BUILDING COMPLICATED INDEX MODELS AND BOOLEAN ALGEBRAS

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Abstract. We build models using an indiscernible model sub-structures of ${}^{\kappa\geq}\lambda$ and related more complicated structures. We use this to build various Boolean algebras.

Date: May 23, 2023.

2020 *Mathematics Subject Classification.* Primary: 03E05, 03C55; Secondary: 03C45.

Key words and phrases. model theory, set theory, non-structure, Boolean Algebras, rigid, endo-rigid, number of non-isomorphic, length of Boolean Algebras.

For versions up to 2019, the author thanks Alice Leonhardt for the beautiful typing. In the latest version, the author thanks an individual who wishes to remain anonymous for generously funding typing services, and thanks Matt Grimes for the careful and beautiful typing. For their partial support of this research, the author would like to thank: an NSF-BSF 2021 grant with M. Malliaris, NSF 2051825, BSF 3013005232 (2021/10-2026/09); and for various grants from the BSF (United States Israel Binational Foundation), the Israel Academy of Sciences and the NSF via Rutgers University.

This paper is number 511 in the author's publication list; it has existed (and been occasionally revised) for many years. It was mostly ready in the early nineties, and was made public to some extent. This was written as Chapter VII of the book [Shearb], which hopefully will materialize some day, but in the meantime it is [Shec]. The intention was to have [Sheh] (revising [She86]) for Ch.I, [AGSb] for Ch.II, [Shei] for Ch.III, [Sheara] for Ch.IV, [Shek] for Ch.V, [Shea] for Ch.VI, [Shec] for Ch.VII, [Sheg] (a revision of [She85]) for Ch.VIII, [Shed] for the appendix, and probably [She04], [Shee], [Shef], and [Shej]. References like [Shec, 0.1=Lx2] means that x2 is the label of Lemma 0.1 in [Shec].

The reader should note that the version in my website is usually more up-to-date than the one in the mathematical archive.

§ 0. Introduction

We continue [Shei], [Sheara], and [Sheb] (improving [She87, III]) on the one hand, and [She83] on the other. A starting idea was that the "many pairwise non-isomorphic models" proofs in Chapters VII and VIII of [She78], [She90] (and earlier [She71], [She74], [She75]) can be generalized to many contexts — in particular, to building Boolean algebras (as in [She75], [She83]).

In [Shei], [Sheb] we build the so-called "strongly unembeddable sequence of index models" $\langle I_\alpha : \alpha < \lambda\rangle$, and from there build 'many models' or 'models with few automorphisms' (or endomorphisms: e.g. for abelian groups and — our central point here — Boolean algebras) as was done earlier in [She83].

The index models were mainly linear orders and trees with $\omega+1$ levels. In this paper, we deal with generalizations. (See also [She08].)

We begin with an example that motivates our need to pass beyond the framework of trees with $\omega+1$ levels. Suppose that we are asked to construct a rigid Boolean algebra of cardinality λ. We can take a sequence $\langle I_\alpha : \alpha < \lambda\rangle$ exemplifying that $K^\omega_{\rm tr}$ has the so-called full $(\lambda, \lambda, \aleph_0, \aleph_0)$-bigness property (see [Shei, 2.5=L2.3]). (It says that each I_α is so-called "strongly unembeddable" into $\sum\big\{I_\beta : \beta \in \lambda \setminus \{\alpha\}\big\}$. These exist: e.g. λ is regular and I_α codes S_α, a stationary subset of $\{\delta < \lambda : {\rm cf}(\delta) = \aleph_0\}$, with $\langle S_\alpha : \alpha < \lambda\rangle$ pairwise disjoint.)

Now build a Boolean algebra ${\rm BA}(I_\alpha)$ for each α. We then construct a rigid Boolean algebra $\mathbf{B}_\lambda$ by choosing an increasing continuous sequence $\langle \mathbf{B}_\alpha : \alpha \le \lambda\rangle$, where $\mathbf{B}_0$ is trivial and $\mathbf{B}_{\alpha+1}$ is obtained from $\mathbf{B}_\alpha$ by "planting" a copy of ${\rm BA}(I_\alpha)$ below $a_\alpha \in \mathbf{B}_\alpha$, and our bookkeeping will ensure that $\mathbf{B}_\lambda \setminus \{0\} = \{a_\alpha : \alpha < \lambda\}$. This seems to be a reasonable strategy, and it works (see a little more below). Now suppose, moreover, that we are asked to construct a complete Boolean algebra $\mathbf{B}$ of cardinality λ with no non-trivial one-to-one endomorphism. We should assume that $\lambda^{\aleph_0} = \lambda$ (as the cardinality of any complete Boolean algebra satisfies this equality) and it is natural to demand in addition that $\mathbf{B}$ satisfies the ccc. It is not hard to modify the construction above so that $\mathbf{B}_\lambda$ has the ccc, so let $\mathbf{B}$ be its completion.

Assume toward a contradiction that $f : \mathbf{B} \to \mathbf{B}$ is a non-trivial, one-to-one endomorphism. We can find $a \in \mathbf{B} \setminus \{0\}$ with $a \cap f(a) = 0$ and $\alpha < \lambda$ such that $a = a_\alpha$. Then I_α is embedded in $\mathbf{B} \restriction a_\alpha$ in some sense, say by $\eta \mapsto a^\alpha_\eta$. Hence $\eta \mapsto f(a^\alpha_\eta)$ is a similar embedding into $\mathbf{B} \restriction f(a)$ that is constructed from $\langle I_\beta : \beta \ne \alpha\rangle$ alone. It seems reasonable that the demand "I_α strongly unembeddable into $\sum\{I_\beta : \beta \ne \alpha\}$" in the sense of Definition [Shei, 2.5=L2.3] can be used to deduce a contradiction; this works in the case above (i.e. without the completion demand). However in the present case $f(a^\alpha_\eta)$ is not in general a member of $\mathbf{B}_\lambda$, but rather is a countable union $\bigcup\limits_{n<\omega} b^\alpha_{\eta,n}$ of members of $\mathbf{B}_\lambda$. We would like to find an appropriate unembeddability condition of I_α into $\sum\limits_{\beta\ne\alpha} I_\beta$ to handle this complication. At some price, our original notion can be modified to handle this complication when η has finite length, but not when η has length ω. Instead, in this latter case, we replace it by an "approximation" $b^\alpha_{\eta,n(\alpha,\eta)} > 0$: this was part of the motivation of having the definition "strongly finitary on P^I_ω" in [Shei, 2.5=L2.3]. Previously, we could use demands like "$a^\alpha_{\eta\restriction\ell} \ge a^\alpha_\eta$" but now we have to use demands like $a^\alpha_\nu \cap a^\alpha_\eta = 0$, $\ell g(\eta) = \omega$, $\ell g(\nu) < \omega$, but such demands tend to contradict the ccc.

Our solution is to replace subtrees of ${}^{\omega\ge}\lambda$ by index sets I of the form

$$I = I' \cup \big\{(\eta \restriction n)\hat{\ }\langle\alpha_\ell\rangle : n < \omega,\ \eta \in I',\ \eta(n) = (\alpha_0, \alpha_1) \text{ and } \ell \in \{0, 1\}\big\},$$

where $I' \subseteq {}^{\omega}\{(\alpha_0, \alpha_1) : \alpha_0 < \alpha_1 < \lambda\}$, and choose $\mathrm{BA}(I)$ to be generated by $\{a^I_\eta : \eta \in I\}$ freely except that

$$\eta \in I' \wedge \eta(n) = (\alpha_0, \alpha_1) \quad \Rightarrow \quad a^I_{\eta \restriction n ^\frown \langle \alpha_0 \rangle} - a^I_{\eta \restriction n ^\frown \langle \alpha_1 \rangle} \geq a^I_\eta.$$

(Actually, to ensure the ccc it is better to use a more complicated variant.) But now the bigness properties have to be proved in this context. For other aims, we use subtrees of ${}^{\omega \geq}2$ of cardinality $\kappa \in [\aleph_1, 2^{\aleph_0})$, originally to deal with number of non-isomorphic models.

In this work we deal with more complicated index sets as motivated above.

In §1 we introduce classes like $K^\omega_{\mathrm{tr}(n)}$, which are close to being trees with $\omega + 1$ levels, together with bigness properties (related to $\psi_{\mathrm{tr}(n)}$) for them. We prove some existence theorems of the form "for many λ there is a sequence $\langle I_\alpha : \alpha < \lambda \rangle$, where each $I_\alpha \in K^\omega_{\mathrm{tr}(n)}$ has cardinality λ and is strongly $\psi_{\mathrm{tr}(n)}$-unembeddable into $\sum\limits_{\beta \neq \alpha} I_\beta$." We also define "super" versions of these bigness properties related to the ones in [Shea, 1.1=L7.1,1.5=L7.3].

In §2 we construct Boolean algebras with few appropriate morphisms for several versions.

In §3 we construct a ccc Boolean algebra of cardinality $2^{\aleph_0}$ of pre-given length (see Definition 3.3) such that any infinite homomorphic image has cardinality $2^{\aleph_0}$. We use a Boolean algebra constructed from a single $I \in K^\omega_{\mathrm{tr}(n)}$ as in §2. As it happens, the complicated $I \in K^\omega_{\mathrm{tr}(h)}$ are not needed, just non-trivial ones. Our point is that $K^\omega_{\mathrm{tr}(h)}$ is not good just for the constructions in §2, it is a quite versatile way to build structures with pre-assumed properties (not to speak of varying the index model).

The main result is (3.6):

(∗) For $\mu \in [\aleph_0, 2^{\aleph_0})$, there is a ccc Boolean algebra $\mathbf{B}$ with length μ (see Definition 3.2 below) such that every infinite homomorphic image of $\mathbf{B}$ is of cardinality $2^{\aleph_0}$.

If μ is a limit cardinal and $\mathrm{cf}(\mu) > \aleph_0$ we can demand the length is not obtained (see Definition 3.2): if $\mathrm{cf}(\mu) = \aleph_0$ this is impossible.

Also, we can replace $\aleph_0$ here by any strong limit cardinal κ of cofinality $\aleph_0$ (see 3.14).

In §4 we deal with trees of the form $S \cup {}^{\omega >}2$, where $S \subseteq {}^{\omega}2$ is of cardinality λ.

Note that §1, §2 are revised versions of parts of [She83] and parallel to [Shea], and §4 is a revised version of parts of [She89]. The results in §2 answer problems of Monk (presented in Oberwolfach 1980).

In §3, we solve a problem of Boolean algebras of Monk on which the author earlier gave a consistency result, using ideas from §2.

§4 supersedes [She78, VIII 1.8] and repeats [She89, 1.2,1.3]. Baldwin [Bal89] has continued [She89, 1.2-1.3]. We can apply this to models of $\varphi \in \mathbb{L}_{\aleph_1, \aleph_0}$, probably using [She99].

Recall that in [She78, Ch.VIII,1.8+1.7(2)], we proved that for pairs of first order complete theories (T, T_1) satisfying the hypothesis of Theorem 4.1 below

$$\dot{\mathbb{I}}(\lambda, T_1, T) \geq \min\{2^\lambda, \beth_2\}.$$

We shall improve the result replacing $\dot{\mathbb{I}}(\lambda, T_1, T)$ by $\dot{I}\dot{E}(\lambda, T_1, T)$. We improve the proof from [She78, VIII 1.8]; in particular, we use the trees U_η defined in Fact 4.9. They are subtrees of ${}^{\omega >}2$ as close to disjoint as we can manage.

We can use trees similar to $({}^{\omega \geq}2, \triangleleft)$ with finite or countable levels and heavier structure (i.e., like pure conditions in forcing notions as in [She92, §2]). As in 1.4(3), we use here a weak form of representation: the amount of similarity depends on the terms and formulas.

We can use such trees as in §2 to build "complicated," rigid-like structures. In [She80, 1.2,1.1(3)] (more in [She79, 1.4, 1.1]) this was done for abelian groups: one step is getting $\mathbb{Z} \subseteq G$ such that G is $\aleph_1$-free of cardinality $\aleph_1$, $\mathbb{Z}$ not a direct summand of G). This was continued in Göbel and Shelah [GS95] and Göbel-Shelah-Ströngmann [GSS03].

Definition 0.1. 1) We say a structure M is atomically $(<\mu)$-stable when: if $A \subseteq M$ and $|A| < \mu$ then the set $\{\mathrm{tp}_{\mathrm{qf}}(\bar{a}, A, M) : \bar{a} \in {}^{\omega>}M\}$ of possible types has cardinality $< \mu$.
2) We may write μ-stable instead of '$(< \mu^+)$-stable.'

§ 1. Trees with structure

We deal here with "relatives" of $K^\omega_{\rm tr}$ which are more complicated, strengthening our ability to carry out our constructions. The existence proofs still work, at least partially.

In this section (and the next) we define and see what we can do for $K^\omega_{\rm ptr}$, $\varphi_{\rm ptr}$, $K^\omega_{{\rm tr}(n)}$, $\varphi_{{\rm tr}(n)}$, $K^\omega_{{\rm tr}(*)}$, $\varphi_{{\rm tr}(*)}$ (which were introduced in [She83]) getting the parallel of [Shea, 2.15=L7.11]. The reason for their introduction was for constructing certain Boolean algebras; we shall deal with these constructions later.

More specifically, [Shei, 2.2=Lf5] defines versions of "I is strongly $\varphi(\bar x, \bar y)$-unembeddable into J" and "K has [full and/or strong] $(\chi, \lambda, \mu, \kappa)$-bigness," so we can apply it to $(K, \varphi) = (K^\kappa_{\rm ptr}, \psi_{\rm ptr})$, or $(K^\kappa_{{\rm tr}(h)}, \psi_{{\rm tr}(h)})$ or $(K^\kappa_{{\rm tr}(h)}, \psi'_{{\rm tr}(h)})$, as defined in Definitions 1.1,1.2 below. But below, essentially we choose more general φ-s represented by $\mathbf{e}$.

The relevant results are obtained by the existence of the super version, as in [Shea] (see Definitions 1.4,1.6).

§ 1(A). The frame.

Definition 1.1. 1) $K^\kappa_{\rm ptr}$ is the class of I such that:

(A) The set of elements of I is, for some linear order J, a subset of

$$\begin{aligned}\mathrm{set}_{{\rm tr}(h)}[J] := \big\{\eta : \ &\eta \text{ is a sequence of length} \le \kappa, \text{ such that if} \\ &i+1 < \ell g(\eta) \text{ then } \eta(i) \text{ has the form } \langle s,t\rangle \text{ with } s <_J t, \\ &\text{and if } i = \ell g(\eta)-1 \text{ then } \eta(i) \in J\big\}.\end{aligned}$$

Also, if $\eta \in I$, $i+1 < \ell g(\eta)$, and $\eta(i) = \langle s,t\rangle$ then $(\eta \restriction i)\hat{}\langle s\rangle \in I$ and $(\eta \restriction i)\hat{}\langle t\rangle \in I$. Furthermore, the empty sequence belongs to I, and if $\delta < \ell g(\eta)$ is a limit ordinal then $\eta \restriction \delta \in I$.

(B) The relations of I are:

- (α) $\eta \trianglelefteq \nu$, meaning 'η is an initial segment of ν' (i.e. $\eta = \nu \restriction \ell g(\eta)$).
- (β) $P_i = \{\eta : \ell g(\eta) = i\}$
- (γ) $<_1 = \big\{(\eta,\nu) : \ell g(\eta) = \ell g(\nu) = i+1,\ \eta(i) <_J \nu(i),\ \eta \restriction i = \nu \restriction i\big\}$
- (δ) $\mathrm{Eq}_i = \{\langle \eta,\nu\rangle : \eta \restriction i = \nu \restriction i\}$
- (ε) $\mathrm{Suc}_L = \{\langle \eta,\nu\rangle : \eta \restriction i = \nu \restriction i,\ i+1 = \ell g(\eta) < \ell g(\nu),\ \nu(i) = \langle s,t\rangle$ and $\eta(i) = s$ for some $i < \kappa$ and $s <_J t\}$
- (ζ) $\mathrm{Suc}_R = \{\langle \eta,\nu\rangle : \eta \restriction i = \nu \restriction i,\ i+1 = \ell g(\eta) < \ell g(\nu),\ \nu(i) = \langle s,t\rangle$ and $\eta(i) = t$ for some $i < \kappa$ and $s <_J t\}$
- (η) An individual constant $\langle\ \rangle$.
- (θ) Functions $\mathrm{Res}^L_\alpha, \mathrm{Res}^R_\alpha$ such that $\mathrm{Res}^L_\alpha(\eta) = (\eta \restriction \alpha)\hat{}\langle s\rangle$ and $\mathrm{Res}^R_\alpha(\eta) = (\eta \restriction \alpha)\hat{}\langle t\rangle$ when $\eta(\alpha) = \langle s,t\rangle$ and $\alpha+1 < \ell g(\eta)$, and $\mathrm{Res}^L_\alpha(\eta) = \mathrm{Res}^R_\alpha(\eta) = \eta$ otherwise.

2) Let

$$\begin{aligned}\psi_{\rm ptr}(x_0,x_1;y_0,y_1) = \bigvee_{i+1<\kappa} \big[&P_{i+1}(x_1) \wedge P_{i+1}(y_1) \wedge P_\kappa(x_0) \wedge (x_0 = y_0) \\ &\wedge \mathrm{Suc}_L(x_1,x_0) \wedge \mathrm{Suc}_R(y_1,y_0) \wedge (x_1 <_0 y_0)\big].\end{aligned}$$

This depends on κ, but we usually suppress this parameter.

3) $I \in K^\kappa_{\rm ptr}$ is standard <u>iff</u> in (1)(A), J is a set of ordinals with the natural order, or at least a well ordering (usually we shall use those).

Definition 1.2. 1) For $h : \kappa \to \omega \setminus \{0\}$, the class $K^\kappa_{\mathrm{tr}(h)}$ is defined like K^κ_{ptr}, but replacing pairs by increasing $h(i)$-tuples at level i. That is,

(A) the set of elements of I is, for some linear order J, a subset of

$$\Big\{\eta : \eta \text{ is a sequence of length } \le \kappa, \\ \text{for } i+1 < \ell g(\eta),\ \eta(i) \text{ has the form } \langle s_0, \dots, s_{h(i)-1}\rangle \\ \text{such that } s_0 <_J s_1 <_J \dots <_J s_{h(i)-1} \text{ and} \\ \text{for } i+1 = \ell g(\eta),\ \eta(i) \in J\Big\}.$$

Also, if $\eta \in I$, $i+1 < \ell g(\eta)$, $m < h(i)$ and $\eta(i) = \langle s_0, \dots, s_{h(i)-1}\rangle$ then $(\eta \restriction i)\char`^\langle s_m\rangle \in I$. Furthermore, the empty sequence belongs to I, and if $\delta < \ell g(\eta)$ is a limit ordinal then $\eta \restriction \delta \in I$.

(B) The relations of I are:

(α) $\eta \trianglelefteq \nu$, which holds iff $\eta = \nu \restriction \ell g(\eta)$.

(β) $P_i := \{\eta : \ell g(\eta) = i\}$

(γ) $<_1 := \{\langle \eta, \nu\rangle : \ell g(\eta) = \ell g(\nu) = i+1,\ \eta(i) <_J \nu(i),\ \eta \restriction i = \nu \restriction i\}$

(δ) $\mathrm{Eq}_i = \{\langle \eta, \nu\rangle : \eta \restriction i = \nu \restriction i\}$

(ε) For $m < h(i)$ and $i < \kappa$:

$$\mathrm{Suc}_{i,m} = \{\langle \eta, \nu\rangle : \eta \restriction i = \nu \restriction i,\ \ell g(\eta) = i+1, \\ \nu(i) = \langle s_0, \dots, s_{h(i)-1}\rangle, \eta(i) = s_m\}$$

(ζ) An individual constant $\langle\ \rangle$.

(θ) Functions Res^m_α such that $\mathrm{Res}^m_\alpha(\eta) = (\eta \restriction \alpha)\char`^\langle s_m\rangle$ when

$$\eta(\alpha) = \langle s_0, \dots, s_{h(\alpha)-1}\rangle,\ \alpha < \ell g(\eta) \text{ and } m < h(\alpha).$$

If $\ell g(\eta) \le \alpha$ then we stipulate $\mathrm{Res}^m_\alpha(\eta) = \eta$. If $n \ge h(\ell g(\eta(\alpha)))$ or $\ell g(\eta) = \alpha + 1 \wedge \eta(\alpha) = s_0$ we stipulate

$$\mathrm{Res}^n_\alpha(\eta) = (\eta \restriction \alpha)\char`^\langle s_0\rangle.$$

2) $\psi_{\mathrm{tr}(h)}(\bar x; \bar y)$, where $\bar x = (x_0, x_1)$, $\bar y = (y_0, y_1)$, is[1]

$$(x_0 = y_0) \wedge\ P_\kappa(y_0) \wedge \bigvee_{i<\kappa} \big[P_{i+1}(x_1) \wedge P_{i+1}(y_1) \wedge (x_1 <_1 y_1) \wedge \\ \mathrm{Suc}_{i,0}(x_0, x_1) \wedge \mathrm{Suc}_{i,h(i)-1}(y_1, y_0)\big]$$

3) We define $\psi'_{\mathrm{tr}(h)}$ as follows:

$$\bigvee_{i<\kappa} \Big(x_0 = y_0 \wedge P_\kappa(y_0) \wedge \bigwedge_{\ell=1}^{h(i)-1} (x_{\ell+1} = y_\ell) \wedge \\ \bigwedge_{\ell=1}^{h(i)} \big[P_{i+1}(x_\ell) \wedge P_{i+1}(y_\ell) \wedge \mathrm{Res}^\ell_i(x_0) = x_\ell \wedge \mathrm{Res}^\ell_i(y_0) = y_\ell\big]\Big)$$

so if $\alpha = \sup(\mathrm{rang}(h))$ then $\bar x = \langle x_\ell : \ell < 1 + \alpha\rangle$, $\bar y = \langle y_\ell : \ell < 1 + \alpha\rangle$ (noting $\alpha \le \omega$).

4) If $\bigwedge\limits_{i<\kappa} h(i) = n$ we may write $K^\kappa_{\mathrm{tr}(n)}$, so for $n = 2$ we get K^κ_{ptr} up to some renaming.

[1]Below, the intention is $y_0 \restriction i = x_\ell \restriction i$ and $y_0(i) = \langle x_0(i), \dots, x_{h(i)-1}(n)\rangle$.

If $\bigwedge_{i<\kappa} h(i) = i \mod \omega$ we may write $K^\kappa_{\mathrm{tr}(*)}$. We say "$I \in K^\kappa_{\mathrm{tr}(h)}$ is standard" in case the underlying set J is well ordered (usually a set of ordinals). When we write $\eta(\alpha)(\ell)$, we mean $\eta(\alpha)$ if $\ell g(\eta) = \alpha + 1$ and $\mathrm{Res}^\ell_\alpha(\eta)$ if $\alpha + 1 < \ell g(\eta)$.

Remark 1.3. Here, when dealing with K^ω_{ptr} (or $K^\omega_{\mathrm{tr}(n)}$, $K^\omega_{\mathrm{tr}(*)}$, $K^\omega_{\mathrm{tr}(h)}$; those are parallel cases), we introduce the "super*" version, parallel to Definitions [Shea, 1.1=L7.1, 1.4=L7.2].[2] So the easy case [Shea, 1.6=L7.5] has to be redone, hence claim [Shea, 1.8(2)=L7.5(2)] is no longer of any help and we should prove a parallel. The role of $\bar{\mathbf{e}}$ here corresponds in the role ψ_{tr} in [Shei, §2], [Shea, §1].

Definition 1.4. Let $h : \omega \to \omega \setminus \{0\}$, and $\bar{\mathbf{e}}$ be a function with domain ω, with $\bar{\mathbf{e}}(n)$ an equivalence relation on $\mathcal{P}(h(n))$ satisfying

$$u_1 \ \bar{\mathbf{e}}(n) \ u_2 \ \Rightarrow \ |u_1| = |u_2|.$$

For this definition we identify a set (of natural numbers or ordinals) with an increasing sequence enumerating it. Defining $\bar{\mathbf{e}}$ we may ignore classes which are singleton; see clause (5) on default values.

1) For $I \in K^\omega_{\mathrm{tr}(h)}$, $J \in K^\omega_{\mathrm{tr}(h')}$ and cardinals μ, κ we say I is (μ,κ)-super-$\bar{\mathbf{e}}$-unembeddable into J (for $K^\omega_{\mathrm{tr}}(h)$) <u>when</u>:

$(*)^{I,J}_{\mu,\kappa,\bar{\mathbf{e}}}$ For every large enough regular cardinal χ, $x \in \mathcal{H}(\chi)$, for a fixed well ordering $<^*_\chi$ of the set $\mathcal{H}(\chi)$ and $f_1 : I \to {}^{\kappa>}J$, there are $\langle M_n, N_n : n < \omega\rangle$ such that:

- (i) $M_n \prec N_n \prec M_{n+1} \prec (\mathcal{H}(\chi), \in, <^*_\chi)$
- (ii) $M_n \cap \mu = N_n \cap \mu$ and $\kappa \subseteq M_0$.
- (iii) I, J, μ, κ, h, x belong to M_0.
- (iv) There is $\eta \in P^I_\omega$ such that for every n we have $\eta \restriction n \in M_n$. Also, for n large enough, for $\ell < h(n)$, we have $\mathrm{Res}^\ell_n(\eta) \in N_n \setminus M_n$ and they realize the same Dedekind cut by $<^I_1$ on
$$\{\nu \in I \cap M_n : \nu, \mathrm{Res}^0_n(\eta) \text{ are } <^I_1\text{-comparable}\}.$$
This is equivalent to "$\mathrm{Res}^0_n(\eta)$, $\mathrm{Res}^1_n(\eta), \ldots, \mathrm{Res}^{h(n)-1}_n(\eta)$ realize the same Dedekind cut on $\{(\eta \restriction n)\hat{\ }\langle s\rangle \in I : s \in M_n\}$."
(Recall that $<^I_1$ linearly orders $\{(\eta \restriction n)\hat{\ }\langle s\rangle : s\} \cap I$.)
- (v) For η as above: if $h(n) > 1$ and $u_1 \ \bar{\mathbf{e}}(n) \ u_2$ then
  - (α) If $\ell_1 \in u_1 \wedge \ell_2 \in u_2 \wedge |u_1 \cap \ell_1| = |u_2 \cap \ell_2|$ then $f_1(\mathrm{Res}^{\ell_1}_n(\eta))$ and $f_1(\mathrm{Res}^{\ell_2}_n(\eta))$ have the same length.
  - (β) The sequences $\nu_{\eta,n,u_1}, \nu_{\eta,n,u_2} \in {}^{\kappa>}J$ realize the same atomic type over $J \cap M_n$ in J, where for $u \subseteq h(n)$ we let $\nu_{\eta,n,u}$ be the concatenation of the sequences $f_1(\mathrm{Res}^\ell_n(\eta))$ for $\ell \in u$.
- (vi) For every $\nu \in P^J_\omega$,
$$\Big(\bigcup_{n<\omega} M_n\Big) \cap \bigcup\{\mathrm{Res}^\ell_n(\nu) : \ell < h(n),\ n < \omega\}$$
is included in some M_m.

2) For $I, J \in K^\omega_{\mathrm{tr}(h)}$ and cardinals μ, κ, we say[3] that I is (μ,κ)-super-$\bar{\mathbf{e}}$-unembeddable$'$ into J (for $K^\omega_{\mathrm{tr}(h)}$) <u>when</u>:

[2]And see more versions in [Shea, 1.5=L7.3, 1.6=L7.3A].

[3]This is helpful in constructing Boolean algebras as in §2 in more cardinals without using Definition 1.4(1) (or even $\psi'_{\mathrm{rt}(h)}$), but this is the minor variant and the reader can ignore it.

$(*)'_{I,J,\mu,\aleph_0}$ For every large enough χ and $x \in \mathcal{H}(\chi)$, for a fixed well ordering $<^*_\chi$ of $\mathcal{H}(\chi)$, there exists M such that:

(i) $M \prec (\mathcal{H}(\chi), \in, <^*_\chi)$

(ii) $x \in M$

(iii) M is countable.

(iv) There is $\eta \in P^I_\omega$ such that

$$m < \omega \wedge \ell < h(n) \Rightarrow \mathrm{Res}^\ell_n(\eta) \in M$$

and for every function $f \in M$ from ${}^{\omega >}I$ to μ, for infinitely many n, we have:

⊛ If $\ell'_0 < \ldots < \ell'_{k-1} < h(n)$, $\ell''_0 < \ldots < \ell''_{k-1} < h(n)$, and $\{\ell'_0, \ldots, \ell'_{k-1}\}\ \bar{\mathbf{e}}(n)\ \{\ell''_0, \ldots, \ell''_{k-1}\}$ then

$$f\big(\langle \mathrm{Res}^{\ell'_i}_n(\eta) : i < k\rangle\big) = f\big(\langle \mathrm{Res}^{\ell''_i}_n(\eta) : i < k\rangle\big).$$

(v) If $\nu \in P^J_\omega$ then either $\nu \in M$ or, for some $k < \omega$, we have $\nu \restriction k \in M$ and $\nu \restriction (k+1) \notin M$.

3) Let $\bar{\mathbf{e}}_0$ be defined by

$$\bar{\mathbf{e}}_0(n) = \big\{\big(\{\ell\}, \{k\}\big) : \ell, k < h(n)\big\}.$$

Let $\bar{\mathbf{e}}_1$ be defined by $\big\{\big(\{0, \ldots, h(n)-2\}, \{1, \ldots, h(n)-1\}\big)\big\}$. Let $\bar{\mathbf{e}}_2$ be defined by

$$\bar{\mathbf{e}}_2(n) = \big\{\big(\{0, \ldots \lfloor h(n)/2\rfloor - 1\}, \{\lfloor h(n)/2\rfloor, \ldots, 2\lfloor h(n)/2\rfloor - 1\}\big)\big\}.$$

If $\bar{\mathbf{e}} = \bar{\mathbf{e}}_0$ we may omit the subscript.

4) For $\langle I_\xi : \xi \in W\rangle$, W a set of ordinals, $I_\xi \in K^\omega_{\mathrm{tr}(h)}$, standard for simplicity, letting

$$\zeta(*) = \sup\big(W \cup \{\eta(0)(\ell) : \eta \in \bigcup\{I_\xi : \xi \in W\}\}\big) + 1$$

we define $\sum\limits_{\xi \in W} I_\xi \in K^\omega_{\mathrm{tr}(h)}$ as $\big\{\langle\ \rangle\big\} \cup \big\{\langle \xi\rangle \underset{\zeta(*)}{\otimes} \eta : \xi \in W \text{ and } \eta \in I_\xi\big\}$. (On $\otimes$, see below.)

Remark 1.5. 1) We can also define this for trees with more than ω levels (as in Definitions 1.1, 1.2) but we feel we have enough parameters anyhow.

2) Recall $\xi \underset{\zeta(*)}{\otimes} \eta$ is $\langle\ \rangle$ if $\eta = \langle\ \rangle$, and is $\langle \zeta(*) \times \xi + \eta(0), \eta(1), \eta(2), \ldots\rangle$ otherwise.

Definition 1.6. 1) $K^\omega_{\mathrm{tr}(h)}$ has the $(\chi, \lambda, \mu, \kappa)$-*super-$\bar{\mathbf{e}}$-bigness property* when there are standard $I_\zeta \in K^\omega_{\mathrm{tr}(h)}$ for $\zeta < \chi$ with $|I_\zeta| = \lambda$ such that I_ζ is (μ, κ)-super-$\bar{\mathbf{e}}$-unembeddable into I_ε for each $\zeta \neq \varepsilon < \chi$.

