MORE RAMSEY THEORY FOR HIGHLY CONNECTED MONOCHROMATIC SUBGRAPHS

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ABSTRACT. An infinite graph is said to be highly connected if the induced subgraph on the complement of any set of vertices of smaller size is connected. We continue the study of weaker versions of Ramsey Theorem on uncountable cardinals asserting that if we color edges of the complete graph we can find a large highly connected monochromatic subgraph. In particular, several questions of Bergfalk, Hrušák and Shelah [5] are answered by showing that assuming the consistency of suitable large cardinals the following are relatively consistent with ZFC:

- $\kappa \to_{hc} (\kappa)^2_\omega$ for every regular cardinal $\kappa \geq \omega_2$,
- $\neg\mathsf{CH} + \aleph_2 \to_{hc} (\aleph_1)^2_\omega$.

Building on a work of Lambie-Hanson [14], we also show that

- $\aleph_2 \to_{hc} [\aleph_2]^2_{\omega,2}$ is consistent with $\neg\mathsf{CH}$.

To prove these results, we use the existence of ideals with strong combinatorial properties after collapsing suitable large cardinals.

1. Introduction

The paper studies weak versions of the Ramsey Theorem on uncountable cardinals. Following [5] we say that a graph $G = (X, E)$ is *highly connected* if for every $Y \subset X$ of cardinality strictly smaller than $|X|$, the subgraph $(X \setminus Y, E \cap [X \setminus Y]^2)$ is connected. Given cardinal numbers $\theta \leq \lambda \leq \kappa$,

$$\kappa \to_{hc} (\lambda)^2_\theta$$

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denotes the statement that for every $c : [\kappa]^2 \to \theta$ there is an $\xi \in \theta$ and $A \in [\kappa]^\lambda$ such that $(A, c^{-1}(\xi) \cap [A]^2)$ is highly connected, in which case we shall say that A *is highly connected in color* ξ.

The original motivation for studying this partition relation came from the study of the higher derived limits in forcing extensions (see [6, 4, 2] and [17]) and is related in spirit to the *partition hypotheses* of [2] and another weak Ramsey property concerning the so-called *topological* K_κ (first studied by Erdős and Hajnal in [8]) considered by Komjáth and Shelah in [13].

This paper continues the study initiated in [5] and further investigated in [3, 14] as to for which $\theta < \lambda \leq \kappa$ the $\to_{hc}$ arrow holds. Among the facts proved in [5] are:

- For every infinite cardinal κ and a natural number n, $\kappa \to_{hc} (\kappa)^2_n$,
- If $\kappa \leq 2^\theta$ then $\kappa \not\to_{hc} (\kappa)^2_\theta$, and
- If $\lambda = \lambda^\theta$ then $\lambda^+ \to_{hc} (\lambda)^2_\theta$,

in particular, $2^\omega \not\to_{hc} (2^\omega)^2_\omega$. On the other hand, it was shown there that assuming the existence of a weakly compact cardinal, consistently $2^{\omega_1} \to_{hc} (2^{\omega_1})^2_\omega$. However, in that model, 2^{ω_1} is weakly inaccessible, being a former large cardinal in a forcing extension by a poset satisfying small chain condition.

In light of the results mentioned above, following natural questions were raised in [5] and [3]:

Question 1.1.

(1) Is it consistent that $\kappa \to_{hc} (\kappa)^2_\omega$ holds for an accessible κ? For example, when κ is $\aleph_2$ or $\aleph_{\omega+1}$?
(2) Is $\aleph_2 \to_{hc} (\aleph_1)^2_\omega$ equivalent to the Continuum Hypothesis?

Note that by a result of Lambie-Hanson [14], if $\kappa \to_{hc} (\kappa)^2_\omega$ holds, then $\square(\kappa)$ necessarily fails. In particular, if $\aleph_2 \to_{hc} (\aleph_2)^2_\omega$ were to hold, we will need to use at least a weakly compact cardinal and if $\aleph_{\omega+1} \to_{hc} (\aleph_{\omega+1})^2_\omega$ were to hold, then we will need to use significantly stronger large cardinals.

Here we answer the first question in the positive and the second question in the negative by analyzing *remnants of large cardinal properties* on smaller cardinals after suitable forcing, often the Levy or Mitchell collapse, in the form of existence of ideals having strong combinatorial properties.

We finish this section with a few more definitions and notations. Let κ be a regular uncountable cardinal

Definition 1.2. Fix $k \in \omega$. We let

$$\kappa \to [\kappa]^2_{\omega,k}$$

to abbreviate that for any $c : [\kappa]^2 \to \omega$, there exist $H \in [\omega_2]^{\aleph_2}$ and $K \in [\omega]^k$ such that $(H, c^{-1}(K)) \cap [H]^2)$ is highly connected.

The following is a more refined variation of the highly connected partition relations, conditioned on the lengths of the paths.

Definition 1.3. Fix $n \in \omega \cup \{\omega\}$. Let $\kappa \to_{hc,<n} (\kappa)^2_\omega$ abbreviate $\kappa \to_{hc} (\kappa)^2_\omega$ via paths of length $< n$. More precisely, it asserts: for any $c : [\kappa]^2 \to \omega$, there exists $A \in [\kappa]^\kappa$ and $i \in \omega$ such that for any $C \in [A]^{<\kappa}$ and $\alpha, \beta \in A - C$, there exist $l < n$ and a path $\langle \gamma_k : k < l+1 \rangle \subset A - C$ with $\gamma_0 = \alpha$ and $\gamma_l = \beta$ such that for all $j < l$, $c(\gamma_j, \gamma_{j+1}) = i$.

The organization of the paper is

(1) In Section 2, we establish the consistency of $\aleph_2 \to_{hc} (\aleph_1)^2_\omega$ and $\neg\mathsf{CH}$,
(2) in Section 3, we isolate and investigate 2-precipitous ideals (see Definition 3.1) whose existence implies $\kappa \to_{hc} [\kappa]^2_{\omega,2}$,
(3) in Section 4, we demonstrate two methods of constructing 2-precipitous ideals and show that $\aleph_2 \to_{hc} [\aleph_2]^2_{\omega,2} + 2^\omega \geq \omega_2$ is consistent,
(4) in Section 5, we prove the consistency of $\aleph_2 \to_{hc} (\aleph_2)^2_\omega$,
(5) in Section 6, we sketch how to use large cardinals to establish the consistency of the ideal hypothesis used in Section 5,
(6) in Section 7, we show we cannot improve the result in Section 6 by making the lengths of the paths required to connect vertices shorter,
(7) finally in Section 8, we finish with some open questions.

2. The consistency of $\aleph_2 \to_{hc} (\aleph_1)^2_\omega + \neg\mathsf{CH}$

We call an ideal I on ω_1 is $\aleph_1$-*proper with respect to* $S \subset P_{\aleph_2}(H(\theta))$ where θ is a large enough regular cardinal if for any $M \in S$ and $X \in M \cap I^+$, there exists an extension $Y \subset_I X$ such that Y is (M, I^+)-generic. Namely, for any dense $D \subset I^+$ with $D \in M$ and any $Y' \subset_I Y$, there exists $X \in D \cap M$ such that $Y' \cap X \in I^+$.

Lemma 2.1. *If I is $\aleph_1$-proper with respect to $\{M\}$ with $M \prec H(\theta)$ of size $\aleph_1$ containing I then whenever $Y \in I^+$ is a (M, I^+)-generic condition, the following holds:*
for any $E \subset I^+$ in M, if there is some $Y' \in E$ such that $Y \subset_I Y'$, then there exists $X \in E \cap M$ such that $Y \cap X \in I^+$.

