simpler.

Definition 7.3. We say $(\bar{N}, \bar{a}, \bar{I})$ is a decreasing pair for M <u>when</u> for some n:

- (a) $\bar{N} = \langle N_\ell : \ell \le n\rangle$ is $\le_{\mathfrak{s}}$-increasing
- (b) $N_\ell \le_{\mathfrak{s}} M, N_\ell \in K^{\mathfrak{s}}_{\lambda_{\mathfrak{s}}}$
- (c) $\bar{a} = \langle a_\ell : \ell < n\rangle$
- (d) $(N_\ell, N_{i+1}, a_\ell) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$
- (e) $\bar{\mathbf{I}} = \langle \mathbf{I}_\ell : \ell \le n\rangle$
- (f) $\mathbf{I}_\ell$ is independent in (N_ℓ, M)
- (g) $\mathbf{I}_\ell \subseteq \{c \in M : \mathrm{ortp}(c, N_\ell, M) \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(N_\ell)$ is $\perp N_k$ if $k < \ell\}$
- (h) if $N_\ell \le_{\mathfrak{s}} N \le_{\mathfrak{s}} M, b \in M \backslash N_0 \backslash \mathbf{I}_\ell, \mathrm{ortp}(b, N, M)$is $\pm N_\ell$ but N_k for $k < \ell$ <u>then</u> b depends on $\mathbf{I}_\ell$ in (N_ℓ, M).

<u>Attempt to prove decomposition</u>

We assume dimensional continuity to prove decomposition. If we like to get rid of "M is λ^+_{gs}-saturated", we assume we have a somewhat weaker version $\mathfrak{s}_*$ of $\mathfrak{s}$ with $\lambda_{\mathfrak{s}_*} < \lambda_{\mathfrak{s}}$ and $\le_{\mathfrak{s}_*}$-represent N_0 is $\langle N_{0,i} : i < \lambda_{\mathfrak{s}}\rangle$ and work with it. Assuming CH, $|T| = \aleph_0$ fine. Without dimensional discontinuity we call nice $(\bar{N}, \bar{a}, \bar{\mathbf{I}})$ of length $\le \kappa_{\mathfrak{s}}$!

* * *

Definition 7.4. We say $\mathbf{d} = (I, N, \bar{a}, \bar{\mathbf{I}})) = (I_{\mathbf{d}}, \bar{N}_d, \bar{a}_{\mathbf{d}}, \bar{\mathbf{I}}_{\mathbf{d}})$ is a partial decomposition of when:

$\boxplus$ (a) $I \subseteq {}^{\omega>}\mathrm{Ord}$ is closed under initial segments

(b) $\bar{N} = \langle N_\eta : \eta \in I \rangle$ so $N_\eta = N_{\mathbf{d},\eta}$

(c) $\bar{a} = \langle a_\eta : \eta \in I \backslash \{<>\} \rangle$ so $a_\eta = a_{\mathbf{d},\eta}$

(d) $\bar{\mathbf{I}} = \langle \mathbf{I}_\eta : \eta \in I \rangle$ so $\mathbf{I}_\eta = \mathbf{I}_{\mathbf{d},\eta}$

(e) if $\eta \in I$ then $(\langle N_{\eta \restriction \ell} : \ell \leq \ell g(\eta) \rangle, \langle \bar{a}_{\eta \restriction (\ell+1)} : \ell < \ell g(\eta) \rangle, \langle \mathbf{I}_{\eta \restriction \ell} : \ell \leq \ell g(\eta) \rangle)$ is nice in M

(f) if $\eta \in I$ then $\langle a_{\eta \char`^ \langle \alpha \rangle} : \eta \char`^ \langle \alpha \rangle \in I \rangle$ is a sequence of members of $\mathbf{I}_\eta$ with no repetitions.

Definition 7.5. Let $\leq_\mu$ be the following two-place relation on the set of decompositions of M:

$\bar{\mathbf{d}}_1 \leq_M \mathbf{d}_2$ iff

(a) $I_{\mathbf{d}_1} \subseteq \mathbf{I}_{d_1}$

(b) $\bar{N}_{\mathbf{d}_1} = \bar{N}_{\mathbf{d}_2} \restriction I_{d_1}$

(c) $\bar{a}_{\mathbf{d}_1} = \bar{a}_{\mathbf{d}_2} \restriction (I_{\mathbf{d}_1} \backslash \{< j\})$

(d) $\bar{\mathbf{I}}_{\mathbf{d}_1} = \bar{\mathbf{I}}_{d_2} \restriction I_{\mathbf{d}_1}$

§ 8. III, Analysis of dimension for $\mathbf{P}$

Hypothesis 8.1. 1) $\mathfrak{s}$ is a very good type-full (μ,λ,κ)-NF-frame (see 6.7), or is full good (μ,λ,κ)-frame$_{\mathfrak{s}}$ - see Definition and we sometimes use type-full (6.4) and $\perp = \underset{\rm wk}{\perp}$); semi-continuous will mean $\le_{\mathfrak{s}}$-semi-continuous; the "p base on $\bar{a}$" should be used to justify $\perp = \underset{\rm wk}{\perp}$.
2) $\mathfrak{C}$ is a $\mathfrak{s}$-monster or we use a place $K_{\mathfrak{a}}$ (or K_A and $K_{(M,\bar{a})}$, see Definition 1.6(5)).

Definition 8.2. Let $K_{\mathfrak{a}}$ be a place, $\mathbf{P}$ is a A-based family of types (see 1.10 closed under parallelism (add to Definition in §6), i.e., with the monster version this means:

- (a) $A \subseteq \mathfrak{C}, M$ vary on $M \le_{\mathfrak{k}} \mathfrak{C}$ such that $A \subseteq M$ or just $\mathfrak{k}_A$ is well defined
- (b) $\mathbf{P} \subseteq \cup\{\mathscr{S}^{\rm bs}(M) : M \in \mathfrak{k}_A$, i.e., $A \subseteq M \le_{\mathfrak{k}} \mathfrak{C}\}$
- (c) every automorphism of $\mathfrak{C}$ over A maps $\mathbf{P}$ onto itself

here we add

- (d) if $p_\ell \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_\ell)$ for $\ell = 1,2, M_\ell \in K_{\mathfrak{a}}$ and $p_1 \| p_2$ then $p_1 \in \mathbf{P} \Leftrightarrow p_2 \in \mathbf{P}$ (desirable here, not in weight check!).

1) $\mathbf{P}$ is $\mathfrak{a}$-dense when ($\mathbf{P}, A$ are as in part (0))

- $(*)$ if $M \in K_{\mathfrak{a}}$ and $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ then $(\exists q \in \mathbf{P} \cap \mathscr{S}^{\rm bs}(M))[p \underset{\rm wk}{\pm} q]$.

2) We say $\langle M_i, a_j : i \le \alpha, j < \alpha\rangle$ is a $\mathbf{P}$-primeness sequence when:

- (a) M_i is $\le_{\mathfrak{s}}$-increasing semi-continuous [i.e., if i is a limit ordinal then M_i is $\mathfrak{s}$-prime over $\cup\{M_j : j < i\}$, see Definition 6.8(5)]
- (b) $(M_i, M_{i+1}, a_i) \in K^{3,\rm pr}_{\mathfrak{s}}$
- (c) $\mathrm{ortp}_{\mathfrak{s}}(a_i, M_i, M_{i+1}) \in \mathbf{P}$.

3) We may say $\langle M_i, a_j, p_j : i \le \alpha, j < \alpha\rangle$ is a $\mathbf{P}$-prime sequence when in addition

- (d) $p_i = \mathrm{ortp}_{\mathfrak{s}}(a_i, M_i, M_{i+1})$.

Definition 8.3. Let $\mathbf{P}$ be a $\mathbf{a}$-based family.
1) $\mathbf{P}^\perp = \{p : p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M), M \in K_{\mathfrak{a}}$ and $p \perp \mathbf{P}$, see below$\}$.
2) $p \perp \mathbf{P}$ means that for some $M : p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M), M \in K_{\mathfrak{a}}$ and if $M \le_{\mathfrak{s}} N, q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N), p \in \mathscr{S}^{\rm bs}(M)$ is a non-forking extension of q then $p' \perp q$.

Claim 8.4. *1) If $\mathbf{P}$ is an $\mathfrak{a}$-based family of types then:*

- (a) *$\mathbf{P}^\perp$ is an $\mathfrak{a}$-based family of types*
- (b) *$\mathbf{P} \cup \mathbf{P}^\perp$ is dense.*

2) Any Boolean combination of $\mathfrak{a}$-based families of types in an $\mathfrak{a}$-based family of types.

Proof. 1) Clause (a): (is immediate).

Mainly we should check that $\mathbf{P}^+$ is closed under parallelism. Let $N_\ell \in K_A, p_\ell \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N_\ell)$ for $\ell = 1,2, N_1 \le_{\mathfrak{s}} N_2, p_2$ is a non-forking extension of p_1. We should prove $p_1 \in \mathbf{P}^+ \Leftrightarrow p_2 \in \mathbf{P}^\perp$. The direction $\Rightarrow$ is obvious. For $\Leftarrow$ use amalgamation, etc.

Clause (b) should be clear (weak density).
2) Easy. $\square_{8.4}$

Claim 8.5. *Assume* $\mathbf{P}$ *is a reasonable A-based, dense.*
1) If $M \in K_{\mathfrak{a}}$ *and* $q \in \mathscr{S}(M)$, *then we can find a* $\mathbf{P}$*-primeness sequence* $\langle M_i, a_j : i \leq \alpha, j < \alpha \rangle$ *with* $M_0 = M$ *such that* q *is realized in* M_α.
2) If $M \leq_{\mathfrak{s}} N$ *and* $M \in K_{\mathfrak{a}}$ *then for some* $\mathbf{P}$*-primeness sequence* $\langle M_i, a_j : i \leq \alpha, j < \alpha \rangle$ *we have* $M_0 = M$ *and* $N \leq_{\mathfrak{s}} M_\alpha$.
3) In part (2) if $\|N\| \leq \chi \in [\lambda, \mu)$ *and* $\chi = \chi^{<\kappa}$ *then we can demand* $\alpha \leq \chi$. *Similarly in part (1)* without loss of generality $\|M_\alpha\| \leq \|N\|^{<\kappa}$.

Proof. 1) We can find N such that $M \leq_{\mathfrak{s}} N$ and $a \in N$ realizes q; we let $A = \{a\}$ and apply 2).
2) Let $\chi = \|N\|^{<\kappa}(< \mu)$. Now try to choose (M_i, N_i) and then a_i by induction on $i < \chi^+$ such that (here we use "$\mathfrak{s}$ is full").

⊛ (a) $M_0 = M$
(b) $M_i \in K^{\mathfrak{s}}_{\leq \chi}$ is $\leq_{\mathfrak{s}}$-increasing $\leq_{\mathfrak{s}}$-semi-continuous with i
(c) N_i is $\leq_{\mathfrak{s}}$-increasing semi-continuous with i
(d) $N_0 = N$
(e) $M_i \leq_{\mathfrak{s}} N_i$
(f) $\langle M_\varepsilon, a_\zeta : \varepsilon \leq i, \zeta < i \rangle$ form a $\mathbf{P}$-primeness sequence
(g) if $i = j + 1$ then for some $c_i \in N_i \backslash M_i$ we have $\text{ortp}_{\mathfrak{s}}(c_i, M_i, N_i) \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M_i)$ and $\text{ortp}_{\mathfrak{s}}(c_i, M_{i+1}, N_{i+1})$ forks over M_i.