2) $K^\omega_{\mathrm{tr}(h)}$ has the *full* $(\chi, \lambda, \mu, \kappa)$-super-$\bar{\mathbf{e}}$-bigness property when there are standard $I_\zeta \in K^\omega_{\mathrm{tr}(h)}$ for $\zeta < \chi$, $|I_\zeta| = \lambda$ such that I_ζ is (μ, κ)-super-$\bar{\mathbf{e}}$-unembeddable into $J_\zeta = \sum\limits_{\varepsilon < \chi, \varepsilon \neq \zeta} I_\varepsilon$ for each $\zeta < \chi$.

3) We may also add superscripts to distinguish slightly different super-bigness properties: $\mathrm{super}^{\mathrm{nr}}$ will with be used for the properties defined in parts (1) and (2) above; $\mathrm{super}^{\mathrm{vr}}$ will be almost identical, but we replace unembeddable by unembeddable′ (i.e. in Definition 1.4 we replace $(*)^{I_\zeta, J_\zeta}_{\mu,\kappa}$ by $(*)'_{I_\zeta, J_\zeta, \mu, \kappa, \theta}$).

We may replace λ by $\bar{\lambda} = (\lambda_0, \lambda_1)$ if '$|I_\xi| = \lambda$' is replaced by

$$\text{‘}|I_\xi| = \lambda_0, \quad \big|\{\eta \in I_\xi : \ell g(\eta) < \omega\}\big| = \lambda_1.\text{’}$$

4) Whenever we state a theorem, definition, or claim which does not depend on the specific version of bigness, we will write ‘super^x.’

Remark 1.7. Also, $K^\omega_{\rm tr}$ can be brought into the framework above as a specific case (i.e. h is constantly 1).

Claim 1.8 (Monotonicity). *For every given h we have:*
1) If $K^\omega_{{\rm tr}(h)}$ has the [full] $(\chi_1,\lambda_1,\mu,\kappa)$-super $\bar{\mathbf{e}}$-bigness properties, $\chi_2 \le \chi_1$ and $\lambda_2 \ge \lambda_1$, then $K^\omega_{{\rm tr}(h)}$ has the [full] $(\chi_2,\lambda_2,\mu,\kappa,\theta)$-super $\bar{\mathbf{e}}$-bigness property; similarly for super.
2) If $K^\omega_{{\rm tr}(h)}$ has the full $(\chi,\lambda,\mu,\kappa)$-super $\bar{\mathbf{e}}$-bigness property and $\chi_1 = \min\{\chi,\lambda\}$ then $K^\omega_{{\rm tr}(h)}$ has the $(2^{\chi_1},\lambda,\mu,\kappa)$-super $\bar{\mathbf{e}}$-bigness. Similarly for super.

Proof. 1) Straightforward.
2) Similar to [Shea, 1.8(2)=L7.5], but we elaborate.
If $\langle I_\alpha : \alpha < \chi\rangle$ exemplifies "$K^\omega_{{\rm tr}(h)}$ has the full $(\chi,\lambda,\mu,\kappa,\theta)$-super^x $\bar{\mathbf{e}}$-bigness property," $\chi_1 = \min\{\chi,\lambda\}$ and $h(0) = n(*)$, then we let $J_A = \sum\{I_\alpha : \alpha \in A\}$ for $A \subseteq \chi_1$ (see Definition 1.4(4)).
Let $\langle A_\alpha : \alpha < 2^{\chi_1}\rangle$ be such that $A_\alpha \subseteq \lambda$, and $\alpha \ne \beta \Rightarrow A_\alpha \not\subseteq A_\beta$. Now $\langle J_{A_\alpha} : \alpha < 2^{\chi_1}\rangle$ exemplifies "$K^\omega_{{\rm tr}(h)}$ has the $(2^{\chi_1},\lambda,\mu,\kappa,\theta)$-super^x $\bar{\mathbf{e}}$-bigness property." $\square_{1.8}$

On the [full] strong $(\chi,\lambda,\mu,\kappa)$-bigness property (and strongly finitary version) see [Shei, 2.5=L2.3]; by 1.9 below, for $\psi_{tr(h)}$ from Definition 1.2(2) it is a consequence of the super version and as in [Shei], [Shea] it is useful.

Claim 1.9. *If $K^\omega_{{\rm tr}(h)}$ has the [full] $(\chi,\lambda,\mu^{<\kappa},2^{<\kappa})$-super-bigness property and $\chi \le \lambda$ then $K^\omega_{{\rm tr}(h)}$ has the [full] strong $(\chi,\lambda,\mu,\kappa)$-bigness property for $\psi_{{\rm tr}(h)}$ for functions f which are strongly finitary on P_ω.*

Proof. The result follows by the definitions and 1.10 below. $\square_{1.9}$

Analogously to [Shea, 1.9=L7.5A], we have:

Claim 1.10. *If $(*)^{I,J}_{\mu_1,\kappa_1}$ (where $\mu_1 = \mu^{<\kappa}$, $\kappa_1 = 2^{<\kappa}$, $\{I,\ J\} \subseteq K^\omega_{{\rm tr}(h)}$ are standard[4]) and $h \in {}^\omega\omega$, then I is strongly $(\mu,\kappa,\psi_{{\rm tr}(h)})$-unembeddable into J for embeddings which are strongly finitary on P^I_ω.*

Proof. Recalling 1.4(3) we have $\bar{\mathbf{e}} = \bar{\mathbf{e}}_0$. Without loss of generality I, J are subsets of ${}^{\omega\ge}\big(\bigcup\limits_{n<\omega}({}^n\theta)\big)$ for some cardinal θ, and let $<^*$ be a well ordering of $\mathscr{M}_{\mu,\kappa}[J]$ (which respects being a subterm). Suppose f is a function from I into $\mathscr{M}_{\mu,\kappa}(J)$, so for $\eta \in I$,

$$f(\eta) = \sigma_\eta(\nu_{\eta,0},\dots,\nu_{\eta,i},\dots)_{i<\alpha_\eta}$$

for some term σ_η, ordinal $\alpha_\eta < \kappa$, $\nu_{\eta,i} \in J$ and f is strongly finitary on P_ω, i.e.,

$$\eta \in P^I_\omega \ \Rightarrow\ \alpha_\eta < \omega \wedge [\sigma_\eta \text{ has finitely many subterms}].$$

Let χ be regular large enough, and define for $\eta \in P^I_\omega$,

$$g(\eta) = \big\{\alpha : \text{the } \alpha\text{-th element by } <^* \text{ is a subterm of } f(\eta)\big\}$$

(so we use "the strongly finitary" only so that $g(\eta)$ is finite).
Let $\langle M_n, N_n : n < \omega\rangle$ be as in the conclusion of Definition 1.4(1) and let $\eta \in P^I_\omega$ be as in clause (iv) of Definition 1.4(1). Let $m = \alpha_\eta$ and $\nu_\ell = \nu_{\eta,\ell} \in J$. Apply clause (v) of Definition 1.4(1) to each ν_ℓ. For $\ell < m$ define

$$k_\ell = \min\big\{k \le \omega :\ \text{if } k < \max(\omega, \ell g(\nu_\ell)) \text{ then } \nu_\ell \restriction (k+1) \notin \bigcup_{n<\omega} M_n\big\}.$$

[4]See Definition 1.4(1).

If $k_\ell = \omega$ then by clause (v), for some $n(\ell) < \omega$, we have $\{\nu_\ell \restriction k : k < \omega\} \subseteq M_{n(\ell)}$. If $k_\ell < \omega$ clearly for some $n(\ell) < \omega$ we have:

(∗) $\nu_\ell \restriction k_\ell \in M_{n(\ell)}$ and $\ell g(\nu_\ell) > k_\ell \Rightarrow \nu_\ell \restriction (k_\ell + 1) \notin \bigcup_{n<\omega} M_n$, and if $\nu_\ell(k_\ell) = \langle \alpha_{\ell,k_\ell,i} : i < h(k_\ell)\rangle$ and $i < h(k_\ell)$, $\alpha_{\ell,k_\ell,i} \notin M_{n(\ell)}$ then:
  (i) $\alpha_{\ell,k_\ell,i} \notin \bigcup_{n<\omega} M_n$, hence $n < \omega \Rightarrow \alpha_{\ell,k_\ell,i} \notin N_n$.
  (ii) $z_\ell := \min_{<^I_1}\{y \in \bigcup_{n<\omega} M_n : (\nu \restriction k_\ell)\hat{}\langle y\rangle \in J$ and $(\nu \restriction k_\ell)\hat{}\langle \alpha_{\ell,k_\ell,i}\rangle <^I_1 y\}$ belongs to $M_{n(\ell)}$. (We can arrange that there are such y-s or allow ∞ as a value.)

Let n_* be such that $\max\{n(0), \dots, n(m-1)\} \le n_* < \omega$ and

$$\ell g(\nu_\ell) < \omega \;\Rightarrow\; \bigcup\{\mathrm{Rang}(\nu_\ell(k)) : k < \ell g(\nu_\ell)\} \cap \bigcup_{k<\omega} M_k \subseteq M_{n(*)}.$$

Let $y_\ell = \eta$ (for $\ell < \omega$) and $x_\ell = (\eta \restriction n_*)\hat{}\langle \alpha_\ell\rangle$ for $\ell < h(n_*)$, where $\eta(n_*) = \langle \alpha_\ell : \ell < h(n)\rangle$ (and $x_{h(n_*)+\ell} = x_0$) and the rest should be clear. $\square_{1.10}$

Lemma 1.11. *1) K^ω_{ptr} and $K^\omega_{\mathrm{tr}(h)}$ (when $h \in {}^\omega(\omega \setminus \{0,1\})$) have the full $(\lambda, \lambda, \mu, \kappa)$-super–bigness property* <u>*when*</u>*:*

$\oplus_0$ *λ regular, $\lambda > \mu \ge \kappa$, and $\lambda > \mu^\kappa$.*

2) $K^\omega_{\mathrm{tr}(h)}$ has the full $(\lambda, \lambda, \mu, \kappa)$-super–bigness property <u>*when*</u>*:*

$\oplus_1$ *$\lambda > \mu \ge \kappa$ and $\lambda^{\aleph_0} = \lambda$.*

3) Above, we can deduce that $K^\omega_{\mathrm{tr}(h)}$ has the full $(\lambda, \lambda, \mu, \kappa)$-$\psi_{\mathrm{tr}(h)}$-bigness property.

Proof. Similar to [Shea, §1], but we shall prove it in §1B. (In fact, we can prove more as in [Shea].) $\square_{1.11}$

Claim 1.12. *1) Let $I \in K^\omega_{\mathrm{tr}(h)}$.* <u>*Then*</u> *$I$ is atomically μ-stable* <u>*iff*</u>

(A) For $n < \omega$ and $\eta \in P^I_{n+1}$, the linear order $(\{\nu \in P^I_{n+1} : \nu \restriction n = \eta \restriction n\}, <^I_1)$ is atomically μ-stable (i.e., for every subset of cardinality $\le \mu$ only $\le \mu$ many Dedekind cuts are realized).

(B) For any $I' \subseteq I$ with $|I'| \le \mu$, the set

$$\{\eta \in P^I_\omega : n < \omega \wedge \ell < h(n) \;\Rightarrow\; \mathrm{Res}^\ell_n(\eta) \in I'\}$$

has cardinality $\le \mu$.

2) For $\mu = \mathrm{cf}(\mu) > \aleph_0$, "atomically $(< \mu)$-stable" is characterized similarly (for $\mu = \chi^+$, this means "atomically χ-stable").

3) If $I \in K^\omega_{\mathrm{tr}(h)}$ is standard, $\mu = \mathrm{cf}(\mu)$, and $[\alpha < \mu \Rightarrow |\alpha|^{\aleph_0} < \mu]$ <u>*then*</u> *I is atomically $(< \mu)$-stable.*

4) The family of "atomically $(< \mu)$-stable $I \in K^\omega_{\mathrm{tr}(h)}$" is closed under well ordered sums.

Proof. 1) Let $J \subseteq I$ be of cardinality $\le \mu$. Without loss of generality

$\boxtimes_1$ $\eta \in J \wedge n < \ell g(\eta) \wedge \ell < h(n) \;\Rightarrow\; \mathrm{Res}^\ell_n(\eta) = (\eta \restriction n)\hat{}\big\langle(\eta(n))(\ell)\big\rangle \in J$,

$\boxtimes_2$ if $\eta \in P^I_\omega$ and

$$(\forall \ell, n)\big[\ell < h(n) \wedge n < \omega \;\Rightarrow\; \mathrm{Res}^\ell_n(\eta) \in J\big]$$

<u>then</u> $\eta \in J$ (see clause (b) of the assumption).

Let $J' = \{\eta \restriction \ell : \eta \in J,\ \ell < \ell g(\eta)\}$, and for $\nu \in J'$ let

$$J^*_\nu = \{\eta : \eta \in I,\ \eta \notin J,\ \ell g(\eta) \geq \ell g(\nu) + 1 \text{ and } \nu \triangleleft \eta\}.$$

So

(∗) $\langle J^*_\nu : \nu \in J' \rangle$ is a partition of $I \setminus J$.

For $\eta \in I \setminus J$ let $k(\eta) = \max\{k : \eta \restriction k \in J\}$. It is well defined (and $< \omega$) by $\boxtimes_2$ above, and clearly $\eta \in J^*_{\eta \restriction k(\eta)}$.

We now observe:

⊗ If $n < \omega$, $\bar\eta' = \langle \eta'_\ell : \ell < n \rangle$, $\bar\eta'' = \langle \eta''_\ell : \ell < n \rangle$, and $\eta'_\ell,\ \eta''_\ell \in I$, then a sufficient condition for $\mathrm{tp}_{\mathrm{qf}}(\bar\eta', J, I) = \mathrm{tp}_{\mathrm{qf}}(\bar\eta'', J, I)$ is:

(a) If $\eta'_\ell \in J$ or $\eta''_\ell \in J$ <u>then</u> $\eta'_\ell = \eta''_\ell$.

(b) $\ell g(\eta'_\ell) = \ell g(\eta''_\ell)$

(c) If $\eta'_\ell \notin J$ (equivalently, $\eta''_\ell \notin J$) <u>then</u> $k(\eta'_\ell) = k(\eta''_\ell)$ — call it k_ℓ — and $\eta'_\ell \restriction k_\ell = \eta''_\ell \restriction k_\ell$.

(d) for $\ell_1, \ell_2 < n < \omega$ and $k < \omega$, we have

(α) $\eta'_{\ell_1} \restriction k = \eta'_{\ell_2} \restriction k \Leftrightarrow \eta''_{\ell_1} \restriction k = \eta''_{\ell_2} \restriction k$

(β) If both conditions in (α) hold, $k < \ell g(\eta'_\ell) \wedge k < \ell g(\eta'_{\ell_2})$, $m_1, m_2 < h(k)$, and for $i = 1, 2$ we have

$$k + 1 < \ell g(\eta'_{\ell_i}) \wedge t'_i = (\eta'_{\ell_i}(k))(m_i) \wedge t''_i = (\eta''_{\ell_i}(k))(m_i)$$

or

$$k + 1 = \ell g(\eta'_{\ell_i}) \wedge t'_i = \eta'_{\ell_i}(k) \wedge t''_i = \eta''_{\ell_i}(k)$$

<u>then</u>

$$(\eta'_{\ell_1} \restriction k)\hat{\ }\langle t'_1 \rangle <^I_1 (\eta'_{\ell_1} \restriction k)\hat{\ }\langle t'_2 \rangle \Leftrightarrow (\eta'_{\ell_1} \restriction k)\hat{\ }\langle t''_1 \rangle <^I_1 (\eta''_{\ell_1} \restriction k)\hat{\ }\langle t''_2 \rangle$$

(e) If (α) <u>then</u> (β), where:

(α) $\eta'_\ell \in I^*_\nu$, $\eta'_\ell \restriction k \in J'$, and $\eta'_\ell \restriction (k+1) \notin J'$ (hence similarly for η''_ℓ).
Second, $\nu \triangleleft \rho \in J$ and $m_1, m_2 < h(\ell g(\nu))$.
Third, we have $\bullet_1$ or $\bullet_2$, where
$\bullet_1$ $k + 1 < \ell g(\eta'_\ell) \wedge t' = (\eta'_\ell(k_\ell))(m_1) \wedge t'' = (\eta''_\ell(k_\ell))(m_1)$
$\bullet_2$ $k + 1 = \ell g(\eta'_\ell) \wedge t' = \eta'_\ell(k) \wedge t'' = \eta''_\ell(k)$
Lastly, $k+1 < \ell g(\rho) \wedge s = (\rho(k))(m_2)$ or $k+1 = \ell g(\rho) \wedge s = \rho(k)$.

(β) $\bullet_1$ $\nu\hat{\ }\langle s \rangle <^I_1 \nu\hat{\ }\langle t' \rangle \ \Leftrightarrow\ \nu\hat{\ }\langle s \rangle <^I_1 \nu\hat{\ }\langle t'' \rangle$
$\bullet_2$ $s = t' \Leftrightarrow s = t''$.

It is easy to check that this is true. Also, ⊗ defines the equivalence relation (equality of q.f. types in I over J) as various pieces of information being the same. Now in all cases we have $\leq \mu$ choices (for clauses (d), (e) in ⊗, recall clause (A) in the assumption) so we are done.

2) Similarly.

3) Follows, as well orders are atomically μ-stable.

4) Straight. $\square_{1.12}$

Claim 1.13. *If $I \in K^\omega_{\mathrm{tr}(h)}$ is standard and λ satisfies $(\forall \alpha < \lambda)\big[|\alpha|^{\aleph_0} < \lambda\big]$ <u>then</u> I is $(< \lambda)$-atomically stable.*

Proof. Obvious by 1.12(1) for λ successor and by 1.12(2) for λ a limit cardinal.

§ 1(B). Existence Proofs.

Lemma 1.14. The h-fold simple B.B. Lemma.

Assume $\lambda, \kappa \geq \aleph_2$, $J = (\lambda, <)$, $h : \chi \to \omega \setminus \{0\}$, $I \in K^{\kappa}_{\mathrm{tr}(h)}$ as in Definition 1.2 for J, and let $P^I_\kappa = \{\eta : \ell g(\eta) = \kappa\}$, $P^I_{<\kappa} = \bigcup_{i<\kappa} P^I_i$, $\eta(i) \in \mathrm{inc}_{h(i)}(J)$, and $S = \mathcal{H}_{<\aleph_0}(\lambda)$.

1) There are functions f_η for $\eta \in P^I_\kappa$, and pairwise disjoint $Y_\varepsilon \subseteq P^I_\kappa$ for $\varepsilon < \kappa$ such that:

- (i) $\mathrm{dom}(f_\eta) = \big\{(\mathrm{Res}^\ell_i)^I(\eta) : i < \kappa,\ \ell < h(i)\big\}$. *That is,*

$\mathrm{dom}(f_\eta) = \big\{\eta \restriction j : j < i \textit{ not successor}\big\} \cup \big\{(\eta \restriction j)^\frown\big\langle \eta(j)(\ell)\big\rangle : j+1 < i,\ \ell < h(j)\big\}$.

- (ii) $\mathrm{rang}(f_\eta) \subseteq S$
- (iii) *If f is a function from $P^I_{<\kappa}$ into S and g is a function from $P^I_{<\kappa}$ into some $\gamma < \kappa$, and $\varepsilon < \lambda$, then for some $\eta \in Y_\varepsilon \subseteq P^I_\kappa$, we have:*
  - $\bullet_1$ $f_\eta \subseteq f$
  - $\bullet_2$ $g \restriction \{(\mathrm{Res}^\ell_i)^I(\eta) : \ell < h(i)\}$ *is constant for each $i < \kappa$.*

2) In clause (iii), assume further that we are also given $\big\langle (h_i, \theta_i, g_i) : i < \kappa\big\rangle$ such that $k_i < h(i)$, $g_i : P^I_{i+1} \to \theta_i$, and $\lambda \to (h(i))^{k_i}_{\theta_i}$. Then we can add

- $\bullet_3$ $g \restriction \big[\{(\mathrm{Res}^\ell_i)^I(\eta) : \ell < h(i)\}\big]^{k_i}$ *is constant for each $i < \kappa$.*

Remark 1.15. 1) Quoting 1.14 in [AGSa, Th 3.14, Def 3.13], note that:

- (a) κ, λ there correspond to $\aleph_0$ and λ here.
- (b) $\Lambda_{<\omega}, \Lambda_\omega$ there correspond to $P^I_{<\kappa}, P^I_\kappa$ here.
- (c) g_η, g, f, λ there correspond to f_η, f, g, γ here.

2) We can allow finite λ, but then we would have to add the condition

$$(h(i) - 1) \cdot \gamma < \lambda.$$

Proof. 1) Let $\langle W_{s,\varepsilon} : s \in S,\ \varepsilon < \lambda\rangle$ be a partition of λ into $|S \times \lambda|$-many sets, each of cardinality λ. For $i \leq \kappa$, let $\Lambda_i = \{\eta \restriction i : \eta \in P^I_\kappa\}$ and choose, by induction on i:

- $(*)_i$ f_η, for $\eta \in \Lambda_i$, such that:
  - (a) $\mathrm{dom}(f_\eta) = \big\{\eta \restriction j : j < i \text{ not successor}\big\} \cup \big\{(\eta \restriction j)^\frown\big\langle \eta(j)(\ell)\big\rangle : j+1 < i,\ \ell < h(j)\big\}$
  - (b) $\mathrm{rang}(f_\eta) \subseteq S$
  - (c) If $\nu \triangleleft \eta$ then $f_\nu \subseteq f_\eta$.
  - (d) If $j+1 < i$ and $\ell < h(j)$ then $f_\eta\big((\eta \restriction j)^\frown\langle \eta(j)(\ell)\rangle\big)$ is $s(\ell)$ provided that $\eta(j+1)(0) \in W_{s,\varepsilon}$, where s is a sequence of length $h(j)$, and is zero otherwise.
  - (e) If $j = 0$ or j is limit $< i$, and $\eta(j)(0) \in W_{s,\varepsilon}$ for some s, then that s will be $f_\eta(\eta \restriction j)$.

So $\langle f_\eta : \eta \in P^I_i\rangle$ is well defined for $i \leq \kappa$, and it obviously satisfies clauses $(i), (ii)$ of the desired conclusion. What about clause (iii)?

Fix ε. Assume $f : P^I_{<\kappa} \to S$ and $g : P^I_{<\kappa} \to \gamma$ for some $\gamma < \lambda$. We choose $\eta_i \in \Lambda_i$ by induction on i.

If $i = 0$ or i is limit, we have no freedom.

If $i = j+1$ and j is not a successor ordinal, then let $s = \big\langle f(\eta \restriction j)\big\rangle$ (so $s \in \mathcal{H}_{<\aleph_0}(\lambda)$), and choose $\rho \in \mathrm{inc}_{h(j)}(W_{s,\varepsilon})$ and let $\eta_i = \eta_j{}^\frown\langle\rho\rangle$.

Lastly, if $i = j+1$ and j is a successor ordinal, then let

$$s = \langle \mathrm{Res}^\ell_{j-1}(\eta) : \ell < h(j-1)\rangle$$

and choose $\rho \in \mathrm{inc}_{h(j-1)}(W_{s,\varepsilon})$ and let $\eta_i = \eta_j{}^\frown\langle\rho\rangle$.

Now it is easy to check the η satisfies clause (iii).

2) Similarly. $\square_{1.14}$

Proof. **Proof of Lemma 1.11**

1,2) **Case 1**: λ regular, $\lambda > \mu \geq \kappa$, $\lambda = \lambda^\kappa > \mu^\kappa$, and $(\forall \theta < \lambda)[\theta^{\aleph_0} < \lambda]$.

$K = K^\omega_{\rm ptr}$ is a special case of $K^\omega_{{\rm tr}(h)}$ with $h \in {}^\omega\{2\}$, so we will restrict ourselves to the case $K = K^\omega_{{\rm tr}(h)}$.

Let $S = \{\delta < \lambda : {\rm cf}(\delta) = \omega\}$, and $\langle S_\zeta : \zeta < \lambda \rangle$ be a sequence of pairwise disjoint stationary subsets of S. Recalling $\lambda \geq \aleph_2$, for each ζ we can find $\overline{C} = \langle C_\delta : \delta \in S_\zeta \rangle$ such that:

$(*)_1$ (a) C_δ is a club of δ.
 (b) ${\rm otp}(C_\delta) = \omega$
 (c) $\overline{C}$ guesses clubs.

$(*)_2$ For $\delta \in S_\zeta$, let $\eta_\delta, \nu_\delta \in {}^\omega\lambda$ be defined by:
 (a) $\eta_\delta(n)$ is the $(2n)^{\rm th}$ member of C_δ.
 (b) $\nu_\delta(n)$ is the $(2n+1)^{\rm th}$ member of C_δ.

$(*)_3$ (a) Let $\Lambda_\delta = \big\{\eta \in {}^\omega\delta : \eta(n) \in {\rm inc}_{h(n)}\big([\eta_\delta(n), \nu_\delta(n))\big)\big\}$.
 (b) Let $I^\lambda_{{\rm tr}(h)} \in K^\omega_{{\rm tr}(h)}$ be as in 1.2(1), with its set of elements denoted ${\rm set}_{{\rm tr}(h)}(J)$.
 (c) Let I_ζ be the submodel of $I^\lambda_{{\rm tr}(h)}$ with set of elements

$$\bigcup\{\Lambda_\delta : \delta \in S_\zeta\} \cup P^{I^\lambda_{{\rm tr}(h)}}_{<\kappa}.$$

$(*)_4$ We will show that $\bar{I} = \langle I_\zeta : \zeta < \lambda \rangle$ exemplifies the conclusion.

So let $\zeta(*) < \lambda$, $I := I_{\zeta(*)}$, and $J := \sum\limits_{\zeta \neq \zeta(*)} I_\zeta$. Note

$(*)_5$ Assume $\chi, x, \langle (M_n, N_n) : n < \omega \rangle$ are as in clauses (i)-(iii) of $(*)^{I,J}_{\lambda,\mu,\kappa}$ in Definition 1.4(1).
 (a) If $M_n \cap \lambda \in \lambda$ for all $n < \omega$ and $\delta = \bigcup\{M_n \cap \lambda : n < \omega\} \in S_{\zeta(*)}$ <u>then</u> clause (iv) there holds.
 (I.e. if $\nu \in P^J_\omega$ and $\{\nu \restriction n : n < \omega\} \subseteq \bigcup\limits_m M_m$ <u>then</u> $\{\nu \restriction n : n < \omega\} \subseteq M_m$ for some m.)
 (b) If we add the demand $(\forall \theta < \lambda)[\theta^{\aleph_0} < \lambda]$, then we can add $\bigvee\limits_m [\nu \in M_m]$ (intended for stronger versions of super).

Now if indeed $(\forall \theta < \lambda)[\theta^{\aleph_0} < \lambda]$, we can continue as in the proof of [Shea, 1.11(1)=L7.6]. In particular, we find M_n, N_n as in $(*)_5$. Otherwise, we find M_n, N_n as above and choose $M_* \prec (\mathcal{H}(\chi), \in)$ of cardinality μ such that $[M_*]^\kappa \subseteq M_*$, $\langle (M_n, N_n) : n < \omega \rangle \in M_*$, and $\mu + 1 \subseteq M_*$. Now use $\big\langle (M_n \cap M_*, N_n \cap M_*) : n < \omega \big\rangle$.

Case 2: $\lambda > \chi = \chi^\kappa$ and $2^\chi \geq \lambda$.

We prove the full $(2^\chi, \lambda, \chi, \kappa)$-super bigness property, getting M_n-s such that $(\forall \theta)[\kappa^\theta = \kappa \Rightarrow {}^\theta(M_n) \subseteq M_n]$.

Without loss of generality $\chi \geq \mu$. As in the proof of [Shea, 1.11(2)=L7.6] until the end: the choice of ρ is natural, as in 1.14.

Case 3: $\lambda = 2^\theta$, θ strong limit singular, $\theta > \mu$, ${\rm cf}(\theta) = \aleph_0$.

Let $\lambda_n \in (\mu, \lambda)$ be increasing with n. Let $\langle M^*_\alpha : \alpha < \lambda \rangle$ list the elements of

$$\begin{aligned}\mathcal{M} := \big\{M : M &\text{ has universe } \mathcal{H}(\theta) \text{ and expands } (\mathcal{H}(\theta), \in) \\ &\text{such that } \tau_M \subseteq \mathcal{H}_{<\kappa^+}(\theta) \text{ and } \{\lambda_n : n < \omega\} \subset M\big\}\end{aligned}$$

such that each model of $\mathcal{M}$ appears λ times in the sequence.

Now choose $\mathbf{s}_\alpha = \langle (M_{\alpha,n}, N_{\alpha,n}) : n < \omega \rangle$ by induction on $\alpha < 2^\lambda$:

(∗) (a) $M_{\alpha,n} \prec N_{\alpha,n} \prec M_{\alpha,n+1} \prec M^*_\alpha$
(b) $[M_{\alpha,n}]^{<\kappa} \subseteq M_{\alpha,n}$, $[N_{\alpha,n}]^{<\kappa} \subseteq M_{\alpha,n}$.
(c) $\mu + 1 \subseteq M_{\alpha,n}$
(d) $\|M_{\alpha,n}\| = \mu$
(e) If $\beta < \alpha$ then $\bigcup\limits_n M_{\beta,n} \cap \bigcup\limits_n M_{\alpha,n} = M_{\alpha,k}$ for some k.

Why? in the induction step we use the Δ-system lemma for trees.

Case 4: λ strong limit singular, $\mathrm{cf}(\lambda) > \kappa$.
As in [Shea, 1.11(3)=L7.6].

We are done now; why?

Assume, in the proof of 1.11(1), that none of the cases above hold. Let $\theta = \min\{\theta' : 2^{\theta'} \geq \lambda,\ \theta' \geq \mu\}$. As Case 2 does not hold, necessarily $\theta^\kappa \geq \lambda$ and $\chi > \mu$. If $\sigma < \theta$ and $2^\sigma \geq \theta$ then $2^\sigma = (2^\sigma)^\sigma \geq \theta^\kappa \geq \lambda$, so having $\theta' = \theta + \mu$ contradicts the choice of θ. Therefore $\sigma < \theta \Rightarrow 2^\sigma < \theta$, so θ is strong limit. As $\theta^\kappa \geq \lambda$, necessarily $\mathrm{cf}(\theta) \leq \kappa$. Also, $\mu^\kappa = \mu$, hence $\mu < \theta$.

3) Follows by part (1) and Claim 1.10. $\square_{1.11}$

§ 2. Applications to Boolean Algebras

Here we construct some Boolean algebras with "no non-trivial morphism."

We shall mainly use $\mathrm{BA}_{\mathrm{tr}}(I)$, $I \in K^\omega_{\mathrm{tr}}$ for constructing mono-rigid ccc Boolean algebras; $\mathrm{BA}_{\mathrm{tr}(h)}(I)$, $I \in K^\omega_{\mathrm{tr}(h)}$, $h \in {}^\omega(\omega \setminus \{0,1,2\})$ for constructing complete mono-rigid ccc Boolean algebras; and $\mathrm{BA}_{\mathrm{trr}}(I)$, $I \in K^\omega_{\mathrm{tr}}$ for constructing Bonnet–rigid Boolean algebras. In each case, for every I from a relevant family (which exemplifies full bigness in the relevant case), we derive a Boolean algebra $\mathrm{BA}_x(I)$, chosen to fit the proof of the case of rigidity we are interested in (this is Definition 2.1). We then build a Boolean algebra $\mathbf{B}$ of cardinality λ, planting a copy of $\mathrm{BA}_x(I_a)$ below enough elements $a \in \mathbf{B}$ such that $a \neq b \Rightarrow I_a \neq I_b$ (see 2.4). We mainly show that $\mathrm{BA}_{\mathrm{tr}(h)}(I)$ satisfies a strong version of the ccc hence the ccc is preserved (see 2.6), hence the outcome of the construction 2.4 is as required with respect to the ccc, completeness, and cardinality. We then observe the relevant weak representability results (see 2.12). Note that if we consider the completion of a ccc Boolean algebra $\mathbf{B}$ and $\mathbf{B}$ is weakly represented in $\mathscr{M}_{\aleph_0,\aleph_0}(J)$ then its completion is weakly represented in $\mathscr{M}_{\aleph_1,\aleph_1}(J)$. Next (in 2.14) we deal with deducing unembeddability of $\mathrm{BA}_x(I)$ into a Boolean algebra $\mathbf{B}$ which is weakly represented in $\mathscr{M}_{\mu,\kappa}(J)$, the main case is part (2). We deduce as conclusions that there are mono-rigid [complete] Boolean algebras (2.16, 2.17). We then deal with Bonnet rigid Boolean algebras (2.18 till the end).

