THE MONADIC THEORY OF ORDER

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Abstract. We deal with the monadic (second-order) theory of order. We prove all known results in a unified way, show a general way of reduction, prove more results and show the limitation on extending them. We prove (CH) that the monadic theory of the real order is undecidable. Our methods are model-theoretic, and we do not use automaton theory.

This is a slightly corrected version of a very old work.

0. Introduction

The monadic logic is first order logic when we add variables ranging over sets, and allow quantification over them. If pairing functions are available this is essentially second order logic. The monadic theory of a class K of L-models is $\{\psi : \psi$ is a sentence in monadic logic, satisfied by any member of $K\}$.

Here we shall investigate cases where the members of K are linear orders (with one-place predicates).

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Let us review the history. Ehrenfeucht [Ehr61] proved the decidability of the first-order theory of order. Gurevich [Gur64] deduced from it the case of linear order with one-place predicates. Büchi [Büc60] and Elgot [Elg61] proved the decidability of the weak monadic theory (i.e., we can quantify over finite sets) of (the order of) ω, using automaton theory. Büchi continued in this direction, in [Büchi62], showing that also the monadic

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theory (i.e., quantification is possible over arbitrary sets) of ω is decidable; and in [Büc65b] he showed the decidability of the weak monadic theory of ordinals. In [BS73, p. 96]he proved the decidability of the monadic theory of countable ordinals. Rabin [Rab69] proved a very strong and difficult result, implying the decidability of the monadic theory of countable orders. Büchi [BS73] showed the decidability of the monadic theory of ω_1 and of $\{\alpha : \alpha < \omega_2\}$.

Meanwhile Laüchli [Läu68], using methods of Ehrenfeucht [Ehr59] and Fraisse [Fra56] and continuing works of Galvin (unpublished) and Laüchli and Leonard [LL66], proved the decidability of the weak monadic theory of order. He did not use automaton theory. Pinus [Pin72] strengthened, somewhat, those results. Our results have been announced in [She73a], [She73b]

By our notation Laüchli used $Th^n_{\bar{k}}$ only for $\bar{k} = \langle 1, 1, 1, \ldots \rangle$ (changed for the quantification over finite sets).
Remark: We are not interested here in results without the axiom of choice. See Siefkes [Sil70] which shows that the result on ω is provable in ZF. This holds also for $\alpha < \omega^*$. Litman [Lit76] pointed out some mistakes in [BS73, 6] (theorems without AC); proved connected results, and showed in ZF that ω_1 is always characterizable by a sentence.

In Section 7 we prove (CH) the undecidability of the monadic theory of the real order and of the class of orders, and related problems. It can be read independently, and has a discussion on those problems. Gurevich finds that our proof works also for the lattice of subsets of a Cantor discontinuum, with the closure operation, and similar spaces. Hence Grzegorczy's [Grz51] question is answered (under CH)[1].

Our work continues [Läu68], but for well ordering we use ideas of Büchi and Rabin. We reduce here the decision problem of the monadic theories of some (classes of) orders [e.g., well orderings; the orders which do not embed ω_1 not ω_1^*] to problems more combinatorial in nature. So we get a direct proof for the decidability of countable orders (answering a question of Büchi [BS73, p.35] Our proof works for a wider class, thus showing that the countable orders cannot be characterized in monadic theory, thus answering a question of Rabin [Rab69](p.12). Moreover, there are uncountable orders which have the same monadic theory as the rationals (e.g., dense Specker order; see [Jec03] for their existence; and also some uncountable subsets of the reals). We also show that the monadic theory of $\{\alpha : \alpha < \lambda^+\}$ is recursive in that of λ, generalizing results of Büchi for ω and ω_1. Unfortunately, even the monadic theory of ω_2 contains a statement independent of ZFC. For a set A of ordinals, let $F(A) = \{\alpha : \alpha$ is a limit ordinal of cofinality $> \omega, \alpha < \sup A$, and $\alpha \cap A$ is a stationary subset of $\alpha\}$.

Now Jensen [Jen72] proved the following:

[1]Gurevich meanwhile has proved more and has a paper in preparation.

Theorem 0.1. $(V = L)$. *A regular cardinal κ is weakly compact if and only if for every stationary $A \subsetneqq \kappa$, such that $(\forall \alpha \in A)[\mathrm{cf}(\alpha) = \omega], F(A) \neq \varnothing$.*

As the second part is expressible in the monadic theory of order, the Hanf number of the monadic theory of order is high. Clearly also the monadic theory of the ordinals depends on an axiom of large cardinals.

Now, Baumgartner [Bau76] *shows that if ZFC+ (there is a weakly compact cardinal) is consistent,* then *it is consistent with ZFC that*

(*) *for any stationary $A \subsetneqq \omega_2$, if $(\forall \alpha \in A)[\mathrm{cf}(\alpha) = \omega]$, then $F(A) \neq \varnothing$ (and in fact is stationary).*

So ZFC does not determine the monadic theory of ω_2. This partially answers [Büc65a] *(pp.34-43; p.38, problem 2).*

We can still hope that the number of possible such theories is small, and each decidable, but this seems unlikely. We can also hope to find the sentences true in every model of ZFC. A more hopeful project is to find a decision procedure assuming $V = L$. We show that for this it suffices to prove only the following fact. Let D_{ω_2} be the filter of closed unbounded subsets of ω_2. (Magidor disproves (**) in $V = L$, but it may still be consistent with ZFC.)

(**) *if $A \subsetneqq \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}, F(A) = B \cup C, A$ is stationary,* then *there are A_1, A_2, such that $A = A_1 \cup A_2, A_1 \cap A_2 = \varnothing, A_1, A_2$ are stationary and $F(A_1) = B(\mathrm{mod} D_{\omega_2}), F(A_2) = C(\mathrm{mod} D_{\omega_2})$.*

We prove, in fact, more: that the monadic theory of ω_2 and the first order theory of $\langle \underline{P}(\omega_2)/D_{\omega_2}, \cap, \cup, F\rangle$ are recursive one in the other.

Conjecture 0.2. $(V = L)$. The monadic theory of ω_2 (and even ω_n) is decidable.

Conjecture 0.3. ($V = L+$ there is no weakly compact cardinal). The monadic theory of well orders is decidable.

Laüchli and Leonard [LL66] define a family $\underline{M}$ of orders as follows: It is the closure of $\{1\}$ by

(1) $M + N$,
(2) $M \cdot \omega$ and $M \cdot \omega^*$,
(3) $\sum^*_{i<n} M_i$ which is $\sum_{a \in Q} M_a$ and $\{a \in Q : M_a = M_i\}$ is a dense subset of the rationals, and each $M_a \in \{M_i : i < n\}$.

(See Rosenstein [Ros69] and Rubin [Rub74] for generalization.)

Läuchli [Läu68] proved that every sentence from the weak monadic language of order has a countable model if and only if it has a model in $\underline{M}$. Easy checking of Section 4 shows this holds also for the monadic language. On the other hand, looking at the definition of $\underline{M}$, we can easily see that for every $M \in \underline{M}$ there is a monadic sentence ψ such that $M \models \psi$, and $\|N\| \leqq \aleph_0, N \models \psi$ imply $N \cong M$.

In this way we have a direct characterization of $\underline{M}$.

Theorem 0.4. *$M \in \underline{M}$ if and only if M is countable and satisfies some monadic sentence which is $(\leqq \aleph_0)$-categorical.*

Also for other classes whose decidability we prove, we can find subclasses analogous to $\underline{M}$. This theorem raises the following question:

Conjecture 0.5. For every $N \in \underline{M}$ there is a monadic sentence ψ such that $M \models \psi$ implies that M and N have the same monadic theory. (It suffices to prove this for the rational order.)

Related questions are:

Conjecture 0.6. There is a monadic sentence ψ such that $R \models \psi$ and $M \models \psi$ imply that M and R have the same monadic theory.[2]

Conjecture 0.7. There is an order M which has the same monadic theory as R, but is not isomorphic to R.[3]

Conjecture 0.8. There are orders with the same monadic theories, whose completions do not have the same monadic theories.[4]

The characterization of $\underline{M}$ gives us also

Conclusion 0.9. The question whether a sentence in the first-order (or even monadic) theory of order is $(\leqq \aleph_0)$-categorical (or $\aleph_0$-categorical) is decidable.

A natural question is whether the monadic theory of $\mathfrak{M}$ is more "complex" than that of the ordinals (the orders in $\mathfrak{M}$ are countable unions of scattered types; see Laver [Lav71, §3], which includes results of Galvin). To answer this, we have the

Definition 0.10. For a model M with relations only, let $M^\sharp$ be the following model:

(i) its universe is the set of finite sequences of elements of M;
(i) its relations are
  (a) $<$, where $\bar{a} < \bar{b}$ means $\bar{a}$ is a initial segment of $\bar{b}$,
  (b) for each n-place predicate R from the language of M, $R^{M^\sharp} = \{\langle\langle a_1, \dots, a_{m-1}, b^1\rangle, \langle a_1, \dots, a_{m-1}, b^2\rangle, \dots, \langle a_1, \dots, a_{m-1}, b^n\rangle\rangle : a_i, b^i$ are elements of M, and $M \models R[b^1, \dots, b^n]\}$.

The author suggested a generalization of Rabin's automaton from [Rab69], proved the easy parts: the lemmas on union and intersection, and solved the emptiness problem. Then J.Stup elaborated those proofs, and proved the complementation lemma. Thus a generalization of the theorem and proof of [Rab69] gives

Theorem 0.11. *The monadic theory of $M^\sharp$ is recursive in the monadic theory of M.*

Thus, using [Lav71, §3] *notation, we get, e.g.,*

[2]Confirmed by Gurevich
[3]Refuted by Gurevich
[4]Confirmed by Gurevich

Conclusion 0.12. The monadic theory of $\{M : M \in \mathfrak{M}, \|M\| \leqq \lambda\}$ is recursive in the monadic theory of λ.

Because by Section 2 the monadic theory of $\sigma_{\lambda^+,\lambda^+}$ is recursive in the monadic theory of λ, by 0.6 the monadic theory of $\eta_{\lambda^+,\lambda^+}$ is recursive in the monadic theory of λ, and so we finish, as by [Lav71, 3.2(iv),3.4] $\eta_{\lambda^+,\lambda^+}$ is a universal member of $\{M \in \mathfrak{M} : \|M\| \leqq \lambda\}$.

Also useful are the following (Le Tourneau [LT68] proved parts (1),(2) at least):[5]

Theorem 0.13. *Let L be a language with one one-place function symbol, equality and one place predicates.*

(1) *The monadic theory of L is decidable.*
(2) *If a monadic sentence ψ of L has a model, it has a model of cardinality $\leqq \aleph_0$.*
(3) *In (2) we can find $n = n(\psi) < \aleph_0$ and a model M such that $|\{b \in |M| : f(b) = a\}| \leqq n$ for any $a \in |M|$.*

This is because, if M_λ is the model whose universe is λ, and whose language contains equality only, in $M_\lambda^\sharp$ we can interpret a universal L-model (see Rabin [Rab69]*). This implies (1). Note that all M_λ (λ an infinite cardinal) have the same monadic theory. This proves (2). For (3) note that if $M_{\aleph_0} \models \psi$, then for all big enough $n, M_n \models \psi$.*

Remark (1): Rabin [Rab69] prove the decidability of the countable Boolean algebras, in first-order logic expanded by quantification over ideals. By the Stone representation theorem, each countable Boolean algebra can be represented as the Boolean algebra generated by the intervals of a countable order. By the method of Section 3 we can prove that the theory of countable linear orders in monadic logic expanded by quantification over such ideals, is decidable, thus reproving Rabin's result. (The only points is that methods of Section 2 apply.)

Conjecture 0.14. The monadic theory of orders of cardinality $\leqq \aleph_1$ is decidable when $\aleph_1 < 2^{\aleph_0}$.

Conjecture 0.15. The theory of Boolean algebras of cardinality $< \lambda$ or in first-order logic expanded by allowing quantification over ideals is decidable when $\lambda \leqq 2^{\aleph_0} (\lambda = \aleph_2 \leqq 2^{\aleph_0})$.

Remark: We can prove Conclusion 0.7 by amalgamating the methods of Section 4,5, and 6.

[5] Le Tourneau only claimed the result. Lately also Routenberg and Vinner proved this theorem.

1. Ramsey theorem for additive coloring

A *coloring* of a set I is a function f from the set of unordered pairs of distinct elements of I, into a finite set T of colors. We write $f(x,y)$ instead of $f(\{x,y\})$, assuming usually that $x<y$. The coloring f is additive if for $x_i<y_i<z_i\in I$ $(i=1,2)$.

$$f(x_1,y_1)=f(x_2,y_2);f(y_1,z_1)=f(y_2,z_2)$$

imply $f(x_1,z_1)=f(x_2,z_2)$. In this case a (partial) operation $+$ is defined on T, such that for $x<y<z\in I, f(x,z)=f(x,y)+f(y,z)$. A set $J\subseteqq I$ is homogeneous (for f) if there is a $t_0\in T$ such that for every $x<y\in J, f(x,y)=t_0$.

Ramsey's theorem [Ram29] states, in particular, that if we color an infinite set with a finite set of colors, then there is an infinite homogeneous subset. This theorem has many generalization and applications. It was used in [Büchi62] for a coloring which was, in fact, additive. Using an idea of Rabin, Büchi [BS73, 12, p.58] offered an alternative proof (using, in fact, additivity) and in [BS73, 6.2, p.111] straightforwardly generalized it to ω_1 (the result for ω_1 is not true for coloring in general). We give the natural extension to arbitrary ordinals (which is immediate, and included for completeness) and a parallel theorem for dense orders.

Theorem 1.1. *If δ is a limit ordinal, f an additive coloring of δ (by a set T of n colors), then there is an unbounded homogeneous subset J of δ.*

Remarks:

(1) If the cofinality of δ is $\geqq\omega_1$ we can assume that if $a,b<c', f(a,c')=f(b,c')$, then $a,b<c\in J$ implies $f(a,c)=f(b,c)$.
(2) Instead of $|T|<\aleph_0$, we need assume only $|T|<\mathrm{cf}(\delta)$.

Conclusion 1.2. Under the condition of 1.1, there are a closed unbounded subset J of δ, and $J_k, J^\ell, 1\leqq k,\ell\leqq|T|$ and $t^\ell_k\in T$ such that $J=\cup_k J_k=\cup_\ell J^\ell$, the J_k's are disjoint, the J^ℓ's are disjoint, and if $a<b\in J, a\in J_k, b\in J^\ell$ then $f(a,b)=t^\ell_k$.

Theorem 1.3. *If f is an additive coloring of a dense set I, by a finite set T of n colors, then there is an interval of I which has a dense homogeneous subset.*

Conclusion 1.4. Under the hypothesis of 1.3, there is an interval (a,b) of I, and $(a,b)=\cup^{|T|}_{k=1}J_k=\cup^{|T|}_{\ell=1}J^\ell$ and colors $t^\ell_k\in T$ such that for $x<y, x\in J_k, y\in J^\ell, f(x,y)=t^\ell_k$.

Remark: We can choose the J_0, J_k, J^ℓ's so that they are definable by first-order formulas with parameters in the structure $(\delta,<,f)$ (or $(I,<,f)$).

Proof of Theorem 1.1: Define: For $x,y\in\delta, x\sim y$ if there is a z such that $x,y<z<\delta$, and $f(x,z)=f(y,z)$; clearly this implies by the additivity of f that for any $z', z < z' < \delta, f(x,z') = f(y,z')$. It is easy to verify that $\sim$ is an equivalence relation with $\leqq |T|$ equivalence classes. So there is at least one equivalence class I, which is an unbounded subset of δ. Let x_0 be the first element of I. Let, for $t \in T, I_t = \{y : x_0 \neq y \in I, f(x_0,y) = t\}$. Clearly $I - \{x_0\} = \cup_{t\in T} I_t$, hence for some s, I_s is an unbounded subset of δ. Let $\langle a_i : i < \mathrm{cf}(\delta)\rangle$ be an increasing unbounded sequence of elements of δ. Define by induction on i elements $y_i \in I$. If for all $j < i (i < \mathrm{cf}(\delta), y_j$ have been defined, let $y_i < \delta$ be such that $y_i > y_j, y_i > a_j, y_i > x_0$ and $f(x_0,y_i) = f(y_j,y_i)$ for any $j < i$, and $y_i \in I_s$. Now $J = \{y_i : i < \mathrm{cf}(\delta)\}$ is the desired set. Clearly it is unbounded. If $y_j < y_i$ (hence $j < i$) then

$$f(y_j,y_i) = f(x_0,y_i) = s.$$

So J is homogeneous.

Proof of Conclusion 1.2: If the cofinality of δ is $\aleph_0$, then the J from 1.1 is also closed (trivially). So assume $\mathrm{cf}(\delta) > \aleph_0$, let $T = \{t_1,\dots,t_n\}$, and let J, y_j be as defined in the proof of 1.1; and let J^* be the closure of $\{y_{j+1} : j < \mathrm{cf}(\delta)\}$. Then $J^* = \{y^j : j < \mathrm{cf}(\delta)\}$ is increasing, continuous, and $y^{j+1} = y_{j+1}$. Let $J' = \{y^j : j \text{ is a limit ordinal}\}$,

$J_k = \{y^j : j$ is a limit ordinal, $f(y^j, y^{j+1}) = t_k\}$,
$J^\ell = \{y^j : j$ is a limit ordinal, and $(\forall i < j)(\exists \alpha)(i < \alpha < j \wedge f(y^{\alpha+1}, y^j) = t_\ell)$
but this does not fold for any $\ell' < \ell\}$.

Now clearly $J' = \cup_k J_k = \cup_\ell J^\ell$, and if $x \in J_k, z \in J^\ell, x < z$ then $x = y^i, z = y^j, i < j, i, j$ are limit ordinals and there is an $\alpha, i < \alpha < j$, such that $f(y^{\alpha+1}, y^j) = t_\ell$. Hence

$$\begin{aligned} f(x,z) &= f(y^i,y^j) = f(y^i,y^{i+1}) + f(y^{i+1},y^{\alpha+1}) + f(y^{\alpha+1},y^j) \\ &= t_k + f(y_{i+1},y_{\alpha+1}) + t_\ell = t_k + s + t_\ell \overset{\text{def}}{=} t_k^\ell. \end{aligned}$$

Clearly all the demands are satisfied.

