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RAMSEY PARTITIONS OF METRIC SPACES

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Abstract. We investigate the existence of metric spaces which, for any coloring with a fixed number of colors, contain monochromatic isomorphic copies of a fixed starting space K. In the main theorem we construct such a space of size $2^{\aleph_0}$ for colorings with $\aleph_0$ colors and any metric space K of size $\aleph_0$. We also give a slightly weaker theorem for countable ultrametric K where, however, the resulting space has size $\aleph_1$.

1. Introduction

Recall that the standard Hungarian arrow notation

$$\kappa \to (\lambda)^{\nu}_{\mu}$$

says that whenever we color ν-sized subsets of κ with μ-many colors there is a homogeneous subset of κ of size λ. The question whether, for a given λ, ν, μ, there is a κ such that the arrow holds has been well studied in Ramsey theory. If $\nu = 1$ the coloring becomes a partition of κ and the question reduces to a simple cardinality argument. However, if we add additional structure into the mix, the question becomes nontrivial. The following definition makes precise what we mean by "adding additional structure":

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Definition. Let $\mathcal{K}$ be a class of structures and κ, λ, μ be cardinals. The arrow

$$\kappa \to_{\mathcal{K}} (\lambda)^1_\mu,$$

is shorthand for the statement that for every $K \in \mathcal{K}$ of size λ there is a $Y \in \mathcal{K}$ of size κ such that for any partition of Y into μ-many pieces one of the pieces contains an isomorphic copy of K.

Note that for a class of structures there are often several natural notions of *contains an isomorphic copy*. So the above notation assumes that the choice of $\mathcal{K}$ includes choosing the notion of *contains an isomorphic copy*. The basic question, given a class $\mathcal{K}$, then becomes whether for every λ, μ there is a κ such that $\kappa \to_{\mathcal{K}} (\lambda)^1_\mu$.

These types of questions have been considered before. For example A. Hajnal and P. Komjáth in [1] and [6] consider the class $\mathcal{G}$ of well-ordered undirected graphs. The notion of "G contains an isomorphic copy of H" is "G contains an induced subgraph graph-isomorphic to H via an order-preserving bijection". For this class they prove

Theorem (Hajnal and Komjáth [1]).

$$2^\kappa \to_{\mathcal{G}} (\kappa)^1_\kappa.$$

J. Nešetřil and V. Rödl consider ([5]) the classes $\mathcal{T}_0$ and $\mathcal{T}_1$ of all T_0 and T_1 topological spaces with homeomorphic embeddings. They prove

Theorem (Nešetřil and V. Rödl [5]). *If $\mathcal{T} = \mathcal{T}_0$ or $\mathcal{T} = \mathcal{T}_1$ then*

$$\kappa^\gamma \to_{\mathcal{T}} (\kappa)^1_\gamma$$

In this paper we will be interested mainly in these questions for metric spaces. There have been some results for metric spaces (see e.g. [3], [4], [10], [9]). Most notably, W. Weiss shows in [9] that there is a limit to what one can prove:

Theorem (Weiss [9]). *Assume that there are no inner models with measurable cardinals. If X is a topological space then there is a coloring of X by two colours such that X doesn't contain a monochromatic homeomorphic copy of the Cantor set.*

Also see [7, 3.8(1), 3.9(3)]: it says that if $2^{\aleph_0} > \aleph_\omega$ and some very weak statement holds (the precise formulation is unimportant here, but it is weak enough that the consistency of its negation is not known) then every Hausdorff space X can be divided into $2^{\aleph_0}$ many sets such that none of which contain a homeomorphic copy of the Cantor set. In particular, this holds in $\mathbf{V}^{\mathbb{P}}$ if $\mathbb{P}$ adds $\geq \beth_\omega$ Cohen reals. See more in [8].

In particular in the class of metric spaces, we can't hope for positive results if $\kappa > \omega$ (but see [8] for a positive result from a supercompact cardinal; more history can be found there). The case $\kappa = \omega$ is not ruled out and, in fact, the main result of this paper, due to the first author, is a positive arrow for this case.

DEFINITION 1.1. Let $\mathcal{M}$ be the class of bounded metric spaces with "X contains an isomorphic copy of Y" being "X contains a subspace which is a scaled copy of Y". (K is a scaled copy of Y if there is a bijection $f: K \to Y$ onto Y and a scaling factor $c \in \mathbb{R}^+$ such that $d_K(x, y) = c \cdot d_Y(f(x), f(y))$.