For $i = 0$ trivial. For $i = j + 1$ if $M_i \neq N_i$ by assumption and Definition 8.2(2) let $c_j \in N_i \backslash M_i$ so $r_j = \text{ortp}_{\mathfrak{s}}(c_j, M_j, N_j) \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M_j)$ hence there $p_j \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M_j) \cap \mathbf{P}$ such that $p_j \pm r_j$. So there are N_i, a_j such that $N_j \leq N_i$ and a_j realizes p_j, but $\{a_j, c_j\}$ is dependent. Choose $M_i \leq_{\mathfrak{s}} N_i$ such that $(M_j, M_i, a_j) \in K^{3,\text{pr}}_{\mathfrak{s}}$. For i limit choose first N_i (using the existence of primes over increasing sequences) then M_i (similarly) and without loss of generality $M_i \leq_{\mathfrak{s}} N_i$ using the Definition of prime. If we have carried the induction we get contradiction to 6.15(2). So for some i we are stuck: M_i is chosen but $N_i = M_i$ so we are done.
3) Follows by the proof of part (2). $\square_{8.5}$

Claim 8.6. *In 8.5 we can add* $\alpha < \kappa_{\mathfrak{s}}$ *if:* $\mathfrak{s}$ *is ufll (and maybe* $\mathbf{P}$ *is dense).*

Proof. Let $\chi = \|M\|^{<\kappa}$, let N,b be such that $M \leq_{\mathfrak{s}} N \in K^{\mathfrak{s}}_{\chi}, b \in N$ realizes q. Now we repeat the proof of 8.5(2) but in ⊛ add

- $\text{ortp}_{\mathfrak{s}}(b, M_i, M_{i+1})$ forks over M_i.

This is possible by the assumption "$\mathbf{P}$ is dense $+\mathfrak{s}$ is full" except when $b \in M_i$. In the end $\langle M_i : i < \kappa \rangle$ is well defined then it is $\leq_{\mathfrak{s}}$-increasing continuous, $\text{ortp}_{\mathfrak{s}}(b, M_\kappa, N_\kappa) \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M_\kappa)$ and it forks over M_i for every $i < \kappa$, contradiction to (c) of (E) of 6.1. $\square_{8.6}$

Claim 8.7. *1) If* $\langle M_i, a_j, p_j : i \leq \alpha, j < \alpha \rangle$ *is a primeness sequence,* $u = \{j : p_j \perp M_0$ *and* $j \in u \Rightarrow p_j$ *does not fork over* $M_0\}$ *and* $q \in \mathscr{S}^{\text{bs}}_{\mathfrak{s}}(M_0)$ *is realized in* M *then* $q \leq^{\text{st}}_{\text{dm}} \{p_j : j \in u\}$.
2) Like (1) but $u = \{j : p_j \perp q\}$.

3) [In 8.5(1) if we assume $(**)^q_{M,\mathbf{P}}$ below we can add $(*)^q_{M,\mathbf{P}}$ to the conclusion where

$(*)^q_{\mathbf{P},M}$ *the type $p_i =: \mathrm{ortp}_{\mathfrak{s}}(a_i, M_i, M_{i+1})$ does not fork over M_0 or q is non-orthogonal to p_i for every $i < \alpha$*

$(**)^q_{\mathbf{P},M}$ *if $M \le_{\mathfrak{s}} N$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(N) \cap \mathbf{P}$ and p is not orthogonal to q (e.g., $p \| p_i$) then p does not fork over M_0 or $p \perp M_0$.*

4) In (3) we can conclude that q is dominated by $\{p_i \restriction M_0 : i < \alpha\}$.
ss 6) 5) We can replace $(**)^q_{\mathbf{P},M}$ in part (3) by:

$\circledast^q_{\mathbf{P},M}$ *if $M \le_{\mathfrak{s}} N_1 <_{\mathfrak{s}} N_2$, then for some $r \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(N_1)$ realized in M_2, there is $p \underset{\mathrm{wk}}{\pm} r$ as in $(**)^q_{\mathbf{P},M}$.*

Proof. The same proof using 6.14(3). □

Discussion 8.8. Below we may think of the case $\bar{a}^i, \bar{a}$ has length $< \kappa_{\mathfrak{s}}$ (if $\mathrm{base}_{\mathfrak{s}}$ is well defined), or we may think $\bar{a}, \bar{a}^i$ list the members of some $N_i \le_{\mathfrak{s}} M, N_i \le_{\mathfrak{s}} M_i$ of cardinality $\lambda_{\mathfrak{s}}$ in this case we can replace "p_i definable over $\bar{a}_i$" by "p_i does not fork over N_i".

The following definition is central here. In the case $\kappa_{\mathfrak{s}} = \aleph_0$ all is clear. Note that even if $\ell g(\bar{\mathbf{a}}) = \|M\|$, we are interested in the case M is brimmed over $\bar{\mathbf{a}}$. Note if we understand $\langle p_i : i < \kappa\rangle / J^{\mathrm{bd}}_\kappa$ then $\mathbf{P}^{\mathfrak{x}}_3 = \mathbf{P}^{\mathfrak{x}}_2$ by earlier claim

Definition 8.9. 1) We say $\mathfrak{x} = \langle M, \bar{\mathbf{a}}, (M_i, N, \bar{\mathbf{a}}^i, p_i, f_i) : i < \kappa\rangle$ is a multi-dimensionality candidate <u>when</u>:

$\circledast(a)$ (α) $M \le_{\mathfrak{s}} M_i$
- (β) $M_i \le_{\mathfrak{s}} N, \bar{\mathbf{a}}$ a sequence of elements of M
- (γ) $\bar{\mathbf{a}}^i$ a sequence of elements of M_i

- (b) $\bigcup\limits_{M}\{M_i : i < \kappa\}$ (i.e. $\langle M_i : i < \kappa\rangle$ is independent over M inside N, see §6 and $M \in K_{\mathfrak{a}}$,
- (c) $f_i : M_0 \to M_i$ is an isomorphism onto M_i
- (d) $f_i \restriction M = \mathrm{id}_M, f_i(\bar{\mathbf{a}}^0) = \bar{\mathbf{a}}^i$
- (e) $\kappa \ge \kappa^1_{\mathfrak{s}}, \kappa = \mathrm{cf}(\kappa)$ or $\kappa > \kappa^1_{\mathfrak{s}} > \ell g(\bar{\mathbf{a}}^i)$
- (f) $p_i \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_i), f_i(p_0) = p_i$
- (g) p_i is based over $\bar{\mathbf{a}}^i$ (see §6 of Definition 6.4(3))
- (h) $\mathrm{ortp}(\bar{\mathbf{a}}_i, M, M_i)$ is based on definable over $\bar{\mathbf{a}}$

1A) In short we say m.d.-candidate. Let $M^{\mathfrak{x}} = M, M^{\mathfrak{x}}_i = M_i$, etc. and the place $\mathfrak{a} = \bar{\mathbf{a}}$ is $(N_{\mathfrak{x}}, \bar{\mathbf{a}}_{\mathfrak{x}})$.
2) We let for $\mathfrak{x}$ as above

- (a) $\mathbf{P}^{\mathfrak{x}}_1 = \{p : p \in \mathscr{S}^{\mathrm{bs}}(N), N \in K_{(M,\bar{\mathbf{a}})}$ and $(\forall^\infty i < \kappa)(p \perp p_i)\}$, that is if $N_{\mathfrak{x}} \le_{\mathfrak{s}} N$ and $p' \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$ is parallel to p then $(\forall^\infty I < \kappa)(p' \perp p)$ where $\forall^* i$ means except $< \kappa(\mathfrak{s})$ many
- (b) $\mathbf{P}^{\mathfrak{x}}_2 = \{p : p \in \mathscr{S}^{\mathrm{bs}}(N), N \in K_{(M,\bar{\mathbf{a}})}$ and $p \perp \mathbf{P}^{\mathfrak{x}}_1\}$ that is $(\mathbf{P}^{\mathfrak{x}}_1)^\perp$
- (c) $\mathbf{P}^{\mathfrak{x}} = \mathbf{P}^{\mathfrak{x}}_0 = \mathbf{P}^{\mathfrak{x}}_1 \cup \mathbf{P}^{\mathfrak{x}}_2$

(d) $\mathbf{P}_3^{\mathfrak{x}} = \{p : p \in \mathscr{S}_{\mathfrak{s}}^{\rm bs}(N), N \in K_{\bar{\mathbf{a}}}$ and p is orthogonal to $\mathbf{P}_2^{\mathfrak{x}}\}$ that is $(\mathbf{P}_2^{\mathfrak{x}})^{\perp}$.

3) We call $\mathfrak{x}$ non-trivial if $p_0^{\mathfrak{x}} \pm p_1^{\mathfrak{x}}$.

Remark 8.10. It is natural to hope $\mathbf{P}_3^{\mathfrak{x}} = \mathbf{P}_1^{\mathfrak{x}}$ but at present we have only $\mathbf{P}_1^{\mathfrak{x}} \subseteq \mathbf{P}_3^{\mathfrak{x}}$.

Observation 8.11. *For a m.d.-candidate $\mathfrak{x}$:*

(a) *if $p_0^{\mathfrak{x}} \perp p_1^{\mathfrak{x}}$ then $\mathbf{P}_1^{\mathfrak{x}} = \cup\{\mathscr{S}_{\mathfrak{s}}^{\rm bs}(N) : N \in K_{\mathfrak{a}}\}$ so $\mathbf{P}_2^{\mathfrak{x}} = \emptyset, \mathbf{P}_3^{\mathfrak{x}} = \mathbf{P}_1^{\mathfrak{x}}$*
(b) *if $i < j < \kappa$ then $p_0^{\mathfrak{x}} \perp p_1^{\mathfrak{x}} \Leftrightarrow p_i^{\mathfrak{x}} \perp p_j^{\mathfrak{x}}$.*

Proof. Clause (a): By 6.2 $\square_{8.11}$

Claim 8.12. *Assume $\mathfrak{x}$ is an m.d.-candidate.*
1) $\mathbf{P}_\ell^{\mathfrak{x}}$ is a $\mathfrak{a}_{\mathfrak{x}}$-based family of types for $\ell = 0, 1, 2, 3$.
2) $\mathbf{P}_{\mathfrak{x}} = \mathbf{P}_1^{\mathfrak{x}} \cup \mathbf{P}_2^{\mathfrak{x}}$ is dense see 8.4].
3) If $\mathbf{P}_2^{\mathfrak{x}} \neq \emptyset$ <u>then</u> there is $q \in \mathbf{P}_2^{\mathfrak{x}} \cap \mathscr{S}_{\mathfrak{s}}^{\rm bs}(M^{\mathfrak{x}})$. [even any $q \in \mathbf{P}_2^{\mathfrak{x}}$ has a $\bar{\mathbf{a}}$-conjugate $q \in \mathbf{P}_2^{\mathfrak{x}} \cap \mathbf{S}_{\mathfrak{s}}^{\rm bs}(M^{\mathfrak{x}})$].

Proof. [The reader can concentrate on the case we use $\mathfrak{s}^{\rm eq}$].
1) Note that for $p \in \mathscr{S}_{\mathfrak{s}}^{\rm bs}(N)$, the truth value of "$p \in \mathbf{P}_1^{\mathfrak{x}}$" is definable over $\cup\{\bar{\mathbf{a}}^j : j \in [i, \kappa)\}$ for any fixed i. But if $N \leq_{\mathfrak{s}} N_1$ and for $p \in \mathscr{S}_{\mathfrak{s}}^{\rm bs}(N_1)$ for some $i < \kappa$, the set $\{\bar{\mathbf{a}}^j : j \in [i, \kappa)\}$ is independent over $(\bar{a}$,base$(p))$. By 8.4(2) also $\mathbf{P}^{\mathfrak{x}} = \mathbf{P}_D^{\mathfrak{x}}$ is based on $\mathfrak{a}$ and lastly $\mathbf{P}_1^{\mathfrak{x}}$ is based on $\bar{\mathbf{a}}$. As for $\mathbf{P}_2^{\mathfrak{x}}$ it is based on $\bar{\mathbf{a}}$ by 8.4(1), clause (a) and its definition. By clause (a) of 8.4 the family $\mathbf{P}_3^{\mathfrak{x}}$ are based on $\bar{\mathbf{a}}$.
2) By clause (b) of 8.4(1).
3) By 6.14(2). $\square_{8.12}$

Claim 8.13. *Assume that $M \leq_{\mathfrak{s}} N$ and $p \in \mathscr{S}_{\mathfrak{s}}^{\rm bs}(N)$. <u>Then</u> we can find a multi-dimensionality candidate $\mathfrak{x}$ such that:*

(a) *$M_{\mathfrak{x}} = M$*
(b) *$M_0^{\mathfrak{x}} = N$*
(c) *$p_0^{\mathfrak{x}} = p$.*

Claim 8.14. *Let $\mathfrak{x}$ be a non-trivial multi-dimensional candidate. <u>Then</u> $\mathbf{P}_2^{\mathfrak{x}}$ is non-empty (hence $\mathbf{P}_2^{\mathfrak{x}} \cap \mathscr{S}_{\mathfrak{s}}^{\rm bs}(M) \neq \emptyset$).*

Proof. Assume it is empty so by 8.12(2) we know that $\mathbf{P}_1^{\mathfrak{x}}$ is dense.

