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Nice $\aleph_1$ generated non-P-points, Part I

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We define a family of non-principal ultrafilters on $\mathbb{N}$ which are, in a sense, very far from P-points. We prove the existence of such ultrafilters under reasonable conditions. In subsequent articles, we intend to prove that such ultrafilters may exist while no P-point exists. Though our primary motivations came from forcing and independence results, the family of ultrafilters introduced here should be interesting from combinatorial point of view too.

1 INTRODUCTION

One of the important notions in general topology and set theory of the reals is that of a P-point. Recall that a *P-point* is a non-principal ultrafilter D on $\mathbb{N}$ with the property that for any countable family $\mathscr{A} \subseteq D$ there is a $B \in D$ almost (modulo finite) included in all $A \in \mathscr{A}$ (cf. Definition 3.6). Concerning these and other special ultrafilters on $\mathbb{N}$, their history and basic applications we refer the reader to the survey article by Blass [1].

One of the reasons why P-points are very important in set theory is that they behave very nicely with respect to forcing. In many applications it is important to preserve P-points by specific forcing notions and by an iterated forcing with countable supports[1] and, for P-points, these issues are very well understood and many results are known in the literature. The following is a list of some of the nice properties of P-points:

(A) There are many forcing notions preserving P-points.
(B) A proper forcing notion preserving being an ultrafilter will also preserve being a P-point.
(C) The property of preserving P-points is preserved in limits of countable support iterations.
(D) We can destroy a P-point by forcing, i.e., ensure it has no extension to a P-point: this allows us to prove the consistency of "there are no P-points".

[1] Here, an ultrafilter *is preserved by forcing* if the ground model ultrafilter generates an ultrafilter in the generic extension (cf. [7, Chapter VI]).

(E) We can destroy some P-points by forcing while preserving others (this is known as "splitting hairs"): this allows us to construct a unique P-point up to isomorphism.

Already the properties (A), (B), and (C) give a controlled way to have ultrafilters generated by $\aleph_1 < 2^{\aleph_0}$ sets (for more details, we refer the reader to [7, Chapters VI & XVIII, § 4]).

The motivating question of this paper and the sequence of papers it starts is whether the theory developed for P-points can be developed for any other classes of ultrafilters. A paradigmatic question is whether property (C) can hold for other classes of ultrafilters:

Question 1.1. Can we find a type of ultrafilters that is preserved by countable support iterations of suitable forcing notions (e.g., proper forcing)? In particular, we are interested in preservation of our ultrafilters at limit stages of countable support iterations: i.e., in a countable support iteration of suitable forcing notions of length $\delta + 1$ where δ is a limit ordinal, if each stage α for $\alpha < \delta$ preserves the relevant property, then stage δ preserves the property.

We suggested this problem in [6, 3.13] and we speculated about it there. [Note that in the situation of Question 1.1, ultrafilters are naturally generated by $\aleph_1$ many sets, as we start with a ground model of CH, and the question of preservation of ultrafilters is only relevant if we add reals (usually, $\aleph_2$ many).] We suspect that Question 1.1 is related to the following question by van Douwen [3], but at present we know neither whether they are really related nor how to answer Question 1.2.

Question 1.2 (van Douwen). Is it consistent that there is no ultrafilter D on $\mathbb{Q}$ such that every $A \in D$ contains a member of D which is a closed set with no isolated points?

Other specific problems that we worked on are:

Question 1.3 (Nyikos). Is it consistent to have an non-principal ultrafilter of character $\aleph_1$, but no P-point?

Question 1.4 (Dow). Is it consistent to have $\mathfrak{u} = \aleph_1$ and P-points, but no P-point of character $\aleph_1$?

This paper starts a series of several papers motivated by these questions. The main points done here (items 1 to 4) and intended in subsequent parts (items 5 to 7) are:

1. We shall define an involved family of sets (really well founded trees) to define a class of ultrafilters;
2. from this, we define ultrafilters analogous to P-points that have no P-point as a quotient (§§ 2 & 3);
3. the ultrafilters are related to a game;
4. such systems exists assuming, e.g., $\diamondsuit_{\aleph_1}$;
5. relevant forcing notions preserve such systems, in particular, we get property (C), i.e., we answer Question 1.1 positively;
6. we have a preservation theorem for the ultrafilters under countable support iterations;
7. as an application, we shall solve Nyikos's Problem 1.3.

In § 4, we describe basic connections to forcing that we intend to use in the independence results in subsequent papers of the series. In Part II of the series (still work in progress), we present these ultrafilters in a more general framework and deal with sufficient conditions for such an ultrafilter to generate an ultrafilter in a suitable generic extension. For the limit case, we intend to continue the proof of preservation theorems in [7], in particular [7, Chapter VI, 1.26, 1.27] and Case A with transitivity of [7, Chapter XVIII, § 3]. For the successor case, we need that the relevant forcing preserves our ultrafilters.

In Part III, we note that the ultrafilters are analogous to selective (i.e., Ramsey) ultrafilters and hope to give a more general framework which also includes P-points.

We should like to note that while it is consistent to have P-points and $\mathfrak{d} > \aleph_1$ (cf. [2] and references there), the existence of ultrafilters as discussed in this paper implies $\mathfrak{d} = \aleph_1$ [1]. However, note that these ultrafilters may be $\aleph_1$-generated in a different sense: they could be unions of $\aleph_1$ families of the form $\mathrm{fil}(B) \cap \wp(\max(B))$.[2] Note that it may be harder (than

[2] We write $\mathrm{fil}(B)$ for the filter generated by B and the co-finite subsets; for $\max(B)$, cf. § 2.

in the P-point case) to build such ultrafilters which are μ-generated for some $\mu > \aleph_1$ (instead of $\aleph_1$-generated) because of the unbounded countable depth involved. We have not considered this question, natural variants of our definition, or generalisations to reasonable ultrafilters (cf. [4, 5, 9]).

2 | SYSTEM OF FILTERS USING WELL FOUNDED TREES

Let $M = (M, <_M)$ be a partial order and B be a subset of M inheriting its order. For $\eta \in B$ we let $B_{\geq\eta} = \{\nu \in B : \eta \leq_M \nu\}$ and $B_{>\eta} = \{\nu \in B : \eta <_M \nu\}$. The set B is called a *branch of* M if it is a maximal chain (linearly ordered subset). We also define the *immediate B-successors* of an element and the set of maximal members of the set B as usual:

$$\mathrm{suc}_B(\eta) = \{\nu \in B : \eta <_M \nu \text{ but for no } \varrho \in B \text{ do we have } \eta <_M \varrho <_M \nu\} \text{ and}$$

$$\max(B) = \{\nu \in B : B \cap M_{>\nu} = \emptyset\}.$$

Two elements $\eta, \nu \in M$ are called $<_M$*-incompatible*, in symbols $\eta \parallel_M \nu$ if they have no common $\leq_M$-upper bound. We say that Y is a *front of* $B \subseteq M$ *if* $Y \subseteq B$ *and every branch of* B meets Y and the members of Y are pairwise $<_M$-incomparable. We write $\mathrm{frt}(B)$ for the set of all fronts of B.

Definition 2.1. Let $M = (M, <_M)$ be a partial order. A countable set $B \subseteq M$ is a *countable well-founded sub-tree of* M if the following conditions (a) to (d) are satisfied.

(a) The structure $(B, <_M \restriction B)$ is a tree with $\leq \omega$ levels and no branch of order type ω (so all chains of B are finite). In particular, B has a $<_M$ minimal element called its *root*, in symbols: $\mathrm{rt}(B)$.
(b) For each $\nu \in B$ the set $\mathrm{suc}_B(\nu)$ is either empty or infinite.
(c) If $\eta, \nu \in B$ have no common $<_M$-upper bound in B, then $\eta \parallel_M \nu$ (i.e., they are incompatible not only in B but even in M).
(d) If ν is not maximal in B and F is a finite subset of $M \setminus M_{\leq\nu}$, then there are infinitely $\sigma \in \mathrm{suc}_B(\nu)$ such that σ is $<_M$-incompatible with all elements of F.

The family of all countable well-founded sub-trees of M is denoted by $\mathrm{CWT}(M)$. For $B \in \mathrm{CWT}(M)$, *the depth of* B is defined recursively by $\mathrm{Dp}(B) := \sup\{\mathrm{Dp}(B_{\geq\eta}) + 1 : \eta \in B \setminus \{\mathrm{rt}(B)\}\}$.

Note that if $B \in \mathrm{CWT}(M)$, then $\mathrm{frt}(B)$ is the family of all maximal sets of pairwise incomparable members of B.

We shall define a natural filter on the set of maximal nodes of every countable well-founded tree B; this filter will naturally induce Rudin-Keisler images on each front of B. First, we introduce two notions of largeness for subtrees: *exhaustive* subtrees correspond to filter sets or "measure 1" sets; *positive* subtrees will correspond to the notion "positive modulo a filter" or "not in the ideal dual to the filter".

