Automorphisms and strongly invariant relations

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Abstract. We investigate characterizations of the Galois connection sInv–Aut between sets of finitary relations on a base set A and their automorphisms. In particular, for $A = \omega_1$, we construct a countable set R of relations that is closed under all invariant operations on relations and under arbitrary intersections, but is not closed under sInvAut.

Our structure (A, R) has an ω-categorical first order theory. A higher order definable well-order makes it rigid, but any reduct to a finite language is homogeneous.

1. Introduction

Our main question is easy to formulate. Let R be a set of finitary relations on a nonempty base set A, and let $\mathsf{Aut}\, R$ denote the set of all automorphisms of the structure $(A\,;\,(\varrho)_{\varrho\in R})$. Conversely, if G is a set of permutations on A, then $\mathsf{sInv}\, G$ denotes the set of all relations σ on A such that all permutations in G are automorphisms of σ. (Formal Definitions follow in the next section.) The question is: How can we characterize the relation sets of the form $\mathsf{sInv}\,\mathsf{Aut}\, R$?

Of course, the operator $\mathsf{sInv}\,\mathsf{Aut}$ is a closure operator, and the operator pair sInv–Aut forms a Galois connection between sets of relations on A and sets of permutations on A. We can reformulate our problem as "Which sets R of relations are *Galois-closed*, i.e., satisfy $R = \mathsf{sInv}\,\mathsf{Aut}\, R$?" or: "Describe the closure operator $\mathsf{sInv}\,\mathsf{Aut}$ *internally*", i.e., without explicit reference to permutations.

Probably the first one who investigated this question in a systematic way was Marc Krasner. Influenced by the Galois connection between permutation groups and field extensions he tried to 'generalize the notion of a field' [7]. Instead of the action of permutations on field elements, he considered the more complex action on relations. For finite base sets A he described the closed sets of relations with the help of some operations on relations. A *logical operation on relations* is an operation, definable by a formula of the first order logic. (For details see the next section.) We call a set of relations a *Krasner algebra* if it is closed under all logical operations. For finite A, the Galois closed sets of relations are exactly the Krasner algebras. (At this point we remark that our notation differs from Krasner's original notation.)

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It is easy to extend this characterization to countable base sets A: In this case the Galois closed relation sets are exactly those Krasner algebras that are additionally closed under arbitrary intersections ($\bigcap$–closed Krasner algebras). This is no longer true for the general case of uncountable sets A. For this case there exists a characterization by R. Pöschel [12, 13] with the help of additional operations of uncountable arity. As we find the use of such operations not very satisfying, we continue to look for better results.

One reason for the existence of $\bigcap$–closed Krasner algebras that are not Galois closed is the fact that first order logic is simply "too weak" to distinguish between sets of different infinite cardinalities. Consequently, it is a natural idea to replace the logical operations by a stronger class of operations. An n-ary operation F on relations is called *invariant*, if the following identity holds for all permutations g and all relations ϱ_1, ϱ_n (with appropriate arities) on A:

$$F(g[\varrho_1], \dots, g[\varrho_n]) = g\,[F(\varrho_1, \dots, \varrho_n)]$$

Clearly, every Galois closed set of relations is $\bigcap$–closed and closed under all invariant operations. It was unknown whether the converse is also true. The problem is: *Does there exist a set of relations that is $\bigcap$–closed and closed under all invariant operations, but not Galois closed for* sInv–Aut*?* ([3, Problem 2.5.2].)

Surprisingly, the answer to this question is *yes*! In the main part of our article, section 3, we give a model theoretical construction of such a set of relations on a base set A of cardinality ω_1.

Finally, in section 4, we give a characterization of the Galois closed relation sets with the help of additional invariant infinitary operations. In contrast to Pöschel's characterization, we restrict these infinite arities to be countable. Section 3 shows that we cannot restrict the arities to be finite, so this seems to be the best possible result.

2. Preliminaries.

Notation. Throughout, let A denote a nonempty base set. Write ω for the set of all natural numbers (and at the same time for the first infinite ordinal). An m-ary relation on A is a subset of A^m, the set of all m-ary relations is denoted by $\mathrm{Rel}^{(m)}(A)$, and $\mathrm{Rel}(A) := \bigcup_{1 \leqslant m \in \omega} \mathrm{Rel}^{(m)}(A)$ is the set of all finitary relations. If $R \subseteq \mathrm{Rel}(A)$, then $R^{(m)} := R \cap \mathrm{Rel}^{(m)}(A)$. We do not distinguish between relations and predicates, therefore $\underline{a} \in \varrho$ and $\varrho(\underline{a})$ have the same meaning. The set of all permutations on A is denoted by $\mathrm{Sym}(A)$. For $g \in \mathrm{Sym}(A)$ and $\underline{a} = (a_1, \dots, a_m) \in A^m$ we put

$$g(\underline{a}) := (g(a_1), \dots, g(a_m)),$$

and for $\varrho \subseteq A^m$ we write

$$g[\varrho] := \{g(\underline{a}) \mid \underline{a} \in \varrho\}.$$

Let $g \in \mathrm{Sym}(A)$ and $\varrho \in \mathrm{Rel}(A)$. We say that g is an *automorphism of* ϱ, or that g *strongly preserves* ϱ, or that ϱ is a *strongly invariant relation for* g, if

$$g[\varrho] = \varrho.$$

This is equivalent to $g[\varrho] \subseteq \varrho$ and $g^{-1}[\varrho] \subseteq \varrho$.

For $R \subseteq \mathrm{Rel}(A)$ and $G \subseteq \mathrm{Sym}(A)$ we define operators $\mathsf{Aut} : \mathscr{P}\mathrm{Rel}(A) \to \mathscr{P}\,\mathrm{Sym}(A)$ and $\mathsf{sInv} : \mathscr{P}\,\mathrm{Sym}(A) \to \mathscr{P}\mathrm{Rel}(A)$:

$$\begin{aligned} \mathsf{Aut}\, R &:= \{g \in \mathrm{Sym}(A) \mid g[\varrho] = \varrho \text{ for all } \varrho \in R\} \\ \mathsf{sInv}\, G &:= \{\varrho \in \mathrm{Rel}(A) \mid g[\varrho] = \varrho \text{ for all } g \in G\} \end{aligned}$$

(For a set X, $\mathscr{P}X$ denotes the set of all subsets of X.)

The operator pair sInv–Aut forms a *Galois connection* between sets of permutations and sets of relations on A, i.e. the following conditions are satisfied:

- $R_1 \subseteq R_2 \;\Rightarrow\; \mathsf{Aut}\, R_2 \subseteq \mathsf{Aut}\, R_1$ and $G_1 \subseteq G_2 \;\Rightarrow\; \mathsf{sInv}\, G_2 \subseteq \mathsf{sInv}\, G_1$
- $R \subseteq \mathsf{sInv}\,\mathsf{Aut}\, R$ and $G \subseteq \mathsf{Aut}\,\mathsf{sInv}\, G$.

Consequently, the operators

$$\mathsf{sInv}\,\mathsf{Aut} : \mathscr{P}\mathrm{Rel}(A) \to \mathscr{P}\mathrm{Rel}(A) \text{ and } \mathsf{Aut}\,\mathsf{sInv} : \mathscr{P}\,\mathrm{Sym}(A) \to \mathscr{P}\,\mathrm{Sym}(A)$$

are closure operators. The sets of relations and the sets of permutations which are closed under these closure operators are called *Galois closed* (with respect to the Galois connection sInv–Aut). *Characterizing a Galois connection* means to describe the Galois closed sets without referring to the connection itself.

In our article we want to find and discuss characterizations of our Galois connection sInv–Aut. There exist many similar Galois connections between sets of relations and sets of different kinds of functions, and they turned out to be useful especially for the investigation of finite mathematical structures. As a general source, we refer to [11] and the list of references given there. Here we are interested in characterizations for infinite base sets A.

The main tool for the description of the closed sets of relations are *operations on relations*. These operations are of the form

$$F : \mathrm{Rel}^{(m_1)}(A) \times \ldots \times \mathrm{Rel}^{(m_n)}(A) \to \mathrm{Rel}^{(m)}(A),$$

with $0 \leqslant n \in \omega$. A set $R \subseteq \mathrm{Rel}(A)$ is *closed under* F if $F(\varrho_1, \ldots, \varrho_n) \in R$ for all $\varrho_i \in R^{(m_i)}$, $1 \leqslant i \leqslant n$.

Special operations on relations are the *logical operations* which can be defined with the help of first order formulas. More exactly: Let $\varphi(P_1, \ldots, P_n; x_1, \ldots, x_m)$ be a formula with predicate symbols P_i (of arity m_i), where all free variables are in $\{x_j \mid 1 \leqslant j \leqslant m\}$. We define

$$L_\varphi(\varrho_1, \ldots, \varrho_n) := \{(a_1, \ldots, a_m) \in A^m \mid \varphi_A(\varrho_1, \ldots, \varrho_n, a_1, \ldots, a_m)\},$$

where $\varphi_A(\varrho_1, \ldots, \varrho_n, a_1, \ldots, a_m)$ means that φ holds in the structure $\langle A\,;\, \varrho_1, \ldots, \varrho_n\rangle$ for the evaluation $x_j := a_j$, $(1 \leqslant j \leqslant m)$.

Examples of logical operations are the Boolean operations intersection $\cap$ and complementation $\mathbf{C}$, defined by the formulas $P_1(x_1, \ldots, x_m) \wedge P_2(x_1, \ldots, x_m)$ and $\neg P(x_1, \ldots, x_m)$.

Properties of Galois closed relation sets.

Definition 2.1. A set $R \subseteq \mathrm{Rel}(A)$ is called a *Krasner algebra (KA) on A*, if R is closed under all logical operations.[1]

If $Q \subseteq \mathrm{Rel}(A)$, then $\langle Q\rangle_{KA}$ denotes the *Krasner algebra generated by Q*, i.e. the least set of relations on A that contains Q and is closed under all logical operations.