Let $N_{\mathfrak{x}} \leq_{\mathfrak{s}} N \in K_{\mathfrak{s}}$ and let p_0^+ is a nonforking extension of p_0 in $\mathscr{S}_{\mathfrak{s}}^{\rm bs}(N)$. By 8.6 there is a $\mathbf{P}_1^{\mathfrak{x}}$-primeness sequence $\langle M_i, a_j : i \leq \alpha, j < \alpha\rangle$ such that $M_0 = N, \alpha < \kappa_{\mathfrak{s}}$ (actually $\alpha < \kappa$ suffice) and some $b \in M_\alpha$ realizes $p_0^{\mathfrak{x}}$; note that $M_i, M_j^{\mathfrak{x}}$ are not directly related.

Now as $\kappa_{\mathfrak{x}}$ is regular $\geq \kappa_{\mathfrak{s}}$ because $\alpha < \kappa^{\mathfrak{x}}$; by the definition of $\mathbf{P}_1^{\mathfrak{x}}$ each $j < \alpha$, for every $\varepsilon < \kappa_{\mathfrak{x}}$ large enough we have $p_\varepsilon^{\mathfrak{x}} \perp {\rm ortp}_{\mathfrak{s}}(a_j, M_j, M_{j+1})$ say for $\varepsilon \in [\zeta_j, \kappa_{\mathfrak{x}})$. As $\kappa_{\mathfrak{x}}$ is regular $> \alpha$, we have $\zeta = \sup\{\zeta_j : j < \alpha\}$ is $< \kappa_{\mathfrak{x}}$. For $\varepsilon \in [\zeta, \kappa^{\mathfrak{x}})$ let $p_{\varepsilon,i}^+ \in \mathscr{S}_{\mathfrak{s}}^{\rm bs}(M_i)$ be the nonforking extension of $p_\varepsilon^{\mathfrak{x}}$ in $\mathscr{S}_{\mathfrak{s}}(M_i)$. Now we can prove by induction on $i \leq \alpha$ that $p_{\varepsilon,i}^+$ is the unique extension of $p_{\varepsilon,0}^{\mathfrak{x}}$ in $\mathscr{S}_{\mathfrak{s}}(M_i)$. As $b \in M_\alpha$ by 6.14(1), ${\rm ortp}_{\mathfrak{s}}(b, M_0, M_\alpha) \in \mathscr{S}_{\mathfrak{s}}^{\rm bs}(M_i)$ is orthogonal to $p_\varepsilon^{\mathfrak{x}}$. As ${\rm ortp}_{\mathfrak{s}}(b, M_0, M_\alpha) = p_0^+$, we can conclude that $p_0^+, p_{\varepsilon,0}^+$ are orthogonal hence $p_0^{\mathfrak{x}}, p_\varepsilon^{\mathfrak{x}}$ are orthogonal which by 8.11(b) is a contradiction to "$\mathfrak{x}$ non-trivial". $\square_{8.14}$

Claim 8.15. *Let $\mathfrak{x}$ be a non-trivial multi-dimensional candidate, $\kappa = \kappa_{\mathfrak{x}}$.*
1) There is $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_{\mathfrak{x}})$ strongly dominated by $\{p^{\mathfrak{x}}_i : i < \kappa\}$.
2) If $N_{\mathfrak{x}} \le_{\mathfrak{s}} M \in K_{\mathfrak{s}}$ and $q \in \mathscr{S}_{\mathfrak{s}}(M) \cap \mathbf{P}^{\mathfrak{x}}_2$ and $p^+_i \in \mathscr{S}_{\mathfrak{s}}(M)$ is a nonforking extension of $p^{\mathfrak{x}}_i$ for $i < \kappa_{\mathfrak{x}}$ then $q \le^{\rm dm}_{\rm st} \{p^+_i : i \in u\}$ for some $u \subseteq \kappa^{\mathfrak{x}}$ [introduction: connect [She90, Ch.V,§5], [She91].*]*
3) Without loss of generality *base*$(q) \subseteq M_{\mathfrak{x}}$*] [We can find $\mathbf{b} = \bar{\mathbf{b}}_\kappa \subseteq M_{\mathfrak{x}}$ such that q is definable over $\bar{\mathbf{b}}$ and $\bar{\mathbf{c}} \subseteq M$ such that $\mathrm{ortp}(\bar{\mathbf{b}}_\kappa, M^{\mathfrak{x}}, M)$ is definable over $\bar{\mathbf{c}}$ and also $\bar{\mathbf{b}}_i \subseteq M$ for $i < \kappa$ such that $\langle \mathbf{b}_i : i < \kappa \rangle$ is indiscernible based on $\bar{\mathbf{c}}$ and $q_i \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M^{\mathfrak{x}})$ is definable over $\bar{\mathbf{b}}_i$ as q is definable over $\bar{\mathbf{b}}$. From this it follows that $q \pm q_i$; moreover, q is strongly dominated by $\{q_i : i < i^*\}$ for some $i^* < \kappa$.]*
4) We can find a m.d.-candidate $\mathfrak{y}$ such that $M_{\mathfrak{x}} \le_{\mathfrak{s}} M_{\mathfrak{y}}, \bar{\mathbf{a}}_{\mathfrak{y}} = \bar{\mathbf{a}}_{\mathfrak{x}}$ and $P^{\mathfrak{y}}$ is parallel to q.

Proof. 1) By 8.14 there is M such that $N_{\mathfrak{x}} \le_{\mathfrak{s}} M$ and $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M) \cap \mathbf{P}^{\mathfrak{x}}_2$. By possibly replacing $\mathfrak{x}$ by a neighborhood, without loss of generality q does not fork over M so by part (2) we have $q \le^{\rm dm}_{\rm st} \{p^{\mathfrak{x}}_i : i \in u\}$, where $u \in [\kappa^{\mathfrak{x}}]^{<\kappa(\mathfrak{s})}$.
2) Let $\mathbf{P}'_3 = \{p \in \mathbf{P}^{\mathfrak{x}}_3$: there is $p' \in \mathscr{S}^{\rm bs}(M)$ parallel to $p\}, \mathbf{P}''_3 = \{p \in \mathbf{P}^{\mathfrak{x}}_3 : p$ is orthogonal to $M\}$.

We may consider using the "not $\kappa_{\mathfrak{s}}$-forking" version.

For $\zeta < \kappa = \kappa_{\mathfrak{x}}$ let $\mathbf{P}^*_\zeta = \mathbf{P}'_3 \cup \mathbf{P}''_3 \cup \{p^+_\varepsilon : \varepsilon < \kappa$ and $\varepsilon > \zeta\}$ and let b realize q. Now

$\circledast_\zeta$ $\mathbf{P}^*_\zeta$ is M-based which is dense above M.

[Why? See in the end of the proof].

Fixing ζ (its value is immaterial), we try by induction on $i < \kappa_{\mathfrak{s}}$ to choose M_i, N_i and a_j for $j < i$ such that:

$\boxplus$ (a) $\langle M_\varepsilon, a_j : \varepsilon \le i, j < i \rangle$ is a $\mathbf{P}^*_\zeta$-primeness sequence
(b) $M_0 = M, M_i \le_{\mathfrak{s}} N_i, N_i$ is $\le_{\mathfrak{s}}$-increasing, $\|N_i\| \le \lambda$
(c) $q = \mathrm{ortp}_{\mathfrak{s}}(b, M_0, N_0)$
(d) $\mathrm{ortp}_{\mathfrak{s}}(b, M_{i+1}, N_{i+1})$ forks over M_i
(e) if $\mathrm{ortp}_{\mathfrak{s}}(a_j, M_j, M_{j+1})$ is a non-forking extension of p^+_ε equivalent of $p^{\mathfrak{x}}_\varepsilon$ for some $\varepsilon \in [\zeta, \kappa)$ then $\mathrm{ortp}_{\mathfrak{s}}(a_j, M, M_{j+1}) \notin \{\mathrm{ortp}_{\mathfrak{s}}(a_i, M, M_{i+1}) : i < j\}$.

So for some $\alpha < \kappa_{\mathfrak{s}}$ we have $\langle M_j : j \le \alpha \rangle$ but we cannot proceed (by the demand of not too long forking). Let $\xi = \sup\{\varepsilon + 1 : \varepsilon < \kappa$ and $\varepsilon = \zeta = 1$ for some $j < \alpha, \mathrm{ortp}(a_j, M_j, M_{j+1})$ is a non-forking extension of $p_\varepsilon\}$. As $\mathbf{P}^*_\zeta$ is dense clearly q is realized by some $b \in M$.

Let $u_0 = \{i < \alpha : \mathrm{ortp}_{\mathfrak{s}}(a_i, M_i, M_{i+1})$ is a nonforking extension of some $p^+_\varepsilon, \varepsilon < \kappa\}$ for $i \in u_0$ let $\varepsilon(i) < \kappa$ be such that a_i realizes $p^+_{\varepsilon(i)}$, so by clause (e) above $\langle \varepsilon(i) : i \in u_0 \rangle$ is without repetitions. Let

$$u_1 = \{i < \alpha : \mathrm{ortp}_{\mathfrak{s}}(a_i, M_i, M_{i+1}) \in \mathbf{P}'_3\}$$

$$u_2 = \{i < \alpha : \mathrm{ortp}_{\mathfrak{s}}(a_i, M_i, M_2) \in \mathbf{P}''_3\}.$$

So $\langle u_0, u_1, u_2 \rangle$ is a partition of α and clearly:

$(*)_1$ $\{a_i : i \in u_0 \cup u_1\}$ is independent in (M, M_α).

[Why? This holds as $i \in u_0 \cup u_1$ implies $p'_i = \text{ortp}_{\mathfrak{s}}(a_i, M_i, M_{i+1})$ does not fork over M. This implication holds because:

- (α) if $i \in u_1$ then p'_i is parallel to some member of $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$, hence p'_i does not fork over M
- (β) if $i \in u_0$ then p'_i is parallel to some p^+_ε but $p_\varepsilon \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$, so we are done.]

$(*)_2$ $i \in u_2 = u \backslash u_0 \cup u_1 \Rightarrow \text{ortp}_{\mathfrak{s}}(a_i, M_i, M_{i+1}) \perp M$

[Why? By the definition of u_2 and $\mathbf{P}''_3$.]

$(*)_3$ $i \in u_1 \Rightarrow \text{ortp}_{\mathfrak{s}}(a_i, M, M_{i+1}) \perp q$.

[Why? As $\text{ortp}_{\mathfrak{s}}(a_i, M, M_{i+1}) \in \mathbf{P}'_3 \subseteq \mathbf{P}^{\mathfrak{r}}_3$ whereas $q \in \mathbf{P}^{\mathfrak{r}}_2$, recalling the definition of $\mathbf{P}^{\mathfrak{r}}_3$.]

Using 8.7(d) by $(*)_1 + (*)_2$ and (a) above we have $q \le \{p_i : i \in u_0 \cup u_1\}$.

By 8.7(2) and $(*)_3$ it follows $q \le^{\rm dm}_{\rm st} \{\text{ortp}_{\mathfrak{s}}(a_i, M, M_{i+1}) : i \in u_0\} = \{p_{\varepsilon(i)} : i \in u_0\}$ but $\langle \varepsilon(i) : \varepsilon(i) \in u_0\rangle$ is without repetition. $\square_{8.15}$

So we are done except one debt: $\circledast_\zeta$.