Definition 2.2. Let $B \in \mathrm{CWT}(M)$. We call B' is an *exhaustive subtree* of B if and only if $B' \in \mathrm{CWT}(M)$, $B' \subseteq B$, $\mathrm{rt}(B') = \mathrm{rt}(B)$, and for all $\nu \in B'$ we have that $\mathrm{suc}_{B'}(\nu) \subseteq \mathrm{suc}_B(\nu)$ and $\mathrm{suc}_B(\nu) \setminus \mathrm{suc}_{B'}(\nu)$ is finite. We let $\mathrm{sb}(B)$ be the set of all exhaustive subtrees B' of B, and we say f *witnesses that* B' *is an exhaustive subtree of* B if $f : B' \setminus \max(B) \longrightarrow [B]^{<\aleph_0}$ *satisfies that* $\nu \in B' \setminus \max(B)$ implies that $\mathrm{suc}_B(\nu) \setminus \mathrm{suc}_{B'}(\nu) \subseteq f(\nu)$. Note that for f being a witness only $f \restriction B'$ matters; in fact, often, only the restriction $f \restriction \{\nu \in B' \mid \exists \eta \in Y : \nu \leq \eta\}$ matters.

Definition 2.3. For antichains Y_1, Y_2 of M we say that Y_2 is *above* Y_1 if and only if

$$(\forall \eta \in Y_2)(\exists \nu \in Y_1)[\nu \leq_M \eta].$$

If Y_2 is above Y_1, we let the projection h_{Y_1,Y_2} be the unique function $h : Y_2 \longrightarrow Y_1$ such that $h(\eta) \leq_M \eta$ for $\eta \in Y_2$.

If $Y_1, Y_2 \in \mathrm{frt}(B)$, then Y_2 *is almost above* Y_1 if and only if for some $B' \in \mathrm{sb}(B)$, $B' \cap Y_2$ is above $B' \cap Y_1$. In this case, we define the projection h_{Y_1,Y_2} as above, but its domain is not Y_2 but the set $\{\eta \in Y_2 : (\exists \nu \in Y_1)(\nu \leq_M \eta)\}$.

Definition 2.3 will be used mainly for $Y_1, Y_2 \in \mathrm{frt}(B)$ where $B \in \mathrm{CWT}(M)$.

For $B \in \mathrm{CWT}(M)$ and $Y \in \mathrm{frt}(B)$ let $E_{B,Y}$ be the filter on Y generated by the family $\{Y \cap B' : B' \in \mathrm{sb}(B)\}$. For $B \in \mathrm{CWT}(M)$ let $\mathrm{psb}_M(B)$ ("p" stands for positive) be the set of *positive subtrees* B' of B which means that $B' \in \mathrm{CWT}(M)$, $B' \subseteq B$, $\mathrm{rt}(B') = \mathrm{rt}(B)$, for all $\nu \in B'$, we have that $\mathrm{suc}_{B'}(\nu) \subseteq \mathrm{suc}_B(\nu)$, and if $\nu \in B' \setminus \max(B)$, then $\mathrm{suc}_{B'}(\nu)$ is an infinite subset of $\mathrm{suc}_B(\nu)$.

Definition 2.4. An antichain $Y \subseteq M$ is an *almost front of* B if for some $B' \in \mathrm{sb}(B)$ the intersection $Y \cap B'$ is a front of B'. Let $\text{alm-frt}(B) = \text{alm-frt}_M(B)$ denote the set of all almost fronts of B. For $Y \in \text{alm-frt}_M(B)$ let

$$\mathrm{fil}_M(Y, B) := \{X \subseteq Y : \text{ for some } B' \in \mathrm{sb}(B) \text{ we have } X \supseteq B' \cap Y\}.$$

The default value of $Y \in \mathrm{frt}(B)$ is $\max(B) = \{\nu \in B : \nu \text{ is } <_M\text{-maximal in } B\}$.

Definition 2.5. Let $\leq^*_M$ be the following binary relation on $CWT(M)$:

$$\begin{aligned} B_1 \leq^*_M B_2 \text{ if and only if } & B_1, B_2 \in \mathrm{CWT}(M),\ \mathrm{rt}(B_1) = \mathrm{rt}(B_2), \text{ and} \\ & \text{for some } B'_2 \in \mathrm{sb}(B_2), \text{ we have that } B'_2 \cap B_1 \in \mathrm{psb}_M(B_1) \\ & \text{and every almost front of } B'_2 \cap B_1 \text{ is an almost front of } B_2. \end{aligned}$$

The tree B'_2 as above will be called a *witness for* $B_1 \leq^*_M B_2$.

Let us remark that if $B, B' \in \mathrm{CWT}(M)$, $B' \subseteq B$ and $\nu \in B'$, then $\mathrm{suc}_B(\nu) \cap B' \subseteq \mathrm{suc}_{B'}(\nu)$, but the two sets do not have to be equal. Note furthermore that in the definitions of both $B' \in \mathrm{sb}(B)$ and $B' \in \mathrm{psb}_M(B)$ we do require that for all $\nu \in B'$, we have that $\mathrm{suc}_B(\nu) \cap B' = \mathrm{suc}_{B'}(\nu)$. This condition implies that if $Y \subseteq B$ is a front of B, then $Y \cap B'$ is a front of B'.

Observation 2.6. Let M be a partial order and $B, B_1, B_2 \in \mathrm{CWT}(M)$.

1. We have that $B_1 \leq^*_M B_2$ if and only if every almost front of B_1 is an almost front of B_2.
2. The relation $\leq^*_M$ is a partial order on $\mathrm{CWT}(M)$.
3. If $B_2 \in \mathrm{psb}_M(B_1)$, then $B_1 \leq^*_M B_2$ and $\mathrm{psb}_M(B_2) \subseteq \mathrm{psb}_M(B_1)$.
4. If $B_2 \in \mathrm{sb}(B_1)$, then $B_2 \in \mathrm{psb}(B_1)$, $\mathrm{sb}(B_2) \subseteq \mathrm{sb}(B_1)$ and $B_1 \leq^*_M B_2 \leq^*_M B_1$.
5. For $B \in \mathrm{CWT}(M)$, $\max(B)$ is a front of B and also $\{\mathrm{rt}(B)\}$ is. If $B \neq \{\mathrm{rt}(B)\}$, then $\mathrm{suc}_B(\mathrm{rt}(B))$ is a front of B.
6. Every front of $B \in \mathrm{CWT}(M)$ is an almost front of B.
7. If $B \in \mathrm{CWT}(M)$ then $\mathrm{Dp}(B)$ is a countable ordinal and $B_{\geq \eta} \in \mathrm{CWT}(M)$ for all $\eta \in B$.
8. If $Y \subseteq B \setminus \{\mathrm{rt}(B)\}$ is a front of B, and $\eta \in \mathrm{suc}_B(\mathrm{rt}(B))$, then $Y \cap B_{\geq \eta}$ is a front of $B_{\geq \eta}$.
9. If Y is an almost front of B and an antichain Z is an almost front of $B_{\geq \eta}$ for every $\eta \in Y \cap B$, then Z is an almost front of B.
10. If $B_1 \leq^*_M B_2$ and Y is a front of B_1, then there is $B'_2 \in \mathrm{sb}(B_2)$ such that $Y \cap B'_2$ is a front of B'_2 and $(B_1)_{\geq \eta} \leq^*_M (B'_2)_{\geq \eta}$ for all $\eta \in Y \cap B'_2$.

Proof. Straightforward. □

Definition 2.7. Let $\mathbf{K}$ be the class of the objects $\mathbf{x} = \langle M_\mathbf{x}, <_{M_\mathbf{x}}, \bar{\mathscr{A}}_\mathbf{x}, \mathscr{A}_\mathbf{x}, \mathscr{B}_\mathbf{x}, \leq_\mathbf{x} \rangle$ satisfying the following properties:

(a) The structure $(M_\mathbf{x}, <_{M_\mathbf{x}}) = (M, <)$ is a partial order with the smallest element $\mathrm{rt}_\mathbf{x} = \mathrm{rt}(\mathbf{x})$. Let $M^-_\mathbf{x} = M_\mathbf{x} \setminus \{\mathrm{rt}_\mathbf{x}\}$,
(b) $\bar{\mathscr{A}}_\mathbf{x} = \bar{\mathscr{A}} = \langle \mathscr{A}_\eta : \eta \in M \rangle = \langle \mathscr{A}^\mathbf{x}_\eta : \eta \in M_\mathbf{x} \rangle$ and $\mathscr{A}_\mathbf{x} = \bigcup \{\mathscr{A}_\eta : \eta \in M^-_\mathbf{x}\}$,
(c) $\mathscr{A}_\eta \subseteq \mathrm{CWT}(M)$, let $\mathscr{A}^-_\eta = \mathscr{A}_\eta \setminus \{\{\eta\}\}$,
(d) $\mathrm{rt}(B) = \eta$ for every $B \in \mathscr{A}_\eta$,
(e) $\mathscr{A}_\eta$ is not empty, in fact $\{\eta\} \in \mathscr{A}_\eta$,
(f) $\mathscr{B}_\mathbf{x} = \mathscr{A}^\mathbf{x}_{\mathrm{rt}_\mathbf{x}} \setminus \{\{\mathrm{rt}_\mathbf{x}\}\}$ and $\leq_\mathbf{x}$ is a directed partial order on $\mathscr{B}_\mathbf{x}$,
(g) $B_1 \leq_\mathbf{x} B_2$ implies $B_1 \leq^*_M B_2$ and, of course, $B_1, B_2 \in \mathscr{B}_\mathbf{x}$,
(h) if $\nu \in B \in \mathscr{A}_\eta$ then $B \cap M_{\geq \nu} \in \mathscr{A}_\nu$.

