

CATEGORICITY AND AMALGAMATION FOR AEC, AND κ MEASURABLE

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ABSTRACT. In the original version of this paper, we assume a theory T that the logic $\mathbb{L}_{\kappa, \aleph_0}$ is categorical in a cardinal $\lambda > \kappa$, and κ is a measurable cardinal. There we prove that the class of models of T of cardinality $< \lambda$ (but $\geq |T| + \kappa$) has the amalgamation property under a natural order; this is a step toward understanding the character of such classes of models.

In this revised version we replaced the class of models of T by \mathfrak{k} , an AEC (abstract elementary class) which has LS-number $< \kappa$, or at least which behave nicely for ultrapowers by D , a normal ultra-filter on κ .

Presently sub-section §1A deals with $T \subseteq \mathbb{L}_{\kappa^+, \aleph_0}$ (and so does a large part of the introduction and little in the rest of §1), but otherwise, all is done in the context of AEC.

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Annotated Content

§0 Introduction, pg. 4.

§1 Preliminaries, pg. 6.

[In §1A we review materials on fragments \mathcal{F} of $\mathbb{L}_{\kappa, \aleph_0}$ (including the theory T) and basic model-theoretic properties (Tarski-Vaught property and L.S.), and we define amalgamation. In §1B we move to AEC $\mathfrak{k} = (\mathfrak{K}, \leq_{\mathfrak{k}})$ which is our main framework now and spell out the connection. In §1C, D we deal with indiscernibles and E.M. models, then we deal with limit ultrapowers which are suitable (for $\mathbb{L}_{\kappa, \aleph_0}$ and for our AECs) and in particular ultralimits. Next, we introduce a notion basic for this paper $M \preceq_{\text{nice}} N$ if there is a $\leq_{\mathfrak{k}}$ -embedding of N into suitable ultralimit of M extending the canonical one.]

§2 The amalgamation property for regular categoricity, pg. 15.

[We get amalgamation in $(K_{\lambda}, \leq_{\mathfrak{k}})$ when one of the extensions is nice in 2.1. We prove that if \mathfrak{k} is categorical in the regular $\lambda > \text{LST}_{\mathfrak{k}} + \kappa$, then $(K_{<\lambda}, \leq_{\mathfrak{k}})$ has the amalgamation property. For this, we show that nice extension (in $K_{<\lambda}$) preserves being a non-amalgamation basis. We also start investigating (in 2.6) the connection between extending the linear order I and the model $\text{EM}(I)$: $I \subseteq_{\text{nice}} J \Rightarrow \text{EM}(I) \leq_{\text{nice}} \text{EM}(J)$; and give sufficient condition for $I \subseteq_{\text{nice}} J$ (in 2.7). From this, we get in K_{λ} a model such that any submodel of an expansion is a \leq_{nice} -submodel (in 2.9, 2.12(2)), and conclude the amalgamation properly in $(K_{<\lambda}, \leq_{\mathfrak{k}})$ when λ is regular (in 2.11) and something for singulars in 2.12.]

§3 Toward removing the assumption of regularity from the existence of universal extensions, pg. 20.

[The problem is that $\text{EM}(\lambda)$ has many sub-models which “sit” well in it and we can prove that there are many amalgamation bases but we need to get this simultaneously. First in 3.1 we show that, if $\langle M_i : i < \theta^+ \rangle$ is $\leq_{\mathfrak{k}}$ -increasing continuous sequence of models from K_{θ} , then for a club of $i < \theta^+$, $M_i \leq_{\text{nice}} \bigcup \{M_j : j < \theta^+\}$. In Definition 3.6, we define nice models (essentially, every reasonable extension is nice). Next (in 3.4) we show that nice models are dense in K_{θ} . Also (by 3.5) many embeddings are nice and (in 3.6) we show that being nice implies being amalgamation base. Then we define a universal extension of $M \in K_{\theta}$ in K_{∂} (Definition 3.7), we prove existence over a model in 3.10 and after preparation prove existence (3.13, 3.14).]

§4 (θ, ∂) -saturated models, pg. 25.

[If $M_i \in K_{\theta}$ for $i \leq \partial$ is increasing continuous, M_{i+1} universal over M_i , each M_i -nice, then M_{∂} is (θ, ∂) -saturated over M_0 . We show existence (and uniqueness). We connect this to more usual saturation and prove that (θ, ∂) -saturation implies niceness (in 4.10).]

§5 The amalgamation property for $K_{<\lambda}$, pg. 30.

[After preliminaries we prove that for $\theta \leq \lambda$ (and $\theta \geq \text{LST}(\mathfrak{k}) + \kappa$ of course) every member of K_{θ} can be extended to one with many nice submodels,

this is done by induction on θ using the niceness of (θ_1, ∂_1) -saturated models. Lastly, we conclude that every $M \in K_{<\lambda}$ is nice hence $K_{<\lambda}$ has the amalgamation property.]

§ 0. INTRODUCTION

The main result of this paper is a proof of the following theorem:

Theorem 0.1. *Suppose that T is a theory in a fragment of $\mathbb{L}_{\kappa, \aleph_0}$ where κ is a measurable cardinal. If T is categorical in the cardinal $\lambda > \kappa + |T|$, then $\mathcal{K}_{<\lambda}$, the class of models of T of power strictly less than λ (but $\geq \chi = \kappa + |T|$), has the amalgamation property (see Definition 1.11 (1)(2)).*

The interest in this theorem stems in part from its connection with the study of categoricity spectra. For a theory T in a logic \mathcal{L} let us define $\text{Cat}(T)$, the *categoricity spectrum of T* , to be the collection of those cardinals λ in which T is categorical. In the 1950's Łos conjectured that if T is a countable theory in first-order logic, then $\text{Cat}(T)$ contains every uncountable cardinal or no uncountable cardinal. This conjecture, based on the example of algebraically closed fields of fixed characteristic, was verified by Morley [Mor65], who proved that if a countable first-order theory is categorical in some uncountable cardinal, then it is categorical in every uncountable cardinal. Following advances made by Rowbottom [Row64], Ressayre [Res69] and Shelah [She69], Shelah [She74] proved the Łos conjecture for uncountable first-order theories: if T is a first-order theory categorical in some cardinal $\lambda > |T| + \aleph_0$, then T is categorical in every cardinal $\lambda > |T| + \aleph_0$. It is natural to ask whether analogous results hold for theories in logics other than first-order logic. Perhaps the best-known extensions of first-order logic are the infinitary logics $\mathbb{L}_{\lambda, \kappa}$. As regards theories in $\mathbb{L}_{\kappa, \aleph_0}$, Shelah (see [She83a] and [She83b]) continuing work begun in [She75] introduced the concept of excellent classes: these have models in all cardinalities, have the amalgamation property and satisfy the Łos conjecture. In particular, if φ is an excellent sentence of $\mathbb{L}_{\kappa, \aleph_0}$, then the Łos conjecture holds for φ . Furthermore, under some set-theoretic assumptions (weaker than the Generalized Continuum Hypothesis) if φ is a sentence in $\mathbb{L}_{\kappa, \aleph_0}$ which is categorical in \aleph_n for every natural number n (or even just if φ is a sentence in $\mathbb{L}_{\kappa, \aleph_0}$ with at least one uncountable model not having too many models in each \aleph_n), then φ is excellent. Now, [She87c], [She09c] try to develop classification theory in some non-elementary classes. We cannot expect much for $\mathbb{L}_{\lambda, \kappa}$ for $\kappa > \aleph_0$. Shelah conjectured that if φ is a sentence in $\mathbb{L}_{\kappa, \aleph_0}$ categorical in some $\lambda > \beth_{\omega_1}$, then φ is categorical in every $\lambda > \beth_{\omega_1}$. (Recall that the Hanf number of $\mathbb{L}_{\kappa, \aleph_0}$ is \beth_{ω_1} , so if ψ is a sentence in $\mathbb{L}_{\kappa, \aleph_0}$ and ψ has a model of power $\lambda \geq \beth_{\omega_1}$, then ψ has a model in every power $\lambda \geq \beth_{\omega_1}$, see [Kei71]). There were some who asked why so tardy the beginning. Recent work of Hart and Shelah [HS90] showed that for every natural number k greater than 1 there is a sentence ψ_k in $\mathbb{L}_{\kappa, \aleph_0}$ which is categorical in the cardinals $\aleph_0, \dots, \aleph_{k-1}$, but which has many models of power λ for every cardinal $\lambda \geq 2^{\aleph_{k-1}}$. The general conjecture for $\mathbb{L}_{\kappa, \aleph_0}$ remains open nevertheless. As regards theories in $\mathbb{L}_{\kappa, \aleph_0}$, progress has been recorded under the assumption that κ is a strongly compact cardinal. Under this assumption Shelah and Makkai [MS90] have established the following results for a λ -categorical theory T in a fragment \mathcal{F} of $\mathbb{L}_{\kappa, \aleph_0}$:

- 1) if λ is a successor cardinal and $\lambda > ((\kappa')^\kappa)^+$ where $\kappa' = \max(\kappa, |\mathcal{F}|)$, then T is categorical in every cardinal greater than or equal to $\min(\lambda, \beth_{(2^{\kappa'})^+})$,
- 2) if $\lambda > \beth_{\kappa+1}(\kappa')$, then T is categorical in every cardinal of the form \beth_δ with δ divisible by $(2^{\kappa'})^+$ (i.e. for some ordinal $\alpha > 0$, $\delta = (2^{\kappa'})^+ \cdot \alpha$ (ordinal multiplication)).

In proving theorems of this kind, one has recourse to the amalgamation property which makes possible the construction of analogs of saturated models. In turn, these are of major importance in categoricity arguments. The amalgamation property holds for theories in first-order logic [CK73] and in $\mathbb{L}_{\kappa, \kappa}$ when κ is a strongly

compact cardinal (see [MS90]: although $\prec_{\mathbb{L}_{\kappa,\kappa}}$ fails the Tarski-Vaught property for unions of chains of length κ (whereas $\prec_{\mathbb{L}_{\kappa,\aleph_0}}$ satisfies it), under a categoricity assumption it can be shown that $\prec_{\mathbb{L}_{\kappa,\aleph_0}}$ and $\prec_{\mathbb{L}_{\kappa,\kappa}}$ coincide). However, it is not known in general for theories in $\mathbb{L}_{\kappa,\aleph_0}$ or $\mathbb{L}_{\kappa,\kappa}$ when one weakens the assumption on κ , in particular when κ is just a measurable cardinal. Nevertheless, categoricity does imply the existence of reasonably saturated models in an appropriate sense, and it is possible to begin classification theory. This is why the main theorem of the present paper is of relevance regarding the categoricity spectra of theories in $\mathbb{L}_{\kappa,\aleph_0}$ when κ is measurable.

A sequel to this paper under preparation (which is now [She01b]) tries to provide a characterization of $\text{Cat}(T)$ at least parallel to that in [MS90] and we hope to deal with the corresponding classification theory later. This division of labor both respects historical precedent and is suggested by the increasing complexity of the material. Another sequel deals with abstract elementary classes (in the sense of [She87a]) (see [She01b], [She99] respectively). On more work see [She01a], [She09b].

The paper is divided into five sections. Section 1 is preliminary and notational. In section 2 it is shown that if the theory $T \subseteq \mathbb{L}_{\kappa,\aleph_0}$ or just suitable AEC \mathfrak{K} is categorical in the regular cardinal $\lambda > \kappa + |T|$, then $K_{<\lambda}$ has the amalgamation property. Section 3 deals with weakly universal models, section 4 with $(\theta, \bar{\theta})$ -saturated and $\bar{\theta}$ -saturated models. In section 5 the amalgamation property for $K_{<\lambda}$ is established.

All the results in this paper (other than those explicitly credited) are due to Saharon Shelah.

* * *

On a more recent survey see [Sheb] and a recent one see [SV], in particular on the history of κ -compact AEC.

We had thought that clearly, the proof of [KS96] works for AEC, but the referee of [SS05] asked to do it explicitly. Here we justify [SV, 4.7]. Note that, [KS96, 1.1, 1.2] essentially proves that $(\text{Mod}(T), \prec_T)$ is an AEC omitting Ax IV.

§ 1. PRELIMINARIES

To start things off in this section, let us fix notation, provide basic definitions and well-known facts, and formulate our working assumptions.

The working assumptions in force throughout the paper are these.

Assumption 1.1. The cardinal κ is an uncountable measurable cardinal, and so there is a κ -complete non-principal ultrafilter on κ , we can fix such D .

Assumption 1.2.

- (1) The theory T is a theory in the infinitary logic $\mathbb{L}_{\kappa, \aleph_0}$ and $\chi = \kappa + |T|$.
- (2) \mathfrak{k} is an AEC which is D -compact (see Definition 1.16, 1.41 respectively).

Our main theorem for the logic $\mathbb{L}_{\kappa, \aleph_0}$ is:

Theorem 1.3. *If $T \subseteq \mathbb{L}_{\kappa, \aleph_0}$ is categorical in $\lambda > \kappa + |\tau_T|$ then the class of models of T of cardinality $< \lambda$ but $\geq \kappa + |\tau_T|$ (under the so called $\prec_{\mathcal{F}_\tau}$) has the amalgamation property.*

Proof. Use Theorem 1.18 on AEC which is applicable by 1.20. $\square_{1.3}$

From these assumptions follow certain facts, of which the most important are these.

Fact 1.4. For each model M of T , κ -complete ultrafilter D over I and suitable set G of equivalence relations on $I \times I$ (see Definition 1.32) the limit ultrapower $\text{Op}(M) = \text{Op}(M, I, D, G)$ is a model of T .

Fact 1.5. For each linear order $I = (I, \leq)$ there exists a generalized Ehrenfeucht-Mostowski model $\text{EM}(I)$ of T .

This section is divided into several subsections: in §1A we deal with a theory T in $\mathbb{L}_{\kappa, \aleph_0}$, in §1B we move to AEC \mathfrak{k} showing that the context in §1A is a special case. Then in §1C we deal with EM models. Finally, in §1D we deal with ultra-powers, ultra-limits, and nice sub-models.

§ 1(A). Frame for $\mathbb{L}_{\kappa, \aleph_0}$.

Relevant set-theoretic and model-theoretic information on measurable cardinals can be found in [Jec03], [CK73], and [Dic75]. τ denotes¹ a vocabulary, i.e. a set of finitary relation and function symbols, including equality. $|\tau|$ is the cardinality of the vocabulary τ .

Definition 1.6.

- (1) For cardinals $\kappa \leq \lambda$, $\mathbb{L}_{\lambda, \kappa}$ is the smallest set of (infinitary) formulas in the vocabulary τ which contains all first-order formulas and which is closed under:

- (A) the formation of conjunctions (disjunctions) of any set of formulas of power less than λ , provided that the set of free variables in the conjunctions (disjunctions) has power less than κ ,
- (B) the formation of $\forall \bar{x}\varphi, \exists \bar{x}\varphi$, where $\bar{x} = \langle x_\alpha : \alpha < \alpha_* \rangle$ is a sequence of variables of length $\alpha_* < \kappa$.

- (2) Whenever we use the notation $\varphi(\bar{x})$ to denote a formula in $\mathbb{L}_{\lambda, \kappa}$, we mean that \bar{x} is a sequence $\langle x_\alpha : \alpha < \alpha_* \rangle$. So if $\varphi(\bar{x})$ is a formula in $\mathbb{L}_{\kappa, \aleph_0}$, then \bar{x} is a finite sequence of variables.

¹In the old version it was called “language” and denoted by L .

Notation 1.7.

(1) \mathcal{F} denotes a fragment of $\mathbb{L}(\tau)$, i.e. a set of formulas of $\mathbb{L}(\tau)$ which contains all atomic formulas of τ , and which is closed under negations, finite conjunctions (finite disjunctions), and the formation of subformulas. An \mathcal{F} -formula is just an element of \mathcal{F} .

(2) T is a theory in $\mathbb{L}(\tau)$, so there is a fragment \mathcal{F} of $\mathbb{L}_{\kappa, \aleph_0}$ such that $T \subseteq \mathcal{F}$ and $|\mathcal{F}| < |T|^+ + \kappa$. If not said otherwise, T and \mathcal{F} are fixed.

(3) Models of T (invariably referred to as models) are τ -structures which satisfy the sentences of T . They are generally denoted M, N, \dots , and $|M|$ is the universe of the τ -structure M ; $\|M\|$ is the cardinality of $|M|$.

(4) For a set A , $|A|$ is the cardinality of A . ${}^{<\omega}A$ is the set of finite sequences in A and for $\bar{a} = \langle a_1 \dots a_n \rangle \in {}^{<\omega}A$, $\text{lg}(\bar{a}) = n$ is the length of \bar{a} . Similarly, if $\bar{a} = \langle a_\zeta : \zeta < \delta \rangle$, we write $\text{lg}(\bar{a}) = \delta$, where δ is an ordinal.

(5) For an element R of τ let $\text{val}(M, R)$, or R^M , be the interpretation of R in the τ -structure M .

(6) We ignore models of power less than κ . K is the class of all models of T ;

$$K_\lambda = \{M \in K : \|M\| = \lambda\}, K_{<\lambda} = \bigcup_{\mu < \lambda} K_\mu, K_{\leq \lambda} = \bigcup_{\mu \leq \lambda} K_\mu, K_{[\mu, \lambda)} = \bigcup_{\mu \leq \chi < \lambda} K_\chi.$$

(7) We write $f: M \xrightarrow{\mathcal{F}} N$ (abbreviated $f: M \rightarrow N$) to mean that f is an \mathcal{F} -elementary embedding (briefly, an embedding) of M into N , i.e. f is a function with domain $|M|$ into $|N|$ such that for every \mathcal{F} -formula $\varphi(\bar{x})$, and $\bar{a} \in {}^{<\omega}|M|$ with $\text{lg}(\bar{a}) = \text{lg}(\bar{x})$, $M \models \varphi[\bar{a}]$ iff $N \models \varphi[f(\bar{a})]$, where if $\bar{a} = \langle a_i : i < n \rangle$, then $f(\bar{a}) := \langle f(a_i) : i < n \rangle$.

(8) In the special case where an embedding f is a set-inclusion (so that $|M| \subseteq |N|$), we write $M \prec_{\mathcal{F}} N$ (briefly $M \prec N$) instead of $f: M \xrightarrow{\mathcal{F}} N$ and we say that M is an \mathcal{F} -elementary submodel of N , or N is an \mathcal{F} -elementary extension of M .