Proof of Theorem 1.3: Remember that $J \subseteqq I$ is dense in an interval (a,b) if for every $x, y \in I, a < x < y < b$, there is a $z \in J$ such that $x < z < y$. It is easy to see that if $J \subseteqq I$ is dense in an interval (a,b) and $J = \cup_{k=1}^m J_k$ $(m > 1)$ then there are k and a', b' such that $a < a' < b' < b, 1 \leqq k \leqq m$ and J_k is dense in (a', b').

Define for any $a \in I, J \subseteqq I$

$$F(a,J) = \{t : t \in T, (\forall x > a)(\exists y \in J)(a < y < x \wedge f(a,y) = t)\}.$$

Notice, that since T is finite, for any $a \in I$, and any $J \subseteqq I$ there is a $b, a < b \in I$ such that:

$$t \in F(a,J) \text{ if and only if there is a } y \in J, a < y < b, f(a,y) = t.$$

We define by induction on $m \leqq n2^n + 2$ intervals (a_m,b_m), sets J_m dense in (a_m,b_m), and (for $m > 0$) sets $D_m \subseteqq T$.

For $m = 0$, let (a_0,b_0) be any interval of I, and $J_0 = \{x \in I : a_0 < x < b_0\}$. Suppose $(a_m,b_m), J_m$ are defined. For any $D \subseteqq T$ let $J_m(D) = \{a \in J_m :$

$F(a, J_m) = D\}$. Clearly $J_m = \cup_{D \subseteqq T} J_m(D)$ and as there are only finitely many possible D's ($\leqq 2^n$), there is an interval (a_{m+1}, b_{m+1}) and $D_{m+1} \subseteqq T$ such that $J_m(D_{m+1})$ is dense in (a_{m+1}, b_{m+1}), and $a_m < a_{m+1} < b_{m+1} < b_m$. Let $J_{m+1} = (a_{m+1}, b_{m+1}) \cap J_m(D_{m+1})$. Clearly $J_m \supseteqq J_{m+1}$, and $m > k$ implies $J_k \supseteqq J_m$, and (a_m, b_m) is a subinterval of (a_k, b_k).

As there are only $\leqq 2^n$ possible D_m, there are a $D \subseteqq T$ and $0 \leqq m_0 < \ldots < m_n \leqq n2^n + 1$ such that $D_{m_i+1} = D$. Define, for $0 \leqq k \leqq n, a^k = a_{m_k}, b^k = b_{m_k}, J^k = J_{m_k}$.[6]

It is easy to check that if $0 \leqq k < l \leqq n, x \in J^\ell$ then $x \in J_{m_\ell} \subseteqq J_{m_{k+1}}$, hence $F(x, J_k) = F(x, J_{m_k}) = D_{m_{k+1}} = D$. It is clear that $J^0 \supseteqq J^1 \supseteqq \ldots \supseteqq J^n$.

Choose $x_0 \in J^n$. Then there is $x_1, x_0 < x_0 < x_1 < b^n$, such that $x_0 < y < x_1, y \in J^0$ implies $f(x_0, y) \in F(x_0, J^0) = D$. Hence $t \in D$ if and only if there is $y \in J^{n-1}, x_0 < y < x_1, f(x_0, y) = t$, if and only if there is $y \in J_0, x_0 < y < x_1, f(x_0, y) = t$. Clearly

$$J^n \cap (x_0, x_1) = \cup_{t \in T} \{y : y \in J^n, x_0 < y < x_1, f(x_0, y) = t\}.$$

Hence there are a, b, t_0 such that $x_0 < a < b < x_1$ and

$$J^* = \{y : y \in J^n, a < y < b, f(x_0, y) = t_0\}$$

is dense in (a, b). Clearly $t_0 \in D$.

It is easy to check that for $t, s \in D, t + s$ is defined and $\in D$, so for $t \in D, m \geqq 1$ defined $mt \in T$, by induction on $m : 1t = t, (m+1)t = mt + t$. As T has n elements, $1t_0, 2t_0, \ldots, (n+1)t_0$ cannot be pairwise distinct. So there are $i, j, 1 \leqq i < (i + j) \leqq n + 1$ such that $it_0 = (i + j)t_0$. Define

$$J = \{y : a < y < b, f(x_0, y) = jt_0, y \in J^{n-j+1}\}.$$

We shall show that J is the desired set.

(I) J is dense in (a, b).

Suppose $a < a' < b' < b$, and we shall find $z \in J, a' < z < b'$. As J^* is dense in (a, b) there are $z^n \in J^* \subseteqq J^n, a' < z^n < b'$. We define by downward induction z^k for $n - -j + 1 \leqq k \leqq n$ such that $z^k \in J^k, a' < z^k < b'$. For $k = n, z^k$ is defined. Suppose z^{k+1} is defined, then as $z^{k+1} \in J^{k+1}$ is follows that $F(z^{k+1}, j^k) = D$. As $t_0 \in D$ there is $z^k \in J^k$, such that $z^{k+1} < z^k < b'$ and $f(z^{k+1}, z^k) = t_0$. Clearly

$$x < z^n < z^{n-1} < \ldots < z^{n-j+1},$$

$$f(x_0, z^n) = t_0, \quad f(z^{i+1}, z^i) = t_0.$$

Hence $f(x_0, z^{n-j+1}) = t_0 + \ldots + t_0 = jt_0$, so $z^{n-j+1} \in J, a' < z^{n-j+1} < b'$.

[6] In fact $D_m(T) \supseteqq D_m(T)$, hence we can replace $n2^n + 2$ by $n^2 + 2$.

(II) J is homogeneous.

Suppose $a < y < z < b, y, z \in J$. Then $y \in J^{n-j+1}$. Now define by downward induction $y^k \in J^k$ for $0 \leqq k \leqq i, y \leqq y^k < z$. Let $y^i = y(y^i \in J^i$ because $y^i = y \in J^{n-j+1}$, and as $i+j \leqq n+1, i \leqq n--j+1$ hence $J^{n-j+1} \subseteqq J^i)$. If y^{k+1} is defined then $F(y^{k+1}, J^k) = D$, hence there are $y^k \in J^k, y^{k+1} < y^k < z$ such that $f(y^{k+1}, y^k) = t_0$. It follows that $x_0 < y = y^i < y^{i-1} < \ldots < y^0 < z$ and

$$f(y^k, y^{k-1}) = t_0.$$

Hence

$$f(y, y^0) = f(y^i, y^0) = it_0.$$

So

$$\begin{aligned} f(y,z) &= f(y,y^0) + f(y^0,z) = it_0 + f(y^0,z) \\ &= (i+j)t_0 + f(y^0,z) = jt_0 + it_0 + f(y^0,z) \\ &= f(x_0,y) + f(y,y^0) + f(y^0,z) = f(x_0,z) = jt_0. \end{aligned}$$

This proves the homogeneity of J.

Proof of Conclusion 1.4: Let $(a, b), J$ and t_0 be as in the proof of 1.3. Let $T = \{t_1, \ldots, t_n\}$. Let

$$J_k = \{y : y \in (a,b), t_k \in F(y,J), t_1, \ldots, t_{k-1} \notin F(y,J)\},$$

$$J^\ell = \{y : y \in (a,b), t_\ell \in F'(y,J), t_1, \ldots, t_{\ell-1} \notin F'(y,J)\}$$

where F' is defined just as F is, but for the reversed order.

Clearly $(a, b) = \cup_k J_k = \cup_\ell J^\ell$. Suppose $x < y, x \in J_k, y \in J_\ell$. Then we can find x', y' $x < x' < y' \in J$, such that $f(x, x') = t_k, f(y', y) = t_\ell$. Hence

$$f(x,y) = f(x,x') + f(x',y') + f(y',y) = t_k + t_0 + t_\ell \stackrel{\text{def}}{=} t^\ell_k.$$

2. The monadic theory of generalized sums

Feferman and Vaught [FV59] proved that the first order theory of sum, product, and even generalized products of models depends only on the first-order theories of the models. Their theorem has generalizations to even more general products (see Olmann) and to suitable infinitary languages (L_α, see Malitz [Mal71]).

On the other hand, it is well-known that for second order theory this is false even for sum (as there is a sentence true in the sum of two models if and only if they are isomorphic, for fixed finite language, of course). Also for monadic (second-order) theory this is false for products of models (there is a sentence true in a direct product of two models of the theory of linear order if and only if the orders are isomorphic). We notice here that the monadic theory of generalized sum depends only on the monadic theories of the summands and notice also generalization of known refinement (see

Fraissé [Fra56]). We can prove them using natural generalization of Ehrenfeucht games (see [Ehr61]). Läuchli [Läu68] uses some particular cases of those theorems for the weak monadic theory. As there is no new point in the proofs, we skip them. We should notice only that a subset of sum of models is the union of subsets of the summands. The results of [FV59] can be applied directly by replacing M by $(|M| \cup \underline{P}(M), M, \in)$.

Notation 2.1. L will be first-order language with a finite number of symbols, L^M the corresponding monadic language, $L(M)$ the first-order, language corresponding to the model M, the universe of M, is $|M|$. Let x, y, z be individual variables; X, Y, Z set variables; a, b, c elements; P, Q sets; $\underline{P}(M) = \{P : P \subseteqq |M|\}$. Bar denotes that this is a finite sequence, e.g., $\bar{a}$; $\ell(\bar{a})$ its length, $\bar{a} = \langle \dots, a_i, \dots \rangle_{i<\ell(\bar{a})}$, and let $\bar{a}(i) = a_i$. We write $\bar{a} \in A$ instead of $a_i \in A$ and $\bar{a} \in M$ instead of $\bar{a} \in |M|$. K is a class of $L(K)$ models ($L(K) = L(M)$ for any $M \in K$). Let

$$K^m = \{(M, \bar{P}) : \bar{P} \in \underline{P}(M)^m\}, K^\infty = \cup_{m<\omega} K^m.$$

Let k, ℓ, m, n, p, q, r denote natural numbers.

Definition 2.2. For any L-model $M, \bar{P} \in \underline{P}(M), \bar{a} \in |M|, \Phi$ a finite set of formulas $\varphi(X_1, \dots, x_1, \dots) \in L$, a natural number n, and a sequence of natural numbers $\bar{k}$ of length $\geqq n$, define

$$t = th^n_{\bar{k}}((M, \bar{P}, \bar{a}), \Phi)$$

by induction on n:

For $n = 0$:

$$t = \{\varphi(X_{\ell_1}, \dots, x_{j_1}, \dots) : \varphi(X_1, \dots, x_1, \dots) \in \Phi, M \models \varphi[P_{\ell_1}, \dots, a_{j_1}, \dots]\}.$$

For $n = m + 1$:

$$t = \{th^m_{\bar{k}}(M, \bar{P}, \bar{a}^\frown \bar{b}) : \bar{b} \in |M|^{\bar{k}(m)}\}.$$

Definition 2.3. For any L-model $M, \bar{P} \in \underline{P}(M)$, a finite set Φ of formulas $\varphi(X_1, \dots, x_1 \dots) \in L, n, \bar{k}$ of length $\geqq n+1$, define $T = Th^n_{\bar{k}}(M, \bar{P}), \Phi)$ by induction on n:

For $n = 0$:

$$T = th^1_{\bar{k}}((M, \bar{P}), \Phi).$$

For $n = m + 1$:

$$T = \{Th^m_{\bar{k}}((M, \bar{P}^\frown \bar{Q}), \Phi) : \bar{Q} \in \underline{P}(M)^{\bar{k}(n)}\}.$$

(1) If Φ is the set of atomic formulas we shall omit it and write $Th^n_{\bar{k}}(M, \bar{P})$.
(2) We always assume $\bar{k}(i) \geqq 1$ for any $i < \ell(\bar{k})$, and $\bar{k}(0) \geqq m_R$ if $R \in L(M)$ is m_R-place.
(3) If we write $\bar{k}(i)$ for $i \geqq \ell(\bar{k})$, then we mean 1, and when we omit $\bar{k}$ we mean $\langle \max\{m_R : R \in L(M)\}, 1, \dots \rangle$.
(4) We could have mixed Definition 2.2, and 2.3, and obtained a similar theorem which would be more refined.

Lemma 2.4. (A) *For every formula $\psi(\bar{X}) \in L^M(M)$ there is an n such that from $Th^n_{\bar{k}}(M, \bar{P})$ we can find effectively whether $M \models \psi[\bar{P}]$.*

(B) *For every $L, \bar{k}, n, \Phi \subseteqq L$, and m there is a set $\Psi = \{\psi_\ell(\bar{X}) : \ell < \ell_0(< \omega), \ell(\bar{X}) = m\}(\psi_\ell \in L^M)$ such that for any L-models M, N and $\bar{P} \in \underline{P}(M)^m, \bar{Q} \in \underline{P}(N)^m$ the following hold:*

(a) *$Th^n_{\bar{k}}((N, \bar{Q}), \Phi)$ can be computed from $\{\ell < \ell_0 : N \models \psi_\ell[\bar{Q}]\}$.*

(b) *$Th^n_{\bar{k}}((N, \bar{Q}), \Phi) = Th^n_{\bar{k}}((M, \bar{P}), \Phi)$ if and only if for any $\ell < \ell_0, M \models \psi_\ell[\bar{P}] \Leftrightarrow N \psi_\ell[\bar{Q}]$.*

Proof: Immediate. In (A) it suffices to take for n the quantifier depth of ψ.

Lemma 2.5. (A) *For given $L, n, m, \bar{k}$, each $Th^n_{\bar{k}}(M, \bar{P})$ is hereditarily finite, and we can compute the set of formally possible $Th^n_{\bar{k}}(M, \bar{P}), \ell(\bar{P}) = m, M$ an L-model. The same holds for Φ.*

(B) *If $\bar{\ell}(0) \geqq \bar{k}(0), 1 = p_0 < p_1 < p_2 < \ldots < p_n \leqq m$ and for $1 \leqq i \leqq n$, $\bar{k}(i) \leqq \sum_{p_{i-1} \leqq j \leqq p_i} \bar{\ell}(j)$ then from $Th^m_{\bar{\ell}}((M, \bar{P}), \Phi)$ we can effectively compute $Th^n_{\bar{k}}((M, \bar{P}), \Phi)$.*

(C) *For every $n, \bar{k}, \bar{\ell}$ we can compute m such that from $Th^m_{\bar{\ell}}((M, \bar{P}), \Phi)$ we can effectively compute $Th^n_{\bar{k}}((M, \bar{P}), \Phi)$.*

(D) *Suppose in Definition 2.3 we make the following changes: We restrict ourselves to partition $\bar{P}$, and let $\bar{Q}$ be a partition refining $\bar{P}$, which divides each P_i to $2^{\bar{k}(m)}$ parts. What we get we call $pTh^n_{\bar{k}}((M, \bar{P}), \Phi)$. Then from $pTh^n_{\bar{k}}((M, \bar{P}), \Phi)$ we can effectively compute $Th^n_{\bar{k}}((M, \bar{P}), \Phi)$, and vice versa.*

(E) *Let K, n, Φ be given. If for every $\bar{k}$ there is an $\bar{\ell}$ such that for every $m, M, N \in K^m$,*

$$Th^n_{\bar{\ell}}(M, \Phi) = Th^n_{\bar{\ell}}(N, \Phi) \Rightarrow Th^{n+1}_{\bar{k}}(M, \Phi) = Th^{n+1}_{\bar{k}}(N, \Phi)$$

then for every $m, \bar{k}$ there is an $\bar{\ell}$ such that for any $n', M, N \in K^m$

$$Th^n_{\bar{\ell}}(M, \Phi) = Th^n_{\bar{\ell}}(N, \Phi) \Rightarrow Th^{n'}_{\bar{k}}(N, \Phi) = Th^{n'}_{\bar{k}}(M, \Phi).$$

Remark: This is parallel to elimination of quantifiers.

(F) In (E), if in the hypothesis $\bar{\ell}$ can be found effectively from $\bar{k}$ then in the conclusion, $\bar{\ell}$ can be found effectively from $m, \bar{k}$. If in addition $\{Th^n_{\bar{k}}(M, \Phi) : M \in K^m\}$ is recursive in $\bar{k}, m$ then $\{Th^p_{\bar{k}}(M, \Phi) : M \in K\}$ is recursive in $p, \bar{k}$.

Proof: Immediate.

The following generalizes the ordered sum of ordered sets (which will be our main interest) to the notion of a generalized sum of models. (Parts (1),(2),(3) of the definition are technical preliminaries.)

Definition 2.6. Let L_1, L_2, L_3 be first-order languages, M_i an L_1-model (for $i \in |N|$), N an L_2-model, and we shall define the L_3-model $M = \sum^{\sigma}_{i \in |N|} M_i$ (the generalized sum of the M_i's relative to σ).[7]

(1) An n-condition τ is a triple $\langle E, \Phi, \Psi \rangle$ where:
  (A) E is an equivalence relation on $\{0, 1, \dots, n-1\}$.
  (B) Φ is a finite set of formulas of the form $\varphi(x_{j_1}, \dots, x_{j_k})$ where $j_1, \dots, j_k$ are E-equivalent and $< n$; and $\varphi \in L_1$.
  (C) Ψ is a finite set of formulas of the form $\psi(x_{j_1}, \dots, x_{j_k})$ where $j_1, \dots, j_k < n, \psi \in L_2$.

(2) If $a_0, \dots, a_{n-1} \in \bigcup_{i \in |N|} M_i, \tau = \langle E, \Phi, \Psi \rangle$ is an n-condition, $a_\ell \in M_{i(\ell)}$, then we say $\langle a_0, \dots, a_{n-1} \rangle$ satisfies τ if:
  (A) $i(\ell) = i(m) \Leftrightarrow \ell E m$;
  (B) $\varphi(x_{j_1}, \dots, x_{j_k}) \in \Phi \Rightarrow M_{i(j_1)} \models \varphi[a_{j_1}, \dots, a_{j_k}]$;
  (C) $\psi(x_{j_1}, \dots, x_{j_k}) \in \Psi \Rightarrow N \models \psi[i(j_i), \dots, i(j_k)]$.

