

# IMPLICATIONS OF RAMSEY CHOICE PRINCIPLES IN ZF

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**Abstract.** The Ramsey Choice principle for families of  $n$ -element sets, denoted  $\text{RC}_n$ , states that every infinite set  $X$  has an infinite subset  $Y \subseteq X$  with a choice function on  $[Y]^n := \{z \subseteq Y : |z| = n\}$ . We investigate for which positive integers  $m$  and  $n$  the implication  $\text{RC}_m \Rightarrow \text{RC}_n$  is provable in ZF. It will turn out that beside the trivial implications  $\text{RC}_m \Rightarrow \text{RC}_m$ , under the assumption that every odd integer  $n > 5$  is the sum of three primes (known as ternary Goldbach conjecture), the only non-trivial implication which is provable in ZF is  $\text{RC}_2 \Rightarrow \text{RC}_4$ .

**key-words:** permutation models, consistency results, Ramsey choice, ternary Goldbach conjecture

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## 1 Introduction

For positive integers  $n$ , the Ramsey Choice principle for families of  $n$ -element sets, denoted  $\text{RC}_n$ , is defined as follows: For every infinite set  $X$  there is an infinite subset  $Y \subseteq X$  such that the set  $[Y]^n := \{z \subseteq Y : |z| = n\}$  has a choice function. The Ramsey Choice principle was introduced by Montenegro [5] who showed that for  $n = 2, 3, 4$ ,  $\text{RC}_n \Rightarrow \text{C}_n^-$ , where  $\text{C}_n^-$  is the statement that every infinite family of  $n$ -element sets has an infinite subfamily with a choice function. However, the question of whether or not  $\text{RC}_n \rightarrow \text{C}_n^-$  for  $n \geq 5$  is still open (for partial answers to this question see [2, 3]).

In this paper, we investigate the relation between  $\text{RC}_n$  and  $\text{RC}_m$  for positive integers  $n$  and  $m$ . First, for each positive integer  $m$  we construct a permutation models  $\text{MOD}_m$

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in which  $\text{RC}_m$  holds, and then we show that  $\text{RC}_n$  fails in  $\mathbf{MOD}_m$  for certain integers  $n$ . In particular, assuming the ternary Goldbach conjecture, which states that every odd integer  $n > 5$  is the sum of three primes, and by the transfer principles of Pincus [6], we obtain that for  $m, n \geq 2$ , the implication  $\text{RC}_m \Rightarrow \text{RC}_n$  is not provable in  $\text{ZF}$  except in the case when  $m = n$ , or when  $m = 2$  and  $n = 4$ .

**FACT 1.1.** *The implications  $\text{RC}_m \Rightarrow \text{RC}_m$  (for  $m \geq 1$ ) and  $\text{RC}_2 \Rightarrow \text{RC}_4$  are provable in  $\text{ZF}$ .*

*Proof.* The implication  $\text{RC}_m \Rightarrow \text{RC}_m$  is trivial. To see that  $\text{RC}_2 \Rightarrow \text{RC}_4$  is provable in  $\text{ZF}$ , we assume  $\text{RC}_2$ . If  $X$  is an infinite set, then by  $\text{RC}_2$  there is an infinite subset  $Y \subseteq X$  such that  $[Y]^2$  has a choice function  $f_2$ . Now, for any  $z \in [Y]^4$ ,  $[z]^2$  is a 6-element subset of  $[Y]^2$ , and by the choice function  $f_2$  we can select an element from each 2-element subset of  $z$ . For any  $z \in [Y]^4$  and each  $a \in z$ , let  $\nu_z(a) := |\{x \in [z]^2 : f_2(x) = a\}|$ ,  $m_z := \min \{\nu_z(a) : a \in z\}$ , and  $M_z := \{a \in z : \nu_z(a) = m_z\}$ . Since  $f_2$  is a choice function, we have  $\sum_{a \in z} \nu_z(a) = 6$ , and since  $4 \nmid 6$ , the function  $f : [Y]^4 \rightarrow Y$  defined by stipulating

$$f(z) := \begin{cases} a & \text{if } M_z = \{a\}, \\ b & \text{if } z \setminus M_z = \{b\}, \\ c & \text{if } |M_z| = 2 \text{ and } f_2(M_z) = c, \end{cases}$$

is a choice function on  $[Y]^4$ , which shows that  $\text{RC}_4$  holds. +

## 2 A model in which $\text{RC}_m$ holds

In this section we construct a permutation model  $\mathbf{MOD}_m$  in which  $\text{RC}_m$  holds. According to [1, p. 211 ff.], the model  $\mathbf{MOD}_m$  is a *Shelah Model of the Second Type*.