Definition 2.1. 1) For $I \in K^\omega_{\mathrm{tr}}$ let $\mathrm{BA}_{\mathrm{tr}}(I)$ be the Boolean algebra generated freely by $\{x_\eta : \eta \in I\}$, except that:

$(*)_1$ $\eta \lhd \nu \in P^I_\omega \Rightarrow x_\eta \geq x_\nu$.

2) For $I \in K^\omega_{\mathrm{ptr}}$ let $\mathrm{BA}_{\mathrm{ptr}}(I)$ be the Boolean algebra freely generated by $\{x_\eta : \eta \in I\}$, except that for $\eta \in I$ with $\ell g(\eta) = \omega$, letting $\eta = \langle \langle \alpha_0, \beta_0 \rangle, \dots, \langle \alpha_n, \beta_n \rangle \dots \rangle$, the following holds:

$(*)_2$ For all $n < \omega$, $x_\eta \leq x_{\eta \restriction n \frown \langle \alpha_n \rangle}$ and $x_\eta \cap x_{\eta \restriction n \frown \langle \beta_n \rangle} = 0$.

3) For $h \in {}^\omega(\omega \setminus \{0\})$ and $I \in K^\omega_{\mathrm{tr}(h)}$ let $\mathrm{BA}_{\mathrm{tr}(h)}(I)$ be the Boolean algebra generated freely by $\{x_\eta : \eta \in I\}$, except that for $\eta \in P^I_\omega$ and $n < \omega$, letting $\eta(n) = \langle s_0, \dots, s_{h(n)-1} \rangle$ we have:

$(*)_3$ $x_\eta \leq x_{\eta \restriction n \frown \langle s_0 \rangle}$ and $x_\eta \cap \bigcap\limits_{\ell=1}^{h(n)-1} x_{\eta \restriction n \frown \langle s_\ell \rangle} = 0$.

The second equality is trivial if $h(n) = 1$, so usually $h \in {}^\omega(\omega \setminus \{0,1\})$. If $(\forall n)[h(n) = 1]$ this is like the case of $I \in K^\omega_{\mathrm{tr}}$, and if $(\forall n)[h(n) = 2]$ this is like the case of $I \in K^\omega_{\mathrm{ptr}}$.

4) For $I \in K^\omega_{\mathrm{tr}}$ (or just I is a set of sequences of ordinals closed under initial segments) let $\mathrm{BA}_{\mathrm{trr}}(I)$ be the Boolean algebra freely generated by $\{x_\eta : \eta \in I\}$, except that:

- (A) $x_{\eta \frown \langle \alpha \rangle} \cap x_{\eta \frown \langle \beta \rangle} = 0$ for[5] $\alpha \neq \beta$.
- (B) $x_\eta \leq x_\nu$ for $\nu \lhd \eta$.
- (C) If η has finitely many immediate successors $\{\eta \frown \langle \alpha_\ell \rangle : \ell < k_\eta\}$ and $k_\eta \geq 2$ <u>then</u> $x_\eta = \bigcup \{x_{\eta \frown \langle \alpha_\ell \rangle} : \ell < k_\eta\}$.
- (D) If $\eta \lhd \nu$ and every ρ satisfying $\eta \unlhd \rho \lhd \nu$ has a unique successor, then $x_\eta = x_\nu$.

[5] We are, of course, assuming $\eta \frown \langle \alpha \rangle, \eta \frown \langle \beta \rangle \in I$; similarly in other cases.

5) For $I \in K^\omega_{\mathrm{tr}(h)}$ and $g \in {}^\omega\omega$, $h \in {}^\omega(\omega \setminus \{0,1\})$ satisfying[6] $g \leq h$, we define $\mathrm{BA}_{\mathrm{tr}(h,g)}(I)$ as the Boolean algebra generated freely by x_η ($\eta \in I$), except that:

$(*)_5$ If $\eta \in I$, $\ell g(\eta) = \omega$, $\ell < \omega$, and $\eta(\ell) = \langle \alpha_0, \ldots, \alpha_{k-1}\rangle$ where $k = h(\ell)$, <u>then</u>

(α) $x_\eta \leq \bigcup\limits_{m=0}^{g(\ell)-1} x_{(\eta\restriction\ell)\hat{}\langle\alpha_m\rangle}$

(β) If $g(\ell) < h(\ell) - 1$ then $x_\eta \cap \bigcap\limits_{m=g(\ell)}^{h(\ell)-1} x_{(\eta\restriction\ell)\hat{}\langle\alpha_m\rangle} = 0$.

(Usually we assume $0 < g < h$.)

6) Assume that $h \in {}^\omega(\omega \setminus \{0,1\})$, $\bar{\mathbf{e}}$ an ω-sequence with $\bar{\mathbf{e}}(n) = \{\{u_{1,n}, u_{2,n}\}\}$, where $u_{1,n}, u_{2,n}$ are subsets of $h(n)$ which are not both singletons. For $I \in K^\omega_{\mathrm{tr}(h)}$, we define $\mathrm{BA}_{\mathrm{tr}(h),\bar{\mathbf{e}}}(I)$ as the Boolean algebra freely generated by $\{x_\eta : \eta \in I\}$, except that for $\eta \in P^I_\omega$ and $n < \omega$, letting $\eta(n) = \langle s_0, \ldots, s_{h(n)-1}\rangle$, we have

$(*)'_3$ $x_\eta \leq \bigcup\limits_{\ell \in u_{1,n}} x_{(\eta\restriction n)\hat{}\langle s_\ell\rangle}$ and $x_\eta \cap \bigcup\limits_{\ell \in u_{2,n}} x_{(\eta\restriction n)\hat{}\langle s_\ell\rangle} = 0$.

(We have much freedom in this case).

Notation 2.2. 1) Let $K^\omega_{\mathrm{tr}(h,g)} = K^\omega_{\mathrm{tr}(h)}$ for g, h as in 2.1(3). Note that for $I \in K^\omega_{\mathrm{tr}(h)}$, if $g = 1$ then $\mathrm{BA}_{\mathrm{tr}(h,g)}(I)$ is essentially $\mathrm{BA}_{\mathrm{tr}(h)}(I)$. Also, if $h = 1$ then $K^\omega_{\mathrm{tr}(h)} = K^\omega_{\mathrm{tr}}$ and $\mathrm{BA}_{\mathrm{tr}(h)}(I) = \mathrm{BA}_{\mathrm{tr}}(I)$.
2) When we state a result that holds for tr, ptr, trr, tr(h), or tr(h,g), we will replace the corresponding subscripts with an x. Naturally we define $K^\omega_{\mathrm{trr}} = K^\omega_{\mathrm{tr}}$ and $K^\omega_{\mathrm{tr}(h,g)} = K^\omega_{\mathrm{tr}(h)}$.
3) Note that when we say "a Boolean algebra is freely generated by $X = \{x_i : i \in U\}$, except the set equations . . .," we have 0 and 1 in the Boolean algebra.
4) For a Boolean algebra $\mathbf{B}$ and $a \in \mathbf{B}$, $\mathbf{B} \restriction a$ is the naturally defined Boolean algebra, but $1_{\mathbf{B}\restriction a} = a$. Essentially, we do not consider $1_{\mathbf{B}}$ as an individual constant of $\mathbf{B}$.

Definition 2.3. For Boolean algebras $\mathbf{B}$, $\mathbf{B}_1$ and $a^* \in \mathbf{B}_1 \setminus \{0_{\mathbf{B}_1}\}$, we define the "$\mathbf{B}$-surgery of $\mathbf{B}_1$ at a^*" or "surgery of $\mathbf{B}_1$ at a^* by $\mathbf{B}$", called $\mathbf{B}_2$, as a Boolean algebra extending $\mathbf{B}_1$ such that $\mathbf{B}_2 = [\mathbf{B}_1 \restriction (-a^*)] \times [(\mathbf{B}_1 \restriction a^*) * \mathbf{B}]$, where $\times$ is a direct product and $*$ free product. Alternatively, $\mathbf{B}_2$ can be generated as follows: first make $\mathbf{B}$ disjoint to $\mathbf{B}_1$ (by taking an isomorphic copy) and then $\mathbf{B}_2$ is freely generated by $\mathbf{B}_1 \cup \mathbf{B}$, except the relations

$$\begin{aligned} &0_{\mathbf{B}_1} = 0_{\mathbf{B}} = 0, \\ &a \cap b = c \quad \text{(for } a, b, c \in \mathbf{B}_1 \text{ such that } a \cap b = c \text{ in } \mathbf{B}_1\text{)}, \\ &a \cup b = c \quad \text{(for } a, b, c \in \mathbf{B}_1 \text{ such that } a \cap b = c \text{ in } \mathbf{B}_1\text{)} \end{aligned}$$

$$\begin{aligned} 1_{\mathbf{B}_1} - b = c \quad &\text{(for } b, c \in \mathbf{B}_1 \text{ such that } 1_{\mathbf{B}_1} - b = c \text{ in } \mathbf{B}_1\text{)}, \\ a \cap b = c \quad &\text{(for } a, b, c \in \mathbf{B} \text{ such that } a \cap b = c \text{ in } \mathbf{B}\text{)}, \\ a \cup b = c \quad &\text{(for } a, b, c \in \mathbf{B} \text{ such that } a \cup b = c \text{ in } \mathbf{B}\text{)}, \\ 1_{\mathbf{B}} - b = c \quad &\text{(for } b, c \in \mathbf{B} \text{ such that } 1_{\mathbf{B}} - b = c\text{)} \end{aligned}$$

and

$$1_{\mathbf{B}} = a^*.$$

[6] i.e. $(\forall n)[g(n) \leq h(n)]$.

Construction 2.4. *Let x be one of $\{\mathrm{tr}, \mathrm{ptr}, \mathrm{tr}(h), \mathrm{trr}, \mathrm{tr}(h,g)\}$ and let λ be a cardinal with $\alpha < \lambda^+$ (usually $\alpha = \lambda$, always $\alpha > 0$). The idea is to construct a Boolean algebra by defining an increasing continuous sequence $\mathbf{B}_i$ ($i \le \alpha$), $\mathbf{B}_0$ trivial, and we get $\mathbf{B}_{i+1}$ by a surgery of $\mathbf{B}_i$ at $a^*_i \in \mathbf{B}_i$ by $\mathbf{B}^*_i = \mathrm{BA}_x(I_i)$ (see Definition 2.1 and 2.2(2)), where $|I_i| = \lambda$, $I_i \in K^\omega_x$ and I_i is strongly ψ_x-unembeddable into $\sum\limits_{j<\alpha, j \ne i} I_j$ (or, e.g., super^y-$\bar{\mathbf{e}}$-unembeddable into it, $y \in \{\mathrm{nr}, \mathrm{vr}\}$).*

We denote $\mathbf{B} = \mathbf{B}_\alpha$ by $\mathrm{Sur}_x\langle I_i, a^*_i : i < \alpha\rangle$. Usually we would like to have $\mathbf{B}_\alpha \setminus \{0\} = \{a^*_i : i < \alpha\}$. If there are $\langle I_i : i < \alpha\rangle$ as above and α is divisible by λ then this is clearly possible.

Definition 2.5. 1) A Boolean algebra satisfies the λ-chain condition (or the λ-*cc*) iff there are no λ elements which form an antichain (i.e., they are $\ne 0$ and the intersection of any two is zero).

2) A Boolean algebra satisfies the strong λ-chain condition or the λ-Knaster condition iff among any λ elements there are λ which are pairwise not disjoint.

Claim 2.6. *Let $x \in \{\mathrm{tr}, \mathrm{ptr}, \mathrm{tr}(n), \mathrm{tr}(h), \mathrm{tr}(*)\}$, $I \in K^\omega_x$, λ uncountable regular.*

1) If $x = \mathrm{tr}$ then $\mathrm{BA}_x(I)$ satisfies the strong λ-chain condition.

2) If $x = \mathrm{ptr}$ then $\mathrm{BA}_x(I)$ satisfies the strong $\left(2^{\aleph_0}\right)^+$-chain condition.

3) If $x = \mathrm{tr}(k)$, $k \ge 3$, and $I \in K^\omega_{\mathrm{tr}(k)}$ is standard, then $\mathbf{B} = \mathrm{BA}_{\mathrm{tr}(k)}(I)$ satisfies the strong λ-chain condition; similarly for $K^\omega_{\mathrm{tr}(*)}$, for $K^\omega_{\mathrm{tr}(h)}$ with $h \in {}^\omega(\omega \setminus 3)$, and $K^\omega_{\mathrm{tr}(h,g)}$ (for $h \in {}^\omega(\omega \setminus 3)$ and $g \in {}^\omega\omega$ such that $g \le h$).

Instead of $h \in {}^\omega(\omega \setminus 3)$, we can demand $h \in {}^\omega(\omega \setminus 1)$ and $h(n) \ge 3$ for every large enough n.

4) If $x = \mathrm{ptr}$, $\mathrm{BA}_x(I)$ satisfies the strong λ-chain condition provided that I is atomically $(< \lambda)$-stable; for example, if $(\forall \alpha < \lambda)\left[|\alpha|^{\aleph_0} < \lambda\right]$.

5) If $h, \bar{\mathbf{e}}$ are as in 2.1(6) and for every n large enough, $(*)^n_{\bar{\mathbf{e}}}$ below holds, λ is regular uncountable, and $I \in K^\omega_{\mathrm{tr}(h)}$ then $\mathrm{BA}_{\mathrm{tr}(h),\bar{\mathbf{e}}}(I)$ satisfies the strong λ-chain condition, where:

- $(*)^n_{\bar{\mathbf{e}}}$ *$\bar{\mathbf{e}}(n) = \{(u^n_1, u^n_2)\}$, where $u^n_1, u^n_2 \subseteq \{0, \dots h(n) - 1\}$ are non-empty and not of the same cardinality.*

Remark 2.7. Clearly we can similarly phrase sufficient condition for "any family of λ non-zero elements there is an uncountable subfamily such that any k members of the subfamily have non-zero intersection".

Before we prove 2.6, recall the well known fact: (Here $\mathbf{B}_0 = \{0,1\}$ is the two-element Boolean algebra.)

Fact 2.8. 1) If $\mathbf{B}$ is the Boolean algebra freely generated by $\{x_t : t \in I\}$ except for a set Λ of equations in $\{x_t : t \in I\}$, (so each member of Λ has the form $\sigma(x_{t_0}, \dots, x_{t_{n-1}}) = 0$, where $\sigma(y_0, \dots, y_{n-1})$ is a Boolean term, $t_0, \dots, t_{n-1} \in I$) then, for a Boolean term $\sigma^*(x_{s_0}, \dots, x_{s_{n-1}})$, we have $(\alpha) \Leftrightarrow (\beta)$, where:

- (α) $\mathbf{B} \models \sigma^*(x_{s_0}, \dots, x_{s_{n-1}}) > 0$
- (β) For some function $f : I \to \{0,1\}$, we have:
  - (a) f respects Λ; i.e.

    $$\sigma(x_{t_0}, \dots, x_{t_{m-1}}) \in \Lambda \ \Rightarrow\ \mathbf{B}_0 \models \text{“}0 = \sigma(f(t_0), \dots, f(t_{m-1}))\text{”}.$$
  - (b) $\mathbf{B}_0 \models$ '$\sigma^*(f(s_0), \dots, f(s_{n-1})) = 1$'

2) In fact, if $f : I \to \{0,1\}$ satisfies clause (a) then there is a unique homomorphism $\hat{f}$ from $\mathbf{B}$ into $\mathbf{B}_0$ such that $s \in I \Rightarrow \hat{f}(x_s) = f(s)$.

Now we return to proving 2.6.

Proof. 1) We take $x = \mathrm{tr}$ and check the strong λ-chain conditions. Note that by 2.8 and the definition of $\mathrm{BA}_{\mathrm{tr}}(I)$, we have:

$(*)_1$ $x_{\eta_1} \cap \ldots \cap x_{\eta_k} \cap (-x_{\nu_1}) \cap \ldots \cap (-x_{\nu_m}) = 0$ iff
$$(\exists i, j)\big[\nu_i \lhd \eta_j \in P^I_\omega \vee \nu_i = \eta_j\big].$$

[Why? The 'if' implication is trivial, recalling Definition 2.1(1). For proving the "only if" implication, assume that the second statement holds. Define $f : I \to \{0,1\}$ by $f(\eta) = 1$ iff $(\exists \ell)[\eta = \eta_\ell \vee \eta \lhd \eta_\ell \in P^I_\omega]$; clearly it respects the equations in the definition of $\mathrm{BA}_{\mathrm{tr}}(I)$ and $\hat{f}$ maps $x_{\eta_1} \cap \ldots \cap x_{\eta_k} \cap (-x_{\nu_1}) \cap \ldots \cap (-x_{\nu_n})$ to 1, so by 2.8 we are done.]

Now for $u \in [I]^{<\omega}$, let $x_u = \bigcap\limits_{\eta \in u} x_\eta$ and $x_{-u} = \bigcap\limits_{\eta \in u} (-x_\eta)$. Clearly, if $a \in \mathrm{BA}_x(I) \setminus \{0\}$ then for some $u, v \in [I]^{<\aleph_0}$, we have $0 < x_u \cap x_{-v} \leq a$ (hence u and v are disjoint). In fact, a is a finite union of such elements. To check the strong λ-chain condition it suffices to take $\{(u_i, v_i) : i < \lambda\} \subseteq [I]^{<\aleph_0} \times [I]^{<\aleph_0}$ such that $(\forall i < \lambda)[x_{u_i} \cap x_{-v_i} \neq 0]$, and to find $A \in [\lambda]^\lambda$ such that
$$(\forall i, j \in A)[x_{u_i} \cap x_{-v_i} \cap x_{u_j} \cap x_{-v_j} \neq 0].$$

We may assume that $\langle u_i : i \in A\rangle$ and $\langle v_i : i \in A\rangle$ are Δ-systems (say, with hearts u^*, v^* respectively) so as $u_i \cap v_i = \varnothing$, necessarily
$$u_i \cap v^* = u^* \cap v_i = u^* \cap v^* = \varnothing.$$
We may assume $i \neq j \in A$ implies $u_i \cap v_j = \varnothing$, $u_i \neq u_j$, and $v_i \neq v_j$. We may assume that for some non-zero $m, n < \omega$, for every $i \in A$, we have $|u_i| = m \wedge |v_i| = n$. Say $u_i = \{\eta_{i,\ell} : \ell < m\}$, $v_i = \{\nu_{i,\ell} : \ell < n\}$ (without repetitions) and for each $\ell < m$ the sequence $\langle \eta_{i,\ell} : i \in A\rangle$ is constant or is without repetitions, and similarly $\langle \nu_{i,\ell} : i \in A\rangle$. We may assume

$(*)_2$ $\langle \ell g(\eta_{i,\ell}) : \ell < m\rangle$, $\langle \ell g(\nu_{i,\ell}) : \ell < n\rangle$ is the same for all $i \in A$.

Clearly then, using the Δ-system assumption,

$(*)_3$ For $i \in A$, $\ell < m$, $k < n$ there is at most one $j \in A$ such that $\nu_{j,k} \lhd \eta_{i,\ell} \in P^I_\omega$.

[Why? If we have $\nu_{j,k} \trianglelefteq \eta_{i,\ell} \in P^I_\omega$, note that $\neg(\nu_{i,k} \trianglelefteq \eta_{i,\ell})$ by $(*)_1$, hence $\nu_{j,k} \neq \nu_{i,k}$ so $i \neq j$ and hence $\nu_{j,k} \notin v^*$, and $\nu_{j,k} = \eta_{i,\ell} \restriction \ell g(\nu_{j,k})$. Thus $j \neq j_1 \in A \Rightarrow \ \nu_{j_1,k} \neq \nu_{j,k}$ and hence $j \neq j_1 \in A \Rightarrow \ \nu_{j_1,k} \neq \eta_{i,\ell} \restriction \ell g(\nu_{j,k}) = \eta_{i,\ell} \restriction \ell g(\nu_{j_1,k})$. Hence $j \neq j_1 \in A \Rightarrow \ \neg(\nu_{j_1,k} \trianglelefteq \eta_{i,\ell})$ and we have finished.]

So for $i \in A$, the set
$$w_i := \{j : \text{for some } \ell < m,\ k < n \text{ we have } \nu_{j,k} \lhd \eta_{i,\ell} \in P^I_\omega\}$$
has at most $mn < \aleph_0$ members. So by $(*)_1$ it suffices to find $A' \in [A]^\lambda$ such that $[i \neq j \in A' \ \Rightarrow \ j \notin w_i]$. By Hajnal free subset theorem [Haj62][7] there is[8] such A'.

2) The case $x = \mathrm{ptr}$ is similar, but more complicated. First note

$(*)_4$ Assume $I \in K^\omega_{\mathrm{ptr}}$ and $\mathbf{B} = \mathrm{BA}_{\mathrm{ptr}}(I)$. If $m, n < \omega$, and $\nu_k, \eta_\ell \in I$ for $\ell < m, k < n$ <u>then</u> $\mathbf{B} \models x_{\eta_0} \cap \ldots \cap x_{\eta_{m-1}} \cap (-x_{\nu_0}) \cap \ldots \cap (-x_{\nu_{n-1}}) = 0$ <u>iff</u> at least one of the following conditions holds:
  (a) $(\exists \ell, k < m)[\ell g(\eta_\ell) = \omega \wedge \mathrm{Suc}_R(\eta_k, \eta_\ell)]$
  (b) $(\exists \ell < m)(\exists k < n)[\ell g(\eta_\ell) = \omega \wedge \mathrm{Suc}_L(\nu_k, \eta_\ell)]$

[7] Or see [Shed, 3.14=L4.Ha].

[8] Note that $(-x_{\nu_{j_1,\ell_1}}) \cap (-x_{\nu_{j_2,\ell_2}}) > 0$ always holds.

(c) $(\exists \ell, k < m)(\exists j < \omega)(\exists \alpha, \beta, \gamma)\big[\ell g(\eta_\ell) = \ell g(\eta_k) = \omega \wedge \eta_\ell \restriction j = \eta_k \restriction j \ \wedge \ \eta_\ell(j) = \langle \alpha, \beta\rangle \ \wedge \ \eta_k(j) = \langle \beta, \gamma\rangle\big]$

(d) $(\exists \ell < m)(\exists k < n)[\eta_\ell = \nu_k]$.

[Why? If (a) or (b) or (c) or (d) holds then the intersection is zero by the equations we have imposed defining $\mathrm{BA}_{\mathrm{ptr}}(I)$ in Definition 2.1(2), so the "if" implication holds. Next we prove the other implication, so we assume (a), (b), (c), and (d) fail, and we shall use 2.8. We have to define $f(\rho)$ for $\rho \in I$; we do it by cases.

Case 1: $\ell g(\rho) = \omega$, $\rho \in \{\eta_0, \ldots, \eta_m\}$.
Let $f(\rho) = 1$.

Case 2: $\ell g(\rho) = \omega$, Case 1 does not hold.
Let $f(\rho) = 0$.

Case 3: $\ell g(\rho) = k < \omega$ and for some $\ell < m$, $\ell g(\eta_\ell) = \omega$ and $\mathrm{Suc}_L(\rho, \eta_\ell)$.
Let $f(\rho) = 1$.

Case 4: $\ell g(\rho) = k < \omega$ and for some $\ell < m$, $\ell g(\eta_\ell) = \omega$ and $\mathrm{Suc}_R(\rho, \eta_\ell)$.
Let $f(\rho) = 0$.

Case 5: $\ell g(\rho) < \omega$, $\rho \in \{\eta_\ell : \ell < m\}$.
Let $f(\rho) = 1$.

Case 6: No previous case applies.
Let $f(\rho) = 0$.

First, f is well defined. (I.e. there are no contradictions between cases 3+4, cases 3+5, cases 4+5, as clauses (c), (b), and then (a) of $(*)_4$ fail, respectively.[9]) Second, we show that f respects the equations from Definition 2.1(2); that is, from $(*)_2$ there. If $x_\eta \le x_{\eta\restriction n\hat{}\langle\alpha_n\rangle}$ is an instance of $(*)_2$ of 2.1(2) and f fails it (that is, $f(\eta) = 1$, $f(\eta \restriction n\hat{}\langle\alpha_n\rangle) = 0$) then necessarily by $\ell g(\eta) = \omega$ Case 1 occurs for η, hence Case 3 occurs for $(\eta \restriction n)\hat{}\langle\alpha_n\rangle$. So $f((\eta \restriction n)\hat{}\langle\alpha_n\rangle) = 1$, hence f has to satisfy the equation. Similarly for the other equation in $(*)_2$ of 2.1(2), using Case 4 instead Case 3. Third: $f(x_{\eta_\ell}) = 1$ for $\ell < m$ by Cases 1, 5, and $f(\nu_k) = 0$ for $k < n$ as by failure of clause (d), Case 2 occur if $\ell g(\nu_k) = \omega$, and Case 6 occurs if $\ell g(\nu_k) < \omega$. So by 2.8 we are done proving $(*)_4$]

Let $a_\alpha \in \mathrm{BA}_x(I) \setminus \{0\}$ for $\alpha < \lambda = (2^{\aleph_0})^+$, so as before without loss of generality $a_\alpha = x_{\eta_{\alpha,0}} \cap \ldots \cap x_{\eta_{\alpha,n_\alpha-1}} \cap (-x_{\eta_{\alpha,n_\alpha}}) \cap \ldots \cap (-x_{\eta_{\alpha,m_\alpha-1}}) \neq 0$. Without loss of generality $n_\alpha = n^*$, $m_\alpha = m^*$ and $P^I_\omega \cap \{\eta_{\alpha,\ell} : \ell < m^*\} \neq \varnothing$ (for notational simplicity below). We can define $\eta_{\alpha,\ell}$ (for $m^* \le \ell < \omega$) such that

$$\mathrm{Suc}_L(\rho, \eta_{\alpha,\ell}) \vee \mathrm{Suc}_R(\rho, \eta_{\alpha,\ell}) \Rightarrow \rho \in \{\eta_{\alpha,k} : k < \omega\}$$

Without loss of generality the atomic type of $\langle \eta_{\alpha,\ell} : \ell < \omega\rangle$ in I does not depend on α, and they form a Δ-system: i.e.

$(*)$ $\eta_{\alpha,\ell_1} = \eta_{\beta,\ell_2} \wedge \alpha \neq \beta \ \Rightarrow \ (\forall \alpha_1, \beta_1 < \lambda)[\eta_{\alpha_1,\ell_1} = \eta_{\alpha_1,\ell_2} = \eta_{\beta_1,\ell_1} = \eta_{\beta_1,\ell_2}]$.

Now we apply $(*)_4$: check that each case fails.

3) Without loss of generality we deal with $K^\omega_{\mathrm{tr}(h,g)}$. Let $a_\alpha \neq 0$ $(\alpha < \lambda)$ be non-zero pairwise disjoint elements, let $a_\alpha = \sigma_\alpha(\bar{x}_{\bar\eta_\alpha})$, σ_α a Boolean term, $\bar\eta_\alpha$ a finite sequence from I, (i.e. we write $\bar{x}_{\langle \eta_{\alpha,0},\ldots,\eta_{\alpha,k_\alpha-1}\rangle}$ instead of $\langle x_{\eta_{\alpha,0}}, \ldots, x_{\eta_{\alpha,k_\alpha-1}}\rangle$). Without loss of generality $\sigma_\alpha = \sigma$ and $\bar\eta_\alpha = \langle \eta_{\alpha,0}, \ldots, \eta_{\alpha,k-1}\rangle$ is without repetition, and

$$a_\alpha = \bigcap_{\ell < k(0)} x_{\eta_{\alpha,\ell}} \cap \bigcap_{k(0) \le \ell < k} (-x_{\eta_{\alpha,\ell}})\,.$$

[9] Actually, cases 3+5 cannot contradict.

So there is $n(\alpha) < \omega$ such that $\ell g(\eta_{\alpha,\ell}) < \omega \Rightarrow \ell g(\eta_{\alpha,\ell}) \leq n(\alpha)$, and $\ell g(\eta_{\alpha,\ell(1)}) = \ell g(\eta_{\alpha,\ell(2)}) = \omega$, $\ell(1) \neq \ell(2)$ implies

$$\eta_{\alpha,\ell(1)} \restriction n(\alpha) \neq \eta_{\alpha,\ell(1)} \restriction n(\alpha)$$

and $(\forall n)[n \geq n(\alpha) - 1 \Rightarrow h(n) \geq 3]$.

Without loss of generality, if $m < n(\alpha)$, $\ell g(\eta_{\alpha,i}) > m+1$, $\eta_{\alpha,i}(m) = \langle \gamma_0, \gamma_1, \ldots \rangle$ then $(\eta_{\alpha,i} \restriction m)\hat{\ }\langle \gamma_j \rangle$ belongs to $\{\eta_{\alpha,0}, \eta_{\alpha,1}, \ldots\}$ (for we can change $\bar{\eta}_\alpha$ and σ_α, and then uniformize σ_α, k again).

Now without loss of generality $n(\alpha) = n^*$ for every α, $\ell g(\eta_{\alpha,i}) = \ell_i \leq \omega$, and the truth value of $(\eta_{\alpha,i_1} \restriction m)\hat{\ }\langle \eta_{\alpha,i_1}(m)(m') \rangle = \eta_{\alpha,i_2}$ does not depend on α. Also (by the theorem on Δ-systems) for every $m < k$, $\langle \eta_{\alpha,m} : \alpha < \lambda \rangle$ is constant or is without repetition. Also there is $j_m \leq n^*$ such that $\eta_{\alpha,m} \restriction j_m$ is constant, but $\langle \eta_{\alpha,m}(j_m) : \alpha < \lambda \rangle$ is an indiscernible sequence in I satisfying either $\bullet_1$ or $\bullet_2$, where

- $\bullet_1$ The $\eta_{\alpha,m}(j_m)$ are pairwise distinct tuples of length $h(j_m)$, and $j_m + 1 < \ell_m$.
- $\bullet_2$ The $\eta_{\alpha,m}(j_m)$ are singletons and $j_m + 1 = \ell_m$.

(Recall that $<_1^I$ is a well ordering; that is, we use "I is standard.") It follows that: $i_1, i_2 < k$, $\alpha, \beta, \gamma < \lambda$, $\ell \leq n^*$, and $\eta_{\alpha,i_1} \restriction \ell = \eta_{\beta,i_2} \restriction \ell$ implies

$$\eta_{\alpha,i_1} \restriction \ell = \eta_{\alpha,i_2} \restriction \ell = \eta_{\gamma,i_1} \restriction \ell = \eta_{\gamma,i_2} \restriction \ell.$$

Let $\alpha < \beta < \lambda$, and we shall prove $a_\alpha \cap a_\beta \neq 0$. For notational simplicity let $\alpha = 0$ and $\beta = 1$. Now we shall define a function f from I to the trivial Boolean algebra $\mathbf{B}_0 = \{0, 1\}$.

Let

$\boxplus$ (a) $u = \{\ell < k(0) : \ell g(\eta_{0,\ell}) = \omega,\ \eta_{0,\ell} \restriction n^* = \eta_{1,\ell} \restriction n^*\}$
  (b) For $\ell \in u$, let $n_\ell = \min\{n < \omega : \eta_{0,\ell}(n) \neq \eta_{1,\ell}(n)\} \geq n^*$.
  (c) For $\ell \in u$ and $n \geq n_\ell$, let
   - $\rho_\ell^n = \eta_{0,\ell} \restriction n$
   - $\langle \alpha_{\ell,i}^n : i < h(n) \rangle$ is equal to $\eta_{0,\ell}(n)$.
   - $\langle \beta_{\ell,i}^n : i < h(n) \rangle$ is equal to $\eta_{1,\ell}(n)$.
   - $\Lambda_\ell^n = \{\rho_\ell^n\hat{\ }\langle \alpha_{\ell,i}^n \rangle, \rho_\ell^n\hat{\ }\langle \beta_{\ell,i}^n \rangle : i < h(n)\}$.