When dealing with $M_\mathbf{x}, \bar{\mathscr{A}}_\mathbf{x}$ etc we may omit $\mathbf{x}$ when clear from the context.

Definition 2.8. Let $\mathbf{x} \in \mathbf{K}$ and $\eta \in M_{\mathbf{x}}$.

1. Let $\mathrm{frt}_{\mathbf{x}}(\eta) := \bigcup_{B \in \mathscr{A}^{\mathbf{x}}_{\eta}} \mathrm{frt}(B)$ and $\mathrm{frt}^{-}_{\mathbf{x}}(\eta) := \{Y \in \mathrm{frt}(\eta) : Y \neq \{\eta\}\}$. We write $\mathrm{frt}_{\mathbf{x}} := \mathrm{frt}_{\mathbf{x}}(\mathrm{rt}_{\mathbf{x}})$ and $\mathrm{frt}^{-}_{\mathbf{x}} := \mathrm{frt}^{-}_{\mathbf{x}}(\mathrm{rt}_{\mathbf{x}})$ and define $\text{alm-frt}_{\mathbf{x}}(\eta)$ and $\text{alm-frt}_{\mathbf{x}}$ similarly (cf. Definition 2.4).
2. Let $B \in \mathscr{A}^{\mathbf{x}}_{\eta}$. We define $\mathrm{Fin}(B)$ to be the set $\{f : f$ is a function with domain $B \setminus \max(B)$ such that $f(\nu) \in [\mathrm{suc}_B(\nu)]^{<\aleph_0}$ for all $\nu \in B \setminus \max(B)\}$ and for $f \in \mathrm{Fin}(B)$ we set $A_f := A_{B,f} := \{\eta \in B : (\forall \varrho \in B \setminus \max(B))(\forall \sigma \in \mathrm{suc}_B(\varrho))(\sigma \leq_M \eta$ implies $\sigma \notin f(\varrho))\}$.
3. Assume that $Y \in \text{alm-frt}_{\mathbf{x}}$. We let $D^{\mathbf{x}}_{Y}$ be the family $\{Z \subseteq Y :$ for some $B \in \mathscr{B}_{\mathbf{x}}$ and $B' \in \mathrm{sb}(B)$ we have $Y \in \text{alm-frt}(B)$ and $B' \cap Y \subseteq Z\}$.
4. If $B \in \mathscr{B}_{\mathbf{x}}$, then $D_{\mathbf{x}}(B) = D^{\mathbf{x}}_{\max(B)}$.
5. We let $\mathrm{Dp}_{\mathbf{x}}(\eta) = \sup\{\mathrm{Dp}(B) + 1 : B \in \mathscr{A}^{\mathbf{x}}_{\eta}\}$.

If $\mathbf{x}$ is clear from the context, then we may omit the subscript or superscript $\mathbf{x}$ in the objects defined above.

Let us recall the definition of the Rudin-Keisler order on ultrafilters.

Definition 2.9. Let D_ℓ be an ultrafilter on $\mathscr{U}_\ell$ for $\ell = 1, 2$. We say $D_1 \leq_{\mathrm{RK}} D_2$ if and only if there is a function h whose domain and range are subsets of $\mathscr{U}_2, \mathscr{U}_1$, respectively, such that for all $A \subseteq \mathscr{U}_1$, we have $A \in D_1$ if and only if $\{a \in \mathrm{Dom}(h) : h(a) \in A\} \in D_2$.

Observation 2.10. Assume $\mathbf{x} \in \mathbf{K}$ and let $B, B_1, B_2 \in \mathscr{B}_{\mathbf{x}}$.

1. The singleton $\{\mathrm{rt}_{\mathbf{x}}\}$ is in $\mathrm{frt}_{\mathbf{x}}$ and $D^{\mathbf{x}}_{\{\mathrm{rt}_{\mathbf{x}}\}} = \{\{\mathrm{rt}_{\mathbf{x}}\}\}$.
2. If $B_1 \leq_{\mathbf{x}} B_2$, $f \in \mathrm{Fin}(B_1)$ and $Y \in \text{alm-frt}(B_1)$, then $Y \in \text{alm-frt}(B_2)$ and there is $g \in \mathrm{Fin}(B_2)$ such that $Y \cap A_{B_2,g} \subseteq Y \cap A_{B_1,f}$.
3. If $Y \in \text{alm-frt}(B_\ell)$, $f_\ell \in \mathrm{Fin}(B_\ell)$ (for $\ell = 1, 2$), then there are $B^* \in \mathscr{B}_{\mathbf{x}}$ and $g \in \mathrm{Fin}(B^*)$ such that $B_1 \leq_{\mathbf{x}} B^*$, $B_2 \leq_{\mathbf{x}} B^*$ and $Y \cap A_{B^*,g} \subseteq Y \cap A_{B_1,f_1} \cap A_{B_2,f_2}$.
4. If $Y \in \text{alm-frt}_{\mathbf{x}}$, then $D^{\mathbf{x}}_{Y}$ is a filter on Y.
5. If $B_1 \leq_{\mathbf{x}} B_2$, $Y_1 \in \text{alm-frt}(B_1)$, and $Y_2 = Y_1 \cap B_2$ (hence $Y_2 \in \text{alm-frt}(B_2)$), then $Y_2 \in D^{\mathbf{x}}_{Y_1}$ and $D^{\mathbf{x}}_{Y_2} = D^{\mathbf{x}}_{Y_1} \restriction Y_2$.
6. Assume that $Y_1, Y_2 \in \mathrm{frt}(B)$ and Y_2 is above Y_1. Let $h : Y_2 \to Y_1$ be the (surjective) projection, i.e., $h(\nu_2) = \nu_1$ if and only if $\nu_1 \in Y_1, \nu_2 \in Y_2$, and $\nu_1 \leq_{M_{\mathbf{x}}} \nu_2$. Then $h(D_{Y_2}) = D_{Y_1}$, i.e., $D_{Y_1} = \{A \subseteq Y_1 : h^{-1}[A] \in D_{Y_2}\}$ (so h witnesses $D_{Y_1} \leq_{\mathrm{RK}} D_{Y_2}$).
7. If $B_1 \leq_{\mathbf{x}} B_2$ and $Y_\ell = \mathrm{suc}_{B_\ell}(\mathrm{rt}_{\mathbf{x}})$ for $\ell = 1, 2$, then
   (a) Y_ℓ is a front of B_ℓ and Y_1 almost above Y_2, cf. Definition 2.3, and
   (b) if Y is a front of B_ℓ and it is not $\{\mathrm{rt}_{\mathbf{x}}\}$, then Y is above Y_ℓ.
8. The set $\max(B)$ is the maximal front of B which means that it is above any other.
9. If $\mathbb{Q}$ is an ${}^{\omega}\omega$-bounding forcing and $B \in \mathscr{B}_{\mathbf{x}}$, then for any $B' \in \mathrm{sb}(B)^{\mathbf{V}[\mathbb{Q}]}$ there is $B'' \in (\mathrm{sb}(B))^{\mathbf{V}}$ such that $B'' \subseteq B'$.
10. If F is a finite subset of $M^{-}_{\mathbf{x}}$, $B \in \mathscr{B}_{\mathbf{x}}$, then there is a branch (i.e., a maximal chain) $C \subseteq B$ such that for all $\varrho \in F$ and $\sigma \in C$, we have $\varrho \nleq_M \sigma$).

Proof. Straightforward. □

Definition 2.11. For an (infinite) cardinal κ let $\mathbf{K}_{<\kappa}$ be the class of $\mathbf{x} \in \mathbf{K}$ such that $\|\mathbf{x}\| := |M_{\mathbf{x}}| + \sum\{|\mathscr{A}^{\mathbf{x}}_{\eta}| : \eta \in M_{\mathbf{x}}\} < \kappa$, similarly $\mathbf{K}_{\leq\kappa}$. The relation $\leq_{\mathbf{K}}$ is the following two-place relation on $\mathbf{K}$ (it is a partial order, cf. Observation 2.13 below):[3]

$$\mathbf{x} \leq_{\mathbf{K}} \mathbf{y} \text{ if and only if } M_{\mathbf{x}} \subseteq M_{\mathbf{y}} \text{ (as partial orders),}$$
$$\text{for any } \eta, \nu \in M_{\mathbf{x}} \text{ we have } \nu \parallel_{M_{\mathbf{x}}} \eta \text{ if and only if } \nu \parallel_{M_{\mathbf{y}}} \eta,$$
$$\eta \in M_{\mathbf{x}} \text{ implies } \mathscr{A}^{\mathbf{x}}_{\eta} \subseteq \mathscr{A}^{\mathbf{y}}_{\eta},\ \mathrm{rt}_{\mathbf{y}} = \mathrm{rt}_{\mathbf{x}}, \text{ and } \leq_{\mathbf{x}} = \leq_{\mathbf{y}} \restriction \mathscr{B}_{\mathbf{x}}.$$

[3] Note that $\leq_{\mathbf{x}} = \leq_{\mathbf{y}} \restriction \mathscr{B}_{\mathbf{x}}$ implies that $\mathrm{rt}_{\mathbf{y}} = \mathrm{rt}_{\mathbf{x}}$.