Fix an integer  $m \geq 2$  and let  $\mathcal{L}_m$  be the signature containing the relation symbol  $\text{Sel}_m$ . Let  $\mathsf{T}_m$  be the  $\mathcal{L}_m$ -theory containing the following axiom-schema:

*For all pairwise different  $x_1, \dots, x_m$ , there exists a unique index  $i \in \{1, \dots, m\}$  such that, whenever  $\{b_1, \dots, b_m\} = \{1, \dots, m\}$ ,*

$$\text{Sel}_m(x_{b_1}, \dots, x_{b_m}, x_b) \iff b = i.$$

In other words,  $\text{Sel}_m$  is a selecting function which selects an element from each  $m$ -element set  $\{x_1, \dots, x_m\}$ . In any model of the theory  $\mathsf{T}_m$ , the relation  $\text{Sel}_m$  is equivalent to a function  $\text{Sel}$  which selects a unique element from any  $m$ -element set.

For a model  $\mathbf{M}$  of  $\mathsf{T}_m$  with domain  $M$ , we will simply write  $M \models \mathsf{T}_m$ . Let

$$\tilde{C} = \{M : M \in \text{fin}(\omega) \wedge M \models \mathsf{T}_m\}.$$

Evidently  $\tilde{C} \neq \emptyset$ . Partition  $\tilde{C}$  into maximal isomorphism classes and let  $C$  be a set of representatives. We proceed with the construction of the set of atoms for our permutation model. The next theorem and its proof are taken from [1], with a minor difference which will play an essential role in our work.

**PROPOSITION 2.1.** *Let  $m \in \omega \setminus \{0\}$ . There exists a model  $\mathbf{F} \models \mathbb{T}_m$  with domain  $\omega$  such that*

- *Given a non empty  $M \in C$ ,  $\mathbf{F}$  admits infinitely many submodels isomorphic to  $M$ .*
- *Any isomorphism between two finite submodels of  $\mathbf{F}$  can be extended to an automorphism of  $\mathbf{F}$ .*

*Proof.* The construction of  $\mathbf{F}$  is made by induction. Let  $F_0 = \emptyset$ .  $F_0$  is trivially a model of  $\mathbb{T}_m$  and, for every element  $M$  of  $C$  with  $|M| \leq 0$ ,  $F_0$  contains a submodel isomorphic to  $M$ . Let  $F_n$  be a model of  $\mathbb{T}_m$  with a finite initial segment of  $\omega$  as domain and such that for every  $M \in C$  with  $|M| \leq n$ ,  $F_n$  contains a submodel isomorphic to  $M$ . Let

- $\{A_i : i \leq p\}$  be an enumeration of  $[F_n]^{\leq n}$ ,
- $\{R_k : k \leq q\}$  be an enumeration of all the  $M \in C$  such that  $1 \leq |M| \leq n + 1$ ,
- $\{j_l : l \leq u\}$  be an enumeration of all the embeddings  $j_l : F_n|_{A_i} \hookrightarrow R_k$ , where  $i \leq p$ ,  $k \leq q$  and  $|R_k| = |A_i| + 1$ .

For each  $l \leq u$ , let  $a_l \in \omega$  be the least natural number such that  $a_l \notin F_n \cup \{a_{l'} : l' < l\}$ . The idea is to add  $a_l$  to  $F_n$ , extending  $F_n|_{A_i}$  to a model  $F_n|_{A_i} \cup \{a_l\}$  isomorphic to  $R_k$ , where  $j_l : F_n|_{A_i} \hookrightarrow R_k$ . Define  $F_{n+1} := F_n \cup \{a_l : l \leq u\}$ . In [1],  $F_{n+1}$  is made into a model of  $\mathbb{T}_m$  in a non-controlled way, while here we impose the following: Let  $\{x_1, \dots, x_n\}$  be a subset of  $F_{n+1}$  from which we have not already chosen an element. Suppose that  $s > t$  implies  $x_s > x_t$  (recall that  $F_{n+1}$  is a subset of  $\omega$ ). Then we simply impose  $\text{Sel}(\{x_1, \dots, x_n\}) = x_n$ . The desired model is finally given by  $\mathbf{F} = \bigcup_{n \in \omega} F_n$ .