A set R of relations is called $\bigcap$*–closed*, if

(1) $A^m \in R$ for all $m \in \omega \setminus \{0\}$
(2) R is closed under arbitrary intersections

i.e. for all m and all $Q \subseteq R^{(m)}$ we have $\bigcap Q \in R^{(m)}$. (Here we put $\bigcap \emptyset = A^m$.)

If $Q \subseteq \mathrm{Rel}(A)$, then $\langle Q\rangle_{KA,\bigcap}$ denotes the least $\bigcap$–closed Krasner algebra containing Q.

Clearly, $\langle \_ \rangle_{KA}$ and $\langle \_ \rangle_{KA,\bigcap}$ are closure operators with $\langle Q\rangle_{KA} \subseteq \langle Q\rangle_{KA,\bigcap}$ for all $Q \subseteq \mathrm{Rel}(A)$.

If a set R of relations is $\bigcap$–closed and closed under complementation $\mathbf{C}$ (e.g. if $R = \langle R\rangle_{KA,\bigcap}$), then it is also closed under arbitrary unions.

The next lemma gives the obvious connection between these notions and the Galois closed relation sets. For a proof we refer e.g. to [3].

Lemma 2.2. *If $R \subseteq \mathrm{Rel}(A)$ is Galois closed ($R = \mathsf{sInv}\,\mathsf{Aut}\,R$), then R is a $\bigcap$–closed Krasner algebra, $R = \langle R\rangle_{KA,\bigcap}$. Consequently, for all $Q \subseteq \mathrm{Rel}(A)$ we have $\langle Q\rangle_{KA,\bigcap} \subseteq \mathsf{sInv}\,\mathsf{Aut}\,Q$.*

(Galois closed sets of relations are sometimes called *Krasner clones.* So every Krasner clone is a Krasner algebra, but not vice versa.)

Definition 2.3. Let $R \subseteq \mathrm{Rel}(A)$ and $\underline{a} \in A^m$. We define:

$$\Gamma_R(\underline{a}) := \bigcap \{\varrho \in R^{(m)} \mid \underline{a} \in \varrho\}$$

We collect some properties of Γ.

Lemma 2.4. *Let $R_1, R_2, R \subseteq \mathrm{Rel}(A)$, $G \subseteq \mathrm{Sym}(A)$ and $\underline{a} \in A^m$. Then the following hold.*

(1) $R_1 \subseteq R_2 \Rightarrow \Gamma_{R_2}(\underline{a}) \subseteq \Gamma_{R_1}(\underline{a})$
(2) *$\underline{a} \in \Gamma_R(\underline{a})$ and $\Gamma_R(\underline{a}) \subseteq \varrho$ for all $\varrho \in R^{(m)}$ with $\underline{a} \in \varrho$. Moreover, $\varrho = \bigcup_{\underline{a}\in\varrho} \Gamma_R(\underline{a})$ for all $\varrho \in R$.*
(3) *If R is $\bigcap$–closed, then $\Gamma_R(\underline{a}) \in R$.*
(4) *If R is closed under complementation, then $\{\Gamma_R(\underline{a}) \mid \underline{a} \in A^m\}$ is a partition of A^m and the relation $\underline{a} \sim_R \underline{b} :\iff \underline{a} \in \Gamma_R(\underline{b})$ is an equivalence relation. If R is closed under complementation and $\bigcap$-closed, then $R^{(m)}$ is an atomic Boolean algebra.*

    Note that $\underline{a} \sim_R \underline{b}$ iff there is no relation $\varrho \in R$ separating $\underline{a}$ from $\underline{b}$.
(5) *If R_1 and R_2 are $\bigcap$–closed and closed under complementation, then $R_1 = R_2$ if and only if $\Gamma_{R_1}(\underline{a}) = \Gamma_{R_2}(\underline{a})$ for all m and all $\underline{a} \in A^m$.*

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[1]An older notation is *Krasner algebra of second kind*, see e.g. [11].

(6) $\Gamma_{\mathsf{sInv}\,G}(\underline{a}) = \{g(\underline{a}) \mid g \in \langle G\rangle_{group}\}$, *where* $\langle G\rangle_{group}$ *is the subgroup of* $\mathrm{Sym}(A)$, *generated by* $G \subseteq \mathrm{Sym}(A)$.

Proof. (1)–(5) are direct consequences of the Definitions. For (6), we first note that $\{g(\underline{a}) \mid g \in \langle G\rangle_{group}\}$ contains $\underline{a}$ and is strongly invariant for all $g \in G$. Therefore $\Gamma_{\mathsf{sInv}\,G}(\underline{a}) \subseteq \{g(\underline{a}) \mid g \in \langle G\rangle_{group}\}$. On the other hand, $\mathsf{sInv}\,G$ is $\bigcap$–closed, therefore $\underline{a} \in \Gamma_{\mathsf{sInv}\,G}(\underline{a}) \in \mathsf{sInv}\,G$, and every relation with these properties must contain all $g(\underline{a})$ with $g \in \langle G\rangle_{group}$. Consequently also $\Gamma_{\mathsf{sInv}\,G}(\underline{a}) \supseteq \{g(\underline{a}) \mid g \in \langle G\rangle_{group}\}$. □

Lemma 2.5. *Let* $R \subseteq \mathrm{Rel}(A)$ *be* $\bigcap$*–closed and closed under complementation. Then* $R = \mathsf{sInv}\,\mathsf{Aut}\,R$ *if and only if for all* m *and all* $\underline{a}, \underline{b} \in A^m$ *with* $\underline{a} \sim_R \underline{b}$ *there exists an automorphism* $g \in \mathsf{Aut}\,R$ *with* $\underline{b} = g(\underline{a})$.

Proof. By Lemma 2.4(5) we have $R = \mathsf{sInv}\,\mathsf{Aut}\,R$ iff for all m and for all $\underline{a} \in A^m$ the equality $\Gamma_R(\underline{a}) = \Gamma_{\mathsf{sInv}\,\mathsf{Aut}\,R}(\underline{a})$ holds. By Lemma 2.4(6), this equality is equivalent to $\Gamma_R(\underline{a}) = \{g(\underline{a}) : g \in \mathsf{Aut}\,R\}$. Because of $\Gamma_R(\underline{a}) \in R$ (by Lemma 2.4(3)) this is true if for every $\underline{b} \in \Gamma_R(\underline{a})$ there is an automorphism $g \in \mathsf{Aut}\,R$ with $\underline{b} = g(\underline{a})$. □

Partial automorphisms. The last lemma can be used to find characterizations in some special cases.

Definition 2.6. A partial automorphism f of a relation set $Q \subseteq \mathrm{Rel}(A)$ (or of the structure $\underline{A} = (A; (\sigma)_{\sigma\in Q})$) with *domain* $\mathrm{dom}\, f = A_1 \subseteq A$ and *image* $\mathrm{im}\, f = A_2 \subseteq A$ is a bijective function $f : A_1 \to A_2$, such that for all $\sigma \in Q$, $m = \mathrm{arity}(\sigma)$ and all $a_1, \ldots, a_m \in \mathrm{dom}\, f$, we have: $\sigma(a_1, \ldots, a_m) \Leftrightarrow \sigma(f(a_1), \ldots, f(a_m))$.

A set $Q \subseteq \mathrm{Rel}(A)$ (or the structure $\underline{A} = (A; (\sigma)_{\sigma\in Q})$) is said to be *homogeneous*, if every finite partial automorphism can be extended to an automorphism of Q.

Lemma 2.7. *If* $Q \subseteq \mathrm{Rel}(A)$ *is a homogeneous set of relations, then* $\langle Q\rangle_{KA,\bigcap} = \mathsf{sInv}\,\mathsf{Aut}\,Q$.

Proof. We first claim that for every relation set Q, the relation set $R = \mathsf{sInv}\,\mathsf{Aut}\,Q$ is homogeneous. To see this, notice that R is Galois closed and therefore $\bigcap$-closed and closed under complementation (Lemma 2.2), therefore $\Gamma_R(\underline{a}) \in R$ for all $\underline{a} \in A^m$ (Lemma 2.4(3)). Let f be a finite partial automorphism with $\mathrm{dom}\, f = \{a_1, \ldots, a_m\}$ and $f(\underline{a}) = \underline{b}$. Then $\underline{a} \in \Gamma_R(\underline{a})$ (Lemma 2.4(2)) implies $f(\underline{a}) = \underline{b} \in \Gamma_R(\underline{a})$. Because of $R = \mathsf{sInv}\,\mathsf{Aut}\,R$ and Lemma 2.5 there exists an automorphism $g \in \mathsf{Aut}\,R$ with $g(\underline{a}) = \underline{b}$.

We have $Q \subseteq \langle Q\rangle_{KA,\bigcap} \subseteq \mathsf{sInv}\,\mathsf{Aut}\,Q$, so also $\langle Q\rangle_{KA,\bigcap}$ is homogeneous. So w.l.o.g. $Q = \langle Q\rangle_{KA,\bigcap}$.

Let $\underline{a} = (a_1, \ldots, a_m)$ and $\underline{b} = (b_1, \ldots, b_m)$. We will use Lemma 2.5. So, assuming $\underline{a} \sim_Q \underline{b}$, we have to find an automorphism g with $g(\underline{a}) = \underline{b}$.

If $Q = \langle Q\rangle_{KA,\bigcap}$ and $1 \leq i < j \leq m$, then Q contains the relations $d^{(m)}_{i,j} := \{(a_1, \ldots, a_m) \mid a_i = a_j\}$, defined by the logical formula $x_i = x_j$. If $a_i = a_j$,then $\Gamma_Q(\underline{a}) \subseteq d^{(m)}_{i,j}$ and $\underline{b} \in \Gamma_Q(\underline{a})$ implies $b_i = b_j$. Similar $b_i = b_j$ and $\underline{b} \sim_q \underline{a}$ implies $a_i = a_j$.

We conclude that the map $f := \{(a_i, b_i) \mid i = 1, \ldots, m\}$ is a finite 1-1 map. As no relation in Q separates $\underline{a}$ from $\underline{b}$, f is even a partial automorphism for Q.