Notation 1.8.

(1) $(I, \leq_I), (J, \leq_J)$ are partial orders; we will not bother to subscript the order relation unless really necessary; we write I for (I, \leq) . We say (I, \leq) is directed iff for every i_1 and i_2 in I , there is $i \in I$ such that $i_1 \leq i$ and $i_2 \leq i$. $(I, <)^*$ is the (reverse) linear order $(I^*, <^*)$ where $I^* = I$ and $s <^* t$ iff $t < s$.

(2) A set $\langle M_i : i \in I \rangle$ of models indexed by I is a $\prec_{\mathcal{F}}$ -directed system iff (I, \leq) is a directed partial order and for $i \leq j$ in I , $M_i \prec_{\mathcal{F}} M_j$.

Note that, the union $\bigcup_{i \in I} M_i$ of a $\prec_{\mathcal{F}}$ -directed system $\langle M_i : i \in I \rangle$ of τ -structures is an τ -structure. In fact, more is true.

Fact 1.9.

(1) (Tarski-Vaught property) The union of a $\prec_{\mathcal{F}}$ -directed system $\langle M_i : i \in I \rangle$ of models of T is a model of T , and for every $j \in I$, $M_j \prec_{\mathcal{F}} \bigcup_{i \in I} M_i$.

(2) If M is a fixed model of T such that for every $i \in I$ there is $f_i: M_i \xrightarrow{\mathcal{F}} M$ and for all $i \leq j$ in I , $f_i \subseteq f_j$, then $\bigcup_{i \in I} f_i: \bigcup_{i \in I} M_i \xrightarrow{\mathcal{F}} M$. In particular, if $M_i \prec_{\mathcal{F}} M$ for every $i \in I$, then $\bigcup_{i \in I} M_i \prec_{\mathcal{F}} M$. Let α be an ordinal. A $\prec_{\mathcal{F}}$ -chain of models of length α is a sequence $\langle M_\beta : \beta < \alpha \rangle$ of models such that if $\beta < \gamma < \alpha$, then $M_\beta \prec_{\mathcal{F}} M_\gamma$. The chain is continuous if for every limit ordinal $\beta < \alpha$, $M_\beta = \bigcup_{\gamma < \beta} M_\gamma$.

Fact 1.10.

- (1) (Downward Löwenheim-Skolem Property): Suppose that M is a model of T , $A \subseteq |M|$ and $\max(\kappa + |T|, |A|) \leq \lambda \leq ||M||$. Then there is a model N such that $A \subseteq |N|$, $||N|| = \lambda$ and $N \prec_{\mathcal{F}} M$.
- (2) If N and $M_1 \subseteq M_2$ are τ -models, \mathcal{F} is a fragment of $\mathbb{L}_{\kappa, \aleph_0}$, and $M_\ell \prec_{\mathcal{F}} N$ for $\ell = 1, 2$ then $M_1 \prec_{\mathcal{F}} M_2$.

Now we turn from the rather standard model-theoretic background to the more specific concepts which are central in our investigation.

Definition 1.11.

- (1) Suppose that $<$ is a binary relation on a class K of models (mainly $(K, <) = (K_{\mathfrak{k}}, <_{\mathfrak{k}})$, see below) $\mathcal{K} = \langle K, < \rangle$ has the *amalgamation property* (AP) iff for every $M, M_1, M_2 \in K$, if f_i is an isomorphism from M onto $\text{rng}(f_i)$ and $\text{rng}(f_i) < M_i$ for $i = 1, 2$, then there exist $N \in K$ and isomorphisms g_i from M_i onto $\text{rng}(g_i)$ for $i = 1, 2$ such that $\text{rng}(g_i) < N$ and $g_1 f_1 = g_2 f_2$. The model N is called an *amalgam* of M_1, M_2 over M with respect to f_1, f_2 .
- (2) An τ -structure M is an *amalgamation base* (a.b.) for $\mathcal{K} = \langle K, < \rangle$ iff $M \in K$ and whenever for $i = 1, 2, M_i \in K$ and f_i is an isomorphism from M onto $\text{rng}(f_i)$, $\text{rng}(f_i) < M_i$, then there exist $N \in K$ and isomorphisms g_i ($i = 1, 2$) from M_i onto $\text{rng}(g_i)$ such that $\text{rng}(g_i) < N$ and $g_1 f_1 = g_2 f_2$.

So $\mathcal{K} = \langle K, < \rangle$ has AP iff every model in K is an a.b. for \mathcal{K} .

Example 1.12. Suppose that T is a theory in first-order logic having an infinite model. Define, for M, N in the class $K_{\leq |T| + \aleph_0}$ of models of T of power at most $|T| + \aleph_0$, $M < N$ iff there is an embedding of M into an elementary submodel of N . Then $\mathcal{K}_{\leq |T| + \aleph_0} = \langle K_{\leq |T| + \aleph_0}, < \rangle$ has AP, (see [CK73]).

Example 1.13. Suppose that T is a theory in $\mathbb{L}_{\kappa, \aleph_0}$ and \mathcal{F} is a fragment of $\mathbb{L}_{\kappa, \aleph_0}$ containing T with $|\mathcal{F}| < |T|^+ + \kappa$. Let $<$ be the binary relation $\prec_{\mathcal{F}}$ defined on the class K of all models of T . $M \in K$ is an a.b. for \mathcal{K} iff whenever for $i = 1, 2, M_i \in K$ and f_i is an $\prec_{\mathcal{F}}$ -elementary embedding of M into M_i , there exist $N \in K$ and \mathcal{F} -elementary embeddings g_i ($i = 1, 2$) of M_i into N such that $g_1 f_1 = g_2 f_2$.

Definition 1.14. Suppose that $<$ is a binary relation on a class K of models. Let μ be a cardinal. $M \in K_{\leq \mu}$ is a μ -*counter amalgamation basis* (μ -c.a.b.) of $\mathcal{K} = \langle K, < \rangle$ iff there are $M_1, M_2 \in K_{\leq \mu}$ and isomorphisms f_i from M into M_i such that:

- (a) $\text{rng}(f_i) < M_i$ ($i = 1, 2$),
- (b) there is no amalgam $N \in K_{\leq \mu}$ of M_1, M_2 over M with respect to f_1, f_2 .

Observation 1.15. Suppose that T, \mathcal{F} and $<$ are as in 1.13 and $\kappa + |T| \leq \mu < \lambda$. Note that if there is an amalgam N' of M_1, M_2 over M (for M_1, M_2, M in $K_{\leq \mu}$), then by Fact 1.10 there is an amalgam $N \in K_{\leq \mu}$ of M_1, M_2 over M .

§ 1(B). Replacing T by AEC.

On AEC see [She87a], [She09a] or [Bal09], recall:

Definition 1.16. We say $\mathfrak{k} = (K_{\mathfrak{k}}, \leq_{\mathfrak{k}})$ is a a.e.c. with L.S.T. number $\lambda(\mathfrak{k}) = \text{LS}_{\mathfrak{k}} = \text{LST}(\mathfrak{k})$ may write K for $K_{\mathfrak{k}}$, if it is a class of $\tau_{\mathfrak{k}}$ -models when:

- Ax 0: The holding of $M \in K, N \leq_{\mathfrak{k}} M$ depend on N, M only up to isomorphism, i.e. $[M \in K, M \cong N \Rightarrow N \in K]$ and [if $N \leq_{\mathfrak{k}} M$ and f is an isomorphism from M onto the τ -model $M', f \upharpoonright N$ is an isomorphism from N onto N' then $N' \leq_{\mathfrak{k}} M'$.]
- Ax I: if $M \leq_{\mathfrak{k}} N$ then $M \subseteq N$ (i.e. M is a submodel of N).

- **Ax II:** $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_2$ implies $M_0 \leq_{\mathfrak{k}} M_2$ and $M \leq_{\mathfrak{k}} M$ for $M \in K$.
- **Ax III:** If λ is a regular cardinal, M_i ($i < \lambda$) is a $\leq_{\mathfrak{k}}$ -increasing (i.e. $i < j < \lambda$ implies $M_i \leq_{\mathfrak{k}} M_j$) and continuous (i.e. for every limit ordinal $\delta < \lambda$, $M_\delta = \bigcup_{i < \delta} M_i$) then $M_0 \leq_{\mathfrak{k}} \bigcup_{i < \lambda} M_i$.
- **Ax IV:** If λ is a regular cardinal and M_i (for $i < \lambda$) is $\leq_{\mathfrak{k}}$ -increasing continuous and $M_i \leq_{\mathfrak{k}} N$ for $i < \lambda$ then $\bigcup_{i < \lambda} M_i \leq_{\mathfrak{k}} N$.
- **Ax V:** If $N_0 \subseteq N_1 \leq_{\mathfrak{k}} M$ and $N_0 \leq_{\mathfrak{k}} M$ then $N_0 \leq_{\mathfrak{k}} N_1$.
- **Ax VI:** If $A \subseteq N \in K$ and $|A| \leq \text{LST}(\mathfrak{k})$ then for some $M \leq_{\mathfrak{k}} N$, $A \subseteq |M|$ and $\|M\| \leq \text{LST}(\mathfrak{k})$ (and $\text{LST}(\mathfrak{k})$ is the minimal infinite cardinal satisfying this axiom which is $\geq |\tau|$; the $\geq |\tau|$ is for notational simplicity).

Definition 1.17. We define “ \mathfrak{k} categorical in λ ”, $\mathfrak{k}_{<\lambda}$, “ \mathfrak{k} has amalgamation” “ $M \in K_{\mathfrak{k}}$ is a.b.”, “ M is c.a.b.” naturally (see Definition 1.11, 1.14).

So our main theorem is:

Theorem 1.18. *If \mathfrak{k} is categorical in $\lambda > \text{LS}_{\mathfrak{k}}$ (and for transparency has no models of cardinality $< \text{LS}_{\mathfrak{k}} + \kappa$) then $\mathfrak{k}_{<\lambda}$ has amalgamation.*

Claim 1.19. *Assume \mathfrak{k} is an AEC and $\tau = \tau_{\mathfrak{k}}$. Then;*

There are $\tau_1 = \tau_{\mathfrak{k},1} \supseteq \tau_{\mathfrak{k}}$ of cardinality $|\tau| + \text{LST}_{\mathfrak{k}}$ and a set \mathcal{P} of qf 1-types in \mathbb{L}_{τ_1} such that:

- (A) *a τ -structure M belongs to $K_{\mathfrak{k}}$ iff it can be expanded to a τ_T -model M^+ from K_+ when:*
 - $K_+ = K_{\mathfrak{k}}^+ = \{M : N \text{ a } \tau_T\text{-structure omitting every } p \in \mathcal{P}\}$.
- (B) *If $M^+ \in K_+$ and $M^+ \upharpoonright \tau \leq_{\mathfrak{k}} N$ then there is a $N^+ \in K_+$ expanding N such that $M^+ \subseteq N^+$.*
- (C) *(K_+, \subseteq) is an AEC with $\text{LST}(K_+, \subseteq) = \text{LST}(\mathfrak{k})$.*
- (D) *There is a set $\tau'_1 \subseteq \tau_1$ of cardinality $\text{LST}_{\mathfrak{k}}$ such that $A \subseteq M^+ \in K_+ \Rightarrow \text{cl}_{\tau'_1}(A, M^+) \subseteq M^+$.*
- (E) *Some $\psi \in \mathbb{L}_{\lambda, \aleph_0}$ defines $K_{\mathfrak{k},1}$ where $\lambda = \text{LST}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$.*

Proof. By [She09a, 1.7].

□_{1.19}

Definition 1.20.

Assume T is a theory in $\mathbb{L}_{\kappa, \aleph_0}(\tau_T)$, τ_T determined by T (so $|T| \leq (|\tau_T| + \kappa)^{<\kappa}$) and recall \mathcal{F}_T is the set of formulas $\varphi(x)$ such that $\varphi(\bar{x})$ is a sub-formulas of some sentence $\psi \in T$. We define $\mathfrak{k} = \mathfrak{k}_T$ as follows:

- (A) $K_{\mathfrak{k}}$ is the class of τ_T -models of T of cardinality $\geq \kappa + |T|$.
- (B) $M \leq_{\mathfrak{k}} N$ iff:
 - (a) $M, N \in K_{\mathfrak{k}}$,
 - (b) $M \subseteq N$,
 - (c) $M \leq_{\mathcal{F}} N$ i.e., if $\varphi(\bar{x}) \in \mathcal{F}_T$ (see below, so $\text{lg}(\bar{x})$ is limit and $\bar{a} \in \text{lg}(\bar{x})M$) then $M \models \varphi[\bar{a}]$ iff $N \models \varphi[\bar{a}]$.

Claim 1.21. *If T is a theory in $\mathbb{L}_{\kappa, \aleph_0}(\tau_T)$, then:*

- (A) \mathfrak{k}_T is an AEC.
- (B) $\text{LST}_{\mathfrak{k}_T} = \text{LST}(\mathfrak{k}_T) \leq |T| + \kappa$ (actually $\leq |\mathcal{F}_T|$).
- (C) *If $T \subseteq \mathbb{L}_{\lambda^+, \aleph_0}(\tau_T)$ then $\text{LST}_{\mathfrak{k}} \leq |\tau| + \lambda$.*

Proof. This holds by 1.9 and Fact 1.10.

□_{1.21}

Conclusion 1.22. *To prove our results for T it suffices to prove them for \mathfrak{k}_T .*

Proof. By 1.21 just check the definition.

□_{1.22}

Definition 1.23. We say the AEC \mathfrak{k} is (μ, λ, κ) -representable when there are (τ_1, T_1, Γ) such that:

- (a) $\tau_1 \supseteq \tau_{\mathfrak{k}}$ has cardinality $\leq \lambda$,
- (b) $T_1 \subseteq \mathbb{L}(\tau_1)$ is a first order logic universal theory, so $|T_1| \leq \lambda$,
- (c) Γ is a set of $\leq \mu$ qf-types in $\mathbb{L}(\tau_1)$, each of cardinality $< \kappa$,
- (d) $M \in K_{\mathfrak{k}}$ iff M is the $\tau_{\mathfrak{k}}$ -reduct of some $M_2 \in \text{EC}(T_1, \Gamma)$, where
$$\text{EC}(T_1, \Gamma) = \{N : N \text{ a } \tau_1\text{-model of } T_1 \text{ omitting every } p(x) \in \Gamma\},$$
- (e) $M \leq_{\mathfrak{k}} N$ iff for every $M_1 \in \text{EC}(T_1, \Gamma)$ expanding M , there is $N_1 \in \text{EC}(T_1, \Gamma)$ expanding N and extending M_1 .

Claim 1.24. Let \mathfrak{k} be an AEC. If $\lambda \geq \text{LS}_{\mathfrak{k}} + |\tau_{\mathfrak{k}}|$, then \mathfrak{k} is $(2^\lambda, \lambda, \lambda^+)$ -representable.

Proof. By 1.19, that is, by [She09a]. □_{1.24}

§ 1(C). Indiscernibles and Ehrenfeucht-Mostowski structures.

The basic results on generalized Ehrenfeucht-Mostowski models can be found in [She78] or [She90, VII].

Definition 1.25.

- (1) We recall here some notation. Let \mathbf{I} be a class of models which we call the *index models*. Denote the members of \mathbf{I} by I, J, \dots , etc.
- (2) For $I \in \mathbf{I}$ we say that $\langle \bar{a}_s : s \in I \rangle$ is *indiscernible in M* iff for every $\bar{s}, \bar{t} \in {}^{<\omega} I$ realizing the same atomic type in I , $\bar{a}_{\bar{s}}$ and $\bar{a}_{\bar{t}}$ realize the same type in M (where $\bar{a}_{\langle s_0, \dots, s_n \rangle} = \bar{a}_{s_0} \wedge \dots \wedge \bar{a}_{s_n}$).
- (3) If $\tau \subseteq \tau'$ are vocabularies and Φ is a function with domain including

$$\{\text{tp}_{\text{at}}(\bar{s}, \emptyset, I) : \bar{s} \in {}^{<\omega} I\}$$

and $I \in \mathbf{I}$, we let $\text{GEM}'(I, \Phi)$ be an τ' -model generated by $\bigcup_{s \in I} \bar{a}_s$ such that $\text{tp}_{\text{at}}(\bar{a}_{\bar{s}}, \emptyset, M) = \Phi(\text{tp}_{\text{at}}(\bar{s}, \emptyset, I))$; $\langle \bar{a}_s : s \in I \rangle$ is called the *skeleton*.

- (4) We say that Φ is *proper for \mathbf{I}* if for every $I \in \mathbf{I}$, $\text{GEM}'(I, \Phi)$ is well-defined.
- (5) Let $\text{GEM}(I, \Phi)$ be the τ -reduct of $\text{GEM}'(I, \Phi)$.
- (6) For the purposes of this paper we'll let \mathbf{I} be the class \mathbf{LO} of linear orders and Φ will be proper for \mathbf{LO} and then write EM (instead GEM). For $I \in \mathbf{LO}$ we abbreviate $\text{EM}'(I, \Phi)$ by $\text{EM}'(I)$ and $\text{EM}(I, \Phi)$ by $\text{EM}(I)$.

We first deal with pairs (T, \mathcal{F}) .

Claim 1.26. For each linear order $I = (I, \leq)$ there exists a generalized Ehrenfeucht-Mostowski model $\text{EM}(I)$ of T .

(See Nadel [Nad85] and Dickmann [Dic85] or [She90, VII, §5]).

Let \mathcal{F} be a fragment of $\mathbb{L}_{\kappa, \aleph_0}$. Recall that a theory $T \subseteq \mathcal{F}$ is called a *universal theory in $\mathbb{L}_{\kappa, \aleph_0}$* iff the axioms of T are sentences of the form $\forall \bar{x} \varphi(\bar{x})$, where $\varphi(\bar{x})$ is a quantifier-free formula in $\mathbb{L}_{\kappa, \aleph_0}$.

Definition and Proposition 1.27. Suppose that T is a theory and $T \subseteq \mathcal{F}$ where \mathcal{F} is a fragment of $\mathbb{L}_{\kappa, \aleph_0}$.