We conclude by showing that every isomorphism between finite submodels can be extended to an automorphism of  $\mathbf{F}$ . Let  $i_0 : M_1 \rightarrow M_2$  be an isomorphism of  $\mathbb{T}_m$ -models. Let  $a_1$  be the least natural number in  $\omega \setminus (M_1 \cup M_2)$ . Then  $M_1 \cup M_2 \cup \{a_1\}$  is contained in some  $F_n$  and by construction we can find some  $a'_1 \in \omega$  such that  $\mathbf{F}|_{M_1 \cup \{a_1\}}$  is isomorphic to  $\mathbf{F}|_{M_2 \cup \{a'_1\}}$ . Extend  $i_0$  to  $i_1 : M_1 \cup \{a_1\} \rightarrow M_2 \cup \{a'_1\}$  by imposing  $i_1(a_1) = a'_1$ . Let  $a_2$  be the least integer in  $\omega \setminus (M_1 \cup M_2 \cup \{a_1, a'_1\})$  and repeat the process. The desired automorphism of  $\mathbf{F}$  is  $i = \bigcup_{n \in \omega} i_n$ .  $\dashv$

**REMARK 1.** *Let us fix some notations and terminology. The elements of the model  $\mathbf{F}$  above constructed will be the atoms of our permutation model. Each element  $a$  corresponds to a unique embedding  $j$ . We shall call the domain of  $j$  the ground of  $a$ . Moreover, given two atoms  $a$  and  $b$ , we say that  $a < b$  in case  $a <_\omega b$  according to the natural ordering. Notice that this well ordering of the atoms will not exist in the permutation model.*

Let  $A$  be the domain of the model  $\mathbf{F}$  of the theory  $\mathsf{T}_m$ . To build the permutation model  $\mathbf{MOD}_m$ , consider the normal ideal given by all the finite subsets of  $A$  and the group of permutations  $G$  defined by

$$\pi \in G \iff \forall X \in \text{fin}(\omega), \pi(\text{Sel}(X)) = \text{Sel}(\pi X).$$

**THEOREM 2.2.** *For every positive integer  $m$ ,  $\mathbf{MOD}_m$  is a model for  $\text{RC}_m$ .*

*Proof.* Let  $X$  be an infinite set with support  $S'$ . If  $X$  is well ordered, the conclusion is trivial, so let  $x \in X$  be an element not supported by  $S'$  and let  $S$  be a support of  $x$ , with  $S' \subseteq S$ . Let  $a \in S \setminus S'$ . If  $\text{fix}_G(S \setminus \{a\}) \subseteq \text{sym}_G(x)$  then  $S \setminus \{a\}$  is a support of  $x$ , so by iterating the process finitely many times we can assume that there exists a permutation  $\tau \in \text{fix}_G(S \setminus \{a\})$  such that  $\tau(x) \neq x$ . Our conclusion will follow by showing that there is a bijection between an infinite set of atoms and a subset of  $X$ , namely between  $\{\pi(a) : \pi \in \text{fix}_G(S \setminus \{a\})\}$  and  $\{\pi(x) : \pi \in \text{fix}_G(S \setminus \{a\})\}$ . Suppose towards a contradiction that there are two permutations  $\sigma, \sigma' \in \text{fix}_G(S \setminus \{a\})$  such that  $\sigma(x) = \sigma'(x)$  and  $\sigma(a) \neq \sigma'(a)$ . Then, by direct computation, the permutation  $\sigma^{-1}\sigma'$  is such that  $\sigma^{-1}\sigma'(a) \neq a$  and  $\sigma^{-1}\sigma'(x) = x$ . Let  $b = \sigma^{-1}\sigma'(a)$ . Then  $\{b\} \cup (S \setminus \{a\})$  is a support of  $x$ . By construction of the model, the set  $\{\pi(a) : \pi \in \text{fix}_G(\{b\} \cup (S \setminus \{a\}))\}$  is infinite, from which we deduce that also the set