Because of the homogeneity of Q, f can be extended to an automorphism $g \in \mathsf{Aut}\, Q$. Therefore $\underline{b} = g(\underline{a})$ for some $g \in \mathsf{Aut}\, Q$. □

Relation sets of the form $\mathsf{sInv}\,\mathsf{Aut}\, Q$ are homogeneous. Hence, a $\bigcap$–closed Krasner algebra is Galois closed if and only if it is homogeneous.

The Galois closed permutation sets. We want to have a short look at the other side of our Galois connection. The characterization of the Galois closed permutation sets is well known ([5]) and provides no difficulties. We need an additional closure operator $\mathrm{Loc_o} : \mathscr{P}\,\mathrm{Sym}(A) \to \mathscr{P}\,\mathrm{Sym}(A)$:

$$\mathrm{Loc_o}\, G := \{f \in \mathrm{Sym}(A) \mid (\forall m \in \omega \setminus \{0\})(\forall \underline{a} \in A^m)(\exists g \in G) f(\underline{a}) = g(\underline{a})\}$$

Theorem 2.8. *A set $G \subseteq \mathrm{Sym}(A)$ is Galois closed ($G = \mathsf{Aut}\,\mathsf{sInv}\, G$) if and only if $G = \langle G\rangle_{group}$ and $\mathrm{Loc_o}\, G = G$.*

For the proof we refer to [5]. The operator $\mathrm{Loc_o}$ is a *topological closure operator*, multiplication and inversion of permutations are continuous with respect to the underlying topology. Therefore, the Galois closed automorphism sets are characterized as certain topological groups. For a more detailed discussion we refer to [4, 4.1].

A first characterization of the Galois closed relation sets. In [12] and [13], R. Pöschel characterized the closed sets of relations with the help of infinitary operations. Let I be an arbitrary index set, let $m, m_i \in \omega \setminus \{0\}$ ($i \in I$). For an I-tuple $(\varrho_i)_{i\in I}$ of relations with $\varrho_i \in \mathrm{Rel}^{(m_i)}(A)$ the *strong superposition* with parameters $\underline{a} \in A^m$, $\underline{b}_i \in A^{m_i}$ is defined as follows:

$$\mathrm{sSup}_{\underline{a},(\underline{b}_i)_{i\in I}}(\varrho_i)_{i\in I} := \{g(\underline{a}) \mid g \in \mathrm{Sym}(A) \text{ and } g(\underline{b}_i) \in \varrho_i \text{ for all } i \in I\}$$

A set $R \subseteq \mathrm{Rel}(A)$ is *closed under strong superposition*, if $\mathrm{sSup}_{\underline{a},(\underline{b}_i)_{i\in I}}(\varrho_i)_{i\in I} \in R$ whenever $\varrho_i \in R$ for $i \in I$.

Theorem 2.9. *Let $R \subseteq \mathrm{Rel}(A)$ be $\bigcap$–closed and closed under* **C**. *Then $R = \mathsf{sInv}\,\mathsf{Aut}\, R$ if and only if R is closed under strong superposition.*

Proof. If $f \in \mathrm{Sym}(A)$, then the definition of $\mathrm{sSup}_{\underline{a},(\underline{b}_i)_{i\in I}}$ implies

$$\mathrm{sSup}_{\underline{a},(\underline{b}_i)_{i\in I}}(f[\varrho_i])_{i\in I} = f\left[\mathrm{sSup}_{\underline{a},(\underline{b}_i)_{i\in I}}(\varrho_i)_{i\in I}\right].$$

Therefore every automorphism of $\{\varrho_i \mid i \in I\}$ is an automorphism of the structure $\mathrm{sSup}_{\underline{a},(\underline{b}_i)_{i\in I}}(\varrho_i)_{i\in I}$. Consequently, every Galois closed set of relations is closed under strong superposition.

If R is closed under strong superposition, then we can choose I and $(\underline{b}_i)_{i\in I}$ such that all finite sequences with elements of A occur among the $\underline{b}_i$. (For infinite A, this is possible with $|I| = |A|$.) Then R contains the relation $\mathrm{sSup}_{\underline{a},(\underline{b}_i)_{i\in I}}(\,\Gamma_R(\underline{b}_i)\,)_{i\in I}$ and consequently $\Gamma_R(\underline{a}) \subseteq \mathrm{sSup}_{\underline{a},(\underline{b}_i)_{i\in I}}(\,\Gamma_R(\underline{b}_i)\,)_{i\in I}$. This implies for every $\underline{b} \in \Gamma_R(\underline{a})$ the existence of a permutation $g \in \mathrm{Sym}(A)$ with $\underline{b} = g(\underline{a})$ and with $g(\underline{c}) \in \Gamma_R(\underline{c})$ for all n and all $\underline{c} \in A^n$. This g is an automorphism of R, therefore Lemma 2.5 implies that R is Galois closed. □

As seen in the proof, we can restrict the arities of the strong superpositions to $|I| = |A|$. Nevertheless, to be closed under strong superposition is a very strong condition. It immediately implies the existence of the necessary automorphisms in the sense of Lemma 2.5. Therefore, we continue to find better characterizations.

A Characterization for countable base set A. For finite base set A, the Galois closed relation sets are exactly the Krasner algebras ([7, 8, 9, 11]). This result can be extended to the countable case:

Theorem 2.10. *Let A be a countable or finite set and $R \subseteq \mathrm{Rel}(A)$. Then $R = \mathsf{sInv}\,\mathsf{Aut}\,R$ if and only if R is a $\bigcap$–closed Krasner algebra, $R = \langle R\rangle_{KA,\bigcap}$. Therefore $\langle Q\rangle_{KA,\bigcap} = \mathsf{sInv}\,\mathsf{Aut}\,Q$ for all $Q \subseteq \mathrm{Rel}(A)$.*

Proof. For the proof we refer to [2, 3.3.6.(v)] or to [3, 2.4.4.(i)]. One direction is provided by Lemma 2.2. For the other direction we can use a back & forth construction to obtain the automorphisms that are necessary to apply Lemma 2.5. □

The next example shows that this characterization cannot be extended to uncountable sets.

Example 2.11. Consider the following three countable structures:

(1) $(\mathbb{Q}; <)$ (the rational numbers with the linear order).
(2) The full countable bipartite graph: $(A \cup B; \varrho)$, where A and B are disjoint countable sets, and $\varrho = (A \times B) \cup (B \times A)$.
(3) The countable random graph. (See e.g. [4, 6.4.4].)

Each of these structures $\underline{M} = (M; \varrho)$ has the following properties:

(a) $Th(\underline{M})$, the first order theory of $\underline{M}$, is ω-categorical.
(b) All unary first order formulas $\varphi(x)$ are equivalent (mod $Th(\underline{M})$) to $x = x$ or to $x \neq x$, i.e., the only subsets of $\underline{M}$ that are first order definable without parameters are the empty set and the whole model.
(c) For any uncountable cardinal κ there is a model $\underline{M}_\kappa$ of cardinality κ such that the set

$$\varrho^* := \{x : \text{The set } \{y : \varrho(x, y)\} \text{ is countable}\}$$

is neither empty nor the full model.

In each of these models $\underline{M}_\kappa$, the set R of first order definable relations (without parameters) is clearly a Krasner algebra and is trivially closed under $\bigcap$ (since, by Ryll-Nardzewski's theorem, for any k there are only finitely many k-ary relations in R). But in each model $\underline{M}_\kappa$ the set ϱ^* is a (higher order) definable subset of M_κ, hence $\varrho^* \in \mathsf{sInv}\,\mathsf{Aut}(R) \setminus R$.

This shows that R is not Galois closed.

Invariant operations. In the last example, the logical operations, together with arbitrary intersections, are too weak to provide the closure under $\mathsf{sInv\,Aut}$. In particular, with logical operations it is not possible to distinguish between sets of different infinite cardinality. The next idea to obtain a characterization is to replace the logical operations by a family of stronger operations on relations.

Definition 2.12. An operation $F : \mathrm{Rel}^{(m_1)}(A) \times \ldots \times \mathrm{Rel}^{(m_n)}(A) \to \mathrm{Rel}^{(m)}(A)$ is called *invariant*, if for all $g \in \mathrm{Sym}(A)$ and all $\varrho_i \in \mathrm{Rel}^{m_i}(A)$ the following identity holds:

$$F(g[\varrho_1], \ldots, g[\varrho_n]) = g\,[F(\varrho_1, \ldots, \varrho_n)]$$

If $Q \subseteq \mathrm{Rel}(A)$, then $\langle Q\rangle_{inv}$ denotes the closure of Q under all invariant operations, and $\langle Q\rangle_{inv,\bigcap}$ denotes the least set of relations that is closed under all invariant operations, $\bigcap$–closed and contains the set Q.

(The notations "logical operations" and "invariant operations" are adopted from [6].) The operators $Q \mapsto \langle Q\rangle_{inv}$ and $Q \mapsto \langle Q\rangle_{inv,\bigcap}$ are closure operators, and we have $\langle Q\rangle_{inv} \subseteq \langle Q\rangle_{inv,\bigcap}$ for all $Q \subseteq \mathrm{Rel}(A)$.

We collect some easy properties of the invariant operations.

Lemma 2.13.

(1) *Every logical operation is invariant. In particular the projection operations are logical operations and hence invariant. If A is finite, then every invariant operation is logical.*
(2) *The invariant operations form a clone, i.e. they contain the projection operations, and the superposition of invariant operations is again an invariant operation.*
(3) *If F is invariant and $\varrho_i \in \mathrm{Rel}(A)$ $(1 \leqslant i \leqslant n)$, then*
$$\mathsf{Aut}\{\varrho_1, \ldots, \varrho_n\} \subseteq \mathsf{Aut}\{F(\varrho_1, \ldots, \varrho_n)\}.$$
(4) $R \subseteq \langle R\rangle_{inv} \subseteq \langle R\rangle_{inv,\bigcap} \subseteq \mathsf{sInv\,Aut}\,R$ *for all R. So, if $R = \mathsf{sInv\,Aut}\,R$, then R is $\bigcap$–closed and closed under all invariant operations, $R = \langle R\rangle_{inv,\bigcap}$.*

Proof. For (1) and (2) we refer to [6]. (3) is a direct consequence of 2.12 and (4) is a direct consequence of (3). □

The next lemma shows that invariant operations are sufficient, if we have only finitely many relations.