Now

$\oplus$ For $\ell \in u$ and $n \geq n_\ell$, there is a function $f_\ell^n : \Lambda_\ell^n \to \{0,1\}$ such that:
  (a) If $g(n) > 0$ then
   $\bullet_1$ $\big(\exists i < g(n)\big)\big[f_\ell^n\big(\rho_\ell^n\hat{\ }\langle \alpha_{\ell,i}^n \rangle\big) = 1\big]$
   $\bullet_2$ $\big(\exists i < g(n)\big)\big[f_\ell^n\big(\rho_\ell^n\hat{\ }\langle \beta_{\ell,i}^n \rangle\big) = 1\big]$
  (b) If $f(n) > g(n)$ then
   $\bullet_1$ $(\exists i)\big[(g(n) \leq i < f(n)) \wedge f_\ell^n\big(\rho_\ell^n\hat{\ }\langle \alpha_{\ell,i}^n \rangle\big) = 0\big]$
   $\bullet_2$ $(\exists i)\big[(g(n) \leq i < f(n)) \wedge f_\ell^n\big(\rho_\ell^n\hat{\ }\langle \beta_{\ell,i}^n \rangle\big) = 0\big]$.

Why? The proof is by splitting into cases.

- If $g(n) = 0$ let f_ℓ^n be constantly 0.

[Why is this OK? Now $\oplus$(a) is empty and $\oplus$(b) is trivial, as $f(n) > 0$.]

- If $g(n) = 1$ let f_ℓ^n map $\rho_\ell^n\hat{\ }\langle \alpha_{\ell,0}^n \rangle$ and $\rho_\ell^n\hat{\ }\langle \beta_{\ell,0}^n \rangle$ to 1 and everything else in Λ_ℓ^n to zero.

[Why is this OK? Because $h(n) \geq 3$ so $h(n) - g(n) \geq 2$.]

- If $g(n) \geq 2$ let f_ℓ^n map $\rho_\ell^n\hat{\ }\langle \alpha_{\ell,g(n)}^n \rangle$ and $\rho_\ell^n\hat{\ }\langle \beta_{\ell,g(n)}^n \rangle$ to 1 and everything else to zero.

[Why is this OK? Similar to the previous case.]

Obviously exactly one of the cases hold, so we are done proving $\oplus$.

We define f so that $f(\eta) = 1$ iff one of the following cases occurs:

$\odot$ (a) $\eta = \eta_{j,\ell}$, where $j < 2$ and $\ell < k(0)$.
(b) For some $\ell \in u$ and $n \geq n_\ell$, we have $\eta \in \Lambda^n_\ell$ and $f^n_\ell(\eta) = 1$.
(c) $\ell < k(*)$, $\ell \notin u$, $n \geq n^*$, $j < 2$, and $\eta = (\eta_{j,\ell} \restriction n)\hat{\ }\langle \eta_{j,\ell}(n)(0) \rangle$.

Clearly f is well defined. Also,

(*) If $\ell \in [k(0), k)$ and $j \in \{0, 1\}$ then $f(\eta_{j,\ell}) = 0$.

[Why? Let $\eta = \eta_{j,\ell}$ and assume toward contradiction that (*) fails. There are three possible reasons for $f(\eta_{j,\ell}) = 1$. The first is clause (a) of $\odot$ above; that is, $\eta = \eta_{j(1),\ell(1)}$, where $j(1) \in \{0, 1\}$ and $\ell(1) < k(0)$, but for $j \neq j(1)$ this is impossible by the "cleaning" above, and if $j = j(1)$ this is impossible as $a_j \neq 0$.

The second is clause $\odot$(b); so for some $\ell \in u$ and $n \geq n_\ell$, we have $\eta_{j,\ell} \in \Lambda^n_\ell$, but this implies $\ell g(\eta_{j,\ell}) < \omega$. But we have assumed $\ell g(\eta_{j,\ell}) < \omega \;\Rightarrow\; \ell g(\eta_{j(1),\ell(1)}) \leq n^*$ while

$$(\eta_{j(1),\ell(1)} \restriction i)\hat{\ }\langle \alpha^{j(1),\ell(1)}_m \rangle$$

appears in the sequence $\langle \eta_{j(\ell),\ell} : \ell < k \rangle$, so we have an easy contradiction.

The third is clause $\odot$(c), which is easy as well.

It is enough to prove that there is a homomorphism f from $\mathrm{BA}_{\mathrm{tr}(h,g)}[I]$ to $\{0, 1\}$ such that $\hat{f}(x_\eta) = f(\eta)$ as then we are done because clearly (by (*), and f's definition) $\hat{f}(a_0) = \hat{f}(a_1) = 1$. To prove this we have to show that the identities appearing in the definition of $\mathrm{BA}_x[I]$ are respected by f. Such an identity looks like

$\oplus$ $x_\rho \leq \bigcup\limits_{m=0}^{g(i)} x_{(\rho\restriction i)\hat{\ }\langle\alpha_m\rangle}$ or $x_\rho \cap \bigcap\limits_{m=g(i)+1}^{h(i)-1} x_{(\rho\restriction i)\hat{\ }\langle\alpha_m\rangle} = 0$, where $\rho \in P^I_\omega$ and $\rho(i) = \langle \alpha_0, \ldots, \alpha_{h(i)-1} \rangle$.

If $f(\rho) = 0$ they hold trivially, so we should consider only the case $f(\rho) = 1$. As $\ell g(\rho) = \omega$, necessarily $\rho = \eta_{j(*),\ell(*)}$ for some $j(*) < 2$ and $\ell(*) < k(0)$. (In the other cases in the definition of f where $f(x_\rho) = 1$, the sequence ρ is finite.) If $i < n^*$ then $(\rho \restriction i)\hat{\ }\langle \alpha_m \rangle \in \{\eta_{j(*),\ell} : \ell < k\}$ for every $m < h(i)$; so as $a_j > 0$, by clause (a) of the definition of f and by (*) we can finish. So assume $i \geq n^*$. Now if $\eta_{1-j(*),\ell(*)} \restriction i \neq \rho \restriction i$ then $f((\rho \restriction i)\hat{\ }\langle \alpha_m \rangle)$ is 1 if $m = 0$, and is 0 if $m \neq 0$, so clearly the two equations in ($\oplus$) hold. We are left with case

$$\eta_{1-j(*),\ell(*)} \restriction i = \rho \restriction i \quad (= \eta_{j(*),\ell(*)} \restriction i)$$

and $i \geq n^*$. So we just use the definition of f^i_ℓ.

4) Like part 2).

5) Like part (3). $\square_{2.6}$

Claim 2.9. *1) If $\mathbf{B}_1, \mathbf{B}$ satisfy the strong λ-chain condition, $a^* \in \mathbf{B}_1 \setminus \{0_{\mathbf{B}_1}\}$, and $\mathbf{B}_2$ is the result of a $\mathbf{B}$-surgery of $\mathbf{B}_1$ at a^*, then $\mathbf{B}_2$ satisfies the strong λ-chain condition. If one of $\mathbf{B}_1$, $\mathbf{B}$ satisfies the strong λ-chain condition, and the other only the λ-chain condition, then $\mathbf{B}_2$ satisfies the λ-chain condition.*

2) If $\mathbf{B}_2$ is the result of a $\mathbf{B}$-surgery of $\mathbf{B}_1$ at a^*, then $\mathbf{B}_1 \leq \mathbf{B}_2$ (i.e., $\mathbf{B}_1$ is a subalgebra of $\mathbf{B}_2$, and every maximal antichain of $\mathbf{B}_1$ is a maximal antichain of $\mathbf{B}_2$. This is also called "$\mathbf{B}_2$ is a regular extension of $\mathbf{B}_1$").

Proof. Well known (and easy). $\square_{2.9}$

Claim 2.10. *The relation $\lessdot$ between Boolean algebras is a partial order, and if a sequence $\langle \mathbf{B}_i : i < \alpha \rangle$ is $\lessdot$-increasing continuous* <u>*then*</u> $\mathbf{B}_0 \lessdot \bigcup_{i<\alpha} \mathbf{B}_i$, *and if each* $\mathbf{B}_i$ *satisfies the strong χ-chain condition (for a regular χ),* <u>*then*</u> *so does* $\bigcup_{i<\alpha} \mathbf{B}_i$.

Proof. Well known: Solovay-Tenenbaum [ST71] for the χ-chain condition, and Kunen-Tall [KT79, p.179] for the strong χ-chain condition. $\square_{2.9}$

Claim 2.11. *1) In Construction 2.4, if $|I_i| = \lambda$ (hence $|\mathrm{BA}_x(I_i)| = \lambda$ for $i < \alpha$)* <u>*then*</u> $\|\mathbf{B}_i\| = \lambda$ *for* $0 < i \le \alpha$.

2) In 2.4, if each $\mathrm{BA}_x(I_i)$ satisfies the strong χ-chain condition and χ is regular <u>*then*</u> $\mathbf{B} = \mathrm{Sur}_x\langle I_i, a_i^* : i < \alpha \rangle$ *satisfies the (strong) χ-chain condition.*

3) Assume that in 2.4 we use non-trivial $\mathbf{B}_0$ and $|I_i| = \lambda$. <u>*Then*</u> $\|\mathbf{B}\| = \lambda + \|\mathbf{B}_0\|$. *If in addition $\mathbf{B}_0$ satisfies the λ-cc, and each $\mathrm{BA}_x(I_i)$ satisfies the strong λ-chain condition,* <u>*then*</u> $\mathbf{B}$ *satisfies the λ-cc; if in addition $\mathbf{B}_0$ satisfies the strong λ-cc,* <u>*then*</u> *so does* $\mathbf{B}$.

Proof. 1) Trivial.

2) By 2.5, 2.6, 2.9, 2.10.

3) Similar. $\square_{2.11}$

Lemma 2.12. *1) For the construction in 2.4, $\mathbf{B}_\alpha$ is weakly representable in $\mathscr{M}^*_{\aleph_0,\aleph_0}\big(\sum_{i<\alpha} I_i\big)$ (see Definition* [Shei, 2.4=L2.2(c),(d)]*).*

2) Moreover, $\mathbf{B}_\alpha \restriction (1 - a_i^*)$ is weakly representable in $\mathscr{M}^*_{\aleph_0,\aleph_0}\big(\sum_{j<\alpha, j\neq i} I_j\big)$.

3) If $\mathbf{B}_\alpha$ satisfies the θ-chain condition <u>*then*</u> $\mathbf{B}^c_\alpha$ *(the completion of $\mathbf{B}_\alpha$) can be weakly represented in $\mathscr{M}^*_{\theta,\theta}(\sum_{j<\alpha} I_j)$. This representation can extend the one from 2.12(1).*

4) Similarly for 2.12(2).

5) If in 2.4 we use a non-trivial $\mathbf{B}_0$, we have to adapt. For example, assume $\mathbf{B}_0$ is weakly representable in a relevant way (e.g., for (1) assume $\mathbf{B}_0$ is weakly represented in $\mathscr{M}_{\aleph_0,\aleph_0}(J + \sum_{i<\alpha} I_i)$).

Proof. 1) Define $f(0) = 0$, $f(1) = 1$. Given $b \in \mathbf{B}_\alpha$ not equal to 0 or 1, say that b first appears in $\mathbf{B}_{i+1}$.

Say

$$b = \big(b', \bigcup_{j<m} (c_j \cap d_j)\big)$$

with $b' \in \mathbf{B}_i \restriction (-a_i^*)$ and $c_j \in \mathbf{B}_i \restriction a_i^*, d_j \in \mathrm{BA}_x(I_i)$. Say (by induction hypothesis) $f(b') = x'$, $f(c_j) = x_j$, $f(a_i^*) = x$, $d_j = \sigma_j(x_{\eta_0}, \ldots, x_{\eta_{m-1}})$ where σ is a Boolean term, and $\eta_0, \ldots, \eta_{m-1} \in I_i$.

Then we set

$$f(b) = F_k(x, x', x_0, \ldots, x_{m-1}, \eta_0, \ldots, \eta_{m-1}),$$
$$k \text{ codes } \langle m, n, \sigma_0, \ldots, \sigma_{m-1} \rangle,$$

where F_k is a suitable function symbol. Thus, $f(b)$ codes all the relevant information about b.

2) We may assume that $a_i^* \neq 0, 1$. We go exactly as in (1) up to $\mathbf{B}_i$. For $\alpha > i$, we use $(-a_i^*)$ in place of 1, and working always with $\mathbf{B}_\alpha \restriction (-a_i^*)$. Note that no terms involving I_i appear then.

3) For each $a \in \mathbf{B}^c_\alpha$ we can fix $\kappa < \theta$ and a sequence $\langle b_\gamma : \gamma < \kappa \rangle$ of elements of $\mathbf{B}_\alpha$ such that $a = \bigcup\limits_{\gamma<\kappa} b_\gamma$. Then let $f_\alpha = F(\sigma_\gamma : \gamma < \kappa)$, where $f(b_\gamma) = \sigma_\gamma$ for all $\gamma < \kappa$.

4),5) Similarly. $\square_{2.12}$

Remark 2.13. 1) In 2.15-2.16 below we can omit the 'weak' from representation and the 'strong' from unembeddability.

2) Why weakly represented? As the order of the construction and the choice of the a^*_i play a role in the definition, we can overcome this in various ways but there is no real reason for doing this

Lemma 2.14. *1) Suppose $I \in K^\omega_{\rm tr}$ is strongly $(\aleph_0, \aleph_0, \psi_{\rm tr})$-unembeddable into $J \in K^\omega_{\rm tr}$, and $\mathbf{B}$ is a Boolean algebra weakly representable in $\mathscr{M}_{\aleph_0,\aleph_0}(J)$. Then $\mathrm{BA}_{\rm tr}(I)$ is not embeddable into $\mathbf{B}$.*

2) Suppose $I \in K^\omega_{\rm tr}$ is strongly $(\mu, \kappa, \psi_{\rm tr})$-unembeddable into J for embeddings which are strongly finitary on P^I_ω, and $\mathbf{B}$ a Boolean algebra weakly represented in $\mathscr{M}_{\mu,\kappa}(J)$. Then $\mathrm{BA}_{\rm tr}(I)$ is not embeddable into $\mathbf{B}$.

Proof. 1) Let $g : B \to \mathscr{M}_{\aleph_0,\aleph_0}(J)$ be a weak representation of $\mathbf{B}$ into $\mathscr{M}_{\aleph_0,\aleph_0}(J)$ (with the well ordering $<^*$), and h be an embedding of $\mathrm{BA}_{\rm tr}(I)$ into $\mathbf{B}$. For $\eta \in I$ define $f(\eta) = g(h(x_\eta))$. As I is strongly $(\aleph_0, \aleph_0, \psi_{\rm tr})$-unembeddable into J, there are ν_1, ν_2, η, n such that $\eta \in P^I_\omega$, $\nu_1 = \eta \restriction (n+1)$, $\nu_1 \restriction n = \nu_2 \restriction n$, $\nu_2(n) <^J_1 \nu_1(n)$, $\ell g(\nu_1) = \ell g(\nu_2) = n+1$, and

$$\big\langle f(\nu_1), f(\eta)\big\rangle \approx \big\langle f(\nu_2), f(\eta)\big\rangle \mod \big(\mathscr{M}^*_{\aleph_0,\aleph_0}(J), <^*\big).$$

Hence (because g is a weak representation)

$$h(x_\eta) < h(x_{\nu_1}) \Leftrightarrow h(x_\eta) < h(x_{\nu_2}) \quad \text{(in } \mathbf{B}).$$

But h is an embedding, hence $x_\eta < x_{\nu_1} \Leftrightarrow x_\eta < x_{\nu_2}$ in $\mathrm{BA}_{\rm tr}(I)$, contradicting the definition of $\mathrm{BA}_{\rm tr}(I)$.

2) Similar. $\square_{2.14}$

Lemma 2.15. *1) Suppose I, $J \in K^\omega_{\rm ptr}$ and I is standard, strongly $(\mu, \kappa, \psi_{\rm ptr})$-unembeddable into J by f strongly finitary on P^I_ω. If $\mathbf{B}$ is a Boolean algebra weakly representable in $\mathscr{M}_{\aleph_0,\aleph_0}(J)$ (say, by g), $\mathbf{B} \subseteq \mathbf{B}_1$ dense[10] in $\mathbf{B}_1$, and g_1 extends g and is a weak representation of $\mathbf{B}_1$ in $\mathscr{M}_{\mu,\kappa}(J)$, then $\mathrm{BA}_{\rm ptr}(I)$ is not embeddable into $\mathbf{B}_1$.*

2) Analogously for $K^\omega_{{\rm tr}(h)}$, $\psi_{{\rm tr}(h)}$, $\mathrm{BA}_{{\rm tr}(h)}(-)$ (for $h \in {}^\omega(\omega \setminus 2)$) and $K^\omega_{{\rm tr}(h)}$, $\psi_{{\rm tr}(h,g)}$, $\mathrm{BA}_{{\rm tr}(h,g)}(-)$.

3) If $I \in K^\omega_{{\rm tr}(h)}$ is standard, $(\aleph_0, \aleph_0)$-super^{vr} unembeddable into $J \in K^\omega_{{\rm tr}(h)}$, $\mathbf{B}$ is weakly represented in $\mathscr{M}_{\aleph_0,\aleph_0}(J)$ and satisfies the ccc (for example $\mathrm{Rang}(h) \subseteq [3, \omega)$) then $\mathrm{BA}_{{\rm tr}(h)}(I)$ is not embeddable into the completion of $\mathbf{B}$.

Proof. 1) Suppose $\mathbf{f}$ is an embedding of $\mathrm{BA}_{\rm ptr}(I)$ into $\mathbf{B}_1$. For $\eta \in I$, define $f(\eta)$ as follows: if $\ell g(\eta) < \omega$ then $f(\eta) = g_1(\mathbf{f}(x_\eta))$, whereas if $\ell g(\eta) = \omega$, choose $a_\eta \in \mathbf{B}$, $0 < a_\eta \leq \mathbf{f}(x_\eta)$ (possible as $\mathbf{B}$ is dense in $\mathbf{B}_1$) and let $f(\eta) = g(a_\eta)$. As I is strongly $(\mu, \kappa, \psi_{\rm ptr})$-unembeddable into J by a function f which is strongly finitary on P^I_ω, there are ν_1, ν_2, η, n such that $\eta \in P^I_\omega$, $\nu_1 = \eta \restriction n\hat{\ }\langle\alpha\rangle$, $\nu_2 = \eta \restriction n\hat{\ }\langle\beta\rangle$, $\eta(n) = \langle\alpha, \beta\rangle$, $\alpha < \beta$, and

$$\big\langle f(\nu_1), f(\eta)\big\rangle \approx \big\langle f(\nu_2), f(\eta)\big\rangle \mod (M_{\mu,\kappa}(J), <^*).$$

[10] E.g. $\mathbf{B}_1$ is the completion of $\mathbf{B}$ — the case that interests us.

Hence, as g_1 is a weak representation

$$(*) \qquad \begin{array}{rcl} \mathbf{B}_1 \models \mathbf{f}(a_\eta) < \mathbf{f}(x_{\nu_1}) & \Leftrightarrow & \mathbf{B}_1 \models \mathbf{f}(a_\eta) < \mathbf{f}(x_{\nu_2}), \\ \mathbf{B}_1 \models \mathbf{f}(a_\eta) \cap \mathbf{f}(x_{\nu_1}) = 0 & \Leftrightarrow & \mathbf{B}_1 \models \mathbf{f}(a_\eta) \cap \mathbf{f}(x_{\nu_2}) = 0. \end{array}$$

But in $\mathrm{BA}_{\mathrm{ptr}}(I)$, $x_{\nu_1} \geq x_\eta$, $x_{\nu_2} \cap x_\eta = 0$. Hence, as $\mathbf{f}$ is an embedding,

$$\mathbf{B}_1 \models \text{“}\mathbf{f}(x_{\nu_1}) \geq \mathbf{f}(x_\eta) \wedge \mathbf{f}(x_{\nu_2}) \cap \mathbf{f}(x_\eta) = 0\text{”}.$$

But $0 < a_\eta \leq \mathbf{f}(x_\eta)$, so $\mathbf{f}(x_{\nu_1}) \geq a_\eta$, $\mathbf{f}(x_{\nu_2}) \cap a_\eta = 0$, a contradiction to $(*)$ above.

We have proved that $\mathrm{BA}_{\mathrm{ptr}}(I)$ is not embeddable into $\mathbf{B}_1$.

2) Similar proof (the extra details appear in the proof of part (3)).

3) Note that this is not used. Assume toward contradiction that $\mathbf{f}$ is an embedding of $\mathrm{BA}_{\mathrm{tr}(h)}(I)$ into $\mathbf{B}_1$, the completion of $\mathbf{B}$. Let $g : \mathbf{B} \to \mathscr{M}_{\aleph_0,\aleph_0}(J)$ be a weak representation (say, for the well ordering $<^*$) of $\mathscr{M}_{\aleph_0,\aleph_0}(J)$ which respects subterms. So by 2.4(3) there is $g_1 : \mathbf{B}_1 \to \mathscr{M}_{\aleph_1,\aleph_1}(J)$ which extends g and is a weak representation of $\mathbf{B}_1$ in $(\mathscr{M}_{\aleph_1,\aleph_1}(J), <^*)$. Choose a function $f : I \to \mathscr{M}_{\aleph_1,\aleph_1}(J)$ as in the proof of part (1). Let $x = \langle h, g, g_1, f, I, J, \mathbf{B}, \mathbf{B}_1\rangle$ and let χ be large enough.

As it is assumed in part (3) that “I is $(\aleph_0,\aleph_0)$-super$^{\mathrm{vr}}$ unembeddable into J,” there are M, η as in $(*)'$ of Definition 1.4(2). Let $f(\eta) = \sigma_\eta(x_{\nu_{\eta,0}}, \ldots, x_{\nu_{\eta,k(\eta)-1}})$, where $\nu_{\eta,k} \in J$ are pairwise distinct for $k < k(\eta)$. For each k let $n_k \leq \omega$ be maximal such that $\nu_{\eta,k} \restriction n_k \in M$: it exists by clause (v) in $(*)'$ of Definition 1.4(2). If $n_k < \ell g(\nu_{k,\ell})$ then for each $m < h(n_k)$ let $\nu^*_{k,m} = (\nu_{\eta,k} \restriction n_k)\hat{\ }\langle s_{k,m}\rangle \in M$ be $<^J_1$-minimal such that $\mathrm{Res}^m_{n_k}(\nu_{\eta,k}) <^J_1 \nu^*_{k,m}$. Clearly it exists, except when $\mathrm{Res}^m_{n_k}(\nu_{\eta,k})$ is $<^J_1$-above every $\{(\nu_{\eta,k} \restriction n_k)\hat{\ }\langle s\rangle : s \in M\}$; in that case we let $s_{k,m} = \infty$ with the obvious conventions.

Let $\bar\nu := \langle \nu_k : k < k(\eta)\rangle$. We define

$$Y^* = \big\{\bar\nu : \bar\nu \text{ is similar in } J \text{ to } \langle \nu_{\eta,0}, \ldots, \nu_{\eta,k(\eta)-1}\rangle \text{ over } Z^*\big\}$$

where $Z^* = \big\{\nu_{\eta,k} : \nu_{\eta,k} \in M\big\} \cup \big\{\nu^*_{k,m} : k < k(\eta),\ m < h(k(\eta))\big\}$. Clearly Z^* is a finite subset of M. We define a filter D on Y^*: $Y \in D$ iff there are $\nu'_{k,m} <^J_1 \nu^*_{k,m}$ for all relevant k, m such that if $\langle \nu''_k : k < k(\eta)\rangle$ satisfies $\nu'_{k,m} \leq^J_1 \nu''_{k,m}$ for all relevant k and m then $\langle \nu''_k : k < k(\eta)\rangle \in Y$.

Clearly $(Y^*, D) \in M$, and by weak representability the following function f_1 belongs to M:

$$\mathrm{dom}(f_1) = \{\varrho \in I : \ell g(\varrho) < \omega\}, \quad \mathrm{rang}(f_1) \leq \{0,1\},$$

$$f_1(\varrho) = \begin{cases} 1 & \big\{\bar\nu \in Y^* : \mathrm{BA}_{\mathrm{tr}(h)}(J) \models \mathbf{f}(x_\varrho) \geq \sigma_\eta(x_{\nu_0}, \ldots, x_{\nu_{k(\eta)-1}})\big\} \in D \\ & \quad \underline{\text{iff}} \text{ that set is } \neq \varnothing \mod D \\ 0 & \text{otherwise.} \end{cases}$$

Recall that σ_η is a $\tau_{\aleph_0,\aleph_0}$-term, hence it is $\in M$. So by the choice of M and η, for infinitely many n, (as $\tau = \tau_0$; see Definition 1.4), we have that the truth values of $\mathrm{BA}_{\mathrm{tr}(h)}(J) \models \mathbf{f}(x_{\mathrm{Res}^\ell_n(\eta)}) \geq \sigma_\eta(x_{\nu_{\eta,0}}, \ldots, x_{\nu_{\eta,k(\eta)-1}})$ are the same for all $\ell < h(n)$. As $\mathbf{f}$ is an embedding, $\mathbf{B}_1 \models \mathbf{f}(x_\eta) \geq \sigma_\eta(x_{\nu_{\eta,0}}, \ldots, x_{\nu_{\eta,k(\eta)-1}}) > 0$, and $\mathrm{BA}_{\mathrm{tr}(h)}(I) \models x_{\mathrm{Res}^0_n(\eta)} \geq x_\eta$, we have

$$\mathbf{B}_1 \models \text{“}\mathbf{f}(x_{\mathrm{Res}(\eta^0_n(\eta))}) \geq \mathbf{f}(x_\eta) \geq f(\eta) = \sigma_\eta(x_{\nu_{\eta,0}}, \ldots x_{\nu_\eta, k(\eta)-1} > 0.\text{”}$$

So $f_1(\mathrm{Res}^0_n(\eta)) = 1$, hence by the choice of n we have $\ell < h(n) \Rightarrow f_1(\mathrm{Res}^\ell_n(\eta)) = 1$. So $\mathbf{B}_1 \models$ “$\bigcap\limits_{\ell < h(n)} \mathbf{f}(x_{\mathrm{Res}^\ell_n(\eta)}) \cap f(x_\eta) > 0$”, but $\mathbf{f}$ is an embedding and $\mathrm{BA}_{\mathrm{tr}(h)}(J) \models$ “$0 < f(\eta) \leq \mathbf{f}(x_\eta)$” hence $\mathrm{BA}_{\mathrm{tr}(h)}(I) \models \bigcap\limits_{\ell < n} x_{\mathrm{Res}^\ell_n(\eta)} \cap x_\eta > 0$, contradicting the definition of $\mathrm{BA}_{\mathrm{tr}(h)}(I)$. $\square_{2.15}$

Conclusion 2.16. *Suppose $\lambda > \aleph_0$. Then:*

(A) There is a rigid Boolean algebra $\mathbf{B}$ satisfying the $\aleph_1$-chain condition λ.

(B) Moreover, if $a, b \in \mathbf{B}$ are $\neq 0$, $a - b \neq 0$, then $\mathbf{B} \restriction a$ cannot be embedded into $\mathbf{B} \restriction b$ (hence $\mathbf{B}$ has no one-to one endomorphism $\neq \mathrm{id}$).

(C) Moreover, we can find such $\mathbf{B}_i$ (for $i < 2^\lambda$) with $|\mathbf{B}_i| = \lambda$; and if $a \in \mathbf{B}_i$, $b \in \mathbf{B}_j$ with $i \neq j$ or $a - b \neq 0$ then $\mathbf{B}_i \restriction a$ cannot be embedded into $\mathbf{B}_j \restriction b$.

Proof. We leave it to the reader as the next proof is similar (but here we should use $(\lambda, \lambda, \aleph_0, \aleph_0)$-$\psi_{\mathrm{tr}}$-bigness, Theorem [Shea, 2.20=L7.11], and $x = \mathrm{tr}$ instead of $(\lambda, \lambda, 2^{\aleph_0}, \aleph_1)$-$\psi_{\mathrm{tr}(h)}$-bigness, [Shea, 1.11=L7.6], and $x = \mathrm{tr}(h)$ there respectively. (Also, we have dealt with it in [Shei, 2.16=L2.7]). $\square_{2.16}$

Conclusion 2.17. *1) There is a complete Boolean algebra $\mathbf{B}$ satisfying the ccc, having density λ*

$$\text{(in fact, } a \in \mathbf{B} \setminus \{0\} \ \Rightarrow\ \mathbf{B} \restriction a \text{ has density } \lambda, \text{ so } |\mathbf{B}| = \lambda^{\aleph_0})$$

and monorigid (i.e., every one-to-one endomorphism is the identity) provided that:

$(*)_1$ K^ω_{ptr} has the full strong $(\lambda, \lambda, 2^{\aleph_0}, \aleph_1)$- ψ_{ptr}-bigness property for f strong finitary on P_ω, by standard atomically $(< \aleph_1)$-stable $I \in K^\omega_{\mathrm{ptr}}$.

2) We can replace $(*)_1$ by $(*)_2 \vee (*)_3 \vee (*)_4$, where for some $h \in {}^\omega(\omega \setminus 3)$:

$(*)_2$ λ is as in 1.11(1) or

$(*)_3$ $K^\omega_{\mathrm{tr}(h)}$ has the full strong $(\lambda, \lambda, 2^{\aleph_0}, \aleph_1)$-$\psi_{\mathrm{tr}(h)}$-bigness property or

$(*)_4$ $K^\omega_{\mathrm{tr}(h)}$ has the full super$^{\mathrm{vr}}(\lambda, \lambda, 2^{\aleph_0}, \aleph_1)$-bigness property.

3) Moreover, we can find such $\mathbf{B}_i$ (for $i < 2^\lambda$) satisfying the following: if $a \in \mathbf{B}_i \setminus \{0\}$, $b \in \mathbf{B}_j \setminus \{0\}$, $[i \neq j \vee (i = j \wedge a - b \neq 0_{\mathbf{B}_i})]$, then $\mathbf{B}_i \restriction a$ cannot be embedded into $\mathbf{B}_j \restriction b$.

Proof. We first prove parts (1) and (2). For part (1) let $h \in {}^\omega\omega$ be constantly 2. First note that if f is a one-to-one endomorphism $\neq \mathrm{id}$ of any Boolean algebra $\mathbf{B}$, then there is an element $a \neq 0$ with $a \cap f(a) = 0$. First, choose x with $x \neq f(x)$. If $x \cap -f(x) \neq 0$ we can take $a = x \cap -f(x)$; if $-x \cap f(x) \neq 0$ we can take $a = -x \cap f(x)$. Hence for (1) and (2) we only need to find $\mathbf{B}$ of power λ such that if $a, b \in \mathbf{B}$ are non-zero and $a - b \neq 0$ (and even $a \cap b = 0$), then $\mathbf{B} \restriction a$ cannot be embedded in $\mathbf{B} \restriction b$.