$$L = \{e \in A : \exists \pi \in \text{fix}_G(\{b\} \cup (S \setminus \{a\})) \text{ such that } \pi(x) = x, \pi(a) = e\}$$

is infinite. Now, by assumption there is a permutation  $\tau \in \text{fix}_G(S \setminus \{a\})$  such that  $\tau(x) \neq x$ . Let  $y_0 := \tau(x)$ , with  $c = \tau(a)$  and  $d = \sigma^{-1}\sigma'(c)$ . Then the same argument shows that also

$$R = \{e \in A : \exists \pi \in \text{fix}_G(\{d\} \cup (S \setminus \{a\})) \text{ such that } \pi(x) = y_0, \pi(c) = e\}$$

must be infinite.

First note that in  $L$  there are infinitely many elements with ground  $\{b\} \cup (S \setminus \{a\})$ , which is a support of  $L$ . This is because  $\{b\} \cup (S \setminus \{a\}) \cup \{a\} \subseteq \mathbf{F}$  is a finite model of  $\mathsf{T}_m$  and in the construction of our permutation model we add infinitely many elements  $l \in \omega$  such that  $\{b\} \cup (S \setminus \{a\}) \cup \{l\}$  and  $\{b\} \cup (S \setminus \{a\}) \cup \{a\}$  are isomorphic via an isomorphism  $\delta$  with  $\delta|_{\{b\} \cup (S \setminus \{a\})} = \text{id}|_{\{b\} \cup (S \setminus \{a\})}$  and  $\delta(l) = a$ . We can extend  $\delta$  to an automorphism  $\delta \in \text{fix}_G(\{b\} \cup (S \setminus \{a\}))$ . By definition of  $L$ , since  $\{b\} \cup (S \setminus \{a\})$  is a support of  $x$ , we have that  $l \in L$ .

Similarly, in  $R$  there are infinitely many elements with ground  $\{d\} \cup (S \setminus \{a\})$ , which is a support of  $R$ . This is because  $\{d\} \cup (S \setminus \{a\}) \cup \{c\} \subseteq \mathbf{F}$  is a finite model of  $\mathsf{T}_m$  and in the construction of our permutation model we add infinitely many elements  $l \in \omega$  such that  $\{d\} \cup (S \setminus \{a\}) \cup \{l\}$  and  $\{d\} \cup (S \setminus \{a\}) \cup \{c\}$  are isomorphic via an isomorphism  $\delta$  with  $\delta|_{\{d\} \cup (S \setminus \{a\})} = \text{id}|_{\{d\} \cup (S \setminus \{a\})}$  and  $\delta(l) = c$ . We can extend  $\delta$  to an automorphism  $\delta \in \text{fix}_G(\{d\} \cup (S \setminus \{a\}))$ . By definition of  $R$ , since  $\{c\} \cup (S \setminus \{a\})$  is a support of  $y_0$ , we

have that  $l \in R$ .

Let  $r \in R$  with ground  $\{d\} \cup (S \setminus \{a\})$  and  $p, l \in L$  with ground  $\{b\} \cup (S \setminus \{a\})$  such that  $r \geq p$ ,  $l \geq p$  and  $\min(\{p, q, r\}) > \max(\{d, b\} \cup (S \setminus \{a\}))$ . We want to show that every map

$$\gamma : (S \setminus \{a\}) \cup \{p\} \cup \{l\} \rightarrow (S \setminus \{a\}) \cup \{p\} \cup \{r\}$$

with  $\gamma|_{(S \setminus \{a\}) \cup \{p\}} = \text{id}_{(S \setminus \{a\}) \cup \{p\}}$  and  $\gamma(l) = r$  is an isomorphism of  $\mathbb{T}_m$ -models. Let  $X \subseteq (S \setminus \{a\}) \cup \{p\} \cup \{l\}$ . If  $\{p, l\} \cap X = \emptyset$  we have that  $\gamma(\text{Sel}(X)) = \text{Sel}(\gamma(X))$ . If  $l \in X$  and  $p \notin X$  let  $\pi_l, \pi_r \in \text{fix}_G(S \setminus \{a\})$  with  $\pi_l(a) = l$  and  $\pi_r(a) = r$ . Then  $\pi_r \circ \pi_l^{-1}|_X = \gamma|_X$ . So since  $\pi_r \circ \pi_l^{-1} \in G$  we have  $\gamma(\text{Sel}(X)) = \text{Sel}(\gamma(X))$ . In the last case, when  $\{p, l\} \subseteq X$ , the function selects the greatest element with respect to  $<_\omega$ , given the particular care we took in the construction of the selection function on the set of atoms. So we can extend  $\gamma$  to a function  $\tau' \in \text{fix}_G(\{p\} \cup (S \setminus \{a\}))$  with  $\tau'(l) = r$ .