Lemma 2.14. *Let $Q \subseteq \mathrm{Rel}(A)$ be a finite set. Then $\mathsf{sInv\,Aut}\,Q = \langle Q\rangle_{inv}$.*

Proof. Let $Q = \{\varrho_1, \ldots, \varrho_n\}$ and let $\sigma \in \mathsf{sInv\,Aut}\,R$. We define an invariant operation F with $F(\varrho_1, \ldots, \varrho_n) = \sigma$:

$$F(\sigma_1, \ldots, \sigma_n) := \bigcup\{g[\sigma] \mid g \in \mathrm{Sym}(A) \text{ and } (\forall i = 1 \ldots n) g[\varrho_i] = \sigma_i\}$$

It is easy to verify that F has the desired properties. □

Let $\mathcal{H} : \mathscr{P}Z \to \mathscr{P}Z$ be a closure operator on a set Z. The *algebraic part of $\mathcal{H}$* is the closure operator

$$\mathcal{H}^{alg} : \mathscr{P}Z \to \mathscr{P}Z,\ X \mapsto \bigcup\{\mathcal{H}X_0 \mid X_0 \subseteq X,\ X_0 \text{ finite}\}.$$

$\mathcal{H}$ is *algebraic* if $\mathcal{H} = \mathcal{H}^{alg}$.

Lemma 2.14 shows that the closure operator $Q \mapsto \langle Q \rangle_{inv}$ is the *algebraic part* of $\mathsf{sInv}\,\mathsf{Aut}$. More exactly, for all $Q \subseteq \mathrm{Rel}(A)$ we have

$$\langle Q \rangle_{inv} = \bigcup \{\mathsf{sInv}\,\mathsf{Aut}\, Q_0 \mid Q_0 \subseteq Q \text{ and } Q_0 \text{ is finite}\} = (\mathsf{sInv}\,\mathsf{Aut})^{alg} Q$$

So far, we have the closure operators $\langle \_ \rangle_{KA}$, $\langle \_ \rangle_{inv}$, $\langle \_ \rangle_{KA,\bigcap}$, $\langle \_ \rangle_{inv,\bigcap}$ and $\mathsf{sInv}\,\mathsf{Aut}$. The operators $\langle \_ \rangle_{KA}$ and $\langle \_ \rangle_{inv}$ are algebraic, the other operators are not algebraic if A is infinite. For all $Q \subseteq \mathrm{Rel}(A)$ we have

$$\langle Q \rangle_{KA} \subseteq \begin{matrix} \langle Q \rangle_{inv} \\ \langle Q \rangle_{KA,\bigcap} \end{matrix} \subseteq \langle Q \rangle_{inv,\bigcap} \subseteq \mathsf{sInv}\,\mathsf{Aut}\, Q.$$

For finite base set A all these closure operators coincide. For countable A, the operators $\langle \_ \rangle_{KA}$, $\langle \_ \rangle_{inv}$ and $\langle \_ \rangle_{KA,\bigcap}$ are pairwise distinct – see e.g. [10, Theorem 4], and $\langle \_ \rangle_{KA,\bigcap} = \langle \_ \rangle_{inv,\bigcap} = \mathsf{sInv}\,\mathsf{Aut}$. If A is uncountable, then also $\langle \_ \rangle_{KA,\bigcap} \neq \langle \_ \rangle_{inv,\bigcap}$, as a consequence of the examples in 2.11.

All these properties now lead to the *conjecture*, that the Galois closed sets of relations are exactly the sets of relations that are $\bigcap$–closed and closed under all invariant operations. In order to verify this conjecture, we have to answer one question:

Does there exist a set R of relations that is $\bigcap$–closed and closed under all invariant operations, but is not Galois closed?

This question was formulated as an open problem e.g. in [3, Problem 2.5.2].

Surprisingly, the question has a positive answer and therefore the conjecture above is false. In the next section we will give a model theoretic construction of a relation set R with the mentioned properties.

3. A model theoretic construction

In this section we consider relational models of the form $\underline{M} = (M; (\varrho_m)_{1 \leqslant m \in \omega})$, where $\varrho_m \in \mathrm{Rel}^{(m)}(A)$ for all m. Thus our language $\mathcal{L}$ has exactly one relation symbol for every arity m. (We use the predicate symbols also to denote the corresponding relations.)

If $\underline{M} = (M; (\varrho_m)_{1 \leqslant m \in \omega})$ is such a model, then $\underline{M}^{[m]} := (M; \varrho_1, \ldots, \varrho_m)$ denotes the reduct of $\underline{M}$ to the relations $\varrho_1, \ldots, \varrho_m$.

Our construction is guided by the following Lemma.

Lemma 3.1. *We fix a vocabulary of infinitely many relational symbols $\{\varrho_m \mid m \in \omega\}$.*

Let $\underline{A} = (A\,; (\varrho_m)_{1 \leqslant m \in \omega})$ be an infinite model. Let $\underline{A}^{[m]} = (A; \varrho_1, \ldots, \varrho_m)$.

We assume that the following hold:

1. *The theory $Th(\underline{A})$ is ω-categorical (i.e., has up to isomorphism exactly one countable model).*
2. *For all m, the reduct $\underline{A}^{[m]}$ is homogeneous in the sense of 2.6.*
3. *$\underline{A}$ is rigid, i.e. $\mathsf{Aut}\,\underline{A} = \{\mathrm{id}_A\}$.*

Then, letting $R := \langle \varrho_1, \varrho_2, \dots \rangle_{KA}$ be the set of first order definable relations in $\underline{A}$, we have:

$$\langle \varrho_1, \dots, \varrho_m \rangle_{KA} = \mathsf{sInv}\ \mathsf{Aut}\{\varrho_1, \dots, \varrho_m\}$$

and $R = \bigcup_m \langle \varrho_1, \dots, \varrho_m \rangle_{KA}$, *but*

$$R = \langle R \rangle_{inv,\bigcap} \subsetneq \mathsf{sInv}\ \mathsf{Aut}\, R$$

Proof. First note that R (as well as $\langle \varrho_1, \varrho_2, \dots \varrho_m \rangle_{KA}$) is closed under arbitrary intersections, since (by Ryll-Nardzewski's theorem) there are only finitely many k-ary relations in R, for any k.

We now show that R is also closed under all invariant operations. We have

$$\langle \varrho_1, \varrho_2, \dots, \varrho_m \rangle_{KA} = \langle \varrho_1, \varrho_2, \dots, \varrho_m \rangle_{KA,\bigcap}$$

Trivially,

$$\langle \varrho_1, \varrho_2, \dots, \varrho_m \rangle_{KA} \subseteq \langle \varrho_1, \varrho_2, \dots, \varrho_m \rangle_{inv} \subseteq \mathsf{sInv}\ \mathsf{Aut}\{\varrho_1, \varrho_2, \dots, \varrho_m\},$$

but by (2) and Lemma 2.7 we have

$$\langle \varrho_1, \varrho_2, \dots, \varrho_m \rangle_{KA,\bigcap} = \mathsf{sInv}\ \mathsf{Aut}\{\varrho_1, \varrho_2, \dots, \varrho_m\},$$

so

$$\langle \varrho_1, \varrho_2, \dots, \varrho_m \rangle_{KA} = \langle \varrho_1, \varrho_2, \dots, \varrho_m \rangle_{inv}.$$

As both operators $\langle \_ \rangle_{KA}$ and $\langle \_ \rangle_{inv}$ are algebraic, this yields

$$R = \langle \varrho_1, \varrho_2, \dots \rangle_{KA} = \langle \varrho_1, \varrho_2, \dots \rangle_{inv}$$

hence $R = \langle R \rangle_{inv,\bigcap}$.

Clearly R is countable. As $\mathsf{Aut}\, R = \{\mathrm{id}_A\}$ and so $\mathsf{sInv}\,\mathsf{Aut}\, R = \mathrm{Rel}(A)$, which is an uncountable set. Consequently

$$\langle R \rangle_{inv,\bigcap} \neq \mathsf{sInv}\,\mathsf{Aut}\, R. \qquad \square$$

It remains to prove that a model with properties (1)–(3) exists. Because of Theorem 2.10, such a model cannot be countable. We start by defining the logical theory $\mathcal{T}$ that we want our model to satisfy. First we define an appropriate notion of a *clause*.

Definition 3.2. A *literal in the variables* $x_0, \dots, x_n$ $(n \in \omega)$ is a formula of the form

$$\varrho_m(x_{i_1}, \dots, x_{i_m}) \text{ (unnegated) or } \neg\varrho_m(x_{i_1}, \dots, x_{i_m}) \text{ (negated)}$$

such that $1 \leqslant m \leqslant n+1$, $\{i_1, \dots, i_m\} \subseteq \{0, \dots, n\}$, the $i_1, \dots, i_m$ are pairwise distinct and $0 \in \{i_1, \dots, i_m\}$.

A *clause in* $x_0, \dots, x_n$ is a conjunction K of literals in $x_0, \dots, x_n$, such that no literal will appear twice, and no literal appears in negated and unnegated form.

Please note that there are only finitely many clauses in $x_0, \dots, x_n$. The variable x_0 plays a special role — it has to appear in every literal.

Now we formulate our theory $\mathcal{T}$:

Definition 3.3. $\mathcal{T}$ consists of (the universal closures of) the following formulas: Firstly, for all $1 \leqslant m \in \omega$ we have:

(T1) $\varrho_m(x_1, \ldots, x_m) \to \bigwedge_{1\leqslant i<j\leqslant m} x_i \neq x_j$

Secondly, for all $n \in \omega$ and all clauses $K = K(x_0, \ldots, x_n)$ in $x_0, \ldots, x_n$ we take the formula:

(T2) $\bigwedge_{1\leqslant i<j\leqslant n} x_i \neq x_j \to (\exists x_0)\, K(x_0, \ldots, x_n)$

Informal Discussion 3.4. *We will see below that the theory $\mathcal{T}$ is complete and ω-categorical. Our aim is to construct an uncountable model $\underline{M}$ of this theory on the base set ω_1 in which the well-order $(\omega_1, <)$ is definable (by a formula in higher order logic). To help us achieve this aim, we use the following "recommendation":*

$\varrho(x_1, x_2, \ldots, x_n)$ **should** *hold iff* $x_1 < x_2 < \cdots < x_n$

However, this is just a recommendation, not a law. In order to also get homogeneity of the restricted models $\underline{M}^{[m]}$, we allow our model to disobey this recommendation, if there is a good reason for it. A good reason can be the desire to satisfy an axiom of our theory, or to extend a partial automorphism.