Definition 2.12. If $\langle \mathbf{x}_\alpha : \alpha < \delta \rangle$ is a $\leq_{\mathbf{K}}$-increasing sequence we define $\mathbf{x}_\delta = \bigcup\{\mathbf{x}_\alpha : \alpha < \delta\}$, the union of the sequence, by $M_{\mathbf{x}_\delta} = \bigcup\{M_{\mathbf{x}_\alpha} : \alpha < \delta\}$ as partial orders and $\mathscr{A}^{\mathbf{x}_\delta}_\eta = \bigcup\{\mathscr{A}^{\mathbf{x}_\alpha}_\eta : \alpha < \delta$ satisfies $\eta \in M_{\mathbf{x}_\alpha}\}$ and $\leq_{\mathbf{x}_\delta} = \bigcup\{\leq_{\mathbf{x}_\alpha} : \alpha < \delta\}$.

Observation 2.13. It is easy to see that the relation $\leq_{\mathbf{K}}$ is a partial order and that this order is closed under chains, i.e., whenever $\langle \mathbf{x}_\alpha : \alpha < \delta \rangle$ is $\leq_{\mathbf{K}}$-increasing, we can define $\mathbf{x}_\delta$ as the union of the sequence. It is then clear that $\mathbf{x}_\delta$ is a $\leq_{\mathbf{K}}$-lub of the sequence and $\|\mathbf{x}_\delta\| \leq \sum \|\{\|\mathbf{x}_\alpha\| : \alpha < \delta\}$.

Definition 2.14. Let $\mathbf{x} \in \mathbf{K}$. We say that

(a) $\mathbf{x}$ is *fat* if and only if whenever $B \in \mathscr{B}_{\mathbf{x}}$ and $B' \in \mathrm{sb}(B)$, then there is $B'' \in \mathrm{sb}(B')$ such that $B'' \in \mathscr{B}_{\mathbf{x}}$ and $B \leq_{\mathbf{x}} B''$;

(b) $\mathbf{x}$ is *big* if and only if whenever $B \in \mathscr{B}_{\mathbf{x}}$ and $\mathbf{c} : \max(B) \longrightarrow \{0,1\}$, then for some $B' \in \mathscr{B}_{\mathbf{x}}$ we have that $B' \in \mathrm{psb}_{M_{\mathbf{x}}}(B) \cap \mathscr{B}_{\mathbf{x}}$, $B \leq_{\mathbf{x}} B'$ and $\mathbf{c} \restriction \max(B')$ is constant;

(c) $\mathbf{x}$ is *large* if and only if whenever $B \in \mathscr{B}_{\mathbf{x}}$ and $\mathbf{c}$ is a function with domain $\max(B)$, then for some $B' \in \mathrm{psb}_{M_{\mathbf{x}}}(B) \cap \mathscr{B}_{\mathbf{x}}$ and a front Y of B' we have $B \leq_{\mathbf{x}} B'$ and for all $\eta, \nu \in \max(B')$, we have that $\mathbf{c}(\eta) = \mathbf{c}(\nu)$ if and only if there is a $\varrho \in Y$ such that $\varrho \leq_{M_{\mathbf{x}}} \eta$ and $\varrho \leq_{M_{\mathbf{x}}} \nu$;

(d) $\mathbf{x}$ is *full* if and only if whenever $B \in \mathscr{A}^{\mathbf{x}}_\eta$, $\eta \neq \mathrm{rt}_{\mathbf{x}}$ and $B' \in \mathrm{psb}_{M_{\mathbf{x}}}(B)$, then $B' \in \mathscr{A}^{\mathbf{x}}_\eta$.

3 | CONSTRUCTION OF ULTRA-SYSTEMS

Lemma 3.1. The set $\mathbf{K}_{\leq\aleph_0}$ is non-empty.

Proof. Define $\mathbf{x}$ so that $M_{\mathbf{x}} = \{\eta_*\}$, $\mathscr{A}^{\mathbf{x}}_{\eta_*} = \{\{\eta_*\}\}$, $\mathrm{rt}_{\mathbf{x}} = \eta_*$. Now it is easy to check. □

Lemma 3.2. If $\mathbf{x} \in \mathbf{K}$ and $\eta \in M_{\mathbf{x}}$ satisfies $|\mathscr{A}^{\mathbf{x}}_\eta| = 1$, i.e., $\mathscr{A}^{\mathbf{x}}_\eta = \{\{\eta\}\}$, then for some $\mathbf{y} \in \mathbf{K}$ we have $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$, $|\mathscr{A}^{\mathbf{y}}_\eta| > 1$ and $\|\mathbf{y}\| \leq \|\mathbf{x}\| + \aleph_0$.

Proof. Let $\langle \eta_n : n < \omega \rangle$ be pairwise distinct objects not belonging to $M_{\mathbf{x}}$. We define $\mathbf{y}$ as follows: We let $M_{\mathbf{y}} := M_{\mathbf{x}} \cup \{\eta_n : n < \omega\}$ and $\nu <_{M_{\mathbf{y}}} \varrho$ if and only if $\nu <_{M_{\mathbf{x}}} \varrho$ or $\nu \leq_{M_{\mathbf{x}}} \eta$ and there is an n such that $\varrho = \eta_n$). The set $\mathscr{A}^{\mathbf{y}}_\nu$ is defined by the following case distinction: If $\nu \in M_{\mathbf{x}} \backslash \{\eta\}$, then $\mathscr{A}^{\mathbf{y}}_\nu := \mathscr{A}^{\mathbf{x}}_\nu$; if $\nu = \eta$, then $\mathscr{A}^{\mathbf{y}}_\nu := \{\{\eta\}, \{\eta_n : n < \omega\} \cup \{\eta\}\}$; and if $\nu = \eta_n$, then $\mathscr{A}^{\mathbf{y}}_\nu := \{\{\eta_n\}\}$. Finally, if $\eta \neq \mathrm{rt}_{\mathbf{x}}$, then $\leq_{\mathbf{y}} := \leq_{\mathbf{x}}$; otherwise (i.e., if $\eta = \mathrm{rt}_{\mathbf{x}}$), it is determined by $\{\eta\} \leq_{\mathbf{y}} \{\eta_n : n < \omega\} \cup \{\eta\}$. Now check. □

Lemma 3.3. 1. If $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$, then for some $\mathbf{y} \in \mathbf{K}_{\leq\aleph_0}$ we have $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ and in $\mathscr{B}_{\mathbf{y}}$ there is a $\leq_{\mathbf{y}}$-maximal member.

2. If $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$ and some $B \in \mathscr{B}_{\mathbf{x}}$ is $\leq_{\mathbf{x}}$-maximal, then for some $\mathbf{y} \in \mathbf{K}_{\leq\aleph_0}$ and $B' \in \mathscr{B}_{\mathbf{y}}$ we have $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ and $B <_{\mathbf{y}} B'$.
3. If $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$, $\eta \in M_{\mathbf{x}}$, $B_1 \in \mathscr{A}^{\mathbf{x}}_\eta$, $B_2 \in \mathrm{psb}_{M_{\mathbf{x}}}(B_1)$ and if $\eta = \mathrm{rt}_{\mathbf{x}}$ implies that B_1 is $\leq_{\mathbf{x}}$-maximal, then there is $\mathbf{y} \in \mathbf{K}_{\leq\aleph_0}$ such that $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ and $B_2 \in \mathscr{A}^{\mathbf{y}}_\eta$.
4. If $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$, $B_1 \in \mathscr{B}_{\mathbf{x}}$ and $B_2 \in \mathrm{sb}(B_1)$, then there is $\mathbf{y} \in \mathbf{K}_{\leq\aleph_0}$ such that $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ and $B_2 \in \mathscr{B}_{\mathbf{y}}$.

Proof. The proof of (2), (3), & (4) is straightforward; cf. also Lemmas 3.4 & 3.5 below. We therefore only need to prove (1).

If in $(\mathscr{B}_{\mathbf{x}}, \leq_{\mathbf{x}})$ there is a maximal member, then we let $\mathbf{y} = \mathbf{x}$. Otherwise, as it is directed (cf. clause (f) of Definition 2.7) and $\|\mathbf{x}\| \leq \aleph_0$ (because $\mathbf{x} \in \mathbf{K}_{\leq\aleph_0}$), there is a strictly $\leq_{\mathbf{x}}$-increasing cofinal sequence $\langle B_n : n < \omega \rangle$. Let $Y_n = \mathrm{suc}_{B_n}(\mathrm{rt}_{\mathbf{x}})$.

Note that for each $m_1 < m_2$, the set $Y_{m_1} \cap B_{m_2}$ is an almost front of B_{m_2} (so also it is almost above Y_{m_2}). Hence for $m_1 < m_2 \leq n$ we have that $Y_{m_1} \cap B_n$ is an almost front of B_n which is almost above $Y_{m_2} \cap B_n$. Consequently we may choose $B^*_n \in \mathrm{sb}(B_n)$ such that each $Y_\ell \cap B^*_n$ is a front of B^*_n and $Y_\ell \cap B^*_n$ is above $Y_{\ell+1} \cap B^*_n$ (for all $\ell < n$). Moreover, we may also require that

$$\text{for each } \ell < n \text{ and } \eta \in Y_\ell \cap B^*_n \text{ we have } (B_\ell)_{\geq\eta} \leq^*_{M_{\mathbf{x}}} (B^*_n)_{\geq\eta} \tag{3.1}$$

(remember Observation 2.6 10).