Proof. Proof of $\circledast_\zeta$

Towards contradiction assume $r \in \mathscr{S}_{\mathfrak{s}}(N), M \le_{\mathfrak{s}} N \in K_{\mathfrak{s}}$ and r is orthogonal to every member of $\mathbf{P}^*_\zeta$. As $r \perp p_i$ for $i \in [\zeta, \kappa)$ clearly $r \in \mathbf{P}^{\mathfrak{r}}_1$, so by the definitions of $\mathbf{P}^{\mathfrak{r}}_1, \mathbf{P}^{\mathfrak{r}}_2$ we have $r \perp \mathbf{P}^{\mathfrak{r}}_2$ hence $r \in \mathbf{P}^{\mathfrak{r}}_3$. $\square$

Case 1: $r \perp M$.

So $r \in \mathbf{P}''_3 \subseteq \mathbf{P}^*_\zeta$.

Case 2: $r \pm M$.

We can find $r' \in \mathscr{S}(M)$ which is a good enough "reflection of r over $\bar{\mathbf{a}}_{\mathfrak{r}}$", hence $r \pm r'$ but still $r' \in \mathbf{P}^{\mathfrak{r}}_3$. How? Recalling that $\mathbf{P}^{\mathfrak{r}}_3$ is $\mathfrak{a}$-based, $A \subseteq M$ small enough, this is proved as in 6.14 but we elaborate.
3) Again as in 6.14.

Conclusion 8.16. *1) If $\mathbf{P}$ is $\mathfrak{a}$-based, $M \in K_{\mathfrak{a}}$ and $\mathbf{P}$ is dense, $q_* \in \mathscr{S}_{\mathfrak{s}}(M)$ and $(\forall p \in \mathbf{P})(p \pm q_* \Rightarrow p$ does not fork over $M)$, then there are $\alpha < \kappa_{\mathfrak{s}}, p_i \in \mathbf{P} \cap \mathscr{S}_{\mathfrak{s}}(M)$ for $i < \alpha$ such that $p_i \pm q^*$ for $i < \alpha$ and q^* is weakly dominated by $\{p_i : i < \alpha\}$.*
2) Assume $\mathbf{P} \subseteq \mathscr{S}^{\rm bs}(M)$ is based on some small $A \subseteq M, q_* \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ and if $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ is orthogonal to $\mathbf{P}$ then it is orthogonal to q_*. Then we can find $\langle p_i : i < \alpha\rangle, \alpha < \kappa_{\mathfrak{s}}$ as in part (1).

Proof. 1) Let $\mathbf{P}^* =: \mathbf{P}_1 \cup \mathbf{P}_2 \cup \mathbf{P}_3$ where

$$\mathbf{P}_1 = \{p : p \text{ is parallel to some } p' \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M) \text{ which belong to } \mathbf{P}\}$$

$$\mathbf{P}_2 = \{p : p \perp \mathbf{P}_1 \text{ and } p \text{ is parallel to some } p' \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)\}$$

$$\mathbf{P}_3 = \{p : p \perp M\}.$$

We would like to apply 8.7(3) or 8.7(1). Clearly $\mathbf{P}^*$ is based on M and $(**)$ of 8.7(3) holds.

The main point is to prove that $\mathbf{P}^*$ is dense above M. So let $M \leq_{\mathfrak{s}} N$ and $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N)$. If $q \perp M$ then $q \in \mathbf{P}_3 \subseteq \mathbf{P}^*$ so O.K. If $q \pm \mathbf{P}_1$ then there is $p_0 \in \mathbf{P} \subseteq \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ and $p \in \mathscr{S}_{\mathfrak{s}}(N)$ an extension of p_0 which does not fork over M such that $p \pm q$, clearly $p \in \mathbf{P}_1 \subseteq \mathbf{P}^*$ so O.K. Hence we are left with the case $q \pm M, q \perp \mathbf{P}_1$.

If $q \pm \mathbf{P}_2$ then we can find $p' \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ orthogonal to $\mathbf{P}$ but not to p, so $p' \restriction N \in \mathbf{P}_2$ is not orthogonal to q.

So assume $q \perp \mathbf{P}_2$. We can find $N^+, \langle f_i : i < \kappa\rangle, \langle N_i : i < \kappa\rangle$ such that $N_0 = N, M \leq_{\mathfrak{s}} N_i \leq_{\mathfrak{s}} N^+, \langle N_i : i < \kappa\rangle$ is independent over M inside N^+, f_i an isomorphism from N_0 onto N_i over M. Let $q_i = f_i(q)$, so for some $\mathfrak{x}, M^{\mathfrak{x}} = M, p^{\mathfrak{x}}_i = q_i$. If $\mathfrak{x}$ is trivial, then $i < j \Rightarrow q_i \perp q_j$ then by earlier claim , $q_i \perp M$, contradiction to a statement above. So $\mathfrak{x}$ is non-trivial hence by 8.15(1) there is $q' \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ dominated by $\{q_i : i < \kappa\}$ not orthogonal to each q_i. For $\ell = 1,2$ as $i < \kappa \Rightarrow q_i \perp \mathbf{P}_\ell$ by 6.14 also $q' \perp \mathbf{P}_\ell$. By the definition of $\mathbf{P}_2$ as $q' \in \mathscr{S}_{\mathfrak{s}}(M)$ we have $q' \in \mathbf{P}_2$ but $q' \perp \mathbf{P}_2$, contradiction.
2) Let $\mathbf{P}' = \{p : p$ parallel to some $p' \in \mathbf{P} \cap \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ <u>or</u> p is orthogonal to $M\}$. Now $\mathbf{P}'$ is M-based and it is dense above M (as if $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M), M \leq_{\mathfrak{s}} N$, either $q \perp M$ so $q \in M'$ or by 6.14(2) there is $q' \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M), q' \pm q_*, [q' \perp \mathbf{P} \cap \mathscr{S}^{\rm bs}(M) \Leftrightarrow q \perp \mathbf{P} \cap \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)]$.
So applying (1) we are done. $\square_{8.16}$

Claim 8.17. *1) If $p_1, p_2 \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ are not orthogonal <u>then</u> some $r \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ is weakly dominated by p_1 and weakly dominated by p_2.*
2) If $p, q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ and $(\forall r \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M))(r \perp p \Rightarrow r \perp q)$ <u>then</u> p weakly dominates q (and, of course, the inverse is trivial).
3) If $p, q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ and every $r \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ weakly dominated by q is not orthogonal to p <u>then</u> p weakly dominates q.

Proof. 1) We shall rely on parts (2) + (3). For $\ell = 1,2$ let $\{q^\ell_i : i < \alpha_\ell\}$ be a maximal set of pairwise orthogonal types from $\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$, each weakly dominated by p_ℓ and orthogonal to $p_{3-\ell}$. So $\alpha_\ell < \kappa_{\mathfrak{s}}$ by 6.14(x).

Let $\mathbf{P}_0 = \{q : q$ orthogonal to $p_1\}$

$$\mathbf{P}_1 = \{q : q \text{ parallel to some } q^1_i, i < \alpha\}.$$

Clearly $\mathbf{P}_0 \cup \mathbf{P}_1$ is A-based for some $A \subseteq M, |A| < \kappa_{\mathfrak{s}}$. First assume that there is no $r \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ orthogonal to $\mathbf{P}_0 \cup \mathbf{P}_1$. By 8.16(2) there are $\alpha < \kappa_{\mathfrak{s}}$ and $r_i \in (\mathbf{P}_0 \cup \mathbf{P}_1) \cap \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ for $i < \alpha$ such that $i < \alpha \Rightarrow r_i \pm p_1$ and p_1 is weakly dominated by $\{r_i : i < \alpha\}$. Necessarily $i < \alpha \Rightarrow r_i \in \{q^1_i : i < \alpha\}$ hence p_1 is weakly dominated by $\{q^1_i : i < \alpha_1\}$. But $i < \alpha_1 \Rightarrow q^1_i \perp p_2$ hence by 6.14(x) $p_1 \perp p_2$, a contradiction.

Second, assume that there is $r \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ orthogonal to $\mathbf{P}_0 \cup \mathbf{P}_1$. As $r \perp \mathbf{P}_0$, by part (2) below r is weakly dominated by p_1. As r is also orthogonal to $\mathbf{P}_1$, it satisfies the first two demands on $q^1_{\alpha_1}$, so by $\{q^1_i : i < \alpha_1\}$ maximality, clearly r is not orthogonal to p_2. By the maximality of $\{q^1_i : i < \alpha_1\}$ clearly if r_1 is weakly dominated by r then $r_1 \pm p_2$. Hence by part (3), r is weakly dominated by p_2, so r is as required.
2) Let $\mathbf{P}_0 =: \{r : r \perp p\}, \mathbf{P}_1 = \{p' : p'$ parallel to $p\}$, so trivially $\mathbf{P}_0 \cup \mathbf{P}_1$ is dense M-based. Hence by 8.16(1) there are $\{r_i : i < \alpha\} \subseteq (\mathbf{P}_0 \cup \mathbf{P}_1) \cap \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$ which weakly

dominates q and $i < \alpha \Rightarrow r_i \pm q$. But every $r \in \mathbf{P}_0$ is orthogonal to p hence by the assumption of part (2), is orthogonal to q, hence $i < \alpha \Rightarrow r_i \in \mathbf{P}_0 \cup \mathbf{P}_1 \backslash \mathbf{P}_0 = \mathbf{P}_1$, so $i < \alpha \Rightarrow r_i \| p$. As $i < \alpha \Rightarrow r_i \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ we get $i < \alpha \Rightarrow p_i = p$. So q is dominated by p.
3) Let $\mathbf{P}_0 =: \{r : r \perp q\}, \mathbf{P}_1 = \{p' : p' \| p\}$. Now $\mathbf{P}_0 \cup \mathbf{P}_1$ is M-based. Also $\mathbf{P}_0 \cup \mathbf{P}_1$ is dense.
[Why? Assume r is orthogonal to $\mathbf{P}_0 \cup \mathbf{P}_1, r \perp \mathbf{P}_0$ which means: if r' is orthogonal to q then it is orthogonal to r (by the definition of $\mathbf{P}_0$) hence by part (2) we know that r is weakly dominated by q. But by the assumption of part (3) we know r is not orthogonal to p hence r is not orthogonal to $\mathscr{P}_1$, contradiction.]

By 8.16 there are $\alpha < \kappa_{\mathfrak{s}}$ and $r_i \in (\mathbf{P}_0 \cup \mathbf{P}_1) \cap \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ such that q is dominated by $\{r_i : i < \alpha\}$ and $i < \alpha \Rightarrow r_i \pm q$. So $r_i \notin \mathbf{P}_0$ hence $r_i \in \mathbf{P}_1$, i.e., $r_i = p$, so p weakly dominates q as required. $\square_{8.17}$

Conclusion 8.18. *Assume $M_0 \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} N$ and $p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$ is orthogonal to M_0 but not to M_1. Then there is $q \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_1)$ orthogonal to M_0 but not to p.*

Proof. Let $\kappa = \kappa_{\mathfrak{s}}$.

Without loss of generality N is brimmed over M_1, so without loss of generality N is brimmed over N_0 such that p does not fork over N_0 and $N_0 \le N$; let $\bar{a}_0$ list N_0, so there are $\bar{a}_i \in {}^{\kappa(\mathfrak{s})>}N$ for $i \in [1, \kappa(\mathfrak{s}))$ such that $\langle \bar{a}_i : i < \kappa_{\mathfrak{s}} \rangle$ is an indiscernible sequence over M_1 based on $\mathrm{ortp}_{\mathfrak{s}}(\bar{a}_0, M_1, N_1)$ and let $p_i \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$ be definable over $\bar{a}_i$ as p was definable over $\bar{a}$. As $p_0 = p \perp M_0$ clearly $i < \kappa \Rightarrow p_i \perp M_0$ and as $p_0 \pm M_1$ clearly ($p_i \pm M$ and) $i < j < \kappa_{\mathfrak{s}} \Rightarrow p_i \pm p_j$. So by 8.15(1) there is $q \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_1)$ such that q is dominated by $\{p_i : i < \kappa_{\mathfrak{s}}\}$. Now as q is dominated by a set of types orthogonal to M_0 by 6.14(x) also q is orthogonal to M_0. Also as q is dominated by $\{p_i : i < \kappa_{\mathfrak{s}}\}$ for some $i, q \perp p_i$, but by the choice of the $\bar{a}_i, p_i$ this holds for every i in particular $q \pm p_0 = p$, so we are done. $\square_{8.18}$

Part IV

§ 9. For weakly successful to NF-frames

Discussion 9.1. We may like to see that in (G) or 6.7, (b) is redundant, imitating earlier proofs.
Should we add $K^{3,\mathrm{qr}}_{\mathfrak{s}}$?