To keep track of the cases where the recommendation is not followed, we construct an auxiliary function $h : M \to \omega$, and we will demand the following "law", which is a relaxed version of the "recommendation":

For all sufficiently long tuples $(x_1, \ldots, x_n)$:
$\varrho(x_1, x_2, \ldots, x_n)$ **must hold** *if $x_1 < x_2 < \cdots < x_n$, and* **must not hold** *otherwise*

Here, "sufficiently long" is defined as: $n > \max(h(x_1), \ldots, h(x_n))$.

Thus, whenever we violate our recommendation at a tuple $(x_1, \ldots, x_n)$, we will define a sufficiently large value of h at one of the points $x_1, \ldots, x_n$.

Before we investigate the theory $\mathcal{T}$, we want to examine some technical definitions and lemmas. ω_1 denotes the first uncountable ordinal, $\omega_1 = \{\alpha \mid \alpha < \omega_1\}$. ω_1 is well-ordered by $<$. The universes of all our models will be subsets of ω_1.

Definition 3.5. Let $\underline{M} = (M; (\varrho_m)_{1\leqslant m\in\omega})$ be a model, let $h : M \circ\!\!\to \omega$ be a partial function, $n \in \omega \setminus \{0\}$ and let $\underline{a} = (a_1, \ldots, a_n) \in M^n$. We say that $\underline{a}$ is *a weak n-tuple for h* if $\operatorname{dom} h \cap \{a_1, \ldots, a_n\} \neq \emptyset$ and

$$\max h(\underline{a}) := \max\{h(a_i) \mid a_i \in \operatorname{dom} h\} < n.$$

All other n-tuples are called *strong for h.*

Now let $\underline{N} = (N; (\sigma_m)_{1\leqslant m\in\omega})$ and $\underline{M} = (M; (\varrho_m)_{1\leqslant m\in\omega})$ be models with $N \subset M \subseteq \omega_1$. Let $h : M \circ\!\!\to \omega$ be a partial function with $\operatorname{dom} h = M \setminus N$. We write $\underline{N} \sqsubset_h \underline{M}$ if

(1) $\underline{N} \leqslant \underline{M}$ ($\underline{N}$ is a submodel of $\underline{M}$), and
(2) for all n-tuples $(a_1, a_2, \ldots, a_n) \in M^n$ that are weak for h the following condition holds:

$$\varrho_n(a_1, \ldots, a_n) \iff a_1 < a_2 < \ldots < a_n$$

(Here $<$ is the well-ordering on ω_1.)

We write $\underline{N} \sqsubset \underline{M}$ if $\underline{N} \sqsubset_h \underline{M}$ for some partial function h with $\operatorname{dom} h = M \setminus N$.

Lemma 3.6.

(1) *The relation $\sqsubset$ is transitive.*
(2) *If $\underline{M_0} \sqsubset \underline{M_1} \sqsubset \cdots$ is a chain of length ω, and $\underline{M_\omega}$ is the directed union of this chain (i.e., $M_\omega = \bigcup_{n\in\omega} M_n$, and each $\underline{M}_n$ is also a submodel of $\underline{M_\omega}$), then $\underline{M_j} \sqsubset \underline{M_\omega}$ for all $j < \omega$.*
(3) *Similarly, if $(\underline{M_i})_{i\leq\alpha}$ (where α is a limit ordinal) is a continuous chain of models (i.e., $M_i \leq M_j$ for all $i \leq j \leq \alpha$, and for each limit $\delta \leq \alpha$ we have $M_\delta = \bigcup_{i<\delta} M_i$), and*

$$\forall i < \alpha : \ \underline{M_i} \sqsubset \underline{M_{i+1}},$$

then $\underline{M_j} \sqsubset \underline{M_\alpha}$ for all $j < \alpha$.

Proof. We prove (1): Assuming $\underline{M_1} \sqsubset_{h_1} \underline{M_2}$ and $\underline{M_2} \sqsubset_{h_2} \underline{M_3}$ for some $h_1 : M_2 \setminus M_1 \to \omega$ and $h_2 : M_3 \setminus M_2 \to \omega$, we have to show $\underline{M_1} \sqsubset \underline{M_3}$.
Put $h := h_1 \cup h_2 : M_3 \setminus M_1 \to \omega$. If $\underline{a} \in (M_3^n \setminus M_1^n)$ is weak for h, then either $\underline{a} \in (M_2^n \setminus M_1^n)$ or one of the a_i belongs to $M_3 \setminus M_2$. In the first case, $\max h_1(\underline{a}) = \max h(\underline{a}) < n$, and $\underline{a}$ is weak for h_1. Therefore ($\underline{M_2}$ is a submodel of $\underline{M_3}$),

$$\varrho_n^3(\underline{a}) \iff \varrho_n^2(\underline{a}) \iff a_1 < a_2 < \ldots < a_n.$$

In the second case, $\max h_2(\underline{a}) \leqslant \max h(\underline{a}) < n$, therefore $\underline{a}$ is weak for h_2 and $\varrho_n^3(\underline{a}) \iff a_1 < \ldots < a_n$.
The proofs of (2) and (3) are similar. □

The next technical lemma provides the basic step in our construction.

Lemma 3.7. *Let $\underline{M_0} = (M_0; (\varrho_m^0)_{1\leqslant m\in\omega})$ be a countable model of our language $\mathcal{L}$, such that $M_0 \subseteq \omega_1$ and $\varrho_m(x_1, \ldots, x_m) \to \bigwedge_{1\leqslant i<j\leqslant m} x_i \neq x_j$ holds for all m. Moreover, let $\pi_0 : M_0 \circ\!\!\to M_0$ be a partial (finite or infinite) automorphism of the reduct $\underline{M_0}^{[s]}$ for some $s \in \omega$ and let $\alpha \in \omega$.*

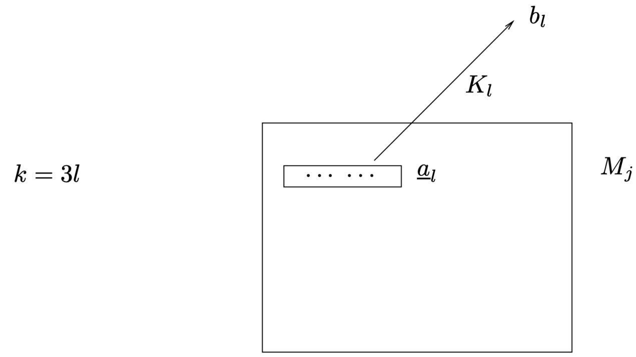
Then there exists a countable model $\underline{M_\omega} = (M_\omega; (\varrho_m^\omega)_{1\leqslant m\in\omega})$ with $M_0 \subset M_\omega \subset \omega_1$, $\alpha \in M_\omega$ and a total automorphism $\pi_\omega : M_\omega \circ\!\!\to M_\omega$ of the s–reduct $\underline{M_\omega}^{[s]}$ such that the following hold:

(1) $\underline{M_0} \sqsubset \underline{M_\omega}$
(2) *$\underline{M_\omega}$ is a model of the theory $\mathcal{T}$.*
(3) *π_ω extends π_0*

Proof. We construct $\underline{M_\omega}$ as the union of a chain of countable models $\underline{M_j}$ with $j \in \omega$. We will have $\underline{M_j} \sqsubset \underline{M_{j+1}}$ for all $j \in \omega$, and for every j we will have a partial automorphism π_j of $\underline{M_j}^{[s]}$ such that π_{j+1} extends π_j, and $\operatorname{dom}(\pi_{j+1}) \cap \operatorname{im}(\pi_{j+1}) \supseteq M_j$.

We explain the step from M_j to M_{j+1}. Let $M_j = \{a_i \mid i \in \omega\}$ be an enumeration of the elements of M_j. (The elements a_i are not necessarily in the order, given by $<$ in ω_1.) Let $\mathcal{K}$ be the following set:

$$\mathcal{K} := \{(n, \underline{a}, K) \mid n \in \omega, \underline{a} \in M_j^n \text{ and } K \text{ a clause in } x_0, \ldots, x_n\}$$



This set is countable. Let $(n_l, \underline{a}_l, K_l)_{l\in\omega}$ be an enumeration of this set.

Let $B \subseteq \omega_1 \setminus M_j$ be a countable set of ordinals, and let $B = \{b_k \mid k \in \omega\}$ be an enumeration of B. (Again, this enumeration need not necessarily follow the well-order $<$ on ω_1.) We put $M_{j+1} := M_j \cup B$, and we have to define the relations ϱ_m^{j+1} and the partial function π_{j+1}. Moreover, we must define a function $h : B \to \omega$, in order to establish the relation $\underline{M_j} \sqsubset_h \underline{M_{j+1}}$.

For all m and all m-tuples $\underline{a} \in M_j^m$ we define:

$$\varrho_m^{j+1}(\underline{a}) :\Longleftrightarrow \varrho_m^j(\underline{a})$$

This makes sure that $\underline{M_j} \leqslant \underline{M_{j+1}}$. Moreover, we define $\neg\varrho_m^{j+1}(c_1, \dots, c_m)$ for all $c_1, \dots, c_m \in M_{j+1}$ with $|\{c_1, \dots, c_m\}| < m$.

For every $k \in \omega$ we will conduct a special task, where we define the value $h(b_k)$, define a partial function $p_k : M_{j+1} \circ\!\!\to M_{j+1}$ such that p_{k+1} always extends p_k. We start with $p_0 := \pi_j$, and finally we will have $\pi_{j+1} := \bigcup_k p_k$. Moreover, in every step we define the truth values of $\varrho_m^{j+1}(\underline{a})$ for some tuples $\underline{a}$.