Fix a list $\langle \varrho_\ell : \ell < \omega \rangle$ of all members of $M_{\mathbf{x}}$ (possibly with repetitions). By induction on $n < \omega$, choose ν_n such that

$$\nu_n \in Y_n \cap B^*_n = \mathrm{suc}_{B^*_n}(\mathrm{rt}_{\mathbf{x}}) \tag{3.2}$$

$$\text{if } \ell < n, \text{ then } \nu_n, \nu_\ell \text{ are } <_{M_{\mathbf{x}}}\text{-incompatible } (i.e., \nu_\ell \parallel_{M_{\mathbf{x}}} \nu_n), \tag{3.3}$$

$$\text{if } \ell < n \text{ and } \varrho_\ell \neq \mathrm{rt}_{\mathbf{x}}, \text{ then } \varrho_\ell \parallel_{M_{\mathbf{x}}} \nu_n. \tag{3.4}$$

[Why is the choice possible? By the demand (d) of Definition 2.1 applied to $\nu = \mathrm{rt}_{\mathbf{x}}$ and $F = \{\nu_\ell, \varrho_\ell : \ell < n\} \setminus \{\mathrm{rt}_{\mathbf{x}}\}$.]
We define

$$B^* = \{\mathrm{rt}_{\mathbf{x}}\} \cup \bigcup \{B^*_n \cap (M_{\mathbf{x}})_{\geq \nu_n} : n < \omega\}.$$

This set B^* is clearly a countable well-founded tree, $B^* \in \mathrm{CWT}(M_{\mathbf{x}})$ with root $\mathrm{rt}_{\mathbf{x}}$ and $\mathrm{suc}_{B^*}(\mathrm{rt}_{\mathbf{x}}) = \{\nu_n : n < \omega\}$.

[Why? It should be clear that conditions (a) and (b) of Definition 2.1 hold, $\mathrm{rt}(B^*) = \mathrm{rt}_{\mathbf{x}}$ and $\mathrm{suc}_{B^*}(\mathrm{rt}_{\mathbf{x}}) = \{\nu_n : n < \omega\}$. To verify clause (c) suppose $\eta, \nu \in B^*$ are $<_{M_{\mathbf{x}}}$-incomparable. Then both $\eta \neq \mathrm{rt}_{\mathbf{x}}$ and $\nu \neq \mathrm{rt}_{\mathbf{x}}$, so $\eta, \nu \in \bigcup_{n<\omega} (B^*)_{\nu_n}$. If, for some n, we have $\eta, \nu \in B^*_n \cap (M_{\mathbf{x}})_{\geq \nu_n}$, then they are $<_{M_{\mathbf{x}}}$-incompatible as $B^*_n \subseteq B_n$ and B_n satisfies Definition 2.1(c). Otherwise, for some distinct ℓ, n we have $\eta \in B^*_\ell \cap (M_{\mathbf{x}})_{\geq \nu_\ell}$ and $\nu \in B^*_n \cap (M_{\mathbf{x}})_{\geq \nu_n}$. Now, if we could find $\varrho \in M_{\mathbf{x}}$ such that $\varrho \geq_{M_{\mathbf{x}}} \eta$ and $\varrho \geq_{M_{\mathbf{x}}} \nu$, then ν_ℓ, ν_n would be compatible contradicting (3.3), so B^* indeed satisfies clause (c) of Definition 2.1. Finally, to verify (d) suppose $\nu \in B^* \setminus \max(B^*)$ and $F \subseteq M_{\mathbf{x}} \setminus (M_{\mathbf{x}})_{\leq \nu}$ is finite. If $\nu_n \leq_{M_{\mathbf{x}}} \nu$ for some n, then the properties of B^*_n apply. So suppose $\nu = \mathrm{rt}_{\mathbf{x}}$. Choose m so that $F \subseteq \{\varrho_\ell : \ell < m\}$ and use condition (3.4) to argue that for all $n \geq m$ and $\varrho \in F$ we have $\nu_n \parallel_{M_{\mathbf{x}}} \varrho$.]

We also have that $B \leq^*_{M_{\mathbf{x}}} B^*$ for all $B \in \mathscr{B}_{\mathbf{x}}$.

[Why? Since $\leq^*_{M_{\mathbf{x}}}$ is a partial order and by the choice of B_n, it is enough to show that for each $n < \omega$ we have $B_n \leq^*_{M_{\mathbf{x}}} B^*$, i.e., that every almost front of B_n is an almost front of B^*. To this end suppose that $Z \subseteq B_n$ is an almost front of B_n for some $n < \omega$. If $Z = \{\mathrm{rt}_{\mathbf{x}}\}$, then there is nothing to do, so suppose $Z \subseteq B_n \setminus \{\mathrm{rt}_{\mathbf{x}}\}$, i.e., $Z \subseteq \bigcup \{(B_n)_{\geq \varrho} : \varrho \in Y_n\}$. Plainly, the set $X = \{\varrho \in Y_n : Z \text{ is not an almost front of } (B_n)_{\geq \varrho}\}$ is finite and hence for some $m > n$ we have $X \subseteq \{\varrho_\ell : \ell < m\}$. Then for every $k > m$ we have

(a) The element ν_k is incompatible with every $\nu \in X$;
(b) The set $Y_n \cap (B^*_k)_{\geq \nu_k}$ is a front of $(B^*_k)_{\geq \nu_k}$;
(c) $(B_n)_{\geq \eta} \leq^*_{M_{\mathbf{x}}} (B^*_k)_{\geq \eta}$ for every $\eta \in Y_n \cap (B^*_k)_{\geq \nu_k}$ (by (3.1));
(d) the set $Z \cap (B_n)_{\geq \eta}$ is an almost front of $(B_n)_{\geq \eta}$ for every $\eta \in Y_n \cap (B^*_k)_{\geq \nu_k}$, and thus
(e) the set $Z \cap (B^*_k)_{\geq \eta}$ is an almost front of $(B^*_k)_{\geq \eta}$ for every $\eta \in Y_n \cap (B^*_k)_{\geq \nu_k}$.
(f) Finally, Z is an almost front of $(B^*_k)_{\geq \nu_k}$ (by Observation 2.6 9 and (b) & (e)).

Since $\mathrm{suc}_{B^*}(\mathrm{rt}_{\mathbf{x}}) = \{\nu_k : k < \omega\}$, we know that $\{\nu_k : m < k < \omega\}$ is an almost front of B^*. Therefore, by Observation 2.6 9 and (f), we conclude that Z is an almost front of B^*.]

Lastly, we define $\mathbf{y}$ by $(M_{\mathbf{y}}, <_{M_{\mathbf{y}}}) := (M_{\mathbf{x}}, <_{M_{\mathbf{x}}})$, $\mathscr{A}^{\mathbf{y}}_\nu = \mathscr{A}^{\mathbf{x}}_\nu$ if and only if $\nu \in M_{\mathbf{x}} \setminus \{\mathrm{rt}_{\mathbf{x}}\}$, and $\mathscr{A}^{\mathbf{y}}_{\mathrm{rt}_{\mathbf{x}}} = \mathscr{A}^{\mathbf{x}}_{\mathrm{rt}_{\mathbf{x}}} \cup \{B^*\}$, and $B_1 \leq_{\mathbf{y}} B_2$ if and only if $B_1 \leq_{\mathbf{x}} B_2$ or $B_1 \in A^{\mathbf{y}}_{\mathrm{rt}_{\mathbf{x}}} \wedge B_2 = B^*$. It should be clear that $\mathbf{y} \in \mathbf{K}_{\leq \aleph_0}$ is as required. □

Lemma 3.4. Assume that $\mathbf{x} \in \mathbf{K}_{\leq \aleph_0}$ and $B \in \mathscr{B}_{\mathbf{x}}$ is $\leq_{\mathbf{x}}$-maximal. Then for some $\mathbf{y} \in \mathbf{K}_{\leq \aleph_0}$ and $B' \in \mathscr{B}_{\mathbf{y}}$ we have $\mathbf{x} \leq \mathbf{y}$, $M_{\mathbf{x}} = M_{\mathbf{y}} = M$, $B' \in \mathscr{B}_{\mathbf{y}}$ is $\leq_{\mathbf{y}}$-maximal, and if $\nu \in B' \setminus \max(B')$ and $\varrho \in M \setminus M_{\leq \nu}$, then for all but finitely many $\sigma \in \mathrm{suc}_{B'}(\nu)$ we have $\varrho \parallel_M \sigma$.