Let  $\pi_r \in \text{fix}_G(S \setminus \{a\})$  such that  $\pi_r(a) = r$  and  $\pi_r(x) = y_0$ . Let  $\pi_l \in \text{fix}_G(S \setminus \{a\})$  with  $\pi_l(a) = l$  and  $\pi_l(x) = x$ . Then we have that  $\pi_r^{-1} \circ \tau' \circ \pi_l(a) = a$  which implies that  $\pi_r^{-1} \circ \tau' \circ \pi_l(x) = x$  because the function fixes  $S$ . So

$$\tau'(x) = \tau' \circ \pi_l(x) = \pi_r(x) = y_0. \quad (1)$$

Now let  $\pi_p \in \text{fix}_G(S \setminus \{a\})$  with  $\pi_p(a) = p$  and  $\pi_p(x) = x$ . Since  $S$  is a support of  $x$ ,  $\pi_p(S) = \{p\} \cup (S \setminus \{a\})$  is also a support of  $\pi_p(x) = x$ . Therefore,

$$\tau'(x) = x.$$

This is a contradiction to (1). So we showed that for all  $\sigma, \sigma' \in \text{fix}_G(S \setminus \{a\})$ ,  $\sigma(x) = \sigma'(x)$  implies  $\sigma(a) = \sigma'(a)$ , from which we get the desired bijection.  $\dashv$

### 3 For which $n$ is $\text{MOD}_m$ a model for $\text{RC}_n$ ?

The following result shows that for positive integers  $m, n$  which satisfy a certain condition, the implication  $\text{RC}_m \Rightarrow \text{RC}_n$  is not provable in  $\text{ZF}$ . Assuming the ternary Goldbach conjecture, it will turn out that all positive integers  $m, n$  satisfy this condition, except when  $m = n$ , or when  $m = 2$  and  $n = 4$ .

**PROPOSITION 3.1.** *Let  $m, n \in \omega$ . Assume that, for some  $k \in \omega$ ,  $n$  can be written as a sum  $n = \sum_{i \in k} n_i$  with each  $n_i \neq 1$ , such that for all possible finite sequences  $(m_i)_{i \in k}$  satisfying*

- $0 \leq m_i \leq n_i$
- $m_i = 0$  or  $\text{gcd}(n_i, m_i) > 1$

we have that  $m \neq \sum_{i \in k} m_i$ . Then the implication  $\text{RC}_m \Rightarrow \text{RC}_n$  is not provable in ZF.

*Proof.* We show that in  $\text{MOD}_m$ ,  $\text{RC}_n$  fails. Assume towards a contradiction that  $\text{RC}_n$  holds in  $\text{MOD}_m$  and let  $S$  be a support of a selection function  $f$  on the  $n$ -element subsets of an infinite subset  $X$  of the set of atoms  $A$ .

By the construction in Proposition 2.1, given any model  $N$  of  $\mathbb{T}_m$  extending  $S$ , we can find a submodel of  $X \cup S$  isomorphic to  $N$ .

Our conclusion can hence follow from finding a model  $M$  of  $\mathbb{T}_m$  which extends  $S$  with  $|M \setminus S| = n$  and such that  $M$  admits an automorphism  $\sigma$  which fixes pointwise  $S$  and which does not have any other fixed point, since then  $\sigma(f(M \setminus S)) \neq f(M \setminus S)$  but  $\sigma(M \setminus S) = M \setminus S$ . We start with the following claim:

**Claim.** Given a cyclic permutation  $\pi$  on some set  $P$  of cardinality  $|P| = q$ , if a non-trivial power  $\pi^r$  of  $\pi$  fixes a proper subset  $P'$  of  $P$ , then  $\gcd(|P'|, |P|) > 1$ .