Hypothesis 9.2. $\mathfrak{s}$ is a weakly successful good (μ, λ, κ)-frame with primes over chains.

Our aim is to define $\mathrm{NF}_{\mathfrak{s}}$ and prove that it satisfies the relevant properties, as in [She09c, §7]. Also $\mathfrak{s}^+$ will have the density for $K^{3,\mathrm{pr}}_{\mathfrak{s}(*)}$.

Claim 9.3. *If $M_\ell \leq_{\mathfrak{s}} M$ and $p \in \mathscr{S}^{\mathrm{bs}}(M_\ell)$ for $\ell = 1, 2$, then there is $r \in \mathscr{S}^{\mathrm{bs}}(N)$ dominated by both.*

Proof. For $\ell = 1, 2$ let $\mathbf{P}_\ell = \{r : r \in \mathscr{S}(N), r$ orthogonal to $p_{3-\ell}$. $\square_{9.3}$

Claim 9.4. *1) Let $\mathbf{P}$ be A-based and $\mathbf{P}^\perp = \{p \in \mathscr{S}^{\mathrm{bs}}(M) : p \perp \mathbf{P}\}$.*

§ 10. Strong stability, Weak form of superstability

The reader can think of the first order case.

Definition 10.1. For stable T let $\kappa_{\rm ict}(T)$ be the minimal κ such that if $\mathbf{I} \subseteq \mathfrak{C}_T$ is independent over $A, \bar b \in {}^{\omega>}\mathfrak{C}$ (we allow $\mathbf{I}$ to consist of infinite sequences, too) <u>then</u> for some $\mathbf{J} \subseteq \mathbf{I}$ of cardinality $< \kappa$ the set $\mathbf{I}\backslash\mathbf{J}$ is independent over $(A \cup \mathbf{J} \cup \{\bar b\}, A)$.
2) For stable T let $\kappa_{\rm ict}(T)$ is defined similarly except in the end we require just $\bar b \in \mathbf{I}\backslash\mathbf{J} \Rightarrow \mathrm{ortp}(\bar c, A + \bar b)$ does not fork over A.

Remark 10.2. Are there suitable $K^{\rm bs}_\kappa$-templates. The case $K^{\rm tr}_\omega$ is not clear.

Definition 10.3. 1) We say p is pseudo regular (or 1-reg) when: there is $(M, N, a) \in K^{3,\rm pr}_{\mathfrak{s}}$ such that:

- (a) $\mathrm{ortp}(a, M, N)$ is parallel to p
- (b) (M, N, a) is pseudo regular which means there is no $\mathbf{J}$ such that $(M, N, \mathbf{J}) \in K^{3,\rm bs}_{\mathfrak{s}}$ and $|\mathbf{J}| \geq 2$.

2) We say p is almost regular or 2-regular when for every $(M, N, a) \in K^{3,\rm pr}_{\mathfrak{s}}$ representing p and $b \in N\backslash M$ we have $(M, N, b) \in K^{3,\rm pr}_{\mathfrak{s}}$.

Claim 10.4. *In Definition 10.3, the choice of (M, N, a) is immaterial.*

Proof. Left to the reader. □

Claim 10.5. $[\kappa_{\rm jct}(\mathfrak{s}) = \aleph_0]$.
1) If $(M, N, a) \in K^{3,\rm pr}_{\mathfrak{s}}$ <u>then</u> for some $b \in N\backslash M$ and $N' \leq_{\mathfrak{s}} N$ we have $(M, n, b) \in K^{3,\rm pr}_{\mathfrak{s}}$ is pseudo-regular.
2) Moreover, almost regular.

Proof. 1) We try to choose (b_n, c_n) by induction on n such that:

$\circledast$ (a) $b_n \neq c_n \in N\backslash M\backslash\{b_\ell, c_\ell : \ell < n\}$
(b) $\{b_0, \ldots, a_{n-1}, b_n, c_n\}$ is independent in (M, N).

If we succeed, we get that

$\circledast_1$ $\{a, b_n\} \subseteq N\backslash M$ is not independent in N over M.

$\circledast_2$ $\{b_n : n < \omega\}$ is independent in (M, N).

[Why? By $\circledast(b)$+ the finite character of independence.

So it is enough to carry the induction. If a pair (b_0, c_0) does not exist thta (M, N, a) is pseudo regular, so $b = a$ is O.K. If $b_0, \ldots, b_{n-1}, b_n, c_n$ have been defined let $N^*_s \leq_{\mathfrak{s}} N$ be such that $(M, N_n, c_n) \in K^{3,\rm pr}_{\mathfrak{s}}$ If this triple is pseudo regular choose $b := c_n$ and we are done. Otherwise there is $\mathbf{J}$ such that $(M, N_n, \mathbf{J}) \in K^{3,\rm bs}_{\mathfrak{s}}$ and $|\mathbf{J}_n| \geq 2$.

(Note: $\{b_n, c_n\}$ is independent in (M, N) hence $b_\ell \notin N_n$).

Choose $b_{n+1} \neq c_{n+1} \in \mathbf{J}$, now $\{b_0, \ldots, b_n, b_{n+1}, c_{n+1}\}$ is independent in (M, N_0) so we are done.
2) We try to choose (b_n, c_n, N_n, M_n) by induction on n (can waive c_0) such that

$\circledast$ (a) $b_n \in N\backslash M, M \leq_{\mathfrak{s}} M_n \leq_{\mathfrak{s}} N$ and $(M, M_n, b_n) \in K^{3,\rm pr}_{\mathfrak{s}}$

(b) $b_0 = a, M_0 = N = N_0$ we may use $c_0 = a$ but this is immaterial
(c) $N \leq_{\mathfrak{s}} N_n$ and $\ell < n \Rightarrow N_\ell \leq_{\mathfrak{s}} N_n$
(d) $\{c_1, \dots, c_n, b_n\} \subseteq N_n$ is independent in (M, N_n) and is with no repetition
(e) $\{a, c_\ell\}$ is dependent in (M, N_n)
(f) $\mathrm{ortp}(c_{\ell+1}, N_\ell, N_{\ell+1})$ does not fork over M.

If we succeed to carry the induction, M, N_ω prime over $\langle N_n : n < \omega\rangle, a$ and $\langle c_\ell : \ell \in [1,\omega)\rangle$ contradicts the assumption. [We assume/use: independency has local character.]

For $n = 0$ there is no problem by clause (b). So assume we have chosen $c_1, \dots, c_n, b_n, M_n, N_n$. If $\mathrm{ortp}(b_n, M, M_n)$ is almost regular we are done so there is $b \in M_n \backslash M$ contradicting it, call it b_{n+1} and choose $M_{n+1} \leq_{\mathfrak{s}} M_n$ such that $(M, M_{n+1}, b_{n+1}) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$. By an earlier Claim there is a pair $(N^*_n, M^*_n, \mathbf{J}_n)$ such that

⊠ (a) $N_n \leq_{\mathfrak{s}} N^*_n$
(b) $M \leq_{\mathfrak{s}} M_n \leq_{\mathfrak{s}} M^*_n \leq_{\mathfrak{s}} N^*_n$
(c) $(M, M^*_n, \mathbf{J}_n) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$
(d) $b_{n+1} \in \mathbf{J}$
(e) $\{N_n, M^*_n\}$ is independent over M_n.

Let $\mathbf{J}'_n = \mathbf{J}_n \backslash \{b_{n+1}\}$.

<u>Case 1</u>: $\{a\} \cup \mathbf{J}'_n$ is independent in (M, N^*_n).

Then: $\mathbf{J}'_n$ is independent in (M, N, N^*_n) hence $\mathbf{J}'_n \cup \{b_n\}$ is independent in (M, N^*_n) but $M_n \leq_{\mathfrak{s}} N^*_n$ and $(M, M_n, b_n) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$ hence $\mathbf{J}'_n$ is independent in (M, M_n, M^*_n). Let $M^{**}_n \leq_{\mathfrak{s}} M^*_n$ be such that $(M_{n+1}, M^{**}_n, \mathbf{J}'_n) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$ hence $\mathrm{ortp}(b_n, M^{**}_n, M^*_n)$ does not fork over M_{n+1} and is orthogonal to M hence (see earlier calim) (clear but we can also change definition 10.1), contradiction to the choice of b_{n+1}.

<u>Case 2</u>: Not Case 1.

Let $N_{n+1} = N^*_n$. We can find finite $\mathbf{J}''_n \subseteq \mathbf{J}'_n$ such that $\{a\} \cup \mathbf{J}'_n$ is not independent in (M, N^*_n). We can replace $\mathbf{J}''_n$ by one element and call it c_{n+1}. $\square_{10.5}$

Claim 10.6. $[\kappa_{\mathrm{rct}}(\mathfrak{s}) = \aleph_0]$. *For every $M \in K_{\mathfrak{s}}$ and $p \in \mathscr{S}(M)$ for some $N, \mathbf{J}$ we have:*

⊛$_1$ (a) $(M, N, \mathbf{J}) \in K^{3,\mathrm{pr}}_{\mathfrak{s}}$
(b) $c \in \mathbf{J} \Rightarrow \mathrm{ortp}(c, M, N)$ *is almost regular*
(c) $\mathbf{J}$ *is finite*

⊛$_2$ *p is realized in N.*

Proof. By earlier claim. $\square$

Claim 10.7. $\mathbf{P}_{2-reg}$ *is a basis.*

Definition 10.8. 1) $\mathbf{P}_{\ell-\mathrm{reg}} = \{p : p \in S(N'), N \leq_{\mathfrak{s}} N', p$ is almost ℓ-regular$\}$.
2) $p \in \mathscr{S}^{\mathrm{na}}(M)$ is 3-regular if where: for no $q_1 \perp q_2 \in \mathscr{S}^{\mathrm{na}}(M)$ do we have $q_2 \pm p \perp q_2$.
3) P is 4-regular if it is 3-regular and 2-regular.

Claim 10.9. *1) If q^∞ dominates p^∞ and q is 3-regular then p is 3-regular.*
2) [$\kappa_{\rm rct}(\mathfrak{s}) = \aleph$] $\mathbf{P}_{4-reg}$ is a basis.
3) If $p, q \in \mathscr{S}^{\rm na}_{\mathfrak{s}}(M)$ are 4-regular not orthogonal then p^∞, q^∞ are equivalent.

Proof. Straightforward. $\square_{10.9}$

Remark 10.10. 1) So we have the main gap for when $\kappa_{\rm ut}(T)$ but this is not a natural assumption ??
2) Proof of the minimal J (in 10.1) see earlier claim, etc.

$$\mathbf{J}_0 = \cup\{\mathbf{J}'' \subseteq \mathbf{I}'' : \begin{array}{ll} (\alpha) & \bar{b} \in \mathbf{J}' \text{ or } \mathbf{J}' \cup \{\bar{b}\} \text{ not independent} \\ (\beta) & \mathbf{J}' \text{ finite} \\ (\gamma) & \mathbf{J}' \text{ minimal under this}\}. \end{array}$$

$\mathbf{J}_{n+1}$ like $\mathbf{J}_0$ for $\mathbf{I} \backslash \{\mathbf{J}_\ell : \ell \leq n\}, (A, A \cup \{\mathbf{J}_\ell : \ell \leq n\}$.
$\mathbf{J} = \cup\{\mathbf{J}_n : n < \omega\}$.