(3) The rule, σ is $\langle L_1, L_2, L_3, \sigma^* \rangle$ where σ^* is a function whose domain is the set of predicates of L_3; if R is an n-place predicate in $L_3, \sigma^*(R)$ will be a finite set of n-conditions.

(4) $M = \sum^{\sigma}_{i \in |N|} M_i$ is an L_3-model, whose universe is $\cup_{i \in |N|} |M_i|$, and for every predicate $R \in L_3, R^M = \{\langle a_0, \dots, a_{n-1} \rangle$ satisfies some $\tau \in \sigma^*(R)\}$.

  Let $\Phi(\sigma)$ ($\Psi(\sigma)$) be the set of all formulas $\varphi_j \in L_1(\sigma)$ ($\psi_p \in L_2(\sigma)$) appearing in the $\sigma(R)$'s, $R \in L_3(\sigma)$, and the equality.

Remarks:

(1) We use the convention that $\sum^{\sigma}_{i \in N}(M_i, \bar{P}^i) = (\sum^{\sigma}_{i \in N} M_i, \cup_{i \in N} \bar{P}^i)$ where for $\bar{P}^i = \langle P^i_1, \dots, P^i_m \rangle, \bigcup_i \bar{P}_i = \langle \bigcup_i P^i_1, \dots, \bigcup_i P^i_m \rangle$.
(2) We could have defined the sum more generally, by allowing the universe and the equality to be defined just as the other relations.

Lemma 2.7. *For any $\sigma, n, m, \bar{k}$, if for $\ell = 1, 2, \bar{P}^\ell_1 \in \bar{P}(M^\ell_i)^m$ and for every $i \in N$,*

$$Th^n_{\bar{k}}((M^1_i, \bar{P}^1_i), \Phi(\sigma)) = Th^n_{\bar{k}}((M^2_i, \bar{P}^2_i, \bar{P}^2_i), \Phi(\sigma)),$$

then

$$Th^n_{\bar{k}}(\sum^{\sigma}_{i \in N}(M^1_i, \bar{P}^1_i)) = Th^n_{\bar{k}}(\sum^{\sigma}_{i \in N}(M^2_i, \bar{P}^2_i)),$$

Theorem 2.8. *For any $\sigma, n, m, \bar{k}$ we can find an $\bar{r}$ such that: if $M = \sum^{\sigma}_{i \in N} M_i, t_i = Th^n_{\bar{k}}((M_i, \bar{P}_i), \Phi(\sigma))$, and $Q_t = \{i \in N : t_i = t\}, \ell(\bar{P}_i) = m$, then from $Th^n_{\bar{r}}((N, \dots, Q_t, \dots), \Psi(\sigma))$ we can effectively compete $Th^n_{\bar{k}}(M, \bigcup_i \bar{P}_i)$ (which is uniquely determined).*

Definition 2.9. (1) For a class K of models

$$Th^n_{\bar{k}}(K, \Phi) = \{Th^n_{\bar{k}}(M, \Phi) : M \in K\}.$$

[7] We assume, of course, that the $|M_i|$'s are pairwise disjoint.

(2) The monadic theory of K is the set of monadic sentences true in every model in K.
(3) For any $\bar\sigma, K_1, K_2$, let $C\ell^{\bar\sigma}(K_1, K_2)$ be the minimal class K such that
 (A) $K_1 \subseteq K$,
 (B) if $j < \ell(\bar\sigma), M_i \in K, N \in K_2$ then $\sum^{\bar\sigma(i)}_{i\in|N|} M_i \in K$.

Conclusion 2.10. Suppose $\bar\sigma, n, \bar k, m$ are given. $L_1(\sigma_i) = L_3(\sigma_i) = L, L_2(\sigma_i) = L_2; L, L_2$ are finite and each $\Psi(\sigma_i), \Psi(\sigma_i)$ is a set of atomic formulas. There is an $\bar r$ such that for every K_1, K_2, from $Th^n_{\bar r}(K_2^{\bar r(n+1)}), Th^n_{\bar k}(K_1^m)$ we can effectively compute $Th^n_{\bar k}(K^m)$ where $K = C\ell^{\bar\sigma}(K_1, K_2)$ (remember $K_1^m = \{(M, \bar P) : M \in K_1, \bar P \in \underline{P}(M)^m)$ (K_1 should be a class of L-models, K_2 a class of L_2-models).

Proof: For every $j < \ell(\bar\sigma)$ let $\bar r^j$ relate to $\bar\sigma(j), n, \bar k, m$ just as $\bar r$ relates to σ, n, k, m in Theorem 2.8. Now choose an $\bar r$ such that for every $\ell \leqq n, \bar r(\ell) \geqq r^j(\ell)$.

Let T be the set of formally possible $Th^n_{\bar k}(M, \bar P)$, for M and L-model, $\ell(\bar P) = m$, and we can define $r(n+1) = |T|$. Let $T = \{t(0), \ldots, t(p-1)\}$ (so $p = |T| = r(n+1)$).

Clearly, by the definition of $\bar r^j$, and by (a trivial case of) 2.3(B), if $M = \sum^{\bar\sigma(j)}_{i\in N} M_i, t_i = Th^n_{\bar k}(M_i, \bar P_i), Q_\ell = \{i \in N : t_i = t(\ell)\}, \ell(\bar P_i) = m$, then from $t = Th^n_{\bar r}(N, \ldots, Q_1, \ldots)_{\ell<p}$ we can effectively compute $Th^n_{\bar k}(M, \bigcup_i, \bar P_i)$, and denote it by $g(t)$.

Now define by induction on $\ell, T_\ell \subseteq T$.

Let $T_0 = Th^n_{\bar k}(K_\ell^m)$, and if T_q is defined let T_{q+1} be the union of T_q with the set of $t \in T$ satisfying the following condition:

(*) There is a $t^* \in Th^n_{\bar r}(K_2^{r(n+1)})$ such that $t = g(t^*)$, and if t^* implies that Q_ℓ is not empty, then $t(\ell) \in T_q$.

Remark: Clearly if $t^* = Th^n_{\bar r}(N, \ldots, Q_\ell, \ldots)$ then from t^* we can compute $Th^0_{\bar r}(N, \ldots, Q_\ell, \ldots)$ and hence know whether $Q_\ell \neq \varnothing$.

Clearly $T_0 \subseteq T_1 \subseteq T_2, \ldots \subseteq T$ so, as $|T| = p$, for some $q \leqq p, T_q = T_{q+1}$.

Now let

$$K_* = \{M \in K : \text{ for every } \bar P \in (\underline{P}(|M|)^m Th^n_{\bar k}(M, \bar P) \in T_q\}.$$

Clearly $Th^n_{\bar k}(k_*^m) \subseteq T_q$, and we can effectively find T_q. Now if $N \in K_2, M_i \in K_*$ for $i \in N$, and $M = \sum^{\sigma(j)}_{i\in N} M_i$, then for any $\underline{P} \in \bar P(|M|)^m, Th^n_{\bar k}(M, P) \in T_{q+1} = T_q$ by the definition of T_{q+1}, and $M \in K$ by the definition of K, hence $M \in K_*$. As clearly $K_1 \subseteq K_* \subseteq K$, by the definition of $K = C\ell^{\bar\sigma}(K_1, K_2)$ necessarily $K_* = K$. So it suffices to prove that $Th^n_{\bar k}(K_*^m) \supseteqq T_\ell$. (Take $\ell = q$.) This is done by induction on ℓ.

Lemma 2.11. *If M is a finite model, then for any $\Phi, n, \bar k$ we can effectively compute $Th^n_{\bar k}(M, \Phi)$ from M.*

Remark 2.12. Naturally we can ask whether we can add to (or replace the) monadic quantifiers (by) other quantifiers, without essentially changing the conclusions of this section. It is easily seen that, e.g., the following quantifiers suitable:

(1) $(\exists^f X)$ –there is a finite set X
(2) $(\exists^\lambda X)$ –there is a set $X, |X| < \lambda$ (λ a regular cardinal). when dealing with ordered sums of linear order, also
(3) $(\exists^{wo} X)$ –there is a well-ordered set X
(4) $(\exists_\lambda X)$ –there is a set X, with no increasing not decreasing sequence in it of length λ (λ a regular cardinal).

If we add some of those quantifiers, we should, in the definition of $Th^0_n((M, \bar{P}), \Phi)$ state which Boolean combinations of the P_ℓ's are in the range of which quantifiers. If we e.g., replace the monadic quantifier by $(\exists^\lambda X)$, we should restrict the P's to sets of cardinality $< \lambda$.

Another possible generalization is to generalized products. Let $M = \prod^\sigma_{i \in N} M_i$ (where $L(M_i) = L_1(\sigma), L(N) = L_2(\sigma), L(M) = L_3(\sigma)$) means: $|M| = \prod_{i \in N} |M_i|$, and if $f_1, \dots, f_n \in M, M \models R[f_1, \dots, f_n]$ if and only if $N \models \psi_R[\dots, P_\ell, \dots]$ where

$$P_\ell = \{i \in N : M_i \models \varphi^R_\ell[f_1(i), \dots, f_n(i)]\}$$

(and φ_ℓ is a first order sentence from $L_1(\sigma), \psi_R$ a monadic sentence from $L_3(\sigma)$). Then, of course, we use $Th^n_{\bar{k}}(N, \underline{P}), th^n_{\bar{k}}(M, \bar{a})$. All our theorems generalize easily, but still no application was found.

If not specified otherwise, we restrict ourselves to the class K_{ord} of models of the theory of order (sometimes with one-place relations which will be denoted, e.g., $(M, \bar{P})$). $\sigma = \sigma_{\text{ord}}$ is the ordered sum of ordered sets and is omitted. Therefore $\Psi(\sigma)$ and $\Phi(\sigma)$ are the set of atomic formulas. For the sum of two orders we write $M_1 + M_2$. The ordinals, the reals R, and the rationals Q have their natural orders. If $M = \sum_{i \in |N|} M_i$ we write $Th^n_{\bar{k}}(M, \bar{P}) = \sum_{i \in |N|} Th^n_{\bar{k}}(M_i, \bar{P}_i)$ where $\bar{P} = \bigcup_i \bar{P}_i$. Let $T(n, m, \bar{k})$ be the set of formally possible $Th^n_{\bar{k}}(M, \bar{P}), M$ an order, $\ell(\bar{P}) = m$.

Corollary 2.13. *For any $n, m, \bar{k}$ there is $\bar{r} = \bar{r}(n, m, \bar{k})$ such that if $P_t = \{i \in N : t_i = t\}$ for $t \in T(n, m, \bar{k})$ then $\sum_{i \in N} t_i$ can be effectively computed from $Th^n_{\bar{r}}(N, \dots, P_t, \dots)$.*

3. Simple application for decidability

Using Section 2 we shall prove here some theorems, most of them known. We prove the decidability of the theories of the finite orders, the countable ordinals [BS73] and show that from the monadic theory of λ we can compute effectively the monadic theory of $K = \{\alpha : \alpha < \lambda^+\}$ (this was shown for $\lambda = \omega, \lambda = \omega_1$ in [BS73] We do not try to prove the results on definability and elimination of quantifiers. For finite orders this can be done and the method becomes similar to that of automaton theory. For $\omega, \{\alpha : \alpha < \omega_1\}, \omega_1$ this can be done by using the previous cases (e.g., for ω using the result on the finite orders). We can prove the decidability of the weak monadic theory (with $\exists^f$ only) of the n-successors theory by the method of this section (Doner [Don65] proved it). It would be very interesting if we could have proved in this way that the monadic theory of the 2-successor theory is decidable (Rabin [Rab69] proved it).

In order to use Section 1 we should note

Lemma 3.1. *For any $m, \bar{k}, (N, \bar{P})$, the coloring $f^n_{\bar{k}}$ on N is additive where*

$$f^n_{\bar{k}}(a,b) = Th^n_{\bar{k}}((N,\bar{P})\restriction[a,b)),$$

where $(N,\bar{P})\restriction[a,b)$ is a submodel of $(N,\bar{P})$ with the universe $[a,b) = \{x \in N : a \leqq x < b\}$.

Proof: By lemma 2.7.

Let us list some immediate claims.

Lemma 3.2. (A) *If for any $n, \bar{k}$ we can compute effectively $Th^n_{\bar{k}}(K)$, then the monadic theory of K is decidable; and vice-versa.*

(B) *If the monadic theory of K is decidable then so is the monadic theory of K' where K' is the class of:*
  - (i) *submodels of K,*
  - (ii) *initial segments of orders from K,*
  - (iii) *orders which we get by adding (deleting) first (last) elements from orders of K,*
  - (iv) *converses of orders from K,*
  - (v) *$(M,\bar{P}), M \in K, \bar{P} \in \underline{P}(M)^m$.*

Proof: Immediate.

Theorem 3.3. *The monadic theory of the class K_{fin} of finite orders is decidable.*

Proof: Let K_n be the class of orders of cardinality n; up to isomorphism K_n has only one element, n. Hence by Lemma 2.11 we can compute $Th^n_{\bar{k}}(K_i)$. Hence by Conclusion 2.10, for every $n, \bar{k}$ we can compute $Th^n_{\bar{k}}(K)$ where $K = C\ell(K_1, K_2)$. But clearly K is the class of finite orders. So by 3.2(A) we finish.

Theorem 3.4. *The monadic theory of ω is decidable.*

Proof: We shall compute $\{Th^n_{\bar{k}}(\omega,\bar{P}) : \bar{P} \in \underline{P}(\omega)^m\}$ by induction on n, for every $\bar{k}, m$ simultaneously.

For $n = 0$ is it easy.

Suppose we have done it for $n-1$ and we shall do it for $n, m, \bar{k}$. By the induction hypothesis we can compute $Th^n_{\bar{\ell}}(\omega)$ for every $\bar{\ell}$, in particular for

$\bar{r} = \bar{r}(n, m, \bar{k})$ (see 2.13). Now for any $M = (\omega, P_1, \ldots, P_m)$, by 1.1 we can find an $f^n_{\bar{k}}$-homogeneous set $\{a_i : i < \omega\}(a_i < a_{i+1})$. So letting

$$t = T^n_{\bar{k}}((\omega, \bar{P}){\restriction}[0, a_0)),\ \ s = Th^n_{\bar{k}}((\omega, \bar{P}){\restriction}[a_i, a_j)) \text{ for } i < j;$$

we have

$Th^n_{\bar{k}}(\omega, \bar{P}) = Th^n_{\bar{k}}((\omega, \bar{P}){\restriction}[0, a_0)) + \sum_{i<\omega} Th^m_{\bar{k}}((\omega, \bar{P}){\restriction}[a_i, a_{i+1})) = t + \sum_{i<\omega} s.$

As $Th^n_r(\omega)$ is known, by 2.13, we can compute $Th^n_{\bar{k}}(M, \bar{P})$ from s, t. Now for any $t, s \in Th^n_{\bar{k}}(K^m_{\text{fin}}), s \neq Th^n_{\bar{k}}(0, \bar{P}), \bar{P} \in \underline{P}(\varnothing)^m$, there is an $(\omega, \bar{P})$ such that $Th^n_{\bar{k}}(\omega, \bar{P}) = t + \sum_{i<\omega} s$.

As we know $Th^n_{\bar{k}}(K^m_{\text{fin}})$ by 3.3, and can easily find whether $s \in Th^n_{\bar{k}}(K^m_{\text{fin}}) - -Th^n_{\bar{k}}(\{0\})$, we finish.

Theorem 3.5. (A) *From the monadic theory of λ (λ a cardinal) we can compute effectively the monadic theory of $K = \{\alpha : \alpha < \lambda^+\}$.*

(B) *Moreover every monadic sentence which has model $\alpha < \lambda^+$, has a model $\beta < \lambda^\omega$.*

(C) (i) *For every $\alpha < \lambda^+$ there is a $\beta < \lambda^{\omega+1} + \lambda^\omega$ which has the same monadic theory*

(ii) *if $\mu \leqq \lambda$ and for every regular $\chi \leqq \lambda$ there is a $\chi' \leqq \mu$ such that χ, χ' have the same monadic theory, then we can choose $\beta < \lambda^\omega \mu + \lambda^\omega$.*[8]

(iii) *If we could always find $\chi < \mu$ then $\beta < \lambda^\omega \mu$, and if $\lambda = \omega, \beta < \lambda^\omega + \lambda^\omega$.*[9]

(iv) *Also, for every $\alpha < \lambda^+$, there are $n < \omega, \lambda_1, \ldots, \lambda_n \leqq \lambda$, such that the monadic theory of α is recursive in the monadic theories of $\lambda_1, \ldots, \lambda_n$, and λ_i is a regular cardinal.*

(D) *In general, the bounds in (B),(C) cannot be improved.*

Remark: Büchi [BS73] already proved (B),(C) for $\lambda = \omega$ and (B) for $\lambda + \omega_1$.

Proof:

(A) Define $K_1 = K_2 = \{\alpha : \alpha \leqq \lambda\}$; by 3.2(A)(i) and 3.2(B) we can compute $Th^n_{\bar{k}}(K_i)$ for every $n, \bar{k}$ and $i = 1, 2$ (from the monadic theory of λ, of course). Hence by 2.10 we can compute $Th^n_{\bar{k}}(K')$ for every $n, \bar{k}$ where $K' = C\ell(K_1, K_2)$. Clearly every member of K' is well-ordered and has cardinality $\leqq \lambda$. So up to isomorphism $K' \subseteqq K$. We should prove now only that equality holds. If not, let α by the first ordinal not in K', and $\alpha < \lambda^+$. If α is a successor ordinal, $\alpha - 1 \in K'$; $1, 2 \in K'$ hence $\alpha = (\alpha - 1) + 1 \in K'$, a contradiction. If α is a limit ordinal, its cofinality is $\leqq \lambda$. Let $\alpha = \sum_{i<i_0} \alpha_i, i_0 \leqq \lambda, \alpha_i < \alpha$; then $i_0, \alpha_i \in K'$ so $\alpha \in K'$, a contradiction.

[8] In fact, $\beta < M^{\omega+1} + M^\omega$.