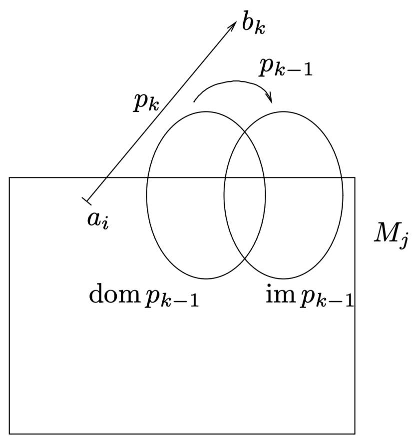
We distinguish three cases of steps k, depending on whether $k \equiv 0, 1$, or 2 mod (3). [A main point will be that the definitions in the various cases do not contradict each other.]

Step k for $k = 3l$: If $l = 0$, then let $p_k := \pi_j$, otherwise $p_k := p_{k-1}$ remains unchanged.

Let $(n_l, \underline{a}_l, K_l)$ be the element of $\mathcal{K}$ with index l. We define $h(b_k) := n_l + 1$. Now, for every literal which occurs in K, we define the truth values in such a way, that $K_l(b_k, \underline{a}_l)$ becomes true. Thus, letting $\underline{a}_l = (a_l(1), \dots, a_l(n_l))$, we define truth values for certain tuples from the set $\{a_l(1), \dots, a_l(n_l), b_k\}^{<\omega} \setminus \{a_l(1), \dots, a_l(n_l)\}^{<\omega}$.

The largest index m of a literal which occurs in K_l is $n_l + 1$. Therefore all these tuples $\underline{c}$ satisfy $\max h(\underline{c}) \geq h(b_k) \geq m$ and are strong for h. [Note that in no previous step have we committed ourselves to the truth value of $\varrho_j(\underline{c})$ for any tuple $\underline{c}$ in which b_k appears.]

$k = 3l + 1$



Step k for $k = 3l+1$: In this case we define $h(b_k) := s$, and we extend the partial function p_{k-1}. If $\operatorname{dom} p_{k-1} \supseteq M_j$, then we simply put $p_k := p_{k-1}$ and we are done.

If $M_j \setminus \operatorname{dom} p_{k-1} \neq \emptyset$, then let $i := \min\{i \mid a_i \notin \operatorname{dom} p_{k-1}\}$. We extend p_{k-1} by defining $p_k(a_i) := b_k$. Then, for all $m \leqslant s$ and all $c_1, \ldots, c_m \in \operatorname{im} p_k$ such that $c_1, \ldots, c_m$ are pairwise distinct and $b_k \in \{c_1, \ldots, c_m\}$, we define the truth value of $\varrho_m^{j+1}(c_1, \ldots, c_m)$. Write $\underline{c}$ for $(c_1, \ldots, c_m)$. If the value of $\varrho^{j+1}(p_k^{-1}(\underline{c}))$ is already known, in particular if $p_k^{-1}(\underline{c}) \in M_j^m$, then we put

$$\varrho_m^{j+1}(\underline{c}) :\Longleftrightarrow \varrho^{j+1}(p_k^{-1}(\underline{c}).$$

If $\varrho^{j+1}(p_k^{-1}(\underline{c}))$ is still not fixed (in particular, then the $p_k^{-1}(c_i)$ cannot all be in M_j), then we define both values, namely we put

$$\varrho_m^{j+1}(\underline{c}), \varrho_m^{j+1}(p_k^{-1}(\underline{c})) :\Longleftrightarrow p_k^{-1}(c_1) < \ldots < p_k^{-1}(c_m).$$

(Here the relation symbol $<$ denotes the well-order of ω_1.)

Because of the definition of $h(b_k)$, the $(c_1, \ldots, c_m)$ are always strong for h and hence are exempted from our "recommendation" 3.4. The other tuples, $p_k^{-1}(\underline{c})$, follow our recommendation anyway, whether or not they are h-strong.

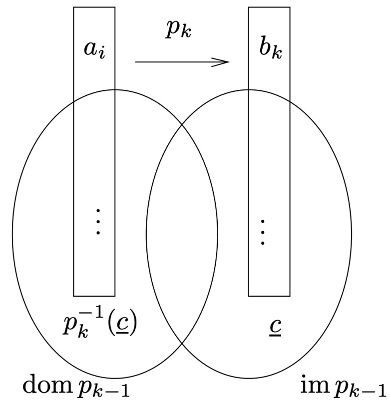
[As before, note that tuples $\underline{c}$ in which b_k appears have never been considered in any previous step $k' < k$.]

Step k for $k = 3l + 2$: This step is similar to the previous step, but this time we take care of $\operatorname{im} p_k$ rather than $\operatorname{dom} p_k$, or in other words: we reverse the roles of p_k and p_k^{-1}. We leave the details to the reader.

By induction we obtain from these steps a model, where the relations ϱ_m^{j+1} are only partially defined. In order to finish the definition, we put

$$\varrho_m^{j+1}(\underline{c}) :\Longleftrightarrow c_1 < c_2 < \ldots < c_m,$$

$k = 3l + 1$,
definition of
$\varrho(\underline{c})$ and $\varrho(p_k^{-1}(\underline{c}))$



whenever the truth value of $\varrho_m^{j+1}(\underline{c})$ has not been defined during the inductive construction.

From the construction, it is now clear that $\underline{M_j} \sqsubset \underline{M_{j+1}}$. Moreover, $\pi_{j+1} := \bigcup_{k\in\omega} p_k$ is a partial automorphism of $\underline{M_{j+1}}^{[s]}$ that extends π_i and satisfies $M_j \subseteq \operatorname{dom}\pi_{j+1}$ and $M_j \subseteq \operatorname{im}\pi_{j+1}$. Moreover, all formulas in $\mathcal{T}$ of the form (T2) in Definition 3.3 are satisfied, whenever $x_1, \dots, x_m \in M_j$.

The countable set $B \subset \omega_1 \setminus M_j$ was arbitrary. So, if the ordinal α is not in M_0, then we can assume that $\alpha \in B$, e.g. for $j = 0$. Consequently we will have $\alpha \in M_\omega$ in the end.

We form the directed union $\underline{M_\omega} := \bigcup_{j\in\omega} \underline{M_j}$ Because of Lemma 3.6(2), we have $\underline{M_0} \sqsubset \underline{M_\omega}$. Moreover, the union $\pi_\omega := \bigcup_{j\in\omega} \pi_j$ is a bijective partial automorphism of $\underline{M_\omega}^{[s]}$, which is everywhere defined and surjective, i.e. it is an automorphism of $\underline{M_\omega}^{[s]}$ which extends π_0.

Finally, if $\psi(x_1, \dots, x_n)$ is a formula of $\mathcal{T}$ and $a_1, \dots, a_n \in M_\omega$, then there exists j with $a_1, \dots, a_n \in M_j$. Consequently $\psi(a_1, \dots, a_n)$ holds in $\underline{M_{j+1}}$ and all other extensions of $\underline{M_j}$, in particular it is true in $\underline{M_\omega}$. Consequently $\underline{M_\omega}$ is a model of $\mathcal{T}$. This finishes the proof.

□

Now we collect some properties of our theory $\mathcal{T}$.

Lemma 3.8. (1) *$\mathcal{T}$ is consistent and has no finite models.*
(2) *$\mathcal{T}$ has the property of elimination of quantifiers. (I.e. every formula is (modulo $Th(\underline{A})$) equivalent to a quantifier free formula.)*
(3) *$\mathcal{T}$ is complete.*
(4) *$\mathcal{T}$ is ω–categorical.*
(5) *If $\underline{M}, \underline{N}$ are models of $\mathcal{T}$ and $\underline{N}$ is a submodel of $\underline{M}$, $\underline{N} \leqslant \underline{M}$, then $\underline{N}$ is an elementary submodel of $\underline{M}$, i.e. for every formula $\varphi(x_1, \dots, x_n)$ and for every $\underline{a} \in N^n$ we have that $\varphi(\underline{a})$ holds in $\underline{N}$ iff it holds in $\underline{M}$.*

Proof. (1) Easy. The consistency of $\mathcal{T}$ is a corollary of Lemma 3.7, and the nonexistence of finite models is ensured by the formulas (T2) in 3.3.

(2) In order to prove that for every formula $\psi(x_1,\dots,x_n)$ there is a quantifier free formula $\varphi(x_1,\dots,x_n)$ with $\mathcal{T}\models(\varphi\leftrightarrow\psi)$, it is sufficient to prove this for formulas ψ of the form $(\exists x_0)K(x_0,x_1,\dots,x_n)$, where K is a clause in $x_0,\dots,x_n$.

For any equivalence relation θ on $\{1,\dots,n\}$ we put:

$$\mu_\theta := (\bigwedge_{(i,j)\in\theta} x_i = x_j)\wedge(\bigwedge_{(i,j)\notin\theta} x_i\neq x_j).$$

Let E_n denote the set of all equivalence relations on $\{1,\dots,n\}$. Then

$$K\leftrightarrow\bigvee_{\theta\in E_n}(K\wedge\mu_\theta).$$

Therefore also

$$(\exists x_0)K(x_0,x_1,\dots,x_n)\leftrightarrow\bigvee_{\theta\in E_n}(\exists x_0)(K\wedge\mu_\theta).$$

It is sufficient to show that every formula $(\exists x_0)(K\wedge\mu_\theta)$ is equivalent to a quantifier free formula. If θ is not the equality relation, then we can replace any variable x_j ($j\geq 1$) by x_i, where i is a representative of the equivalence class of j. If then a variable appears twice in a literal of K, then either the clause becomes false modulo $\mathcal{T}$ (if the literal is unnegated), or the literal can be omitted modulo $\mathcal{T}$ (if it is negated). In the end we obtain either formulas which are true (modulo $\mathcal{T}$) or false (modulo $\mathcal{T}$) or equivalent to a formula of the form

$$(\exists x_0)(K\wedge\bigwedge_{1\leqslant i<j\leqslant n} x_i\neq x_j).$$

$\mathcal{T}$ contains the formula $\bigwedge_{1\leqslant i<j\leqslant n} x_i\neq x_j\to(\exists x_0)K$, therefore

$$\mathcal{T}\models(\bigwedge_{1\leqslant i<j\leqslant n} x_i\neq x_j\leftrightarrow(\exists x_0)(K\wedge\bigwedge_{1\leqslant i<j\leqslant n} x_i\neq x_j).$$

(3) By (2), every closed formula is (mod $\mathcal{T}$) equivalent to **true** or **false**.