Proof. Fix a list $\langle \varrho_\ell : \ell < \omega \rangle$ of all members of $M_{\mathbf{x}}$ (possibly with repetitions). For each $\eta \in B \setminus \max(B)$ by induction on $n < \omega$ we choose $\nu_{\eta,n}$ such that $\nu_{\eta,n} \in \mathrm{suc}_B(\eta)$, $\nu_{\eta,n} \neq \nu_{\eta,k}$ for $k < n$ (and hence $\nu_{\eta,n} \parallel \nu_{\eta,k}$ for $k < n$), and if $k < n$ and $\varrho_k \notin M_{\leq \eta}$, then $\varrho_k \parallel \nu_{\eta,n}$. Next, by downward induction on $\eta \in B$ we define $B_\eta = \bigcup \{B_{\nu_{\eta,n}} : n < \omega\} \cup \{\eta\}$. Lastly we define $\mathbf{y}$ by $(M_{\mathbf{y}}, <_{\mathbf{y}}) := (M_{\mathbf{x}}, <_{\mathbf{x}})$, $\mathscr{A}^{\mathbf{y}}_\eta := \mathscr{A}^{\mathbf{x}}_\eta$ if $\eta \in M_{\mathbf{x}}$ but $\eta \notin B \setminus \max(B)$, and $\mathscr{A}^{\mathbf{y}}_\eta := \mathscr{A}^{\mathbf{x}}_\eta \cup \{B_\eta\}$ if $\eta \in B \setminus \max(B)$, $\mathscr{B}_{\mathbf{y}} := \mathscr{B}_{\mathbf{x}} \cup \{B_{\mathrm{rt}_{\mathbf{x}}}\}$ and for $B', B'' \in \mathscr{B}_{\mathbf{y}}$ we let $B' \leq_{\mathbf{y}} B''$ if and only if $B' \leq_{\mathbf{x}} B''$ or $B'' = B_{\mathrm{rt}_{\mathbf{x}}}$. □

Lemma 3.5. If $\mathbf{x} \in \mathbf{K}_{\leq \aleph_0}$, $Y \in \text{alm-frt}_{\mathbf{x}}$ and $Z \subseteq Y$, then for some $\mathbf{y} \in \mathbf{K}_{\leq \aleph_0}$ we have $\mathbf{x} \leq_{\mathbf{K}} \mathbf{y}$ and either $Z \in D^{\mathbf{y}}_Y$ or $(Y \setminus Z) \in D^{\mathbf{y}}_Y$.

Moreover, if h is a function with domain Y, then above we can demand that for some $B \in \mathscr{B}_{\mathbf{y}}$, $Y \cap B$ is a front of B and for some front Y' of B which is below Y and a one-to-one function h' with domain Y' we have if $\varrho \in Y'$, $\sigma \in Y \cap B$, and $\varrho \leq_{M_{\mathbf{y}}} \sigma$, then $h(\varrho) = h'(\sigma)$. (Note that possibly $Y' = \{\mathrm{rt}_{\mathbf{y}}\}$ and then $h{\restriction}(Y \cap B)$ is constant.)

Proof. We first prove the first claim: by Lemma 3.3 1 without loss of generality there is $B \in \mathscr{B}_{\mathbf{x}}$ such that B is $\leq_{\mathbf{x}}$-maximal in $\mathscr{B}_{\mathbf{x}}$; clearly $Y \cap B$ is an almost front of B and so without loss of generality $Y \subseteq B$. We know that $B[\leq Y] := \{\varrho \in B : (\exists \nu)[\varrho \leq_{M_{\mathbf{x}}} \nu \in Y]\}$ has no ω-branch, so by $<_{M_{\mathbf{x}}}$-downward induction on $\nu \in B[\leq Y]$ we choose $(\mathbf{t}_\nu, Y_\nu)$ such that (where $M = M_{\mathbf{x}}$, of course):

(a) $\mathbf{t}_\nu \in \{0, 1\}$ and if $\mathbf{t}_\nu = 1$, then $Y_\nu \subseteq M_{\geq \nu} \cap Z$; if $\mathbf{t}_\nu = 0$, then $Y_\nu \subseteq M_{\geq \nu} \cap (Y \backslash Z)$,
(b) $Y_\nu = Z \cap B'_\nu$ for some $B'_\nu \in \mathrm{psb}_M(B_{\geq \nu})$,
(c) if $\nu \in Y$, then $Y_\nu = \{\nu\}$ and $\mathbf{t}_\nu$ is the truth value of $\nu \in Z$,
(d) if $\nu \in B[\leq Y] \backslash Y$, then for every finite set $F \subseteq M \backslash M_{\leq \nu}$ there are infinitely many $\sigma \in \mathrm{suc}_B(\nu)$ such that for all $\varrho \in F$ we have that $\varrho \parallel \sigma$) and $\mathbf{t}_\sigma = \mathbf{t}_\nu$, $Y_\nu = \bigcup \{Y_\sigma : \sigma \in \mathrm{suc}_B(\nu)$ and $\mathbf{t}_\sigma = \mathbf{t}_\nu\}$.

This is easily done and so $\mathbf{t}_{\mathrm{rt}_{\mathbf{x}}}$ is well defined. For $\nu \in B[\leq Y]$ we let $B^*_\nu = \{\rho \in B_{\geq \nu}$: By downward induction on $\sigma \in Y_\nu$, we have $\rho \in B^*_\sigma \vee \rho \leq_M \sigma\}$. Now define $\mathbf{y}$ by adding B^*_ν to $\mathscr{A}^{\mathbf{x}}_\nu$ for every $\nu \in B[\leq Y]$, and check.

For the "moreover" part, first note that by Lemmas 3.3(1) & 3.4 we may assume that there is $B \in \mathscr{B}_{\mathbf{x}}$ such that B is $\leq_{\mathbf{x}}$-maximal, the set Y is a front of B, and if $\nu \in B \backslash \max(B)$ and $\varrho \in M \backslash M_{\leq \nu}$, then for all but finitely many $\sigma \in \mathrm{suc}_B(\nu)$ we have $\varrho \parallel_M \sigma$.

Now note that if $h' : Y' \longrightarrow A$, $Y' \in \mathrm{frt}(B')$, $Z = \{\eta \in B' : \mathrm{suc}_{B'}(\eta) \subseteq Y'\}$ is a front of B' and $h'{\restriction}\mathrm{suc}_{B'}(\eta)$ is one-to-one for all $\eta \in Z$, then we can find $B'' \in \mathrm{psb}_M(B)$ such that $h'{\restriction}B'' \cap Y'$ is one-to-one. So we may continue similarly as in the first part of the proof. □

Let us recall the following definition.

Definition 3.6 (P-points and Q-points). Let D be a nonprincipal ultrafilter on a countable set $\mathrm{Dom}(D)$. We say D is a *Q-point* if whenever f is a finite-to-one function with domain $\mathrm{Dom}(D)$, then $f{\restriction}A$ is one-to-one for some $A \in D$. We say that D is a *P-point* if for each sequence $\langle A_n : n < \omega \rangle$ of sets from D there is an $A \in D$ such that $A \backslash A_n$ is finite for each $n < \omega$.

We can conclude the main result of this section.

Theorem 3.7. Assume CH. There is a $\mathbf{x} \in \mathbf{K}$ such that:

(a) (α) $\mathscr{A}^{\mathbf{x}}_\eta \neq \{\{\eta\}\}$ for $\eta \in M_{\mathbf{x}}$,
(β) $\mathscr{B}_{\mathbf{x}} = \mathscr{A}^{\mathbf{x}}_{\mathrm{rt}(\mathbf{x})} \backslash \{\{\mathrm{rt}_{\mathbf{x}}\}\}$ is $\aleph_1$-directed under $\leq_{\mathbf{x}}$,
(b) if $Y \in \mathrm{frt}^-_{\mathbf{x}}$, then
(α) $D^{\mathbf{x}}_Y$ is a non-principal ultrafilter on Y, and
(β) $D^{\mathbf{x}}_Y$ is a Q-point, cf. Definition 3.6,
(c) if $B_1 \in \mathscr{B}_{\mathbf{x}}$, then for some $B_2 \in \mathscr{B}_{\mathbf{x}}$ we have $B_1 \leq_{\mathbf{x}} B_2$ and $B_1 \cap \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}}) = \emptyset$, moreover[4]

$$(\forall \sigma \in \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}}))(\exists^\infty \varrho \in \mathrm{suc}_{B_1}(\mathrm{rt}_{\mathbf{x}}))[\sigma \leq_{M_{\mathbf{x}}} \varrho].$$

(d) $\mathbf{x}$ is fat, big, large, and full (cf. Definition 2.14).

Proof. We choose $\mathbf{x}_\alpha \in \mathbf{K}_{\leq \aleph_0}$ by induction on $\alpha < \aleph_1$ so that if $\beta < \alpha < \aleph_1$, then $\mathbf{x}_\beta \leq_{\mathbf{K}} \mathbf{x}_\alpha$, and for each successor α, there is a $\leq_{\mathbf{x}_\alpha}$-maximal element in $\mathscr{B}_{\mathbf{x}_\alpha}$. We use a bookkeeping device to ensure largeness and bigness: for $\alpha = 0$ we use Lemma 3.1; for α limit we use Definition 2.12 & Observation 2.13; if $\alpha = \beta + 1$, β is limit, then we use the first part of Lemma 3.5 (and the instructions from our bookkeeping device) to take care of the bigness; if $\alpha = \beta + 2$, β is limit, then we

[4] This is not a serious addition: as always, the number of $\sigma \in \mathrm{suc}_{B_2}(\mathrm{rt}_{\mathbf{x}})$ failing this condition is finite.

use the "moreover" part of Lemma 3.5 (and the instructions from our bookkeeping device) to take care of the largeness, if $\alpha = \beta + 3$, β is limit, then we use Lemma 3.3(3) & (4) (and the instructions from our bookkeeping device) to ensure that at the end $\mathbf{x}$ is fat and full; if $\alpha = \beta + k$, β is limit, $4 \leq k < \omega$, then we ensure clause (c). In the end we let $\mathbf{x} = \bigcup_{\alpha<\aleph_1} \mathbf{x}_\alpha$. Then $\mathbf{x}$ is fat, big, large and $\mathscr{B}_\mathbf{x}$ is $\aleph_1$-directed. Note that clause (b)(β) follows from the largeness. □

Definition 3.8. We say that $\mathbf{x} \in \mathbf{K}$ is *nice* if it satisfies conditions (a)–(d) of Theorem 3.7. The class of all nice $\mathbf{x}$ is denoted by $\mathbf{K}_\mathrm{n}$; it is called *reasonable* if it satisfies (a), (c) of Theorem 3.7. Let $\mathbf{K}_\mathrm{r}$ be the set of all $\mathbf{x} \in \mathbf{K}$ which are reasonable and let $\mathbf{K}_\mathrm{u}$ be the set of $\mathbf{x} \in \mathbf{K}_\mathrm{r}$ for which clause (b)(α) of Theorem 3.7 holds.