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Proof. Define a dense subset $D_E \subset I^+$ in M as follows: $A \in D_E$ iff either there exists $B \in E$, $A \subset_I B$ or for all $B \in E$, $A \cap B =_I \emptyset$. By the hypothesis, there exists $X' \in D_E \cap M$ such that $Y \cap X' \in I^+$. Note that there exists $X \in E$ such that $X' \subset_I X$ since $X' \cap Y' \neq_I \emptyset$ and $Y \subset_I Y' \in E$. The elementarity of M guarantees the existence of such $X \in M$. □

If I is $\aleph_2$-saturated then I is $\aleph_1$-proper with respect to a closed unbounded subset of $P_{\aleph_2}(H(\theta))$ for sufficiently large θ. There are many models where ω_1 carries a σ-complete $\aleph_2$-saturated ideal and CH fails. For example, they are both consequences of Martin's Maximum [9].

Proposition 2.2. *If there exists a σ-complete $\aleph_1$-proper ideal on ω_1 with respect to a stationary subset of $\{X \in P_{\aleph_2}(H(\theta)) : \sup X \cap \omega_2 \in \mathrm{cof}(\omega_1)\}$ for some large enough θ, then $\aleph_2 \rightarrow_{hc} (\aleph_1)^2_\omega$.* [1]

Proof. Given $c : [\omega_2]^2 \rightarrow \omega$, we will find $A \in [\omega_1]^{\aleph_1}$ and $B \in [\omega_2 - \omega_1]^{\aleph_1}$ satisfying the following properties: there exists $k \in \omega$ such that

(1) for any $\alpha_0, \alpha_1 \in A$, there are uncountably many $\beta \in B$ such that $c(\alpha_0, \beta) = k = c(\alpha_1, \beta)$, and
(2) for any $\beta_0 \in B$, there are uncountably many $\alpha \in A$ satisfies that $c(\alpha, \beta_0) = k$.

Claim 2.3. $A \cup B$ is highly connected witnessed by k.

Proof of the claim. Let C be the countable set of vertices being removed. If $\alpha_0, \alpha_1 \in A$, then it follows immediately by the first requirement that there is some $\beta \in B - C$ such that $c(\alpha_0, \beta) = c(\alpha_1, \beta) = k$.

Let us check the case when $\alpha \in A$ and $\beta \in B$. We can find some large enough $\alpha' \in A - C$ such that $c(\alpha', \beta) = k$. After that we find some $\beta' \in B - C$ such that $c(\alpha', \beta') = k = c(\alpha, \beta')$. Then α is connected to β via the k-path: $\alpha, \beta', \alpha', \beta$.

If $\beta_0, \beta_1 \in B$, then we can easily reduce to the previous case by finding some large enough $\alpha_0 \in A$ such that $c(\alpha_0, \beta_0) = k$. Apply the previous analysis to $\alpha_0 \in A, \beta_1 \in B$. □

We proceed to find A, B, k as above. For each $\alpha \in \omega_2 - \omega_1$ and $i \in \omega$, let

$$X_{\alpha,i} = \{\gamma \in \omega_1 : c(\gamma, \alpha) = i\}.$$

Let $M \prec H(\theta)$ of size $\aleph_1$ contain I, c with $\sup M \cap \omega_2 \in \mathrm{cof}(\omega_1)$ and let $Y \in I^+$ be (M, I^+)-generic. Let $\rho \in \omega_2 - \sup M \cap \omega_2$. By the

[1]Originally we used the hypothesis that ω_1 carries a countably complete $\aleph_2$-Knaster ideal. Stevo Todorcevic pointed out that our proof should work from a weaker saturation hypothesis, such as $\aleph_2$-saturation.

σ-completeness of I, find some $k \in \omega$ such that $A = Y \cap X_{\rho,k} \in I^+$, which is still (M, I^+)-generic. Finally, let us define B recursively. Let $\langle a^i = (a^i_0, a^i_1) : i < \omega_1 \rangle$ enumerate $[A]^2$ with unbounded repetitions. Suppose we have defined $\langle \beta_j : j < \alpha \rangle$ for some $\alpha < \omega_1$ satisfying that for all $j < \alpha$,

(1) $a^j \subset X_{\beta_j,k}$ and
(2) $X_{\beta_j,k} \cap A \in I^+$.

It is clear if the construction is good for all $\alpha < \omega_1$, then $A, B = \{\beta_j : j < \omega_1\}$ and k are as desired.

Suppose we are at the α-th step of the construction and let us find β_α maintaining the same requirements. Let $\bar{\beta} = \min M - \sup_{j<\alpha} \beta_j < \sup M \cap \omega_2$. Consider $E = \{X_{\beta,k} \in I^+ : a^\alpha_0, a^\alpha_1 \in X_{\beta,k}, \beta > \bar{\beta}\}$. In particular, $E \in M$ and $A \subset_I X_{\rho,k} \in E$. By Lemma 2.1, there is $X_{\beta_\alpha,k} \in M \cap E$ such that $A \cap X_{\beta_\alpha,k} \in I^+$. $\square$

3. 2-precipitous ideals on κ and $\kappa \rightarrow_{hc} [\kappa]^2_{\omega,2}$

Chris Lambie-Hanson [14] showed that adding weakly compact many Cohen reals forces that $2^\omega \rightarrow_{hc} [2^\omega]^2_{\omega,2}$, in contrast with the ZFC fact that $2^\omega \not\rightarrow_{hc} (2^\omega)^2_\omega$. He also demonstrated that such partition relations already have non-trivial consistency strength, by showing that $\square(\kappa)$ implies $\kappa \not\rightarrow_{hc} [\kappa]^2_{\omega,<\omega}$.

In this section, we investigate the ideal hypothesis on κ that implies $\kappa \rightarrow_{hc} [\kappa]^2_{\omega,2}$. In particular, such analysis enables us to have more consistent scenarios, such as a model where $2^\omega \geq \omega_2$ and $\aleph_2 \rightarrow_{hc} [\aleph_2]^2_{\omega,2}$ both hold.

Definition 3.1. We say an ideal I on κ is *2-precipitous* if Player Empty does not have a winning strategy in the following game G_I with perfect information: Player Empty and Nonempty take turns playing a $\subset_I$-decreasing sequence of pairs of I-positive sets $\langle (A_n, B_n) : n \in \omega \rangle$ with Player Empty starting the game. Player Nonempty wins iff there exist $\alpha < \beta$ with $\alpha \in \bigcap_{n\in\omega} A_n$ and $\beta \in \bigcap_{n\in\omega} B_n$.

Lemma 3.2. *Fix a dense subset $D \subset P(\kappa)/I$. Player Empty has a winning strategy in G_I iff Player Empty has a winning strategy σ in G_I such that range$(\sigma) \subset \{A - M : A \in D, M \in I\}$.*

Proof. Let us show the nontrivial direction $(\rightarrow)$. Fix a winning strategy σ of Player Empty. The input of σ will be $(I^+ \times I^+)^{<\omega}$, corresponding to the sequence of positive sets Player Nonempty has played so far. Let $\pi : I^+ \rightarrow I^+$ be a map such that $\pi(B) = A - M$ where $(A, M) \in D \times I$ is least (with respect to some fixed well ordering) such that $A - M \subset B$.