[9] In the first case $\beta < M$.

(B) Let us first show that

(*) For every $n, \bar{k}$ there is $q = q(n, \bar{k}) < \omega$ such that if $\alpha, \beta < \lambda^+, \mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$, and α, β are divisible by λ^q, then $Th^n_{\bar{k}}(\alpha) = Th^n_{\bar{k}}(\beta)$.

For $n = 0$ it is immediate, and we prove it for n. By the pigeonhole principle there are $1 < \ell < p \leqq 2|T(n, 0, \bar{k})| + 1$ such that $Th^n_{\bar{k}}(\lambda^\ell) = Th^n_{\bar{k}}(\lambda^p)$. Clearly,

$$\lambda^{\ell+2} = \sum_{i<\lambda} (\lambda^{\ell+1} + \lambda^\ell).$$

Hence

$$\begin{aligned} Th^n_{\bar{k}}(\lambda^{\ell+2}) &= Th^n_{\bar{k}}[\textstyle\sum_{i<\lambda}(\lambda^{\ell+1} + \lambda^\ell)] = \sum_{i<\lambda} Th^n_{\bar{k}}(\lambda^{\ell+1} + \lambda^\ell) \\ &\textstyle\sum_{i<\lambda}[Th^n_{\bar{k}}(\lambda^{\ell+1}) + Th^n_{\bar{k}}(\lambda^\ell)] = \sum_{i<\lambda}[Th^n_{\bar{k}}(\lambda^p) \\ &\textstyle\sum_{i<\lambda} Th^n_{\bar{k}}(\lambda^\ell) = Th^n_{\bar{k}}(\sum_{i<\lambda} \lambda^\ell) = Th^n_{\bar{k}}(\lambda^{\ell+1}). \end{aligned}$$

Hence we prove by induction on $m, \ell < m < \omega$ that $Th^n_{\bar{k}}(\lambda^m) = Th^n_{\bar{k}}(\lambda^{\ell+1})$; choose $q = q(n, \bar{k}) = \ell + 1$. Let $\alpha, \beta < \lambda^+$ be divisible by λ^q and have the same cofinality, and we shall prove $Th^n_{\bar{k}}(\alpha) = Th^n_{\bar{k}}(\beta)$. Clearly it suffices to prove $Th^n_{\bar{k}}(\alpha) = Th^N_{\bar{k}}(\lambda^q\mu)$ where $\mu = \mathrm{cf}(\alpha)$. Let us prove it by induction on α, and let $\alpha = \lambda^q\gamma$. If $\gamma = \gamma_1 + 1$, then for $\gamma_1 = 0$ i is trivial, and for $\gamma_1 > 0$

$$\begin{aligned} Th^n_{\bar{k}}(\alpha) &= Th^n_{\bar{k}}(\lambda^q\gamma_1 + \lambda^q) = Th^n_{\bar{k}}(\lambda^q\gamma_1) + Th^n_{\bar{k}}(\lambda^q) \\ &= Th^n_{\bar{k}}[\lambda^q \circ \mathrm{cf}(\lambda^q\gamma_1)] + Th^n_{\bar{k}}(\lambda^{q+2}) \\ &= Th^n_{\bar{k}}[\lambda^q \circ \mathrm{cf}(\lambda^q\gamma_1) + \lambda^{q+2}] = Th^n_{\bar{k}}(\lambda^{q+2}) = Th^n_{\bar{k}}(\lambda^q \circ \lambda) \\ &= Th^n_{\bar{k}}[\lambda^q \circ \mathrm{cf}(\alpha)]. \end{aligned}$$

If γ is a limit ordinal $\gamma = \sum_{i<\mathrm{cf}(\gamma)} \gamma_i, \gamma_i < \gamma$ a successor,

$$\begin{aligned} Th^n_{\bar{k}}(\alpha) &= Th^n_{\bar{k}}[\lambda^q(\textstyle\sum_{i<\mathrm{cf}(\gamma)} \gamma_i)] = Th^n_{\bar{k}}(\sum_{i<\mathrm{cf}(\gamma)} \lambda^q\gamma_i) \\ &= \textstyle\sum_{i<\mathrm{cf}(\gamma)} Th^n_{\bar{k}}(\lambda^q\gamma_i) \\ &= \textstyle\sum_{i<\mathrm{cf}(\gamma)} Th^n_{\bar{k}}[\lambda^q \circ \mathrm{cf}(\lambda^q\gamma_i)] \\ &\textstyle\sum_{i<\mathrm{cf}(\gamma)} Th^n_{\bar{k}}(\lambda^{q+1}) = \sum_{i<\mathrm{cf}(\gamma)} Th^n_{\bar{k}}(\lambda^q) \\ &= Th^n_{\bar{k}}[\lambda^q \circ \mathrm{cf}(\gamma)]. \end{aligned}$$

So we have proved (*). Let us prove (B). Let $\alpha < \lambda^+$ be a model of a sentence ψ. Choose by 2.2(A),(OR 3.2?) $n, \bar{k}$ such that from $Th^n_{\bar{k}}(\beta)$ we know whether $\beta \models \psi$, and let $q = q(n, \bar{k})$, and let $\alpha = \lambda^q\beta + \gamma, \gamma < \lambda^q$. Then

$$Th^n_{\bar{k}}(\alpha) = Th^n_{\bar{k}}[\lambda^q \circ \mathrm{cf}(\lambda^q\beta) + \gamma], \text{ and } \lambda^q \circ \mathrm{cf}(\lambda^q\beta) + \gamma < \lambda^{q+2}.$$

(C) Divide α by λ^ω so $\alpha = \lambda^\omega\alpha_1 + \alpha_2, \alpha_2 < \lambda^\omega$. Let α'_1 be 1 if α_1 is a successor, and $\mathrm{cf}(\alpha_1)$ otherwise. Then $\lambda^\omega\alpha_1, \lambda^\omega\alpha'_1$ are divisible by $\lambda^{q(n\bar{k})}$ for every $n, \bar{k}$ and have equal cofinality. So by the proof of (B), for every $n, \bar{k}, Th^n_{\bar{k}}(\lambda^\omega\alpha_1) = Th^n_{\bar{k}}(\lambda^\omega\alpha'_1)$. Hence $\lambda^\omega\alpha_1 + \alpha_2, \lambda^\omega\alpha'_1 + \alpha_2$

has the same monadic theory, and $\lambda^\omega\alpha_1'+\alpha_2 < \lambda^\omega\lambda+\lambda^\omega = \lambda^{\omega+1}+\lambda^\omega$. This proves (C)(i).

If $\chi' \leqq \mu$ has the same monadic theory as α_1' then $\lambda^\omega\alpha_1+\alpha_2, \lambda^\omega\alpha_1'+\alpha_2$ and $\lambda^\omega\chi'+\alpha_2$ (which is $< \lambda^\omega\mu+\lambda^\omega$) have the same monadic theories. If $\chi' < \mu$ clearly $\lambda^\omega\chi' + \alpha_2 < \lambda^\omega\mu$.

If $\lambda = \omega$ then $\mathrm{cf}(\lambda)^\omega\alpha_1) = \omega$ in any case, hence $\alpha = \omega^\omega\alpha_1 + \alpha_2$, and $\omega^\omega+\alpha_1 < \omega^\omega+\omega^\omega$ has the same monadic theory. Every $\alpha < \lambda^+$ we can uniquely represent as

$$\alpha = \lambda^\omega\alpha' + \lambda^n\alpha_n + \ldots + \lambda^1\alpha_1 + \alpha_0; \alpha_i < \lambda.$$

The monadic theory of α is recursive in the monadic theories of $\lambda, \mathrm{cf}(\lambda)^\omega\alpha'), \alpha_n, \ldots, \alpha_0$. So we can prove inductively (C)(iv).

(D) Suppose $\lambda > \omega, \lambda$ is regular, and there is a sentence ψ such that $\alpha \models \psi$ if $\alpha = \lambda$. Then there are sentences ψ_n such that $\alpha \models \psi_n$ if and only if $\alpha = \lambda^n$, sentences φ_n such that $\alpha \models \varphi_n$ if and only if α is divisible by λ^n, and sentence φ such that $\alpha \models \varphi$ if $\mathrm{cf}(\alpha) = \lambda$. Then $\lambda^{\omega+1}$ is a model of $\{\varphi, \varphi_n : n < \omega\}$. If α is also a model of $\{\varphi, \varphi_n : n < \omega\}$ then λ^n divides α for every n, hence λ^ω divides α, so $\alpha = \lambda^\omega\beta$. If β is a successor, $\mathrm{cf}(\alpha) = \omega$ but $\alpha \models \varphi$ so β is a limit hence $\mathrm{cf}(\alpha) = \mathrm{cf}(\beta)$, so $\mathrm{cf}(\beta) = \lambda$, so $\beta \geqq \lambda$ hence $\alpha \geqq \lambda^\omega\circ\lambda = \lambda^{\omega+1}$. Similarly $\lambda^{\omega+1} + \lambda^n$ is the smallest model of its monadic theory.

Lemma 3.6. (A) *In 3.5(A) it suffices to know the monadic theory of $\{\mu : \mu$ a regular cardinal $\leqq \lambda\}$. So if λ is singular it suffices to know the monadic theory of $\{\alpha : \alpha < \lambda\}$.*

(B) *For every sentence ψ,*
   (1) *there is a sentence φ (all in the monadic theory of order) such that $\alpha \models \varphi$ if and only if α is a limit and $\mathrm{cf}(\alpha) \models \psi$,*
   (2) *there is a sentence characterizing the first ordinal which satisfies ψ and*
   (3) *for every $n < \omega$ there is φ_n such that $\alpha \models \varphi_n$ if and only if φ is the n^{th} regular cardinal satisfying ψ.*

(C) *There are monadic sentences φ_n such that $\alpha \models \varphi_n$ if and only if $\alpha = \omega_n$. If $V = L$ there are monadic sentences φ_n^1 such that $\alpha \models \varphi_n^1$ if and only if α is the n^{th} weakly compact cardinal.*

Proof:

(A) Immediate by 3.5(C)(iv).

(B) (a) Let φ say that there is no last element, and for any unbounded P there is an unbounded $Q \subseteqq P$ which satisfies ψ (if $\mathrm{cf}(\alpha) \models \neg\psi$ we can choose Q as a set of order-type $\mathrm{cf}(\alpha)$; so $\alpha \models \varphi$. If $\mathrm{cf}(\alpha) \models \neg\psi$, let P be a subset of α of order-type $\mathrm{cf}(\alpha)$; hence any unbounded $Q \subseteqq P$ has order-type $\mathrm{cf}(\alpha)$, so $\alpha \models \neg\varphi$).
   (b) Immediate.

(c) We use (1) and (2) to define φ_n inductively. Let φ_0 say that α is the first ordinal whose cofinality satisfies ψ. Let φ_{n+1} say that α is the first ordinal whose cofinality satisfies $\psi \wedge \neg\varphi_0 \wedge \ldots \wedge \neg\varphi_n$.

(C) For φ_n use (B)(3) for ψ sating α is an infinite ordinal. For φ^1_n use (B)(3) and Theorem 0.1 (of Jensen).

4. The monadic theory of well-orderings

If $a \in (M, \bar{P})$ let

$$th(a, \bar{P}) = \{x \in X_i : a \in P_i\} \cup \{x \notin X_i : a \notin P_i\}$$

(so it is set of formulas).

Let D_α denote the filter of (generated by) the closed unbounded subset of $\alpha, \mathrm{cf}(\alpha) > \omega$.

Lemma 4.1. *If the cofinality of $\alpha > \omega$, then for every $\bar{P} \in \underline{P}(\alpha)^m$ there is a closed unbounded subset J of α such that: for each $\beta < \alpha$, all the models*

$$\{(\alpha, \bar{P}){\restriction}[\beta, \gamma) : \gamma \in J, \mathrm{cf}(\gamma) = \omega, \gamma > \beta\}$$

have the same monadic theory.

Remark: Büchi [BS73, 6.1,p.110] proved Lemma 4.1 for $\alpha = \omega_1$, by a different method.

Proof: For every $n, \bar{k}$ there is, by 1.1, 3.1 a homogeneous unbounded $I^n_{\bar{k}} \subseteqq \alpha$, by the coloring $f^n_{\bar{k}}$ of $(\alpha, \bar{P})$, so there is $t^n_{\bar{k}}$ such that for every $\beta < \gamma \in I^n_{\bar{k}}$, $Th^n_{\bar{k}}((\alpha, \bar{P}){\restriction}[\beta, \gamma)) = t^n_{\bar{k}}$. Let $J^n_{\bar{k}}$ be the set of accumulation points of $I^n_{\bar{k}}$, and $J = \bigcap_{n,\bar{k}} J^n_{\bar{k}}$. Clearly J is a closed and unbounded subset of α.

Let $\beta < \alpha$, and $\beta^n_{\bar{k}}$ be the first ordinal $> \beta$ in $I^n_{\bar{k}}$. Then for any $\gamma \in J, \gamma > \beta, \mathrm{cf}(\gamma) = \omega$, and for every $n, \bar{k}$ we can find $\gamma_\ell \in I^n_{\bar{k}}, \gamma_\ell < \gamma_{\ell+1}, \lim_{\ell\to\omega}\gamma_\ell = \gamma$ and $\gamma_0 = \beta^n_{\bar{k}}$. Therefore

$$\begin{aligned} Th^n_{\bar{k}}((\alpha, \bar{P}){\restriction}[\beta, \gamma)) &= Th^n_{\bar{k}}((\alpha, \bar{P}){\restriction}[\beta, \beta^n_{\bar{k}})) + \textstyle\sum_{\ell<\omega} Th^n_{\bar{k}}((\alpha, \bar{P}){\restriction}[\gamma_\ell, \gamma_{\ell+1})) \\ &= Th^n_{\bar{k}}((\alpha, \bar{P}){\restriction}[\beta, \beta^n_{\bar{k}})) + \textstyle\sum_{\ell<\omega} t^n_{\bar{k}}. \end{aligned}$$

So, $Th^n_{\bar{k}}((\alpha, \bar{P}){\restriction}[\beta, \gamma)$ does not depend on the particular γ.

Definition 4.2. $ATh^n_{\bar{k}}(\beta, (\alpha, \bar{P}))$ for $\beta < \alpha, \alpha$ a limit ordinal of cofinality $> \omega$ is $Th^n_{\bar{k}}((\alpha, \bar{P}){\restriction}[\beta, \gamma))$ for every $\gamma \in J, \gamma > \beta, \mathrm{cf}(\gamma) = \omega$; where J is from Lemma 4.1.

Remark: As D_α is a filter, this definition does not depend on the choice of J.

Definition 4.3. We define $WTh^n_{\bar{k}}(\alpha, \bar{P})$:

(1) if α is a successor or has cofinality ω, it is $\varnothing$,

(2) otherwise we define it by induction on n:
for $n = 0$: $WTh^n_{\bar{k}}(\alpha, \bar{P}) = \{t : \{\beta < \alpha : th(\beta, \bar{P}) = t\}$ is a stationary subset of $\alpha\}$,

for $n+1$: let $WTh^{n+1}_{\bar{k}}(\alpha, \bar{P}) = \{\langle S_1(\bar{Q}), S_2(\bar{Q})\rangle : \bar{Q} \in \underline{P}(\alpha)^{\bar{k}(n+1)}\}$

where

$S_1(\bar{Q}) = WTh^n_{\bar{k}}(\alpha, \bar{P}, \bar{Q})$,
$S_2(\bar{Q}) = \{\langle t, s\rangle : \{\beta < \alpha : WTh^n_{\bar{k}}((\alpha, \bar{P}, \bar{Q}){\restriction}\beta) = t, th(\beta, \bar{P}{}^\frown\bar{Q}) = s\}$

is a stationary subset of $\alpha\}$.

Remark: Clearly, if we replace $(\alpha, \bar{P})$ by a submodel whose universe is a closed unbounded subset of $\alpha, WTh^n_{\bar{k}}(\alpha, \bar{P})$ will not change. Of course $WTh^n_{\bar{k}}(M)$ is well defined for every well-ordered model.

Definition 4.4. Let $\mathrm{cf}(\alpha) > \omega, M = \alpha, \bar{P})$ and we define the model $g^n_{\bar{k}}(M) = (\alpha, g^n_{\bar{k}}(\bar{P}))$.

Let
$$(g^n_{\bar{k}}(\bar{P}))_s = \{\beta < \alpha : s = ATh^n_{\bar{k}}(\beta, M)\}$$
and (when $m = \ell(\bar{P})$)
$$g^n_{\bar{k}}(\bar{P}) = \langle \ldots, (g^n_{\bar{k}}(\bar{P}))_s, \ldots\rangle_{s \in T(n,m,\bar{k})}.$$

Remark:

(1) In $g^n_{\bar{k}}(\bar{P})$ we unjustly omit α, but there will be no confusion.
(2) Remember $T(n, m, \bar{k})$ is the set of formally possible $Th^n_{\bar{k}}(M, \bar{P}), \ell(\bar{P}) = m$.

Lemma 4.5. (A) *$g^n_{\bar{k}}(\bar{P})$ is a partition of α.*
(B) *$g^n_{\bar{k}}(\bar{P}{}^\frown\bar{Q})$ is a refinement of $g^n_{\bar{k}}(\bar{P})$ and we can effectively correlate the parts.*
(C) *$g^{n+1}_{\bar{k}}(\bar{P})$ is a refinement of $g^n_{\bar{k}}(\bar{P})$ and we can effectively correlate the parts.*
(D) *The parallels of Lemma 2.5 for Th, pTh, hold for $WTh, pWTh$.*

Proof: Immediate.

Theorem 4.6. *For every $n, m, \bar{k}$ we can effectively find $\bar{r} = \bar{r}_1(n, m, \bar{k})$ such that: If $\mathrm{cf}(\alpha^i) > \omega, M_i = (\alpha^i, \bar{P}^i), \ell(\bar{P}^i) = m$ for $i = 1, 2$ and $ATh^n_{\bar{k}}(0, M_1) = ATh^n_{\bar{k}}(0, M_2)$ and $WTh^n_{\bar{r}}(g^n_{\bar{k}}(M_1)) = WTh^n_{\bar{r}}(g^n_{\bar{k}}(M_2))$ then $Th^n_{\bar{k}}(M_1) = Th^n_{\bar{k}}(M_2)$.*

Proof: We prove by induction on n.