§ 11. Decomposition

Discussion 11.1. Is this phenomena possible? I.e., can we type $\langle p_{\alpha,i} : \alpha < \lambda^+_{\mathfrak{s}}, i < i^* < \kappa\rangle$ such that ($i^* \geq 2$)

$\boxdot$ (a) $\langle\langle p_{\alpha,i} : i < i^*\rangle : \alpha < \lambda^+_{\mathfrak{s}}\rangle$ indiscernible

(b) $\langle p_{\alpha,i} : i < i^*\rangle$ are pairwise orthogonal

(c) $p_{\alpha,i} \perp p_{\beta,j} \Leftrightarrow (\alpha = \beta \wedge i \neq j)$

(d) by $\leq^{\rm dom}_{\rm wk}, \bar{p}_\alpha = \langle p_{\alpha,i} : i < i^*\rangle$ are pairwise equivalent

(e) for no q do we have $\bigwedge\limits_{\alpha} q \leq^{\rm dom}_{\rm wk} p_{\alpha,0}$.

A Conjecture: The phenomena $\boxdot$ is possible at least in our framework.

Still we believe the main gap holds. At the moment two approaches seem reasonable.

The first

Program B: Add imaginary elements to $M \in K^{\mathfrak{s}}$, getting $K^{\mathfrak{s}^*}$ such that there the phenomena disappears.

Note: after expanding, do we have primes?

But most natural by our research is:

Program C: Define $\mathfrak{s}^+$, derived from $\mathfrak{s}$ such that in $\mathfrak{s}^+$ we have more dichotomy and get the main or repeat in ${\rm Dp}(\mathfrak{s})$ times.

The trees here will be $\subseteq {}^{\kappa(\mathfrak{s})\geq}\mu$.

Hypothesis 11.2.

(a) $\mathfrak{s}$ is a very good (μ,λ,κ)−NF-frame, full, finite base (for $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M)$, ${\rm bas}(p) \in M$ is defined)

(b) in $\mathfrak{s}$ there are NF-prime (though not necessarily uniqueness so prime among the compatible cases; from this we should have decomposition and previously: NF-trees)

(c) $\lambda_* = \lambda^{\mathfrak{s}}$ as just $\lambda_* = \lambda_*^{<\kappa} \in [\lambda_{\mathfrak{s}}, M_{\mathfrak{s}})$, $\mathfrak{s}$ stable in λ_*.

Definition 11.3. Assume $M_0 \leq_{\mathfrak{s}} N, \|M_\ell\| \leq \lambda_* < \chi \leq \|N\|$ and $\mathbf{P} \subseteq \cup\{\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M') : M' \subseteq M_0\}$.

1) Let $\mathbf{F}^1_\chi(\mathbf{P}, M_0, M_1, N)$ be the set of p satisfying:

(a) $p \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_1)$

(b) p is orthogonal to $\mathbf{P}$ (i.e., $p \perp q$ for every $q \in \mathbf{P}$) and is dominated by M_0

(c) $\dim(p, N) = \chi$

(d) if $q \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(M_1)$ is dominated by p then $\dim(q, N) = \dim(p, N)$ (recall $\geq$ always holds).

2) Let $\mathbf{F}^0_{\lambda_*,\chi}(\mathbf{P}, M, N)$ be defined similarly except

$(d)^+$ if $q \in \mathbf{P}_{N,\lambda_*} =: \cup\{\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M'); M' \leq_{\mathfrak{s}} N, \|M'\| \leq \lambda_*\}$ and q is dominated by p then $\dim(q, N) = \dim(p, N)$.

3) $\mathbf{F}^\ell_{\lambda_*}(\mathbf{P},N) = \cup\{\mathbf{F}^\ell_{\lambda_*,\chi}(\mathbf{P},M,N) : \|M\| < \chi \le \|N\|\}$; similarly $F^1_\chi(\mathbf{P},M_0,M_1,N)$ for $\ell = 1$ we omit λ_*.
4) If $M_1 = M_0$ we may omit M_1; if $\lambda_* = \lambda^{\mathfrak{s}}$ we may omit it. We may omit $\mathbf{P}, M_0$, too.

Claim 11.4. *Assume that $\lambda_* < \chi \le \|N\|, p \in \mathbf{F}^0_{\lambda_*,\chi}(N)$. Then there is M_* such that*

(a) $\mathrm{Dom}(p) \le_{\mathfrak{t}} M_* \le_{\mathfrak{t}} N$
(b) $M_* \in K^{\mathfrak{t}}_\chi$
(c) *$p_* \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{t}}(M_*)$, the non-forking extension of p_* in $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{t}}(M_*)$, has a unique extension in $\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$.*

Proof. We try to choose M^*_i, N^*_i, a_i by induction on $i < \lambda^{++}_*$ such that

⊛ (a) $M^*_i \le_{\mathfrak{t}} N$ is $\le_{\mathfrak{t}}$-increasing
(b) $\|M^*_i\| \le \chi$
(c) $M \le_{\mathfrak{t}} M^*_0$
(d) $N^*_i \le_{\mathfrak{t}} M^*_i, N^*_i \le_{\mathfrak{s}} N, N^*_i \in K^{\mathfrak{t}}_{\lambda^+_*}$
(e) N^*_i is $\le_{\mathfrak{t}}$-increasing
(f) $a_i \in N\backslash M^*_i, \mathrm{ortp}_{\mathfrak{t}}(a_i,M^*_i,N)$ is not orthogonal to p
(g) $\mathrm{ortp}_{\mathfrak{t}}(a_i,M^*_i,N)$ does not fork over N^*_i and $a_i \in N^*_{i+1}$
(h) $N^*_0 = M$
(i) if $q \in \mathscr{S}^{\mathrm{bs}}(N^*_i)$ and $\dim(q,N) \le \chi$ then there is $\mathbf{J}_q$, a maximal set independent in (N^*_i,N) which is included in $q(N)$

(alternatively: $N^*_i \le_{\mathfrak{t}} N^{**}_i \le_{\mathfrak{t}} N^*_{i+1}, N^*_i \le M^*_i, N^*_i$ is $\le_{\mathfrak{t}}$-increasing continuous, $N^{**}_i \in K^{\mathfrak{s}}_{\lambda^+_*}$ depends on choice of framework).

For $\ell = 0$ there is no problem. If $\langle M^*_j, N^*_j, N^+_j, a_j : j < i\rangle$ is defined but we cannot choose (M_i,a_i) as above then $M^* = \cup\{M^*_j : j < i\} \cup M$ is as required.

If we have carried the induction, we have $p_i \in \mathscr{S}^{\mathrm{bs}}(N^*_i)$ as $\pm M$ hence there are $\varepsilon < \lambda^{++}_*$ and $q \in \mathscr{S}^{\mathrm{bs}}(N^*_\varepsilon)$ such that $q \le^{\mathrm{dom}}_{\mathrm{wk}} \{p_i : i \in [j,\lambda^{++}_*)$ for every $j < \lambda^{++}_*$ [see §6 find/add citation; anyhow true for f.o. - making the canonical basis indiscernible].

Now easy contradiction. $\square$

Claim 11.5. *Assume $M \le_{\mathfrak{s}} N, N$ is $\|M\|^+$-saturated and $\|M\| < \chi_i \le \|N\|$ for $i = 0,1,2$ and $\chi_1 < \chi_2$ and $\mathbf{P} \subseteq \cup\{\mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M') : M' \le_{\mathfrak{s}} M\}$.*
1) $\mathbf{F}^\ell_\chi(\mathbf{P},M,N) \subseteq \mathbf{F}^\ell_\chi(\mathbf{P},M,N)$ for $\ell < 2$ and $\mathbf{F}^0(\mathbf{P},M,N) \subseteq \mathbf{F}^1(\mathbf{P},M,N)$.
2) If $p_i \in \mathbf{F}^0_{\chi_i}(\mathbf{P},M,N)$ for $i = 0,1$ then $p_1 \perp p_2$.
3) If $p \in \mathbf{F}^0_\chi(\mathbf{P},M,N), q \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ and $\dim(q,N) > \chi$ then $p \perp q$.
4) In (2), (3) we can use $\mathbf{F}^1$.

Proof. 1) By the definition.
2) By 3).
3) Let $M' \le_{\mathfrak{s}} M, M' \in K^{\mathfrak{s}}_{\lambda_{\mathfrak{s}}}$ be such that p,q does not fork over M'. If the desired conclusion fails by 8.17(1) there is $r \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M')$ satisfying $M' \le_{\mathfrak{s}} N, \|M'\| = \lambda_{\mathfrak{s}}$ such that r dominated by both p and q. By the latter $\dim(r,N) \ge \dim(q,N) > \chi$ and by the former there is no such r.

4) Because in the proof of part (3), without loss of generality $M' = M$. $\square_{11.5}$

Claim 11.6. *Assume*

(a) $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} N$
(b) $\mathbf{P} \subseteq \cup\{\mathscr{S}^{\rm bs}_{\mathfrak{s}}(M') : M' \leq_{\mathfrak{s}} M_0\}$
(c) $\mathbf{P}^* = \cup\{p \in \mathscr{S}^{\rm bs}(M'), M' \leq_{\mathfrak{s}} N, \|M'\| \leq \lambda_*\}$.

Then

(α) $\mathbf{F}^0_\chi(\mathbf{P}, M_0, N)$ *is dense in* $\mathbf{P}^*$
(β) *moreover if* $p \in \mathbf{P}^*$ *and* $\dim(p, N) = \chi > \lambda_*$ *then* p *dominates some* $q \in \mathbf{F}^0_{\lambda_*,\chi}(\mathbf{P}, M_0, N)$
(γ) $\mathbf{F}^0_\chi(\mathbf{P}, N)$ *is dominated by* $\mathbf{F}^0_\chi(\mathbf{P}, M_0, N), \mathbf{P}_{N,\lambda_*}$ *in the sense that, i.e., if* $p \in \mathbf{P}_{N,\lambda_*}$ *is* $\pm\mathbf{P}^\perp$ *then for some* $q \in \mathbf{F}^0_{\lambda_*}(\mathbf{P}, M, N)$ *is* $\pm p$.

Proof. Clause (α): By Clause (β).

Clause (β): Assume this fails. Let $\{p_i : i < i^*\}$ be a maximal family of pairwise orthogonal types from $\mathscr{P}_{N,\lambda_*}$ each dominated by p with $\dim(p_i, N) > \chi$.

Note that by our assumption toward contradiction, if $q \in \mathscr{P}_{N,\lambda_*}$ is dominated by p, then there is $r \in \mathscr{P}_{N,\lambda^*}$ dominated by q and $\dim(r, N) > \chi$ so $r \perp p_i$ for some i. By 6.14, if $q \in \mathscr{S}^{\rm bs}(M'), M' \leq_{\mathfrak{s}} N' \wedge N \leq_{\mathfrak{s}} N, q$ dominated by p then there is $r \in \mathscr{S}^{\rm bs}(M'), M'' \leq_{\mathfrak{s}} N'' \wedge N' \leq_{\mathfrak{s}} N'', r$ dominated by q and $\bigvee_i r \pm p_i$ hence $\bigvee_i q \pm p_i$.

Let $M' \in p_{N,\lambda_*}$ be such that $\text{Dom}(p), \text{Dom}(p_i) \subseteq M'$, i.e., $p \leq^{\rm dom}_{\rm wk} \{p_i : i < i^*\}$ but this implies (6.14) $\dim(p, N) \geq \min\{\dim(p_i, i) : i < i^*\} \geq \chi^+ > \chi$, contradiction to the assumption on p. $\square_{11.6}$

Definition 11.7. We say $(M, \mathbf{P}, M^*, \mathbf{J}, N^*)$ is an approximation when:

(a) $M \leq_{\mathfrak{s}} M^* \leq_{\mathfrak{s}} N^*$
(b) $(M, M^*, \mathbf{J}) \in K^{3,\rm qr}_{\mathfrak{s}}$
(c) $\mathbf{P} \subseteq \mathscr{S}^{\rm bs}(M), \mathbf{P}_{M,\lambda_*}$
(d) $\mathbf{J} \subseteq \{c \in M^* : \text{ortp}(c, M, N^*)$ is a non-forking extension of some $p \in \mathbf{P}\}$
(e) if $M^* \leq_{\mathfrak{s}} N <_{\mathfrak{s}} N^*$ and $c \in N^* \backslash N$ then $\text{ortp}(c, N, N^*)$ is orthogonal to $\mathbf{P}$.