To prove the claim, notice that  $\pi^r$  is a disjoint union of cycles of the same length  $l = \frac{q}{\gcd(q,r)}$ . Consider the subgroup of  $\langle \pi \rangle$  given by  $\langle \pi^r \rangle$ . Then  $P'$  is a disjoint union of orbits of the form  $\text{Orb}_{\langle \pi^r \rangle}(e)$  with  $e \in P'$ , all of them with the same cardinality  $s$ , with  $s$  being a divisor of  $l = \frac{q}{\gcd(q,r)}$  and hence of  $q$ , from which we deduce the claim.

We now want to show that we can find a model  $M$  of  $\mathbb{T}_m$ , which extends  $S$  with  $|M \setminus S| = n$  and such that it admits an automorphism  $\sigma$  which fixes pointwise  $S$  and acts on  $M \setminus S$  as a disjoint union of  $k$  cycles, each of length  $n_i$  for  $i \in k$ . This can be done as follows. Pick an  $m$ -element subset  $P$  of  $M$  for which  $\text{Sel}(P)$  has not been defined yet. If  $P \cap S \neq \emptyset$  then let  $\text{Sel}(P)$  be any element in  $P \cap S$ . Otherwise, by our assumptions, there is a cycle  $C_j$  of length  $n_j$  for some  $j \in k$  such that  $\gcd(|P \cap C_j|, |C_j|) = 1$ . Define  $\text{Sel}(P)$  as an arbitrarily fixed element of  $P \cap C_j$  and, for all permutations  $\pi$  in the group generated by  $\sigma$ , define  $\text{Sel}(\pi(P)) = \pi(\text{Sel}(P))$ . We need to argue that this is indeed well defined, i.e. that for two permutations  $\pi, \pi' \in \langle \sigma \rangle$  we have that  $\pi(P) = \pi'(P)$  implies  $\pi(\text{Sel}(P)) = \pi'(\text{Sel}(P))$ . Problems can arise only when  $P \cap S = \emptyset$ , in which case we notice that  $\pi(P) = \pi'(P)$  implies  $\pi(P \cap C_j) = \pi'(P \cap C_j)$ , which in turn by the claim implies that  $\pi^{-1} \circ \pi'$  fixes  $P \cap C_j$  pointwise, from which we deduce  $\pi(\text{Sel}(P)) = \pi'(\text{Sel}(P))$ .  $\dashv$

Proposition 3.1 allows us to immediately deduce the following results.

**COROLLARY 3.2.** *If  $m > n$ , then  $\text{RC}_m$  does not imply  $\text{RC}_n$ .*

*Proof.* It is enough to write  $n$  as  $n = \sum_{i \in 1} n_i$  with  $n_0 = n$  and apply Proposition 3.1.  $\dashv$

**COROLLARY 3.3.** *If there is a prime  $p$  for which  $p \mid n$  but  $p \nmid m$ , then  $\text{RC}_m$  does not imply  $\text{RC}_n$ .*

*Proof.* Given the assumption, writing  $n$  as  $n = \sum_{i \in \frac{n}{p}} n_i$ , where each  $n_i = p$ , allows us to directly apply Proposition 3.1.  $\dashv$

Moreover, we can show the following:

**THEOREM 3.4.** *For any positive integers  $m$  and  $n$ , the implication  $RC_m \Rightarrow RC_n$  is provable in ZF only in the case when  $m = n$  or when  $m = 2$  and  $n = 4$ .*

The proof of Theorem 3.4 is given in the following results, where in the proofs we use two well-known number-theoretical results: The first one is Bertrand's postulate, which asserts that for every positive integer  $m \geq 2$  there is a prime  $p$  with  $m < p < 2m$ , and the second one is ternary Goldbach conjecture (assumed to be proven by Helfgott [4]), which asserts that every odd integer  $n > 5$  is the sum of three primes.