There are a (canonically constructed) finitary vocabulary τ_{sk} and a universal theory T_{sk} in $\mathbb{L}_{\kappa, \aleph_0}$ such that:

- (0) $\tau \subseteq \tau_{\text{sk}}, |\tau_{\text{sk}}| \leq |\mathcal{F}| + \aleph_0$,
- (1) The τ -reduct of any τ_{sk} -model of T_{sk} is a model of T ,

- (2) Whenever N_{sk} is an τ_{sk} -model of T_{sk} and M_{sk} is a substructure of N_{sk} , then $M_{\text{sk}} \upharpoonright \tau \prec_{\mathcal{F}} N_{\text{sk}} \upharpoonright \tau$,
- (3) Any τ -model of T can be expanded to an L_{sk} -model of τ_{sk} ,
- (4) If $M \prec_{\mathcal{F}} N$, then there are τ_{sk} -expansions $M_{\text{sk}}, N_{\text{sk}}$ of M, N respectively such that M_{sk} is a substructure of N_{sk} and N_{sk} is a model of T_{sk} ,
- (5) To any \mathcal{F} -formula $\varphi(\bar{x})$, those corresponds a quantifier-free formula $\varphi^{\text{qf}}(\bar{x})$ of $\mathbb{L}_{\kappa\omega}(\tau_{\text{sk}})$ such that:

$$T_{\text{sk}} \vdash \forall \bar{x} ((\varphi(\bar{x}) \leftrightarrow \varphi^{\text{qf}}(\bar{x})).$$

But now we use the AEC framework.

Claim 1.28.

(1) Assume \mathfrak{k} is an AEC, $\mu = 2^{\text{LST}(\mathfrak{k})}$. If \mathfrak{k} has a model of cardinality $\geq \beth_{\mu^+}$ (or just model of cardinality $\geq \beth_{\alpha}$ for every $\alpha < \mu^+$) then there is Φ such that:

- (a) Φ is as in the beginning of §1B,
- (b) $\tau_{\Phi} = \tau_{\mathfrak{k},1}$,
- (c) $\text{EM}(I)$ has cardinality $\text{LST}_{\mathfrak{k}} + |I|$.

(2) In particular,

- (a) $\text{EM}'(I)$ is a τ_1 -model,
- (b) $\text{EM}(I) = \text{EM}'(I) \upharpoonright \tau$ belongs to K ,
- (c) (follows) if $I \subseteq J$ then $\text{EM}(I) \leq_{\mathfrak{k}} \text{EM}(J)$, both models from K of cardinality $|I| + \text{LST}(\mathfrak{k})$.

Proof. As in [She09a], i.e., by 1.24. □_{1.28}

§ 1(D). Limit ultrapowers, iterated ultrapowers and nice extensions.

An important technique we shall use in studying the categoricity spectrum of a theory in $\mathbb{L}_{\kappa, \aleph_0}$ or suitable AECs is the limit ultrapower. It is convenient to record here the well-known definitions and properties of limit and iterated ultrapowers (see Chang and Keisler [CK73], Hodges-Shelah [HS81]) and then to examine nice extensions of models.

Definition 1.29. Suppose that M is an τ -structure, I is a non-empty set, D is an ultrafilter on I , and G is a filter on $I \times I$. For each $g \in {}^I M$, let

- (a) $\text{eq}(g) := \{\langle i, j \rangle \in I \times I : g(i) = g(j)\}$, and
- (b) $g/D := \{f \in {}^I M : g = f \text{ Mod } D\}$ where

$$g = f \text{ Mod } D \text{ iff } \{i \in I : g(i) = f(i)\} \in D.$$

Let $\prod_{D/G} {}^I M := \{g/D : g \in {}^I M \text{ and } \text{eq}(g) \in G\}$. Note that $\prod_{D/G} {}^I M$ is a non-empty subset of $\prod_D {}^I M = \{g/D : g \in {}^I M\}$ and is closed under the constants and functions of the ultrapower $\prod_D M$ of M modulo D . The limit ultrapower $\prod_{D/G} M$ of the τ -structure M (with respect to I, D, G) is the substructure of $\prod_D M$ whose universe is the set $\prod_{D/G} {}^I M$. The canonical map d from M into $\prod_{D/G} M$ is defined by $d(a) = \langle a_i : i \in I \rangle / D$, where $a_i = a$ for every $i \in I$. Note that the limit ultrapower $\prod_{D/G} M$ depends only on the equivalence relations which are in G , i.e. if \mathbf{E} is the set of all equivalence relations on I and $G \cap \mathbf{E} = G' \cap \mathbf{E}$, where G' is a filter on $I \times I$, then $\prod_{D/G} M = \prod_{D/G'} M$.

Definition 1.30. Let M be an τ -structure, $\langle Y, < \rangle$ a linear order and, for each $y \in Y$, let D_y be an ultrafilter on a non-empty set I_y and $\bar{I} = \langle I_y : y \in Y \rangle$, $\bar{D} = \langle D_y : y \in Y \rangle$. Write $H = \prod_{y \in Y} I_y$.

(1) Let $E = \prod_{y \in Y} D_y$ be the set of $s \subseteq H$ such that there are $y_1 < \dots < y_n$ in Y satisfying:

- (a) for all $i, j \in H$, if $i \upharpoonright \{y_1, \dots, y_n\} = j \upharpoonright \{y_1, \dots, y_n\}$ then $i \in s$ iff $j \in s$,
- (b) $\{\langle i(y_1), \dots, i(y_n) \rangle : i \in s\} \in D_{y_1} \times \dots \times D_{y_n}$.

(2) The iterated ultrapower $\prod_{\bar{D}} |M|$ or $\prod_E |M|$ of the set $|M|$, noting E is a filter on H , is the set $\{f/E : f : H \rightarrow |M| \text{ and for some finite } Z_f \subseteq Y \text{ for all } i, j \in H, \text{ if } i \upharpoonright Z_f = j \upharpoonright Z_f, \text{ then } f(i) = f(j)\}$.

(3) The iterated ultrapower $\prod_E M$ of the τ -structure M with respect to $\langle D_y : y \in Y \rangle$ is the τ -structure whose universe is the set $\prod_E |M|$; for each n -ary predicate symbol R of L , $R^{\prod_E M}(f_1/E, \dots, f_n/E)$ iff $\{i \in H : R^M(f_1(i), \dots, f_n(i))\} \in E$; for each n -ary function symbol F of L , $F^{\prod_E M}(f_1/E, \dots, f_n/E) = \langle F^M(f_1(i), \dots, f_n(i)) : i \in H \rangle / E$.

(4) The canonical map $d : M \rightarrow \prod_E M$ is defined as usual by:

$$d(a) = \langle a : i \in H \rangle / E.$$

Remark 1.31.

(1) Every ultrapower is a limit ultrapower: take $G = P(I \times I)$ and note that $\prod_D M = \prod_{D/G} M$.

(2) Every iterated ultrapower is a limit ultrapower.

[Why? let the iterated ultrapower be defined by $\langle Y, < \rangle$ and $\langle \langle I_y, D_y \rangle : y \in Y \rangle$ (see Definition 1.30). For $Z \in [Y]^{<\omega}$, let $A_Z = \{(i, j) \in H \times H : i \upharpoonright Z = j \upharpoonright Z\}$. Note that $\{A_Z : Z \in [Y]^{<\omega}\}$ has the finite intersection property and hence can be extended to a filter G on $H \times H$. Now for any model M we have $\prod_E M \cong \prod_{D/G} M$ for every filter D over H extending E under the map $f/E \rightarrow f/D$.]

Definition 1.32. Suppose that M is an τ -structure, D is an ultrafilter on a non-empty set I , and G is a suitable set of equivalence relations on I , which means:

- (i) if $e \in G$ and e' is an equivalence relation on I coarser than e , then $e' \in G$,
- (ii) G is closed under finite intersections,
- (iii) if $e \in G$, then $D/e = \{A \subseteq I/e : \bigcup_{x \in A} x \in D\}$ is a κ -complete ultrafilter on I/e which, for simplicity, has cardinality κ .

Then $\text{Op}(M, I, D, G)$ is the limit ultrapower $\prod_{D/\hat{G}} M$ where \hat{G} is the filter on $I \times I$ generated by G . When clear from the context One abbreviates $\text{Op}(M, I, D, G)$ by $\text{Op}(M)$, and one writes f_{Op} for the canonical map $d : M \rightarrow \text{Op}(M)$.

Note that,

Observation / Convention 1.33.

(1) For any τ -structure N , f_{Op} is an $\leq_{\mathfrak{k}}$ -elementary embedding of N into $\text{Op}(N)$ and in particular $f_{\text{Op}} : N \xrightarrow{\mathfrak{k}} \text{Op}(N)$.

(2) Since f_{Op} is canonical, one very often identifies N with the τ -structure $\text{rng}(f_{\text{Op}})$ which is an \mathfrak{k} -elementary substructure of $\text{Op}(N)$, and one writes $N \leq_{\mathfrak{k}} \text{Op}(N)$. In particular for any model $M \in K$ and $\text{Op}, f_{\text{Op}} : M \rightarrow_{\mathfrak{k}} \text{Op}(M)$ (briefly written $M \leq_{\mathfrak{k}} \text{Op}(M)$) so that $\text{Op}(M)$ is a model from K too.

(3) Remark that if D is a κ -complete ultrafilter on I and G is a filter on $I \times I$, then $\text{Op}(M, I, D, G)$ is well defined.

(4) Suitable limit ultrapower means one using a suitable triple.

More information on limit and iterated ultrapowers can be found in [CK73] and [HS81].

Observation 1.34. *Given θ -complete ultrafilters D_1 on I_1 , D_2 on I_2 and suitable filters G_1 on $I_1 \times I_1$, G_2 on $I_2 \times I_2$ respectively, there exist a θ -complete ultrafilter D on a set I and a filter G on $I \times I$ such that:*

$$\text{Op}(M, I, D, G) = \text{Op}(\text{Op}(M, I_1, D_1, G_1), I_2, D_2, G_2)$$

and (D, G, I) is κ -complete. Also iterated ultrapower (along any linear order) which each iterand being ultrapower by κ -complete ultrafilter, gives a suitable triple (in fact, even iteration of suitable limit ultrapowers is a suitable ultrapower).

Definition 1.35. Suppose that K is a class of τ -structures and $< = <_{\mathcal{K}}$ is a binary relation on K (usually $(K, <) = (K_{\mathfrak{k}}, <_{\mathfrak{k}})$). For $M, N \in K$, write $f: M \leq_{\mathcal{K}}^{\text{nice}} N$ to mean (if $<$ is clear from the context we may write $f: M \xrightarrow{\text{nice}} N$ and, if $f = \text{id}_M$ we may write $M \leq_{\text{nice}} N$):

- (a) f is an isomorphism from M into N and $\text{rng}(f) < N$. Which means $f(M) < N$, where $f(M)$ is the model M' with universe $\text{rng}(f)$ such that f is an isomorphism from M into M' ,
- (b) there are a set I , an ultrafilter D on I , a suitable set G of equivalence relations on I (so Definition 1.32 clause (i), (ii), (iii) holds), and an isomorphism g from N into $\text{Op}(M, I, D, G)$ such that $\text{rng}(g) < \text{Op}(M, I, D, G)$ and $gf = f_{\text{Op}}$, where f_{Op} is the canonical embedding of M into $\text{Op}(M, I, D, G)$. f is called a $<$ -nice embedding of M into N . Of course, one writes $f: M \xrightarrow{\text{nice}} N$ and says that f is a nice embedding of M into N when $<$ is clear from the context.

Example 1.36.

(1) Consider T, \mathcal{F} and $\mathcal{K} = \langle K, < \rangle = (K, <_{\mathcal{K}})$ as set up in 1.13. In this case $f: M \xrightarrow{\text{nice}} N$ holds iff $f: M \xrightarrow{\mathcal{F}} N$ and for some suitable $\langle I, D, G \rangle$ and some $g: N \xrightarrow{\mathcal{F}} \text{Op}(M, I, D, G)$, $gf = f_{\text{Op}}$.

Abusing notation one writes $M \xrightarrow{\text{nice}} N$ to mean that there are f, g and Op such that $f: M \xrightarrow{\text{nice}} N$ using g and Op . IF NOT SAID OTHERWISE, $<$ is $<_{\mathfrak{k}}$. We may also write $M \leq_{\text{nice}} N$, and for linear orders we use $I \subseteq_{\text{nice}} J$.

Example 1.37. Let \mathbf{LO} be the class of linear orders and let $(I, \leq_I) < (J, \leq_J)$ mean that $(I, \leq_I) \subseteq (J, \leq_J)$, i.e. (I, \leq_I) is a suborder of (J, \leq_J) . If $f: (I, \leq_I) \xrightarrow{\text{nice}} (J, \leq_J)$, then identifying isomorphic orders, one has $(I, \leq_I) \subseteq (J, \leq_J) \subseteq \text{Op}(I, \leq_I)$ and we may write $(I, \leq_I) \subseteq_{\text{nice}} (J, \leq_J)$.

Observation 1.38. Assume that \mathfrak{k} is as in 1.20. Suppose further $M \leq_{\text{nice}} N$ and $M \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} N$ where $M, M', N \in K$. Then $M \leq_{\mathfrak{k}} M'$.

Proof. For some f, g and Op , $f: M \xrightarrow{\mathfrak{k}} N$, $g: N \xrightarrow{\mathfrak{k}} \text{Op}(M)$ and $gf = f_{\text{Op}}$. Now $g: M' \xrightarrow{\mathfrak{k}} \text{Op}(M)$ (since $M' \leq_{\mathfrak{k}} N$) and $gf = f_{\text{Op}}$ so that $M \leq_{\text{nice}} M'$. $\square_{1.38}$

Observation 1.39. *Suppose that $\langle M_i : i \leq \delta \rangle$ is a continuous increasing chain and for each $i < \delta$, $M_i \underset{\text{nice}}{\leq} M_{i+1}$. Then for every $i < \delta$, $M_i \underset{\text{nice}}{\leq} M_\delta$.*

Proof. Like the proof of 1.31(2). For each $i < \delta$, there is a (I_i, D_i, G_i) as in Definition 1.32 which witnesses $M_i \underset{\text{nice}}{\leq} M_{i+1}$. Let $I \stackrel{\text{def}}{=} \prod_{i < \delta} I_i$, $G = \{e : e \subseteq I \times I \text{ and for some } n < \omega \text{ and } \alpha_1 < \dots < \alpha_n < \delta, \text{ and } e_1 \in G_{\alpha_1}, \dots, e_n \in G_{\alpha_n} \text{ we have: for every } x, y \in I, \text{ such that } (x(\alpha_\ell), y(\alpha_\ell)) \in e_\ell \text{ for } \ell = 1, \dots, n \text{ we have } (x, y) \in e\}$. D will be any ultrafilter on I such that: if $n < \omega$ and $\alpha_1 < \dots < \alpha_n < \delta$, $e_1 \in G_{\alpha_1}, \dots, e_n \in G_{\alpha_n}$, e_ℓ is an equivalence relation on I_{α_ℓ} for $\ell = 1, \dots, n$ and $A \in (D_{\alpha_1}/e_{\alpha_1}) \times \dots \times (D_{\alpha_n}/e_{\alpha_n})$, then the set $\{x \in I : \langle x(\alpha_1)/e_{\alpha_1}, \dots, x(\alpha_n)/e_{\alpha_n} \rangle \in A\}$ belongs to D . We leave the rest to the reader. $\square_{1.39}$

Claim 1.40. *For every model M and $\lambda \geq \kappa + \text{LS}_\mathfrak{k} + \|M\|$ there is N such that $M \underset{\text{nice}}{\leq} N$, $M \neq N$ and $\|N\| = \lambda$.*

Proof. As κ is measurable. $\square_{1.40}$

Definition 1.41.

(1) Assume D is an ultra-filter on κ . For an AEC $\mathfrak{k} = (K_\mathfrak{k}, \leq_\mathfrak{k})$ we say \mathfrak{k} is *D-compact* when:

- (a) if $M \in K_\mathfrak{k}$ then the ultra-power M^κ/D belongs to $K_\mathfrak{k}$,
- (b) moreover, the canonical embedding of M into M^κ/D is a $\leq_\mathfrak{k}$ -embedding,
- (c) if $M \leq_\mathfrak{k} N$ then the canonical embedding of M^κ/D into N^κ/D is a $\leq_\mathfrak{k}$ -embedding,
- (d) \mathfrak{k} has a model of cardinality $\geq \kappa$ (or at least of cardinality σ where D is not σ -complete).

(2) Assume $(Y, \bar{I}, \bar{D}, H, E)$ is as in 1.30. For an AEC \mathfrak{k} we say \mathfrak{k} is *D-compact⁺* when:

- (a) if $M \in K_\mathfrak{k}$ and $\pi_E M \in K_\mathfrak{k}$,
- (b) moreover, the canonical embedding of M into $\pi_E M$ is a $\leq_\mathfrak{k}$ -embedding,
- (c) if $M \leq_\mathfrak{k} N$ then the canonical embedding of $\pi_E M$ into $\pi_E N$ is a $\leq_\mathfrak{k}$ -embedding.

Claim 1.42.

- (1) If \mathfrak{k} is a AEC and $|\tau_\mathfrak{k}| + \text{LS}(\mathfrak{k}) < \kappa$ then \mathfrak{k} is *D-compact* and even *D-compact⁺*.
- (2) If \mathfrak{k} is (μ, λ, κ) -representable, then \mathfrak{k} is *D-compact* (recall D is a normal ultrafilter on κ).
- (3) Also the claim on Op generalizes.

Proof.

(1) By 1.24 and part (c).

(2), (3) Easy. $\square_{1.42}$

For the rest of this work,

Hypothesis 1.43. (1) \mathfrak{k} is a *D-compact* AEC with $\text{LS}_\mathfrak{k} = \chi \geq \kappa$, D a κ -complete ultra filter on κ and $K = K_\mathfrak{k}$.

(2) $\Phi, \mathbf{a} = \langle \bar{a}_s : s \in I \rangle$ are as in 1.41.

(3) Finally, $\lambda > \kappa + |T|$ usually denotes a power in which T is categorical.