For $\mathbf{x} \in \mathbf{K}$ we say that $\mathscr{I} \subseteq \mathscr{A}_\mathbf{x}$ (cf. Definition 2.7(b)) is $\mathbf{x}$-*dense* if and only if for every $B_1 \in \mathscr{B}_\mathbf{x}$ there is B_2 such that $B_1 \leq_\mathbf{x} B_2 \in \mathscr{B}_\mathbf{x}$, and if $A \subseteq M_\mathbf{x} \backslash \{\mathrm{rt}_\mathbf{x}\}$ is finite, then for some ν we have $\nu \in \mathrm{suc}_{B_2}(\mathrm{rt}_\mathbf{x})$, $(B_2)_{\geq\nu} \in \mathscr{I}$, and for all $\varrho \in A$, we have $\varrho \parallel \nu$. For $\mathbf{x} \in \mathbf{K}$ we say $\mathscr{I}$ is $\mathbf{x}$-*open* if $\mathscr{I} \subseteq \mathscr{A}_\mathbf{x}$ and if $B_1 \in \mathscr{I}$, then $\mathrm{sb}(B_1) \cap \mathscr{A}_\mathbf{x} \subseteq \mathscr{I}$.

We call $\mathbf{x} \in \mathbf{K}_\mathrm{r}$ *good* if whenever $\mathscr{I}$ is $\mathbf{x}$-dense, $\mathbf{x}$-open, and $B_1 \in \mathscr{B}_\mathbf{x}$, then for some $B_2 \in \mathscr{B}_\mathbf{x}$ we have $B_1 \leq_\mathbf{x} B_2$ and $(B_2)_{\geq\eta} \in \mathscr{I}$ for all but finitely many $\eta \in \mathrm{suc}_{B_2}(\mathrm{rt}_\mathbf{x})$. Let $\mathbf{K}_\mathrm{g}$ be the class of good elements of $\mathbf{K}_\mathrm{r}$.

Finally, we call $\mathbf{x} \in \mathbf{K}$ is *ultra* if it is both nice and good, i.e., $\mathbf{K}_\mathrm{ut} := \mathbf{K}_\mathrm{g} \cap \mathbf{K}_\mathrm{n}$ is the set of elements that are ultra.

Theorem 3.9. Assume $\diamondsuit_{\aleph_1}$. Then there exists an ultra $\mathbf{x} \in \mathbf{K}$.

Proof. We repeat the proof of Theorem 3.7 but at limit stages $\delta < \aleph_1$ we use additionally $\diamondsuit_{\aleph_1}$ to take care of the additional demand $\mathbf{x} \in \mathbf{K}_\mathrm{g}$ here. We are given a limit ordinal $\delta < \aleph_1$ and a set $\mathscr{J} \subseteq \mathscr{A}_{\mathbf{x}_\delta}$ such that for some $\mathbf{y} \in \mathbf{K}$ with $\mathbf{x}_\delta \leq \mathbf{y}$ and some $\mathscr{I} \subseteq \mathscr{A}_\mathbf{y}$ we have that the set $\mathscr{I}$ is dense open in $\mathscr{A}_\mathbf{y}$, satisfies $\mathscr{J} = \mathscr{I} \cap \mathscr{A}_{\mathbf{x}_\delta}$, and there is a countable elementary submodel $N \prec \mathbf{H}_{\aleph_2}$ with $(\mathbf{y}, \mathscr{I}) \in N$ and $(\mathbf{x}_\delta, \mathscr{J}) = (\mathbf{y} \restriction N, \mathscr{I} \cap N)$, so $M_{\mathbf{x}_\delta} = M_\mathbf{y} \restriction N$, etc.

Let $\langle B^0_\ell : \ell < \omega \rangle$ be an increasing cofinal subset of $(\mathscr{B}_{\mathbf{x}_\delta}, \leq_{\mathbf{x}_\delta})$. For every ℓ there is $B^1_\ell \in \mathscr{B}_{\mathbf{x}_\delta}$ such that $B^0_\ell \leq_{\mathbf{x}_\delta} B^1_\ell$, and for every finite $A \subseteq M_{\mathbf{x}_\delta} \backslash \{\mathrm{rt}(\mathbf{x}_\delta)\}$ there is $\nu \in \mathrm{suc}_{B^1_\ell}(\mathrm{rt}(\mathbf{x}_\delta))$ such that for all $\varrho \in A$, we have $\varrho \parallel \nu$ and $(B^1_\ell)_{\geq\nu} \in \mathscr{I}$. Clearly, for every ℓ for some $k(\ell) > \ell$ we have $B^1_\ell \leq_{\mathbf{x}_\delta} B^0_{k(\ell)}$. We can choose $\langle \ell_n : n < \omega \rangle$ so that $k(\ell_n) < \ell_{n+1}$. Let $B_n = B^1_{\ell_n}$. We continue as in Lemma 3.3 1 using the $\langle B_n : n < \omega \rangle$ and, when choosing ν_n, demanding additionally that $(B_n)_{\geq\nu_n} \in \mathscr{I}$. (Note that $(B_n)_{\geq\nu_n} \in \mathscr{I}$ implies $(B^*_n)_{\geq\nu_n} \in \mathscr{I}$ for B^*_n as there.) □

Proposition 3.10. Assume $\mathbf{x} \in \mathbf{K}_\mathrm{n}$.

(i) If $B \in \mathscr{B}_\mathbf{x}$ and $Y_1, Y_2 \in \mathrm{frt}(B)$ and Y_2 is above Y_1, then $h^\mathbf{x}_{Y_2,Y_1}$ exemplifies $D^\mathbf{x}_{Y_1} \leq_{\mathrm{RK}} D^\mathbf{x}_{Y_2}$.

(ii) The family $\{D^\mathbf{x}_Y : Y \in \mathrm{frt}^-_\mathbf{x}\}$ is $\geq_{\mathrm{RK}}$-directed (even $\aleph_1$ directed).

(iii) If $Y \in \text{alm-frt}^-_\mathbf{x}$, then there is no P-point that is Rudin-Keisler reducible to $D^\mathbf{x}_Y$.

Proof. Claim (i) follows from Observation 2.10 6 and claim (ii) follows from (i) and the directedness of $\mathscr{B}_\mathbf{x}$. We shall prove (iii): Let $B_1 \in \mathscr{B}_\mathbf{x}$ be such that $B_1 \cap Y$ is an almost front of B_1. Suppose that $h : Y \longrightarrow \mathbb{N}$ is such that $h^{-1}[\{n\}] = \emptyset$ mod $D^\mathbf{x}_Y$ for every n, hence there is $A_n \in \mathscr{B}_\mathbf{x}$ which witnesses this. Assume towards contradiction that $h(D^\mathbf{x}_Y)$ is a P-point; without loss of generality h is onto $\mathbb{N}$. As $\mathscr{B}_\mathbf{x}$ is $\aleph_1$-directed we may pick $B_2 \in \mathscr{B}_\mathbf{x}$ such that $A_n \leq_\mathbf{x} B_2$ (for all $n < \omega$) and $B_1 \leq_\mathbf{x} B_2$.

As $\mathbf{x}$ is large, we may apply the Definition 2.14 of large to the pair (B_2, h') where $h'(\eta) = h(\nu)$ when $\nu \leq_{M_\mathbf{x}} \eta \in \max(B)$ and zero if there is no such ν. So there are B_3, Y_3 such that $B_2 \leq_\mathbf{x} B_3$, Y_3 is a front of B_3 below $Y \cap B_3$, and for $\eta, \nu \in Y \cap B_3$ we have that $h(\eta) = h(\nu)$ if and only if there is a $\varrho \in Y_3$ such that $\varrho \leq_{M_\mathbf{x}} \eta \wedge \varrho \leq_{M_\mathbf{x}} \nu$). Let $Z = \mathrm{suc}_{B_3}(\mathrm{rt}_\mathbf{x})$. If $Y_3 = \{\mathrm{rt}_\mathbf{x}\}$, then for some n we have $h^{-1}[\{n\}] \in D^\mathbf{x}_Y$, a contradiction. Therefore $Y_3 \neq \{\mathrm{rt}_\mathbf{x}\}$ and thus $\mathrm{rt}_\mathbf{x} \notin Y_3$, so Y_3 is above Z. Clearly, $D^\mathbf{x}_Z \leq_{\mathrm{RK}} h(D^\mathbf{x}_Y)$ and hence $D^\mathbf{x}_Z$ is a P-point.