Now let $\langle I_\alpha : \alpha < \lambda \rangle$ exemplify the full strong $(\lambda, \lambda, 2^{\aleph_0}, \aleph_1)$-$\psi_{\mathrm{tr}(h)}$-bigness property for f strongly finitary on P_ω; such a sequence exists by $(*)_1$ or $(*)_2$ or $(*)_3$ or $(*)_4$ by 1.11(1), 1.9 for any $h \in {}^\omega(\omega \setminus 3)$. Let $\mathbf{B} = \mathrm{Sur}_x\langle I_\alpha, a^*_\alpha : \alpha < \lambda \rangle$ be as in the construction 2.4 for $x = \mathrm{tr}(h)$, such that $\mathbf{B} \setminus \{0\} = \{a^*_\alpha : \alpha < \lambda\}$. Then by 2.11(1), $|\mathbf{B}| = \lambda$. By 2.6(3), 2.6(4), each $\mathrm{BA}_{\mathrm{tr}(h)}(I_\alpha)$ satisfies the strong $\aleph_1$-cc, hence by 2.11 the Boolean algebra $\mathbf{B}$ satisfies the $\aleph_1$-chain condition. Let $\mathbf{B}^*$ be its completion. Now let $a, b \in \mathbf{B}^*$ be non-zero, with $c = a - b \neq 0$. Toward contradiction, suppose f is an embedding of $\mathbf{B}^* \restriction a$ into $\mathbf{B}^* \restriction b$. Then $f(c) \cap c = 0$, and $f \restriction (\mathbf{B} \restriction c)$ is an embedding of $\mathbf{B}^* \restriction c$ into $\mathbf{B}^* \restriction f(c)$. But $\mathbf{B}$ is dense in $\mathbf{B}^*$ hence $a^*_\alpha \leq c$ for some α, hence $\mathrm{BA}_{\mathrm{tr}(h)}(I_\alpha)$ is embeddable into $\mathbf{B}^* \restriction c$, hence into $\mathbf{B}^* \restriction f(c)$, hence into $\mathbf{B}^* \restriction (-c) = \mathbf{B}^* \restriction (-a^*_\alpha)$. But by 2.12(3), $\mathbf{B} \restriction (-a^*_\alpha)$ is weakly representable in $\mathscr{M}^*_{2^{\aleph_0},\aleph_1}(\sum_{\beta \neq \alpha, \beta < \lambda} I_\beta)$. This contradicts 2.15 when we assume $(*)_4$.

For part (3) let $\langle I_{\alpha,\beta} : \alpha, \beta < \lambda \rangle$ rename $\langle I_\alpha : \alpha < \lambda \rangle$. We shall choose, for $\xi < 2^\lambda$, functions f_ξ, g_ξ from λ to λ and $A_\xi \in [\lambda]^\lambda$ such that g_ξ is one-to-one, $\mathrm{Rang}(f_\xi) = A_\xi$, $(\forall \alpha \in A_\xi)(\exists^\lambda \beta < \lambda)[f_\xi(\beta) = 1]$, and $\xi_1 \neq \xi_2 \Rightarrow A_{\xi_1} \not\subseteq A_{\xi_2}$. For $\xi < 2^\lambda$, let $\mathbf{B}^\xi$ be constructed as $\mathrm{Sur}_x\langle I_{f_\xi(\alpha), g_\xi(\alpha)}, a^\xi_\alpha : \alpha < \lambda \rangle$. For simplicity, assume that for some ξ, for every $a \in \mathbf{B}^\xi \setminus \{0\}$ and $\zeta \in A_\xi$, we have $a^\xi_\alpha = a$ and $f_\xi(\alpha) = \zeta$. Let $\mathbf{B}^{\xi,*}$ be the completion of $\mathbf{B}^\xi$. As g_ξ is one-to-one clearly $\mathbf{B}^\xi$ satisfies the demand in (2), and as $\xi \neq \zeta < 2^\lambda \Rightarrow A_\xi \not\subseteq A_\zeta$ the demands in (3) also hold. $\square_{2.17}$

Conclusion 2.18. *1) For $\lambda > \aleph_0$, there is a Boolean algebra $\mathbf{B}$ of cardinality λ with no non-trivial endomorphism onto itself. Moreover, it is Bonnet rigid (defined below).*

2) We can find such $\mathbf{B}_i$ (for $i < 2^\lambda$) such that for $i, j < 2^\lambda$, $a \in \mathbf{B}_i \setminus \{0\}$, $b \in \mathbf{B}_j \setminus \{0\}$ there is no embedding of $\mathbf{B}_i \restriction a$ into a homomorphic image of $\mathbf{B}_j \restriction b$ except when $i = j \wedge a \leq b$.

We prove it later.

Remark 2.19. We shall use Boolean algebras built from cases of $\mathrm{BA}_{\mathrm{trr}}(I)$ (see Definition 2.1(4)) hence they have no long chains. We can go in the inverse direction using Boolean algebras built from orders — using, for example, $\mathrm{LO}(I)$ the linear order with elements $\{x_\eta, y_\eta : \eta \in I\}$ such that:

(A) $\ell g(\eta) < \omega$ implies $x_\eta < y_\eta$, $y_{\eta\hat{}\langle\alpha\rangle} < x_{\eta\hat{}\langle\beta\rangle}$ for $\alpha < \beta$, and $x_{\eta\restriction n} < x_\eta < y_\eta < y_{\eta\restriction n}$ for $n < \ell g(\eta)$.

(B) $\ell g(\eta) = \omega$ implies $x_{\eta\restriction n} < x_\eta = y_\eta < y_{\eta\restriction n}$ for $n < \omega$.

In such cases we need a parallel to Lemma 2.23, which is true.

We make some preparations to the proof of 2.18.

Definition 2.20. A Boolean algebra $\mathbf{B}$ is called Bonnet–rigid <u>iff</u> there are no Boolean algebra $\mathbf{B}'$ and homomorphisms $\mathbf{f}_\ell : \mathbf{B} \to \mathbf{B}'$ (for $\ell = 0, 1$) such that $\mathbf{f}_0$ is one-to-one and $\mathbf{f}_1$ is onto $\mathbf{B}'$, except when $\mathbf{f}_0 = \mathbf{f}_1$.

Observation 2.21. *1) If $\mathbf{B}$ is Bonnet–rigid <u>then</u> it has no onto endomorphism $\neq \mathrm{id}_{\mathbf{B}}$.*

2) A Boolean algebra $\mathbf{B}$ is Bonnet–rigid <u>if</u>:

(∗) *For no disjoint non-zero $a, b \in \mathbf{B}$ is there an embedding of $\mathbf{B} \restriction a$ into a homeomorphic image of $\mathbf{B} \restriction b$.*

Proof. 1) Otherwise choose $\mathbf{B}' = \mathbf{B}$, $\mathbf{f}_0$ the identity, and $\mathbf{f}_1$ the given endomorphism.

2) Towards contradiction, assume $\mathbf{f}_\ell : \mathbf{B} \to \mathbf{B}'$ (for $\ell = 0, 1$) contradict Bonnet–rigidity. First, suppose $\mathbf{f}_1$ is not one-to-one, so for some $a \in \mathbf{B}$, $a \neq 0$, $\mathbf{f}_1(a) = 0$.

For any $b \in \mathbf{B}$, $\mathbf{f}_1(b - a) = \mathbf{f}_1(b) - \mathbf{f}_1(a) = \mathbf{f}_1(b)$. So $\mathbf{B}'$ is a homomorphic image of $\mathbf{B} \restriction (1 - a)$ and $\mathbf{B} \restriction a$ can be embedded into it, so we are finished.

Second, assume $\mathbf{f}_1$ is one-to-one. Then $\mathbf{f}_1$ is an isomorphism from $\mathbf{B}$ onto $\mathbf{B}'$ hence $\mathbf{f}_1^{-1}\mathbf{f}_0 : \mathbf{B} \to \mathbf{B}$ is an embedding (well defined as f_1 is one to one and onto). It is not the identity (otherwise $\mathbf{f}_0 = \mathbf{f}_1$) so for some $a \in \mathbf{B}$, the elements $a, \mathbf{f}_1^{-1}\mathbf{f}_0(a)$ are disjoint and non-zero; choose $b = \mathbf{f}_1^{-1}\mathbf{f}_0(a)$. $\square_{2.21}$

To prove 2.18, we shall use $\mathrm{BA}_{\mathrm{trr}}(I)$ (see Definition 2.1(4)). Note:

Claim 2.22. *1) The only atoms of $\mathrm{BA}_{\mathrm{trr}}(I)$ are x_η, where η ($\in I$) has no immediate successor, or at least*

(∗) *For all $\nu_1, \nu_2 \in I$, we have $\eta \lhd \nu_1 \wedge \eta \lhd \nu_2 \Rightarrow \nu_1, \nu_2$ are $\lhd$-comparable.*

2) The set $\{x_\eta : \eta \in I\}$ is a dense subset of $\mathrm{BA}_{\mathrm{trr}}(I)$.

Proof. Check. $\square_{2.22}$

Lemma 2.23. *If* $\mathbf{B}$ *is a homomorphic image of* $\mathbf{B}_0 = \mathrm{BA}_{\mathrm{trr}}(I)$, *then* $\mathbf{B}$ *is isomorphic to some* $\mathrm{BA}_{\mathrm{trr}}(J)$, J *weakly representable in* $\mathscr{M}_{\aleph_0,\aleph_0}(I)$ *hence* $\mathbf{B}$ *is weakly representable in* $\mathscr{M}_{\aleph_0,\aleph_0}(I)$.

Proof. So let $\mathbf{J}$ be an ideal of $\mathbf{B}_0$ such that $\mathbf{B}$ is isomorphic to $\mathbf{B}_0/\mathbf{J}$. Let

$$I_1 = \{\eta \in I : x_\eta \notin \mathbf{J}\};$$

I_1 is an approximation to J. (Clearly I_1 is closed under initial segments by 2.1(4)(b).) Let

$$\begin{aligned} A_0 = \big\{\eta \in I_1 : \ & \eta \text{ has } < \aleph_0 \text{ immediate successors in } I_1, \text{ say} \\ & \eta\hat{\ }\langle\alpha_\ell\rangle \ \text{ for } \ell < m, \ \text{ and } \big(x_\eta - \bigcup_\ell x_{\eta\hat{\ }\langle\alpha_\ell\rangle}\big) \in \mathbf{J}\big\}, \\ A_1 = \big\{\eta \in I_1 : \ & \eta \text{ has } < \aleph_0 \text{ immediate successors in } I_1, \text{ say} \\ & \eta\hat{\ }\langle\alpha_\ell\rangle \ \text{ for } \ell < m, \ \text{ and } \big(x_\eta - \bigcup_\ell x_{\eta\hat{\ }\langle\alpha_\ell\rangle}\big) \notin \mathbf{J}\big\}, \\ A_3 = \big\{(\eta,\nu) : \ & \eta \in A_0, \ \eta \lhd \nu \in I_1, \ \ell g(\nu) \text{ is limit}, \ x_\eta - x_{\nu\restriction i} \in \mathbf{J}, \\ & \text{when } \ell g(\eta) \le i < \ell g(\nu) \text{ and for no } \eta' \lhd \eta \text{ does } (\eta',\nu) \\ & \text{have those properties}\big\}, \end{aligned}$$

and let $A_4 = \big\{(\eta,\nu) \in A_3 : x_\eta - x_\nu \notin \mathbf{J}\big\}$.

Now for $\eta \in I$ let $\alpha_\eta = \min\{\alpha : \eta\hat{\ }\langle\alpha\rangle \notin I\}$.

Put

$$J = I_1 \cup \ \{\eta\hat{\ }\langle\alpha_\eta\rangle : \eta \in A_1\} \cup \ \{\eta\hat{\ }\langle\alpha_\eta + 1\rangle : (\eta,\nu) \in A_4\}.$$

Now $\mathrm{BA}_{\mathrm{trr}}(J)$ is isomorphic to $\mathbf{B}$, and the lemma should be clear. $\square_{2.23}$

Now we can turn to

Proof. **Proof of 2.18**:
1) Let $\langle I_\alpha : \alpha < \lambda\rangle$ exemplify that K^ω_{tr} has the full strong $(\lambda,\lambda,\aleph_0,\aleph_0)$-bigness property, I_α standard.

Without loss of generality:

$(*)_1$ $\alpha \ne \beta \ \Rightarrow \ I_\alpha \cap I_\beta = \{\langle\ \rangle\}$

$(*)_2$ If $\nu \in I_\alpha$ then for some η we have $\nu \trianglelefteq \eta \in I_\alpha$ and $\ell g(\eta) = \omega$.

We construct as in 2.4, using $\mathrm{BA}_{\mathrm{trr}}(I_\alpha)$ (i.e., $x = \mathrm{trr}$ there) but making the surgeries on atoms only, getting $\mathbf{B} = \mathrm{Sur}\langle I_\alpha, a^*_\alpha : \alpha < \lambda\rangle$. Looking at the construction, it is clear that $\mathbf{B} = \mathrm{BA}_{\mathrm{trr}}(I^*)$, where

$$\begin{aligned} I^* = \big\{\eta_1\hat{\ }\eta_2\hat{\ }\ldots\hat{\ }\eta_n : n < \omega, \ \eta_\ell \in I_{\alpha_\ell} \text{ for some } \alpha_\ell < \lambda, \text{ and for } \ell < n \\ \text{we have } \ell g(\eta_\ell) = \omega \text{ and } a^*_{\alpha_\ell + 1} \text{ is } x_{\eta_\ell}\big\}. \end{aligned}$$

By 2.21(2), it suffices to prove:

$(**)$ If a, b are disjoint non-zero and $\mathbf{B}'$ is a homomorphic image of $\mathbf{B} \restriction b$ then $\mathbf{B} \restriction a$ cannot be embedded into $\mathbf{B}'$.

Suppose $(**)$ fails and a, b, $\mathbf{B}'$ exemplify this. By Claim 2.22 and $(*)$, there is $\eta \in I^*$ with $x_\eta \le a$ and $\ell g(\eta)$ limit, and let α be such that $a^*_\alpha = x_\eta$. Clearly $\mathbf{B}'$ is also a homomorphic image of $\mathbf{B} \restriction (1 - x_\eta)$, hence by 2.23 it is weakly representable in $\mathscr{M}^*_{\aleph_0,\aleph_0}\big(\sum\limits_{j<\lambda, j\ne\alpha} I_j\big)$ and $\mathbf{B}' \cong \mathrm{BA}_{\mathrm{trr}}(I^+)$ for some I^+ weakly representable in $\mathscr{M}_{\aleph_0,\aleph_0}\big(\sum\limits_{j<\lambda, j\ne\alpha} I_j\big)$.

We can conclude:

$(***)$ $\mathrm{BA}_{\mathrm{trr}}(I_\alpha)$ is weakly representable in $\mathscr{M}_{\aleph_0,\aleph_0}(\sum\limits_{j<\lambda, j\neq\alpha} I_j)$.

But from this the contradiction is trivial (we could avoid the "weakly").
2) No new point. $\square_{2.18}$

§ 3. Arbitrary length of a Boolean Algebra with no small infinite homomorphic image

We recall the definition of the length (and length$^+$) of a Boolean algebra (Definition 3.2). Our aim is to construct a Boolean algebra of cardinality continuum with no infinite homomorphic image of smaller cardinality. Toward this, for a Boolean algebra $\mathbf{B}$, an ω-sequence $\langle a_n : n < \omega \rangle$ of pairwise disjoint members of $\mathbf{B} \setminus \{0_{\mathbf{B}}\}$ and $I \in K^{\omega}_{\mathrm{tr}(h)}$, we define in Definition 3.3 an extension $\mathbf{B}' = \mathrm{ba}[\mathbf{B}, \bar{a}, I]$ of $\mathbf{B}$. We shall use it for h with $\langle h(n) : n < \omega \rangle$ going to infinity. The properties we need are that $\mathbf{B} \lessdot \mathbf{B}'$, $\|\mathbf{B}'\| \le 2^{\aleph_0}$, and $\mathbf{B}'$ satisfies the ccc.[11] Moreover, a stronger version of $\mathbf{B} \lessdot \mathbf{B}'$ holds (see 3.4(5)).

Also, if $\mathbf{f}$ is a homomorphism from $\mathbf{B}'$ into any Boolean algebra $\mathbf{B}''$ satisfying $n < \omega \Rightarrow \mathbf{f}(a_n) > 0$ (in $\mathbf{B}''$) then $\mathbf{B}'$ has at least $2^{\aleph_0}$ elements (see inside the proof of 3.6). Theorem 3.6 is the main result: if $\mu \in [\aleph_1, 2^{\aleph_0}]$ then some ccc Boolean algebra $\mathbf{B}$ of cardinality $2^{\aleph_0}$ and length μ has no infinite homomorphic image of cardinality $< 2^{\aleph_0}$. For this we take care of every antichain $\langle a_n : n < \omega \rangle$ by an extension $\mathrm{ba}[-, \bar{a}, I]$. We start with a ccc Boolean algebra of length and cardinality μ. In this framework we need to show that the length has not increased by the construction. For this we prove, by induction on the length of the construction, that for any family of μ^+ finite sequences from the Boolean algebra and $m < \omega$, there is a subfamily of μ^+ finite sequences which is an indiscernible set. We may like to consider a limit $\mu \in [\aleph_1, 2^{\aleph_0})$ and ask above that its length is μ but the supremum is not obtained; by a similar construction (of length $2^{\aleph_0} \times \mu$) we get such a Boolean algebra, provided that $\mathrm{cf}(\mu)$ is uncountable (see 3.10). If $\mathrm{cf}(\mu) = \aleph_0$ this is impossible (see 3.12). We then generalize the results, replacing $\aleph_0$ by any strong limit κ of cofinality $\aleph_0$.

Convention 3.1. h will be from ${}^{\omega}(\omega \setminus \{0\})$ and for simplicity ${}^{\omega}(\omega \setminus \{0, 1\})$. Actually $h = 2$ suffices,[12] <u>but</u> if we like to have the ccc we'd better use $h \ge 3$.

Definition 3.2. For a Boolean algebra $\mathbf{B}$ let

$$\begin{aligned} \mathrm{length}(\mathbf{B}) &= \sup\{|A| : A \subseteq \mathbf{B},\ A \text{ is linearly ordered by } <_{\mathbf{B}}\}, \\ \mathrm{length}^+(\mathbf{B}) &= \sup\{|A|^+ : A \subseteq \mathbf{B},\ A \text{ is linearly ordered by } <_{\mathbf{B}}\}. \end{aligned}$$

Definition 3.3. For a Boolean Algebra $\mathbf{B}^*$, $\bar{a} = \langle a_n : n < \omega \rangle \subseteq \mathbf{B}^* \setminus \{0_{\mathbf{B}^*}\}$ such that $\bigwedge_{n<m} a_n \cap a_m = 0$, and $I \in K^{\omega}_{\mathrm{tr}(h)}$, we define a Boolean Algebra $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I]$ as follows.

It is freely generated by $\mathbf{B}^* \cup \{x_\eta : \eta \in I\}$, except for the following equalities:

(a) All the equalities which $\mathbf{B}^*$ satisfies, and $x_\eta \le 1_{\mathbf{B}^*}$.

(b) If $n < \omega$ is even, $k = h(n) - 1$, $\eta \in P^I_\omega$, $\nu = \eta \restriction n$, and $\eta(n) = \langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \rangle$, then

$$a_n - \bigcup_{\ell < k/2} \big((x_{\nu\hat{\ }\langle \alpha_{2\ell} \rangle} - x_{\nu\hat{\ }\langle \alpha_{2\ell+1} \rangle} \big) \le x_\eta.$$

(c) If $n < \omega$ is odd, $k = h(n) - 1$, $\eta \in P^I_\omega$, $\nu = \eta \restriction n$, and $\eta(n) = \langle \alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_{k-1} \rangle$ then

$$\Big(a_n \cap \bigcap_{\ell < k/2} \big(1 - (x_{\nu\hat{\ }\langle \alpha_{2\ell} \rangle} - x_{\nu\hat{\ }\langle \alpha_{2\ell+1} \rangle})\big) \Big) \cap x_\eta = 0.$$

(d) $x_{\langle\ \rangle_I} = 0$ ($\langle\ \rangle_I$ is the root of I).

[11] See 3.4(1),(3), 3.5, and inside the proof of 3.6.

[12] I.e. $(\forall n)[h(n) = 2]$, so using $K^{\omega}_{\mathrm{ptr}}$.

Claim 3.4. *1) For $\mathbf{B}^*, \bar{a}$, I as in Definition 3.3,* $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I]$ *is an extension of* $\mathbf{B}^*$ *(so the equalities do not cause members of* $\mathbf{B}^*$ *to become identified, and of course* $1_{\mathrm{ba}[\mathbf{B}^*,\bar{a},I]} = 1_{\mathbf{B}^*}$*).*
2) For $I_1, I_2 \in K^\omega_{\mathrm{tr}(h)}$, $\mathbf{B}^*, \bar{a}$ as in Definition 3.3, if $I_1 \subseteq I_2$ then $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$ *is a subalgebra of* $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2]$.
3) In (1), $\mathbf{B}^* \lessdot \mathrm{ba}[\mathbf{B}^*, \bar{a}, I]$.
4) In (2), if also $I_1 \subseteq^* I_2$ (which means that $I_1 \subseteq I_2$ and $\eta \in P^{I_2}_\omega \setminus I_1 \ \Rightarrow\ \bigvee\limits_{n,\ell} \mathrm{Res}^\ell_n(\eta) \notin I_1$) then $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \lessdot \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2]$.
5) In (4), for every non-zero $c \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2]$ there is d^* such that:

- (*i*) $c \le d^* \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$
- (*ii*) *If $0 < b \le d^*$ and $b \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$ then $c \cap b \ne 0$.*

Proof. 1) It is a particular case of (2) for $I_1 = \{\langle\,\rangle\}$, $I_2 = I$.
2) Let $d^* \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \setminus \{0\}$. We would like to prove that $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2] \models d^* \ne 0$; by the definition of these two Boolean algebras (see 3.3), this suffices. Clearly, without loss of generality, for some $\alpha(*) \le \omega$ we have:

$$\alpha(*) < \omega \wedge d^* \le a_{\alpha(*)} \text{ or } \alpha(*) = \omega \wedge (\forall n)[d^* \cap a_n = 0].$$

Now we shall define a function $\mathbf{f} : \mathbf{B}^* \cup \{x_\eta : \eta \in I_2\} \to \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \restriction d^*$, which will map all the equations appearing in the definition of $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2]$ to ones satisfied in $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \restriction d^*$ and maps d^* to itself; this suffices.

Now we define $\mathbf{f} = \mathbf{f}^{d^*}$ as follows:

- (A) For $b \in \mathbf{B}^*$, $\mathbf{f}(b) = b \cap d^*$ (or more exactly, the interpretation of $b \cap d^*$ in $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$).
- (B) For $\eta \in I_1$, $\mathbf{f}(x_\eta) = x_\eta \cap d^*$.
- (C) If $\eta \in P^{I_2}_\omega$, $\eta \notin I_1$, let

$$\mathbf{f}(x_\eta) = \begin{cases} d^* & \text{if } \alpha(*) \text{ is even (including } \alpha(*) = \omega), \\ 0 & \text{if } \alpha(*) \text{ is odd (and } < \omega). \end{cases}$$

- (D) For $\eta \in I_2 \setminus I_1$ such that (C) does not apply, let $\mathbf{f}(x_\eta) = 0$.

Now check: the main point being that the equations in clauses (b)+(c) of Definition 3.3 hold trivially by the present choice in clause (C).
3) Again, it suffices to prove this for the context of (2); i.e. to prove (4).
4) The proof of part (2) above will suffice, provided that we are given $c \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2] \setminus \{0\}$ and we then find $d^* \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$, $d^* \ne 0$, such that we can construct a function $\mathbf{f}$ as there satisfying that the homomorphism $\hat{\mathbf{f}}$ which $\mathbf{f}$ induces from $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2]$ into $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \restriction d^*$ (which is the identity on the latter by its definition) will satisfy $\hat{\mathbf{f}}(c) \ge d^*$. Now as we can decrease c, without loss of generality $c \notin \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$ and c has the form

$$(*) \quad c = d \cap \bigcap_{\ell < m_0} x_{\eta_\ell} \cap \bigcap_{\ell \in [m_0, m)} (1 - x_{\eta_\ell}),$$

with $m_0 < m$, $d \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \setminus \{0\}$, $\eta_\ell \in I_2 \setminus I_1$ for $\ell < m$. We shall show more than is necessary here (but it will be used in part (5)):

- ⊡ If $0 < d' \le d$ and $d' \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$ then some d^* satisfying $0 < d^* \le d'$, $d^* \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$ is as required. (I.e. there is $\mathbf{f}$ as in the previous proof.)

Choose d^* and $\alpha(*) \leq \omega$ satisfying:

(*i*) $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1] \models 0 < d^* \leq d'$

(*ii*) $d^* \leq a_{\alpha(*)} \wedge \alpha(*) < \omega$ or $\bigwedge_{n<\omega} d^* \cap a_n = 0 \wedge \alpha(*) = \omega$.

For $k < m$ let

$$I^*_{2,k} = I_1 \cup \{\eta_k\} \cup \{\mathrm{Res}^\ell_i(\eta_k) : \mathrm{Res}^\ell_i(\eta_k) \text{ is well defined}\}$$

and for $k \leq m$ let $I_{2,k} = \bigcup\{I^*_{2,\ell} : \ell < k\} \cup I_1$. Easily

$$I_1 = I_{2,0} \subseteq^* I_{2,1} \subseteq^* \ldots \subseteq^* I_{2,m} \subseteq I_2.$$

Clearly $c \in \mathrm{ba}[\mathbf{B}, \bar{a}, I_{2,m}]$ and it suffices to prove that $\mathbf{B}^* \lessdot \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_{2,m}]$, so without loss of generality $I_2 = I_{2,m}$. If $\ell g(\eta_k) < \omega$, we can add the $\mathrm{Res}^\ell_i(\eta_k)$ to $I_{2,k}$ one by one.

As $\lessdot$ is transitive, and by part (2) without loss of generality $m = 1$, and one of the following occurs:

(A) $I_2 \setminus I_1 = \{\eta_0\}$ and $\ell g(\eta_0) < \omega$.

(B) $I_2 \setminus I_1 \subseteq \{\eta_0, \mathrm{Res}^\ell_n(\eta_0) : n_0 \leq n < \omega,\ \ell < h(n)\}$ and $\ell g(\eta_0) = \omega$.

In case (A), let $\mathbf{f}(x_{\eta_0})$ be d^* if $m_0 = 1$, and let $\mathbf{f}(x_{\eta_0})$ be 0 if $m_0 = 0$ (and $\mathbf{f}(b) = b \cap d^*$ if $b \in \mathrm{ba}[\mathbf{B}^*, \bar{a}.I_1]$). In case (B), if $\alpha(*) = \omega$ then act similarly; i.e. define $\mathbf{f}(x_\nu) = d^*$ for $\nu \in I_2 \setminus I_1$ if $m_0 = 1$, and 0 if $m_0 = 0$ for $\eta \in I_2 \setminus I_1$. In case (B), if $\alpha(*) < \omega$, by repeated use of case (A), without loss of generality[13]

$$\big(\forall n \leq \alpha(*)\big)\big(\forall \ell < h(n)\big)\big[\mathrm{Res}^\ell_n(\eta_0) \in I_1\big].$$

Let

$$\begin{array}{ll} \mathbf{f}(b) = b \cap d^* & \text{for } b \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1], \\ \mathbf{f}(x_{\eta_0}) = d^* & \text{if } \alpha(*) \text{ is even}, \\ \mathbf{f}(x_{\eta_0}) = 0 & \text{if } \alpha(*) \text{ is odd}, \\ \mathbf{f}(x_{\mathrm{Res}^\ell_n(\eta_0)}) = 0 & \text{whenever } n < \omega,\ \ell < h(n) \text{ and } \mathrm{Res}^\ell_n(\eta_0) \notin I_1. \end{array}$$

Now check.

5) Again without loss of generality I_1, I_2 satisfy (A) or (B) from the proof of (4) (use 3.4(2) and the transitivity of the conclusion) and even c is in the subalgebra of $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I_2]$ generated by $\{x_{\eta_0}\} \cup \mathrm{ba}[\mathbf{B}^*, \bar{a}, I_1]$. Note also that if $c = c_1 \cup c_2$ it suffices to prove the conclusion for c_1 and for c_2.

So without loss of generality $(*)$ in the proof of part (4) holds, so $c \leq d$, and by the proof of part (4), d is as required. $\square_{3.4}$

Claim 3.5. *Assume $h \geq 3$ or just $h(n) \geq 3$ for n large enough. If $\mathbf{B}^*, \bar{a}, I$ are as in Definition 3.3, I standard, $\lambda = \mathrm{cf}(\lambda) > \aleph_0$ and $\mathbf{B}^*$ satisfies the [strong] λ-cc, <u>then</u> $\mathrm{ba}[\mathbf{B}^*, \bar{a}, I]$ satisfies the [strong] λ-cc.*

Proof. Let $c_i \in \mathrm{ba}[\mathbf{B}^*, \bar{a}, I]$ for $i < \lambda$, $c_i \neq 0$. Without loss of generality c_i has the form

$$c_i = d_i \cap \bigcap_{\ell < m_{i,0}} x_{\eta_{i,\ell}} \cap \bigcap_{\ell \in [m_{i,0}, m_{i,1})} (1 - x_{\eta_{i,\ell}}),$$

where $\eta_{i,\ell} \in I$, $d_i \in \mathbf{B}^* \setminus \{0\}$. Without loss of generality $d_i \leq a_{n_i}$ for some $n_i < \omega$ or $n_i = \omega \wedge \bigwedge_{n<\omega} d_i \cap a_n = 0$. Without loss of generality $m_{i,0} = m_0$, $m_{i,1} = m_1$, $\ell g(\eta_{i,\ell}) = n_\ell$, $n_i = n^*$, and $\langle \eta_{i,\ell} : \ell < m_1 \rangle$ is without repetition for every i.

Also letting $k_i < \omega$ be the minimal k such that

[13] But we do not have to use it.

(∗) (a) $\ell g(\eta_{i,\ell}) < \omega \Rightarrow \ \ell g(\eta_{i,\ell}) \le k$
(b) $n^* < \omega \Rightarrow n^* < k$
(c) $\ell_1 < \ell_2 < m_1 \Rightarrow \eta_{i,\ell_1} \restriction k \ne \eta_{i,\ell_2} \restriction k$
(d) $(\forall n)[n \ge k \Rightarrow h(n) \ge 3]$

and without loss of generality $k_i = k^*$; if $\ell g(\eta_{i,\ell}) = \omega$, $k < k^*$, $\ell < h(k)$ then $\mathrm{Res}^\ell_k(\eta_{i,\ell}) \in \{\eta_{i,m} : m < m_{i,1}\}$.

By the Δ-system argument, without loss of generality

(∗) If $i \ne j < \lambda$, $k \le k^*+1$, and $m', m'' < m_1$, $\ell', \ell'' < h(k)$ and $\mathrm{Res}^{\ell'}_k(\eta_{i,m'}) = \mathrm{Res}^{\ell''}_k(\eta_{j,m'',j})$, then for every $\alpha, \beta < \lambda$ we have
$$\mathrm{Res}^{\ell'}_k(\eta_{\alpha,m'}) = \mathrm{Res}^{\ell''}_k(\eta_{\alpha,m''}) = \mathrm{Res}^{\ell'}_k(\eta_{\beta,m'}) = \mathrm{Res}^{\ell''}_k(\eta_{\beta,m''}).$$

We can now check, (similarly to 2.6). $\square_{3.5}$

Theorem 3.6. *Let $\aleph_1 \le \mu < 2^{\aleph_0}$. There is a Boolean Algebra $\mathbf{B}$ such that:*

(A) $\mathbf{B}$ has cardinality $2^{\aleph_0}$ and satisfies the ccc (and even the strong λ-cc if $\lambda = \mathrm{cf}(\lambda) > \aleph_0$).
(B) $\mathbf{B}$ has length μ (i.e. there is in $\mathbf{B}$ a chain of length μ but no chain of length μ^+).

Moreover:

$(B)^+$ If $n, m < \omega$ and $\bar c^\zeta \in {}^m\mathbf{B}$ for $\zeta < \mu^+$ then for some $Y \in [\mu^+]^{\mu^+}$ (i.e. $Y \subseteq \mu^+$ of cardinality μ^+), the sequence $\langle \bar c^\zeta : \zeta \in Y\rangle$ is a (qf, n)-indiscernible set in the Boolean algebra $\mathbf{B}$ (see 3.7(2) below).
(C) Every infinite homomorphic image of $\mathbf{B}$ has cardinality $2^{\aleph_0}$.