Such π exists since D is dense in $P(\kappa)/I$. Consider $\sigma' = \pi \circ \sigma$. Clearly, the range of $\sigma' \subset \{A - M : A \in D, M \in I\}$. To see that it is a winning strategy for Player Empty, Suppose $\langle (A_n, B_n) : n \in \omega \rangle$ is a play such that Player Empty plays according to σ'. Notice that $\langle (A'_n, B'_n) : n \in \omega \rangle$, where $(A'_n, B'_n) = (A_n, B_n)$ when n is odd and $(A'_n, B'_n) = \sigma(\langle (A_{2k-1}, B_{2k-1}) : 2k-1 < n \rangle)$ is a legal play where Player Empty is playing according to σ. As a result, there do not exist $\alpha < \beta$ such that $\alpha \in \bigcap_{n \in \omega} A'_n = \bigcap_{n \in \omega} A_n$ and $\beta \in \bigcap_{n \in \omega} B'_n = \bigcap_{n \in \omega} B_n$. Therefore, σ' is a winning strategy for Player Empty. □

Theorem 3.3. *If κ carries a uniform normal 2-precipitous ideal, then $\kappa \rightarrow_{hc} [\kappa]^2_{\omega,2}$.*

Proof. Fix a normal uniform 2-precipitous ideal I on κ and a coloring $c : [\kappa]^2 \to \omega$. We say a pair of I-positive sets (B_0, B_1) is *(i,j)-frequent* if for any I-positive sets $B'_0 \subset B_0$, $B'_1 \subset B_1$, there are

- $\alpha < \beta$ with $\alpha \in B'_0, \beta \in B'_1$ such that $c(\alpha, \beta) = i$ and
- $\beta' < \alpha'$ with $\beta' \in B'_1$, $\alpha' \in B'_0$ such that $c(\beta', \alpha') = j$.

Claim 3.4. There exists a pair of I-positive sets (B_0, B_1) and $i, j \in \omega$ such that (B_0, B_1) is (i,j)-frequent.

Proof of the Claim. Starting with a positive pair (A_0, A_1), we find some $i \in \omega$ and positive $(C_0, C_1) \subset (A_0, A_1)$ such that (C_0, C_1) satisfies the first requirement of the (i,j)-frequent-ness, namely, for all positive sets $C'_0 \subset C_0, C'_1 \subset C_1$ there are $(\alpha, \beta) \in C'_0 \otimes C'_1$[2] satisfies that $c(\alpha, \beta) = i$. Suppose for the sake of contradiction that such (C_0, C_1) and i do not exist. We define a strategy σ for Player Empty: they start by playing $(A^0, B^0) =_{def} (A_0, A_1)$. At stage $2i$, with the game played so far is $\langle (A^k, B^k) : k < 2i \rangle$, by the hypothesis, there are positive $(A', B') \subset (A^{2i-1}, B^{2i-1})$ such that no $(\alpha, \beta) \in A' \otimes B'$ satisfy $c(\alpha, \beta) = i$. Player Empty then plays $(A^{2i}, B^{2i}) = (A', B')$. Since by the hypothesis of I, Player Empty does not have a winning strategy. Therefore, there is a play $\langle (A^n, B^n) : n \in \omega \rangle$ where Player Empty plays according to the strategy σ but in the end, there are $(\alpha, \beta) \in \bigcap_{n \in \omega} A^n \otimes \bigcap_{n \in \omega} B^n$. However, if $c(\alpha, \beta) = k$, then at stage $2k$, the strategy of Empty makes sure $c'' A^{2k} \otimes B^{2k}$ omits $\{k\}$, which is a contradiction.

Finally, we repeat the previous argument with input (C_1, C_0) in place of (A_0, A_1) to find positive $(B_1, B_0) \subset (C_1, C_0)$ and $j \in \omega$ satisfying the second part of the (i,j)-frequentness, as desired. □

[2]Given A, B two sets of ordinals we let $A \otimes B = \{(\alpha, \beta) \in A \times B : \alpha < \beta\}$.

Fix (B_0, B_1) that is (i,j)-frequent. We strengthen the conclusion using the normality of I. Recall for any positive $S \in I^+$, $I^* \restriction S$ denote the dual filter of I restricted on S.

Claim 3.5. For any I-positive $B_0' \subset B_0, B_1' \subset B_1$,

- $\{\alpha \in B_0 : \{\beta \in B_1' : c(\alpha,\beta) = i\} \in I^+\} \in I^* \restriction B_0$,
- $\{\beta' \in B_1 : \{\alpha' \in B_0' : c(\beta',\alpha') = j\} \in I^+\} \in I^* \restriction B_1$.

Proof of the claim. Let us just show the first part. The proof of the second part is identical. Suppose for the sake of contradiction, $B^0 =_{def} \{\alpha \in B_0 : B^1_\alpha =_{def} \{\beta \in B_1' : c(\alpha,\beta) = i\} \in I\} \in I^+$. Since I is normal, $B^1 = \bigtriangledown_{\alpha \in B^0} B^1_\alpha \in I$. Applying the assumption that (B_0, B_1) is (i,j)-frequent to B^0 and $B_1' - B^1$, we get $(\alpha,\beta) \in B^0 \otimes (B_1' - B^1)$ such that $c(\alpha,\beta) = i$. However, by the definition of B^1, $\beta \in B^1$, which is a contradiction. □

Applying Claim 3.5, we find $B_0^* \in I^* \restriction B_0, B_1^* \in I^* \restriction B_1$ be such that

(1) for any $\alpha \in B_0^*$, $\{\beta \in B_1^* : c(\alpha,\beta) = i\} \in I^+$ and
(2) for any $\beta' \in B_1^*$, $\{\alpha' \in B_0^* : c(\beta',\alpha') = j\} \in I^+$.

Let us check that $(B_0^* \cup B_1^*, c^{-1}(\{i,j\}))$ is a highly connected subgraph of size $\aleph_2$. Given $C \in [B_0^* \cup B_1^*]^{\leq \aleph_1}$, $\alpha, \beta \in B_0^* \cup B_1^* - C$, we need to find an (i,j)-valued path connecting them in $B_0^* \cup B_1^* - C$. Consider the following cases.

- $\alpha \in B_0^*, \beta \in B_1^*$: let $A_\alpha = \{\gamma \in B_1^* : c(\alpha,\gamma) = i\} \in I^+$ and let $B_\beta = \{\eta \in B_0^* : c(\beta,\eta) = j\} \in I^+$. Since (B_0^*, B_1^*) is (i,j)-frequent, we can find $(\gamma,\eta) \in (A_\alpha - C) \otimes (B_\beta - C)$ such that $c(\gamma,\eta) = j$. Then the path α,γ,η,β is as desired.
- $\alpha, \beta \in B_0^*$ or $\alpha, \beta \in B_1^*$: we can reduce to the previous case by moving either α or β to the other side using an edge of c-color either i or j.

□

4. The consistency of the existence of a 2-precipitous ideal

In this section we discuss two forcing constructions for a 2-precipitous ideal on κ. The first is cardinal preserving and the second involves collapsing cardinals. First let us record some characterizations of 2-precipitous ideals analogous to those of precipitous ideals in [12].

Definition 4.1. *A tree T of maximal antichains* of $P(\kappa)/I \times P(\kappa)/I$ is a sequence of maximal antichains $\langle \mathcal{A}_n : n \in \omega \rangle$ of $P(\kappa)/I \times P(\kappa)/I$ such that $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$ for each $n \in \omega$. A branch through T is a decreasing sequence of conditions $\langle b_n : n \in \omega \rangle$ such that $b_n \in \mathcal{A}_n$.