§ 2. THE AMALGAMATION PROPERTY FOR REGULAR CATEGORICITY

The main aim of this section is to show that if K is categorical in the regular cardinal $\lambda > \text{LS}_{\mathfrak{k}}$, then $\mathfrak{k}_{<\lambda} = \langle K_{<\lambda}, \leq_{\mathfrak{k}} \rangle$ has the amalgamation property (AP) (Definition 1.11 (1)). Categoricity is not presumed if not required.

Recall 1.43 is assumed.

Lemma 2.1. *Suppose that $\chi \leq \mu \leq \lambda$, $M, M_1, M_2 \in K_{\leq \mu}$, $f_1: M \xrightarrow[\text{nice}]{\rightarrow} M_1$, $f_2: M \xrightarrow[\mathfrak{k}]{\rightarrow} M_2$. Then there is an amalgam $N \in K_{\leq \mu}$ of M_1, M_2 over M with respect to f_1, f_2 . Moreover, there are $g_\ell: M_\ell \xrightarrow[\mathfrak{k}]{\rightarrow} N$ for $\ell = 1, 2$ such that $g_1 f_1 = g_2 f_2$ hence $\text{rng}(g_2 f_2) = \text{rng}(g_1 f_1)$ and $g_1: M_1 \xrightarrow[\text{nice}]{\rightarrow} N$.*

Proof. There are g and Op such that $g: M_1 \xrightarrow[\mathfrak{k}]{\rightarrow} \text{Op}(M)$, $g f_1 = f_{\text{Op}}$. Now, f_2 induces an $\leq_{\mathfrak{k}}$ -elementary embedding f_2^* of $\text{Op}(M)$ into $\text{Op}(M_2)$ such that $f_2^* f_{\text{Op}} = f_{\text{Op}} f_2$. Let $g_1 = f_2^* g$ and $g_2 = f_{\text{Op}} \upharpoonright M_2$. By Fact 1.10 one finds $N \in K_{\leq \mu}$ such that $\text{rng}(g_1) \cup \text{rng}(g_2) \subseteq N \leq_{\mathfrak{k}} \text{Op}(M_2)$. Now N is an amalgam of M_1, M_2 over M with respect to f_1, f_2 since $g_1 f_1 = f_2^* g f_1 = f_2^* f_{\text{Op}} = f_{\text{Op}} f_2 = g_2 f_2$. The last phrase in the lemma is easy by properties of Op . $\square_{2.1}$

Lemma 2.2. *Suppose that $M \in K_{\leq \mu}$ is a μ -c.a.b., $\chi \leq \mu < \lambda$. Then $N \in K_{<\lambda}$ is a $\|N\|$ -c.a.b. whenever $f: M \xrightarrow[\text{nice}]{\rightarrow} N$.*

Proof. Suppose that $g: N \xrightarrow[\mathfrak{k}]{\rightarrow} \text{Op}(M)$, $g f = f_{\text{Op}}$. M is a μ -c.a.b., so for some $M_i \in K_{\leq \mu}$ and $f_i: M \xrightarrow[\mathfrak{k}]{\rightarrow} M_i$ ($i = 1, 2$) there is no amalgam of M_1, M_2 over M w.r.t. f_1, f_2 . Let f_i^* be the $\leq_{\mathfrak{k}}$ -elementary embedding from $\text{Op}(M)$ into $\text{Op}(M_i)$ defined by f_i (note that $f_i^* f_{\text{Op}} = f_{\text{Op}} f_i$, $i = 1, 2$). Choose N_i of power $\|N\|$ such that $M_i \cup \text{rng}(f_i^* g) \subseteq N_i \leq_{\mathfrak{k}} \text{Op}(M_i)$. Note that $f_i^* f: N \xrightarrow[\mathfrak{k}]{\rightarrow} N_i$. It suffices to show that there is no amalgam of N_1, N_2 over N w.r.t. $f_1^* g, f_2^* g$.

Well, suppose that one could find an amalgam N^* and $h_i: N_i \xrightarrow[\mathfrak{k}]{\rightarrow} N^*$, $i = 1, 2$, with $h_1(f_1^* g) = h_2(f_2^* g)$. Using Fact 1.10 choose M^* , $\|M^*\| \leq \mu$, $M^* \leq_{\mathfrak{k}} N^*$, $\text{rng}(h_1 f_{\text{Op}} \upharpoonright M_1) \cup \text{rng}(h_2 f_{\text{Op}} \upharpoonright M_2) \subseteq |M^*|$. Set $g_i = h_i f_{\text{Op}} \upharpoonright M_i$, for $i = 1, 2$, and note that:

$$\begin{aligned} g_1 f_1 &= h_1 f_{\text{Op}} f_1 = h_1 f_1^* f_{\text{Op}} = h_1 f_1^* g f = h_2 f_2^* g f = h_2 f_2^* f_{\text{Op}} \\ &= h_2 f_{\text{Op}} f_2 = g_2 f_2 \end{aligned}$$

In other words, M^* is an amalgam of M_1, M_2 over M w.r.t. f_1, f_2 -contradiction. It follows that N is a $\|N\|$ -c.a.b. $\square_{2.2}$

Corollary 2.3. *Suppose that $\chi \leq \mu < \lambda$. If $M \in K_\mu$ is a μ -c.a.b., then there exists $M^* \in K_\lambda$ such that:*

(*) $M \leq_{\mathfrak{k}} M^*$ and for every $M' \in K_{<\lambda}$, if $M \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} M^*$, then M' is a $\|M'\|$ -c.a.b.

Proof. As $\|M\| \geq \kappa$, for some appropriate Op one has $\|\text{Op}(M)\| \geq \lambda$, and by Fact 1.10 one finds $M^* \in K_\lambda$ such that $M \subseteq M^* \leq_{\mathfrak{k}} \text{Op}(M)$. Let's check that M^* works in (*). Take $M' \in K_{<\lambda}$, $M \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} M^*$; so $M \xrightarrow[\text{nice}]{\leq} M'$ since $M^* \leq_{\mathfrak{k}} \text{Op}(M)$; hence by 2.2, M' is a $\|M'\|$ -c.a.b. $\square_{2.3}$

Theorem 2.4. *Suppose that \mathfrak{k} is λ -categorical, $\lambda = \text{cf}(\lambda) > \chi$. If $K_{<\lambda}$ fails AP, then there is $N^* \in K_\lambda$ such that for some continuous increasing $\leq_{\mathfrak{k}}$ -chain $\langle N_i \in K_{<\lambda} : i < \lambda \rangle$ of models,*

- (1) $N^* = \bigcup_{i < \lambda} N_i$,
 (2) for every $i < \lambda$, $N_i \not\leq_{\text{nice}} N_{i+1}$ (and so $N_i \not\leq_{\text{nice}} N^*$).

Proof. $\mathfrak{k}_{<\lambda}$ fails AP, so for some $\mu < \lambda$ and $M \in K_{\leq \mu}$, M is a μ -c.a.b. By 2.2 and 1.40 Without loss of generality $M \in K_\mu$. Choose by induction a continuous strictly increasing $\leq_{\mathfrak{k}}$ -chain $\langle N_i \in K_{<\lambda} : i < \lambda \rangle$ as follows:

$N_0 = M$; at a limit ordinal i , take the union; at a successor ordinal $i = j+1$, if there is $N \in K_{<\lambda}$ such that $N_j \leq_{\mathfrak{k}} N$ and $N_j \not\leq_{\text{nice}} N$, choose $N_i = N$, otherwise choose for N_i any non-trivial $\leq_{\mathfrak{k}}$ -elementary extension of N_j of power less than λ . $\square_{2.4}$

Claim 2.5. $(\exists j_0 < \lambda)(\forall j \in (j_0, \lambda))(N_j \text{ is a } \|N_j\| \text{-c.a.b.})$.

Proof. Suppose not. So one has a strictly increasing sequence $\langle j_i : i < \lambda \rangle$ such that for each $i < \lambda$, N_{j_i} is not a $\|N_{j_i}\|$ -c.a.b. Let $N_* = \bigcup_{i < \lambda} N_{j_i}$. So $\|N_*\| = \lambda$. Applying 2.3 one can find $M^* \in K_\lambda$ such that whenever $M' \in K_{<\lambda}$ and $M \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} M^*$, then M' is a $\|M'\|$ -c.a.b.

Since \mathfrak{k} is λ -categorical, there is an isomorphism g of N_* onto M^* . Let $N = g^{-1}(M)$ and $M_i = g(N_{j_i})$ for $i < \lambda$. $\|N\| = \mu < \text{cf}(\lambda) = \lambda$, so there is $i_0 < \lambda$ such that $N \subseteq N_{j_{i_0}}$.

In fact $N_{j_{i_0}}$ is a $\|N_{j_{i_0}}\|$ -c.a.b. [Otherwise, consider $M_{j_{i_0}}$. Since $M \leq_{\mathfrak{k}} M_{j_{i_0}} \leq_{\mathfrak{k}} M^*$ and $\|M_{j_{i_0}}\| < \lambda$, $M_{j_{i_0}}$ is a $\|M_{j_{i_0}}\|$ -c.a.b., so there are $f_\ell : M_{j_{i_0}} \xrightarrow{\mathcal{F}} M'_\ell$, ($\ell = 1, 2$), with no amalgam of M'_1, M'_2 over $M_{j_{i_0}}$ w.r.t. f_1, f_2 . If $N_{j_{i_0}}$ is not a $\|N_{j_{i_0}}\|$ -c.a.b., then one can find an amalgam $N^+ \in K_{\leq \|N_{j_{i_0}}\|}$ of M'_1, M'_2 over $N_{j_{i_0}}$ w.r.t. $f_1 g, f_2 g$ such that $h_\ell : M'_\ell \xrightarrow{\mathfrak{k}} N^+$ and $h_1(f_1 g) = h_2(f_2 g)$; so $h_1 f_1 = h_2 f_2$ and N^+ is thus an amalgam of M'_1, M'_2 over $M_{j_{i_0}}$ w.r.t. f_1, f_2 , $\|N^+\| \leq \|N_{j_{i_0}}\| = \|M_{j_{i_0}}\|$ -contradiction.] This contradicts the choice of $N_{j_{i_0}}$. So the claim is correct.

It follows that for each $j \in (j_0, \lambda)$ there are N_j^1, N_j^2 in $K_{<\lambda}$ and $f_\ell : N_j \rightarrow_{\mathfrak{k}} N_j^\ell$ such that no amalgam of N_j^1, N_j^2 over N_j w.r.t. f_1, f_2 exists. By 2.2 for some $\ell \in \{1, 2\}$, $N_j \not\leq_{\text{nice}} N_{j+1}^\ell$. So by the inductive choice of $\langle N_{j+1} : j < \lambda \rangle$, $\forall j \in (j_0, \lambda)(N_j \not\leq_{\text{nice}} N_{j+1})$. Taking $N^* = \bigcup_{j_0 < j < \lambda} N_j$, one completes the proof (of course for $j_0 < j < \lambda$, $N_j \not\leq_{\text{nice}} N^*$: if $N_j \leq_{\mathfrak{k}} N^* \leq_{\mathfrak{k}} \text{Op}(N_j)$, then by 1.38 $N_j \leq_{\text{nice}} N_{j+1}$ -contradiction). $\square_{2.5}$

Theorem 2.6. Suppose that $(I, <_I), (J, <_J)$ are linear orders and I is a suborder of J . If $(I, <_I) \subseteq_{\text{nice}} (J, <_J)$, then $\text{EM}(I) \preceq_{\text{nice}} \text{EM}(J)$.

Proof. Without loss of generality for some cardinal μ , ultrafilter D on μ and suitable set G , a filter on $\mu \times \mu$, $(I, <_I) \subseteq (J, <_J) \subseteq \text{Op}((I, <_I), \mu, D, G) = \text{Op}(I, <)$, and $|\text{Op}(I, <)| = \{f/D : f \in {}^\mu I, \text{eq}(f) \in G\}$ where $\text{eq}(f) = \{(i, j) \in \mu \times \mu : f(i) = f(j)\}$. So for each $t \in J$, there exists $f_t \in {}^\mu I$ such that $t = f_t/D$. Note that if $t \in I$, then $f_t/D = f_{\text{Op}(t)}$ so that without loss of generality for all $i < \mu$, $f_t(i) = t$. Define a map h from $\text{EM}(J)$ into $\text{Op}(\text{EM}(I))$ as follows. An element of $\text{EM}(J)$ has the form

$$\sigma^{\text{EM}'(J)}(x_{t_1}, \dots, x_{t_n})$$

where $t_1, \dots, t_n \in J$, σ an τ -term. Define, for $t \in J$, $g_t \in {}^\mu \text{EM}(I)$ by $g_t(i) = x_{f_t(i)}$. Note that $f_t(i) \in I$, so that $x_{f_t(i)} \in \text{EM}(I)$ and so $g_t/D \in \text{Op}(\text{EM}(I))$. Let $h(\sigma^{\text{EM}'(J)}(x_{t_1}, \dots, x_{t_n})) = \sigma^{\text{Op}(\text{EM}'(I))}(g_{t_1}/D, \dots, g_{t_n}/D)$ which is an element in

$\text{Op}(\text{EM}(I))$. The reader is invited to check that h is an $\leq_{\mathfrak{t}}$ -elementary embedding of $\text{EM}(J)$ into $\text{Op}(\text{EM}(I))$. So $\text{EM}(I) \leq_{\mathfrak{t}} \text{EM}(J)$.

Finally note that if $a = \sigma^{\text{EM}'(I)}(x_{t_1}, \dots, x_{t_n}) \in \text{EM}(I)$, $t_1, \dots, t_n \in I$, then $h(a) = \sigma^{\text{Op}(\text{EM}'(I))}(g_{t_1/D}, \dots, g_{t_n/D}) = \sigma^{\text{Op}(\text{EM}'(I))}(\langle x_{t_1} : i < \mu \rangle / D, \dots, \langle x_{t_n} : i < \mu \rangle / D) = f_{\text{Op}}(\sigma^{\text{EM}'(I)}(x_{t_1}, \dots, x_{t_n})) = f_{\text{Op}}(a)$. Thus $\text{EM}(I) \underset{\text{nice}}{\leq} \text{EM}(J)$. $\square_{2.6}$

Criterion 2.7. Suppose that $(I, <)$ is a suborder of the linear order $(J, <)$. We have $(I, \leq) \underset{\text{nice}}{\subseteq} (J, <)$ when:

- (*) for every $t \in J \setminus I$,
- (\aleph) $\text{Op}((I, <) \upharpoonright \{s \in I : (J, <) \models s < t\}) = \kappa$,

or

- (\beth) $\text{Op}((I, <)^* \upharpoonright \{s \in I : (J, <)^* \models s <^* t\}) = \kappa$

Notation 2.8. $(I, <)^*$ is the (reverse) linear order $(I^*, <^*)$ where $I^* = I$ and $(I^*, <^*) \models s <^* t$ iff $(I, <) \models t < s$.

Proof. Let us list some general facts which facilitate the proof.

Fact (A): Let $\underline{\kappa}$ denote the linear order $(\kappa, <)$ where $<$ is the usual order $\epsilon \upharpoonright \kappa \times \kappa$. If $J_1 = \underline{\kappa} + J_0$, then $\underline{\kappa} \underset{\text{nice}}{\subseteq} J_1$ (+ is addition of linear orders in which all elements in the first order precede those in the second).

Fact (B): If $\underline{\kappa} \subseteq (I, <)$, $\underline{\kappa}$ is unbounded in $(I, <)$ and $J_1 = I + J_0$, then $I \underset{\text{nice}}{\subseteq} J_1$.

Fact (C): If $I \underset{\text{nice}}{\subseteq} J$, then $I + J_1 \underset{\text{nice}}{\subseteq} J + J_1$.

Fact (D): $I \underset{\text{nice}}{\subseteq} J$ iff $(J, <)^* \underset{\text{nice}}{\subseteq} (I, <)^*$.

Fact (E): If $\langle I_\alpha : \alpha \leq \delta \rangle$ is a continuous increasing sequence of linear orders and for $\alpha < \delta$, $I_\alpha \underset{\text{nice}}{\subseteq} I_{\alpha+1}$, then $I_\alpha \underset{\text{nice}}{\subseteq} I_\delta$.

Now using these facts, let us prove the criterion. Define an equivalence relation E on $J \setminus I$ as follows: tEs iff t and s define the same Dedekind cut in $(I, <)$. Let $\{t_\alpha : \alpha < \delta\}$ be a set of representatives of the E -equivalence classes. For each $\beta \leq \delta$, define

$$I_\beta = J \upharpoonright \left\{ t : t \in I \vee \bigvee_{\alpha < \beta} tEt_\alpha \right\}$$

so $I_0 = I$, $I_\delta = J$ and $\langle I_\alpha : \alpha \leq \delta \rangle$ is a continuous increasing sequence of linear orders. By Fact (E), to show that $I \underset{\text{nice}}{\subseteq} J$, it suffices to show that $I_\alpha \underset{\text{nice}}{\subseteq} I_{\alpha+1}$ for each $\alpha < \delta$.

Fix $\alpha < \delta$. Now t_α belongs to $J \setminus I$, so by (*), (\aleph) or (\beth) holds. By Fact (D), it is enough to treat the case (\aleph). So without loss of generality $\text{Op}((I, <) \upharpoonright \{s \in I : (J, <) \models s < t_\alpha\}) = \kappa$.

Let

$$I_\alpha^a = \{t \in I_\alpha : t < t_\alpha\}$$

$$I_\alpha^b = \{t \in I_{\alpha+1} : t \in I_\alpha^a \vee tEt_\alpha\}$$

$$I_\alpha^c = \{t \in I_\alpha : t > t_\alpha\}.$$

Note that $I_\alpha = I_\alpha^a + I_\alpha^c$ and $I_{\alpha+1} = I_\alpha^b + I_\alpha^c$. Recalling Fact (C), it is now enough to show that $I_\alpha^a \subseteq_{\text{nice}} I_\alpha^b$. Identifying isomorphic orders and using (N), one has that κ is unbounded in I_α^a and $I_\alpha^b = I_\alpha^a + (I_\alpha^b \setminus I_\alpha^a)$ so by Fact (B), $I_\alpha^a \subseteq_{\text{nice}} I_\alpha^b$ as required. $\square_{2.7}$

Of the five facts, we prove (A), (B), and (E) as (C) and (D) are obvious.