Claim 11.8. *Assume*

(a) $(M, \mathbf{P}_0, M^*, \mathbf{J}_0, N^*)$ *is an approximation where* $\mathbf{P} = \mathscr{S}^{\rm bs}(M)$ *(or* $\mathbf{P}_0 \cup \mathbf{P}_1$ *if you prefer)*
(b) $\mathbf{P}_1 = \{p \in \mathscr{S}^{\rm bs}(M) : p \perp \mathbf{P}_0\}$
(c) N^* *is* λ_*^+*-saturated.*

Then *we can find* $\mathbf{J}_1, M$ *such that*

(α) $(M, \mathbf{P}_0 \cup \mathbf{P}_1, M^{**}, \mathbf{J}_0 \cup \mathbf{J}_1, N^*)$ *is an approximation*
(β) $\mathbf{J}_1 \subseteq \{c \in N : \text{ortp}(c, M, N^*) \in \mathbf{P}_1\}$.

For each $p \in \mathbf{P}_{N,\lambda_*}$ *choose a maximal family* $\{q_{p,i} : i < i_p\}$ *of pairwise orthogonal types from* $\mathbf{F}^0_{\lambda_*,\chi}(\mathbf{P}, M, N)$ *each dominated by* p *so* $i_p < \kappa_{\mathfrak{s}}$*;* without loss of generality $\langle q_{p,i} : i < i_p\rangle$ *depend just on* $p/11$*. Let* $N \in K^{\mathfrak{s}}_{\lambda_*}$ *be such that* $M \le_{\mathfrak{s}} N \le_{\mathfrak{s}} N^*$ *and* $p \in \mathscr{S}^{\rm bs}(M_*), I < i_p \Rightarrow \mathrm{Dom}(q_{p,i}) \le_{\mathfrak{s}} N$*, so* without loss of generality $q_{p,i} \in \mathscr{S}^{\rm bs}_{\mathfrak{s}}(N)$*. For each such* $q \in \mathscr{S}^{\rm bs}(N)$ *let* $\mathbf{J}_{q,i}$ *be a maximal subset of* $q(N)$ *independent in* (M, N) *and let* $\langle b_{q,\alpha} : \alpha < \chi_p\rangle$ *list* $\mathbf{J}_p$ *so* $\chi_q = \dim(q, N)$ *If* $q \in \mathscr{S}^{\rm bs}(N) \cap \mathbf{F}_{\lambda_*,N}(N)$ *then let* $N^*_q \in K^{\mathfrak{t}}_\chi$ *be such that* $N \le_{\mathfrak{t}} N^*_q \le_{\mathfrak{t}}, \chi(q) = \dim(q, N)$ *be such that the non-forking extension* $q^\oplus$ *of* q *in* $N^*_q, q^\otimes$ *has a unique extension in* $\mathscr{S}^{\rm bs}(N)$*.*

Let $\langle N^*_{q,\alpha} : \alpha \le \chi(q)\rangle$ *is* $\le_{\mathfrak{t}}$*-increasing continuous with union* $N^*_q, \|N^*_{q,\alpha}\| \le \lambda_* + |\alpha|$*. Let* $\langle a_{q,\alpha} : \alpha < \chi_q\rangle$ *list* N^*_q *and let* $\langle a^*_\alpha : \alpha < \lambda_*\rangle$ *list* N*.*

Let $\langle p_i : 9 < i < i(*)\rangle$ *list* $\mathscr{S}^{\rm bs}(N)$ *(can use less). Now we try to choose* $M_i, N_i, \mathbf{I}_i$ *by induction on* α *such that*

$\circledast$ (a) $M_\alpha \le_{\mathfrak{t}} M^*$ *is increasing continuous,* $M_\alpha \in K^{\mathfrak{t}}_{\le\lambda_*+|\alpha|}$

(b) $N_\alpha \le_{\mathfrak{t}} N^*$ *is increasing continuous* $N_\alpha \in K^{\mathfrak{t}}_{\le\lambda_*+|\alpha|}$

(c) $\mathbf{I}_\alpha \subseteq \{c \in N : \mathrm{ortp}_{\mathfrak{t}}(c, M, N^*) \in \mathbf{P}_1\}$ *is increasing continuous* $|I_\alpha| \le \lambda_* + |\alpha|$

(d) $(M_\alpha, N_\alpha, \mathbf{I}_\alpha) \in K^{3,\rm vq}_{\mathfrak{t}}$ *and even* $\in K^{3,\rm cn}_{\mathfrak{t}}$ *(recall cn stands for constructible, it is unreasonable to say prime as if* $\mathfrak{t}$*-models of a stable* T*, there are no primes)*

(e) *if* $N \nsubseteq N_\alpha$ *<u>then</u> if possible under the present restriction, for some* $\zeta < \lambda_*, \mathrm{ortp}(a^*_\zeta, N_{\alpha+1}, N)$ *forks over* N_α

(f) *if* $\alpha = i(*) \times \beta + i, N \subseteq N_\alpha$ *and* $i < i(*)$*, <u>then</u> if possible under the present restrictions, for some* $\zeta < \chi(p_i)$*,* $\mathrm{ortp}_{\mathfrak{t}}(a_{q,\alpha}, N_{\alpha+1}, N)$ *forks over* N_α*.*

So for some $\alpha(*), (M_\alpha, N_\alpha, \mathbf{I}_\alpha)$ *is defined iff* $\alpha \le \alpha(*)$ *hence* $\mathrm{NF}(M_\alpha, M^*, N_\alpha, N^*)$*. Let* $M^{**} \le_{\mathfrak{s}} N^*$ *be prime over* $N_\alpha \cup M^*, \mathbf{J}_1 = \mathbf{I}_{\alpha(*)}$ *and let* $M^\oplus \le_{\mathfrak{s}} N^*$ *be such that* $(M^*, M^\otimes, \mathbf{J}_1) \in K^{3,\rm qr}_{\mathfrak{s}}$ *we shall show that they are as required. The main point is to prove the following is impossible*

$\boxdot$ $M^{**} \le_{\mathfrak{s}} N' <_{\mathfrak{s}} N^*, c \in N^* \backslash N', \mathrm{ortp}_{\mathfrak{s}}(c, N', N^*) \pm M$*.*

We first prove

$\boxdot_1$ *if* $\alpha < \lambda^+_*$ *and* $N \nsubseteq N_\alpha$ *then in* $\circledast(e)$ *there is such* ζ*.*

[Why? Note that $\mathrm{NF}_{\mathfrak{t}}(M_\alpha, M^*, N_\alpha, N^*)$ *by earlier claim. Let* $N^+_\alpha \le_{\mathfrak{s}} N$ *be prime over* $M^* \cup N_\alpha$*. If* $N \subseteq N^+_\alpha$ *we can easily find* $(M_{\alpha+1}, N_{\alpha+1})$ *such that:* $N_{\alpha+1} \cap N \backslash N_\alpha \ne \emptyset, (M_{\alpha+1}, N_{\alpha+1}, \mathbf{I}_\alpha) \in K^{3,\rm cn}_{\mathfrak{t}}, M_\alpha \le_{\mathfrak{t}} M_{\alpha+1} \le_{\mathfrak{t}} M^*, N_\alpha \le_{\mathfrak{t}} N_{\alpha+1} \le_{\mathfrak{t}} N^+_\alpha$*, and we are done. Otherwise, let* $c \in N \backslash N^+_\alpha$*, so* $\mathrm{ortp}(c, N^+_\alpha, N) \perp \mathbf{P}_0$*. If* $\mathrm{ortp}(c, N^+_\alpha, N) \perp M$ *then let* $N^\oplus_\alpha \le_{\mathfrak{s}} N^*$ *be such that* $(N^+_\alpha, N^\oplus_\alpha, c) \in K^{3,\rm pr}_{\mathfrak{s}}$ *and proceed as in the case* $N \subseteq N^+_\alpha$*. So assume* $\mathrm{ortp}(c, N^+_\alpha, N) \pm M, \mathrm{ortp}(c, N^+_\alpha, N) \perp \mathbf{P}_0$*, hence for some* $p \in \mathscr{P}_1, p \pm \mathrm{ortp}(c, N^+_\alpha, N)$*. As* $p \perp \mathbf{P}_0$ *the non-forking extension* p^+ *of* p *in* $\mathscr{S}^{\rm bs}(N^+_\alpha)$ *is* λ^+_**-isolated, but* N^* *is* λ^+_**-saturated?? hence there is* $b \in N$ *realizing* p^+ *such that* $\{b, c\}$ *is not independent over* N^+_α*. Now we can proceed as in the case* $N \subseteq N^+_\alpha$ *getting* $b \in N_{\alpha+1}, \mathrm{ortp}_{\mathfrak{t}}(c, N_{\alpha+1}, N^*)$ *forks over* N_α*. So we are done.]*

$\boxdot_2$ $N \subseteq N_{\alpha(*)}$ *for some* $\alpha(*) < \lambda_*^+$.

[Why? By $\boxdot_1$.*]*

$\boxdot_3$ (a) *if* $q = q_j \in \mathscr{S}^{\rm bs}(N)$ *is* $\perp \mathbf{P}_0$ *and* $q \in \mathbf{F}^0_{**,\chi}(N^*)$ *and* $N \subseteq N_\alpha, \alpha < \chi, \alpha = j \mod i(*)$, *then in clause (f) of* $\circledast$ *there is such* ζ

(b) *if* $\chi(q_i) \le \alpha$ *then* $N_{q_i} \subseteq N_\alpha$.

[Why? We prove it by induction on χ. *Let us prove clause (a), then clause (b) follows as in* $\boxdot_2$ *so let* $N^+_\alpha \le_{\mathfrak s} N^*$ *be prime over* $M^* \cup N_\alpha$, *exists as* $\mathrm{NF}_{\mathfrak t}(M_\alpha, M^*, N_\alpha, N^*)$ *see earlier claim; if* $N_q \subseteq N^+_\alpha$ *we are done as in the proof of* $\boxdot_1$, *so let* $b \in N_q \backslash N^+_\alpha$. *As there* $\mathrm{ortp}_{\mathfrak s}(b, N^+_\alpha, N^*) \perp \mathbf{P}$ *and* without loss of generality $\mathrm{ortp}_{\mathfrak s}(b, N^+_\alpha, N^*) \perp M$. *So there is* $p \in \mathbf{P}_1$ *not orthogonal to it. Hence there is* $q_* \in \mathscr{S}^{\rm bs}(N)$ *dominated by* p *not orthogonal to* $\mathrm{ortp}_{\mathfrak s}(b, N^+_\alpha, N^*)$ *such that* $q_* \in \mathbf{F}^0_{\lambda_*,*}(N)$ *and let* χ_* *be such that* $q_* \in \mathbf{F}^0_{\lambda_*,\chi_*}(N)$. *So necessarily* $N_{q_*} \nsubseteq N_\alpha$ *hence* $\chi_* > \alpha$.*]*

We first try to analyze the case of quite saturated models.