For $n = 0$, it is easy to check that $Th^n_{\bar{k}}(M_i) = ATh^n_{\bar{k}}(0, M_i)$ hence the theorem is trivial.

Suppose we have proved the theorem for n, and we shall prove it for $n+1$. Suppose $\bar{Q}^1 \in \underline{P}(\alpha^1)^{\bar{k}(n+1)}$, and we shall find $\bar{Q}^2 \in \underline{P}(\alpha^2)^{\bar{k}(n+1)}$ such that $Th^n_{\bar{k}}(\alpha^1, \bar{P}^1, \bar{Q}^1) = Th^n_{\bar{k}}(\alpha^2, \bar{P}^2, \bar{Q}^2)$; be the symmetry in the hypothesis this is sufficient. Let $g^n_{\bar{k}}(\bar{P}^1 \frown \bar{Q}^1) = \bar{Q}^{*1}, g^{n+1}_{\bar{k}}(\bar{P}^1) = \bar{P}^{*1}, g^{n+1}_{\bar{k}}(\bar{P}^2) = \bar{P}^{*2}$. Define $\bar{r}(n+1) = \ell(g^n_{\bar{k}}(\bar{P}^1 \frown \bar{Q}^{-1})) = \ell(\bar{Q}^{*1})$ and $\bar{r} \restriction (n+1) = r_1(n, m + \ell(\bar{P}^1), \bar{k})$.

By the assumptions and Definition 4.3, there is $\bar{Q}^{*2} \in \underline{P}(\alpha^2)^{\bar{k}(n+1)}$ such that (for our $n, \bar{r}$ and $\alpha^2, \bar{P}^{*2}$; $\alpha^1, \bar{P}^{*1}$), $S_\ell(\bar{Q}^{*1}) = S_\ell(\bar{Q}^{*2})$ for $\ell = 1, 2$. (The notation is inaccurate, but should be clear.) So, for $\ell = 1$, we get $WTh^n_{\bar{r}}(\alpha^1, \bar{P}^{*1}, \bar{Q}^{*1}) = WTh^n_{\bar{r}}(\alpha^2, \bar{P}^{*2}, \bar{Q}^{*2})$, and without loss of generality $0 \in Q^{*1}_s \leftrightarrow 0 \in Q^{*2}_s$. (From now on we can replace $\bar{r}$ by $\bar{r} \restriction (n+1)$.) So by Lemma 4.3, for $\ell = 1, 2$, $\bar{Q}^{*\ell}$ is a partition of α^ℓ refining $\bar{P}^{*\ell}$, hence for every $\beta < \alpha^\ell$ there is a unique $s_\ell(\beta)$ such that $\beta \in Q^{*\ell}_{s_\ell(\beta)}$.

Now, for $\ell = 1, 2$, choose a closed unbounded subset J_ℓ of α^ℓ such that:

(0) every member of J_ℓ which is not an accumulation point of J_ℓ, has cofinality ω,

(1) for any s, if $Q^{*\ell}_s$ is not a stationary subset of α^ℓ then $Q^*_s \cap J_\ell = \varnothing$,

(2) if $\beta < \gamma < \alpha^\ell$; $\mathrm{cf}(\gamma) = \omega$ then
$Th^{n+1}_{\bar{k}}((\alpha^\ell, \bar{P}^\ell) \restriction [\beta, \gamma)) = ATh^{n+1}_{\bar{k}}(\beta, (\alpha^\ell, \bar{P}^\ell))$ (use Lemma 4.1),

(3) for every $\gamma \in J_\ell, \mathrm{cf}(\gamma) = \omega$,
$$Th^{n+1}_{\bar{k}}((\alpha^\ell, \bar{P}^\ell) \restriction [0, \gamma)), ATh^{n+1}_{\bar{k}}(0, (\alpha^\ell, \bar{P}^\ell)),$$

(4) if $Q^{*\ell}_s \cap J_\ell \neq \varnothing, \beta \in J_\ell$
then there are $\gamma \in J_\ell, \gamma > \beta, s_\ell(\gamma) = s$ such that $\{\xi \in J_\ell : \beta \leqq \xi \leqq \gamma\}$ is finite,

(5) for any s, t, if $\{\beta < \alpha^\ell : t = WTh^n_{\bar{r}}((\alpha^\ell, \bar{Q}^{*\ell}) \restriction \beta), s = Th(\beta, \bar{Q}^{*\ell})\}$ is not a stationary subset of J_ℓ, then it is disjoint to J_ℓ.

Remark: Note that (5) just strengthens (1).

Now we define $\bar{Q}^2$ by parts. That is, for every $\beta < \gamma \in J_2 \cup \{0\}, \gamma$ is the successor of β in J_2, we define $\bar{Q}^2 \restriction [\beta, \gamma)$ such that
$$s_2(\beta) = Th^n_{\bar{k}}((\alpha^2, \bar{P}^2 \frown \bar{Q}^2) \restriction [\beta, \gamma)).$$
This is possible as by definition of $s_2(\beta), \beta \in Q^{*2}_{s_\ell(\beta)}$, hence
$$s_i(\beta) \in ATh^{n+1}_{\bar{k}}(\beta, (\alpha^2, \bar{P}^2)).$$
We now prove

(*) if $\beta < \gamma \in J_2 \cup \{0\}, \mathrm{cf}(\gamma) = \omega$, then
$$s_2(\beta) = Th^n_{\bar{k}}((\alpha^2, \bar{P}^2, \bar{Q}^2) \restriction [\beta, \gamma)).$$

We prove it by induction on γ for all β.

(i) By (0) the first $\gamma > \beta_1, \gamma \in J_2$ has cofinality ω, and by the definition of $\bar{Q}^2(*)$ is satisfied.

(ii) Let $\beta < \xi < \gamma, \xi \in J_2$, for no $\zeta \in J_2, \xi < \zeta < \gamma$, has cofinality ω. Then by the induction hypothesis $Th^n_{\bar{k}}((\alpha^2, \bar{P}_2, \bar{Q}^2) \restriction [\beta, \xi)) = s_2(\beta)$ and
$$Th^n_{\bar{k}}((\alpha^2, \bar{P}^2, \bar{Q}^2) \restriction [\xi, \gamma)) = s(\xi).$$

We should now show that $s_2(\beta) + s_2(\xi) = s_2(\beta)$. So it suffice to find $\beta' < \xi' < \gamma' \in J_1, s_1(\beta') = s_2(\beta), \mathrm{cf}(\xi') = \omega = \mathrm{cf}(\gamma'), s_1(\xi') = s_2(\xi')$; and by the definition of $\alpha^2, Q^{*1}_{s_2(\beta)}$ is a stationary subset of α^1, hence for some $\beta' \in J_1, \beta' \in Q^{*1}_{s_2(\beta)}$ hence $s_2(\beta') = s_2(\beta)$. As $\xi \in J_2$,

$$\{\zeta \in Q^{*2}_{s_2(\xi)} : WTh^n_{\bar r}(\alpha^2, \bar P^{*2}, \bar Q^{*2}) = \varnothing\}$$

is stationary, hence we can find $\xi' \in J_1, ch(\xi') = \omega, s_2(xi') = s_2(\xi)$.

(iii) If γ is an accumulation point of J_2 the proof is similar to that of (ii). Choose $\xi_m, m < \omega, \beta < \xi_m < \xi_{m+1} < \gamma, \lim_m \xi_m = \gamma, \mathrm{cf}(\xi_m) = \omega$, and $s_2(\xi_m) = s_2(\xi_{m+1})$ (use (4)). Then

$$\begin{aligned} Th^n_{\bar k}(\alpha^2, \bar P^2, \bar Q^2){\restriction}[\beta, \gamma)) &= Th^n_{\bar k}((\alpha^2, \bar P^2, \bar Q^2){\restriction}[\beta, \xi^0)) \\ &\quad + \textstyle\sum_{m<\omega} Th^n_{\bar k}((\alpha^2, \bar P^2, \bar Q^2){\restriction}[\xi_m, \xi_{m+1})) \\ &= s_2(\beta) + \textstyle\sum_{m<\omega} s_2(\xi_0). \end{aligned}$$

We should prove this sum is $s_2(\beta)$, and this is done as in (ii).

(iv) There are $\xi \in J, \beta < \xi < \gamma, \gamma$ the successor of ξ in J_2 and $\mathrm{cf}(\xi) > \omega$. As before we can find $\beta' < \xi' < \gamma' \in J_1, s_1(\beta') = s_2(\beta), WTh^n_{\bar r}((\alpha^1, \bar P^{1*}){\restriction}\xi') = WTh^n_{\bar r}((\alpha^2, \bar P^{*2}){\restriction}\xi), s_1(\xi') = s_2(\xi), \mathrm{cf}(\xi') > \omega, \mathrm{cf}(\gamma') = \omega$. So clearly

$$Th^n_{\bar k}((\alpha^2, \bar P^2, \bar Q^2){\restriction}[\xi, \gamma)) = s_2(\xi) = s_2(\xi') = Th^n_{\bar k}((\alpha^1, \bar P^2.\bar Q^1){\restriction}[\xi', \gamma')).$$

Now also

$$Th^n_{\bar k}((\alpha^2, \bar P^2, \bar Q^2){\restriction}[\beta, \xi)) = Th^n_{\bar k}((\alpha^1, \bar P^1, \bar Q^1){\restriction}[\beta', \xi'))$$

by the induction hypothesis on n and on γ.

So we have proved (*) and $g^n_{\bar k}((\alpha^2, \bar P^2, \bar Q^2)) = (\alpha^2, \bar Q^{*2})$.

Now by the induction hypothesis on n it follows that $Th^n_{\bar k}(\alpha^1, \bar P^1, \bar Q^1) = Th^n_{\bar k}(\alpha^2, \bar P^2, \bar Q^2)$.

Theorem 4.7. *If $\mathrm{cf}(\alpha) > \omega$,*

$$t_1 = WTh^n_r(g^n_{\bar k}(\bar P)), t_2 = ATh^n_{\bar k}(0, (\alpha, \bar P)),\ \bar r = \bar r_1(n, \ell(\bar P), \bar k),$$

then we can effectively compute $Th^n_{\bar k}(\alpha, \bar P)$ form t_1, t_2.

Proof: The proof is similar to that of 4.4.

Conclusion 4.8. If λ is a regular cardinal, and we know $ATh^n_{\bar k}(0, \lambda)$, $WTh^n_{\bar r}(\lambda)$, $(\bar r = r_1(n, 0, \bar k))$, then we can compute $Th^n_{\bar k}(\lambda)$.

Lemma 4.9. *If λ is a regular cardinal $> \omega, \bar r = r(n, 0, \bar k)$, then, letting $T_1 = \{Th^n_{\bar r}(\mu) : \omega < \mu < \lambda, \mu$ a regular cardinal$\}, T_2 = \{Th^n_{\bar r}(\alpha) : \alpha < \lambda\}$, we can compute effectively $ATh^n_{\bar k}(0, \lambda)$ from T_1; and we can compute T_1 effectively from T_2.*

Proof: Let $T = \{t_1, \ldots, t_n\}$, and if $t_i = Th^n_r(\mu)$ let $t'_i = Th^n_{\bar{k}}(\mu^q), q = q(n, \bar{k})$, (we can compute it effectively: see the proof of 3.5(B) for the definition of $q(n, \bar{k})$) and let $t = t'_1 + \ldots + t'_\ell$, then

$$\sum_{m<\omega} t = t\omega = ATh^n_{\bar{k}}(0, \lambda).^{10}$$

Conclusion 4.10. Let λ be a regular cardinal. If the monadic theory of $\{\alpha : \alpha < \lambda\}$, and $\{WTh^n_{\bar{k}}(\lambda) : n, \bar{k}\}$ are given then we can compute effectively the monadic theory of λ.

Lemma 4.11. *For a regular $\lambda, \{WTh^n(\lambda) : n < \omega\}$ and the first-order theory of $M^\lambda = (\underline{P}(\lambda)/D_\lambda, \cup, \cap, -, \varnothing, 1, \ldots, R^\lambda_t, \ldots)$ are recursive one in the other, where $R^\lambda_t(P, \bar{Q})$ holds if and only if*
$\{\beta < \lambda : \beta \in P,$ and for some $n, t = WTh^n((\lambda, \bar{Q})\restriction\beta)\} \neq \varnothing(\text{mod } D_\lambda)$.

Remark: Note that for every t there is at most one possible n.

Proof: Immediate, similar to the proof of Lemma 2.4.

Conclusion 4.12. If the monadic theory of $\{\alpha : \alpha < \lambda\}$ and the first-order theory of M^λ are decidable, then so is the monadic theory of λ.

Using 4.12 we can try to prove the decidability of the monadic theory of λ by induction on λ.

For $\lambda = \omega$ we know it by 3.4.

For $\lambda = \omega_1$ the $R^{\omega_1}_t$'s are trivial, (because each $\beta < \omega_1$ is a successor or $\text{cf}(\beta) = \aleph_0$, hence by Definition 4.4(1), $R^{\aleph_1}_t(P, \bar{Q})$ holds if and only if $t = \varnothing$). So it suffices to prove the decidability of $(\underline{P}(\omega_1)/D_{\omega_1}, \cap, \cup, -, \varnothing, 1)$. But by Ulam [Ula30] this is an atomless Boolean algebra, so its theory is decidable. Hence we reprove the theorem of Büchi [BS73].

Conclusion 4.13. The monadic theory of ω_1 is decidable.

Now we can proceed to $\lambda = \omega_2$. Looking more closely at the proof for ω_1, we see that $WTh^n_{\bar{k}}(\omega_1, \bar{P})$ can be computed from the set of atoms in the Boolean algebra generated by the P_i which are stationary subsets of ω_1; and we can replace ω_1 by any ordinal of cofinality ω_1. So all the $R^{\omega_2}_t$ can be defined by the function F/D_{ω_2},

$$F(I) = \{\alpha < \omega_2 : \text{cf}(\alpha) = \omega_1, \alpha \backslash I \cap \omega_2 \notin D_\alpha\}.$$

Conclusion 4.14. The first order theory of

$$M^{\omega_2}_1 = (\underline{P}(\omega_2)/D_{\omega_2}, \cap, \cup, -, \varnothing, 1, F/D_{\omega_2})$$

is decidable if and only id the monadic theory of ω_2 is decidable.

Notice that $F(I \cup J) = F(I) \cup F(J)$, and that for $M^{\omega_2}_1$ to have a decidable theory, it suffices that it have elimination of quantifiers. For this it suffices

[10] The second phrase is immediate by 3.6(3).

(*) for any stationary $A \subsetneqq \{\alpha < \omega_2 : \mathrm{cf}(\alpha) = \omega\}$ and B, C such that $F(A) = B \cup C$ there are stationary $A', B', A = A' \cup B', A' \cap B' = \varnothing, F(A') = A(\mathrm{mod} D_{\omega_2})$ and $F(B') = B(\mathrm{mod} D_{\omega_2})$.

Conjecture 4.15. (*) is consistent with ZFC.

5. From orders to uniform orders

An equivalence relation E on an ordered set N is *convex* if xEy, $x < z < y \in N$, implies xEy, i.e., every equivalence class is convex. On $N/E = \{\alpha/E : a \in N\}$ a natural ordering is defined. If J is a convex of a model $(M, \bar{P})$ then $th(J, \bar{P})$ is $\langle \ell, s_1, s_2 \rangle$ such that if there is no last (first) element in $J, s_2 = 1$ $(s_1 = 1)$, if b is the last (first) element, $s_2 = th(b, \bar{P})$ $(s_1 = th(b, \bar{P}))$ (for definition, see the beginning of Section 4) and $\ell = \min(|J|, 2)$.

Definition 5.1. (1) $\kappa(M)$ is the first cardinal κ, such that neither κ nor κ^* is embeddable in M.
(2) $\kappa(K)$ is l.u.b. $\{\kappa(M) : M \in K\}$.

Definition 5.2. We define for every $n, \bar{k}$, the class $U^n_{\bar{k}}$ and $UTh^n_{\bar{k}}((M, \bar{P}))$ for $M \in U^n_{\bar{k}}$

(1) $U^n_{\bar{k}} = \{(M, \bar{P}) : M$ is dense order with no first nor last element and there are t_0 and a dense $I \subsetneqq |M|$ such that for every $a < b \in I$:
$$t_0 = Th^n_{\bar{k}}((M, \bar{P}) {\restriction} (a, b)) \text{ and } th(a, \bar{P}) = th(b, \bar{P})\}.$$
Now we define $UTh^n_{\bar{k}}(M, \bar{P})$ be induction on n.
(2) $UTh^0_{\bar{k}}(M, \bar{P}) = Th^0_{\bar{k}}(M, \bar{P})$.
(3) $UTh^{n+1}_{\bar{k}}(M, \bar{P}) = \langle S_1, S_2, \mathrm{com} \rangle$ where
   (A) $S_1 = \{UTh^n_{\bar{k}}(M, \bar{P}, \bar{Q}) : \bar{Q} \in \underline{P}(M)^{\bar{k}(n+1)}, (M, \bar{P}, \bar{Q}) \in U^n_{\bar{k}}\}$,
   (B) Before we define S_2, we make some conventions:
      (α) $T_1(T_2)$ is the set of formally possible $th(J, \bar{P}^1), J \neq \varnothing$, and $\ell(\bar{P}^1) = \ell(\bar{P}), (\ell(\bar{P}^1) = \ell(\bar{P}) + \bar{k}(n+1))$;
      (β) $T_3 = \{\langle \ell, s_1, t, s_2 \rangle : \langle \ell, s_1, s_2 \rangle \in T_2, t \in T(n, \ell(\bar{P}) + \bar{k}(n+1), \bar{k})$ and $\ell = 1$ if and only if t is the "theory" of the empty model$\}$;
      (γ) If $\langle \ell, s_1, s_2 \rangle \in T_1, \langle \ell', s'_1, t, s'_2 \rangle \in T_3$ then $\langle \ell, s_1, s_2 \rangle \leqq \langle \ell', s'_1, t, s'_2 \rangle$ when: $\ell = \ell'$ and $s_1 = 1 \Leftrightarrow s'_1 = 1, s_2 = 1 \Leftrightarrow s'_2 = 1$ and $s_1 \neq 1 \to s_1 \subsetneqq s'_1, s_2 \neq 1 \to s_2 \subsetneqq s'_2$;
      (δ) At last let $\bar{r} = \bar{r}(n, \ell(\bar{P}), \bar{k})$ be from 2.13, $S_2 = \{UTh^n_{\bar{r}}(M/E, \bar{P}^*, \bar{Q}^*) :$ E a non-trivial convex equivalence relation over $|M|, (M/E, \bar{P}^*, \bar{Q}^*) \in U^n_{\bar{r}}, \bar{P}^* = \langle \ldots, P^*_t, \ldots \rangle_{t \in T_1}$, where $P^*_t = \{a/E : a \in |M|, th(a/E, \bar{P}) = t\}$ and $Q^* = \langle \ldots, Q^*_t, \ldots \rangle_{t \in T_3}$ is a partition of $|M|/E$ refining $\bar{P}^*$ and $\emptyset \neq Q^*_{t(1)} \subsetneqq P^*_t$ implies $t(1) \leqq t\}$.
   (C) Com is $+$ if M is a complete order, and $--$ otherwise.