**PROPOSITION 3.5.** *If  $m$  is prime and  $n \neq m$ , then the implication  $RC_m \Rightarrow RC_n$  is not provable in ZF*

*Proof.* Given Corollary 3.3, we can assume that  $n = m^k$  for some natural number  $k > 1$ . Let  $p$  be a prime such that  $m < p < 2m$ , whose existence is guaranteed by Bertrand's postulate. Then clearly  $m \nmid n - p$ , from which we get that the decomposition  $n = p + (n - p)$  satisfies the assumptions of Proposition 3.1.  $\dashv$

**PROPOSITION 3.6.** *If  $n$  is odd and  $m \neq n$ , then the implication  $RC_m \Rightarrow RC_n$  is not provable in ZF.*

*Proof.* By the ternary Goldbach conjecture, let us write  $n$  as sum of three primes  $n = p_1 + p_2 + p_3$ . Given Proposition 3.5, we can assume that  $m = p_1 + p_2$ , since otherwise the decomposition  $n = p_1 + p_2 + p_3$  would satisfy the assumptions of Proposition 3.1.

We first deal with the case in which  $p_1 = p_2 = p_3$  holds, for which we rename  $p = p_1$ . By hand we can exclude the case  $p = 2$ , and now we want to show that the decomposition  $n = n_0 + n_1 = (3p - 2) + 2$  can be used in Proposition 3.1. Notice that  $\gcd(3p - 2, 2p - 2) \in \{1, p\}$ , from which we deduce that necessarily if  $m = m_0 + m_1$  as in Proposition 3.1, then  $m_1 = 0$ . To conclude this first case, it suffices to notice that, since  $p$  is a prime grater than 2,  $\gcd(3p - 2, 2p)$  necessarily equals 1.

We can now assume that it is not true that  $p_1 = p_2 = p_3$ . Since  $n$  is odd,  $p_1 + p_2 \nmid p_3$ . If  $p_3 \nmid p_1 + p_2$ , then the decomposition  $n = n$  actually satisfies Proposition 3.1. So, without loss of generality let us assume that  $p_3 < p_1$  and  $p_3 \mid p_1 + p_2$ . By  $p_3 \mid p_1 + p_2$  we deduce that  $p_2 \neq p_3$ , and we now consider the decomposition  $n = n_0 + n_1 = (p_2 + p_3) + p_1$ . We can't have  $m_1 = p_1$  since  $\gcd(p_2, p_2 + p_3) = 1$ . On the other hand, we can't even have  $m_1 = 0$  since  $p_1 + p_2 > p_2 + p_3$ , which proves that the assumptions of Proposition 3.1 are satisfied.  $\dashv$

**PROPOSITION 3.7.** *Let  $m > 2$  be an even natural number and  $k \in \omega$  such that  $2^k + 1$  is prime. If  $n = m + 2^k$ , then the implication  $RC_m \Rightarrow RC_n$  is not provable in ZF.*

*Proof.* We consider the decomposition  $n = n_0 + n_1 = (m - 1) + (2^k + 1)$ . It directly follows from the assumptions of the proposition that in order to write  $m = m_0 + m_1$

as in Proposition 3.1, since  $n_0 < m$ , necessarily  $m_1 = 2^k + 1$ , from which we deduce  $m_0 = m - 2^k - 1$ . This immediately gives a contradiction in the case  $2^k + 1 > m$ , so let us assume  $2^k + 1 < m$ . We get again a contradiction by the fact that  $\gcd(m_0, n_0) = \gcd(m - 2^k - 1, m - 1) = \gcd(2^k, m - 1) = 1$ , where we used that  $m$  is even. We can hence conclude that Proposition 3.1 can be applied.  $\dashv$

**PROPOSITION 3.8.** *Let  $m$  and  $n$  be even natural numbers such that there is an odd prime  $p$  with  $m < p < n$  and  $n > p + 1$ . Then the implication  $\text{RC}_m \Rightarrow \text{RC}_n$  is not provable in ZF.*

*Proof.* If  $n = p + 3$  or  $n = p + 5$  the decomposition  $n = p + (n - p)$  directly allows us to apply Proposition 3.1. Otherwise, by the ternary Goldbach conjecture, write  $n - p$  as sum of three primes  $n - p = p_1 + p_2 + p_3$ . Consider now the decomposition  $n = \sum_{i \in A} n_i = p + p_1 + p_2 + p_3$ . In order to write  $m = \sum_{i \in A} m_i$ , necessarily  $m_0 = 0$ . If  $n - p < m$  we can already conclude that Proposition 3.1 can be applied. Otherwise, we find ourselves in the assumptions of Proposition 3.6, which again allows us to conclude that  $\text{RC}_m$  does not imply  $\text{RC}_n$ .  $\dashv$

The following result deals with all the remaining cases and completes the proof of Theorem 3.4.