(4) Modulo the theory $\mathcal{T}$, there are only finitely many quantifier-free formulas in the variables $x_1,\dots,x_n$, namely, Boolean combinations of atomic formulas $\varrho_m(x_{i_1},\dots,x_{i_m})$, for $i_1,\dots,i_m\in\{1,\dots,n\}$ and $m\leq n$. (Note that formulas $\varrho_m(x_{i_1},\dots,x_{i_m})$ for $m>n$ and $i_1,\dots,i_m\leq n$ are automatically false mod $\mathcal{T}$, because of (T1) in Definition 3.3.
This implies ω-categoricity, by Ryll-Nardzewski's theorem. [Actually, we do not need ω-categoricity itself for our construction, we only need the fact that there are only finitely many first order definable k-ary relations, for any k.]

(5) This is a consequence from the fact that $\mathcal{T}$ has elimination of quantifiers. □

The results in Lemma 3.8 make sure that the condition 3.1(1) is satisfied for every model of $\mathcal{T}$. Now we construct a model $\underline{A}$, such that also the conditions 3.1(2) and (3) are satisfied.

We will obtain $\underline{A}$ as a directed union over an uncountable chain of models, $\underline{A} := \bigcup_{i\in\omega_1} \underline{M_i}$, such that every $\underline{M_i}$ is a model of $\mathcal{T}$, $\underline{M_i} \sqsubset \underline{M_{i+1}}$ and $i \in M_{i+1} \subset \omega_1$ for all $i \in \omega_1$. Because of Lemma 3.8(4), this is an elementary chain, therefore $\underline{A}$ is again a model of $\mathcal{T}$. Because of $i \in M_{i+1}$ for all $i \in \omega_1$ and $M_i \subseteq \omega_1$, the carrier set $A = \bigcup_{i\in\omega_1} M_i$ is $A = \omega_1$.

In order to obtain a model with homogeneous s–reducts, we have to make sure that certain partial automorphisms can be extended to automorphisms. For this reason, we use a triply-indexed family $(\pi_{n,i,j})_{n\in\omega,i,j\in\omega_1,i\leqslant j}$ of partial automorphisms.

First we explain, what the $\pi_{n,i,i}$ are. If $\underline{M}$ is a countable model, then there are only countable many pairs (p,s) with $s \in \omega$ and p a finite partial automorphism of the s–reduct $\underline{M}^{[s]}$. Therefore there exists an enumeration $(p_n, s_n)_{n\in\omega}$ of all these finite partial automorphisms with corresponding s. Now, for $i \in \omega$ and $\underline{M} = \underline{M_i}$ we put $\pi_{n,i,i} := p_n$, and $s_{n,i} := s_n$. Therefore

(**) $(\pi_{n,i,i})_{n\in\omega}$ is a list of all finite partial automorphisms of all possible reducts $\underline{M_i}^{[s]}$.

The $\pi_{n,i,j}$ with $i < j$ will be extensions of $\pi_{n,i,i}$.

Now we explain how to construct the models $\underline{M_j}$ and sequences of partial automorphisms $(\pi_{n,i,j} : i \leq j)$ by transfinite induction on $j \in \omega_1$). This construction will use the usual 'bookkeeping–argument' to take care of $\omega_1 \times \omega$ many tasks in ω_1 steps. Let $\omega_1 = \bigcup_{n\in\omega,i\in\omega_1} C_{n,i}$ be a partition of ω_1 into pairwise disjoint sets $C_{n,i}$, such that $|C_{n,i}| = \omega_1$ for all (n,i) and $\min C_{n,i} \geq i$.

If $j = 0$, then let $\underline{M_0}$ be a countable model of $\mathcal{T}$ with $M_0 \subseteq \omega_1$. (The existence of such a model is clear from Lemma 3.7.) The $\pi_{n,0,0}$ are defined as in (**).

If j is a limit ordinal, then put $\underline{M_j} := \bigcup_{i<j} \underline{M_i}$. (As a directed union of an elementary chain of models of $\mathcal{T}$, this is again a model of $\mathcal{T}$.) The $\pi_{n,j,j}$ are defined as in (**), and $\pi_{n,i,j} := \bigcup_{i\leqslant l<j} \pi_{n,i,l}$.

For a successor ordinal $j+1$ we use Lemma 3.7:

(1) We define $\underline{M}_{j+1}$ as follows. Let (n,i) be the pair with $j \in C_{n,i}$. According to the Lemma, there exists a model $\underline{M_{j+1}}$ with $j \in M_{j+1} \subset \omega_1$ and $\underline{M_j} \sqsubset \underline{M_{j+1}}$, and there exists an extension of $\pi_{n,i,j}$ to a partial automorphism $\bar{\pi}$ of $\underline{M_{j+1}}^{[s_{n,i}]}$ with $M_j \subseteq \operatorname{dom} \bar{\pi}$ and $M_j \subseteq \operatorname{im} \bar{\pi}$.

(2) We let $\pi_{n,i,j+1}$ be the partial automorphism $\bar{\pi}$ from (1).

(3) The $\pi_{n,j+1,j+1}$ are defined as in (**), enumerating all finite partial automorphisms of reducts of $\underline{M_{j+1}}$.

(4) For all (n,i) such that $j \notin C_{n,i}$, we put $\pi_{n,i,j+1} := \pi_{n,i,j}$.

It is easy to verify by transfinite induction, that the $\pi_{(n,i,j)}$ are always partial automorphisms of $\underline{M_j}^{[s_{n,i}]}$, and that $\pi_{n,i,j}$ extends $\pi_{n,i,k}$ for all k with $i \leqslant k < j$.

As mentioned above, we put $\underline{A} := \bigcup_{i\in\omega_1} \underline{M_i}$.

Lemma 3.9. *If $s \in \omega$, then every finite partial automorphism π of $\underline{A}^{[s]}$ can be extended to an automorphism of $\underline{A}^{[s]}$.*

Proof. The function π is finite, therefore there exists $i \in \omega$ with $\operatorname{dom} \pi \cup \operatorname{im} \pi \subseteq M_i$. $\underline{M_i}$ is a submodel of $\underline{A}$, therefore π is a finite partial automorphism of $\underline{M_i}^{[s]}$. Consequently, $\pi = \pi_{n,i,i}$ for some n with $s = s_{n,i}$. We define $\pi' := \bigcup\{\pi_{n,i,j} : j \in \omega_1, j \geq i\}$. The function π' is a partial isomorphism of $\underline{A}^{[s]}$. By our construction, we have $M_j \subseteq \operatorname{dom} \pi'$ for all $j \in C_{n,i}$, i.e. $\operatorname{dom} \pi \supseteq \bigcup_{j \in C_{n,i}} M_j = \omega_1$. The same holds for $\operatorname{im} \pi'$, therefore π' is a total automorphism of $\underline{A}^{[s]}$. □

It remains to show that $\underline{A}$ has the property (3) in Lemma 3.1.

Lemma 3.10. $\underline{A} = (A; (\varrho_m)_{m \in \omega \setminus \{0\}})$ *is rigid, i.e.* $\mathsf{Aut}\,\underline{A} = \{\mathrm{id}_A\}$.

Proof. Let S be a countable subset of A, $x, y \in S$ and $h : A \setminus S \to \omega$ be a function. Then we define that $E(x, y, S, h)$ is true iff for all m and for all $\underline{a} = (a_1, \ldots, a_m) \in A^m \setminus S^m$ the following holds:

If $(\varrho_m(\underline{a}) \wedge \max h(\underline{a}) < m \wedge (\exists i, j \in \{1, \ldots, m\})(x = a_i \wedge y = a_j))$,
then $i < j$

We claim $x < y \iff (\exists S)(\exists h)E(x, y, S, h)$.

- Proof of "⇒":] Let $i \in \omega_1$ be the least ordinal with $x, y \in M_i$. Let $S := M_i$. We have $\underline{M_i} \sqsubset \underline{A}$, therefore $\underline{M_i} \sqsubset_h \underline{A}$ for some $h : A \setminus S \to \omega$. If $\underline{a} \in A^m \setminus S^m$, is weak for h, then $\varrho_m(\underline{a}) \iff a_1 < a_2 < \ldots < a_m$. Therefore, if $x = a_i$, $y = a_j$ and $x < y$, then $i < j$.
- Proof of "⇐": Let S be a countable subset of A and let $h : A \setminus S \to \omega$ be a function such that $E(x, y, S, h)$.

  Since ω_1 has uncountable cofinality, and S is countable, there must be some $i < \omega_1$ with $S \subseteq M_i$.

  Let i be the least ordinal with $S \subseteq M_i$. Let $p \in A \setminus M_i$. (Then also $p \in A \setminus S$.) We have $\underline{M_i} \sqsubset_{h_i} \underline{A}$ for some $h_i : A \setminus M_i \to \omega$. Let $m := \max\{h(p), h_i(p)\} + 3$ and choose $z_1, \ldots, z_{m-3} \in S$, pairwise distinct and distinct from x and y. Let $\underline{a} = (a_1, \ldots, a_m)$ be the m-tuple consisting of the elements of $\{p, x, y, z_1, \ldots, z_{m-3}\}$ in the ordering according to $<$, i.e. $a_1 < a_2 < \ldots < a_m$.

  Find i, j such that
  $$x = a_i, \; y = a_j.$$
  Thus,
  $$i < j \iff x < y.$$
  First we note that $\max h_i(\underline{a}) < m$, so $\underline{a}$ is weak for h_i. As $\underline{M_i} \sqsubset \underline{M}$, we must have $\varrho_m(\underline{a})$.