Lemma 5.3. (A) *From $Th^{n+2}_{\bar{k}}(M,\bar{P})$ we can check whether $(M,\bar{P}) \in U^n_{\bar{k}}$ and compute $UTh^n_{\bar{k}}(M,\bar{P})$.*

(B) *Also the parallel to 2.3 holds.*

Lemma 5.4. *For every dense $N \in K, \|N\| > 1, n, \bar{k}$, there is a convex submodel M of N which belongs to $U^n_{\bar{k}}, \|M\| > 2$.*

Proof: By Theorem 1.3, and 2.7

Lemma 5.5. *Suppose N is a dense order, $\kappa(N) \leqq \aleph_1$; $I \subsetneqq |N|$ is a dense subset, and for every $a < b \in I, t_0 = Th^n_{\bar{k}}((N,P)\restriction[a,b))$. Then there is t_1 such that*

(1) *for every $a < b \in |N|, t_1 = Th^n_{\bar{k}}((N,\bar{P})\restriction(a,b))$.*

(2) *Moreover for every convex $J \subsetneqq |N|$, with no first nor last element, $t_1 = Th^n_{\bar{k}}((N,\bar{P})\restriction J)$.*

Proof: Clearly it suffices to prove (2). Choose $a_0 \in J \cap I$. Now define $a_n, 0 < n < \omega$ such that $a_n \in J \cap I, a_n < a_{n+1}$ and $\{a_n : n < \omega\}$ is unbounded in J (this is possible as $\kappa(N) \leqq \aleph_1$). Now define similarly, $a_n \in J \cap I, n$ a negative integer so that $a_{n-1} < a_n < a_0$ and $\{a_n : n$ is a negative integer$\}$ is unbounded from below in J.

So, letting Z be the integers,

$$Th^n_{\bar{k}}((N,\bar{P})\restriction J) = \sum_{n\in Z} Th^n_{\bar{k}}((N,\bar{P})\restriction[a_n,a_{n+1})) = \sum_{n\in Z} t_0 \overset{def}{=} t_1.$$

Theorem 5.6. *Let M be an order, $\kappa(M) \leqq \aleph_1$.*

(A) *Knowing t and that $t = UTh^n_{\bar{k}}(M,\bar{P}), (M,\bar{P}) \in U^n_{\bar{k}}$ we can effectively compute $F(t) = Th^n_{\bar{k}}(M,\bar{P})$.*

(B) *If $(M^i, barP^i) \in U^n_{\bar{k}}$ for $i = 1, 2$, and $UTh^n_{\bar{k}}(M^1,\bar{P}^1) = UTh^n_{\bar{k}}(M^2,\bar{P}^2)$ then $Th^n_{\bar{k}}(M^1,\bar{P}^1) = Th^n_{\bar{k}}(M^2,\bar{P}^2)$.*

Proof: Clearly (A) implies (B). So we prove (A) by induction on n.

For $n = 0$ it is trivial.

Suppose we have proved the theorem for n, and we shall prove it for $n+1$.

Let $UTh^{n+1}_{\bar{k}}(M,\bar{P}) = \langle S_1, S_2, \text{com}\rangle$. We should find

$$T = \{Th^n_{\bar{k}}(M,\bar{P},\bar{Q}) : \bar{Q} \in \underline{P}(M)^{\bar{k}(n+1)}\}.$$

If $t \in S_1$, then for some $\bar{Q} \in \underline{P}(M)^{\bar{k}(n+1)}, (M,\bar{P},\bar{Q}) \in U^n_{\bar{k}}$ and $t = UTh^n_{\bar{k}}(M,\bar{P},\bar{Q})$, hence, by the induction hypothesis $F(t) = Th^n_{\bar{k}}(M,\bar{P},\bar{Q})$, so $F(t) \in T$. We can conclude that $T' = \{F(t) : T \in S_1\} \subseteqq T$.

Now if $t^* \in S_2$, then there is a convex equivalence relation E on M, such that $t^* = UTh^n_{\bar{r}}(M/E,\bar{P}^*,\bar{Q}^*)$ where the conditions of S_2 are satisfied. If $Q^*_{\langle \ell,s_1,t,s_2\rangle} \neq \varnothing$, and $\ell > 1$ implies $t \in T$ then we can define $\bar{Q} \in \underline{P}(M)$ such that for $a/E \in Q^*_{\langle \ell,s_1,t,s_2\rangle}$:

(1) $UTh^n_{\bar{k}}((M,\bar{P},\bar{Q})\restriction \mathrm{int}(a/E)) = t$,
(2) $th(a/E,\bar{Q}) = \langle \ell, s_1, s_2\rangle$.

Remark: (1) can be done because by Lemma 5.5(2) if $\mathrm{int}(a/E) \neq \varnothing$ then

$$Th^{n+1}_{\bar{k}}((M,\bar{P})\restriction \mathrm{Int}(a/E)) = Th^{n+1}_{\bar{k}}(M,\bar{P}) = T.$$

Now clearly knowing t^* we can compute

$$S(t^*) = \{t : Q^*_{\langle \ell,s_1,t,s_2\rangle} \neq \varnothing, t \neq Th^n_{\bar{k}}(\varnothing), \text{ for some } s_1, s_2\}$$

where $\bar{Q}^*$ is an above. We can also compute $G(t) = Th^n_{\bar{k}}(M,\bar{P},\bar{Q})$. We know that $t \in S_2, S(t) \subseteqq T$, imply $G(t) \in T$.

We know also that if

(i) $t = Th^n_{\bar{k}}((M,\bar{P})\restriction\{a\})$ for some $a \in M$, and

(ii) $t_1, t_2 \in T$,

then: $\sum_{0\leqq n}(t_1 + t) \in T$ and $\sum_{\substack{n<0\\ n\in Z}}(t+t_2) \in T, t_1 + t + t_2 \in T$ and if com is $-, t_1 + t_2 \in T$ (where Z is the set of integers) (we use the facts that M is dense, $\kappa(M) \leqq \aleph_1$).

Now let T^* be the minimal subset of $T(n,\ell(\bar{P}),\bar{k})$ such that

(a) $T^* \supseteqq T'$,
(b) $t \in S_2, S(t) \subseteqq T^*$ imply $G(t) \in T^*$,
(c) if $t_1, t_2 \in T^*, t = Th^n_{\bar{k}}((M,P)\restriction\{a\})$ then $t_1 + t + t_2 \in T^*$;
(d) if $t_2 \in T^*, t_1 = Th^n_{\bar{k}}((M < \bar{P})\restriction\{a\})$ for some $a \in M$ then

$$\sum_{0\leqq n<\omega}(t_2 + t_1) \in T^*, \sum_{\substack{n\leqq 0\\ n\in Z}}(t_1 + t_2) \in T^*;$$

(e) if $t_1, t_2 \in T_2$, com is $--$ then $t_1 + t_2 \in T^*$.

It is easy to see that as S_1, S_2 are given and $T(n,\ell(\bar{P}),\bar{k})$ is (hereditarily) finite and known, we can effectively compute T^*. So it suffices to prove that $T = T^*$ but as clearly $T^* \subseteqq T$ it suffices to prove:

$$t \in T \Rightarrow t \in T^*.$$

As $t \in T$, there is $\bar{Q} \in \underline{P}(M)^{\bar{k}(n+1)}$ such that $t = Th^n_{\bar{k}}(M,\bar{P},\bar{Q})$. Define the equivalence relation E on $M : aEb$ if and only if $a = b$ or, without loss of generality we assume that $a < b$, for every $a', b' \in M, a \leqq a' < b' \leqq b, Th^n_{\bar{k}}(M,\bar{P},\bar{Q})\restriction(a',b')) \in T^*$. It is easy to check that E is a convex equivalence relation over M. Now we shall show that if $a \in M$, $\mathrm{int}(A/E) \neq \varnothing$ then $Th^n_{\bar{k}}((M,\bar{P},\bar{Q})\restriction \mathrm{int}(a/E))$ belongs to T^*. Choose $a_0 \in a/E$, and then define $a_n, n \geqq 0$ such that $a_n < a_{n+1}, \{a_n : 0 \leqq n < \omega\}$ is unbounded in $\mathrm{int}(a/E)$. Without loss of generality $th(a_n, \bar{P}{}^\frown\bar{Q}) = s_0$ for every $n > 0$. Hence

$$\begin{aligned}&Th^n_{\bar{k}}((M,\bar{P}{}^\frown\bar{Q})\restriction\{x \in \mathrm{int}(a/E) : a_0 < x\})\\ &= \textstyle\sum_{0\leqq n<\omega}[Th^n_{\bar{k}}((M,\bar{P},\bar{Q})\restriction(a_n,a_{n+1})) + Th^n_{\bar{k}}((M,\bar{P},\bar{Q})\restriction\{a_{n+1}\})].\end{aligned}$$

By the definition of $E, Th^n_{\bar{k}}((M,\bar{P},\bar{Q})\restriction(a_n,a_{n+1}))\in T^*$, hence by (d),

$$Th^n_{\bar{k}}((M,\bar{P},\bar{Q})\restriction\{x\in \mathrm{int}(a/E): a_0<x\})\in T^*.$$

Similarly,

$$Th^n_{\bar{k}}((M,\bar{P},\bar{Q})\restriction\{x\in \mathrm{int}(a/E): x<a_0\})\in T^*.$$

So by (c),

$$Th^n_{\bar{k}}(M,\bar{P},\bar{Q})\restriction \mathrm{int}(a/E))\in T^*.$$

Similarly, by (c),(e) in M/E there are no two successive elements, so M/E is a dense order.

Define $\bar{P}^*=\langle\dots,P_{\langle\ell,s_1,s_2\rangle},\dots\rangle, \bar{Q}^*=\langle\dots,Q^*_{\langle\ell,s_1,t,s_2\rangle},\dots\rangle$ such that

(1) $a/E\in P_{\langle\ell,s_1,s_2\rangle}$ if and only if $th(a/E,\bar{P})=\langle\ell,s_1,s_2\rangle$,
(2) $a/E\in Q^*_{\langle\ell,s_1,t,s_2\rangle}$ if and only id $Th^n_{\bar{k}}((M,\bar{P},\bar{Q})\restriction\mathrm{int}(a/E))=t$; and $th(a/E,\bar{P}{}^\frown\bar{Q})=\langle\ell,s_1,s_2\rangle$.

By Lemma 5.2, $(M/E,\bar{P}^*,\bar{Q}^*)$ either has only one element or it has an interval $(a/E,b/E)\neq\varnothing$ such that $(M/E,\bar{P}^*,\bar{Q}^*)\restriction(a/E,b/E)\in U^n_{\bar{r}}$.

Now we prove aEb and so show that this case does not occur and E has one equivalence relation, hence $Th^n_{\bar{k}}(M,\bar{P},\bar{Q})\in T^*$ and so we shall finish.

Let $a\leqq a'<b'\leqq b$, then let

$$\begin{aligned}J_2&=\{c\in M: a'/E<c/E<b'/E\},\\ J_1&=\{c\in M: a'<c\in\mathrm{int}(a'/E)\},\\ J_3&=\{c\in M: b'>c\in\mathrm{int}(b'/E)\}.\end{aligned}$$

By (b), $Th^n_{\bar{k}}((M,\bar{P},\bar{Q})\restriction J_2)\in T^*$; by (d) $Th^n_{\bar{k}}((M,\bar{P},\bar{Q})\restriction J_i)\in T^*$ for $i=1,3$. Hence by (c) and (e) $Th^n_{\bar{k}}((M,\bar{P},\bar{Q})\restriction(a',b'))\in T^*$. So aEb, and we finish.

Theorem 5.7. (A) *If $\kappa(K)\leqq\aleph_1$, and for every $M\in K$, there is $N\in K\cap U^{n+1}$ extending M, then from $UTh^{n+1}_{\bar{k}}(K)=\{UTh^{n+1}_{\bar{k}}(M): M\in K\cap U^{n+1}_{\bar{k}}\}$, we can compute $Th^n_{\bar{k}}(K)$. Hence if $UTh^n(K)$ is recursive in n, then the monadic theory of K is decidable.*

(B) *Suppose $\kappa(K)\leqq\aleph_1, K$ is closed under $M+N, \sum_{n<\omega}M, \sum_{\substack{n\in Z\\ n\leqq 0}}M_n, \sum_{i\in Q}M_i$ are convex submodels and division by convex equivalence relations. Then from $UTh^n_{\bar{r}}(K)$ $(\bar{r}=r(n,0,\bar{k}))$ we can compute $Th^n_{\bar{k}}(K)$. Hence if $UTh^n(K)$ is recursive in n, then the monadic theory of K is decidable.*

Proof:

(A) Immediate.
(B) Essentially the same as the proof of 5.4.

Remark: Of course there are other versions of (B), e.g., for a class of complete orders.

6. Applications of Section 5 to dense orders

Definition 6.1. K_S is the class of orders M such that no submodel of M is isomorphic to ω_1 or ω_1^* or an uncountable subset of the reals[11]

Lemma 6.2. (A) *K_S satisfies the hypothesis of 5.7(B). Also no member of K_S is complete, except the finite ones.*
(B) *K_S has uncountable members, but $M \in K_S$ implies $\|M\| \leqq \aleph_1$.*

Proof:
(A) Immediate.
(B) The Specker orders. See e.g., [Jec71][12] for existence.

Theorem 6.3. (A) *The monadic theory of K_S is decidable.*
(B) *All dense order from K_S, with no first nor last element, have the same monadic theory.*

Proof: We shall show that for $(M, \bar{P}) \in U^0(K), \bar{P}$ a partition, $pUTh^1(M, \bar{P})$ can be computed from $pUTh^0(M, \bar{P})$ (hence the former uniquely determine the latter). Then by the parallel to Lemma 2.5, clause (B) follows immediately and (A) follows by5.7(B).

So let $t = pUTh^0(M, \bar{P})$ be given; that is, we know that $\bar{P}$ is a partition of M to dense or empty subsets, $M \in U^0$, hence M is dense with no first and no last element, $M \in K$, and we know $\{i : P_i \neq \varnothing\}$. So without loss of generality. $P_i \neq \varnothing$ for every i and also $M \neq \varnothing, P_i$ is dense. Let $pTh^1(M, \bar{P}) = \langle S_1, S_2, \mathrm{com}\rangle$, so we should compute com, S_1, S_2.

Part (1) com: As $M \in K$, and as clearly the rational order is embeddable in M, M cannot be complete.

Part (2) S_1: It suffices to prove that any dense subset P of M can be split into two disjoint dense subsets of M.

So we shall prove more.

(*) If M is a dense order, $I \subseteqq |M|$ is a dense subset, *then* we can partition I to two dense subsets of M. That is, there are $J_1, J_2, I = J_1 \cup J_2, J_1 \cap J_2 = \varnothing$ and J_1, J_2 are dense subsets of M.

We define a equivalence relation E on $I : aEb$ if, $a = b$ or there are $a_0 < a, b < b_0$ and $a_0 < a' < b' < b_0$ implies $|\{c \in: a' < c < b'\}| = |\{c \in I : a < c < b\}|$ (and they are infinite by assumption). Now for every E-equivalence class a/E with more than one element, let $\lambda = |\{a \in I : b' < a < c'\}|$ for every $b' \leqq c' \in a/E$.

Case I: $|a/E| = \lambda > 0$.

Then let $\{\langle b_i, c_i\rangle : i < \lambda\}$ be an enumeration of all pairs $\langle b, c\rangle$ such that $b, c \in a/E, b < c$. Define by induction on $i < \lambda, a_i^1, a_i^2 \in a/E$. If we have

[11] Those are the Specker orders; we get them from Aronszajn trees.
[12] There is some overlapping between S_1 and S_2.

defined them for $j < i$, choose

$$a_i^1 \in \{d \in I : b_i < d < c_i\} \backslash \{a_j^2 : j < i\},$$
$$a_i^2 \in \{d \in I : b_i < d < c_i\} \backslash \{a_j^1 : j \leqq i\}.$$

By cardinality considerations this is possible. Define $J_1(a/E) = \{a_i^1 : i < \lambda\}$.

Case II: $\lambda < |a/E|$.

Then clearly $|a/E| = \lambda^+$, and we can partition a/E into λ^+ convex subsets $A_i, i < \lambda^+$, each of power λ. So on each we can define $J_1(A_1)$ such that $J_1(A_i), A_i \backslash J_1(A_i)$ are dense subsets of A_i. Let $J_1(a/E) = \bigcup_{i<\lambda^+} J_1(A_i)$.

Case III: $\lambda = 0$, so $|a/E| = 1$.

Let $J_1(a/E) = \varnothing$. Let $J_1 = \bigcup_{a \in I} J_1(a/E), J_2 = I \backslash J_1$.

It is easy to check that J_1, J_2 are the desired subsets.