Proof of Fact (A). Since κ is measurable, there is a κ -complete uniform ultrafilter D on κ [see J]. For every linear order J_0 (or J_0^*) there is $\text{Op}_{I,D}(-)$, the iteration of I ultrapowers $(-)^{\kappa}/D$, ordered in the order J_0 (or J_0^*), giving the required embedding (use 1.34). \square

Proof of Fact (B). Since $\kappa \subseteq I$ and using Fact (A), we know that there is an operation Op such that the following diagram commutes:

$$\begin{array}{ccccc}
 & & I & & \\
 & \nearrow \text{id} & & \nwarrow \text{Canonical} & \\
 \kappa & \xrightarrow{\text{Canonical}} & \text{Op}(\kappa) & \xrightarrow{\text{Natural}} & \text{Op}(I) \\
 & \searrow \text{id} & \uparrow \text{Fact}(A) & & \uparrow \\
 & & \kappa + J_0 & \xrightarrow{\text{id}} & I + J_0
 \end{array}$$

Chasing through the diagram, we obtain the required embedding. \square

Proof of Fact (E). Apply 1.39 to the chain $\langle I_\alpha : \alpha \leq \delta \rangle$. \square

Fact 2.9. Suppose that $\lambda \geq \kappa$. There exist a linear order $(I, <_I)$ of power λ and a sequence $\langle A_i \subseteq I : i \leq \lambda \rangle$ of pairwise disjoint subsets of I , each of power κ such that $I = \bigcup_{i \leq \lambda} A_i$ and,

(*) if $\lambda \in X \subseteq \lambda + 1$, then $I \upharpoonright \bigcup_{i \in X} A_i \subseteq_{\text{nice}} I$.

Proof. Let $I = (\lambda + 1) \times \kappa$ and define $<_I$ on I : $(i_1, \alpha_1) <_I (i_2, \alpha_2)$ iff $i_1 < i_2$ or $(i_1 = i_2 \text{ and } \alpha_1 > \alpha_2)$. For each $i \leq \lambda$, let $A_i = \{i\} \times \kappa$. Check (*) of 2.7: suppose that $\lambda \in X \subseteq \lambda + 1$. Write $I_X = I \upharpoonright (\bigcup_{i \in X} A_i)$. To show that $I_X \subseteq_{\text{nice}} I$, one employs Criterion 2.7. Consider $t \in I - I_X$, say $t = (i, \alpha)$ (note that $\alpha < \kappa$ and $i < \lambda$, since $\lambda \in X$) and $i \notin X$. Let $j = \min(X - i)$; note that j is well-defined, since $\lambda \in X - i$, and $j \neq i$. For every $\beta < \kappa$, one has $t <_I (j, \beta)$ and $(j, \beta) \in I_X$. Also if $s \in I_X$ and $t <_I s$, then for some $\beta < \kappa(j, \beta) <_I s$. Thus $\langle (j, \beta) : \beta < \kappa \rangle$ is a cofinal sequence in $(I_X \upharpoonright \{s \in I : t <_I s\})^*$. By the criterion, $I_X \subseteq_{\text{nice}} I$. $\square_{2.9}$

Theorem 2.10. Suppose that $\kappa = \text{cf}(\delta) \leq \delta < \lambda$. Then $\text{EM}(\delta) \subseteq_{\text{nice}} \text{EM}(\lambda)$.

Proof. By Fact (B) of 2.7, one has that $\delta \subseteq_{\text{nice}} \lambda$; so by 2.6, $\text{EM}(\delta) \subseteq_{\text{nice}} \text{EM}(\lambda)$. $\square_{2.10}$

Now let's turn to the main theorem of this section.

Theorem 2.11. Suppose that T is categorical in the regular cardinal $\lambda > \chi$. Then $\mathfrak{k}_{<\lambda}$ has the amalgamation property.

Proof. Suppose that $\mathfrak{k}_{<\lambda}$ fails AP. Note that $\|\text{EM}(\lambda)\| = \lambda$. Apply 2.4 to find $M^* \in K_\lambda$ and $\langle M_i : i < \lambda \rangle$ satisfying 2.4 (1) and 2.4 (2). Since T is λ -categorical, $M^* \cong \text{EM}(\lambda)$, so without loss of generality $\text{EM}(\lambda) = \bigcup_{i < \lambda} M_i$. $C = \{i < \lambda : M_i =$

$\text{EM}(i)\}$ is a club of λ . Choose $\delta \in C, \text{cf}(\delta) = \kappa$. By 2.10, $\text{EM}(\delta) \underset{\text{nice}}{\leq} \text{EM}(\lambda)$, so $M_\delta \underset{\text{nice}}{\leq} M^*$. But of course by 2.4(2) $M_\delta \not\leq M^*$ -contradiction. $\square_{2.11}$

Theorem 2.12. *Suppose that K is categorical in $\lambda > \chi$. Then:*

(1) *K has a model M of power λ such that if $N \leq_{\mathfrak{t}} M$ and $\|N\| < \lambda$, then there exists N' such that:*

- (α) $N \leq_{\mathfrak{t}} N' \leq_{\mathfrak{t}} M$,
- (β) $\|N'\| = \|N\| + \chi$,
- (γ) $N' \underset{\text{nice}}{\leq} M$.

(2) *K has a model M of power λ and an expansion M^+ of M by at most χ functions such that if $N^+ \subseteq M^+$, then $N^+ \upharpoonright \tau \underset{\text{nice}}{\leq} M$.*

Proof. Let $\langle I, \langle A_i : i \leq \lambda \rangle \rangle$ be as in 2.9. Let $M = \text{EM}(I)$. Suppose that $N \leq_{\mathfrak{t}} M, \|N\| < \lambda$. Then there exists $J \subseteq I, |J| < \lambda$ such that $N \subseteq \text{EM}(J)$ hence $N \underset{\text{nice}}{\leq} \text{EM}(J) \underset{\text{nice}}{\leq} \text{EM}(I)$. So there is $X \subseteq \lambda + 1, \lambda \in X, |X| < \lambda$ such that $J \subseteq \bigcup_{i \in X} A_i$. Note that $|\bigcup_{i \in X} A_i| \leq |X|\kappa < \lambda$. Now $N' = \text{EM}(I \upharpoonright \bigcup_{i \in X} A_i)$ is as required, since $I \upharpoonright \bigcup_{i \in X} A_i \underset{\text{nice}}{\leq} I$ and so by 2.6 $\text{EM}(I \upharpoonright (\bigcup_{i \in X} A_i)) \underset{\text{nice}}{\leq} \text{EM}(I)$. This proves (1).

(2) We expand $M = \text{EM}(I)$ as follows:

- (a) by all functions of $\text{EM}'(I)$,
- (b) by the unary functions $f_\ell (\ell < n)$ which are chosen as follows: we know that for each $b \in M$ there is σ_b an τ_1 -term (τ_1 -the vocabulary of $\text{EM}'(I)$) and $t(b, 0) < t(b, 1) < \dots < t(b, n_{\sigma_b} - 1)$ from I such that

$$b = \tau_b(x_{t(b,0)}, x_{t(b,1)}, \dots, x_{t(b, n_{\sigma_b}-1)})$$

(it is not unique, but we can choose one; really if we choose it with n_b minimal it is almost unique). We let

$$f_\ell(b) = \begin{cases} x_{t(b,\ell)}, & \text{if } \ell < n_{\tau_b}, \\ b, & \text{if } \ell \geq n_{\tau_b}. \end{cases}$$

- (c) by unary functions g_α, g^α for $\alpha < \kappa$ such that if $t < s$ are in I , $\alpha = \text{otp}[(t, s)_I^*]$ then $g^\alpha(x_t) = x_s, \bigvee_{\beta < \kappa} g_\beta(x_s) = x_t$ (more formally $g^\alpha(x_{(i,\beta)}) = x_{(i,\beta+\alpha)}$ and $g_\alpha(x_{(i,\beta)}) = x_{(i,\alpha)}$) in the other cases $g^\alpha(b) = b, g_\alpha(b) = b$.
- (d) by individual constants $c_\alpha = x_{(\lambda,\alpha)}$ for $\alpha < \kappa$.

Now suppose N^+ is a submodel of M^+ and N its τ -reduct. Let $J \stackrel{\text{def}}{=} \{t \in I : x_t \in N\}$, now J is a subset of I of cardinality $\leq \|N\|$ as for $t \neq s$ from J , $x_t \neq x_s$. Also if $b \in N$ by clause (b), $x_{t(b,\ell)} \in N$ hence $b \in \text{EM}(J)$; on the other hand if $b \in \text{EM}(J)$ then by clause (b) we have $b \in N$; so we can conclude $N = \text{EM}(J)$. So far this holds for any linear order I .

By clause (c) $J = \bigcup_{i \in X} A_i$ for some $X \subseteq \lambda + 1$, and by clause (d), $\lambda \in X$.

Now $\text{EM}(J) \underset{\text{nice}}{\leq} \text{EM}(I) \neq M$ by 2.9. $\square_{2.12}$

§ 3. TOWARDS REMOVING THE ASSUMPTION OF REGULARITY FROM THE
EXISTENCE OF UNIVERSAL EXTENSIONS

In §2 we showed that $\mathfrak{k}_{<\lambda}$ has the amalgamation property when T is categorical in the regular cardinal $\lambda > \chi$. We now study the situation in which λ is not assumed to be regular.

Our problem is that while we know that most submodels of $N \in K_\lambda$ sit well in N (see 2.12(2)) and that there are quite many $N \in K_{<\lambda}$ which are amalgamation bases, our difficulty is to get those things together: constructing $N \in K_\lambda$ as $\bigcup_{i<\lambda} N_i$, $N_i \in K_{<\lambda}$ means N has $\leq_{\mathfrak{k}}$ -submodels not included in any N_i .

Theorem 3.1. *Suppose that T is categorical in λ and $\chi \leq \theta < \lambda$. If $\langle M_i \in K_\theta : i < \theta^+ \rangle$ is an increasing continuous $\leq_{\mathfrak{k}}$ -chain, then:*

$$\left\{ i < \theta^+ : M_i \leq_{\text{nice}} (\bigcup_{j<\theta^+} M_j) \right\} \in D_{\theta^+}.$$

Remark 3.2.

- (1) We cannot use 2.12(1) as possibly λ has cofinality $< \chi$.
- (2) Recall that D_{θ^+} is the closed unbounded filter on θ^+ .

Proof. Write $M_{\theta^+} = \bigcup_{i<\theta^+} M_i$. Choose an operation Op such that for all $i < \theta^+$, $\|\text{Op}(M_i)\| \geq \lambda$. Let $M_i^* = \text{Op}(M_i)$. Applying Fact 1.10 for non-limit ordinals, 1.9 for limit ordinals, one finds inductively an increasing continuous $\leq_{\mathfrak{k}}$ -chain $\langle N_i : i \leq \theta^+ \rangle$ such that for $i < \theta^+$, $M_i \leq_{\mathfrak{k}} N_i \leq_{\mathfrak{k}} M_i^*$, $\|N_i\| = \lambda$, $N_{\theta^+} = \bigcup_{i<\theta^+} N_i$.

Note that $\|N_{\theta^+}\| = \theta^+ \cdot \lambda = \lambda$.

Since T is λ -categorical, $N_{\theta^+} \cong \text{EM}(I)$ where 2.9 furnishes I of power λ . By 2.12(2), there is an expansion $N_{\theta^+}^+$ of N_{θ^+} by at most $\kappa + |T|$ functions such that if $A \subseteq |N_{\theta^+}^+|$ is closed under the functions of $N_{\theta^+}^+$, then $(N_{\theta^+}^+ \upharpoonright \tau_{\mathfrak{k}}) \upharpoonright A \leq_{\text{nice}} N_{\theta^+}$.

Choose a set A_i and an ordinal j_i , by induction on $i < \theta^+$, satisfying:

- (1) $A_i \subseteq |N_{\theta^+}|$, $|A_i| \leq \theta$; $\langle A_i : i < \theta^+ \rangle$ is continuous increasing,
- (2) $\langle j_i : i < \theta^+ \rangle$ is continuous increasing,
- (3) A_i is closed under the functions of $N_{\theta^+}^+$,
- (4) $A_i \subseteq |N_{j_{i+1}}|$,
- (5) $|M_i| \subseteq A_{i+1}$.

This is possible: for zero or limit ordinals unions work; for $i+1$ choose j_{i+1} to satisfy (2) and (4), and A_{i+1} to satisfy (1), (3) and (5).

By (2), $C = \{i < \theta^+ : i \text{ is a limit ordinal and } j_i = i\}$ is a club of θ^+ i.e. $C \in D_{\theta^+}$.

Fix $i \in C$. Note that $|M_i| \subseteq A_i$ and $A_i \subseteq |N_i|$ (since $|M_i| = \bigcup_{j<i} |M_j| \subseteq \bigcup_{j<i} A_{j+1} = A_i = \bigcup_{i'<i} A_{i'} \subseteq \bigcup_{i'<i} |N_{j_{i'+1}}| = N_{j_i} = N_i$ (using (5), (1), (4), (2) and $j_i = i$)) and so $M_i \leq_{\mathfrak{k}} (N_{\theta^+}^+ \upharpoonright \tau_{\mathfrak{k}}) \upharpoonright A_i \leq_{\mathfrak{k}} N_i \leq_{\mathfrak{k}} M_i^* = \text{Op}(M_i)$, so that $M_i \leq_{\text{nice}} (N_{\theta^+}^+ \upharpoonright \tau_{\mathfrak{k}}) \upharpoonright A_i$. However by (3) and the choice of N_{θ^+} and $N_{\theta^+}^+$ one has also that $(N_{\theta^+}^+ \upharpoonright \tau_{\mathfrak{k}}) \upharpoonright A_i \leq_{\text{nice}} N_{\theta^+}$. So by transitivity of \leq_{nice} , one obtains $M_i \leq_{\text{nice}} N_{\theta^+}$.

Finally remark that $M_{\theta^+} \leq_{\mathfrak{k}} N_{\theta^+}$ since $M_i \leq_{\text{nice}} N_i \leq_{\mathfrak{k}} N_{\theta^+}$ for every $i < \theta^+$. Hence

$$C \subseteq \left\{ i < \theta^+ : M_i \leq_{\text{nice}} M_{\theta^+} \right\} \in D_{\theta^+}. \quad \square_{3.1}$$

Definition 3.3. Suppose that $\theta \in [\chi, \lambda)$ and $M \in K_\theta$. M is nice iff whenever $M \leq_{\mathfrak{k}} N \in K_\theta$, then $M \leq_{\text{nice}} N$. (The analogous $\leq_{\mathfrak{k}}$ -elementary embedding definition runs: M is nice iff whenever $f: M \rightarrow N \in K_\theta$ then $f: M \xrightarrow{\text{nice}} N$).

Theorem 3.4. Suppose that T is categorical in λ and $M \in K_\theta, \theta \in [\chi, \lambda)$. Then there exists $N \in K_\theta$ such that $M \leq_{\mathfrak{k}} N$ and N is nice.

Proof. Suppose otherwise. We'll define a continuous increasing $\leq_{\mathfrak{k}}$ -chain $\langle M_i \in K_\theta : i < \theta^+ \rangle$ such that for $j < \theta^+$:

$$(*)_j \quad M_j \not\leq_{\text{nice}} M_{j+1}.$$

For $i = 0$, put $M_0 = M$; if i is a limit ordinal, put $M_i = \bigcup_{j < i} M_j$; if $i = j + 1$, then since 3.4 is assumed to fail, M_{j+1} exists as required in $(*)_j$ (otherwise M_i works as N in 3.4). But now $\langle M_i : i < \theta^+ \rangle$ yields a contradiction to 3.1, since $C = \{i < \theta^+ : M_i \leq_{\mathfrak{k}} \bigcup_{j < \theta^+} M_j\} \in D_{\theta^+}$ by 3.1 so that choosing j from C one has $M_j \leq_{\text{nice}} M_{j+1}$ by 1.38, contradicting $(*)_j$. $\square_{3.4}$

Theorem 3.5. Suppose that \mathfrak{k} is categorical in λ and $\theta \in [\chi, \lambda)$. If $M \in K_\theta$ is nice and $f: M \rightarrow N \in K_{\leq \lambda}$, then $f: M \xrightarrow{\text{nice}} N$.

Proof. Choosing an appropriate Op and using Fact 1.10 one finds N_1 such that $N \leq_{\mathfrak{k}} N_1$ and $\|N_1\| = \lambda$. Find $M'_1 \leq_{\text{nice}} N_1$ by 2.12(2) such that $\text{rng}(f) \subseteq |M'_1|$, $\|M'_1\| = \theta$. So $M'_1 \leq_{\mathfrak{k}} N_1$ and therefore $N_1 \restriction \text{rng}(f) \leq_{\mathfrak{k}} M'_1$. M is nice, so $f: M \xrightarrow{\text{nice}} M'_1$. Now $M'_1 \leq_{\text{nice}} N_1$, therefore $f: M \xrightarrow{\text{nice}} N_1$. So there are Op and $g: N_1 \rightarrow \text{Op}(M)$ satisfying $gf = f_{\text{Op}}$. Since $N \leq_{\mathfrak{k}} N_1$ it follows that $f: M \xrightarrow{\text{nice}} N$ as required. $\square_{3.5}$

Corollary 3.6. Suppose that $M \in K_\theta$ is nice, $\theta \in [\chi, \lambda)$. Then M is an a.b. in $\mathfrak{k}_{\leq \lambda}$ i.e. if $f_i: M \rightarrow M_i, M_i \in K_{\leq \lambda} (i = 1, 2)$, then there exists an amalgam $N \in K_{\leq \lambda}$ of M_1, M_2 over M w.r.t. f_1, f_2 .

Proof. By 3.5 $f_i: M \xrightarrow{\text{nice}} M_i (i = 1, 2)$. Hence by 2.1 there is an amalgam $N \in K_{\leq \lambda}$ of M_1, M_2 over M w.r.t. f_1, f_2 . $\square_{3.6}$

Definition 3.7. Suppose that $\theta \in [\chi, \lambda)$ and ∂ is a cardinal.

(1) A model $M \in K_\theta$ is ∂ -universal iff for every $N \in K_\partial$, there exists an $\leq_{\mathfrak{k}}$ -elementary embedding $f: N \rightarrow M$. M is universal iff M is $\|M\|$ -universal.