The proof by Jech and Prikry [12] (see also [10, Proposition 2.7]) essentially gives the following.

Theorem 4.2 ([12]). *I is 2-precipitous if for any pair of positive sets (C_0, C_1) and a tree T of maximal antichains $\langle \mathcal{A}_n : n \in \omega \rangle$ below (C_0, C_1), there exists a sequence $\langle (A_n, B_n) : n \in \omega \rangle$ such that*

1. *$\langle (A_n, B_n) : n \in \omega \rangle$ is a branch through the tree T, and*
2. *there exist $\alpha < \beta$ such that $\alpha \in \bigcap_{n\in\omega} A_n$ and $\beta \in \bigcap_{n\in\omega} B_n$.*

Remark 4.3. *Suppose we are given a dense subset $D \subset P(\kappa)/I$. By Lemma 3.2, it is without loss of generality to assume $\{(C_0, C_1)\} \cup \bigcup_{n\in\omega} \mathcal{A}_n \subset \{A - M : A \in D, M \in I\} \times \{A - M : A \in D, M \in I\}$.*

For the rest of this section, we will apply Remark 4.3 liberally. Also it turns out that suppressing the I part does not affect the reasoning. Therefore, to avoid cumbersome notations, we will further assume that $\mathcal{A}_n \subset D \times D$ for all $n \in \omega$. Given a partial order $\mathbb{P}$, we denote the complete Boolean algebra generated by $\mathbb{P}$ as $\mathbb{B}(\mathbb{P})$.

Proposition 4.4. *If I is a κ-complete normal ideal on κ and $P(\kappa)/I \simeq \mathbb{B}(Add(\omega, \lambda))$ for some λ, then I is 2-precipitous.*

Proof. Let $\pi : \mathbb{B}(Add(\omega, \lambda)) \to P(\kappa)/I$ be an isomorphism. For each $r \in Add(\omega, \lambda)$, let $X_r = \pi(r)$. Here we identify $Add(\omega, \lambda)$ as a dense subset of $\mathbb{B}(Add(\omega, \lambda))$, $D = \{X_r : r \in Add(\omega, \lambda)\}$.

Suppose for the sake of contradiction, I is not 2-precipitous. By Theorem 4.2, we can find the witnessing (C_0, C_1) and a tree T of maximal antichains below (C_0, C_1). Note that since $P(\kappa)/I \times P(\kappa)/I$ satisfies c.c.c, each $\mathcal{A}_n \subset D \times D$ is countable. Find $r_0, r_1 \in Add(\omega, \lambda)$ such that $C_i = X_{r_i}$ for $i < 2$.

Let $G \subset P(\kappa)/I$ be generic containing C_0. Then in $V[G]$, there is a generic elementary embedding $j : V \to M$ which can be taken to be the ultrapower embedding with respect to the added generic V-ultrafilter extending the dual filter of I.

Consider $T'_n = \{B^* : \exists (A^*, B^*) \in j(\mathcal{A}_n), \kappa \in A^*\}$. Note that $\langle T'_n : n \in \omega \rangle \in M$ since $V[G] \models {}^\omega M \subset M$. Note that $T'_n \subset j''V$. This follows from the fact that each $\mathcal{A}_n$ is countable, hence $j(\mathcal{A}_n) = j''\mathcal{A}_n$.

Claim 4.5. In M, T'_n is a maximal antichain below $j(C_1)$ for the poset $j(P(\kappa)/I)$.

Proof of the claim. Suppose not. By the product lemma, $\mathcal{B} = \{B : \exists (A, B) \in \mathcal{A}_n, A \in G\}$ is a maximal antichain for $(P(\kappa)/I)^V \simeq \mathbb{B}(Add(\omega, \lambda))$ below X_{r_1} in $V[G]$. We can enumerate

$$\mathcal{B} = \langle X_{p_n} : n \in \omega, r_n \in Add(\omega, \lambda) \rangle.$$

In particular, $\langle p_n : n \in \omega\rangle$ is a maximal antichain for $Add(\omega,\lambda)$ below r_1 in $V[G]$.

If $\langle j(X_{p_n}) : n \in \omega\rangle$ is not a maximal antichain in $j(P(\kappa)/I) \simeq j(\mathbb{B}(Add(\omega,\lambda)))$, then there exists a condition $r \in Add(\omega, j(\lambda))$ such that $X^*_r =_{\mathrm{def}} j(\pi)(r)$ is incompatible with any condition in the set $\{j(X_{p_n}) : n \in \omega\}$. This means r is incompatible with any condition in $\{j(p_n) : n \in \omega\}$. Since $j(p_n) = j''p_n$, we may assume $r \in Add(\omega, j''\lambda)$. Let $r^* = j^{-1}(r)$. Then $r^* \in Add(\omega,\lambda)/r_1$ is incompatible with any condition in $\{p_n : n \in \omega\}$. This contradicts with the fact that $\langle p_n : n \in \omega$ is a maximal antichain subset of $Add(\omega,\lambda)$ below r_1 in $V[G]$. $\square$

Let $H \subset j(P(\kappa)/I)$ be generic over $V[G]$ containing $j(D)$. Since $M \models j(P(\kappa)/I)$ is a $j(\kappa)$-complete and $\aleph_1$-saturated ideal, H gives rise an ultrapower embedding $k : M \to N$ with critical point $j(\kappa)$. Consider $b = \{(A_n, B_n) : (\kappa, j(\kappa)) \in k(j(A_n)) \times k(j(B_n))\}$. By Claim 4.8, H meets T'_n for all $n \in \omega$. As a result, $k \circ j(b) = k \circ j''b$ is a branch $\langle (A^*_n, B^*_n) : n \in \omega\rangle$ through $k(j(T))$ in $V[G*H]$ with $(\kappa, j(\kappa)) \in \bigcap_{n\in\omega} A^*_n \otimes \bigcup_{n\in\omega} B^*_n$. Since N is well-founded, there is such a branch in N. By the elementarity of $k\circ j$, T has a branch $\langle (A_n, B_n) : n \in \omega\rangle$ in V satisfying that there are $\alpha < \beta$ such that $\alpha \in \bigcap_{n\in\omega} A_n$ and $\beta \in \bigcap_{n\in\omega} B_n$. $\square$

It is easy to see that if $P(\kappa)/I$ has a σ-closed dense subset, then I is 2-precipitous. However, in this case, it is necessary that $2^\omega < \kappa$.

The second construction gives a scenario where κ is a small uncountable cardinal (like $\aleph_2$) as well as CH fails. In particular, such ideal can be constructed using the Mitchell collapse [16]. [3]

Recall the Mithcell forcing from [16] (the representation of the forcing here is due to Abraham, see [1] and [7, Section 23]) $\mathbb{M}(\omega,\lambda)$ consists of conditions of the form (p, r) where $p \in Add(\omega,\lambda)$ and r is a function on λ of countable support such that for any $\alpha < \lambda$, $\Vdash_{Add(\omega,\alpha)} r(\alpha)$ is a condition in $Add(\omega_1, 1)$. The order is that $(p_2, r_2) \leq (p_1, r_1)$ iff $p_2 \supset p_1$, $supp(r_2) \supset supp(r_1)$, and for any $\alpha \in supp(r_1)$, $p_2 \restriction \alpha \Vdash_{Add(\omega,\alpha)} r_2(\alpha) \leq_{Add(\omega_1,1)} r_1(\alpha)$.