Remark 3.7. 1) Note that $(B)^+ \Rightarrow (B)$; for it $m = 1$ suffices, for this constant h is OK below, but the proof here is simpler.
2) Let $\bar{\mathbf{c}} = \langle \bar c^\zeta : \zeta \in Y\rangle$ be a sequence of m-tuples from a model M (for example, a Boolean algebra) and Δ a set of formulas in $\mathbb{L}(\tau_M)$. We say $\bar{\mathbf{c}}$ is an (Δ, n)-indiscernible set iff for any $\zeta_0, \dots, \zeta_{n-1}$ from Y with no repetitions and $\xi_0, \dots, \xi_{n-1}$ from Y with no repetitions, the Δ-type of $\bar c^{\zeta_0}\hat{}\dots\hat{}\bar c^{\zeta_{n-1}}$ in M is equal to the Δ-type of $\bar c^{\xi_0}\hat{}\dots\hat{}\bar c^{\xi_{n-1}}$ in M. For Δ the set of quantifier free formulas we write qf.

Proof. Let $h : \omega \to \omega$ be, for example, $h(n) = 2n+2$.

Let $I_\beta \in K^\omega_{\mathrm{tr}(h)}$ be standard for $\beta < 2^{\aleph_0}$, have cardinality continuum, and be such that:

$(*)_{I_\beta}$ For every $f : I_\beta \to \theta$, $\theta < 2^{\aleph_0}$, for some $\eta \in P^{I_\beta}_\omega$, for every $n < \omega$,
$$\big(\forall \ell < h(n)\big)\big[f(\mathrm{Res}^0_n(\eta)) = f(\mathrm{Res}^\ell_n(\eta))\big]$$
$$\text{(i.e. } \eta(m) = \big\langle \alpha_\ell : \ell < h(n)\big\rangle \Rightarrow \big|\big\{f(\eta \restriction n\hat{}\langle \alpha_\ell\rangle) : \ell < h(n)\big\}\big| = 1.)$$

[Why do such I-s exist? The full tree will serve; that is, we let
$$I_\beta = \big\{\langle \bar\alpha^\ell : \ell < \gamma\rangle : \gamma \le \omega,\ \bar\alpha^\ell \text{ an increasing sequence of length } h(\ell) \text{ from } 2^{\aleph_0}, \text{ except in the case } 0 < \gamma < \omega \wedge \ell = \gamma - 1; \text{ then we demand } \bar\alpha^\ell \text{ is just an ordinal} < 2^{\aleph_0}\big\}.$$

This is as required, as for any $f : I_\beta \to \theta$ we can choose a sequence $\bar\alpha_\ell = \langle \beta_{\ell,0}, \dots, \beta_{\ell,h(\ell)-1}\rangle$ by induction on $\ell < \omega$, where $\beta_{\ell,0} < \ldots < \beta_{\ell,h(\ell)-1} < 2^{\aleph_0}$ and $f(\langle \bar\alpha^0, \dots, \bar\alpha^{\ell-1}, \beta_{\ell,i}\rangle)$ does not depend on i. This is possible as $2^{\aleph_0} > |\mathrm{rang}(f)|$. So I_β-s as required in $(*)_{I_\beta}$ indeed exist.]

We shall now construct $\mathbf{B}_\alpha$ (for $\alpha \le 2^{\aleph_0}$) and $\bar{a}^\alpha = \langle a^\alpha_n : n < \omega\rangle$ such that:

(I) (a) $\mathbf{B}_0$ is a subalgebra of $\mathcal{P}(\omega)$ of cardinality μ with a chain of cardinality μ satisfying the ccc (even the strong $\aleph_1$-cc).
[E.g. let A be a set of μ reals, let h be a one to one function from ω onto the rationals and $\mathbf{B}$ is the Boolean algebra of subset of ω generated by $\big\{\{n : h(n) < a\} : a \in A\big\}$. Clearly $\mathfrak{B}$ has a linearly ordered subset of cardinality μ, e.g. its set of generators. Of course, its length is not $> \mu$ as its cardinality is μ.]

(b) $\mathbf{B}_\alpha$ is increasing continuous, of cardinality $2^{\aleph_0}$ if $\alpha > 0$.

(c) $\bar{a}^\alpha$ is an ω-sequence of pairwise disjoint non-zero elements of $\mathbf{B}_\alpha$.

(d) If $\alpha < 2^{\aleph_0}$, $a_n \in \mathbf{B}_\alpha \setminus \{0_{\mathbf{B}_\alpha}\}$, and $\bigwedge\limits_{n \ne m} a_n \cap a_m = 0$ then for $2^{\aleph_0}$ many ordinals α, we have $\bigwedge\limits_{n<\omega} a_n = a^\alpha_n$.
[You can demand that $\{a^\alpha_n : n < \omega\}$ is a maximal antichain; it does not matter.]

(e) $\mathbf{B}_{\alpha+1} = \mathrm{ba}[\mathbf{B}_\alpha, \bar{a}^\alpha, I_\alpha]$ (We denote the x_η by x^α_η for $\eta \in I_\alpha$.)

There is no problem to do the bookkeeping, and $\mathbf{B}_\alpha \subseteq \mathbf{B}_{\alpha+1}$ by 3.4(1). We shall show that $\mathbf{B} := \mathbf{B}_{2^{\aleph_0}}$ is as required. Obviously $\mathbf{B}$ has cardinality $2^{\aleph_0}$.

By 3.4(3) clearly $\mathbf{B}_\alpha \lessdot \mathbf{B}_{\alpha+1}$, so we can prove by induction on α that $\beta < \alpha \;\Rightarrow\; \mathbf{B}_\beta \lessdot \mathbf{B}_\alpha$, by 2.9, 2.10. We can also prove by induction on α that $\mathbf{B}_\alpha$ satisfies the $\aleph_1$-cc (even the strong λ-cc when $\lambda = \mathrm{cf}(\lambda) > \aleph_0$): the successor stage is proved by 3.5, the limits steps by 2.10. So demand (A) from 3.6 holds. If $\mathbf{f}$ is a homomorphism from $\mathbf{B}$ onto some $\mathbf{B}'$ with $\aleph_0 \le \|\mathbf{B}'\| < 2^{\aleph_0}$ then there are $b_n \in \mathbf{B}' \setminus \{0\}$ pairwise disjoint. Now for some $a_n \in \mathbf{B}$, $\mathbf{f}(a_n) = b_n$ and without loss of generality $\bigwedge\limits_{n \ne m} a_n \cap a_m = 0$ (otherwise use $a'_n = a_n \setminus \bigcup\limits_{m<n} a_m$). Hence for every infinite co-infinite $Y \subseteq \omega$. for some $\alpha = \alpha_Y$:

$$\{a^\alpha_{2n} : n < \omega\} = \{a_n : n \in Y\} \quad \text{and} \quad \{a^\alpha_{2n+1} : n < \omega\} = \{a_n : n \in \omega \setminus Y\}.$$

Now define $g : I_\alpha \to \mathbf{B}'$ by $g(\eta) = \mathbf{f}(x^\alpha_\eta)$, so by the choice of the I_α-s (i.e. by $(*)_{I_\beta}$) for some $\eta = \eta_Y \in P^{I_\alpha}_\omega$, for every n, letting $\eta(n) = \langle \alpha_0, \alpha_i, \dots, \alpha_{h(n)-1}\rangle$, we have

$$\bigwedge_{\ell<h(n)} f(x^\alpha_{\eta\restriction n\hat{}\langle\alpha_\ell\rangle}) = f(x^\alpha_{\eta\restriction n\hat{}\langle\alpha_0\rangle}).$$

Hence $\mathbf{f}\big(x^\alpha_{\eta\restriction n\hat{}\langle\alpha_\ell\rangle} - x^\alpha_{\eta\restriction n\hat{}\langle\alpha_{\ell+1}\rangle}\big) = 0_{\mathbf{B}'}$ for $\ell < h(n) - 1$ and hence

$$\mathbf{f}\big(a^\alpha_n \cap \bigcap_{\ell<\frac{h(n)-1}{2}} \big(1 - (x^\alpha_{\eta\restriction n\hat{}\langle\alpha_{2\ell}\rangle} - x^\alpha_{\eta\restriction n\hat{}\langle\alpha_{2\ell+1}\rangle})\big)\big) = \mathbf{f}(a^\alpha_n)$$

and

$$\mathbf{f}\big(a^\alpha_n - \bigcup_{\ell<\frac{h(n)-1}{2}} \big(x^\alpha_{\eta\restriction n\hat{}\langle\alpha_{2\ell}\rangle} - x^\alpha_{\eta\restriction n\hat{}\langle\alpha_{2\ell+1}\rangle}\big)\big) = \mathbf{f}(a^\alpha_n).$$

Hence (see Definition 3.3)

$$\begin{aligned} n \text{ is even} \quad &\Rightarrow \quad \mathbf{B}' \models \mathbf{f}(a^\alpha_n) \le \mathbf{f}(x^\alpha_\eta),\\ n \text{ is odd} \quad &\Rightarrow \quad \mathbf{B}' \models \mathbf{f}(a^\alpha_n) \cap \mathbf{f}(x^\alpha_\eta) = 0.\end{aligned}$$

Therefore,

$$\begin{aligned} m \in Y \quad &\Rightarrow \quad \text{for some even } n,\ a^\alpha_n = a_m \quad \Rightarrow \quad \mathbf{B}' \models b_m \le \mathbf{f}(x^\alpha_\eta),\\ m \in \omega \setminus Y \quad &\Rightarrow \quad \text{for some odd } n,\ a^\alpha_n = a_m \quad \Rightarrow \quad \mathbf{B}' \models b_m \cap \mathbf{f}(x^\alpha_\eta) = 0.\end{aligned}$$

As this occurs for every infinite co-infinite $Y \subseteq \omega$, for some $\alpha = \alpha_Y$, and $\eta = \eta_Y$, clearly we get $2^{\aleph_0}$ many distinct members of $\mathbf{B}'$ (simply, the $\mathbf{f}(x_{\eta_Y})$), a contradiction. So demand **(C)** of 3.6 holds.

What about the length, i.e, clauses **(B)** and $\mathbf{(B)}^+$? For **(B)**, first note that $\mathbf{B}_0$ has a chain of cardinality μ and hence so does $\mathbf{B}$. If $J \subseteq \mathbf{B}$ is a chain, $|J| = \mu^+$, then $(\mathbf{B})^+$ gives a contradiction and even the "weakly indiscernible sequence" version does because as $\mathbf{B} \models$ ccc, it has no subset of order type μ^+ or $(\mu^+)^*$; but the variant of $(\mathbf{B})^+$ implies just this ($m = 1$ suffices).

So it suffices to prove that clause $\mathbf{(B)}^+$ holds for $\mathbf{B}_\alpha$ by induction on α.

Case 1: $\alpha = 0$.

Trivial (can get $\bar{c}^\zeta$ constant on $Y \in [\mu^+]^{\mu^+}$).

Case 2: α is limit, $\text{cf}(\alpha) \neq \mu^+$.

For some $\beta < \alpha$,

$$Y_1 = \{\zeta < \mu^+ : \bar{c}^\zeta \subseteq \mathbf{B}_\beta\} \in [\mu^+]^{\mu^+},$$

(note that if $\text{cf}(\alpha) < \mu^+$, then we can get $Y_1 = \mu^+$) and use the induction hypothesis.

Case 3: $\text{cf}(\alpha) = \mu^+$.

Let $\langle \beta_\varepsilon : \varepsilon < \mu^+ \rangle$ be an increasing continuous sequence with limit α. Let $n, m, \langle \bar{c}^\zeta : \zeta < \mu^+ \rangle$ be given. Without loss of generality $\bar{c}^\zeta = \langle c^\zeta_\ell : \ell < m \rangle$ is a partition of $1_{\mathbf{B}_\alpha}$ (i.e., $\ell_1 \neq \ell_2 \Rightarrow c^\zeta_{\ell_1} \cap c^\zeta_{\ell_2} = 0$ and $1_{\mathbf{B}_\alpha} = \bigcup\limits_{\ell<m} c^\zeta_\ell$). For each $\zeta < \mu^+$, we can find $a^\zeta_\ell, b^\zeta_\ell \in \mathbf{B}_{\beta_\zeta}$ such that:

(a) $a^\zeta_\ell \leq c^\zeta_\ell \leq b^\zeta_\ell$

(b) $(0 < x \leq b^\zeta_\ell - a^\zeta_\ell) \wedge x \in \mathbf{B}_{\beta_\zeta} \Rightarrow (x \cap c^\zeta_\ell - a^\zeta_\ell \neq 0) \wedge (x - c^\zeta_\ell \neq 0)$.

[Why? By use of 3.4(5)). If ζ is limit then for some $f(\zeta) < \zeta$ we have $\{a^\zeta_\ell, b^\zeta_\ell : \ell < m\} \subseteq \mathbf{B}_{f(\beta_\zeta)}$. By Fodor lemma for some $\varepsilon(*) < \mu^+$ and a stationary set $S \subseteq \mu^+$, we have $\bigwedge\limits_{\zeta \in S} f(\zeta) = \varepsilon(*)$.

So

(c) $\varepsilon \in S \Rightarrow \{a^\varepsilon_\ell, b^\varepsilon_\ell : \ell < m\} \subseteq \mathbf{B}_{\beta_{\dot{\varepsilon}(*)}}$.

Also without loss of generality

(d) If $\varepsilon < \zeta \in S$ then $\{c^\varepsilon_\ell : \ell\} \subseteq \mathbf{B}_{\beta_\zeta}$.

Now apply the induction hypothesis on $\mathbf{B}_{\beta_{\dot{\varepsilon}(*)}}$ and $\langle \bar{a}^\zeta \char`^ \bar{b}^\zeta : \zeta < \mu^+ \rangle$, where $\bar{a}^\zeta = \langle a^\zeta_\ell : \ell < m \rangle$, $\bar{b}^\zeta := \langle b^\zeta_\ell : \ell < m \rangle$.

So there is $Y \in [S]^{\mu^+}$ such that $\langle \bar{b}^\zeta : \zeta \in Y \rangle$ is an (n, qf)-indiscernible set. So let $\zeta_0 < \ldots < \zeta_{n-1}$ be from S and for $k \leq n$ let $\mathbf{B}'_k$ be the subalgebra of $\mathbf{B}$ generated by $X_k = \{\bar{b}^{\zeta_i}_\ell, b^{\zeta_i}_\ell : i < n,\ \ell < m\} \cup \{\bar{c}^{\zeta_i}_\ell : i < k,\ \ell < m\}$. We understand $\mathbf{B}'_0$ by the choice of Y and we can understand $\mathbf{B}'_n$ by clauses (c) + (d) above.

Case 4: $\alpha = \beta + 1$.

Let $n, m < \omega$ and $\bar{c}^\zeta \in {}^m(\mathbf{B}_{\beta+1})$ for $\zeta < \mu^+$ be given, $\bar{c}^\zeta = \langle c^\zeta_\ell : \ell < m \rangle$. So there are $k_{\zeta,0} = k(\zeta, 0) < \omega$, $k_{\zeta,1} = k(\zeta, 1) < \omega$ and $b^\zeta_0, \ldots, b^\zeta_{k_{\zeta,0}-1} \in \mathbf{B}_\beta$, $\eta^\zeta_0, \ldots, \eta^\zeta_{k_{\zeta,1}-1} \in I_\beta$ and Boolean terms σ^ζ_ℓ (for $\ell < m$) such that

$$c^\zeta_\ell = \sigma^\zeta_\ell(b^\zeta_0, \ldots, b^\zeta_{k_{\zeta,0}-1}, x_{\eta^\zeta_0}, \ldots, x_{\eta^\zeta_{k(\zeta,1)}}).$$

Without loss of generality $\langle \eta^\zeta_\ell : \ell < k_{\zeta,1}\rangle$ is a Δ-system.

Without loss of generality $k_{\zeta,0} = k_0$, $k_{\zeta,1} = k_1$, $\sigma^\zeta_\ell = \sigma_\ell$ and $\ell g(\eta^\zeta_\ell) = m_\ell \le \omega$ for every $\zeta < \mu^+$.

Also, there is $k_{\zeta,2} < \omega$ such that:

(II) (α) $\ell g(\eta^\zeta_\ell) < \omega \Rightarrow \ell g(\eta^\zeta_\ell) < k_{\zeta,2}$

(β) $\eta^\zeta_{\ell_1} \ne \eta^\zeta_{\ell_2} \Rightarrow \eta^\zeta_{\ell_1} \restriction k_{\zeta,2} \ne \eta^\zeta_{\ell_2} \restriction k_{\zeta,2}$

(γ) $2n+2 < k_{\zeta,2}$.

Without loss of generality $\bigwedge\limits_\zeta k_{\zeta,2} = k_2$.

Without loss of generality the statement $(*)$ (with k_2 here for k^* there and is $> n$) from the proof of 3.5 holds (essentially being a Δ-system), i.e.

$(*)$ If $i \ne j < \lambda$, $k \le k_2+1$, and $m', m'' < m_1$, $\ell', \ell'' < h(k)$ and $\mathrm{Res}^{\ell'}_k(\eta_{m',i}) = \mathrm{Res}^{\ell''}_k(\eta_{m'',j})$, then for every $\alpha, \beta < \lambda$ we have:

$$\mathrm{Res}^{\ell'}_k(\eta_{m',\alpha}) = \mathrm{Res}^{\ell''}_k(\eta_{m'',\alpha}) = \mathrm{Res}^{\ell'}_k(\eta_{m',\beta}) = \mathrm{Res}^{\ell''}_k(\eta_{m'',\beta}).$$

Let $\bar b^\zeta = \langle b^\zeta_\ell : \ell < k_0\rangle$. By the induction hypothesis, without loss of generality $\left\langle \bar b^\zeta{}^\frown\langle a^\beta_\ell : \ell \le k_2\rangle : \zeta < \mu^+\right\rangle$ is (qf, n)-indiscernible and without loss of generality the sequence $\left\langle \langle \eta^\zeta_\ell \restriction (k_2+1) : \ell < k_1\rangle : \zeta < \mu^+\right\rangle$ is indiscernible (sequence of finite sequences of ordinals).

To finish the proof of 3.6 it suffices to observe 3.8 below. $\square_{3.6}$

Observation 3.8. *If $\mathbf{B}^* = \mathrm{ba}_2[\mathbf{B}, \bar a, I]$, $n^* < \omega$, $I^0 = \{\eta \in I : \ell g(\eta) \le n^*\}$, $Z \subseteq P^I_\omega$, and for every $\nu \in Z$ and $n \ge n^*$ the set*

$$\{\nu' \restriction (n+1) : \nu' \in Z,\ \nu' \restriction n = \nu \restriction n\}$$

has $< \lfloor h(n)/2 \rfloor$ elements, <u>then</u> $\{x_\eta : \eta \in Z\}$ is independent in $\mathbf{B}^*$ over $\mathbf{B}^0 := \mathrm{ba}_2[\mathbf{B}, \bar a, I^0]$, except the equations $c^+_\eta \le x_\eta \wedge c^-_\eta \cap x_\eta = 0$ for $\eta \in Z$, where

$$\begin{aligned} c^+_\eta &= \bigcup\Big\{a_{2n} - \bigcup\{x_{\mathrm{Res}^{2\ell}_{2n}(\eta)} - x_{\mathrm{Res}^{2\ell+1}_{2n}(\eta)} : \ell < h(n)/2\} : 2n < n^*\Big\},\\ c^-_\eta &= \bigcup\Big\{a_{2n+1} - \bigcup\{x_{\mathrm{Res}^{2\ell}_{2n+1}(\eta)} - x_{\mathrm{Res}^{2\ell+1}_{2n+1}(\eta)} : 2\ell+1 < h(2n+1)\} : \\ &\qquad 2n+1 < n^*\Big\}. \end{aligned}$$

(Note: $\eta_1 \restriction n^* = \eta_2 \restriction n^* \ \Rightarrow\ (c^+_{\eta_1}, c^-_{\eta_1}) = (c^+_{\eta_2}, c^-_{\eta_2})$.)

Proof. Let $\mathbf{f}_0$ be any function with domain $X = \{x_\eta : \eta \in Z\}$ such that $\mathbf{f}_0(x_\eta) \in \{c^+_\eta, 1_\mathbf{B} - c^-_\eta\}$, and let

$$J^1 = I^0 \cup X \cup \{\mathrm{Res}^\ell_n(\nu) : \nu \in Z,\ \ell < h(n),\ n < \omega\}.$$

Clearly, by 3.4(2) it suffices to find a homomorphism from $\mathbf{B}_1 := \mathrm{ba}[\mathbf{B}, \bar a, J^1]$ into $\mathbf{B}^0$ extending $\mathrm{id}_{\mathbf{B}^0} \cup \mathbf{f}_0$. For this it suffices to find a mapping $\mathbf{f}$ from $\mathbf{B} \cup \{x_\eta : \eta \in J^1\}$ into $\mathbf{B}^0$ extending $\mathrm{id}_{\mathbf{B}^0}, \mathbf{f}_0$, and $\mathrm{id}_{\{x_\eta : \eta \in I^0\}}$, and preserving the equations defining $\mathrm{ba}[\mathbf{B}, \bar a, J^1]$. As $\mathbf{f} \restriction \mathbf{B}$, $\mathbf{f} \restriction \{x_\eta \in I : \ell g(\eta) \le n^*\}$, and $\mathbf{f} \restriction \{x_{(\eta,\varrho)} : \eta \in Z\}$ are defined, and

$$J^1 = \bigcup_{n \in [n^*, \omega)} \{x_\eta : \eta \in Z_n\} \cup X \cup Z$$

where $Z_n = \{\eta \in J^1 : \ell g(\eta) = n+1\}$, it will suffice to choose $\mathbf{f} \restriction \{x_\eta : \eta \in Z_n\}$ for each $n \in [n^*, \omega)$ to finish the definition of $\mathbf{f}$.

Let $Y_n = \{\nu \restriction n : \nu \in Z_n\}$, and for $\eta \in Y_n$ let $X_{n,\eta} = \{\nu \in Z_n : \nu \restriction n = \eta\}$. Clearly $\langle X_{n,\eta} : \eta \in Y_n\rangle$ is a partition of Z_n.

For $\eta \in Y_n$ let $\mathscr{P}_{n,\eta} = \{\nu \restriction (n+1) : \nu \in Z,\ \nu \restriction n = \eta\}$ and

$$\mathcal{S}_{n,\eta} = \big\{(\rho \restriction n)\hat{\ }\langle \rho(n)(\ell)\rangle : \ell < k(n)\big\}.$$

By the assumption on Z, for every $\eta \in Y_n$ the set $\mathscr{P}_{n,\eta}$ has $< h(n)/2$ elements. Now:

(∗) For $\eta \in Y_n$ there is a function $f_\eta : \mathcal{S}_{n,\eta} \to \{0_{\mathbf{B}^*}, 1_{\mathbf{B}^*}\}$ such that if $\nu \in \mathscr{P}_{n,\eta}$ is equal to $\eta\hat{\ }\langle \alpha_0, \dots, \alpha_{h(n)-1}\rangle$ then for some $\ell < (h(n)-1)/4$ we have $f_\eta(x_{\eta\hat{\ }\langle\alpha_{2\ell}\rangle}) = 1_{\mathbf{B}^*}$ and $f_\eta(x_{\eta\hat{\ }\langle\alpha_{2\ell+1}\rangle}) = 0_{\mathbf{B}^*}$.

[Why is this possible? By finite cardinality considerations.]

Now define $\mathbf{f} \restriction Z_n$ as follows: if $\nu \in Z_n$ then $\nu \in \mathscr{P}_{n,\eta}$ for some $\eta \in Y_n$, and so we let $\mathbf{f}(x_\nu) = f_\eta(x_\nu)$.

Now check. $\square_{3.8}$

Discussion 3.9. 1) In the proof of clause $(B)^+$, the successor case we use the fact that $h(n)$ converges to ∞, as when the level increases we need more $\eta \in P^{I_\beta}_\omega$ to see non-freeness.

2) The proof there for limit α uses just "$\langle \mathbf{B}_i : i \le 2^{\aleph_0}\rangle$ is $\lessdot$-increasing continuous with projections" (i.e. 3.4(5)), and the induction hypothesis.

3) We can vary the construction in some ways. We can demand that each $\bar{a}^\alpha$ is a maximal antichain — no difference so far. We may like to use $\langle I_\beta : \beta < 2^{\aleph_0}\rangle$ such that I_β is not super unembeddable into $\sum_{\gamma \ne \beta} I_\gamma$. We can construct our Boolean algebra to be monorigid (i.e., with no one-to-one endomorphism), and even get $2^{2^{\aleph_0}}$ such Boolean algebras, no one embeddable to another: even restricting to non-zero elements, even not embeddable into the completion of another. To carry this out we need the following for $\lambda = 2^{\aleph_0}$: there is $\bar{I} = \langle I_\alpha : \alpha < \lambda\rangle$ exemplifying that $K^\omega_{\mathrm{tr}(h)}$ has the full $(\lambda, \lambda, \aleph_1, \aleph_1)$-super bigness property, such that for at least one β, I_β satisfies $(*)_{I_\beta}$ from the beginning of the proof of 3.6. Now such a $\bar{I}$ does exist (with $(*)_{I_\beta}$ for every β); this may be elaborated elsewhere.

4) Of course the proof works for $\mu = 2^{\aleph_0}$.

5) We can separate some parts of the proof to independent claims. We can ask for "$\mathbf{B}$ has length μ, but no chain of cardinality μ" (i.e. the supremum is not obtained) for μ limit. It is natural to demand $\mathrm{cf}(\mu) > \aleph_0$. Next, we address this.

Claim 3.10. *1) Assume $2^{\aleph_0} \ge \mu$ and $\aleph_0 < \kappa = \mathrm{cf}(\mu) < \mu$. Then there is a Boolean algebra $\mathbf{B}$ such that $|\mathbf{B}| = 2^{\aleph_0}$, $\mathbf{B}$ has no homomorphic image of cardinality $\in [\aleph_0, 2^{\aleph_0})$, and $\mathrm{length}(\mathbf{B}) = \mu$, but the supremum is not obtained (i.e. $\mathrm{length}^+(\mathbf{B}) = \mu$ and every infinite homomorphic image $\mathbf{B}'$ of $\mathbf{B}$ has length $\ge \mu$).*

2) Similarly, but slightly modifying the assumption to $\aleph_0 < \kappa = \mathrm{cf}(\mu) = \mu$.

Proof. Like 3.6.

1) Let $\mu = \sum_{i<\kappa} \mu_i$ with $\langle \mu_i : i < \kappa\rangle$ be increasing continuous and $\kappa < \mu_i < \mu$. For $\varepsilon < \kappa$, let $\mathbf{B}^\varepsilon$ be a subalgebra of $\mathcal{P}(\omega)$ of cardinality μ_ε and length μ_ε. Let $\langle I_\alpha : \alpha < 2^{\aleph_0} \times \kappa\rangle$ be as in the proof of 3.6. We define $\mathbf{B}_\alpha$ (for $\alpha \le 2^{\aleph_0} \times \kappa$) similarly to the proof of 3.6. Specifically:

$\mathbf{B}_0 = \mathbf{B}^0$, the trivial Boolean algebra,
$\mathbf{B}_{2^{\aleph_0}\times\varepsilon+1}$ is the free product $\mathbf{B}_{2^{\aleph_0}\times\varepsilon} * \mathbf{B}^\varepsilon$,
$\mathbf{B}_\alpha$ is increasing continuous in α,
$\mathbf{B}_{\alpha+1} = \mathrm{ba}[\mathbf{B}_\alpha, \bar{a}^\alpha, I_\alpha]$ for $\alpha < 2^{\aleph_0} \times \kappa,\ \alpha \notin \big\{2^{\aleph_0} \times \varepsilon : \varepsilon < \kappa\big\}$,

where $\bar{a}_\alpha$ is $\langle a_{\alpha,n} : n < \omega\rangle$, $a_{\alpha,n} \in \mathbf{B}_\alpha$, $a_{\alpha,n} > 0$, $n_1 \ne n_2 \Rightarrow a_{\alpha,n_1} \cap a_{\alpha,n_2} = 0$. The choice of the $\bar{a}_\alpha$-s (i.e. the bookkeeping) is as in the proof of 3.6 above.

So, by the proof of 3.6:

(∗) If $0 < \alpha < 2^{\aleph_0} \times \kappa$ then
- (α) $\mathbf{B}_\alpha$ satisfies the strong λ-cc if $\lambda = \mathrm{cf}(\lambda) > \aleph_0$.
- (β) $\mathbf{B}_{1+\alpha}$ has length $\mu_0 + \sum\{\mu_\varepsilon : 2^{\aleph_0} \times \varepsilon < \alpha\} < \mu$.
- (γ) If $\alpha = 2^{\aleph_0} \times \varepsilon$ with ε a successor ordinal, then $\mathbf{B}_\alpha$ has no homomorphic image of cardinality $\in [\aleph_0, 2^{\aleph_0})$.
- (δ) If $\alpha < \beta \leq 2^{\aleph_0} \times \kappa$ and $b \in \mathbf{B}_\beta \setminus \{0_{\mathbf{B}_\beta}\}$ then for some $a \in \mathbf{B}_\alpha$ we have $\mathbf{B}_\beta \models b \leq a$ and if $\mathbf{B}_\alpha \models 0 < a' \leq a$ then $a' \cap b > 0_{\mathbf{B}_\beta}$.

[Note: for clause (β) we use the proof of $(\mathrm{B})^+$ of 3.6. For $\alpha = 2^{\aleph_0} \times \varepsilon + 1$ for clause (δ) we have a new clause, but easy one].

It follows that

(∗∗) $\mathbf{B} = \mathbf{B}_{2^{\aleph_0} \times \kappa}$ has length μ.

Now we just need to show

(∗∗∗) For $J \subseteq \mathbf{B}$ (with $|J| = \mu$) a chain we get a contradiction.

Let $\mathbf{B}^*_\varepsilon = \mathbf{B}_{2^{\aleph_0} \times \varepsilon}$. Let $c_\alpha \in J$ (for $\alpha < \mu$) be pairwise distinct.

By clause $(*)(\delta)$, for each $\varepsilon < \kappa$ and $\alpha < \mu^+_\varepsilon$ we can find $b^\varepsilon_\alpha \in \mathbf{B}^*_\varepsilon$ such that:

(a) $c_\alpha \leq b^\varepsilon_\alpha$
(b) $0 < x \leq b^\varepsilon_\alpha \wedge x \in \mathbf{B}^*_\varepsilon \;\Rightarrow\; x \cap c_\alpha \neq 0$

Note:

(c) b^ε_α is unique, and
(d) $c_\alpha \leq c_\beta \;\Rightarrow\; b^\varepsilon_\alpha \leq b^\varepsilon_\beta$.

As $\mathbf{B}^*_\varepsilon$ has length $\leq \mu_\varepsilon$ and J is a chain, necessarily for some $Y_\varepsilon \subseteq \mu^+_\varepsilon$ with $|Y_\varepsilon| = \mu^+_\varepsilon$ we have

(e) $b^\varepsilon_\alpha = b^\varepsilon$ for $\alpha \in Y_\varepsilon$.

We can apply clause $(*)(\delta)$ to $-c_\alpha$ (for $\alpha \in Y_\varepsilon$ and $\mathbf{B}^*_\varepsilon$, and possibly shrinking Y_ε) to get $a^\varepsilon_\alpha \in \mathbf{B}^*_\varepsilon$ such that:

(f) $(-c_\alpha) \leq a^\varepsilon_\alpha$ and $0 < x \leq a^\varepsilon_\alpha \wedge x \in \mathbf{B}^*_\varepsilon \Rightarrow x \cap (-c_\alpha) \neq 0$.

As above, without loss of generality, shrinking Y_ε further we get

(g) $a^\varepsilon_\alpha = a^\varepsilon$ for $\alpha \in Y_\varepsilon$.