Claim 11.9. *If* $M_0, \mathbf{P}, N^*$ *satisfies* $\circledast_1$, *then* $M_1, N_1, \mathbf{J}$ *satisfies* $\circledast_2$ *where*

$\circledast_1 = \circledast^1_{M_0,N}$ $\quad M_0 \le_{\mathfrak s} N^*, M_0 \in K_\lambda, \mathbf{P} \subseteq \mathscr{S}^{\rm bs}(M)$ *and* N^* *is* λ_*^+*-saturated or just* λ^+*-saturated.*

$\circledast_2 = \circledast^2_{M_0,M_1,N_1,\mathbf{J},N^*}$ $\quad$ *the following holds*

(a) $M_0 \le_{\mathfrak s} M_1 \le_{\mathfrak s} N_1 \le_{\mathfrak s} N^*$,
(b) $\mathbf{J}$ *is independent in* (M_1, N_1)
(c) *if* $c \in \mathbf{J}$ *then* $p = \mathrm{ortp}(c, M_1, N^*) \in \mathbf{F}^0(\mathbf{P}, M_0, N^*)$
(d) $(M_1, N_1, \mathbf{J}) \in K^{3,\rm qr}_{\mathfrak s}$
(e) *if* $q \in \mathscr{S}^{\rm bs}(N_1)$ *does not fork over* M_1 *and is* $\perp \mathbf{P}$ *then* q *has a unique extension in* $\mathscr{S}_{\mathfrak s}(N^*)$
(f) *if* $p \in \mathbf{F}^0(\mathbf{P}, M_0, N^*)$ *then* $|\{c \in \mathbf{J} : \mathrm{ortp}_{\mathfrak s}(c, M_0, N) = p\}| = \dim(p, N^*)$
(g) $(M_0, M_1, \mathbf{J}' \cup \mathbf{J}'') \in K^{3,\rm vq}_{\mathfrak s}$ *where for* $c \in \mathbf{J}$ *we have* $c \in \mathbf{J}' \Leftrightarrow \mathrm{ortp}_{\mathfrak s}(c, M_0, N^*) \in \mathbf{F}^0(\mathbf{P}, M_0, N^*), c \in \mathbf{J}'' \Rightarrow \mathrm{ortp}_{\mathfrak s}(c, M_0, N^*) \perp \mathbf{F}^0(\mathbf{P}, M_0, N^*)$.

Proof. We choose by induction on $i < (\lambda^3)^+$ a triple $(M_i, N_i, \mathbf{J}_i, \mathbf{J}'_i, \mathbf{J}''_i)$ such that:

(a) $M_i \le_{\mathfrak s} N^*$ is from $K_{\lambda^{\mathfrak s}}$
(b) M_i is $\le_{\mathfrak s}$-increasing continuous
N_i is $\le_{\mathfrak s}$-increasing continuous, $M_i \le_{\mathfrak s} N_i \le_{\mathfrak s} N^*$
(c) if $i = j+1, q \in \mathscr{S}^{\rm bs}_{\mathfrak s}(M_i), q \perp \mathbf{P}$ and there is (M', r) such that $M' \le_{\mathfrak s} N^*$ of cardinality λ and $r \in \mathscr{S}^{\rm bs}_{\mathfrak s}(M')$ is dominated by q and $\dim(p, N^*) < \dim(q, N^*)$ then there is such $r \in \mathscr{S}^{\rm bs}_{\mathfrak s}(M_i)$ (hence also if $\mathrm{cf}(i) \ge \kappa_{\mathfrak s}$ this holds)
(d) if $j < i$ then:
  (α) $\mathbf{J}_j \backslash (\mathbf{J}_j \cap M_i) \subseteq \mathbf{J}_i$
  (β) $\mathrm{NF}_{\mathfrak s}(M_i \cap_j, M_i, N_j, N^*)$,
  (γ) $(M_j, M_i \cap N_j, \mathbf{J}_j \cap M_i) \in K^{3,\rm qr}_{\mathfrak s}$.
(e) $(M_i, N_i, \mathbf{J}_i) \in K^{3,\rm vr}_{\mathfrak s}$
(f) $\mathbf{J}$ is a maximal subset of $N^* \backslash M_i$ which is independent in (M_i, N^*) and satisfies clause (c)

(g) if $\mathrm{cf}(i) = \kappa$ then there is a triple (N_i^-, a_i, c_i, q_i) such that:
  (α) $M_i \le_{\mathfrak{s}} N_i^- \le_{\mathfrak{s}} N_i$
  (β) $N_i^- \in K_\lambda$
  (γ) $b_i, c_i \in N^* \backslash N_i$
  (δ) $q_i = \mathrm{ortp}(c_i, N_i, N^*)$ does not fork over M_i and is as in (c)
  (ε) $\mathrm{ortp}_{\mathfrak{s}}(b_i, N_i, N^*) \pm q$
  (ζ) $\mathrm{ortp}_{\mathfrak{s}}(b_i, N_i, N^*)$ does not fork over N_i^-'
  (θ) $N_i^- \subseteq N_{i+1}$

(h) $\mathbf{J}'_i \subseteq \{c \in N^* : \mathrm{ortp}_{\mathfrak{s}}(c, M_0, N) \in \mathbf{F}^0, (\mathbf{P}, M_0, N^*)\}$ increases with i

(i) $\mathbf{J}''_i \subseteq \{c \in N^* : \mathrm{ortp}_{\mathfrak{s}}(c, M_0, N^*)$ orthogonal to $\mathbf{F}^0(\mathbf{P}, M_0, N^*)$

(j) if $i = j + 2$ then $(M_0, M_i, \mathbf{J}'_i, \mathbf{J}'_i) \in K_{\mathfrak{s}}^{3,\mathrm{uq}}$

(k) $\mathbf{J}_0$ satisfies clause (f) of the claim.

So for some $i(*) \le (\lambda^{\mathfrak{s}})^+$, we have defined for i iff $i < i(*)$.

Case 1: We succeed to carry the induction, i.e., $i(*) = (\lambda^{\mathfrak{s}})^*$.

Let $\bar{a}_i = \langle a^i_\varepsilon : \varepsilon < \lambda\rangle \in {}^{\lambda+2}(N^*)$ list $N_i^- \cup \{b_i, c_i\}, \{a^i_{4\varepsilon} : \varepsilon < \lambda\}$ list $b_i, c_i \notin \{a^i_\varepsilon : \varepsilon \ne 1, 2\}, N_i^- \cap M_i, (a^i_{\lambda+}, a^i_{\lambda+1}) = (b_i, c_i)$. By x.x. for some stationary $S \subseteq \{i < (2^\lambda)^+ : \mathrm{cf}(i) = \lambda)$ the sequence $\langle \bar{a}_i : i \in S\rangle$ is convergent and indiscernible over $M_{j(*)}$.

Let $j(*)$ be minimal such that $\mathrm{otp}(S \cap j(*)) \ge \kappa$. By 8.18 we get a contradiction to the maximality of $\mathbf{J}_{j(*)}$.

Case II: for some limit δ we have defined for all $i < \delta$.

Case III: $i(*) = 0$.

Trivial.

Case IV: $i(*) = j + 1, \mathrm{cf}(j) \ne \lambda^+$.

Easy.

Case V: $i(*) = j + 1, \mathrm{cf}(j) = \lambda^+$. $\square_{11.9}$

We prove that $(M_j, N_j, \mathbf{J}_j)$ are as required.

Claim 11.10. *Assume* $\mathfrak{s}$ *has* NDOP.

Assume $\circledast^2_{M_0,M_1,N_1,\mathbf{J}_1,N^*}$ *and* N^* *is* $\|M_1\|^+$*-saturated,* $\|M_1\| = \|M_1\|^{<\kappa(\mathfrak{s})}$.

Then we can find $N_0, \mathbf{J}_0$ *such that:*

(α) $\mathrm{NF}_{\mathfrak{s}}(M_0, M_1, N_0, N_1)$

(β) $(M_0, N_0, \mathbf{J}_0) \in K_{\mathfrak{s}}^{3,\mathrm{qr}}$

(γ) $c \in \mathbf{J}_0 \Rightarrow \mathrm{ortp}_{\mathfrak{s}}(c, M_0, N_0) \in \mathbf{F}^*(\mathbf{P}, M_0, N^*)$.

Proof. Let $\Theta_\ell = \{\dim(p, N^*) : p \in \mathbf{F}^*(\mathbf{P}, M_\ell)\}$. Fix for the time being $\chi \in \Theta_\ell$, let $\mathbf{P}_\chi = \mathbf{F}^0_\chi(\mathbf{P}, M_0, M_1, N), \mathbf{P}^*_\chi = \mathbf{F}^0_\chi(\mathbf{P}, M_0, N), \mathbf{J}^1_\chi = \{c \in \mathbf{J}_1 : \mathrm{ortp}_{\mathfrak{s}}(c, M_1, N^*) \in \mathbf{P}_\chi\}$. Let $\mathbf{J}^1_\chi = \cup\{\mathbf{J}^1_{\chi,\varepsilon} : \varepsilon < \chi\}$ where $\mathbf{J}^1_{\chi,\varepsilon}$ is increasing with ε.

First Proof: Now we choose by induction on i an element (of sequences of length $< \kappa_{\mathfrak{s}}$) $c_{\chi,i}$ such that:

(*i*) $c_{\chi,i} \in N^* \backslash M_0 \backslash \{c_{\chi,j} : j < i\}$

(*ii*) $\mathrm{ortp}_{\mathfrak{s}}(c_{\chi,i}, M_0, N^*) \in \mathscr{P}^*_\chi$

(*iii*) $\{c_{\chi,j} : j \le i\}$ is independent in (M_0, N^*)

(*iv*) $\varepsilon_\chi(c_{\chi,i}) \le \varepsilon(c)$ for any c satisfying (i), (ii), (iii) where $\varepsilon_\chi(c) = \varepsilon_\chi(c, N^*) = \mathrm{Min}\{\varepsilon \le \chi$: if $\varepsilon < \chi$ then c is not given over $(M_0, M_1 \cup \mathbf{J}^1_{\chi,\varepsilon})$

$\boxdot_1$ $\varepsilon_\chi(c_j) \le \varepsilon_\chi(c_i)$ for $j < i$.
[Why? Trivially.]

$\boxdot_2$ for every ε the set $\{i : \varepsilon_\chi(c_i) \le \varepsilon\}$ is an ordinal $i[\varepsilon] < (\|M_1\| + |\varepsilon|)^+$
[Why? By one of the basic properties of dimension.]

$\boxdot_3$ if $N^* \le_{\mathfrak{s}} N^+, c \in N^+ \backslash M_0 \backslash \{c_{\chi,i} : i < \chi\}$ and $\mathrm{ortp}_{\mathfrak{s}}(c, M_0, N^+) \in \mathbf{P}^*_\chi$ and $\{c_{\chi,i} : i < \chi\} \cup \{c\}$ is independent in (M_0, N^+) then $\varepsilon_\chi(c, N^+) = \chi$.
[Why? Assume c is a counterexample that $\varepsilon_\chi(c, N^+) = \zeta < \chi$ and let $i(*) = i[\zeta]$.

Now

Second proof of 11.10:

Problems :

1) Try to use 11.6 for 11.10.

2) The problem is (a) or (b) where

(*a*) the existence of q dominated by M_1 orthogonal to $M_0, (M_0, M_1, a) \in K^{3,\mathrm{uq}}_{\mathfrak{s}}$ and no $r \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_1)$ is dominated by q, so possibly $\dim(q, N) > \sup\{\mathrm{dom}(p, N) : p \in \mathscr{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)\}$.

Moreover, if $M_1 \le M_2, (M_0, M_2, a) \in K^{3,\mathrm{uq}}_{\mathfrak{s}}$ nothing changes. There is a hidden independence property, but some cases seemingly has just one more unary function so there is a decomposition

(*b*) instead M_0, M we have $\langle M_i : i \le \delta\rangle$ semi continuous, $\delta < \kappa_{\mathfrak{s}}$ and we look at q dominated by M_δ orthogonal to M_i for $i < \delta$ (needed only if we like to have decomposition to trees $\subseteq {}^{\kappa >}\mu$.

3) If NDOP and for some $\{M_i : i < \alpha\}$ independent over M the prime model over $\cup\{M_i : i < \alpha\}$ is not minimal, then any logic fix-length-game + quantifiers on dimension does not characterize models up to isomorphism.

4) Have imaginary elements for

(*a*) p/parallelism

(*b*) $p/\mathbf{E}, p_1\mathbf{E}p_2$ iff they always have the same dimension (i.e., each dominated the other).

5) conjecture: if p is dominated by M but dominate no one in $\mathscr{S}^{\mathrm{bs}}(M)$ then on some p/E there is a group hence is of depth 0 (and even p^∞ is?).

6) Groups: see what I wrote to Lessman.

7) Take NDOP from 705 in [She09e]: if $\mathfrak{s}$ is excellent exactly up to $n+1$ then $\mathfrak{s}^+$ is excellent exactly up to n or up to $n+1$. Help in §12 but need lower. But: does $EM(I)$ help?

Define slim if $EM(I, \Phi)$ is without order this as dividing line. $\square$

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