Define R to be the poset consisting of countably supported functions r with domain λ such that for each $\alpha \in supp(r)$, $r(\alpha)$ is an $Add(\omega,\alpha)$-name for a condition in $Add(\omega_1, 1)$. The order of R is the following: $r_1 \leq_R r_2$ iff $supp(r_1) \supset supp(r_2)$ and for any $\alpha \in supp(r_2)$, $\Vdash_{Add(\omega,\alpha)} r_1(\alpha) \leq_{Add(\omega_1,1)} r_2(\alpha)$.

The following are standard facts about this forcing (see [7]):

[3] We thank Spencer Unger for his suggestion on the relevance of the Mitchell collapse.

(1) $\mathbb{M}(\omega, \lambda)$ projects onto $Add(\omega, \lambda)$ by projecting onto the first coordinate,
(2) $Add(\omega, \lambda) \times R$ projects onto $\mathbb{M}(\omega, \lambda)$ by the identity map.

Remark 4.6. *Whenever $(p_2, r_2) \leq_{\mathbb{M}(\omega,\lambda)} (p_1, r_1)$, there exists $r_2' \in R$ with $dom(r_2') = dom(r_2)$ such that $r_2' \leq_R r_1$ and $p_2 \restriction \alpha \Vdash_{Add(\omega,\alpha)} r_2(\alpha) = r_2'(\alpha)$ for any $\alpha \in dom(r_2)$. In other words, (p_2, r_2) and (p_2, r_2') are equivalent conditions. We will use this fact freely in the following proofs.*

Note that the poset R has the property that any countable decreasing sequence has a greatest lower bound.

Proposition 4.7. *If $P(\kappa)/I \simeq \mathbb{B}(\mathbb{M}(\omega, \lambda))$ for some λ, then I is 2-precipitous.*

Proof. Let $\pi : \mathbb{M}(\omega, \lambda) \to D$ be an isomorphism where $D \subset P(\kappa)/I$ is a dense subset. For any $(p, r) \in \mathbb{M}(\omega, \lambda)$, let $X_{p,r}$ denote $\pi(p, r)$. Assume for the sake of contradiction that I is not 2-precipitous. Fix a winning strategy σ for Player Empty in the game G_I. We may assume σ satisfies the conclusion of Lemma 3.2 applied to D. To avoid cumbersome notations, we will assume for simplicity that σ outputs elements from $D \times D$. See the paragraph after Remark 4.3. We will use σ to construct a tree of antichains $T = \langle \mathcal{A}_n : n \in \omega \rangle$ below $\sigma(\emptyset) = (A, B) = (X_{p_a^{-1}, r_a^{-1}}, X_{p_b^{-1}, r_b^{-1}}) = \mathcal{A}_{-1}$ satisfying additional properties:

(1) $\mathcal{A}_{n+1}$ refines $\mathcal{A}_n$,
(2) $\mathcal{A}_n = \langle (a_i^n, b_i^n) : i < \gamma_n \rangle \subset D$ is countable (note that it is *not maximal* anymore),
(3) for each $n \in \omega, i \in \gamma_n$, there are unique $(p_{i,a}^n, r_{i,a}^n), (p_{i,b}^n, r_{i,b}^n) \in \mathbb{M}(\omega, \lambda)$ such that $X_{p_{i,a}^n, r_{i,a}^n} = a_i^n$ and $X_{p_{i,b}^n, r_{i,b}^n} = b_i^n$,
(4) for any n and $i < j \in \gamma_n$, $(r_{j,a}^n, r_{j,b}^n) \leq_{R \times R} (r_{i,a}^n, r_{i,b}^n)$,
(5) for any (r_0, r_1) lower bound for $\langle (r_{i,a}^n, r_{i,b}^n) : i \in \gamma_n \rangle$, we have that $\mathcal{A}_n \downarrow (r_0, r_1) =_{def} \{X_{p_{i,a}^n, r_0}, X_{p_{i,b}^n, r_1} : i \in \gamma_n\}$ is a maximal antichain in $(A \cap X_{\emptyset, r_0}, B \cap X_{\emptyset, r_1})$,
(6) for any n and i, j, $(r_{j,a}^{n+1}, r_{j,b}^{n+1}) \leq_{R \times R} (r_{i,a}^n, r_{i,b}^n)$,
(7) for any branch $\langle (A_n, B_n) : n \in \omega \rangle$ through T, there do not exist $\alpha < \beta$ such that $\alpha \in \bigcap_{n \in \omega} A_n$ and $\beta \in \bigcap_{n \in \omega} B_n$.

Assuming that the construction of such T is possible, let us derive a contradiction. Let (r_a, r_b) be the greatest lower bound in $R \times R$ for $\langle \langle (r_{i,a}^n, r_{i,b}^n) : i \in \gamma_n \rangle : n \in \omega \rangle$. By property (5), we know that for each n, $\mathcal{A}_n \downarrow (r_a, r_b)$ is a maximal antichain below $(X_{\emptyset, r_a} \cap A, X_{\emptyset, r_b} \cap B)$ as a subset of $(P(\kappa)/I)^V$ in $V[G]$.

Force over V to get a generic $G \subset P(\kappa)/I$ over V containing $X_{\emptyset,r_a} \cap A$. Using G, we find an elementary embedding $j : V \to M$ with critical point κ. In $V[G]$, consider T' consisting of $\mathcal{A}'_n = \{j(C) : \kappa \in j(D), (D,C) \in \mathcal{A}_n \downarrow (r_a, r_b)\}$. Notice that by property (5) and the product lemma, $\mathcal{A}^*_n = \{C : j(C) \in \mathcal{A}'_n\}$ is a maximal antichain below $X_{\emptyset,r_b} \cap B$.

Claim 4.8. For each $n \in \omega$, $\mathcal{A}'_n \subset j(P(\kappa)/I)$ is a maximal antichain below $j(B) \cap X_{\emptyset,j(r_b)}$ in $V[G]$ (or in M, since $V[G] \models {}^\omega M \subset M$).

Proof of the claim. Otherwise, we can find $(p, r^*) \in j(\mathbb{M}(\omega,\lambda))$ below $(\emptyset, j(r_b))$ such that $X^*_{p,r^*} =_{\text{def}} j(\pi)((p,r^*)) \subset j(B) \cap X_{\emptyset,j(r_b)}$ and X^*_{p,r^*} is incompatible with any element in $\mathcal{A}'_n$. By changing to an equivalent condition if necessary, we may assume that $r^* \leq_{j(R)} j(r_b)$. As a result, $p \perp_{j(Add(\omega,\lambda))} j(p^n_{k,b})$ for all $k \in \omega$. Consider $p' = j^{-1}(p) \in Add(\omega,\lambda)$. Then $p' \perp_{Add(\omega,\lambda)} p^n_{k,b}$ for all $k \in \omega$. As a result, X_{p',r_b} is incompatible with each element in $\mathcal{A}^*_n$, but $X_{p',r_b} \cap B \cap X_{\emptyset,r_b} \in I^+$, which is a contradiction to the fact that $\mathcal{A}^*_n$ is a maximal antichain below $X_{\emptyset,r_b} \cap B$. $\square$

Let $H \subset j(P(\kappa)/I)$ be generic over V containing $j(B \cap X_{\emptyset,r_b})$. Then in $V[G * H]$, we can form an elementary embedding $k : M \to N$ with critical point $j(\kappa)$. Consider $b = \{(A_n, B_n) : (\kappa, j(\kappa)) \in j(A_n) \otimes k(j(B_n)), (A_n, B_n) \in \mathcal{A}_n, n \in \omega\}$. By Claim 4.8, $k \circ j''b \in V[G * H]$ is a branch through $k(j(T))$ violating property (7) as witnessed by $(\kappa, j(\kappa))$. Since N is a well-founded inner model of $V[G*H]$, there is such a branch in N. By the elementarity of $k \circ j$, there is such a branch in V through T violating property (7), which is a contradiction.