Part (3) S_2: By (2) it suffices to find to possible $UTh^0(M/E, \bar{P}^*)$, where $\bar{P}^* = \langle \dots, P^*_{\langle \ell, s_1, s_2 \rangle}, \dots \rangle, P^*_{\langle \ell, s_1, s_2 \rangle} = \{a/E : th(a/E, \bar{P}) = \langle \ell, s_1, s_2 \rangle\}$, and $(M/E, \bar{P}^*) \in U^0(K)$; so $W_E = \{\langle \ell, s_1, s_2 \rangle : P^*_{\langle \ell, s_1, s_2 \rangle} \neq \varnothing\}$ contain all relevant information. Clearly $W_E \neq \varnothing$ and $\langle \ell, s_1, s_2 \rangle \in W_E \Rightarrow \ell > 0$ and we can also discard the case $\langle \ell, s_1, s_2 \rangle \in W_E \Rightarrow \ell = 1$. Also if $\langle \ell, s_1, s_2 \rangle \in W_E$, then $\langle \ell, s_1, s_2 \rangle$ is formally possible.

Suppose W satisfies all those conditions, and we shall find a suitable E such that $W_E = W$. Let $W = \{\langle \ell^i, s_1^i, s_2^i \rangle : i < q < \omega\}$. Choose a $J \subseteqq |M|$, countably dense in itself, unbounded in M from above and from below, such that each $P_j \cap J$ is a dense subset of J, and for no $a \in |M| \backslash J$ is there a first (last) element in $\{b \in J : b > a\}$ ($\{b \in J ; b < a\}$). J defines $2^{\aleph_0}$ Dedekind cuts, but as $M \in K$, only $\leqq \aleph_0$ of them are realized. Let $\{a_n : n < \omega\}$ be a set of representatives from those cuts (that is, for every $a \in |M| \backslash J$ there is $n < \omega$ such that $[a, a_n]$ or $[a_n, a]$ is disjoint to J). Let $J = \{b_n : n < \omega)$. Now we define by induction on n a set H_n of convex disjoint subsets of M, such that:

(a) $H_n \subseteqq H_{n+1}; H_n$ is finite.
(b) If $I_1 \neq I_2 \in H_n$ then $I_1 < I_2$ or $I_2 < I_1$ and between them there are infinitely many members of J.
(c) If $I \in H_n, I$ has no last element, then for every $a \in |M| \backslash J, a > I$, there is $b \in J, I < b < a$, and also $J \cap I$ is unbounded in I.
(d) The same holds for the converse order.
(e) If $I_1 < I_2 \in H_n, i < q$ then there are $I \in H_{n+1}, th(I, \bar{P}) = \langle \ell^i, s_1^i, s_2^i \rangle$.[13]
(f) $a_n, b_n \in \bigcup \{I : I \in H_n\}$.
(g) If $I \in H_n$ has a first (last) element then this element belongs to J. It is not hard to define the H_n's. Clearly $\bigcup_n \bigcup_{I \in H_n} I = |M|$. So define E as follows:

aEb if and only if $a = b$ or for some $n < \omega, I \in H_n, a, b \in I$.

[13] Also, $I_1 < I < I_2$, and $I_0 \in H_n$ implies $th(I_0, \bar{P}) \in W$.

It is not hard to check that $W_E = W$. So we finish the proof.
Along similar lines we can prove

Theorem 6.4. *Suppose M is a dense order with no first nor last elements, M is a submodel of the reals, and for every perfect set P of reals, $P \cap |M|$ is countable, or even $< 2^{\aleph_0}$. Then the monadic theory of M is the monadic theory of rationals.*

Remark 1: We can integrate the results of 6.3, 6.4. Always some M satisfies the hypothesis of 6.4. If $2^{\aleph_0} > \aleph_1$, any dense $M \subsetneqq R, |M| < 2^{\aleph_0}$, and if $2^{\aleph_0} = \aleph_1$, the existence can be proved.

Remark 2: In 6.4 we can demand less of $|M|$: For all countable, disjoint and dense sets $Y_1, \dots Y_n (n < \omega)$ there is a perfect set P of reals such that Y_i is dense in P for $1 \leqq i \leqq n$ and $P \cap |M|$ is $< 2^{\aleph_0}$ (see Section 7 for definition).

The proof of 5.4 is easily applied to the monadic theory of the reals. (We should only notice that R is complete.)

Conclusion 6.5. If we can compute the $UTh^n(R)$ for $n < \omega$ then the monadic theory of the real order is decidable.

Remark: Similar conclusions hold if we add to the monadic quantifier (or replace it by) $(\exists^{<\aleph_1} X)$ (i.e., there is a countable X). Notice that if E is a convex equivalence relation over R, then $\{a/E : |a/E| > 1\}$ is countable.

Grzegorczk [Grz51] asked whether the lattice of subsets of reals with the closure operation has a decidable theory. One of the corollaries of Rabin [Rab69] is that the theory of the reals with quantification over closed sets, and quantification over F_σ sets is decidable.

By our methods we can easily prove

Theorem 6.6. *The reals, with quantifications over countable sets, has a decidable theory. (We can replace "X countable" by "$|X| < 2^{\aleph_0}$" or "$(\forall P)$ (P closed nowhere dense $\to |P \cap X| < 2^{\aleph_0}$)").*

As every closed set is a closure of a countable set, this proves again the result of Rabin [Rab69] *concerning Grzegorczk's question. We can also prove by our method Rabin's stronger results, but with more technical difficulties.*

7. Undecidability of the monadic theory of the real order

Our main theorem here is

Theorem 7.1. (A) *(CH) The monadic theory of the real order is undecidable.*
(B) *(CH) The monadic theory of order is undecidable.*

Theorem 7.2. *(CH) The monadic theory of $K_n = \{(R, Q_1, \dots, Q_n) : Q_i \subsetneqq R\}$, where the set quantifier ranges over countable sets, $1 \leqq n$, is undecidable. (We can even restrict ourselves to sets of rationals.)*

Let $2^{\leqq\omega}$ be the set of sequences of ones and zeros of length $\leqq \omega$; let $\leqq$ be a partial ordering of $2^{\leqq\omega}$ meaning that it is an initial segment, $\prec$ the lexicographic order.

Theorem 7.3. (A) *CH The monadic theory of $(2^{\leqq\omega}, \leqq, \prec)$ is undecidable.*

(B) *(CH) The monadic theory of $K_n = \{(2^{\leqq\omega}, \leqq, \prec, Q_1, \ldots, Q_n) : Q_i \subseteqq 2^{\leqq\omega}\}$, where the set quantifier ranges over sets, $1 \leqq n$, is undecidable. (We can even restrict ourselves to subsets of $2^{<\omega}$).*

Instead of the continuum hypothesis, we can assume only:

(*) *"The union of $< 2^{\aleph_0}$ sets of the first category in not R". This is a consequence of Martin's axiom (see* [Jec03]*) hence weaker than CH, but also its negation is consistent, (see Hechler* [Hec73] *and Mathias* [Mat74] *and Solovay* [Sol70]*). Aside from countable sets, we can use only a set constructible from any well-ordering of the reals. Remember that by Rabin* [Rab69] *quantification over closed and F_σ sets gives us still a decidable theory.*

Conjecture 7A: The monadic theory of $(2^{\leqq\omega}, \leqq, \prec)$, where the set quantifier ranges over Borel sets only, is decidable.

This should be connected to the conjecture on Borel determinacy (see Davis [Dav64], Martin [Mar70] and Paris [Par72]).[14] This conjecture implies

Conjecture 7B: The monadic theory of the reals, where the set quantifier ranges over Borel sets, is decidable (by Rabin [Rab69]).

Conjecture 7C: We can prove 7.1-7.3 in ZFC.

Theorems 7.1(A),(B),7.3(A) answer well known problems (see e.g., Büchi [BS73, p.38, Problem 1,2a,2b,4a]. Theorem 7.3(B) answers a question of Rabin and the author.

Unless mentioned otherwise, we shall use CH or (*).

Notation: R denotes the reals. A *prefect* set is a closed, nowhere dense set of reals, with no isolated points and at least two points (this is a somewhat deviant definition). We use P to denote prefect sets. Let x be an inner point of P if $x \in P$, and for every $\epsilon > 0, (x--\epsilon, x) \cap P \neq \varnothing, (x, x+\epsilon) \cap P \neq \varnothing$. Let $D \subseteq\subseteq R$ be dense in P if for every inner point $x < y$ of P, there is an inner $z \in P \cap D, x < z < y$. Note that if D is dense in P, P is the closure of $P \cap D$. Real intervals will be denoted by (a, b) where $a < b$, or by I; (a, b) is an interval of P if in addition a, b are inner points if P.

Lemma 7.4. *Let J be an index-set, the D_i $(i \in J)$ countable dense subsets of R, and $D = \bigcup_{i \in J} D_i$; and for every $P, |D \cap P| < 2^{\aleph_0}$. Then there is $Q \subseteq R \backslash D, Q = Q\{D_i : i \in J\}$, such that*

[14] Meanwhile Martin [Mar75] proved the Borel determinacy.

(A) *if* $P \cap D \subseteqq D_i$ $(i \in J)$ *and* D_i *is dense in* P *(*P *is, of course, prefect) then* $|P \cap Q| < 2^{\aleph_0}$*;*
(B) *if for no (interval)* I *of* P*, and* $i \in J, P \cap D \cap I \subseteqq D_i$ *but* D *is dense in* P *then* $P \cap Q \neq \varnothing$*.*

Proof: Let $\{P_\alpha : 0 < \alpha < 2^{\aleph_0}\}$ be any enumeration of the perfect sets. We define $x_\alpha, \alpha < 2^{\aleph_0}$ by induction on α.

For $\alpha = 0, x_\alpha \in R$ is arbitrary.

For any $\alpha > 0$, if P_α does not satisfy the assumptions of (B) then let $x_\alpha = x_0$ and if P_α satisfies the assumptions of (B) let $x_\alpha \in P_\alpha \backslash \bigcup\{P_\beta : \beta < \alpha, (\exists i \in J)(P_\beta \cap D \subseteqq D_i \text{ and } D \text{ is dense in } P_\beta)\} = D$.

This is possible because for any β, i, if $P_\beta \cap D \subseteqq D_i, D$ is dense in $P_\beta, P_\beta \cap P_\alpha$ is a closed nowhere dense subset of P_α. As otherwise for some interval I of $P_\alpha, P_\beta \cap P_\alpha$ is dense in P_α, so by the closeness of $P_\beta \cap P_\alpha, P_\beta \cap P_\alpha \cap I = P_\alpha \cap I$; therefore

$$D_i \supseteqq P_\beta \cap D \supseteqq P_\alpha \cap I \cap D,$$

a contradiction of the assumption on P_α. So by (*) and the hypothesis $|P_\alpha \cap D| < 2^{\aleph_0}$ there exists such x_α.

Now let $Q = \{x_\alpha : \alpha < 2^{\aleph_0}\}$. If P satisfies the assumption of (A), then $P \in \{P_\alpha : 0 < \alpha < 2^{\aleph_0}\}$. Hence for some $\alpha, P = P_\alpha$, hence $P \cap D \subseteqq \{x_\beta : \beta < \alpha\}$, so $|P \cap D| < 2^{\aleph_0}$. If $P = P_\alpha$ satisfies the assumption of (B) then $x_\alpha \in P_\alpha, x_\alpha \in Q$, hence $P_\alpha \cap Q \neq \varnothing$. So we have proved the lemma.

Lemma 7.5. *There is a dense* $D \subseteqq R$ *and* $\{D_i : i \in J\}, |J| = 2^{\aleph_0}$ *such that*

(1) $|D \cap P| < 2^{\aleph_0}$ *for every perfect* P*.*
(2) *The* D_i *are pairwise disjoint.*
(3) $D_i \subseteqq D, D_i$ *is dense.*

Proof: Let $\{P_\alpha : \alpha < 2^{\aleph_0}\}$ enumerate the perfect subsets of R, and let $\{I_n : n < \omega\}$ enumerate the rational intervals of R, and if $\alpha = \delta + n$ ($n < \omega, \delta$ a limit ordinal) choose $x_\alpha \in I_n \backslash \bigcup_{\beta<\alpha} P_\beta \backslash \{x_\beta : \beta < \alpha\}$ and let $D = \{x_\beta : \beta < 2^{\aleph_0}), D_\alpha = \{x_{\omega\alpha+n} : n < \omega\}$.

Notation: J will be an index set; $[J]^n = \{U : U \subseteqq J, |U| = n\}$, and if D_i is defined for $i \in J$, let $D_U = \bigcup_{i \in U} D_i$. Subsets of $[J]^n$, i.e., symmetric n-place relations over J, are denoted by S; and if we know $\{D_i : i \in J\}, Q_S$ will by $Q\{D_U : U \in S \cup [J]^{n-1}\}$ from 7.4.

Definition 7.6. Let $\varphi_n(X, D, Q, I^*)$ be the monadic formula saying

(A) X is a dense set in I^* and $X \subseteqq D$.
(B) For every interval $I \subseteqq I^*$, and sets $Y_i, \ldots, i = 1, n+1$, if $Y_i \cap I \subseteqq X$ and the Y_i are pairwise disjoint and each Y_i is dense in I then there is a perfect set $P, P \cap Q = \varnothing$, and each $Y_i \cap I$ is dense in P.

Remark: We can represent the interval I_0 as a convex set.

Lemma 7.7. *Let $D, \{D_i : i \in J\}$ be as in 7.5, I^* an interval, $S \subseteqq]J]^n, Q_S = Q\{D_U : U \in S \cup [J]^{n-1}\}$ as in 7.4. Then for any set $X \subseteqq R, R \models \varphi_n[X, D, Q_S, I^*]$ if and only if*

(A) *X is dense in $I^*, X \subseteqq D$,*
(B) *for any interval $I \subseteqq I^*$ there is a subinterval I_1 and $U \in S \cup [J]^{n-1}$ such that $X \cap I_1 \subseteq D_U$.*

Proof: (I) Suppose $R \models \varphi_n[X, D, Q_S, I^*]$. Then by (A) from Definition 7.6, X is dense in $I^*, X \subseteqq D$ so (A) from here is satisfies. To prove (B)let $I \subseteqq I^*$ be an interval, and suppose that for no subinterval I_1 of I and for no $U \in S \cup [J]^{n-1}$, does $X \cap I_1 \subseteqq D_U$ hold, and we shall get a contradiction. Now we define by induction on $\ell, 1 \leqq \ell \leqq n+1$, distinct $i(\ell) \in J$ and intervals $I^\ell, 0 \leqq \ell \leqq n$ so that $I^0 = I, I^{\ell+1} \subseteqq I^\ell$, and $X \cap D_{i(\ell)} \cap I^\ell$ is dense in I^ℓ.

If we succeed, in Definition 7.6(B), choose I^{n+1} as I, and $X \cap D_{i(\ell)} \cap I^{n+1}$ as Y^ℓ. So necessarily by φ_n's definition there is a perfect P such that $X \cap D_{i(\ell)} \cap I^{\ell+1}$ is dense in P for $\ell = 1, n+1$, and $P \cap Q_S = \varnothing$. But this contradicts Lemma 7.4(B) by the definition of Q_S. So for some $\ell < n+1$ we cannot find appropriate $i(\ell+1), I^{\ell+1}$. So if we let $Y = (X - \bigcup_{k \leq \ell} D_{i(k)}) \cap I^\ell$, for no $I^+ \subseteqq I^\ell$ and no $i \in J$ is $Y \cap D_i \cap I^+$ dense; i.e., for every $i \in J, Y \cap D_i$ is nowhere dense.

If $\ell = n$, but $\{i(1), \ldots, i(n)\} \notin S$ let $D_{i(n)} \cap X \cap I^\ell = Y^1_n \cup Y^1_{n+1}$, where Y^1_n, Y^1_{n+1} are dense subsets of I^ℓ, and $Y^1_k = X \cap D_{i(k)} \cap I^\ell$, and we get a contradiction as before.

If Y is not dense in I^ℓ, it is disjoint to some $I^+ \subseteqq I^\ell$, $X \cap I^+$, so $X \cap I^+ \subseteqq \bigcup_{k<\ell} D_{i(k)}$. So $U = \{i(0), \ldots, i(\ell) \in S \cup [J]^{n-1}, X \cap I^+ \subseteqq D_U$, contradicting an assumption we made in the beginning of the proof. Hence Y is dense in I^ℓ.

As $(\forall i \in J) Y \cap D_i$ is nowhere dense also for every finite $U ssu \cup \subseteqq J, Y \cap D_U$ is nowhere dense. So we can chose inductively distinct $i_m \in J$ and distinct $x_m \in Y \cap D_{i_m}$ such that $\{x_{(n+3)m+k} : m < \omega\}$ are dense subsets of I^ℓ, for $0 \leqq k < n+2$. If we let $Y^2_k = \{x_{(n+3)m+k} : m < \omega\}$ for $k \leqq n+1$, by Definition 7.6 there is a perfect P, such that Y^2_k is dense in $P, P \cap Q = \varnothing$, and we get contradiction by 7.4(B) and the choice of the x_m's.

As all the ways give a contradiction, we finish one implication.

(II) Now we want to prove that $R \models \varphi_n[X, D, Q, I^*]$ assuming the other side.

Clearly $X \subseteqq D$, and X is dense in I^* (by condition (A) of Lemma 7.7). So condition (A) in Definition 7.6 holds. For condition (B) of that definition let $I \subseteqq I^*$ be an interval, $Y_k \cap I \subseteqq X, Y_k$ dense in I for $k = 1, \ldots, n+1$ and $k \neq \ell \Rightarrow Y_k \cap Y_\ell = \varnothing$. We should find a perfect P such that $P \cap Y_k$ is dense in P and $P \cap Q = \varnothing$. We can choose a $U \in S \cup [J]^{n-1}$ and $I_1 \subseteqq I$ so that $X \cap I_1 \subseteqq D_U$ (by the hypothesis). Choose a perfect P such that each Y_k is dense in P. As D is as in 7.4, either case gives $|P \cap D| < 2^{\aleph_0}$.