**PROPOSITION 3.9.** *Let  $m$  and  $n$  be even natural numbers with  $3 \leq \frac{n}{2} \leq m < n$  such that if there is a prime  $p$  with  $m < p < n$ , then  $p = n - 1$ . Then the implication  $\text{RC}_m \Rightarrow \text{RC}_n$  is not provable in ZF.*

*Proof.* By Bertrand's postulate, let  $p$  be a prime with  $\frac{n}{2} < p < n$ . This implies by the assumption  $\frac{n}{2} < p < m$  or  $p = n - 1$ . If we are in the latter case, apply again Bertrand's postulate to find a further prime  $\frac{n}{2} - 1 < p' < n - 2$  (notice that by our assumption we have  $2 \leq \frac{n}{2} - 1$ ). Since  $m$  is not prime we necessarily have  $p' \neq m$ , which together with the present assumptions makes us able to assume without loss of generality that  $\frac{n}{2} < p < m$ . By Proposition 3.7 we can assume  $n - m > 4$ , which implies  $n - p > 5$ . Since by the ternary Goldbach conjecture we can write we can write  $n = p + p_1 + p_2 + p_3$ , notice that by the fact that  $n$  and  $m$  are even, we can assume that  $m - p$  equals some odd prime  $p'$ , since otherwise we could already apply Proposition 3.1 with the decomposition  $n = p + p_1 + p_2 + p_3$  or Proposition 3.7 with the same pair  $(m, n)$ . Now, either writing  $n = p + (n - p)$  satisfies the assumptions of Proposition 3.1, or  $n - p$  is a multiple of  $p'$ . We distinguish two cases, namely when  $n - p$  is a power of  $p'$  and when it is not. In the second case, let  $p''$  be a prime distinct from  $p'$  such that  $p'' \mid n - p$ . Writing  $n$  as the sum  $n = n_0 + \sum_{i \in \frac{n-p}{p''}} n_i = p + \sum_{i \in \frac{n-m}{p''}} p''$  satisfies the hypothesis of Proposition 3.1, as  $n - p < m$  and hence if  $m = m_0 + \sum_{i \in \frac{n-m}{p''}} m_i$  then  $m_0 = p$ . For the last case, without loss of generality assume that  $p_1 + p_2 + p_3 = p_1^k$  for some natural number  $k > 1$ . Notice that  $p_3 \neq p_1$ , since otherwise we would have  $p_2 = p_1^k - 2p_1 = p_1(p_1^{k-1} - 2)$ , which is a contradiction, and similarly we obtain  $p_2 \neq p_1$ . We finally assume wlog that  $p_2 > p_1$ ,



which allows us to conclude that the sum  $n = p + p_2 + (p_1 + p_3)$  satisfies the assumptions of Proposition 3.1, concluding the proof.  $\dashv$

For the sake of completeness, we summarise the proof of our main theorem:

*Proof of Theorem 3.4.* Let  $m$  and  $n$  be two distinct positive integers.

$$\text{ZF} \vdash \text{RC}_m \Rightarrow \text{RC}_n \xrightarrow{\text{Cor. 3.2}} m < n \xrightarrow{\text{Prp. 3.6}} n \text{ is even} \xrightarrow{\text{Cor. 3.3}} m \text{ is even}$$

Now, if  $m$  and  $n$  are both even, we have the following two cases:

$$\begin{aligned} m < \frac{n}{2} &\xrightarrow{\text{Prp. 3.8}} \text{ZF} \not\vdash \text{RC}_m \Rightarrow \text{RC}_n \\ m \geq \frac{n}{2} \geq 3 &\xrightarrow[\text{Prp. 3.8}]{\text{Prp. 3.9}} \text{ZF} \not\vdash \text{RC}_m \Rightarrow \text{RC}_n \end{aligned}$$

Thus, by Fact 1.1, the implication  $\text{RC}_m \Rightarrow \text{RC}_n$  is provable in ZF if and only if  $m = n$  or  $m = 2$  and  $n = 4$ .  $\dashv$

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