THEOREM 1.2. $2^\omega \to_{\mathcal{M}} (\omega)^1_\omega$.

In fact the theorem we prove is much stronger: for every countable metric space any $\aleph_1$-saturated metric space X works.

The original motivation of the second author for considering these arrows comes from a problem of M. Hrušák stated in ([2]):

QUESTION. *Does ZFC prove that there is a non σ-monotone metric space of size $\aleph_1$?*

If one could replace 2^ω by $\aleph_1$ in the above arrow, this would give a positive answer. In fact, for a positive answer it would be sufficient to consider the class $\mathcal{M}$ with isomorphic copies being Lipschitz images, which seems to be much weaker.

The paper is organized as follows. In the second section we prove the main result and in the third section we discuss what can be proved for the restricted class of ultrametric spaces. We finish the introduction by recalling some definitions and facts for the benefit of the reader.

DEFINITION 1.3. 1) A *metric space* is a pair (X, ρ) where $\rho: X \times X \to \mathbb{R}$ is a *metric* (on X), i.e. it satisfies, for all $x, y, z \in X$,

(a) $\rho(x, y) \geq 0$ and $\rho(x, y) = 0 \iff x = y$;

(b) $\rho(x, y) = \rho(y, x)$; and

(c) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$.

2) The third condition is called the *triangle inequality*. If it is strengthened to

$$\forall x, y, z \in X, \ \rho(x, z) \leq \max\{\rho(x, y), \rho(y, z)\}$$

then we say that the space is *ultrametric*.

3) In the remainder of this paper, we may abuse notation slightly and refer to the metric space (X, ρ) as X.

DEFINITION 1.4. A metric space (X, ρ) is $\aleph_1$-saturated if for any at most countable $Y \subseteq X$ and any function $f: Y \to \mathbb{R}^+$ satisfying the triangle inequality

$$(*) \ \ f(x) + f(y) \geq \rho(x, y) \ \& \ f(x) + \rho(x, y) \geq f(y) \ \& \ f(y) + \rho(x, y) \geq f(x)$$

for all $x, y \in Y$ there is $p \in X$ such that $\rho(x,p) = f(x)$ for all $x \in Y$.

NOTE 1.5. There is a standard way to see X as a structure for a language with countably many binary predicates $\{R_q : q \in \mathbb{Q}\}$: namely, interpret the predicate $R_q(x,y)$ as $\rho(x,y) \leq q$. Then the space X is $\aleph_1$-saturated if

(1) it contains a copy of every finite metric space,

(2) given any finite metric spaces $Y_1 \subseteq Y_2$ with $|Y_2 \setminus Y_1| = 1$ and an isometry $\pi\colon Y_1 \to X$, the isometry can be extended to Y_2, and

(3) every bounded 1-type $\subseteq \{R(x,a), R(b,x) : a, b \in A\}$, for A a countable subset of X is realized.

The following is standard and is included here for the convenience of the reader.

OBSERVATION 1.6. There is an $\aleph_1$-saturated metric space of size 2^ω.

PROOF. Let $\{(Y_\alpha, f_\alpha) : \alpha < 2^\omega\}$ be an enumeration of all pairs such that $Y_\alpha \in [2^\omega]^{\leq\omega}$ and $f_\alpha\colon Y_\alpha \to \mathbb{R}^+$ with each pair appearing cofinally often. By induction define a sequence $\langle d_\alpha : \alpha < 2^\omega\rangle$ such that

(1) $d_\alpha \subseteq d_\beta$ for all $\alpha < \beta < 2^\omega$;

(2) d_α is a metric on α; and

(3) if $Y_\alpha \subseteq \alpha$ and (Y_α, f_α) satisfies $(*)$ of Definition 1.4 and there is no $\beta < \alpha$ such that $d_\alpha(y,\beta) = f_\alpha(y)$ for all $y \in Y_\alpha$ then $d_{\alpha+1}(y,\alpha) = f_\alpha(y)$ for all $y \in Y_\alpha$.