As the length of $\mathbf{B}^*_\varepsilon$ is $\leq \mu_\varepsilon < \mu^+_\varepsilon = |Y_\varepsilon|$, for some $\alpha \in Y_\varepsilon$ we have $c_\alpha \notin \mathbf{B}^*_\varepsilon$; as

$$a^\varepsilon_\alpha \geq -c_\alpha, \quad b^\varepsilon_\alpha \geq c_\alpha, \quad b^\varepsilon_\alpha \in \mathbf{B}^*_\varepsilon \quad \text{and} \quad a^\varepsilon_\alpha \in \mathbf{B}^*_\varepsilon,$$

necessarily $a^\varepsilon_\alpha \cap b^\varepsilon_\alpha \neq 0$.

Hence:

(h) $b^\varepsilon \cap a^\varepsilon \neq 0$.

Let $g(\varepsilon) = \min\{\zeta < \varepsilon : a^\varepsilon, b^\varepsilon \in \mathbf{B}^*_\zeta\}$, so for limit ε, $g(\varepsilon) < \varepsilon$. Hence on some stationary $S \subseteq \kappa$ and $\zeta(*) < \kappa$ the function $g \restriction S$ is constantly $\zeta(*)$, and without loss of generality

$$\zeta < \varepsilon \in S \Rightarrow \big|\{\alpha \in Y_\zeta : c_\alpha \in \mathbf{B}^*_\varepsilon\}\big| = \mu^+_\zeta.$$

As $\mathbf{B}$ satisfies the ccc we can find $\varepsilon_1 < \varepsilon_2$ in S such that

$$b^{\varepsilon_1} \cap a^{\varepsilon_1} \cap b^{\varepsilon_2} \cap a^{\varepsilon_2} \neq 0.$$

Choose $\alpha \in Y_{\varepsilon_1}$ such that $c_\alpha \in \mathbf{B}_{\varepsilon_1}$ and $\beta \in Y_{\varepsilon_2}$. Now $\{c_\alpha, c_\beta\}$ is independent: a contradiction.

2) Similarly. $\square_{3.10}$

Remark 3.11. We may further ask: is the restriction "$\mathrm{cf}(\mu) > \aleph_0$" in (3.10) necessary?

Observation 3.12. *Assume that the infinite Boolean algebra* $\mathbf{B}$ *has the length* μ, $\mathrm{cf}(\mu) = \aleph_0$. *Then the length is obtained.*

Proof. Let $\dot{\mathcal{I}} = \{b \in \mathbf{B} : \mathrm{length}(\mathbf{B} \restriction b) < \mu\}$.

Easily

$$b_1 \leq b_2 \wedge b_2 \in \dot{\mathcal{I}} \Rightarrow b_1 \in \dot{\mathcal{I}}.$$

Also clearly $\dot{\mathcal{I}}$ is closed under unions. [Why? If $b_1, b_2 \in \dot{\mathcal{I}}$, $b = b_1 \cup b_2 \notin \dot{\mathcal{I}}$ then there is a chain $\langle c_t : t \in J\rangle$, J a linear order of cardinality μ, $(s <_J t \Rightarrow c_s <_{\mathbf{B}} c_t)$ and $c_t \leq b$.

Let

$$\mathbf{E}_l = \{(t,s) \in J \times J : c_t \cap b_l = c_s \cap b_l\}.$$

Then $\mathbf{E}_l$ is a convex equivalence relation on J; if $|J/\mathbf{E}_l| = \mu$ then $\{c_t \cap b_l : t \in J\}$ exemplifies $b_l \notin \dot{\mathcal{I}}$, a contradiction. So $|J/\mathbf{E}_l| < \mu$. Hence $\mathbf{E} = \mathbf{E}_1 \cap \mathbf{E}_2$ is a convex equivalence relation with $\leq |J/\mathbf{E}_1| \times |J/\mathbf{E}_2| < \mu$ classes, but as $b = b_1 \cup b_2$ it is the equality.]

If $\mathbf{B}/\dot{\mathcal{I}}$ is infinite then we can find $\langle a_n/\dot{\mathcal{I}} : n < \omega\rangle$ pairwise disjoint non-zero. Now $b_n := a_n - \bigcup\limits_{\ell<n} a_\ell$ are pairwise disjoint members of $\mathbf{B}$ not in $\dot{\mathcal{I}}$. Let $\mu = \sum\limits_{n<\omega} \mu_n$, $\mu_n < \mu$. Let $\langle c^n_t : t \in J_n\rangle$ be an increasing chain in $\mathbf{B} \restriction b_n$, $|J_n| \geq \mu_n$ (note that we can invert J_n). Let $J = \sum\limits_{n<\omega} J_n$ (without loss of generality, $n < m \Rightarrow J_n \cap J_m = \varnothing$) and for $t \in J_n$ let $c^*_t = b_0 \cup \ldots \cup b_{n-1} \cup c^n_t$. Now $\langle c^*_t : t \in J\rangle$ exemplifies that the length is obtained. So $\mathbf{B}/\dot{\mathcal{I}}$ is finite, so without loss of generality $\dot{\mathcal{I}}$ is a maximal ideal. Try to choose $a_n \in \dot{\mathcal{I}}$ satisfying $\bigwedge\limits_{\ell<n} a_n \cap a_\ell = 0$ such that $\mathrm{length}(\mathbf{B} \restriction a_n) > \mu_n$. If we succeed, then we may repeat the proof for the case "$\mathbf{B}/\dot{\mathcal{I}}$ is infinite," hence we necessarily fail. Hence for some n (replacing $\mathbf{B}$ by $\mathbf{B} \restriction -(a_0 \cup \ldots \cup a_{n-1})$) we have

$$b \in \dot{\mathcal{I}} \Rightarrow \mathrm{length}(\mathbf{B} \restriction b) \leq \mu_n.$$

Let $J \subseteq \mathbf{B}$ be linearly ordered, $|J| > \mu^+_n$. Possibly shrinking J, without loss of generality $J \subseteq \dot{\mathcal{I}} \vee J \subseteq \mathbf{B} \setminus \dot{\mathcal{I}}$. As we can replace J by $\{1_{\mathbf{B}} - b : b \in J\}$ without loss of generality $J \subseteq \dot{\mathcal{I}}$, so for some $b \in J$ we have $|\{c \in J : c \leq b\}| \geq \mu^+_n$, and hence $\mathrm{length}(\mathbf{B} \restriction b) \geq \mu^+_n$, a contradiction. $\square_{3.12}$

Remark 3.13. We may wonder if we can replace $\aleph_0$ in 3.10 by another cardinals. Most natural are κ strong limit of cofinality ω.

Claim 3.14. *Assume* $\kappa \leq \mu < 2^\kappa$, κ *strong limit and* $\mathrm{cf}(\kappa) = \aleph_0$. *Then there is a Boolean algebra* $\mathbf{B}$ *such that:*

(α) $|\mathbf{B}| = 2^\kappa$
(β) $\mathbf{B} \models$ *ccc*
(γ) $\mathbf{B}$ *has length* μ *(and satisfies clause* $(B)^+$ *of 3.6)*
(δ) $\mathbf{B}$ *has no homomorphic image* $\mathbf{B}'$ *with* $|\mathbf{B}'| \in [\kappa, 2^\kappa)$.

Proof. Let $h \in {}^\omega\omega$ be $h(n) = 2(n+1)$. Let $\mathbf{B}_0 \subseteq \mathcal{P}(\kappa)$ have cardinality μ and length μ,

$$I^0_\alpha = \big\{\eta : \eta \text{ is an } \omega\text{-sequence, } \eta(n) \text{ is an increasing sequence of ordinals } < 2^\kappa \text{ of length } h(n)\big\},$$

and

$$I_\alpha = I^0_\alpha \cup \{\mathrm{Res}^\ell_n(\eta) : n < \omega,\ \ell < h(n),\ \eta \in I^0_\eta\}$$

so $|I_\alpha| = 2^\kappa$. Let $\mathbf{B}_{\alpha+1} = \mathrm{ba}[\mathbf{B}_\alpha, \bar{a}_\alpha, I_\alpha]$, $\mathbf{B}_\alpha$ increasing continuous for $\alpha \leq 2^\kappa$. (Again, $\bar{a}_\alpha$ is an ω-sequence of pairwise disjoint non-zero elements of $\mathbf{B}_\alpha$ such that each sequence appears 2^κ times.)

Again, for $\alpha < \beta$, $\mathbf{B}_\alpha \lessdot \mathbf{B}_\beta$ (and even the conclusion of 3.4(5) holds). The proof that $\mathbf{B} := \mathbf{B}_{(2^\kappa)}$ satisfies 3.14(α), (β), (γ) is as in the proof of 3.6.
For (δ) we need 3.15 below. $\square_{3.14}$

Observation 3.15. *Assume that* κ *is a strong limit cardinal of countable cofinality.*
1) If $\mathbf{B}'$ *is a Boolean Algebra of cardinality* $\geq \kappa$ *but* $< 2^\kappa$ *then:*

(a) *There are pairwise disjoint non-zero* b_n *(for* $n < \omega$*) in* $\mathbf{B}'$ *such that*
($*$) *for no* $c \in \mathbf{B}'$, $\bigwedge\limits_{n<\omega} (b_{2n} \leq c) \wedge \bigwedge\limits_{n<\omega} (b_{2n+1} \cap c = 0)$.

2) For a Boolean algebra $\mathbf{B}'$, *a sufficient condition for* $\mathbf{B}'$ *to satisfy* (a) *(i.e., the existence of a sequence* $\langle b_n : n < \omega\rangle$ *of pairwise disjoint elements of* $\mathbf{B}'$ *satisfying* ($*$) *above) is:*

(b) $\mathbf{B}'$ *has cardinality* $< 2^\kappa$ *and there are* $b_n \in \mathbf{B}'$ *such that* $\bigwedge\limits_{n<m} b_n \cap b_m = 0$ *and* $\kappa = \liminf\limits_n |\mathbf{B} \restriction b_n|$.

We first prove that 3.15 suffices (for finishing the proof of 3.14). Toward contradiction assume that $\mathbf{B}'$ is a Boolean algebra of cardinality $< 2^\kappa$ but $\geq \kappa$, and $\mathbf{B}'$ is a homomorphic image of $\mathbf{B}$. If clause (a) is satisfied by $\mathbf{B}'$, then the proof is very similar to the earlier proof of 3.6: for a homomorphism $\mathbf{f} : \mathbf{B} \to \mathbf{B}'$ from $\mathbf{B}$ onto $\mathbf{B}'$ we can find pairwise disjoint $a_n \in \mathbf{B}$ (for $n < \omega$) such that $f(a_n) = b_n$. So, for some α we have $\bar{a}_\alpha = \langle a_n : n < \omega\rangle$, and we repeat the relevant part of 3.6. Using clauses (b),(c) of Definition 3.3 we get a contradiction. We are left with proving 3.15. First, the second part.

Proof. **Proof of Observation 3.15(2)**:

We can find $\bar{c}^\zeta = \langle c^\zeta_n : n < \omega\rangle$, $c^\zeta_n \in \mathbf{B}'$, $c^\zeta_n \leq b_n$ for $\zeta < 2^\kappa$ such that the sequences $\langle c^\zeta_n : n < \omega\rangle$ are pairwise distinct for $\zeta < 2^\kappa$. For each ζ let $b^\zeta_{2n} = c^\zeta_n$, $b^\zeta_{2n+1} = b_n - c^\zeta_n$, so if clause **(a)** fails then for every $\zeta < 2^\kappa$ there is $y_\zeta \in \mathbf{B}'$ such that for every $n < \omega$ we have

$$b^\zeta_{2n} \leq y_\zeta, \quad b^\zeta_{2n+1} \cap y_\zeta = 0.$$

So $\bigwedge\limits_n y_\zeta \cap b_n = c^\zeta_n$ and hence $\zeta < \xi < 2^\kappa \Rightarrow\ y_\zeta \neq y_\xi$, which contradicts $|\mathbf{B}'| < 2^\kappa$.

Proof of Observation 3.15(1):

Assume that the conclusion fails. For a cardinal μ, let

$$\dot{\mathcal{I}}_\mu = \dot{\mathcal{I}}_\mu[\mathbf{B}'] := \{b \in \mathbf{B}' : \mathbf{B}' \restriction b \text{ has cardinality} < \mu\}.$$

Clearly it is an ideal of $\mathbf{B}'$ increasing with μ and $1_{\mathbf{B}'} \in \dot{\mathcal{I}}_\mu \Leftrightarrow \mu > |\mathbf{B}|$. If $\mathbf{B}'/\dot{\mathcal{I}}_\kappa[\mathbf{B}']$ is infinite then we can easily get condition **(B)** of part (2), and we are done. If it is finite, but $\dot{\mathcal{I}}_\mu[\mathbf{B}] \neq \dot{\mathcal{I}}_\kappa[\mathbf{B}']$ for every $\mu < \kappa$, then let $\kappa = \sum\limits_{n<\omega} \mu_n$, $\mu_n < \mu_{n+1}$, and choose $b_n \in \dot{\mathcal{I}}_\kappa[\mathbf{B}'] \setminus \dot{\mathcal{I}}_{\mu_n}[\mathbf{B}']$. But $\dot{\mathcal{I}}_\kappa[\mathbf{B}'] = \bigcup\limits_{\mu<\kappa} \dot{\mathcal{I}}_\mu[\mathbf{B}']$, so $\bigwedge\limits_n \bigvee\limits_m b_n \in \dot{\mathcal{I}}_{\mu_m}[\mathbf{B}']$. So without loss of generality $b_n \in \dot{\mathcal{I}}_{\mu_{n+1}}[\mathbf{B}'] \setminus \dot{\mathcal{I}}_{\mu_n}[\mathbf{B}']$ and hence $\langle b_n - \bigcup\limits_{l<n} b_l : n < \omega\rangle$ are as required. We are left with the case that for some $\mu(*) < \kappa$,

$$\dot{\mathcal{I}} := \dot{\mathcal{I}}_{\mu(*)}[\mathbf{B}'] = \dot{\mathcal{I}}_\kappa[\mathbf{B}']$$

and without loss of generality $\dot{\mathcal{I}} = \dot{\mathcal{I}}_{\mu(*)}[\mathbf{B}']$ is a maximal ideal.

Without loss of generality $2^{\mu(*)} < \mu_n < \mu_{n+1}$ for $n < \omega$. Let $b_i \in \dot{\mathcal{I}}$ (for $i < \kappa$) be distinct (these exist as $|\mathbf{B}'| \geq \kappa$ and $\dot{\mathcal{I}}$ is a maximal ideal of $\mathbf{B}'$). By the proof of Erdős–Tarski theorem, without loss of generality $\langle b_i : i < \kappa\rangle$ are non-zero pairwise disjoint.

[Why? For example, apply the Δ-system lemma to

$$\big\{\{x : x \leq b_i\} : i < (2^{\mu_n})^+\big\},$$

and get $Y_n \subseteq (2^{\mu_n})^+$ of cardinality $(2^{\mu_n})^+$ and a set A_n of cardinality $\leq 2^{\mu(*)}$ such that

$$i, j \in Y_n \wedge i \neq j \ \Rightarrow\ \{x : x \leq b_i\} \cap \{x : x \leq b_j\} = A_n.$$

So $|A_n| \leq \mu(*)$. Pick $Y'_n \subseteq Y_n$ of cardinality $(2^{\mu_n})^+$ such that

$$i, j \in Y'_n \wedge i \neq j \ \Rightarrow\ \{x : x \leq b_i\} \cap \bigcup_{m<n} A_m = \{x : x \leq b_j\} \cap \bigcup_{m<n} A'_m,$$

where $A'_n = \big\{x : (\exists i \in Y'_n)[x \leq b_i]\big\}$. Let $i(n) = \min(Y'_n)$. Then

$$X_n = \{x_i - x_{i(n)} : i \in Y_n,\ i > i(n)\} \subseteq \mathbf{B} \setminus \{0\}$$

is an antichain, and $\bigcup\limits_n X_n$ is as required.]

Let

$$\mathscr{P}_0 = \big\{Y \in [\kappa]^{\aleph_0} : \text{there is } b \in \dot{\mathcal{I}} \text{ such that } (\forall i \in Y)[b_i \leq b]\big\}.$$

This is a subset of $[\kappa]^{\aleph_0}$ of cardinality $\leq |\dot{\mathcal{I}}| \cdot \mu(*)^{\aleph_0} \leq |\mathbf{B}'| + \kappa = |\mathbf{B}'|$, but $[\kappa]^{\aleph_0} = 2^\kappa > |\mathbf{B}'|$, so there is $Y_0 \in [\kappa]^{\aleph_0} \setminus \mathscr{P}_0$.

Let

$$\mathscr{P}_1 = \big\{Y \in [\kappa]^{\aleph_0} : Y \subseteq \kappa \setminus Y_0 \text{ and } (\exists b \in \dot{\mathcal{I}})(\forall i \in Y)[b_i \leq b]\big\}.$$

By cardinality considerations as above there is $Y_1 \in [\kappa]^{\aleph_0} \setminus \mathscr{P}_1$ disjoint to Y_0. By assumption above (i.e., clause **(a)** fails) there is $b \in \mathbf{B}'$ such that $\bigwedge\limits_{i\in Y_0} b_i \leq b$ and $\bigwedge\limits_{i\in Y_1} b_i \leq (1-b)$. If $b \in \dot{\mathcal{I}}$ we get contradiction to the choice of Y_0, if not then $1_{\mathbf{B}} - b \in \dot{\mathcal{I}}$ contradicts the choice of Y_1. Hence the observation holds and hence the Observation 3.15 is proven. Hence Claim 3.14 is proven. $\square_{3.15}$

Remark 3.16. In other words 3.15 says

- $(*)$ If κ is strong limit, $\mathrm{cf}(\kappa) = \aleph_0$ and $\mathbf{B}$ is a Boolean algebra of cardinality $\geq \kappa$ with $\aleph_1$-separation (i.e., **(a)** of the observation fails) then $|\mathbf{B}| \geq 2^\kappa$.

§ 4. Using subtrees of $({}^{\omega\geq}2, \triangleleft)$ and theories unstable in $\aleph_0$

Theorem 4.1. *Suppose $T \subseteq T_1$ are first order theories, T_1 is countable, T is complete, superstable but $\aleph_0$-unstable. Then for $\lambda > \aleph_0$ we have*

$$\dot{I}\dot{E}(\lambda, T_1, T) \geq \min\{2^\lambda, \beth_2\}.$$

Remark 4.2. The reader is not required to know anything on superstable theories, just to believe a result quoted below. So we can just assume $(*)$ from the proof.

Proof. The assumption that the theory is superstable and not totally transcendental ($= \aleph_0$-stable) is used to obtain $m_a, m_b < \omega$ and a countable set of definable (without parameters) equivalence relations $\{\varphi_n(\bar{x};\bar{y}) : n < \omega\} \subseteq \mathbb{L}(\tau_T)$ such that:[14]

$(*)$ (i) $\ell g(\bar{x}) = \ell g(\bar{y}) = m_a + m_b$

(ii) If M is a model of T and $\bar{a} \in {}^{m_a}|M|$ then the set $\{\bar{a}{}^\frown\bar{b}/\varphi_n : \bar{b} \in {}^{m_b}|M|\}$ is finite.

(iii) If for $\ell = 1, 2$, $\ell g(\bar{a}_\ell) = m_a$, $\ell g(\bar{b}_\ell) = m_b$, and $(\bar{a}_1{}^\frown\bar{b}_1)\ \varphi_n\ (\bar{a}_2{}^\frown\bar{b}_2)$ then $\bar{a}_1 = \bar{a}_2$.

(iv) φ_{n+1} refines φ_n: i.e. for every $n < \omega$, $\bar{x}\ \varphi_{n+1}\ \bar{y}$ implies $\bar{x}\ \varphi_n\ \bar{y}$.

(v) There are (in some model M of T) $\bar{c}_\eta$ for $\eta \in {}^{\omega>}2$ such that:

$$\big[\ell g(\eta) \geq n \wedge \ell g(\nu) \geq n \ \text{ implies } \ \bar{c}_\eta\ \varphi_n\ \bar{c}_\nu \Leftrightarrow \eta \restriction n = \nu \restriction n\big],$$
$$\bar{c}_\eta \restriction m_a = \bar{c}_\nu \restriction m_a, \quad \ell g(\bar{c}_\eta) = m_a + m_b.$$

The existence of this set of equivalence relations was proved in Chapter III, 5.1-5.3 of both [She78] and [She90].

Clearly, without loss of generality we may expand the theory T_1. Let

$$\{c_\ell : \ell < m_1\} \cup \{c_{\eta,\ell} : \ell \in [m_1, m_1 + m_b] \text{ and } \eta \in {}^{\omega>}2\}$$

be new constants in T_1. We let $\bar{c}_\eta = \langle c_\ell : \ell < m_a\rangle{}^\frown\langle c_{\eta,\ell} : \ell \in [m_a, m_a + m_b)\rangle$. and suppose

$$T_1 \supseteq \big\{(\bar{c}_\eta\ \varphi_n\ \bar{c}_\nu) : \ell g(\eta), \ell g(\nu) \geq n,\ \eta \restriction n = \nu \restriction n\big\} \cup$$
$$\big\{\neg(\bar{c}_\eta\ \varphi_n\ \bar{c}_\nu) : \ell g(\eta), \ell g(\nu) \geq n,\ \eta \restriction n \neq \nu \restriction n\big\}.$$

Also without loss of generality, suppose that T_1 has Skolem functions (so the axioms saying it has Skolem functions belong to T_1).

We will use the following fact. [For a sequence $\bar{\eta}$ let $\bar{\eta} = \big\langle \bar{\eta}[\ell] : \ell < \ell g(\bar{\eta})\big\rangle$ and $\bar{a}_{\bar{\eta}} = \bar{a}_{\bar{\eta}[0]}{}^\frown\bar{a}_{\bar{\eta}[1]}{}^\frown\bar{a}_{\bar{\eta}[2]}\ldots$.] $\square_{4.1}$

Fact 4.3. 1) Suppose

(A) $T \subseteq T_1$ are first order theories, T complete and superstable, unstable in $|T_1|$, $\tau = \tau(T)$ and $\tau_1 = \tau(T_1)$, and T_1 has Skolem functions.

(B) τ_1 is countable, or at least MA_μ holds for $\mu = |T_1|$.

(C) φ_n (for $n < \omega$), m_a, m_b are as in $(*)$ above, and $m_* := m_a + m_b$.

(D) $\varphi_n \in \tau$ is a $(2m_*)$-place predicate,

$$\Delta = \{\varphi_n : n < \omega\}, \quad \tau_1^+ = \tau_1 \cup \{d_n : n < \omega\},$$

$\tau \subseteq \tau_1$, and $|\tau_1| \leq \mu < 2^{\aleph_0}$.

Then there are M_1, $\bar{a}_\eta$ $(\eta \in {}^\omega 2)$ such that:

(α) M_1 is a model of T_1 and $\varphi_n^{M_1}$ is an equivalence relation such that $\varphi_{n+1}^{M_1}$ refines $\varphi_n^{M_1}$.

[14] We may write $\bar{x}\ \varphi_n\ \bar{y}$ instead of $\varphi_n(\bar{x},\bar{y})$).

(β) $\tau(M_1) = \tau_1^+$, $\{\bar a_\eta : \eta \in {}^\omega 2\} \subseteq {}^{m_*}|M_1|$, and

$$\ell g(\eta) \geq n \wedge \ell g(\nu) \geq n \ \Rightarrow\ \big[\eta \restriction n = \nu \restriction n \Leftrightarrow (\bar a_\eta\ \varphi_n\ \bar a_\nu)\big].$$

For $\bar\eta \in {}^m({}^\omega 2)$, let $\bar a_{\bar\eta} = \bar a_{\eta_0}{}^\frown \ldots {}^\frown \bar a_{\eta_{m-1}}$.

(β)$_1$ $\bar a_\eta \restriction m_a = \bar a_\nu \restriction m_a = \langle c_\ell^{M_1} : \ell < m_a\rangle$, $\ell g(\bar a_\eta) = m_*$, and if $n < \omega$, $\ell g(\bar a) = m_a < m_*$ <u>then</u> $\big|\{\bar a{}^\frown \bar b/\varphi_m : \bar b \in {}^{m_b}(M_1)\}\big| < k_m$.

(γ) For every formula $\varphi(\bar x)$ from $\mathbb{L}(\tau_1)$ such that m_* divides $\ell g(\bar x)$, there is η_φ such that for $n \in [\eta_\varphi, \omega)$:

$(*)^1_{\varphi,n}$ If $\bar\eta$, $\bar\nu \in {}^m({}^\omega 2)$, $\ell g(\bar\eta) = \ell g(\bar\nu) = m = \dfrac{\ell g(\bar x)}{m_a + m_b}$ (so $\ell g(\bar a_{\bar\eta}) = \ell g(\bar x)$), and

$$\big\langle \eta_\ell \restriction \eta : \ell < \ell g(\bar\eta)\big\rangle = \big\langle \nu_\ell \restriction n : \ell < \ell g(\bar\nu)\big\rangle$$

is without repetitions, then $M_1 \models \varphi[\bar a_{\bar\eta}] = \varphi[\bar a_{\bar\nu}]$.

(δ) $\langle d_n : n < \omega\rangle$ is an indiscernible sequence over $\{\bar a_\eta : \eta \in {}^\omega 2\}$ in $M_1 \restriction \tau_1$.

(δ)$^+$ $d_n \neq d_m$ for $n \neq m$.

2) If $M_1, \tau, \tau_1, \tau_1^+, m_a, m_b, \langle \varphi_n : n < \omega\rangle$ are as in (α), (β), (β)$_1$, (γ), (δ), (δ)$^+$ above and $\mu = \aleph_0$ (or at least MA$_\mu$) <u>then</u>, replacing ${}^\omega 2$ by a subtree, replacing $\langle \varphi_n : n < \omega\rangle$ by a sub-sequence and renaming, decreasing M_1, we can add to part (1):

(γ)$^+$ For every sequence of terms $\bar\sigma(\bar x)$ from τ_1^+, if $m \times (m_a + m_b) = \ell g(\bar x)$, $m_a + m_b = \ell g(\bar\sigma)$, $\bar\sigma(\bar x) \restriction m_a = (\bar\sigma \restriction m_a)(\bar x \restriction m_d)$, $m_e < m_a$, $m_d = m_e \times (m_a + m_b)$, [i.e. $\bar\sigma(\bar a_{\bar\eta}) \restriction m_a = (\bar\sigma \restriction m_a)(\bar a_{\bar\eta \restriction m_e})$ for $\bar\eta \in {}^m({}^\omega 2)$], <u>then</u> there exists $n_{\bar\sigma} < \omega$ such that:

(a) For $n \geq n_{\bar\sigma}$ and $\bar\eta, \bar\nu \in {}^m({}^\omega 2)$ with no repetitions, $\bar\eta \restriction m_e = \bar\nu \restriction m_e$, we have:
- If $\ell \neq k \Rightarrow \bar\eta[\ell] \restriction n \neq \bar\eta[k] \restriction n$ and $(\forall \ell < m)\big[\bar\eta[\ell] \restriction n = \bar\nu[\ell] \restriction n\big]$ <u>then</u> for every $\bar\rho \in {}^m({}^\omega 2)$, $\bar\rho \restriction m_e = \bar\eta \restriction m_e$ implies

$$\big(\bar\sigma(\bar a_{\bar\eta})\ \varphi_n\ \bar\sigma(\bar a_{\bar\rho})\big) \Leftrightarrow \big(\bar\sigma(\bar a_\nu)\ \varphi_n\ \bar\sigma(\bar a_{\bar\rho})\big).$$

(b) For $n \geq \eta_{\bar\sigma}$ and $\bar\eta, \bar\nu \in {}^m({}^n 2)$ each with no repetition and

$$\bar\eta \restriction m_e = \bar\nu \restriction m_e,$$

we have:
- If there are $k \geq n$ and $\bar\eta_1, \bar\nu_1 \in {}^m({}^\omega 2)$ such that $\neg\varphi_k(\bar\sigma(\bar a_{\bar\eta_1}), \bar\sigma(\bar a_{\bar\nu_1}))$, for $\ell < m$, $\bar\eta_1[\ell] \restriction n = \bar\eta[\ell]$, $\bar\nu_1[\ell] \restriction n = \bar\nu[\ell]$, and

$$(\forall \ell, i < m)\big[\bar\eta_1[\ell] = \bar\nu_1[i] \Leftrightarrow \bar\eta[\ell] = \bar\nu[i]\big],$$

<u>then</u> for every $\bar\eta^*, \bar\nu^* \in {}^m({}^\omega 2)$ satisfying $\bar\eta^*[\ell] \restriction n = \bar\eta[\ell]$, $\bar\nu^*[\ell] \restriction n = \bar\nu[\ell]$ (for each $\ell < m$) and

$$(\forall \ell, i < m)\big[\bar\eta^*[\ell] = \bar\nu^*[i] \Leftrightarrow \bar\eta[\ell] = \bar\nu[i]\big]$$

we have $\neg\big[\bar\sigma(\bar a_{\bar\eta^*})\ \varphi_n\ \bar\sigma(\bar a_{\bar\nu^*})\big]$.

Remark 4.4. 1) This is the only place where countability (or MA$_{|\tau_1|}$) is used.
2) For alternative proof see 4.13.

Proof. 1) If we ignore (δ)$^+$ (so can have $d_n = d_0$) use Theorem [She78, Ch.VII,3.7]. In general, use [She78, Ch.VII,Ex.3.1]. What if T_1 is uncountable but MA$_\mu$? (The reader may ignore this proof or see the proof of 4.13.)

Let $\mathbb{P}$ be the forcing notion of adding $\lambda = \beth_{(2^\mu)^+}$ Cohen reals, $\langle \underset{\sim}{\eta}_i : i < \lambda\rangle$, $\underset{\sim}{\eta}_i \in {}^\omega 2$. Let $\chi = (2^\lambda)^+$ and let

$$\Vdash_{\mathbb{P}} \text{“}\underset{\sim}{M} \text{ is a model of } T_1, \text{ the Skolem hull of } \{x_i : i < \lambda\},\ \bar x_i\ \varphi_m\ \bar c_{\underset{\sim}{\eta}_i \restriction m}\text{”}.$$

By the Omitting Type Theorem[15] there are $\mathfrak{B}_1 \prec \mathfrak{B}_2$ with $\mathfrak{B}_1 \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ and $\|\mathfrak{B}_1\| = \mu$ such that T_1, P, $\underset{\sim}{M}$, $\langle x_i : i < \lambda\rangle$ belong to $\mathfrak{B}_1$. Also in $\mathfrak{B}_2$, $\langle a_\rho : \rho \in {}^\omega 2\rangle$ is an indiscernible sequence over $\mathfrak{B}_1$, and $\mathfrak{B}_2 \models$ "$\underset{\sim}{a}_i$ is an ordinal $= \lambda$".

Note that any set which $\mathfrak{B}_2$ considers a maximal antichain of $\mathbb{P}^{\mathfrak{B}_2}$ really is so. Now we can naturally apply MA_μ.