(2) A model $M_2 \in K_\partial$ is ∂ -universal over the model M_1 (and one writes $M_1 \preceq_{\partial\text{-univ}} M_2$) iff $M_1 \leq_{\mathfrak{k}} M_2$ and whenever $M_1 \leq_{\mathfrak{k}} M'_2 \in K_\partial$, then there exists an $\leq_{\mathfrak{k}}$ -elementary embedding $f: M'_2 \rightarrow M_2$ such that $f \restriction M_1$ is the identity. (The embedding version runs: there exists $h: M_1 \rightarrow M_2$ and whenever $g: M_1 \rightarrow M'_2 \in K_\partial$, then there exists $f: M'_2 \rightarrow M_2$ such that $fg = h$.) M_2 is universal over M_1 ($M_1 \preceq_{\text{univ}} M_2$) iff M_2 is $\|M_2\|$ -universal over M_1 .

(3) M_2 is ∂ -universal over M_1 in M iff $M_2 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M$, $\|M_1\| \leq \partial$ and whenever $M'_2 \in K_\partial$ and $M_1 \leq_{\mathfrak{k}} M'_2 \leq_{\mathfrak{k}} M$, then there exists an $\leq_{\mathfrak{k}}$ -elementary embedding $f: M'_2 \rightarrow M_2$ such that $f \restriction M_1$ is the identity. M_2 is universal over M_1 in M iff M_2 is $\|M_2\|$ -universal over M_1 in M .

(4) M_2 is weakly ∂ -universal over M_1 (written $M_1 \prec_{\partial-\text{wu}} M_2$) iff $M_1 \leq_{\mathfrak{k}} M_2 \in K_{\partial}$ and whenever $M_2 \leq_{\mathfrak{k}} M'_2 \in K_{\partial}$, then there exists an $\leq_{\mathfrak{k}}$ -elementary embedding $f: M'_2 \rightarrow M_2$ such that $f \upharpoonright M_1$ is the identity. (The embedding version is: there exists $h: M_1 \rightarrow_{\mathfrak{k}} M_2$ and whenever $g: M_2 \rightarrow_{\mathfrak{k}} M'_2 \in K_{\partial}$, then there exists $f: M'_2 \rightarrow_{\mathfrak{k}} M_2$ such that $h = fgh$ (written $h: M_1 \rightarrow_{\partial-\text{wu}} M_2$)). M_2 is weakly universal over $M_1 (M_1 \preceq_{\text{wu}} M_2)$ iff M_2 is $\|M_2\|$ -weakly universal over M_1 .

(Note that “ M_2 is ∂ -universal over M_1 ” does not necessarily implies “ M_2 is weakly ∂ -universal over M_1 ” as possibly $\|M_2\| > \partial$).

Remark 3.8. In $\mathfrak{k}_{<\lambda}$, if M_1 is an a.b., then weak universality over M_1 is equivalent to universality over M_1 .

Proof. Suppose that $h: M_1 \rightarrow_{\text{wu}} M_2$ and $g: M_1 \rightarrow_{\mathfrak{k}} M'_2 \in K_{\|M_2\|}$. Since M_1 is an a.b. there exist a model N and $h': M_2 \rightarrow_{\mathfrak{k}} N$, $g': M'_2 \rightarrow_{\mathfrak{k}} N$ satisfying $h'h = g'g$. By Fact 1.10 without loss of generality $\|N\| = \|M_2\|$. Since M_2 is weakly universal over M_1 , there exists $h'': N \rightarrow_{\mathfrak{k}} M_2$, $h = h''h'h$. Let $f = h''g': M'_2 \rightarrow M_2$, and note that $fg \upharpoonright M_1 = h''g'g = h''h'h = h$, so that M_2 is universal over M_1 . $\square_{3.8}$

Remark 3.9. For any model M , universality over M implies weak universality over M .

Lemma 3.10. *Suppose that \mathfrak{k} is categorical in $\lambda, \theta \in [\chi, \lambda)$. If $M \in K_{\theta}$ and $M \leq_{\mathfrak{k}} N \in K_{\lambda}$, then there exists $M^+ \in K_{\theta}$ such that:*

- (a) $M \leq_{\mathfrak{k}} M^+ \leq_{\mathfrak{k}} N$,
- (b) M^+ is universal over M in N .

Proof. We choose I such that:

- (*) (a) I is a linear order of cardinality λ ,
- (b) if $\theta \in [\aleph_0, \lambda)$, $J_0 \subseteq I$, $|J_0| = \theta$ then there is J_1 satisfying $J_0 \subseteq J_1 \subseteq I$, $|J_1| = \theta$, and for every $J^* \subseteq I$ of cardinality $\leq \theta$ there is an order-preserving (one to one) mapping from $J_0 \cup J^*$ into $J_0 \cup J_1$ which is the identity on J_0 .

Essentially the construction follows Laver [Lav71] and [She87b, Appendix], see more in [Shea]; but for our present purpose let $I = (\omega^>\lambda, <_{\ell\text{ex}})$; given θ and J_0 we can increase J_0 so without loss of generality $J_0 = \omega^>A$, $A \subseteq \lambda$, $|A| = \theta$. Define an equivalence relation E on $I \setminus J_0$: $\eta E \nu \Leftrightarrow (\forall \rho \in J_0)(\rho <_{\ell\text{ex}} \eta \equiv \rho <_{\ell\text{ex}} \nu)$, easily it has $\leq \theta$ equivalence classes, so let $\{\eta_i^*: i < i^* \leq \theta\}$ be a set of representatives each of minimal length, so $\eta_i^* \upharpoonright (\lg \eta_i^* - 1) \in J_0$, $\eta_i^* (\lg \eta_i^* - 1) \in \lambda \setminus A$.

Let $J_1 = I \cup \{\eta_i^* \hat{\nu}: \nu \in \omega^>\theta \text{ and } i < i^*\}$, so clearly $J_0 \subseteq J_1 \subseteq I$, $|J_1| = \theta$. Suppose $J_0 \subseteq J \subseteq I$, $|J| \leq \theta$, and we should find the required embedding h . As before without loss of generality $J = \omega^>B$, $|B| = \theta$ and $A \subseteq B$. Now $h \upharpoonright J_0 = \text{id}_{J_0}$ so it is enough to define $h \upharpoonright (J_1 \cap (\eta_i^*/E))$, hence it is enough to embed $J_1 \cap (\eta_i^*/E)$ into $\{\eta_1^* \hat{\nu}: \nu \in \omega^>\theta\}$ (under $<_{\ell\text{ex}}$).

Let $\gamma = \text{otp}(B)$, so it is enough to show $(<^{\omega}\gamma, <_{\ell\text{ex}})$ can be embedded into $\omega^>\theta$, where of course $|\gamma| \leq \theta$. This is proved by induction on γ .

Since T is λ -categorical and $\text{EM}(I)$ is a model of T of power λ , there is an isomorphism g from $\text{EM}(I)$ onto N . It follows from (*) that $M^+ = g''(\text{EM}(J)) \in K_{\theta}$ satisfies (1) and (2). (Analogues of (1) and (2) are checked in more detail in the course of the proof of 3.14.) $\square_{3.10}$

Lemma 3.11. *Suppose that \mathfrak{k} is categorical in $\lambda, \theta \in [\chi, \lambda)$, and $\langle M_i \in K_\theta : i < \theta^+ \rangle, \langle N_i \in K_\lambda : i < \theta^+ \rangle$ are continuous $\leq_{\mathfrak{k}}$ -chains such that for every $i < \theta^+$ we have $M_i \leq_{\mathfrak{k}} N_i$. Then there exists $i(*) < \theta^+$ such that $(i(*), \theta^+) \subseteq C = \{i < \theta^+ : M_{i+1} \text{ can be } \leq_{\mathfrak{k}}\text{-elementarily embedded into } N_i \text{ over } M_0\}$.*

Proof. Apply 3.10 for $M_0 \in K_\theta$ and $N_{\theta^+} = \bigcup_{i < \theta^+} N_i \in K_\lambda$ (noting that $M_0 \leq_{\mathfrak{k}} N_0 \leq_{\mathfrak{k}} N_{\theta^+}$) to find $M^+ \in K_\theta$ such that $M_0 \leq_{\mathfrak{k}} M^+ \leq_{\mathfrak{k}} N_{\theta^+}$ and M^+ is universal over M_0 in N_{θ^+} .

For some $i(*) < \theta^+, M^+ \subseteq N_{i(*)}$ and so $M^+ \leq_{\mathfrak{k}} N_{i(*)}$. If $i \in (i(*), \theta^+)$, then $M_{i+1} \in K_\theta$ and $M_0 \leq_{\mathfrak{k}} M_{i+1} \leq_{\mathfrak{k}} N_{i+1} \leq_{\mathfrak{k}} N_{\theta^+}$, so there is an $\leq_{\mathfrak{k}}$ -elementary embedding $f : M_{i+1} \xrightarrow{\mathfrak{k}} M^+$ and $f \upharpoonright M_0$ is the identity. Now $M^+ \leq_{\mathfrak{k}} N_{i(*)} \leq_{\mathfrak{k}} N_i$, so $f : M_{i+1} \xrightarrow{\mathfrak{k}} N_i$. Hence $(i(*), \theta^+) \subseteq C$ as required. $\square_{3.11}$

Theorem 3.12. *Suppose that \mathfrak{k} is categorical in $\lambda, \theta \in [\chi, \lambda), M \in K_\theta$. Then there exists $M^+ \in K_\theta$ such that:*

- (\aleph) $M \leq_{\mathfrak{k}} M^+$ and M^+ is nice,
- (\beth) M^+ is weakly universal over M .

Proof. Define by induction on $i < \theta^+$ continuous $\prec_{\mathcal{F}}$ -chains $\langle M_i \in K_\theta : i < \theta^+ \rangle, \langle N_i \in K_\lambda : i < \theta^+ \rangle$ such that:

- (0) $M_0 = M$,
- (1) $M_i \leq_{\mathfrak{k}} N_i$,
- (2) if $(*)_i$ holds, then M_{i+1} cannot be $\leq_{\mathfrak{k}}$ -elementarily embedded into N_i over M_0 , where $(*)_i$ is the statement:

($*)_i$ there are $M' \in K_\theta$ and $N' \in K_\lambda$ such that $M_i \leq_{\mathfrak{k}} M', N_i \leq_{\mathfrak{k}} N', M' \leq_{\mathfrak{k}} N'$ and M' cannot be $\leq_{\mathfrak{k}}$ -elementarily embedded into N_i over M_0 ,

- (3) $M_{i+1} \underset{\text{nice}}{\leq} N_{i+1}$.

This is possible. N_0 is obtained by an application of Fact 1.10 to an appropriate $\text{Op}(M_0)$ of power at least λ . At limit stages, continuity dictates that one take unions. Suppose that M_i has been defined. If $(*)_i$ does not hold, by 2.12(2) there is $M'' \in K_\theta, M_i \leq_{\mathfrak{k}} M'' \underset{\text{nice}}{\leq} N_i$. Let $M_{i+1} = M'', N_{i+1} = N_i$. If $(*)_i$ does hold for M', N' , let $N_{i+1} = N'$; note that by 2.10(2) there exists $M'' \in K_\theta, M' \leq_{\mathfrak{k}} M'' \underset{\text{nice}}{\leq} N'$; now let $M_{i+1} = M''$. Note that in each case, (3) is satisfied.

Find $i(*) < \theta^+$ and C as in 3.11 and choose $i \in C$. By (1) $M_{i+1} \leq_{\mathfrak{k}} N_{i+1}$ so by 3.10 there exists $M^- \in K_\theta$ such that $M_{i+1} \leq_{\mathfrak{k}} M^- \leq_{\mathfrak{k}} N_{i+1}$ and M^- is weakly universal over M_{i+1} in N_{i+1} . By 3.4 one can find $M^+ \in K_\theta$ such that $M^- \leq_{\mathfrak{k}} M^+$ and M^+ is nice. So M^+ satisfies (\aleph). It remains to show that M^+ is weakly universal over M . Suppose not and let $g : M^+ \xrightarrow{\mathfrak{k}} M^* \in K_\theta$ where M^* cannot be $\leq_{\mathfrak{k}}$ -elementarily embedded in M^+ over M hence cannot be $\leq_{\mathfrak{k}}$ -elementarily embedable in M^- over M , hence in N_{i+1} over M . $M_{i+1} \leq_{\mathfrak{k}} M^* \in K_\theta$ and by (3) $M_{i+1} \underset{\text{nice}}{\leq} N_{i+1} \in K_\lambda$, so by 2.1 there is an amalgam $N^* \in K_\lambda$ of M^*, N_{i+1} . The existence of M^*, N^* implies that $(*)_{i+1}$ holds since M^* cannot be $\leq_{\mathfrak{k}}$ -elementarily embedded into N_{i+1} over M_0 , hence M_{i+2} cannot be $\leq_{\mathfrak{k}}$ -elementarily embedded into N_{i+1} in contradiction to the choice of i as by 3.10 $i+1$ is in C . $\square_{3.12}$

Corollary 3.13. *If \mathfrak{k} is categorical in $\lambda, \theta \in [\chi, \lambda)$ and $M \in K_\theta$ is an a.b. (e.g. M is nice, see 2.1), then there exists $M^+ \in K_\theta$ such that:*

- (\aleph) $M \leq_{\mathfrak{k}} M^+$ and M^+ is nice,
- (\beth) M^+ is universal over M .

Proof. 3.12 and 3.7 (5). □_{3.13}

Corollary 3.14. *Suppose that T is categorical in λ and $\theta \in [\chi, \lambda)$. Then there is a nice universal model $M \in K_\theta$.*

Proof. By 3.4 it suffices to find a universal model of power θ , noting that universality is preserved under $\leq_{\mathfrak{k}}$ -elementary extensions in the same power. As in the proof of 3.10, there is a linear order $(I, <_I)$ of power λ and $J \subseteq I, |J| = \theta$, such that:

- (*) $(\forall J' \subseteq I)$ (if $|J'| \leq \theta$, then there is an order-preserving injective map g from J' into J).

□_{3.14}

Claim 3.15. $\text{EM}(J) \in K_\theta$ is universal.

Proof. $\text{EM}(J)$ is a model of power θ since $\max(|J|, \chi) \leq \theta$ and $\theta = |J| \leq \|\text{EM}(J)\|$. $\text{EM}(J)$ is universal. Suppose that $N \in K_\theta$. Applying Fact 1.10 to a suitably large $\text{Op}(N)$ find $M \in K_\lambda, N \leq_{\mathfrak{k}} M$, so that by λ -categoricity of $\mathfrak{k}, M \cong \text{EM}(I)$. There is a surjective $\leq_{\mathfrak{k}}$ -elementary embedding $h: N \rightarrow_{\mathfrak{k}} N' \leq_{\mathfrak{k}} \text{EM}(I)$ and there exists $J' \subseteq I, |J'| \leq \|N'\| + \chi = \theta$, such that $N' \subseteq \text{EM}(J')$. So by (*) there is an order preserving injective map g from J' into J . Now g induces an $\leq_{\mathfrak{k}}$ -elementary embedding \hat{g} from $\text{EM}(J')$ into $\text{EM}(J)$. Let $f = \hat{g}h$, then $f: N \rightarrow_{\mathfrak{k}} \text{EM}(J)$ is as required. □_{3.15}

Theorem 3.16. *Suppose that \mathfrak{k} is categorical in λ , $\theta \in [\kappa + |T|, \lambda)$, $N \in K_{<\lambda}$ is nice, $M \in K_\theta$ and $M \leq_{\text{nice}} N$. Then M is nice.*

Proof. Let $B \in K_\theta, M \leq_{\mathfrak{k}} B$. Show that $M \leq_{\text{nice}} B$. Well, since $M \leq_{\text{nice}} N$ and $M \leq_{\mathfrak{k}} B$, by 2.1 there exists an amalgam $M^* \in K_{<\lambda}^{\text{nice}}$ of N, B over M . Without loss of generality by 1.15 $\|M^*\| = \|N\|$. N is nice, hence $N \leq_{\text{nice}} M^*$. Since $M \leq_{\text{nice}} N$, it follows by 1.34 that $M \leq_{\text{nice}} M^*$. Since $M \leq_{\mathfrak{k}} B \leq_{\mathfrak{k}} M^*$, it follows by 1.38 that $M \leq_{\text{nice}} B$. □_{3.16}

§ 4. (θ, ∂) -SATURATED MODELS

In this section we define notions of saturation which will be of use in proving amalgamation for \mathfrak{k}_λ .

Definition 4.1. Suppose that ∂ is a limit ordinal, $\aleph_0 \leq \partial \leq \theta \in [\chi, \lambda)$.

(1) An τ -structure M is (θ, ∂) -saturated iff:

- (a) $\|M\| = \theta$,
- (b) there exists a continuous $\leq_{\mathfrak{k}}$ -chain $\langle M_i \in K_\theta : i < \partial \rangle$ such that:
 - (i) M_0 is nice and universal,
 - (ii) M_{i+1} is universal over M_i ,
 - (iii) M_i is nice, and,
 - (iv) $M = \bigcup_{i < \partial} M_i$.

(2) M is θ -saturated iff M is $(\theta, \text{cf}(\theta))$ -saturated.

(3) M is (θ, ∂) -saturated over N iff M is (θ, ∂) -saturated as witnessed by a chain $\langle M_i : i < \partial \rangle$ such that $N \subseteq M_0$.

The principal facts established in this section connect the existence, uniqueness and niceness of (θ, ∂) -saturated models.

Theorem 4.2. Suppose that \mathfrak{k} is categorical in λ and $\partial \leq \theta \in [\chi, \lambda)$. Then:

- (1) there exists a (θ, ∂) -saturated model M ,
- (2) M is unique up to isomorphism,
- (3) M is nice.

Proof. One proves (1), (2) and (3) simultaneously by induction on ∂ .

Ad (1). Choose a continuous $\leq_{\mathfrak{k}}$ -chain $\langle M_i \in K_\theta : i < \partial \rangle$ of nice models by induction on i as follows. For $i = 0$, apply 3.14 to find a nice universal model $M_0 \in K_\theta$. For $i = j + 1$, note that M_j is an a.b. by 3.6 (since M_j is nice), hence by 3.13 there exists a nice model $M_i \in K_\theta, M_j \leq_{\mathfrak{k}} M_i, M_i$ universal over M_j . For limit i , let $M_i = \bigcup_{j < i} M_j$. Note that by the inductive hypothesis (3) on ∂ for $i < \partial$, since M_i is (θ, i) -saturated, M_i is nice. Thus $M = \bigcup_{i < \partial} M_i$ is (θ, ∂) -saturated (witnessed by $\langle M_i : i < \partial \rangle$). Note that M is universal, since $\langle M_i : i < \partial \rangle$ is continuous and M_0 is universal.