  We also have $\max h(\underline{a}) < m$, so $\varrho_m(\underline{a})$ implies $i < j$. So $x < y$.

Let $\bar{E}(x, y) :\iff (\exists S)(\exists h)E(x, y, S, h)$. If π is an automorphism of $\underline{A}$ we write $\pi[h]$ for the map h' satisfying $h'(\pi(x)) = h(x)$. Clearly $E(x, y, S, h) \iff E(\pi(x), \pi(y), \pi[S], \pi[h])$, hence
$$\bar{E}(x, y) \iff \bar{E}(\pi(x), \pi(y)).$$

Consequently every automorphism of $\underline{A}$ has to preserve the well-ordering $<$ of ω_1. But the only order automorphism of $<$ is id_A. $\square$

In Lemmas 3.8, 3.9 and 3.10 we have verified all properties of $\underline{A}$, required in Lemma 3.1. We can now formulate our main result.

Theorem 3.11. *The structure $\underline{A} = (A\,;\,(\varrho_m)_{m\in\omega})$ has the properties 3.1(1)–(3). Consequently, the set $R = \{\varrho_m \mid m \in \omega\}$ satisfies*

$$\langle R\rangle_{KA} = \langle R\rangle_{inv,\bigcap} \neq \mathsf{sInv}\,\mathsf{Aut}\,R.$$

Remark 3.12. The structure $\underline{A}$ that we constructed in 3.2–3.10 has cardinality $\aleph_1$. A similar construction can be carried out to yield a model of any cardinality κ with cofinality $cf(\kappa) \geq \omega_1$. We leave the details to the reader.

4. A characterization with invariant operations of countable arity

The results of the last section show that the closure under $\langle\_\rangle_{inv,\bigcap}$ is too weak to provide the closure under $\mathsf{sInv}\,\mathsf{Aut}$. The question remains, what we should add to obtain an appropriate characterization. Similar as in Theorem 2.9 we will add operations with infinite arity. In contrast to Theorem 2.9, we need only operations with countable arities.

Lemma 4.1. *Let $m \in \omega \setminus \{0\}$ and let $Q \subseteq \mathrm{Rel}^{(m)}(A)$ be closed under complementation. Then there exists a relation $\varrho \in \langle Q\rangle^{(2m)}_{KA,\bigcap}$ with $\mathsf{Aut}\,Q = \mathsf{Aut}\{\varrho\}$.*

Proof. Q is closed under $\mathbf{C}$, therefore the set

$$M := \{\Gamma_Q(\underline{a}) \mid \underline{a} \in A^m\}$$

is a partition of A^m. M can be well-ordered, so let $(\gamma_i)_{i<\kappa}$ be a corresponding enumeration of the elements of M. We put

$$\varrho := \bigcup_{i\leqslant j<\kappa} \gamma_i \times \gamma_j$$

Then $\varrho \in \langle Q\rangle^{(2m)}_{KA,\bigcap}$. This implies $\varrho \in \mathsf{sInv}\,\mathsf{Aut}\,Q$ and therefore $\mathsf{Aut}\{\varrho\} \supseteq \mathsf{Aut}\,Q$.

Now let $g \in \mathsf{Aut}\{\varrho\}$. If $\underline{a}, \underline{b} \in \gamma_i$, then $(\underline{a}, \underline{b}) \in \varrho$ and $(\underline{b}, \underline{a}) \in \varrho$, hence $(g(\underline{a}), g(\underline{b})) \in \varrho$ and $(g(\underline{b}), g(\underline{a})) \in \varrho$. The γ_i are pairwise disjoint, therefore there are unique ordinals $l, k < \kappa$ with $g(\underline{a}) \in \gamma_l$, $g(\underline{b}) \in \gamma_k$. Now $(\underline{a}, \underline{b}) \in \varrho$ implies $l \leqslant k$ and $(\underline{b}, \underline{a}) \in \varrho$ implies $k \leqslant l$, i.e. $l = k$. Consequently all tuples in γ_i are transformed by g to tuples in γ_k. Therefore there exists a function $g_0 : \kappa \to \kappa$ with $g[\gamma_i] \subseteq \gamma_{g_0(i)}$.

g^{-1} is also an automorphism of ϱ, and it is easy to see that the corresponding function $g_0' : \kappa \to \kappa$ has to be the inverse of g_0. Consequently, g_0 is a permutation on κ.

The permutation g_0 preserves the well-order $<$ on κ, because of

$$i < j \Rightarrow \gamma_i \times \gamma_j \subseteq \varrho \Rightarrow \gamma_{g_0(i)} \times \gamma_{g_0(j)} \subseteq \varrho \Rightarrow g_0(i) < g_0(j).$$

The well-order $\langle \kappa, <\rangle$ has only the trivial order automorphism, therefore $g_0 = \mathrm{id}_\kappa$.

We obtain $g[\gamma_i] \subseteq \gamma_i$ and (because g^{-1} is also an automorphism) $g^{-1}[\gamma_i] \subseteq \gamma_i$, i.e. $g[\gamma_i] = \gamma_i$. But then, g is an automorphism for all relations in M, and therefore also for all relations in Q. This yields $\mathsf{Aut}\{\varrho\} \subseteq \mathsf{Aut}\, Q$, and this finishes the proof. □

As a consequence of this Lemma, every possible automorphism group appears already as the automorphism group of an at most countable set of relations.

Lemma 4.2. *For every set $R \subseteq \mathrm{Rel}(A)$ there exists an at most countable set $R_0 \subseteq \mathrm{Rel}(A)$ with $\mathsf{Aut}\, R = \mathsf{Aut}\, R_0$. Moreover, if R is a $\bigcap$–closed Krasner algebra, then we can choose $R_0 \subseteq R$.*

Proof. If R is not closed under $\mathbf{C}$, then we put $R' := R \cup \{\mathbf{C}\sigma \mid \sigma \in R\}$. Then $\mathsf{Aut}\, R = \mathsf{Aut}\, R'$, therefore we can assume w.l.o.g. that R is closed under complementation. By 4.1 there are relations $\varrho_m \in \mathrm{Rel}^{(2m)}(A)$ with $\mathsf{Aut}\, R^{(m)} = \mathsf{Aut}\{\varrho_m\}$. Consequently:

$$\mathsf{Aut}\, R = \bigcap_{1\leqslant m\in\omega} \mathsf{Aut}\, R^{(m)} = \bigcap_{1\leqslant m\in\omega} \mathsf{Aut}\{\varrho_m\} = \mathsf{Aut}\{\varrho_m \mid 1 \leqslant m \in \omega\}$$

The second part in Lemma 4.1 implies that the ϱ_m can be chosen from the $\bigcap$–closed Krasner algebra, generated by R. □

Now we define our additional operations.

Definition 4.3. An *invariant operation with countable arity* is an operation of the form

$$F : \prod_{1\leqslant i\in\omega} \mathrm{Rel}^{(m_i)}(A) \to \mathrm{Rel}^{(m)}(A)$$

($m_i \in \omega \setminus \{0\}$), such that for all $(\varrho_i)_{1\leqslant i\in\omega} \in \prod_{1\leqslant i\in\omega} \mathrm{Rel}^{(m_i)}(A)$ and all $g \in \mathrm{Sym}(A)$ we have

$$F(g[\varrho_i])_{1\leqslant i\in\omega} = g\,[F(\varrho_i)_{1\leqslant i\in\omega}]$$

If $Q \subseteq \mathrm{Rel}(A)$, then $\langle Q\rangle_{\omega-inv}$ is the closure of Q under all invariant operations with countable arity, and $\langle Q\rangle_{\omega-inv,\bigcap}$ is the least set of relations which is closed under all invariant operations with countable arity and $\bigcap$–closed.

(It is clear that $\langle\_\rangle_{\omega-inv}$ and $\langle\_\rangle_{\omega-inv,\bigcap}$ are closure operators.) Similar as in Lemma 2.13 and Lemma 2.14, we can verify the following properties:

Lemma 4.4.

(1) *If $R \subseteq \mathrm{Rel}(A)$ is Galois closed, $R = \mathsf{sInv}\,\mathsf{Aut}\, R$, then R is $\bigcap$–closed and closed under all invariant operations with countable arity, $R = \langle R\rangle_{\omega-inv,\bigcap}$.*
(2) *If $Q \subseteq \mathrm{Rel}(A)$ is countable or finite, then $\langle Q\rangle_{\omega-inv} = \mathsf{sInv}\,\mathsf{Aut}\, Q$.*

Now we can formulate our characterization of the Galois closed sets of relations.

Theorem 4.5. *Let R be a $\bigcap$–closed Krasner algebra. Then R is Galois closed, $R = \mathsf{sInv}\,\mathsf{Aut}\, R$, if and only if $\mathsf{sInv}\,\mathsf{Aut}\, R_0 \subseteq R$ for every countable subset R_0 of R.*

In particular, a set $R \subseteq \mathrm{Rel}(A)$ is Galois closed if and only if it is $\bigcap$–closed and closed under all invariant operations with countable arity, $R = \langle R\rangle_{\omega-inv,\bigcap}$. For all $Q \subseteq \mathrm{Rel}(A)$ we have $\langle Q\rangle_{\omega-inv,\bigcap} = \mathsf{sInv}\,\mathsf{Aut}\, Q$.

Proof. Clearly, $R_0 \subseteq R$ and $R = \mathsf{sInv}\,\mathsf{Aut}\,R$ implies $\mathsf{sInv}\,\mathsf{Aut}\,R_0 \subseteq \mathsf{sInv}\,\mathsf{Aut}\,R = R$ for every subset R_0 of R. Vice versa, if $\mathsf{sInv}\,\mathsf{Aut}\,R_0 \subseteq R$ for all countable subsets, then we can choose the special subset $R_0 \subseteq R$ with $\mathsf{Aut}\,R = \mathsf{Aut}\,R_0$ of Lemma 4.2. Then we obtain:

$$R \subseteq \mathsf{sInv}\,\mathsf{Aut}\,R = \mathsf{sInv}\,\mathsf{Aut}\,R_0 \subseteq R,$$

i.e., R is Galois closed.

Then the other statements are consequences of Lemma 4.4. □

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