The only nontrivial part is guaranteeing (3) for successors. So assume $Y_\alpha \subseteq \alpha$ and that $(*)$ is satisfied and for each $\beta < \alpha$ there is $y \in Y_\alpha$ such that $d_\alpha(\beta,y) \neq f_\alpha(y)$. Extend d_α to $d_{\alpha+1}$ by defining

$$d_{\alpha+1}(\beta,\alpha) = \inf\ \big\{ d_\alpha(\beta,y) + f_\alpha(y) : y \in Y_\alpha \big\}, \quad d_{\alpha+1}(\alpha,\alpha) = 0.$$

Then clearly both (1) and (3) are satisfied. To show that (2) is satisfied it is enough to show that $d_{\alpha+1}(\beta,\alpha) > 0$ for all $\beta < \alpha$. Assume this is not the case for some $\beta < \alpha$. By assumption there is $y \in Y_\alpha$ such that

$$0 < |f_\alpha(y) - d_\alpha(\beta,y)| = \varepsilon.$$

Since $d_{\alpha+1}(\beta,\alpha) = 0$ we can find $z \in Y_\alpha$ such that $d_\alpha(\beta,z) + f_\alpha(z) < \varepsilon/2$. There are two cases, both leading to a contradiction: if $f_\alpha(y) > d_\alpha(\beta,y)$ then $d_\alpha(z,y) < d_\alpha(\beta,y) + \varepsilon/2$ so

$$d_\alpha(z,y) + f_\alpha(z) < d_\alpha(\beta,y) + \varepsilon = f_\alpha(y)$$

contradicting $(*)$. On the other hand if $f_\alpha(y) < d_\alpha(\beta,y)$ then $d_\alpha(z,y) \geq d_\alpha(\beta,y) - d_\alpha(\beta,z) = f_\alpha(y) + \varepsilon - d_\alpha(\beta,z) > f_\alpha(y) + \varepsilon/2 \geq f_\alpha(y) + f_\alpha(z)$ again contradicting $(*)$. This completes the inductive definition. Finally we show that $(2^\omega, d_{2^\omega})$ is $\aleph_1$-saturated. Fix an at most countable $Y \subseteq 2^\omega$ and an $f\colon Y \to \mathbb{R}^+$. Find $\alpha < 2^\omega$ such that $Y \subseteq \alpha$ and $(Y,F) = (Y_\alpha, f_\alpha)$. But then the existence of p in 1.4 is guaranteed by (3) above. $\square$

2. The metric case

Proposition 2.1. *Assume* (K, d) *is a countable bounded metric space and* $(X, \rho) = \bigcup_{n<\omega} X_n$ *is a countable partition of an* $\aleph_1$*-saturated metric space. Then there is an* $n < \omega$ *such that* X_n *contains a scaled copy of* (K, d).

Proof. First fix an enumeration $\{z_k : k < \omega\}$ of K and, aiming towards a contradiction, assume there is no scaled monochromatic copy of K in X. We shall use the following notation: given an (at most) countable $Y \subseteq X$ and a function $f : Y \to \mathbb{R}^+$ as in Definition 1.4, let

$$B(Y, f) = \big\{ p \in X : (\forall y \in Y)[d(p, y) = f(y)] \big\}.$$

By our assumption $B(Y, f) \neq \emptyset$. We shall inductively construct an increasing sequence $\{Y_n : n < \omega\}$ of finite subsets of X and functions $\{f_n : n < \omega\}$ such that

(1) $f_n \subseteq f_{n+1}$; and

(2) $f_n : Y_n \to \mathbb{R}^+$ satisfies $(*)$; and

(3) $B(Y_n, f_n) \cap X_i = \emptyset$ for each $i < n$.

(4) Y_n is nonempty, and $\sup\{d(x_1, x_2) : x_1, x_2 \in Y_n\} \leq 2 \cdot \sup\{d_K(x_1, x_2) : x_1, x_2 \in K\}$.

Let $Y_0 = f_0 = \emptyset$. Assume now that we have constructed Y_n, f_n and choose an arbitrary positive $c \in \mathbb{R}^+$ such that $i, j < \omega \wedge y \in Y_N \Rightarrow c \cdot d(z_i, z_j) < f_n(y)/2^{n+1}$. (We can choose c because (K, d_K) is bounded and Y_n is finite.) We try to choose $z'_i \in B(Y_n, f_n) \cap X_n$ by induction on $i < \omega$ such that $j < i \Rightarrow \rho(z'_j, z'_i) = c \cdot d(z_j, z_i)$. If we succeed then we are done. So without loss of generality there is some k such that $\langle z'_i : i < k \rangle$ is well defined but we cannot choose z'_k. Let $K'_n = \{z'_i : i < k\}$ be this copy and let $Y_{n+1} = Y_n \cup K'_n$. Finally extend f_n to Y_{n+1} by defining