Ad (2). As ∂ is a limit ordinal standard back-and-forth argument shows that if M and N are (θ, ∂) -saturated models, then M and N are isomorphic.

Ad (3). By the uniqueness (i.e. by Ad(2)) it suffice to prove that some (θ, ∂) saturated model is nice. Suppose that M is (θ, ∂) -saturated. We'll show that M is nice.

If $\text{cf}(\partial) < \partial$, then M is also $(\theta, \text{cf}(\partial))$ -saturated and hence by the inductive hypothesis (3) on ∂ for $\text{cf}(\partial)$, M is nice. So we'll assume that $\text{cf}(\partial) = \partial$. Choose a continuous $\leq_{\mathfrak{k}}$ -chain $\langle M_i \in K_\theta : i < \theta^+ \rangle$ such that: M_0 is nice and universal (possible by 3.14); if M_i is nice, then $M_{i+1} \in K_\theta$ is nice and universal over M_i (possible by 3.6 and 3.13); if M_i is not nice (so necessarily i is a limit ordinal), then $M_{i+1} \in K_\theta, M_i \leq_{\mathfrak{k}} M_{i+1}$ and $M_i \not\leq_{\text{nice}} M_{i+1}$. By 3.1 and 1.38 there is a club C of θ^+ such that if $i \in C$, then $M_i \leq_{\text{nice}} M_{i+1}$. So by the choice of $\langle M_i : i < \theta^+ \rangle$, if $i \in C$, then M_i is nice. Choose $i \in C, i = \sup(i \cap C), \text{cf}(i) = \partial$. It suffices to show that M_i is (θ, ∂) -saturated (for then by (2) M_i is isomorphic to M and so M is nice). Choose a continuous increasing sequence $\langle \alpha_\zeta : \zeta < \partial \rangle \subseteq C$ such that

$i = \bigcup \{\alpha_\zeta : \zeta < \partial\}$ (recall that $i = \sup(i \cap C)$, $\text{cf}(i) = \partial$). Now $M_i = \bigcup_{\zeta < \partial} M_{\alpha_\zeta}$. Of course M_{α_0} is universal (since M_0 is universal and $M_0 \leq_{\mathfrak{k}} M_{\alpha_0}$), $M_{\alpha_{\zeta+1}}$ is universal over M_{α_ζ} since $M_{\alpha_{\zeta+1}}$ is universal over $M_{\alpha_\zeta}(\kappa)$ and $M_{\alpha_\zeta} \leq_{\mathfrak{k}} M_{\alpha_{\zeta+1}} \leq_{\mathfrak{k}} M_{\alpha_{\zeta+1}}$. Also M_{α_ζ} is nice for each $\zeta < \partial$ since $\alpha_\zeta \in C$. Hence M_i is (θ, ∂) -saturated. $\square_{4.2}$

Remark 4.3. Remember that by 3.16, if T is categorical in λ , $\theta \in [\chi, \lambda)$, $N \in K_{<\lambda}$ is nice, $M \in K_\theta$ and $M \leq_{\text{nice}} N$. Then M is nice.

Theorem 4.4. *Suppose that \mathfrak{k} is categorical in λ , $\chi \leq \theta < \theta^+ < \lambda$. If $\langle M_i \in K_\theta : i < \theta^+ \rangle$ is a continuous $\leq_{\mathfrak{k}}$ -chain of nice models such that M_{i+1} is universal over M_i for $i < \theta^+$, then $\bigcup_{i < \theta^+} M_i$ is (θ^+, θ^+) -saturated.*

Proof. Write $M = \bigcup_{i < \theta^+} M_i$. Note that if $\langle M'_i \in K_\theta : i < \theta^+ \rangle$ is any other continuous $\leq_{\mathfrak{k}}$ -chain of nice models such that M'_{i+1} is universal over M'_i , then $\bigcup_{i < \theta^+} M'_i \cong M$ (use again the back and forth argument).

By Theorem 4.2 there exists a (θ^+, θ^+) -saturated model N which is unique and nice. In particular $\|N\| = \theta^+$ and there exists a continuous $\leq_{\mathfrak{k}}$ -chain $\langle N_i \in K_{\theta^+} : i < \theta^+ \rangle$ such that:

- (i) N_0 is nice and universal,
- (ii) N_{i+1} is universal over N_i ,
- (iii) N_i is nice,
- (iv) $N = \bigcup_{i < \theta^+} N_i$.

It suffices to prove that M and N are isomorphic models.

Without loss of generality $|N| = \theta^+$. By Fact 1.10 $C_1 = \{\delta < \theta^+ : N \restriction \delta \leq_{\mathfrak{k}} N\}$ contains a club of θ^+ . By 3.1 there exists a club C_2 of θ^+ such that for every $\delta \in C_2$, $N \restriction \delta \leq_{\text{nice}} N$. Since $\{|N_i| : i < \theta^+\}$ is a continuous increasing sequence of subsets of θ^+ , it follows that $C_3 = \{\delta < \theta^+ : \delta \subseteq |N_\delta|\}$ is a club of θ^+ . Hence there is a club C_4 of θ^+ such that $C_4 \subseteq C_1 \cap C_2 \cap C_3 \cap [\theta, \theta^+)$. Note that for $\delta \in C_4$ one has $N \restriction \delta \leq_{\text{nice}} N$, $|N \restriction \delta| = \delta \subseteq |N_\delta|$ and $N_\delta \leq_{\mathfrak{k}} N$, so that $N \restriction \delta \leq_{\mathfrak{k}} N_\delta \leq_{\mathfrak{k}} N$ and so by 1.38 $N \restriction \delta \leq_{\text{nice}} N_\delta$. $\langle N_\delta : \delta \in C_4 \rangle$ is a continuous increasing $\prec_{\mathcal{F}}$ -chain, $N_\delta \in K_{\theta^+}$ and $N \restriction \delta \in K_\theta$.

By 3.16 $N \restriction \delta$ is nice since N_δ is nice (by (iii)). So by 3.13 $N \restriction \delta$ has a nice $\leq_{\mathfrak{k}}$ -extension $B_\delta \in K_\theta$ which is universal over $N \restriction \delta$, without loss of generality $N \restriction \delta \leq_{\mathfrak{k}} B_\delta \leq_{\mathfrak{k}} N$.

[Why? since $N \restriction \delta \leq_{\mathfrak{k}} B_\delta$ (in fact $N \restriction \delta \leq_{\text{nice}} B_\delta$) and $N \restriction \delta \leq_{\text{nice}} N_\delta$, by 2.1 there exists an amalgam $A_\delta \in K_{\leq \theta^+}$ of B_δ, N_δ over $N \restriction \delta$. Let $f_\delta : B_\delta \xrightarrow[\mathfrak{k}]{\text{nice}} A_\delta$ be a witness.

But $N_{\delta+1}$ is universal over N_δ (by (ii)), so A_δ can be $\leq_{\mathfrak{k}}$ -elementarily embedded into $N_{\delta+1}$ over N_δ (say by g_δ), hence B_δ can be $\leq_{\mathfrak{k}}$ -elementarily embedded into N (using $g_\delta f_\delta$).]

Let $C_5 = \{\delta \in C_4 : \text{if } \alpha \in C_4 \cap \delta, \text{ then } |B_\alpha| \subseteq \delta\}$. Note that C_5 is a club of θ^+ since $\|B_\alpha\| = \theta$. [For $\alpha \in C_4$, let $E_\alpha = (\sup |B_\alpha|, \theta^+) \cap C_4$, $E_\alpha = \theta^+$ for $\alpha \notin C_4$ and let E be the diagonal intersection of $\langle E_\alpha : \alpha < \theta^+ \rangle$, i.e. $E = \{\delta < \theta^+ : (\forall \alpha < \delta)(\delta \in E_\alpha)\}$. Note that E is a club of θ^+ and $C_5 \supseteq E \cap C_4$ which is a club of θ^+ .] Thus $\langle N \restriction \delta : \delta \in C_5 \rangle$ is a continuous $\leq_{\mathfrak{k}}$ -chain of nice models, each of power θ . If $\delta_1 \in C_5$ and $\delta_2 = \min(C_5 \setminus (\delta_1 + 1))$, then $N \restriction \delta_1 \leq_{\mathfrak{k}} B_{\delta_1} \leq_{\mathfrak{k}} N \restriction \delta_2$. Hence $N \restriction \delta_2$ is universal over $N \restriction \delta_1$ (since B_{δ_1} is universal over $N \restriction \delta_1$). Let $\{\delta_i : i < \theta^+\}$ enumerate C_5

and set $M'_i = N \restriction \delta_i$. Note that $N = \bigcup_{i < \theta^+} M'_i$. Then $\langle M'_i \in K_\theta : i < \theta^+ \rangle$ is a continuous $\leq_{\mathfrak{t}}$ -chain of nice models, M'_{i+1} is universal over M'_i . Therefore N and M are isomorphic (as said in the beginning of the proof), as required. $\square_{4.4}$

Notation 4.5. $\Theta = \{\bar{\theta} : \bar{\theta} = \langle \theta_i : i < \delta \rangle \text{ is a (strictly) continuous increasing sequence of cardinals, } \chi < \theta_0, \delta < \theta_0 \text{ (a limit ordinal), } \bigcup_{i \leq \delta} \theta_i \leq \lambda\}$ and $\Theta^- = \{\bar{\theta} \in \Theta : \sup \theta_i < \lambda\}$.

Remark 4.6. Let $\theta = \sup_i \bar{\theta}$ for $\bar{\theta} \in \Theta$. Then θ is singular, since $\text{cf}(\theta) \leq \delta < \theta_0 \leq \theta$.

Definition 4.7. Let $\bar{\theta} \in \Theta$. A model M is $\bar{\theta}$ -saturated iff there is a continuous $\leq_{\mathfrak{t}}$ -chain $\langle M_i \in K_{\theta_i} : i < \delta \rangle$ such that $M = \bigcup_{i < \delta} M_i$, M_i is nice and M_{i+1} is θ_{i+1} -universal over M_i .

Definition 4.8. Suppose that $\bar{\theta} \in \Theta$. $\text{Pr}(\bar{\theta})$ holds iff every $\bar{\theta}$ -saturated model is nice.

Remark 4.9. (1) If $\bar{\theta}_1, \bar{\theta}_2 \in \Theta$, $\text{rng}(\bar{\theta}_1) \subseteq \text{rng}(\bar{\theta}_2)$, $\sup \text{rng}(\bar{\theta}_1) = \sup \text{rng}(\bar{\theta}_2)$, and M is $\bar{\theta}_2$ -saturated, then M is $\bar{\theta}_1$ -saturated.

(2) For $\bar{\theta} \in \Theta^-$ and $\text{Pr}(\bar{\theta}')$ whenever $\bar{\theta}' \in \Theta$ is a proper initial segment of $\bar{\theta}$, there is a $\bar{\theta}$ -saturated model and it is unique.

Theorem 4.10. Suppose that $\bar{\theta} \in \Theta^-$ and for every limit ordinal $\alpha < \text{lg}(\bar{\theta})$, $\text{Pr}(\bar{\theta} \restriction \alpha)$. Then $\text{Pr}(\bar{\theta})$.

Proof. By 4.9(1) and the uniqueness of $\bar{\theta}$ -saturated models (4.9.2), without loss of generality one may assume that $\text{otp}(\bar{\theta}) = \text{cf}(\sup(\bar{\theta}))$. Now by 4.6 $(\text{cf}(\theta))^+ < \theta$, so by [She93] 1.15 + Fact 1.10(1)] there exists $\langle S, \langle C_\alpha : \alpha \in S \rangle \rangle$ such that:

- (α) $S \subseteq \theta^+$ is a set of ordinals; $0 \notin S$,
- (β) $S_1 = \{\alpha \in S : \text{cf}(\alpha) = \text{cf}(\theta)\}$ is a stationary subset of θ^+ ,
- (γ) if $\alpha \in S$ is a limit ordinal then $\alpha = \sup C_\alpha$ and if $\alpha \in S$ then $\text{otp}(C_\alpha) \leq \text{cf}(\theta)$,
- (δ) if $\beta \in C_\alpha$, then $\beta \in S$ and $C_\beta = C_\alpha \cap \beta$,
- (ϵ) C_α is a set of successor ordinals.

[Note that the existence of $\langle S, \langle C_\alpha : \alpha \in S \rangle \rangle$ is provable in ZFC.]

Without loss of generality $S \setminus S_1 = \bigcup \{C_\alpha : \alpha \in S_1\}$. We shall construct the required model by induction, using $\langle C_\alpha : \alpha \in S \rangle$. Remember $\bar{\theta} = \langle \theta_\zeta : \zeta < \text{cf}(\theta) \rangle$. Let us start by defining by induction on $\alpha < \theta^+$ the following entities: M_α , $M_{\alpha\xi}$ (for $\alpha < \theta^+$, $\xi < \text{cf}(\theta)$), and N_α (only when $\alpha \in \bigcup_{\beta \in S} C_\beta$) such that:

- (A1) $M_\alpha \in K_\theta$,
- (A2) $\langle M_\alpha : \alpha < \theta^+ \rangle$ is a continuous increasing $\leq_{\mathfrak{t}}$ -chain of models,
- (A3) $M_{\alpha+1}$ is nice, and if M_α is not nice, then $M_\alpha \not\leq_{\text{nice}} M_{\alpha+1}$,
- (A4) $M_\alpha \neq M_{\alpha+1}$,
- (A5) $M_{\alpha+1}$ is weakly universal over M_α ,
- (B1) $M_\alpha = \bigcup_{\xi < \text{cf}(\theta)} M_{\alpha\xi}$, $\|M_{\alpha\xi}\| = \theta_\xi$, if $\alpha \in S_1$, $\beta \in C_\alpha$, $\gamma \in C_\alpha$, $\beta < \gamma$, then:
- (B2) $N_\beta \leq_{\mathfrak{t}} M_\beta$,
- (B3) $\|N_\beta\| = \theta_{\text{otp}(C_\beta)}$,
- (B4) $(\forall \xi < \text{otp}(C_\beta))(M_{\beta\xi} \subseteq N_\gamma)$,

(B5) N_β is nice,

(B6) N_γ is $\theta_{\text{otp}(C_\gamma)}$ -universal over N_β .

There are now two tasks at hand. First of all, we shall explain how to construct these entities (THE CONSTRUCTION, below). Then we shall use them to build a nice $\bar{\theta}$ -saturated model (PROVING $\text{Pr}(\bar{\theta})$, below). From the uniqueness of $\bar{\theta}$ -saturated models it will thus follow that $\text{Pr}(\bar{\theta})$ holds.

THE CONSTRUCTION: we consider several cases:

Case (i): $\beta = 0$. Choose $M_0 \in K_\theta$ and $\langle M_{0\xi} \in K_\theta : \xi < \text{cf}(\theta) \rangle$ with $M_0 = \bigcup_{\xi < \text{cf}(\theta)} M_{0\xi}$ using Fact 1.10. There is no need to define N_0 since $0 \notin C_\alpha$.

Case(ii): β is a limit ordinal. Let $M_\beta = \bigcup_{\gamma < \beta} M_\gamma$ and choose $\langle M_{\beta\xi} : \xi < \text{cf}(\theta) \rangle$ using Fact 1.10. Again there's no call to define N_β since C_α is always a set of successor ordinals.

Case (iii): β is a successor ordinal, $\beta = \gamma + 1$. Choose $M'_\gamma \in K_\theta$ such that $M_\gamma \leq_{\text{t}} M'_\gamma$ and if possible $M_\gamma \not\leq_{\text{nice}} M'_\gamma$; without loss of generality M'_γ is weakly universal over M_γ . If $\beta \notin S$, then define things as above, taking into account (A3). The definitions of $M_\beta, M_{\beta\xi}$ present no special difficulties. Now suppose that $\beta \in S$. The problematic entity to define is N_β .

If $C_\beta = \emptyset$, choose for N_β any nice submodel (of power $\theta_{\text{otp}(C_\beta)}$) of M_γ .

If $C_\beta \neq \emptyset$, then first define $N_\beta^- = \bigcup_{\gamma \in C_\beta} N_\gamma$. Note that N_β^- is nice. [If C_β has a last element β' , then $N_\beta^- = N_{\beta'}$ which is nice; if C_β has no last element, then $N_\beta^- = \bigcup_{\gamma \in C_\beta} N_\gamma$ is $\bar{\theta} \upharpoonright \text{otp}(C_\beta)$ -saturated, and, by the hypothesis of the theorem, $\text{Pr}(\bar{\theta} \upharpoonright \text{otp}(C_\beta))$, so N_β^- is nice.] Also $N_\beta^- \preceq_{\mathcal{F}} M_\gamma$. If $\text{otp}(C_\beta)$ is a limit ordinal we let $N_\beta = N_\beta^-$ and $M_\beta = M'_\gamma$, so we have finished, so assume $\text{otp}(C_\beta)$ is a successor ordinal. To complete the definition of N_β , one requires a Lemma (the proof of which is similar to 3.12, 3.13):

- (*) if $A \subseteq M \in K_\theta$, $|A| \leq \theta_j < \theta$, then there exist a nice $M^+ \in K_\theta$, $M \leq_{\text{t}} M^+$, and nice models $N^*, N^+ \in K_{\theta_j}$, $A \subseteq N^* \leq_{\text{t}} N^+ \leq_{\text{t}} M^+$ and N^+ is universal over N^* .

Why is this enough? Use the Lemma with $M = M'_\beta$ and $A = N_\beta^- \cup \bigcup_{\substack{\xi < \text{otp}(C_\beta) \\ \gamma \in C_\beta}} M_{\gamma\xi}$

to find N^*, N^+, M^+ and choose N^+, M^+ as N_β, M_β respectively.

Now, why (*) holds? The proof of (*) is easy as M'_β is nice.