Let us turn to the construction of T. We will construct T recursively by levels. Let $\mathcal{A}_{-1} = \sigma(\emptyset) = (A, B) = (X_{p_a^{-1}, r_a^{-1}}, X_{p_b^{-1}, r_b^{-1}})$. To avoid excessive repetitions, we will assume that all the conditions from $\mathbb{M}(\omega,\lambda) \times \mathbb{M}(\omega,\lambda)$ extend $((p_a^{-1}, r_a^{-1}), (p_b^{-1}, r_b^{-1}))$.

Let us first define $T(0) = \mathcal{A}_0$. Recursively, suppose we have defined $\mathcal{A}_{0,<\eta} = \langle (X_{p^0_{i,a}, r^0_{i,a}}, X_{p^0_{i,b}, r^0_{i,b}}) : i < \eta \rangle$ (partially) satisfying property (4). Let (t_0, t_1) be a lower bound for $\langle (r^0_{i,a}, r^0_{i,b}) : i < \eta \rangle$ in $R \times R$. If there exists $(q_0, t'_0) \leq (p_a^{-1}, t_0), (q_1, t'_1) \leq (p_b^{-1}, t_1)$ such that $(X_{q_0,t'_0}, X_{q_1,t'_1})$ is incompatible with any element in $\mathcal{A}_{0,<\eta}$, let $(Y^0_{\eta,a}, Y^0_{\eta,b})$ be one such $(X_{q_0,t'_0}, X_{q_1,t'_1})$. Then we define $(X_{p^0_{\eta,a}, r^0_{\eta,a}}, X_{p^0_{\eta,b}, r^0_{\eta,b}})$ be $\sigma(\langle \emptyset, (Y^0_{\eta,a}, Y^0_{\eta,b}) \rangle)$. Notice that this process must stop at some countable stage $\gamma_0 < \omega_1$ since $\{(p^0_{i,a}, p^0_{i,b}) : i < \gamma_0\}$ is an antichain in $Add(\omega,\lambda) \times Add(\omega,\lambda)$ below (p_a^{-1}, p_b^{-1}), which satisfies the countable chain condition. Let us verify all the properties are satisfied. Properties (1), (2), (3), (4), (6) are satisfied by the construction. Property

(7) is not relevant at this stage. Property (5) is satisfied since we only stop when the process described above cannot be continued, which is exactly saying property (5) is satisfied.

In general, the definition of $\mathcal{A}_{n+1}$ is very similar to the construction above. Basically, for each $(C_0, C_1) \in \mathcal{A}_n$, we repeat the process above with (C_0, C_1) playing the role of $\mathcal{A}_{-1}$. One difference, in order to satisfy property (6), is that at the beginning of the construction, we let (h_0, h_1) be the lower bound in $R \times R$ for $\langle (r^n_{i,a}, r^n_{i,b}) : i < \gamma_n \rangle$ and work below $((p_a^{-1}, h_0), (p_b^{-1}, h_1))$ in $\mathbb{M}(\omega, \lambda) \times \mathbb{M}(\omega, \lambda)$.

Finally, to see that property (7) is satisfied, notice that any branch b through T corresponds to a play of the game G_I where Player Empty is playing according to their winning strategy σ. More precisely, b is the sequence of sets played by Player Empty according to σ in a play of the game G_I. As a result, the winning condition of Player Empty ensures (7) is satisfied. $\square$

Corollary 4.9. *It is consistent that $\aleph_2 \rightarrow_{hc} [\aleph_2]^2_{\omega,2}$ and $2^\omega \geq \omega_2$.*

Proof. Let κ be a measurable cardinal. Then in $V^{\mathbb{M}(\omega,\kappa)}$, $2^\omega \geq \omega_2$ and there is an ideal satisfying the hypothesis of Proposition 4.7 (see for example [7, Theorem 23.2]). Apply Proposition 4.7 and Theorem 3.3. $\square$

5. σ-closed ideals and monochromatic highly connected subgraphs

In this section, we prove the following theorem.

Theorem 5.1. *Suppose a regular cardinal κ carries a countably complete uniform ideal I such that there exists a dense collection $H \subset I^+$ that is σ-closed. Then $\kappa \rightarrow_{hc} (\kappa)^2_\omega$. Moreover, $\kappa \rightarrow_{hc,<4} (\kappa)^2_\omega$ holds.*

Fix an ideal I as in the hypothesis of Theorem 5.1. It is worth comparing such ideal hypothesis with the one from the previous section:

- we do not insist that I is normal any more,
- we impose a stronger requirement that the ideal has a σ-closed dense subset (any such ideal is 2-precipitous).

Fix a coloring $c : [\kappa]^2 \rightarrow \omega$. Given $B_0, B_1 \in I^+$ and $i \in \omega$, we say (B_0, B_1) *is i-frequent* if for any positive $B_0' \subset B_0$ and positive $B_1' \subset B_1$, it is the case that $\{\alpha \in B_0' : \{\beta \in B_1' : c(\alpha, \beta) = i\} \in I^+\} \in I^+$.

Remark 5.2. *Equivalently, (B_0, B_1) is i-frequent if for any positive $B_1' \subset B_1$, it is the case that $\{\alpha \in B_0 : \{\beta \in B_1' : c(\alpha, \beta) = i\} \in I^+\} \in I^* \restriction B_0$.*

Claim 5.3. There exists a positive $B \in I^+$ and $i \in \omega$ such that for any positive $B' \subset B$, there are positive $B_0, B_1 \subset B'$ such that (B_0, B_1) is i-frequent.

Proof. In the following proof, to avoid repetitions, whenever we mention a positive set, we implicitly assume the positive set in the σ-closed dense collection H.

Suppose otherwise for the sake of contradiction. For $A \in I^+$ and $j \in \omega$, let $(*)_{A,j}$ abbreviate the assertion: there are positive sets $B_0, B_1 \subset A$ such that (B_0, B_1) is j-frequent. By the hypothesis, we can recursively define $\langle B'_k \in I^+ : k \in \omega \rangle$ such that

- $B'_0 = \kappa$,
- for any $k \in \omega$, $B'_{k+1} \subset B'_k$ and $\neg(*)_{B'_{k+1},k}$.

Let $B' = \bigcap_{k\in\omega} B'_k$. By the σ-closure of I, we have that $B' \in I^+$. Then it satisfies that for any $i \in \omega$, such that there are no positive $B_0, B_1 \subset B'$ such that (B_0, B_1) is i-frequent.