$$f_{n+1}(z'_i) = c \cdot d(z_i, z_k).$$

We need to check that f_{n+1} satisfies $(*)$. Let $x, y \in \operatorname{dom}(f_{n+1})$. The condition is easily seen to be satisfied separately on Y_n (i.e. when $x, y \in Y'_n$) by the inductive hypothesis and on K'_n (i.e. when $x, y \in K_n$) because it is defined from a metric. So without loss of generality let $y \in Y_n$ and $x \in K'_n$, so $x = z'_i$ for some $i < k$. Since $K'_n \subseteq B(Y_n, f_n)$, by definition $\rho(x, y) = \rho(z'_i, y) = f_n(y)$. But then $(*)$ is clearly satisfied (the triangle is isosceles and the two legs are longer than the base by the choice of c).

Finally, we show that the inductive construction has to stop at some point (thus there has to be a scaled copy of K in some X_n). Let $Y = \bigcup_{n<\omega} Y_n$ and $f = \bigcup_{n<\omega} f_n$. Then $B(Y, f)$ is nonempty (because X is $\aleph_1$-saturated) and $B(Y, f) \subseteq B(Y_n, f_n)$ for each $n < \omega$ (since $Y_n \subseteq Y$ and $f_n = f \restriction Y_n$). But then $B(Y, f) \cap X_n = \emptyset$ for each $n < \omega$—a contradiction. $\square$

3. The ultrametric case

As noted in the introduction, the second author's original motivation for studying these questions was the special case

$$\aleph_1 \to_{\mathcal{M}} (\aleph_0)^1_{\aleph_0}$$

for the class of bounded metric spaces. Unfortunately, this arrow probably does not hold in ZFC. However a modified version of this arrow holds for the class of rational ultrametric spaces.

DEFINITION 3.1. 1) A metric space X is called *rational* if $x, y \in X \Rightarrow \rho(x, y) \in \mathbb{Q}$.

2) Repeating Definition 1.3, an *ultrametric* space is a metric space that satisfies the strong triangle inequality

$$\rho(x, z) \le \max\{\rho(x, y), \rho(y, z)\}.$$

3) Given $(X, \le)$ a tree and $x, y \in X$, let $\Delta(x, y)$ be the $\le$-maximal z such that $z \le x \wedge z \le y$.

4) A tree T is θ-branching iff the set of immediate successors of each element of T is of size θ.

THEOREM 3.2. *There is a rational ultrametric space (M, d) of size $\aleph_1$ such that for every coloring of M by countably many colors M contains isometric monochromatic copies of every finite rational ultrametric space.*

This theorem is both a strengthening and a weakening of the above arrow. On the one hand we get a universal space for all copies. The price we have to pay is to restrict the copies to size $< \aleph_0$. The proof of the theorem is split into two parts. We first prove that each finite ultrametric space can be represented as a special kind of a tree. Then we use a standard rank-type argument to show that whenever the tree ${}^{<\omega}\omega_1$ is colored by countably many colors it contains monochromatic copies of all finite trees.

Before continuing with the proof of the first part we recall the following basic observation about ultrametric spaces.

FACT 3.3. Let (X, ρ) be an ultrametric space. Then every triangle is isosceles. Moreover, the base is never longer than the sides. Formally:

$$(\forall T \in [X]^3)(\exists \{a, b\} \subset T, c \in T \setminus \{a, b\})\big(\rho(a, b) \le \rho(a, c) = \rho(b, c)\big)$$

DEFINITION. A metric space (X, ρ) is a *rational tree space* if there is an ordering $\le$ which makes X a tree and a nonincreasing function $h : X \to \mathbb{Q}^+$ such that, for distinct $x \neq y \in X$,

$$\rho(x, y) = \inf\left\{ h(z) : z \le x \ \&\ z \le y \right\}.$$

We will also call the triple $(X, \leq, h)$ a rational tree space. The metric space (X, ρ) is a *rational branch space* if it is a subspace of a rational tree space (T, ρ) with all nodes of X being branches (leaf nodes) of (T, ρ). It is a *regular rational branch space* if, moreover, each node of X has the same height and the function h_T is constant on the levels of T.