2) Satisfy requirement (a) by letting $\varphi^\ell_n(\bar{x}{}^\frown \bar{z}) := E_n\big(\bar{x}{}^\frown \bar{z}, F_\ell(\bar{z}){}^\frown \bar{z}\big)$ for $\ell < \ell^*_n < \omega$, where $F_\ell \in \tau^+$ are such that $\{F_\ell(\bar{x}) : \ell < \ell^*_n\}$ is a complete set of representatives for $\{\bar{x}{}^\frown \bar{z}/\varphi_n : \bar{x}\}$, possibly with repetitions. (Remember T_1 has Skolem functions and there is ℓ^*_n which does not depend on $\bar{z}$ by compactness). Requirement (b) is fulfilled by trimming the perfect tree and renaming. $\square_{4.3}$

Claim 4.5. *For M_1, $\bar{a}_\eta$ $(\eta \in {}^\omega 2)$, φ_n as in the conclusion of 4.3 we can conclude:*

$\otimes$ *$\underline{\text{If}}$ $\nu \neq \rho$ are from ${}^\omega 2$, $\bar{\eta}_\nu = \langle \eta_{\nu,\ell} : \ell < \ell(*)\rangle$, $\bar{\eta}_\rho = \langle \eta_{\rho,\ell} : \ell < \ell(*)\rangle$, $\bar{x} = \langle x_\ell : \ell < \ell(*)\rangle$, $\bar{\sigma}(\bar{x}) = \langle \sigma_m(\bar{x}) : m < m(*)\rangle$, $\nu \restriction k = \rho \restriction k$, $\eta_{\nu,\ell} \restriction k = \eta_{\rho,\ell} \restriction k$, $\langle \eta_{\nu,\ell} : \ell < \ell(*)\rangle$ with no repetitions, $k > n_{\bar{\sigma}}$, and*

$$\bigwedge_{n<\omega} \big[\bar{a}_\nu\ \varphi_n\ \bar{a}_\rho \Leftrightarrow \bar{\sigma}(\bar{a}_{\bar{\eta}_\nu})\ \varphi_n\ \bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})\big]$$

(moreover, the Δ-type of $\bar{a}_\nu{}^\frown \bar{a}_\rho$ and $\bar{\sigma}(\bar{a}_{\bar{\eta}_\nu}){}^\frown \bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})$ (in M) are equal for every n) $\underline{\text{then}}$ $\ell g(\nu \cap \rho) \in \big\{\ell g(\eta_{\nu,\ell} \cap \eta_{\rho,\ell}) : \ell < \ell(*)\big\}$.

Proof. Assume not.

Let $n = \ell g(\rho \cap \nu)$. Then $\varphi_n(\bar{a}_\rho, \bar{a}_\nu) \wedge \neg\varphi_{n+1}(\bar{a}_\rho, \bar{a}_\nu)$. We suppose first (for didactic reasons) for the sake of contradiction that for every $\ell < n_0$ we have

$$\bar{\eta}_\nu[\ell] \neq \bar{\eta}_\rho[\ell] \ \Rightarrow\ \ell g(\bar{\eta}_\nu[\ell] \cap \bar{\eta}_\rho[\ell]) < n.$$

By the equality of types $\neg\varphi_{n+1}\big(\bar{\sigma}(\bar{a}_{\bar{\eta}_\rho}), \bar{\sigma}(\bar{a}_{\bar{\eta}_\nu})\big)$, now we can deduce by Fact 4.3(2) and the assumption that the conclusion of $(\otimes)$ fails, that $\neg\varphi_{n+1}\big(\bar{\sigma}(\bar{a}_{\bar{\eta}_\rho}), \bar{\sigma}(\bar{a}_{\bar{\eta}_\nu})\big)$. Again, by the equality of types $\neg\varphi_n(\bar{a}_\rho, \bar{a}_\nu)$, a contradiction to $\varphi_n(\bar{a}_\rho, \bar{a}_\nu)$.

Now we deal with the general case, i.e., we assume

$(*)$ $(\forall \ell < n_0)\big[\ell g(\bar{\eta}_\nu[\ell] \cap \bar{\eta}_\rho[\ell]) \neq n\big]$.

We shall derive a contradiction.

Define $\bar{\eta} \in {}^{n_0}({}^\omega 2)$:

$$\bar{\eta}[\ell] = \begin{cases} \bar{\eta}_\rho[\ell] & \text{if } \bar{\eta}_\nu[\ell] \restriction n \neq \bar{\eta}_\rho[\ell] \restriction n, \\ \bar{\eta}_\nu[\ell] & \text{otherwise.} \end{cases}$$

Clearly $\bar{\sigma}(\bar{a}_\eta) \restriction m_a = \bar{\sigma}(\bar{a}_{\eta_\rho}) \restriction m_a = \bar{\sigma}(\bar{a}_{\eta_\nu}) \restriction m_a$ and $\bar{\eta} \restriction m_e = \bar{\eta}_\nu \restriction m_e = \bar{\eta}_\rho \restriction m_e$, and also $\bar{\eta}$ is with no repetition and $\langle \bar{\eta}[\ell] \restriction n : \ell < n_0\rangle$ are pairwise distinct.

Since, by the definition of $\bar{\eta}$, for which $\bar{\eta}[\ell] \restriction n = \bar{\eta}_\rho[\ell] \restriction n$, using $(*)$ we obtain

$$\eta[\ell] \restriction (n+1) = \eta_\rho[\ell] \restriction (n+1).$$

Let $\bar{b} = \bar{\sigma}(\bar{a}_{\bar{\eta}})$. By reflexivity of the equivalence relation we have

$$\bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})\ \varphi_{n+1}\ \bar{\sigma}(\bar{a}_{\bar{\eta}_\rho}).$$

By Fact 4.3(1), $\bar{\sigma}(\bar{a}_{\bar{\eta}})\ \varphi_{n+1}\ \bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})$; i.e. $\bar{b}\ \varphi_{n+1}\ \bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})$. Finally,[16] using transitivity of the equivalence relation we have $\neg\varphi_{n+1}\big(\bar{b}, \bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})\big)$.

By the definition of $\bar{\eta}$, for every $\ell < n_0$ we have

$$\bar{\eta}[\ell] = \bar{\eta}_\nu[\ell] \ \text{ or }\ \ell g(\bar{\eta}[\ell] \cap \eta_\nu[\ell]) < n.$$

[15] See, e.g., [She90, Ch.VII,§5].

[16] As $\neg\big(\bar{\sigma}(\bar{a}_{\bar{\eta}_\nu})\ \varphi_{n+1}\ \bar{\sigma}(\bar{a}_{\bar{\eta}_\rho})\big)$.

But since $n > k_0$, clearly

$$\big|\{\bar\eta[\ell] \restriction k_0 : \ell < n_0\}\big| = \big|\{\bar\eta_\nu[\ell] \restriction k_0 : \ell < n_0\}\big| = n_0.$$

So by Fact 4.3(2), as $\neg\big(\bar b\ \varphi_{n+1}\ \bar\sigma(\bar a_{\bar\eta_\nu})\big)$ (see above), we have $\neg\big(\bar b\ \varphi_n\ \bar\sigma(\bar a_{\bar\eta_\nu})\big)$. But $\bar b\ \varphi_n\ \bar\sigma(\bar a_{\bar\eta_\rho})$ (see above) and $\bar\sigma(\bar a_{\bar\eta_\rho})\ \varphi_n\ \bar\sigma(\bar a_{\bar\eta_\rho})$, a contradiction. $\square_{4.5}$

So for proving theorem 4.1 we can assume

Hypothesis 4.6. $M_1, \langle\varphi_n : n < \omega\rangle$, and $\bar a_\eta$ (for $\eta \in {}^\omega 2$) are as in $(\beta) + (\gamma)$ of 4.3(1) and $(\otimes)$ of 4.5.

Lemma 4.7. *Assume $\mu < \lambda \le 2^{\aleph_0}$. We can find $S_\xi \subseteq {}^\omega 2$ for $\xi < 2^{\aleph_0}$, pairwise disjoint, each of cardinality λ, such that*

$\otimes$ *If $\xi < 2^{\aleph_0}$, $f : S_\xi \to {}^{\omega>}\big(\mathscr{M}_{\mu,\omega}(\bigcup_{\zeta\ne\xi} S_\zeta)\big)$ and $\mathbf{n}$ is a function,*

$$\mathbf{n} : \big\{\bar\sigma : (\exists\bar x)\big[\bar\sigma = \langle\sigma_\ell(\bar x) : \ell < \ell^*\rangle\big],\ \sigma_\ell \text{ a term of } \mathbb{L}_{\mu,\aleph_0}(\tau)\big\} \to \omega$$

and τ is the vocabulary of $\mathscr{M}_{\mu,\omega}(\bigcup_{\zeta\ne\xi} S_\xi)$, <u>then</u> we can find m^* (see below) $S^* \subseteq S_\zeta$, $k_0 < \omega$, $n_0 = m_a + m_b < \omega$, a sequence $\bar\sigma(\bar x) = \langle\sigma_\ell(\bar x) : \ell < \ell g(\bar\sigma)\rangle$, with $\ell g(\bar x) = n_0$, $\langle\bar\eta_\nu : \nu \in S^*\rangle$ and $\bar\eta_0 \in {}^{n_0}({}^\omega 2)$ with the following properties. Letting $\eta_{\nu,\ell} = \bar\eta_\nu[\ell]$:

- *(A) $\eta \ne \nu \in S^* \ \Rightarrow\ \ell g(\eta \cap \nu) > k_0$*
- *(B) For $\nu \in S^*$ we have $\ell g(\bar\eta_\nu) = n_0$, $(\forall \ell < n_0)\,[\bar\eta_{\nu,\ell} \restriction k_0 = \bar\eta_{0,\ell} \restriction k_0]$, and $\{\bar\eta_{\nu,\ell} \restriction k_0 : \ell < n_0\} \cup \{\nu \restriction k_0\}$ are pairwise distinct.*
- *(C) $k_0 > \mathbf{n}(\bar\sigma)$*
- *(D) For each $\ell < n_0$, either $\{\bar\eta_{\nu,\ell} : \nu \in S^*\} = \{\bar\eta_{0,\ell}\}$ or $\{\bar\eta_{\nu,\ell} : \nu \in S^*\}$ are pairwise distinct.*
- *(E) The sets $\{\ell g(\nu_1 \cap \nu_2) : \nu_1 \ne \nu_2 \text{ from } S^*\}$ and*
  $$\{\ell g(\eta_{\nu_1,\ell_1} \cap \eta_{\nu_2,\ell_2}) : \nu_1, \nu_2 \in S^* \text{ and } \ell_1, \ell_2 < n_0\}$$
  are disjoint.
- *(F) For every $\nu \in S^*$, $f(\nu) = \bar\sigma(\bar\eta_\nu)$ (i.e. equal to $\big\langle\sigma_\ell(\langle\eta_{\nu,n} : n < n_0\rangle) : l < m^*\big\rangle$).*
- *(G) For $\nu_1 \ne \nu_2 \in S^*$, we have*
  $$\eta_{\nu_1,\ell} = \eta_{\nu_2,\ell} \Leftrightarrow \ell < m_a \Leftrightarrow \eta_{\nu_1,\ell} = \eta_{0,\ell}.$$
- *(H) S^* is μ^+-large. (We say that $S \subseteq {}^\omega 2$ is χ-large <u>iff</u> for every $n < \omega$ and $\nu \in S$ we have $\big|\{\rho \in S : \rho \restriction n = \nu \restriction n\}\big| \ge \chi$.) We can replace μ^+-large by λ-large if $\mathrm{cf}(\lambda) > \aleph_0$.*
- *(I) $\nu_1, \nu_2 \in S^* \wedge \eta_{\nu_1,\ell_1} = \eta_{\nu_2,\ell_2}$ implies $\ell_1 = \ell_2$.*
- *(J) For $\eta \in \bigcup_\xi S_\xi$, let $\xi(\eta)$ be the unique ξ such that $\eta \in S_\xi$. Now, if $\xi(\eta_{\nu_1,\ell_1}) = \xi(\eta_{\nu_2,\ell_2})$ with $\ell_1, \ell_2 < n_0$ and $\nu_1 \ne \nu_2 \in S^*$, then*
  $$\nu \in S^* \Rightarrow \xi(\eta_{\nu_1,\ell_1}) = \xi(\eta_{\nu,\ell_1}) = \xi(\eta_{\nu,\ell_2}).$$

Remark 4.8. 1) This claim is a version of the "unembeddability" results;[17] well, they are necessarily somewhat weaker than in §1 here.
2) Of course, we can replace $\bigcup_{\zeta\ne\xi} S_\zeta$ by $\sum_{\zeta\ne\xi} S_\zeta$.

For proving 4.7 we will use the following combinatorial fact, which is slightly stronger than Sierpiński's lemma on almost disjoint sets of integers:

[17] See Definitions in [Shea, §2], results (for example) in VI, and here in §1 for the tree ${}^{\omega\ge}2$.

Fact 4.9. There are $W(*)$, $\{W_\eta : \eta \in {}^\omega 2\}$, and $\{U_\eta : \eta \in {}^\omega 2\}$ such that for all $\eta \in {}^\omega 2$:

(A) $W(*), W_\eta$ are infinite subsets of ω.

(B) U_η is a perfect tree; i.e. $U_\eta \subseteq {}^{\omega >}2$ is downward closed, $\langle\ \rangle \in U_\eta$, and

$$(\forall \rho \in U_\eta)(\exists \nu \in U_\eta)\big[\rho \trianglelefteq \nu \wedge \nu\char`^\langle 0\rangle \in U_\eta \wedge \nu\char`^\langle 1\rangle \in U_\eta\big].$$

(C) If $\rho, \nu \in U_\eta$, $\rho \neq \nu$, and $\ell g(\rho) = \ell(\nu)$ <u>then</u> $\ell g(\rho \cap \nu) \in W_\eta$, where $\rho \cap \nu$ is the largest common initial segment of ρ and ν; i.e.

$$\ell g(\rho \cap \nu) := \max\{n < \omega : \rho \restriction n = \nu \restriction n\}.$$

(D) For all $\eta_1 \neq \eta_2 \in {}^\omega 2$ and every $\rho \in U_{\eta_1}$, $\nu \in U_{\eta_2}$, there are three possibilities:
  (a) $\ell g(\rho \cap \nu) \in W_{\eta_1} \cap W_{\eta_2}$
  (b) $\ell g(\rho \cap \nu) \in W(*)$ and $\big(\forall \ell < \ell g(\rho \cap \nu)\big)\big[\ell \in W_{\eta_1} \equiv \ell \in W_{\eta_2}\big]$.
  (c) $\rho \trianglelefteq \nu$ or $\nu \triangleleft \rho$.

(E) $W(*) \cap W_\eta = \varnothing$

(F) For distinct η, ν from ${}^\omega 2$, we have:
  (a) $W_\eta \cap W_\nu$ is finite (in fact, an initial segment of both).
  (b) If $\ell \in W(*)$ is above $W_\eta \cap W_\nu$ <u>then</u> $U_\eta \cap U_\nu$ is finite, contained in ${}^{\ell' >}2$ if $\ell < \ell' \in W_\eta \cup W_\nu$, and has no splitting of level $\geq \ell$; i.e.

$$\neg(\exists \rho \in {}^{\omega >}2)\big[\ell g(\rho) \geq \ell \wedge \{\rho\char`^\langle 0\rangle, \rho\char`^\langle 1\rangle\} \subseteq U_\eta \cap U_\nu\big].$$

  (c) If $\ell \in W(*)$ and $\ell < \sup(W_\eta \cap W_\nu)$ <u>then</u> $U_\eta \cap {}^{\ell \geq}2 = U_\nu \cap {}^{\ell \geq}2$.

Proof. By induction on n, define $k(n) = k_n < \omega$, the set $W_n(*) \subseteq k(n)$ and the sets $U_\eta \subseteq {}^{k(n) \geq}2$, $W_\eta \subseteq k(n)$, such that in the end (this imposes natural restrictions on them):

$$\eta \in {}^\omega 2 \ \Rightarrow\ W_\eta \cap k_n = W_{\eta \restriction n}, \quad U_\eta \cap {}^{k(n) \geq}2 = U_{\eta \restriction n}, \quad W(*) \cap k(n) = W_n(*).$$

For $n = 0$, let $k_0 = 0$, $W_n(*) = \varnothing$ and $W_\eta = \varnothing$, $U_\eta = \varnothing$ for $\eta \in {}^n 2$. For the induction step, choose $k^1(n) = k(n) + n + 1$ and for $\eta \in {}^n 2$ let

$$U^1_\eta = U_\eta \cup \{\nu\char`^(\eta \restriction \ell) : \nu \in U_\eta \cap {}^{k(n)}2,\ \ell \leq n\}.$$

Thus

$$\big(\forall \nu \in {}^{k(n)}2 \cap U_\eta\big)\big(\exists! \rho \in {}^{k^1(n)}2 \cap U^1_\eta\big)\big[\nu \trianglelefteq \rho\big].$$

Define $W_{n+1}(*) = W_n(*) \cup \big[k(n), k^1(n)\big)$. Fix an enumeration $\{\eta_k : k < 2^{n+1}\}$ of ${}^{n+1}2$. Let $k(n+1) := k^1(n) + 2^{n+1}$. For $\eta \in {}^{n+1}2$, there is a unique $k < 2^n$ such that $\eta = \eta_k$. Let

$$U_{\eta_k} := U^1_{\eta_k \restriction n} \cup \big\{\nu \in {}^{k(n+1) \geq}2 : \nu \restriction k^1(n) \in U^1_{\eta_k \restriction n}, \text{ and for } \ell < 2^n \text{ we have } k^1(n) + \ell < \ell g(\nu) \wedge (\ell \neq 2k+1) \Rightarrow \nu(k^1(n) + \ell) = 0\big\}$$

and $W_{\eta_k} = W_{\eta_k} \cup \{k^1(n) + 2k + 1\}$. It is easy to verify that the construction provides a family of sets as required. $\square_{4.9}$

Proof of Lemma 4.7: Let $W(*)$, U_η, W_η be as in 4.9. Fix an enumeration $\{\eta_\xi : \xi < 2^{\aleph_0}\} = {}^\omega 2$ and let $W^\xi := W_{\eta_\xi}$. Let

$$S_\xi \subseteq \lim(U_{\eta_\xi})\ \big(= \big\{\rho \in {}^\omega 2 : (\forall n < \omega)[\rho \restriction n \in U_{\eta_\xi}]\big\}\big)$$

be of cardinality λ. Fix $\{\rho^\xi_i : i < \lambda\} = S_\xi$, and without loss of generality S_ξ is χ-large.[18]

[18] Recall that we say $S \subseteq {}^\omega 2$ is χ-large if for every $n < \omega$ and $\nu \in S$, $\big|\{\rho \in S : \rho \restriction n = \nu \restriction n\}\big| \geq \chi$. If $\chi \geq (|\tau_1| + \aleph_0)^+$ we may omit it.

Note that for every $S \subseteq {}^{\omega}2$ of cardinality $> \mu$, for some $S_1 \subseteq S$, $|S_1| \leq \mu$ and $S \setminus S_1$ is μ^+-large. Let $U^{\zeta} = U_{\eta_\zeta}$; note that by 4.9(B)+(D), the sets $S_\xi \setminus S$ are pairwise disjoint.

So let $\xi, f, \mathbf{n}$ be as in the assumption of 4.7$\otimes$.

For $\nu \in S_\xi$ let $f(\nu) = \bar{\sigma}_\nu(\bar{\eta}_\nu)$, where $\bar{\sigma}_\nu$ is a finite sequence of terms and $\bar{\eta}_\nu$ is a finite sequence of members of $\bigcup\limits_{\zeta \neq \xi} S_\zeta$ with no repetitions. So there are $S^* \subseteq S_\xi$ which is μ^+-large, and $\bar{\sigma}$, and an integer n_0 such that

$$\nu \in S^* \ \Rightarrow\ \bar{\sigma}_\nu = \bar{\sigma} \wedge \ell g(\bar{\eta}_\nu) = n_0,$$

and without loss of generality, for some $m_a \leq m_b < \omega$, we have $\bar{\sigma}(\bar{\eta}_\nu) \restriction m_a = \bar{\eta}^*$ and

$$\{\eta^*_\ell : \ell < m_a\} \cup \{\eta_{\nu,\ell} : \nu \in S^* \text{ and } \ell \in [m_a, m_b)\}$$

is without repetition (this is possible by the Δ-system argument).

As $S_\xi \cap \bigcup\limits_{\zeta \neq \xi} S_\zeta = \varnothing$, clearly the sequence $\langle \nu \rangle \frown \bar{\eta}_{\bar{\nu}}$ is without repetitions for any $\nu \in S^*$. So for some $k = k_\nu < \omega$ large enough, we have:

- (i) $\langle \nu \restriction k \rangle \frown \langle \eta_{\nu,\ell} \restriction k : \ell < \ell(*) \rangle$ is without repetitions.
- (ii) Letting $\eta_{\nu,\ell} \in S_{\zeta(\nu,\ell)}$, we have $W^\xi \cap W^{\zeta(\eta_{\nu,\ell})} \subseteq \{0, \ldots, k_\nu - 1\}$. Moreover, $k_\nu > \min(W^\xi \setminus W^{\zeta(\eta_\nu,\ell)})$ (remember clause (F) of 4.7).

As we can shrink S^* as long as it is μ^+-large, without loss of generality for some k:

- (iii) $\nu_1 \neq \nu_2 \in S^* \ \Rightarrow\ \ell g(\nu_1 \cap \nu_2) > k$
- (iv) $\nu \in S^* \ \Rightarrow\ k_\nu < k < \omega$.

So for $\nu_1 \neq \nu_2 \in S^*$, on the one hand $\ell g(\nu_1 \cap \nu_2) \in W^\xi \setminus k$ (as $\nu_1, \nu_2 \in S_\xi \subseteq \lim(U_{\eta_\xi})$; see clause (iii) above and 4.9(C)) and on the other hand

$$\ell g(\eta_{\nu_1 \cap \ell}, \eta_{\nu_2,\ell}) \in W(*) \cup U^{\zeta(\nu_1,\ell)} \cup U^{\zeta(\nu_2,\ell)}$$

which is disjoint to $W^\xi \setminus k$. So we have proved clause (E) of 4.7; the other clauses can be checked. $\square_{1.13}$

Claim 4.10. *If clauses (β), (γ), (δ) of 4.3(1) hold, and 4.5($\otimes$) does as well, <u>then</u> for $\lambda \leq 2^{\aleph_0}$:*

$(*)_\lambda$ *There is a family $\mathscr{P}$ of subsets of ${}^{\omega}2$ each of cardinality λ (even their union has cardinality λ) with $|\mathscr{P}| = 2^\lambda$, such that (letting N^1_S be the Skolem Hull of $\{\bar{a}_\eta : \eta \in S\}$ for $S \in \mathscr{P}$) we have:*

- *For $Y_1 \neq Y_2$ from $\mathscr{P}$, $N^1_{Y_1}$ has no Δ-embedding into $N^1_{Y_2}$.*
- *$\|N^1_Y\| = \lambda$ for $Y \in \mathscr{P}$.*

Proof. For $X \subseteq \lambda$, let M^1_X be the Skolem Hull of $\{\bar{a}_\eta : \eta \in \bigcup\limits_{\xi \in X} S_\xi\}$ and $M_X := M^1_X \restriction \tau_T$.

In order to prove the theorem it is enough to assume $X, Y \subseteq \lambda$ and $X \not\subseteq Y$, and show there does not exist an elementary embedding f from M_X into M_Y. Let $\xi \in X \setminus Y$. For the sake of contradiction suppose $f : M_X \to M_Y$ is an elementary embedding, or just one preserving the satisfaction of $\varphi_n, \neg\varphi_n$.

We can represent M_Y in $\mathscr{M}_{\mu,\omega}(\bigcup\limits_{\zeta \neq \xi} S_\zeta)$, and let us define $f' : S_\xi \to \mathscr{M}_{\mu,\omega}(\bigcup\limits_{\zeta \neq \xi} S_\zeta)$ by $f'(\nu) = f(\bar{a}_\nu)$, let $\mathbf{n}$ be essentially as in 4.3, but translated. Apply lemma 4.7 to f' and $\mathbf{n}$, and get S^*, k_0, n_0, m_a, m_b, $\bar{\sigma}$, $\langle \bar{\eta}_\nu : \nu \in S^* \rangle$ as there. Of course n_0, m_a, m_b are predetermined as in 4.3.

So we are done proving 4.10. $\square_{4.10}$

Proof. **Proof of Theorem 4.1**:

When $\lambda \leq 2^{\aleph_0}$, the result follows from 4.10 by 4.5.

So the proof of Theorem 4.1 for the case $\lambda \leq 2^{\aleph_0}$ is completed. How to deal with the case $\lambda > 2^{\aleph_0}$? We just need to use $(\delta)^+$; i.e. use 4.12 (and Definition 4.11) below. $\square_{4.1}$

Definition 4.11. For any cardinal κ and M_1 as in 4.3(1)(β)-$(\delta)^+$, we define a model $M_{1,\kappa}$ as follows: it is a τ_1-model generated by $\{\bar{a}_\eta : \eta \in {}^\omega 2\} \cup \{d_i : i < \kappa\}$ such that for every $n < \omega$, $i_1 < \ldots i_n < \kappa$, and $\eta_1, \ldots, \eta_m \in {}^\omega 2$, the quantifier-free type of $\bar{a}_{\eta_1} {}^\frown \ldots {}^\frown \bar{a}_{\eta_m} {}^\frown \langle d_{i_1}, \ldots, d_{i_n}\rangle$ in $M_{1,\kappa}$ is equal to the quantifier-free type of $\bar{a}_{\eta_1} {}^\frown \ldots {}^\frown \bar{a}_{\eta_m} {}^\frown \langle d_1, \ldots, d_n\rangle$ in M_1. (So if M_1 has Skolem functions then $M_1 = M_{1,\mu}$ and they realize the same types.)

Claim 4.12. *If clauses $(\beta), (\gamma), (\delta), (\delta)^+$ of 4.3(1) hold, and 4.5($\otimes$) does as well, <u>then</u> for $\lambda \geq 2^{\aleph_0}$:*

- $(*)_\lambda$ *There is a family $\mathscr{P}$ of subsets of ${}^\omega 2$ each of cardinality $2^{\aleph_0}$ with $|\mathscr{P}| = \beth_2$ such that, letting N^λ_S be the Skolem Hull of $\{\bar{a}_\eta : \eta \in S\} \cup \{d_i : i < \kappa\}$ in $M_{1,\lambda}$ with $S \in \mathscr{P}$ (so $\|N^\lambda_S\| = \lambda$), we have:*
  - $(*)$ *For $Y_1 \neq Y_2$ from $\mathscr{P}$, $N^1_{Y_1}$ has no Δ-embedding into $N^1_{Y_2}$ (i.e. no function from $N^1_{Y_1}$ into $N^1_{Y_2}$ preserves all the relations $\pm\varphi_n$).*

We may consider using relations φ_n which are not equivalence relations, and we may like to give another proof when $\mu > \aleph_0$ but still MA_μ holds.

Claim 4.13. [*Assume* MA_μ.]

Suppose M_1, τ_1, $\langle \bar{a}_\eta : \eta \in {}^\omega 2\rangle$, φ_n (for $n < \omega$), $\langle d_n : n < \omega\rangle$ satisfy clauses (a), (b), (β), (γ), (δ) of 4.3, and M_1 is a τ_1-model of the complete first order theory T_1. Also suppose $\bar{a}_\eta \in {}^k(M_1)$ for $\eta \in {}^{\omega >}2$ are such that if $n < m < \omega$ and $\eta, \nu \in {}^m 2$ then $\eta \restriction n = \nu \restriction n \Leftrightarrow M_1 \models \bar{a}_\eta \; \varphi_n \; \bar{a}_\nu$. (So φ_n is not necessarily an equivalence relation and $|\tau_1| = \mu$ is not necessary countable).

1) *If we replaced ${}^{\omega \geq}2$ by a perfect subtree (splitting determined by level only) and replaced $\langle \varphi_n : n < \omega\rangle$ by a subsequence, then we could add the statement of 4.5($\otimes$) to the assumptions.*

2) *So the conclusion of 4.10 holds, and if we further assume $(\delta)^+$ of 4.3, the conclusion of 4.12 also holds.*

Proof. We use Carlson and Simpson [CS84].

Let W^* be the set of ω-sequences η from $\{0,1\} \cup \{x_i : i < \omega\}$ such that each x_i appears infinitely often. For $\eta \in W^*$, let

$$W_\eta = \{\nu \in W^* : \eta(\ell) \in \{0,1\} \Rightarrow \nu(\ell) = \eta(\ell),\ \eta(\ell_1) = \eta(\ell_2) \Rightarrow \nu(\ell_1) = \nu(\ell_2)\}.$$

As a set, $W \subseteq W^*$ is *large* if it contains some W_η. Let

$$I_W = \big\{\nu \in {}^{\omega >}2 : \text{for some } \eta \in W, \text{ for every } \ell, \ell_1, \ell_2 < \ell g(\nu), \\ \eta(\ell) \in \{0,1\} \Rightarrow \nu(\ell) = \eta(\ell) \wedge \eta(\ell_1) = \eta(\ell_2) \Rightarrow \nu(\ell_1) = \nu(\ell_2)\big\}.$$

Let

$$\mathrm{lev}(W) = \big\{\ell : \text{for some } \eta \in W,\ \eta(\ell) \notin \{0,1\} \text{ but } \eta(0), \ldots, \eta(\ell-1) \in \{0,1\}\big\}.$$

We say $W_1 \subseteq^* W_2$ if for some n, $\{\nu \restriction [n,\omega) : \nu \in W_1\} \supseteq \{\nu \restriction (n,\omega) : \nu \in W_2\}$. By MA_μ, if $\langle W_i : i < \delta \leq \mu\rangle$ is $\subseteq^*$-decreasing sequence then there is W such that $\bigwedge_i W_i \subseteq^* W$.

By the partition theorem there, if $n < \omega$, $\eta_1, \ldots, \eta_k \in {}^n 2$ are pairwise distinct and $\bar{\sigma}^1$, $\bar{\sigma}^2$ are τ_1^+-terms <u>then</u> we can find large $W_1 \subseteq W$ such that $W_1 \restriction n = W \restriction n$ and:

$\circledast^n_{W_1,\bar\sigma}$ If $n < m \in \mathrm{lev}(W_1)$, $\rho^\nu_\ell \in \mathcal{T}_{W_1} \cap {}^m 2$ for $\ell = 1,\dots,k$, and $\nu_\ell = \eta_\ell{}^\frown \rho_2 \restriction [n,\omega)$, <u>then</u> the truth value of $\bar\sigma^1(\bar a_{\nu_1},\dots,\bar a_{\nu_k})\ \varphi_n\ \bar\sigma^2(\bar a_{\nu_1},\dots,\bar a_{\nu_k})$ is constant.

Repeating it, we can get W_1 such that $\circledast^n_{W_1,\bar\sigma}$ for every n.

(i) Either g is constant $< \min(\mathrm{lev}(W_1) \setminus n)$ or

$$n \in \mathrm{lev}(W_1) \ \Rightarrow\ \big[g(n), n\big) \cap \mathrm{lev}(W_1) = \varnothing.$$

(ii) If $n < m \in \mathrm{lev}(W_1)$ and $\eta_\ell \lhd \nu_\ell \in \mathcal{T}_{W_1} \cap {}^m 2$ then

$$\min\big\{i : \neg\big[\sigma^1(\bar a_{\nu_1},\dots,\bar a_{\nu_k})\ \varphi_i\ \sigma^2(\bar a_{\nu_1},\dots,\bar a_{\nu_k})\big]\big\} = g(m).$$

We apply such reasoning to the following statement: "Given $\eta_1,\dots,\eta_k \in \mathcal{T}_{W_1} \cap {}^n 2$ pairwise distinct and $n < m \in \mathrm{lev}(W_1)$, and assuming $\eta_\ell \lhd \nu^i_\ell \in \mathcal{T}_{W_1} \cap {}^m 2$ for $\ell \in \{0,1,\dots,k\}$ and $i \in \{0,1\}$, we have

$$\bar\sigma(\bar a_{\nu^0_1},\dots,\bar a_{\nu^0_k})\ \varphi_\ell\ \bar\sigma(\bar a_{\nu^1_1},\dots,\bar a_{\nu^1_k})."$$

We get that this depends only on $\ell g(\nu^0_\ell \cap \nu^1_\ell)$ and $\nu^i_\ell\big(\ell g(\nu^0_\ell \cap \nu^1_\ell)\big)$. $\square_{4.13}$

Discussion 4.14. The parallel (for a module $\dot{\mathbb{M}}$) concerning "a surgery at" is extending the ring $\dot{\mathbf{R}}$ to $\dot{\mathbf{R}}^+$; e.g. by $\{x_t : t \in I\}$ freely except some equation involving x and the x_i-s and "below x" is replaced by the ideal generated by those equations.

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