(*) Now we can find perfect $P_\alpha(\alpha < 2^{\aleph_0})$ such that each Y_k $(1 \leqq k \leqq n+1)$ is dense in P_α and $\alpha \neq \beta$ implies $P_\alpha \cap P_\beta \subseteqq \bigcup_{k=1}^{n+1} Y_k$.

Proof of (*): For η a finite sequence of ones and zeros X_η will be a set of closed-open intervals and singletons with endpoints in $\bigcup_{k=1}^{n+1} Y_k$, which are pairwise disjoint. We define X_η by induction on $\ell(\eta)$. Let $X_{\langle\rangle} = \{[a,b)\}$, where $a, b \in Y_1$, and if X_η is defined, for each interval $[a,b) \in X_\eta$, choose a decreasing sequence x_i^a $(i < \omega)$ whose limit is a, and $x_0^a < b$ and $x_i^a \in Y_k$ if and only if $\ell(\eta) = k \bmod n+1, 1 \leqq k \leqq n+1$. Let, for $m = 0, 1$:

$$X_{\eta\frown\langle m\rangle} = \{(x_{i+1}^1, x_i^a) : \text{ for some } b, [a,b) \in X_\eta \text{ and } i = m \bmod 2\}$$
$$\cup\{\{a\} : \text{ for some } b, [a,b) \in X_\eta, \text{ or } \{a\} \in X_\eta\}.$$

For η a sequence of ones and zeros of length ω, $P_\eta = \bigcap_{\ell<\omega}(\bigcup X_{\eta\restriction n})$.

Because $|P \cap D| < 2^{\aleph_0}$ for some $\alpha, P_\alpha \cap D \subseteqq \bigcup_{k=1}^{n+1} Y_k$; so by 7.4 (and the choice of Q's), $|P_\alpha \cap Q_S| < 2^{\aleph_0}$. We can find $P_\alpha^\beta(\beta < 2^{\aleph_0})$ such that each Y_k is dense in P_α^β and $\beta \neq \gamma \Rightarrow P_\alpha^\beta \cap P_\alpha^\gamma \subseteqq \bigcup_{k=1}^{n+1} Y_k$. So for some $\beta, P_\alpha^\beta \cap Q \subseteqq \bigcup_{k=1}^{n+1} Y_k \subseteqq D$, but $Q \subseteqq R\backslash D$ hence $P_\alpha^\beta \cap Q = \varnothing$, and we finish.

Definition 7.8. Let $\psi_n(X, D, Q, I^*)$ be the monadic formula saying

(A) $\varphi_n(X, D, Q, I^*)$,

(B) for any interval $I_1 \subseteqq I^*$, if Y is disjoint to X and dense in I_1 then $\neg\varphi_n(X \cup Y, D, Q, I_1)$.

Lemma 7.9. *Let D, J, D_i, S, Q_S be as in 7.7. Then for any $X \subseteqq R, R \models \psi_n[X, D, Q_S, I^*]$ if and only if*

(A) *X is dense in $I^*, X \subseteqq D$,*

(B) *for any interval $I \subseteqq I^*$ there is a subinterval I_1 and $U \in S \cup \{V \in [J]^{n-1} : (\forall i \in J)(V \cup \{i\} \notin S)\}$ such that $X \cap I_1 = D_U \cap I_1$.*

Proof:

(I) Suppose $R \models \psi_n[X, D, Q_S, I^*]$, then clearly condition (A) holds. For condition (B) let $I \subseteqq I^*$ be an interval. By Definition 7.8(A), $R \models \varphi_n[X, D, Q_S, I^*]$, hence by Lemma 7.7(B), I has a subinterval I_0 such that $X \cap I_0 \subseteqq D_U$ where $U \in S \cup [J]^{n-1}$. If $(D_U\backslash X) \cap I_0$ is somewhere dense, let it be dense in $I_1 \subseteq I_0$, and let $Y = (D_U\backslash X) \cap I_1$, which gives us a contradiction to Definition 7.8(B). If $U \in [J]^{n-1}$, and for some $i \in J, V = U \cup \{i\} \in S$, we can get a similar contradiction by $Y = (D_V\backslash X) \cap I_0$ in the interval I_0 (as $D_i \subseteq D_V\backslash X, Y$ is dense). We can conclude that: $U \in S$ or $U \in [J]^{n-1}$ and $U \cup \{i\} \notin S$ for every $i \in J$ and that $(D_U\backslash X) \cap I_0$ is nowhere dense. Hence for some $I_1 \subseteq I_0, (D_U\backslash X) \cap I_1 = \varnothing$ hence $X \cap I_1 = D_U \cap I_1$.

(II) Now suppose that conditions (A),(B) hold; by Lemma 7.7 it is easy to see that $R \models \psi_n[X, D, Q_S, I^*]$.

Definition 7.10. Let $\chi_1(D, Q, I^*)$ be the monadic formula saying:

(A) D is dense in I^*, I^* an interval;

(B) if $I \subseteq I^*, X, Y$ are dense in I and

$$R \models \psi_1[X, D, Q, I] \wedge \psi_1[Y, D, Q, I]$$

then for some $I_1 \subseteq I$,

$$X \cap Y \cap I = \varnothing \text{ or } X \cap I_1 = Y \cap I_1.$$

Lemma 7.11. (A) *If $D, \{D_i : i \in J\}$, are as in 7.5 then for any interval $I^*, R \models \chi_1[D, Q_J, I^*]$.*

(B) *If $R \models \chi_1[D, Q, I^*]$ then we can find $I \subsetneqq I^*$, and $X_i, i < \alpha_0$ such that*

  (a) *each X_i is a dense subset of I and $R \models \psi_1[X_i, D, Q, I]$,*

  (b) *if $I_0 \subsetneqq I$, and $X \subsetneqq I_0$ is dense in I_0 and $R \models \psi_1[X, D, Q, I_0]$ then there are $i < \alpha$ and $I_1 \subsetneqq I_0$ such that $X \cap I_1 = X_i \cap I_1$.*

(C) *In (B), $|\alpha_0|$ is uniquely defined by D, Q, I.*

Proof:

(A) By 7.9 it is immediate.

(B) Let $\{X_i : i < \alpha\}$ be a maximal family satisfying (1) and (2) for $I = I^*$. If for some interval I there are no subintervals I^1 and dense $X^* \subsetneqq X \cap I^1$ such that $(\forall i < \alpha_0)$ ($X_i \cap X^*$ is nowhere dense)[15] we are finished. Otherwise we can choose inductively on n intervals $I^n \subsetneqq I^*$ disjoint to $\bigcup_{\ell<n} I^\ell$ and $X_n^* \subsetneqq X \cap I^n$ such that $(\forall i < \alpha_0), X_i \cap X_n^*$ is nowhere dense[16], and such that $\bigcup_{n<\omega} I^n$ is dense in I. Then we could have defined $X_{\alpha_0} = \bigcup_{n<\omega} D_n^*$, a contradiction.

(C) Easy.

Definition 7.12. Let $\chi^n(Q_1, D, Q, I^*)$ be the monadic formula saying

(A) D is dense in I^*, which is an interval.

(B) Suppose $I_0 \subsetneqq I^*, X_\ell \subsetneqq I_0 (\ell < n)$ and $R \models \bigwedge_{\ell<n} \psi_1(X_\ell, D, Q, I_0)$.Then there is $I_1 \subsetneqq I_0$ such that for all $I_2 \subsetneqq I_1$

$$R \models \psi_n(\bigcup_{\ell<n} X_\ell, D, Q_1, I_1) \equiv \psi_n(\bigcup_{\ell<n} X_\ell, D, Q_1, I_2).$$

Lemma 7.13. *If $D, \{D_i : i \in J\}$ are as in Lemma 7.5, $S \subsetneqq [J]^n$ then for any interval $I^*, R \models \chi^n[Q_S, D, Q, I^*]$.*

Proof: Immediate.

Theorem 7.14. *The set A_r is recursive in the monadic theory of order; where $A_r = \{\theta : \theta$ is a first order sentence which has an ω-model i.e., a model M such that $(|M|, R_1)$ is isomorphic to $(\omega, x + 1 = y)\}$.*

[15] and $R \models \psi_1[X^*, D, Q, I^1]$.

[16] and $R \models \psi_1[X^*, D, Q, I^n]$.

Conclusion 7.15. True first order arithmetic is recursive in the monadic theory of order.

Proof: It suffices to define for every first order sentence θ, a monadic sentence $G(\theta)$ so that $R \models G(\theta)$ if and only if θ has an ω-models.

By using Skolem-functions and then encoding them by relations, we can define effectively the sentence $G_1(\theta)$ such that θ has an ω-model if and only if $G_1(\theta)$ has an ω-model and

$$G_1(\theta) = (\forall x_1, \ldots, x_{n(0)})(\exists x_{n(0)+1}, \ldots, x_{n(1)})(\bigvee_i \bigwedge_j \theta_{ij}),$$

θ_{ij} is an atomic, or a negation of an atomic, formula; only the relations $R_0, \ldots, R_{n(2)}$ appear in it; R_0 is the equality; and R_i has $m(i)$-places.

Define (where X, Y, D, Q are variables ranging over sets, I is a variable ranging over intervals and x, y are individual variables):

(0) $G_2(X_k = X_\ell) = (\forall I^1 \subseteqq I^*)(\exists I^2 \subseteqq I^1)(X_k \cap I^2 = X_\ell \cap I^2)$,

(1) $G_2[G_\ell(X_{k(1)}, \ldots, X_{k(m(\ell))})] = (\exists Y)(Y \subseteqq D \backslash D^* wedge \bigwedge_{i=1}^{m(\ell)} \psi_2(X_{k(i)} \cup Y, D, Q_i^\ell, I^*)$ (for $\ell < 0$),

(2) $G_2(\theta) = (\forall X_1, \ldots, X_{n(0)})(\exists X_{n(0)+1}, \ldots, X_{n(1)})$
$(\forall I^0 \subseteqq I)(\exists I^* \subseteqq I^0)[\bigwedge_{\ell=1}^{n(0)} \psi_1(X_\ell, D, Q^*, I^*) \bigwedge \bigwedge_{\ell=1}^{n(0)} X_\ell \subseteqq D^*$

$$\rightarrow \bigwedge_{\ell=n(0)+1}^{n(1)} X_\ell \subseteqq D^* \cap \bigwedge_{\ell=n(0)+1}^{n(1)} \psi_1(X_\ell, D, Q^*, I^*) \wedge \bigwedge_i \bigvee_j G_2(\theta_{ij})].$$

(4) Let χ^* be the conjunction of the following formulas:

(α) D, D^* are dense in $I, D^* \subseteqq D$,

(β) $\chi_1(D, Q^*, I)$,

(γ) $\chi^2(Q_\ell^i, D, Q^*, I)$. Let us denote

$$\tilde{R}_1(X, Y, Q_1^1, Q_1^2, I') = (X \subseteqq D^* \wedge Y \subseteqq D^* \wedge X \cap Y = \varnothing \wedge$$
$$\psi_1(X, D, Q^*, I') \wedge \psi_1(Y, D, Q^* \wedge, I') \wedge (\exists Z)[Z \subseteqq D \backslash D^* \wedge \psi_1(Z, D, Q^*, I') \wedge$$
$$\psi_2(X \cup Z, D, Q_1^1, I') \wedge \psi_2(Y \cup Z, D, Q_1^2, I')]$$

and

(δ) $\psi_1(X_0, D, Q^*, I) \wedge X_0 \subseteqq D^* \wedge (\forall Y)[\psi_1(Y, D, Q^*, I) \wedge Y \subseteqq D^* \rightarrow (\exists Y_1)\tilde{R}_1(Y, Y_1)] \wedge (\forall I' \subseteqq I)(\forall Y)\neg\tilde{R}_1(Y, X_0, Q_1^1, Q_1^2, I') \wedge$

$$(\forall Y_1 Y_2 Y_3)(\forall I^0 \subseteqq I)[\tilde{R}_1(Y_1, Y_2, Q_1^1, Q_1^2, I^0) \wedge \tilde{R}_1(Y_1, Y_3, Q_1^1, Q_1^2, I^0)$$
$$\rightarrow (\forall I^1 \subseteqq I^0)(\exists I^2 \subseteqq I^1) Y_2 \cap I^2 = Y_3 \cap I^2].$$

(ϵ) The formula saying that if (δ) holds when we replace Q_1^1, Q_1^2 by $\tilde{Q}_1^1, \tilde{Q}_1^2$ resp. then

$$(\forall X)(\forall Y)(\forall I' \subseteqq I)[\tilde{R}_1(X, Y, Q_1^1, I') \rightarrow \tilde{R}_1(X, Y, \tilde{Q}_1^1, \tilde{Q}_1^2, I')].$$

(5) $G(\theta) = (\exists Q^*, D, D^*, X_0, \ldots, Q_\ell^i, \ldots)(\forall I)[\chi^* \wedge G_3(\theta)]$.

Now we should prove only that θ has an ω-model if and only if $R \models G(\theta)$.

(I) Suppose M is an ω-model 2021-06-14 if and only if $R \vdash G(\theta)$. Let $J = \omega + \omega, D_i (i < \omega + \omega)$ be countable, pairwise disjoint, dense subsets of R. Choose symmetric and reflexive relations S^i_ℓ on $\omega + \omega$ so that

$$M \models R_\ell(x_1, \ldots, x_{k(\ell)}) \Leftrightarrow (\exists y \in \omega + \omega) \bigwedge_{i=1}^{k(\ell)} \langle y, x_i \rangle \in S^i_\ell \wedge y \notin \omega).$$

To prove $R \models G(\theta)$, let $D = \bigcup_{i<\omega+\omega} D_i, D^* = \bigcup_{i<\omega} D, Q^i_\ell = Q_{(S^i_\ell)}, X_0 = D_0$, and $Q^* = Q_{\omega+\omega}$. Let I be any interval. It is not hard to check that under those assignments $R \models x^* \wedge G_3(\theta)$.

(II) Now suppose $R \models G(\theta)$. Let $Q^*, D, D^*, X_0, Q^i_\ell$ be such that $R \models (\forall I)(\chi^* \wedge G_3(\theta))$. BY (4) ($\beta$), clearly $R \models (\forall I)\chi_1(D, Q^*, I)$. Hence by Lemma 7.11(B) there are I and $D_i, i < \alpha$ satisfying (1),(2),(3) from 7.11(B). As $R \models (\forall I)(\chi^* \cap G_3(\theta))$, then in particular $R \models \chi^* \wedge G_3(\theta)$. By (4)($\delta$), $R \models \psi_1(X_0, D, Q^*, I)$, so we can choose $D_0 = X_0$. (See the proof of 7.11.) By (4)(δ) we can also assume that $R \models \tilde{R}_1(D_n, D_{n+1})$ for $n < \omega$. By (4)(ϵ) necessarily $D_i \subseteqq D^* \Leftrightarrow i < \omega$.
Let $\{\bar{j}_\ell : \ell < \omega\}$ enumerate all sequences $j = \langle j(1), \ldots, j(n(0)) \rangle$ of natural numbers. As $R \models G_3(\theta)$ for every $\bar{j}_\ell$ we can choose $X_i = D_{j_\ell(i)}$, and so there is an assignment $X_i \to D^{\ell,i}$ for $n(0) < i \leqq n(1)$ showing that $R \models G_3(\theta)$. So we can define by induction on $n < \omega$ intervals I_n so that: $I_{n+1} \subseteqq I_n, I_0 \subseteqq I$, and for every $n(0) < i \leqq n(1)$ for some $j_n(i) < \alpha_0, D^{\ell,i} \cap I_{n+1} = D_{j_n(i)} \cap I_{n+1}$.
Now we define a model $M : |M| = \omega$, and $M \models R_\ell[j(1), \ldots, j(m(\ell))] \Leftrightarrow$ for some $n, R \models (\exists Y)[Y \subseteqq D \backslash D^* \bigwedge \bigwedge_{i=1}^{m(\ell)} \psi_2(D_{j(i)} \cap Y, D, Q^i, I^n_n)]$.
It is easy to check that $R \models \theta$.

Remark: By some elaboration, we can add to the definition of A_r also the demand

"R_2 is a well-founded two-place relation"

(also for uncountable structures). Thus, e.g., there are sentences θ_n, such that MA implies: $R \models \theta$ if and only if $2^{\aleph_0} = \aleph_n$.

Theorem 7.16. *The set of first-order sentences which has a model, is recursive in the monadic theory of $\{(R, Q) : Q \subseteqq R\}$ where the set-variables range over subsets of the rationals.*

Remark: Notice that a quantification over P such that D is dense in P can be interpreted by a quantification over $P \cap D$, as the property "x in the closure of X" is first-order. Hence φ_n, ψ_n are, in our restricted monadic theory.

By 7.14,7.15, Theorems 7.1,7.2 and 7.3 are in fact immediate. Theorem 7.1(B) can also be proved by the following observation of Litman [Lit76], which is similar to 3.6(B)(1):

Lemma 7.17. *The monadic theory of the real order is recursive in the monadic theory of order.*

Proof: For every monadic sentence θ let $G(\theta)$ be the monadic sentence saying:

"If the set X is completely ordered, is dense and has no first nor last elements then some $Y \subsetneqq X$ has those properties and in addition $(Y, <) \models \theta$."

As every complete dense order contains a subset isomorphic to R, and any complete dense order $\subsetneqq R$ with no first nor last element is isomorphic to R, clearly $R \models G(\theta)$ if and only if θ is satisfied by all orders so our results is immediate.

Conjecture 7D: The monadic theory of R and the (pure) second-order theory of $2^{\aleph_0}$ are recursive in each other.[17]

Conjecture 7E: The monadic theory of $\{R, Q) : Q \subsetneqq R\}$ with the set-quantifiers ranging over subsets of the rationals; and the (pure) second-order theory of $\aleph_0$ are recursive in each other. Gurevich notes that if $V = L$ the intersection of 7D,E holds.

Conjecture 7F: The monadic theory of order and the (pure) second-order theory, are recursive in each other.

In conjectures 7D,E,F use (*) or CH if necessary.

Conjecture 7G: If D_ℓ is a dense subset of R, and for every $P, |P \cap D_\ell| < 2^{\aleph_0}$, for $\ell = 1, 2$ then $(R, D_1), (R, D_2)$ have the same monadic theory.[18]

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