PROPOSITION. *Each finite rational ultrametric space is a regular rational branch space.*

PROOF. Let (X, ρ) be a finite rational ultrametric space. Define a relation $\leq_0$ on X as follows:

$$x \leq_0 y \iff (\forall z \neq x)(\rho(x, z) \geq \rho(y, z))$$

CLAIM. *The relation $\leq_0$ is transitive.*

PROOF OF CLAIM. Let $a \leq_0 b$ & $b \leq_0 c$. We need to show that $a \leq_0 c$. We may assume a, b, c are distinct, otherwise there is nothing to prove. So consider some $z \neq a$. We just need to show that $\rho(a, z) \geq \rho(c, z)$. If $z = c$ then $\rho(c, z) = 0 \leq \rho(a, z)$. If $z = b$, then the inequality follows directly from $b \leq_0 c$. Then, $\rho(b, a) \geq \rho(c, a)$ because $b \leq_0 c$, and $\rho(c, a) = \rho(a, c) \geq \rho(b, c) = \rho(c, b)$ because $a \leq_0 b$. Together we are done. So assume $z \neq b$. Then $\rho(a, z) \geq \rho(b, z) \geq \rho(c, z)$, and so $\rho(a, z) \geq \rho(c, z)$ as promised. The first inequality follows from $a \leq_0 b$ and the second from $b \leq_0 c$. This finishes the proof of the claim. $\square$

CLAIM. *For each $y \in X$ the set $\{a : a \leq_0 y\}$ is linearly (quasi)-ordered by $\leq_0$.*

PROOF OF CLAIM. Assume $a_0, a_1 \leq_0 y$ and, aiming towards a contradiction, assume that $a_0 \not\leq_0 a_1$ and $a_1 \not\leq_0 a_0$. So there must be z_0, z_1 such that $\varepsilon_i = \rho(a_i, z_i) < \rho(z_i, a_{1-i})$ for $i = 0, 1$. Let $\delta = \rho(a_0, a_1)$. Applying Fact 3.3 we get $\delta = \rho(a_1, a_0) = \rho(a_1, z_0)$ (reading the above inequality for $i = 0$) and $\delta = \rho(a_0, a_1) = \rho(a_0, z_1)$ (for $i = 1$). Now consider the triangle a_0, z_0, z_1. We have $\rho(a_0, z_0) < \delta = \rho(a_0, z_1)$ hence by 3.3 we have $\rho(z_0, z_1) = \delta$.

Since $a_i \leq_0 y$, we have $\delta > \rho(a_i, z_i) \geq \rho(y, z_i)$ for $i = 0, 1$. But, again by Fact 3.3, the triangle z_0, z_1, y is impossible. This is a contradiction. $\square$

Consider now the equivalence relation $a \simeq b \iff a \geq b$ & $b \geq a$ and refine the $\leq_0$ order on each equivalence class to an arbitrary linear order. Call the resulting order $\leq$. Since X is finite, it is clear that $(X, \leq)$ is a tree. For $s \in X$ put

$$h(s) = \max\{\rho(s, t) : t \geq s\} \ \ \big(= \max\{\rho(s, t) : t \geq_0 s\}\big)$$

(The second equality follows from the fact that if $a \simeq b$ and $s \neq a, s \neq b$ then $\rho(s, a) = \rho(s, b)$.) Let d be the metric of the tree space $(X, \leq, h)$.

CLAIM. *$d(x,y) \geq \rho(x,y)$ and, if $x \leq_0 y$, then $d(x,y) = \rho(x,y)$.*

PROOF OF CLAIM. Assume first that $x \leq_0 y$. Then $d(x,y) = h(x) \geq \rho(x,y)$ by definition. Moreover if $z \geq_0 x$ then $\rho(z,y) \leq \rho(x,y), \rho(x,z)$ (since $x \leq_0 z$ and $x \leq_0 y$) and, since X is ultrametric, it follows that $\rho(x,y) = \rho(x,z)$. In particular, since the choice of z was arbitrary, $h(x) = \rho(x,y)$, proving the second part of the claim. To finish the proof assume now that x, y are incomparable in $\leq_0$ and let $s = \Delta(x,y)$. Then $h(s) \geq \rho(s,y) \geq \rho(x,y)$ since $s \leq_0 x$. □

Unfortunately, the inequality in the above claim can be strict (e.g. if we consider the subspace of a tree space which results from deleting a level the resulting subspace cannot be a tree space). We need to add a point to the tree for each pair x, y with $\rho(x,y) < d(x,y)$. We will use the following claim:

CLAIM. *Suppose $(Y, \leq, h)$ is a tree space extending $(X, \leq, h)$ such that $d_Y(x,y) \geq \rho(x,y)$ for each $x, y \in X$. Suppose that there are $a, b \in X$, incompatible in $\leq$ with $\rho(a,b) < d_Y(a,b)$. Then there is a tree space Y' extending Y such that $d_{Y'}(x,y) \geq \rho(x,y)$ for each $x, y \in X$ and $\rho(a,b) = d_{Y'}(a,b)$.*

PROOF OF CLAIM. Let $Y' = Y \cup \{p\}$ and extend the order so that $\Delta(a,b) \leq p \leq a, b$. Moreover let $h(p) = \rho(a,b)$. Notice that if $x, y \in X$ and either $x \not\geq a$ & $x \not\geq b$ or $y \not\geq a$ & $y \not\geq b$ or $x \leq y$ or $y \leq x$ then $d_{Y'}(x,y) = d_Y(x,y)$ and there is nothing to prove. So, without loss of generality, assume $x \geq a$ and $y \geq b$. But then $\rho(a,b) \geq \rho(x,b)$ (since $a \leq x$) and $\rho(b,x) \geq \rho(x,y)$ (since $b \leq y$). Since $\Delta(x,y) = \Delta(a,b) = p$ we have $d_{Y'}(x,y) = h(p) = \rho(a,b)$ and this finishes the proof of the claim. □

Using the above claim to iteratively add points we finally arrive at a tree space $(Y, \leq, h)$ such that $d_Y \restriction X = \rho$ which, moreover, has the same distance set as the original X. It is not hard to further enlarge Y to make it a regular rational branch space. The Proposition is proved. □

PROPOSITION. *Assume T is an ω_1-branching tree*[1] *of height $n < \omega$ and $\chi \colon T \to \omega$ is a coloring of the tree by countably many colors. Then there is an ω_1-branching subtree*[2] *of T whose branches (i.e. leaf nodes) have the same color.*

PROOF. Given a color $c < \omega$ and $s \in T$ define

$$G(s,c,0) \iff \big|\{\alpha : \chi(s^\frown\alpha) = c\}\big| = \omega_1$$

and, inductively,

$$G(s,c,m+1) \iff \big|\{\alpha : G(s^\frown\alpha, c, m)\}\big| = \omega_1.$$

[1] see Definition 3.1(4)

[2] meaning the subtree is downward, and all its maximal nodes are maximal nodes of T.

To prove the proposition it is clearly enough to show that there is some $c < \omega$ such that $G(\emptyset, c, \mathrm{ht}(T) - 1)$. Suppose otherwise. Then we can build by induction α_m for $m < \mathrm{ht}(T) - 1$ such that for $m < \mathrm{ht}(T)$ we have

$$(\forall c < \omega)\neg G\big(\langle \alpha_i : i < m\rangle, c, \mathrm{ht}(T) - m\big).$$

For $m = 0$ this is our assumption, working towards contradiction. For $m + 1$ this is again easy. Hence

$$(\forall c < \omega)\neg G\big(\big\langle \alpha_i : i < \mathrm{ht}(T) - 1\big\rangle, c, 0\big)$$

which is impossible since if we let $s = \langle \alpha_i : i < \mathrm{ht}(T) - 1\rangle$ then, since T is ω_1-branching, s must have uncountably many successors of the same color. □

Proof of Theorem 3.2. Let $M = {}^{<\omega}\omega_1$ and define $h_M \colon M \to \mathbb{Q}$ such that for each $\sigma \in M$ and each $q \in [0, h_M(\sigma))$ the set $\{\alpha : h_M(\sigma^\frown\alpha) = q\}$ has size $\aleph_1$. Let d_M be the corresponding metric making M a tree space. Let X be a finite rational metric space, h a decreasing enumeration of its distance set and let $(Y, \leq, h_Y)$ be a tree space witnessing that X is a regular rational branch space. Let $\chi : T \to \omega$ be an arbitrary coloring of M. Consider the subtree $M' = \{s : h \restriction s = d \restriction |s|\}$. Then M' is ω_1-branching. By the previous proposition there is a color c and an ω_1 branching subtree M'' of M' with all branches of color c. We can now build an order-isomorphism of Y into M'' which, by choice of M', preserves h. It follows that M'' contains a monochromatic isometric copy of X. □

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