Recursively construct an ω-sequence of pairs of I-positive sets $\langle (C_k, D_k) : k \in \omega \rangle$ as follows: start with $(B', B') = (C_{-1}, D_{-1})$, since it is not 0-frequent, there are positive (C_0, D_0) such that for all $\alpha \in C_0$, $\{\beta \in D_0 : c(\alpha, \beta) = 0\} \in I$. In general, as (C_i, D_i) is not i+1-frequent, we can find (C_{i+1}, D_{i+1}) such that for all $\alpha \in C_0$, $\{\beta \in D_0 : c(\alpha, \beta) = i + 1\} \in I$. Let $C^* = \bigcap_{i\in\omega} C_i$ and $D^* = \bigcap_{i\in\omega} D_i$. By the σ-closure of I, both C^* and D^* are in I^+. By the σ-completeness of the ideal, we can find some i and $\alpha \in C^*$ such that $\{\beta \in D^* : c(\alpha, \beta) = i\} \in I^+$. However, this contradicts that $\alpha \in C_i$. $\square$

To finish the proof of Theorem 5.1, by repeatedly applying Claim 5.3, we can find $\langle B_n \in I^+ : n \in \omega \rangle$ and $i \in \omega$ such that for any $n < k$, (B_n, B_k) is i-frequent.

Given $n \in \omega$, let $B_n^* \subset B_n$ be the collection of $\alpha \in B_n$ satisfying that for any $k > n$, $\{\beta \in B_k : c(\alpha, \beta) = i\} \in I^+$. Notice that $B_n^* =_I B_n$ by Remark 5.2 and the fact that I is σ-complete.

We claim that $B = \bigcup_{n\in\omega} B_n^*$ is highly connected witnessed by i. Given $\alpha < \beta \in B$ and $C \in [B]^{<\kappa}$, there must be some $n_0, n_1 \in \omega$ such that $\alpha \in B_{n_0}^*$ and $\beta \in B_{n_1}^*$. Find $k > \max\{n_0, n_1\}$. By the hypothesis, $C_0 = \{\gamma \in B_k^* : c(\alpha, \gamma) = i\} \in I^+$ and $C_1 = \{\gamma \in B_{k+1}^* : c(\beta, \gamma) = i\} \in I^+$. As (B_k, B_{k+1}) is i-frequent, we can find $\gamma_0 \in C_0 - C$ and $\gamma_1 \in C_1 - C$ such that $c(\gamma_0, \gamma_1) = i$. As a result, $\alpha, \gamma_0, \gamma_1, \beta$ is the required path of color i.

6. Remarks on the consistency of the ideal hypothesis

For regular cardinal $\lambda \geq \kappa$, if κ be λ-supercompact, we show that in $V^{Coll(\omega_1,<\kappa)}$, λ carries a κ-complete uniform ideal which admits a dense and σ-close collection of positive sets. The construction is due to Galvin, Jech, Magidor [11] and independently Laver [15]. We supply a proof for the sake of completeness.

Let U be a fine normal ultrafilter on $P_\kappa\lambda$. By a theorem of Solovay (see [18, Theorem 14] for a proof), there exists $B \in U$ such that $a \in B \mapsto \sup a$ is injective. Let $j : V \to M \simeq Ult(V, U)$. Let $\delta = \sup j''\lambda$. Let $G \subset Coll(\omega_1, <\kappa)$ be generic over V. It is well-known that if $H \subset Coll(\omega_1, [\kappa, j(\kappa)))$ is generic over $V[G]$, then we can lift j to $j^+ : V[G] \to M[G * H]$ in $V[G * H]$.

In $V[G]$, consider the ideal

$$I = \{X \subset \lambda : \Vdash_{Coll(\omega_1,[\kappa,<j(\kappa)))} \delta \notin j^+(X)\}.$$

The fact that I is κ-complete and uniform is immediate. Let us show that there exists a dense σ-closed collection of positive sets.

For each $r \in Coll(\omega_1, < j(\kappa))^M/G$, there exists a function $f_r : B \to P$ such that $j(f_r)(j''\lambda) = r$. Define $X_r = \{\sup a : a \in B, f_r(a) \in G\}$.

Claim 6.1. $X_r \in I^+$.

Proof. It suffices to check that $r \Vdash \delta \in j^+(X_r)$. Let $H \subset Coll(\omega_1, < j(\kappa))^M/G$ containing r be generic over $V[G]$, then we can lift j to $j^+ : V[G] \to M[H]$. In particular, $H = j^+(G)$. Since $j(f_r)(j''\lambda) = r \in j^+(G)$, we have that $\delta \in j^+(X_r)$. $\square$

Claim 6.2. $X_r \subset_I X_{r'}$ iff $r \leq_{Coll(\omega_1,<j(\kappa))^M/G} r'$.

Proof. If $r \leq r'$, then it is clear that $X_r \subset_I X_{r'}$. For the other direction, suppose for the sake of contradiction that $r \not\leq_{Coll(\omega_1,<j(\kappa))^M/G} r'$. In particular, there is some extension r^* of r that is incompatible with r'. Then $r^* \Vdash \delta \in j^+(X_r) - j^+(X_{r'})$. Hence, $X_r \not\subset_I X_{r'}$. $\square$

As a result,

$$\{X_r : r \in Coll(\omega_1, < j(\kappa))^M/G\}$$

is σ-closed, since $Coll(\omega_1, < j(\kappa))^M/G$ is σ-closed in $V[G]$.

Claim 6.3. For any $X \in I^+$, there exists some r such that $X_r \subset_I X$.

Proof. Let $r \in Coll(\omega_1, < j(\kappa))^M/G$ force that $\delta \in j^+(X)$. We show that $X_r \subset_I X$. Otherwise, there is some r' forcing that $\delta \in j^+(X_r) - j^+(X)$. In particular, r' forces that $j(f_r)(j''\lambda) = r \in j^+(G)$. By the separability of the forcing, $r' \leq_{Coll(\omega_1,<j(\kappa))^M/G} r$. This contradicts with the fact that r forces $\delta \in j^+(X)$. $\square$

Theorem 6.4. (1) *If κ is measurable, then in $V^{Coll(\omega_1,<\kappa)}$, $\aleph_2 \to_{hc} (\aleph_2)^2_\omega$.*
(2) *If κ is supercompact, then in $V^{Coll(\omega_1,<\kappa)}$, for all regular $\lambda \geq \aleph_2$, $\lambda \to_{hc} (\lambda)^2_\omega$.*

Some large cardinal assumption is necessary to establish the consistency of $\kappa \to_{hc} (\kappa)^2_\omega$, as shown in [14].

7. The lengths of the paths

In Section 5, we have shown that if there exists a σ-complete uniform ideal on ω_2 admitting a σ-closed collection of dense positive sets, then $\omega_2 \to_{hc,<4} (\omega_2)^2_\omega$. One natural question is whether we can improve the conclusion to $\omega_2 \to_{hc,<3} (\omega_2)^2_\omega$. In this section, we show that the answer is no, at least not from the same hypothesis.

Remark 7.1. *If there is a σ-closed forcing P such that in V^P, there is a transitive class M and an elementary embedding $j : V \to M$ with critical point κ, then $\kappa \to_{hc} (\kappa)^2_\omega$ holds. Essentially the same proof from Section 5 works.*

Theorem 7.2. *It is consistent relative to the existence of a measurable cardinal that $\aleph_2 \to_{hc,<4} (\aleph_2)^2_\omega$ but $\aleph_2 \not\to_{hc,<3} (\aleph_2)^2_\omega$.*

Proof. Let κ be a measurable cardinal. We will make use of a forcing poset $\mathbb{P}_\kappa$ due to Komjáth-Shelah [13, Theorem 7]. The final forcing will be $Coll(\omega_1,<\kappa) * \mathbb{P}_\kappa$. It follows from the work by Komjáth-Shelah that $\omega_2 \not\to_{hc,<3} (\omega_2)^2_\omega$ in the final model. Let $G \subset Coll(\omega_1,<\kappa) * \mathbb{P}_\kappa$ be generic. By Remark 7.1, it suffices to check that in a further countably closed forcing extension, there exists an elementary embedding $j : V[G] \to M$ with critical point $\omega_2^{V[G]} = \kappa$.