PROVING $\text{Pr}(\bar{\theta})$:

For $\alpha \in S_1$, consider $\langle N_\beta : \beta \in C_\alpha \rangle$. For $\beta, \gamma \in C_\alpha, \beta < \gamma$, one has by (B4)

$$\bigcup_{\xi < \text{otp}(C_\beta)} M_{\beta\xi} \subseteq N_\gamma. \text{ Therefore } M_\beta \subseteq \bigcup_{\gamma \in C_\alpha} N_\gamma.$$

$(M_\beta = \bigcup_{\xi < \text{cf}(\theta)} M_{\beta\xi} = \bigcup_{\xi < \text{cf}(\alpha)} M_{\beta\xi}(\alpha \in S_1))$; for $\xi < \text{cf}(\alpha)$, choose $\gamma \in C_\alpha, \xi < \gamma, \beta < \gamma$; so $M_{\beta\xi} \subseteq N_\gamma$ and $M_\beta \subseteq \bigcup_{\gamma \in C_\alpha} N_\gamma$.

Thus for every $\beta \in C_\alpha$, $M_\beta \subseteq \bigcup_{\gamma \in C_\alpha} N_\gamma$ hence $M_\alpha = \bigcup_{\beta \in C_\alpha} M_\beta \subseteq \bigcup_{\gamma \in C_\alpha} N_\gamma$ (remember $\alpha = \sup(C_\alpha)$ as $\alpha \in S_1$). If $\gamma \in C_\alpha$, then $N_\gamma \leq_{\text{t}} M_\gamma$ (by (B2)), and so $\bigcup_{\gamma \in C_\alpha} N_\gamma \subseteq$

$\bigcup_{\beta \in C_\alpha} M_\beta = M_\alpha$ by continuity. So $M_\alpha = \bigcup_{\beta \in C_\alpha} N_\beta$ hence $\langle N_\beta : \beta \in C_\alpha \rangle$ exemplifies M_α is $\bar{\theta}$ -saturated (remember $\text{Pr}(\bar{\theta} \restriction \delta)$ for every limit $\delta < \text{lg}(\bar{\theta})$). So M_α is $\bar{\theta}$ -saturated for every $\alpha \in S_1$. In other words $\{\alpha < \theta^+ : M_\alpha \text{ is } \bar{\theta}\text{-saturated}\} \supseteq S_1$ and is stationary, so, applying 3.1, there exists $\alpha < \theta^+$ such that M_α is $\bar{\theta}$ -saturated and $M_\alpha \leq_{\text{nice}} \bigcup_{\beta < \theta^+} M_\beta$. Hence by 1.38 $M_\alpha \leq_{\text{nice}} M_{\alpha+1}$ and so, since $M_{\alpha+1}$ is nice (A3), M_α is nice (by 3.16).

We conclude that $\text{Pr}(\bar{\theta})$ holds.

□_{4.10}

To round off this section of the paper, let us make the connection between $\bar{\theta}$ -saturation and $(\theta, \text{cf}(\theta))$ -saturation (Notation follows 4.5–4.10).

Theorem 4.11. *Let $\bar{\theta} \in \Theta^-$ and $\theta = \sup_i(\theta_i)$. Every $\bar{\theta}$ -saturated model is $(\theta, \text{cf}(\theta))$ -saturated.*

Proof. Let $\langle M_\alpha : \alpha < \theta^+ \rangle$ be as in the proof of 4.10. By 3.1 there exists a club C of θ^+ such that for every $\alpha \in C$, $M_\alpha \leq_{\text{nice}} \bigcup_{\beta < \theta^+} M_\beta$ hence by the construction M_α is nice. So if $\alpha, \beta \in C$ and $\alpha < \beta$, then M_β is a universal extension of M_α and for $\gamma = \sup(\gamma \cap C)$, $\gamma \in C$, one has that M_γ is $(\theta, \text{cf}(\gamma))$ -saturated. Choose $\gamma \in S_1 \cap C$ and $\sup(\gamma \cap C) = \gamma$. So M_γ is $(\theta, \text{cf}(\theta))$ -saturated and also $\bar{\theta}$ -saturated (see proof of 4.10). Together we finish.

□_{4.11}

§ 5. THE AMALGAMATION PROPERTY FOR $\mathfrak{k}_{<\lambda}$

Corollaries 5.5 and 5.6 are the goal of this section, showing that every element of $\mathfrak{k}_{<\lambda}$ is nice (5.4) and $\mathfrak{k}_{<\lambda}$ has the amalgamation property (5.5).

Lemma 5.1. *Suppose that $\langle \mu_i : i < \text{cf}(\mu) \rangle$ is a continuous strictly increasing sequence of ordinals, $\mu = \sup_{i < \text{cf}(\mu)} \mu_i$, and $\chi \leq \mu_0 < \mu \leq \lambda$. Then there exist a linear order I of power μ and a continuous increasing sequence $\langle I_i : i < \text{cf}(\mu) \rangle$ of linear orders such that:*

- (a) $\chi \leq |I_i| \leq \mu_i$,
- (b) $\bigcup_{i < \text{cf}(\mu)} I_i = I$,
- (c) every $t \in I_{i+1} \setminus I_i$ defines a Dedekind cut of I_i in which (at least) one side of the cut has cofinality κ .

Proof. Let $I = (\{0\} \times \mu) \cup (\{1\} \times \kappa)$, $I_i = (\{0\} \times \mu_i) \cup (\{1\} \times \kappa)$ ordered by:

$$(i, \alpha)_{<_I} (j, \beta) \text{ iff } i < j \text{ or } (0 = i = j \text{ and } \alpha < \beta) \text{ or } (1 = i = j \text{ and } \alpha > \beta).$$

□_{5.1}

Lemma 5.2. *Suppose that T is categorical in $\lambda > \text{cf}(\lambda)$, $\kappa + \text{LS}_{\mathfrak{k}} < \mu \leq \lambda$. If $M \in K_\lambda$, then there exists a continuous increasing $\prec_{\mathcal{F}}$ -chain $\langle M_i : i < \text{cf}(\lambda) \rangle$ of models such that:*

- (a) $M \leq_{\mathfrak{k}} \bigcup_{i < \text{cf}(\lambda)} M_i$,
- (b) $\| \bigcup_{i < \text{cf}(\lambda)} M_i \| = \lambda$,
- (c) $\kappa + |T| \leq \|M_i\| < \|M_{i+1}\| < \lambda$,
- (d) for each $i < \text{cf}(\lambda)$, $M_i \leq_{\text{nice}} \left(\bigcup_{j < \text{cf}(\lambda)} M_j \right)$.

Proof. As λ is a limit cardinal, choose a continuous increasing sequence $\langle \mu_i : i < \text{cf}(\lambda) \rangle$, $\lambda = \sup_{i < \text{cf}(\lambda)} \mu_i$, $\kappa + |T| \leq \mu_0 < \lambda$. Let $\langle I, \langle I_i : i < \text{cf}(\lambda) \rangle \rangle$ be as in 5.1. By

λ -categoricity of T without loss of generality $M = \text{EM}(\lambda)$. Let $M_i = \text{EM}(I_i)$ for $i < \text{cf}(\lambda)$. Clearly (1), (2), and (3) hold. To obtain (4), observe that by 2.7 and 3.6 it suffices to remark that by demand (3) from 5.1 on $\langle I_i : i < \text{cf}(\lambda) \rangle$ clauses (N) or (Q) in 2.7 holds for each $t \in I \setminus I_i$. □_{5.2}

Theorem 5.3. *For every $\mu \in [\chi, \lambda]$ and $M \in K_\mu$, there exists $M' \in K_\mu$, $M \leq_{\mathfrak{k}} M'$ such that:*

- (*)_{M'} for every $A \subseteq |M'|$, $|A| < \lambda \wedge |A| \leq \mu$, there is $N \in K_{\chi+|A|}$ such that $A \subseteq N \leq_{\mathfrak{k}} M'$ and N is nice.

Proof. The proof is by induction on μ .

Case 1: $\mu = \chi$. By 3.4 there is $M' \in K_\mu$, $M \leq_{\mathfrak{k}} M'$ and M' is nice. Given $A \subseteq |M'|$ let $N = M'$ and note that N is as required in (*)_{M'}.

Case 2: $\chi < \mu$. Without loss of generality, one can replace M by any $\leq_{\mathfrak{k}}$ -extension in K_μ . Choose a continuous increasing sequence $\langle \mu_i : i < \text{cf}(\mu) \rangle$ such that if μ is a limit cardinal it is a strictly increasing sequence with limit μ ; if μ is a successor, use $\mu_i^+ = \mu$ and in both cases $\chi \leq \mu_i < \mu$. Find $\bar{M} = \langle M_i : i < \text{cf}(\mu) \rangle$ such that:

- (a) $M \leq_{\mathfrak{k}} \bigcup_{i < \text{cf}(\mu)} M_i$,

- (b) $\| \bigcup_{i < \text{cf}(\mu)} M_i \| = \mu,$
- (c) $\|M_i\| = \mu_i,$
- (d) $M_i \leq \bigcup_{\text{nice } j < \text{cf}(\mu)} M_j.$

Why does \bar{M} exist? If $\mu = \lambda$ by 5.2, otherwise by 4.4 (μ regular) and 4.11 (μ singular).

Choose by induction on $i < \text{cf}(\mu)$ models L_i^0, L_i^1, L_i^2 in that order such that:

- (α) $M_i \leq_{\mathfrak{t}} L_i^0 \leq_{\mathfrak{t}} L_i^1 \leq_{\mathfrak{t}} L_i^2 \in K_{\mu_i},$
- (β) $j < i \Rightarrow L_j^2 \leq_{\mathfrak{t}} L_i^0,$
- (γ) $(*)_{L_i^1}$ holds, i.e. for each $A \subseteq |L_i^1|$, there is $N \in K_{\leq \kappa + |T| + |A|}$ such that $A \subseteq N \leq_{\mathfrak{t}} L_i^1$ and N is nice (so in particular L_i^1 is nice, letting $A = |L_i^1|$),
- (δ) L_i^2 is nice and μ_i -universal over L_i^1 ,
- (ε) L_i^0 is $\leq_{\mathfrak{t}}$ -increasing continuous,
- (ζ) $L_i^\ell \cap \bigcup_{j < \text{cf}(\mu)} M_j = M_i$ (or use system of $\leq_{\mathfrak{t}}$ -embeddings).

For $i = 0$, let $L_i^0 = M_0$. For $i = j + 1$, note that by 2.1 there is an amalgam $L_i^0 \in K_{\mu_i}$ of M_i, L_j^2 over M_j since $M_j \leq_{\text{nice}} M_i$ and $M_j \leq_{\mathfrak{t}} L_j^2$ (use last phrase of Fact 1.10 for clause (ζ)); actually not really needed. For limit i , continuity necessitates choosing $L_i^0 = \bigcup_{j < i} L_j^0$ (note that in this case $L_i^0 = \bigcup_{j < i} L_j^2$). To choose L_i^1 apply the inductive hypothesis with respect to μ_i, L_i^0 to find L_i^1 so that $L_i^0 \leq_{\mathfrak{t}} L_i^1$ and $(\gamma)(*)_{(L_i^1)}$ holds. To choose L_i^2 apply Lemma 3.10 to $L_i^1 \in K_{\mu_i}$ giving $L_i^1 \leq_{\mathfrak{t}} L_i^2$, L_i^2 is nice and μ_i -universal over L_i^1 (so (δ) holds).

Let $L = \bigcup_{i < \text{cf}(\mu)} L_i^0 = \bigcup_{i < \text{cf}(\mu)} L_i^1 = \bigcup_{i < \text{cf}(\mu)} L_i^2$, and let $L_i = L_i^0$ if i is a limit, L_i^1 otherwise. Now show by induction on $i < \text{cf}(\mu)$ that L_i is nice.

[Why? show by induction on i for $i = 0$ or i successor that $L_i = L_i^1$ hence use clause (γ), if i is limit then L_i is $(\bar{\theta} \upharpoonright i)$ -saturated, hence L_i is nice by 4.8, 4.10.]

Now $\langle L_i : i < \text{cf}(\mu) \rangle$ witnesses that if μ is regular, L is (μ, μ) -saturated by 4.4, if μ is singular, L is $\bar{\mu}$ -saturated; in all cases L is $\bar{\mu}$ -saturated of power μ , hence by the results of section 4 (i.e. 4.8, 4.10) if $\mu < \lambda$ then L is nice. $\square_{5.3}$

Claim 5.4. $M' = L$ is as required.

Proof. $M \leq_{\mathfrak{t}} \bigcup_{i < \text{cf}(\mu)} M_i \leq_{\mathfrak{t}} \bigcup_{i < \text{cf}(\mu)} L_i^0 = L \in K_\mu$. Suppose that $A \subseteq |L|$. If $|A| = \mu$, then necessarily $\mu < \lambda$ and we take $N = L$. So without loss of generality, $|A| < \mu$. If $\mu = \text{cf}(\mu)$ or $|A| < \text{cf}(\mu)$, then there is $i < \text{cf}(\mu)$ such that $A \subseteq L_i^1$ and, by $(\gamma), (*)_{L_i^1}$ holds, so there is $N \in K_{\kappa + \text{LS}(\mathfrak{R}) + |A|}$, $A \subseteq N \leq_{\mathfrak{t}} L_i^1$, N is nice and $N \leq_{\mathfrak{t}} L$ as required. So suppose that $\text{cf}(\mu) \leq |A| < \mu$. Choose by induction on $i < \text{cf}(\mu)$ models N_i^0, N_i^1, N_i^2 in that order such that:

- (α) $N_i^0 \leq_{\mathfrak{t}} N_i^1 \leq_{\mathfrak{t}} N_i^2,$
- (β) $N_i^2 \leq_{\mathfrak{t}} N_{i+1}^0,$
- (γ) $A \cap L_i^0 \subseteq N_i^0 \leq_{\mathfrak{t}} L_i^0,$
- (δ) $N_i^1 \leq_{\mathfrak{t}} L_i^1$ and N_i^1 is nice,
- (ε) $N_i^2 \leq_{\mathfrak{t}} L_i^2, N_i^2$ is nice and universal over N_i^1 ,
- (ζ) N_i^0, N_i^1, N_i^2 have power at most $\min\{\chi + |A|, \mu_i\}$.

For $i = 0$, apply Fact 1.10 for $A \cap L_0^0, L_0^0$; for $i = j + 1$, apply Fact 1.10 to find $N_i^0 \in K_{\mu_i}, (A \cap L_j^0) \cup N_j^2 \subseteq N_i^0 \leq_{\mathfrak{t}} L_i^0$ (in particular $N_j^2 \leq_{\mathfrak{t}} N_i^0$); for limit $i, N_i^0 = \bigcup_{j < i} N_j^0$. To choose N_i^1 , use $(*)_{L_i^1}$ for the set $A_i = N_i^0$ to find a nice $N_i^1 \in K_{\leq \chi + |A|}, N_i^0 \leq_{\mathfrak{t}} N_i^1 \leq_{\mathfrak{t}} L_i^1$. Note that $\|N_i^1\| \leq \mu_i$. Finally to choose N_i^2 note

that by Lemma 3.12 the model N_i^1 has a nice extension N_i^+ (of power $\|N_i^1\|$) weakly universal over N_i^1 . Now N_i^1 is nice, hence N_i^2 is universal over N_i^1 (by 3.7A(5)) and by Lemma 2.1 there is an amalgam N_i of N_i^+, L_i^1 over N_i^1 such that $\|N_i\| \leq \mu_i$. Since L_i^2 is universal over L_i^1 one can find an $\leq_{\mathfrak{t}}$ -elementary submodel N_i^2 of L_i^2 isomorphic to N_i . Let N_i be N_i^0 if i is a limit, N_i^1 otherwise; prove by induction on i that N_i is nice (by Theorem 4.2).

Now $\bigcup_{i < \text{cf}(\mu)} N_i^0$ is an $\leq_{\mathfrak{t}}$ -elementary submodel of L of power at most $\kappa + |T| + |A|$, including A (by (γ)) and $\bigcup_{i < \text{cf}(\mu)} N_i^0$ is $(\chi + |A|, \text{cf}(\mu))$ -saturated, hence (by Theorem 4.2) nice, as required. $\square_{5.4}$

Corollary 5.5. *If K is categorical in λ then every element of $K_{<\lambda}$ is nice.*

Proof. Suppose otherwise and let $N_0 \in K_{<\lambda}$ be a model which is not nice. Choose a suitable Op such that $\|\text{Op}(N_0)\| \geq \lambda$ and by Fact 1.10 find $M_0 \in K_\lambda, N_0 \leq_{\mathfrak{t}} M_0 \leq_{\mathfrak{t}} \text{Op}(N_0)$ i.e. $N_0 \leq_{\text{nice}} M_0$. It follows that:

\boxplus if $N_0 \leq_{\mathfrak{t}} N \leq_{\mathfrak{t}} M_0$ and $N \in K_{<\lambda}$ then N is not nice.

[Why? By 4.3; alternatively, suppose contrariwise that N is nice. So there is $N_1 \in K_{<\lambda}, N_0 \leq_{\mathfrak{t}} N_1, N_0 \leq_{\text{nice}} N_1 \cdot N_0 \leq_{\text{nice}} N$ since $N_0 \leq_{\text{nice}} M_0$ and $N \leq_{\mathfrak{t}} M_0$, hence there is an amalgam $N' \in K_{<\lambda}$ of N_1, N over N_0 . N is nice, so $N \leq_{\text{nice}} N'$; $N_0 \leq_{\text{nice}} N$, so $N_0 \leq_{\text{nice}} N'$ and so $N_0 \leq_{\text{nice}} N_1$ contradiction.]

On the other hand, applying 5.3 for $\mu = \lambda$ there exists $M' \in K_\lambda$ satisfying $(*)_{M'}$. By λ -categoricity of T without loss of generality, $(*)_{M_0}$ holds (see 5.3) and for $A = |N_0|$ yields a nice model $N \in K_{\kappa+|T|+\|N_0\|}$ such that $N_0 \leq_{\mathfrak{t}} N \leq_{\mathfrak{t}} M_0$ contradicting \boxplus . $\square_{5.5}$

Corollary 5.6. *If K is categorical in λ , then $\mathfrak{k}_{<\lambda}$ has the amalgamation property.*

Proof. By 5.2 and the previous corollary. $\square_{5.6}$

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