Towards this, let us recall the definition of $\mathbb{P}_\kappa$: conditions consist of $(S, f, \mathcal{H}, h)$ such that

- $S \in [\kappa]^{\leq\aleph_0}$,
- $f : [S]^2 \to [\omega]^\omega$,
- $\mathcal{H} \subset [S]^\omega$, $|\mathcal{H}| \leq \aleph_0$, for each $H \in \mathcal{H}$, $otp(H) = \omega$ and for any $H, H' \in \mathcal{H}$, $H \cap H'$ is finite.
- if $\alpha \in S$, $H \in \mathcal{H}$ with $\min H < \alpha$, then $|\{\beta \in H : h(H) \in f(\alpha,\beta)\}| \leq 1$.

The order is: $(S', f', \mathcal{H}', h') \leq (S, f, \mathcal{H}, h)$ iff $S' \supset S$, $\mathcal{H}' \supset \mathcal{H}$, $f' \restriction [S]^2 = f$, $h' \restriction \mathcal{H} = h$ and for any $H \in \mathcal{H}' - \mathcal{H}$, $H \not\subset S$. Note that P_κ is a countably closed forcing of size κ.

Let $j : V \to M$ witness that κ is measurable. Let $G * H \subset Coll(\omega_1,<\kappa) * \mathbb{P}_\kappa$. Let $G^* \subset Coll(\omega_1,<j(\kappa))/G*H$ be generic over $V[G*H]$. This

is possible since $Coll(\omega_1, <\kappa) * \mathbb{P}_\kappa$ regularly embeds into $Coll(\omega_1, < j(\kappa))$ with a countably closed quotient (see [7, Theorem 14.3]). As a result, we can lift $j : V[G] \to M[G^*]$. In order to lift further to $V[G * H]$, we need to force $j(P_\kappa)/H$ over $V[G^*]$. It suffices to show that in $V[G^*]$, $j(\mathbb{P}_\kappa)/H$ is countably closed.

Suppose $\langle p_n =_{def} (S_n, f_n, \mathcal{H}_n, h_n) : n \in \omega\rangle \subset j(\mathbb{P}_\kappa)/H$ is an decreasing sequence. We want to show that $q = \bigcup_{n\in\omega} p_n$ is the lower bound as desired. For this, we only need to verify that $q \in j(\mathbb{P}_\kappa)/H$. More explicitly, we need to verify that q is compatible with $h \in H$ for any $h \in H$. For each $p = (S_p, f_p, \mathcal{H}_p, h_p) \in j(P_\kappa)$, let $p \restriction \kappa$ denote the condition: $(S_p \cap \kappa, f_p \restriction [S_p \cap \kappa]^2, \mathcal{H}_p \cap P(\kappa), h_p \restriction (\mathcal{H}_p \cap P(\kappa)))$. It is not hard to check that $p \restriction \kappa \in P_\kappa$ and $p \leq_{j(\mathbb{P}_\kappa)} p \restriction \kappa$.

Claim 7.3. *$p \in j(P_\kappa)/H$ iff $p \restriction \kappa \in H$.*

Proof of the claim. If $p \restriction \kappa \in H$, to see $p \in j(P_\kappa)/H$, it suffices to see that for any $r \leq_{P_\kappa} p \restriction \kappa$ and $r \in H$, r is compatible with p. To check that $r \cup p$ can be extended to a condition, it suffices to check that for any $B \in \mathcal{H}_r - \mathcal{H}_p$, $B \not\subset S_p$ and any $B \in \mathcal{H}_p - \mathcal{H}_r$, $B \not\subset S_r$. To see the former, note that if $B \in \mathcal{H}_r - \mathcal{H}_p$, then $B \in \mathcal{H}_r - \mathcal{H}_{p\restriction\kappa}$, since $r \leq p \restriction \kappa$, $B \not\subset S_p \cap \kappa$. Since $B \subset \kappa$, we have $B \not\subset S_p$. To see the latter, $B \in \mathcal{H}_p - \mathcal{H}_r$ implies that $B \in \mathcal{H}_p - \mathcal{H}_{p\restriction\kappa}$. In particular, $B \cap [\kappa, j(\kappa)) \neq \emptyset$. Hence $B \not\subset S_r$ since $S_r \subset \kappa$.

If $p \in j(P_\kappa)/H$, then for any $h \in H$, h and p are compatible. In particular, since $p \leq p \restriction \kappa$, we know that any $h \in H$ is compatible with $p \restriction \kappa$. This implies that $p \restriction \kappa \in H$. □

To finish the proof, since for each $n \in \omega$, by Claim 7.3 and the fact that $p_n \in j(\mathbb{P}_\kappa)/H$, we have that $p_n \restriction \kappa \in H$. As a result, we must have $q \restriction \kappa \in H$. By Claim 7.3, we have $q \in j(\mathbb{P}_\kappa)/H$. □

8. Open questions

Question 8.1. Starting from the existence of a weakly compact cardinal, can one force that $\aleph_2 \to_{hc} (\aleph_2)^2_\omega$?

The proof in this paper can be adapted to a weaker assumtion of the existence of a weakly Ramsey cardinal. Recall that κ is *weakly Ramsey* if for any κ-model M, there exists a κ-complete M-ultrafilter that is weakly amenable to M, namely, any $\mathcal{F} \in M$, and $|\mathcal{F}| \leq \kappa$, $\mathcal{F} \cap U \in M$. However, being a weakly Ramsey cardinal is much stronger than being a weakly compact cardinal.

Question 8.2. Is $\aleph_2 \to_{hc,<3} (\aleph_2)^2_\omega$ consistent?

Question 8.3. Can one separate $\aleph_2 \to_{hc,<m} (\aleph_2)^2_\omega$ from $\aleph_2 \to_{hc,<n} (\aleph_2)^2_\omega$ where $4 \leq m < n \leq \omega$?

Question 8.4. Is $\aleph_2 \not\to_{hc} (\aleph_1)^2_\omega$ consistent?

The last question concerns the natural generalization of 2-precipitous ideals.

Definition 8.5. Fix a regular uncountable cardinal κ and $k \in \omega$ with $n \geq 3$. We say an ideal I on κ is *n-precipitous* if Player Empty does not have a winning strategy in the following game G^n_I with perfect information: Player Empty and Nonempty take turns playing a $\subset_I$-decreasing sequence of k-tuples of I-positive sets $\langle (A^i_n)_{i<k} : n \in \omega \rangle$ with Player Empty starting the game. Player Nonempty wins iff there exist $\alpha_0 < \alpha_1 < \cdots < \alpha_{k-1}$ such that $\alpha_i \in \bigcap_{n\in\omega} A^i_n$ for all $i < k$.

We have shown that the existence of a uniform normal 2-precipitous ideal on κ implies that $\kappa \geq \omega_2$. A natural question is whether there is a similar phenomenon for larger n:

Question 8.6. Fix $n \geq 3$. Suppose κ carries a uniform normal n-precipitous ideal. Must κ be $\geq \omega_n$?

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