By clauses (c) and (d) of Theorem 3.7 there is $B_4 \in \mathscr{B}_\mathbf{x}$ such that $B_3 \leq_\mathbf{x} B_4$, $B_4 \cap Z$ is a front of B_4 and

$$(\forall\sigma \in \mathrm{suc}_{B_4}(\mathrm{rt}_\mathbf{x}))(\exists^\infty\varrho \in \mathrm{suc}_{B_3}(\mathrm{rt}_\mathbf{x}))[\sigma \leq_{M_\mathbf{x}} \varrho].$$

For each $\sigma \in \mathrm{suc}_{B_4}(\mathrm{rt}_\mathbf{x})$ let $Z_\sigma = \{\varrho \in Z : \sigma \leq_{M_\mathbf{x}} \varrho\}$, so $\langle Z_\sigma : \varrho \in \mathrm{suc}_{B_4}(\mathrm{rt}_\mathbf{x})\rangle$ is a partition of Z, and $Z_\sigma = \emptyset \mod D^\mathbf{x}_Z$ for each σ. But clearly there is no $Z' \in D^\mathbf{x}_Z$ such that $Z' \cap Z_\sigma$ is finite for every $\sigma \in \mathrm{suc}_{B_4}(\mathrm{rt}_\mathbf{x})$, contradiction to "$D^\mathbf{x}_Z$ is a P-point". □

4 | BASIC CONNECTIONS TO FORCING

Definition 4.1. For a forcing notion $\mathbb{Q}$ and $p \in \mathbb{Q}$ we define $\mathfrak{D}^{\mathrm{sb}}_p = \mathfrak{D}^{\mathrm{sb}}_{\mathbb{Q},p}$, the strong bounding game between the null player NU and the bounding player BND as follows:

A play last ω moves, and in the nth move, the player NU gives a (non-empty) tree $\mathscr{T}_n$ with ω levels and no maximal node and a $\mathbb{Q}$-name $\underset{\sim}{F}_n$ of a function with domain $\mathscr{T}_n$ such that $\eta \in \mathscr{T}_n$ implies $p \Vdash_{\mathbb{Q}}$ "$\underset{\sim}{F}_n(\eta) \in \mathrm{suc}_{\mathscr{T}_n}(\eta)$". After that, player BND chooses $\eta_n \in \mathscr{T}_n$. In the end, the player BND wins the play $\langle \mathscr{T}_n, \eta_n : n < \omega\rangle$ if and only if there is $q \in \mathbb{Q}$ above p forcing that

$$(\forall n < \omega)(\exists k < \mathrm{level}(\eta_n))\big(\underset{\sim}{F}_n(\eta_n{\restriction}k) \le_{\mathscr{T}_n} \eta_n \wedge k \text{ is even}\big),$$

where $\eta_n{\restriction}k$ is the unique $\nu \le_{\mathscr{T}_n} \eta_n$ of level k.

The $\mathfrak{D}^{\mathrm{sb}} = \mathfrak{D}^{\mathrm{sb}}_{\mathbb{Q}}$ is defined similarly, but player NU can choose the condition p in their first move.

Definition 4.2. A forcing notion $\mathbb{Q}$ is *strongly bounding* if for every condition $p \in \mathbb{Q}$ player BND has a winning strategy in the game $\mathfrak{D}^{\mathrm{sb}}_{\mathbb{Q},p}$.

Definition 4.3. Let $B \in \mathrm{CWT}({}^{\omega>}\omega, \triangleleft)$. We say $\mathcal{P} \subseteq [\mathbb{N}]^{\aleph_0}$ is *big* if and only if for every $\mathbf{c} : \mathbb{N} \to \{0,1\}$ there is $A \in \mathcal{P}$ such that $\mathbf{c}{\restriction}A$ is constant; we say that a family $\mathscr{B} \subseteq \mathrm{psb}(B)$ is *big in* B if and only if for every $\mathbf{c} : \max(B) \longrightarrow \{0,1\}$ there is $B' \in \mathscr{B}$ such that $\mathbf{c}{\restriction}\max(B')$ is constant; we say that it is *large in* B if and only if for every function $\mathbf{c}$ with domain $\max(B)$ there is $B' \in \mathscr{B}$ and front Y of B' such that for every $\eta, \nu \in \max(B')$ we have $\mathbf{c}(\eta) = \mathbf{c}(\nu)$ if and only if there is a $\varrho \in Y$ such that $\varrho \le_B \nu \wedge \varrho \le_B \eta$).

Theorem 4.4. Let $M = ({}^{\omega>}\omega, \triangleleft)$, $\mathbb{Q}$ be strongly bounding, and $B \in \mathrm{CWT}(M)$. If $\mathbb{Q}$ preserves some non-principal ultrafilter on $\mathbb{N}$ and $p \Vdash$ "$\underset{\sim}{A} \subseteq \max(B)$",[5] then there are $B' \in \mathrm{psb}(B)$ and $q \in \mathbb{Q}$ such that $p \le q$ and $q \Vdash$ "$\max(B') \subseteq \underset{\sim}{\tau}$" or $q \Vdash$ "$\max(B') \subseteq \max(B) \backslash \underset{\sim}{\tau}$".

Proof. We prove this by induction on $\mathrm{Dp}(B)$ (cf. Definition 2.1), for all such Bs. Let $\eta = \mathrm{rt}(B)$.

Case 1: $\mathrm{Dp}(B) = 0$. Trivial, as then $B = \{\eta\}$, i.e., B is a singleton so $B' = B$ can serve.

Case 2: $\mathrm{Dp}_{\mathbf{x}}(B) = 1$. Then $\mathrm{Dp}(B_{\ge\nu}) = 0$ for all $\nu \in B\backslash\{\eta\}$. Now, $|B\backslash\{\eta\}| = \aleph_0$ and we just need to find $p' \in \mathbb{Q}$ above p such that $\{\nu \in B : \nu \ne \eta$ and p' forces $\nu \in \underset{\sim}{A}$ or forces $\nu \notin \underset{\sim}{A}\}$ is infinite. As $\Vdash_{\mathbb{Q}}$ "$([\mathbb{N}]^{\aleph_0})^{\mathbf{V}}$ is big in $\mathbf{V}^{\mathbb{Q}}$" (cf. footnote), this is possible.

Case 3: $\alpha = \mathrm{Dp}(B) > 1$. Let $Y = \mathrm{suc}_B(\eta)$. Then for $\nu \in Y$ we have $\mathrm{Dp}(B_{\ge\nu}) < \alpha$, hence the induction hypothesis applies to $B_{\ge\nu}$. We may assume that if ϱ is not below η, then for all but finitely many $\nu \in Y$ we have $\nu \parallel \varrho$ (cf. the proof of Lemma 3.4). Let $\langle \nu_n : \nu \in \mathbb{N}\rangle$ list Y.

We simulate a play of $\mathfrak{D}^{\mathrm{sb}}_{\mathbb{Q},p}$ in which the player BND uses a winning strategy and the player NU acts so that in the nth move, we have $\mathscr{T}_n = \{\langle B_0, \ldots, B_{k-1}\rangle : k \in \mathbb{N},\ B_\ell \in \mathrm{psb}(B_{\ge\nu_n})$ for $\ell < k$ and $B_{\ell+1} \subseteq B_\ell$ if $\ell + 1 < k\}$, the relation $<_{\mathscr{T}_n}$ is being an initial segment, and $\underset{\sim}{F}_n(\langle B_0, \ldots, B_{k-1}\rangle)$ is $\langle B_0, \ldots, B_{k-1}, B'\rangle$ for some $B' \in \mathrm{psb}(B_{k-1}) \cap \mathbf{V}$ such that either $\max(B') \subseteq \underset{\sim}{A}$ or $\max(B') \cap \underset{\sim}{A} = \emptyset$. There is such a function $\underset{\sim}{F}_n$ because of the induction hypothesis. Clearly we can do this. As the player BND has used a winning strategy, BND has won the play so there is $q \in \mathbb{Q}$ stronger than p and such that $q \Vdash$ "for every n for some even $k < \mathrm{level}_{\mathscr{T}_n}(\eta_n)$ we have $\underset{\sim}{F}_n(\eta_n{\restriction}k) \le_{\mathscr{T}_n} \eta_n$".

Hence by the choice of $(\mathscr{T}_n, \underset{\sim}{F}_n)$, letting $\eta_n = \langle B_{n,0}, \ldots, B_{n,k(n)}\rangle$ we have for some $\langle \underset{\sim}{\mathbf{t}}_n : n \in \mathbb{N}\rangle$ that $B_{n,k(n)} \in \mathrm{psb}(B_{\ge\nu_n})$, that $\underset{\sim}{\mathbf{t}}_n$ is a $\mathbb{Q}$-name of the truth value, that $q \Vdash$ "if $\underset{\sim}{\mathbf{t}}_n = 1$, then $\max(B_{n,k(n)}) \subseteq \underset{\sim}{A}$, and if $\underset{\sim}{\mathbf{t}}_n = 0$, then $\max(B_{n,k(n)}) \cap \underset{\sim}{A} = \emptyset$". Now because $\mathbb{Q}$ preserves some ultrafilter, there is an infinite $\mathscr{U} \subseteq \mathbb{N}$, a truth value $\mathbf{t}$ and a condition r such that $q \le_{\mathbb{Q}} r$ and $r \Vdash$ "$\underset{\sim}{\mathbf{t}}_n = \mathbf{t}$ for $n \in \mathscr{U}$". Lastly, let $B_* = \bigcup\{B_{n,k(n)} : n \in \mathscr{U}\} \cup \{\eta\}$ and clearly B_*, r are as required. □

Theorem 4.5. Let $B \in \mathrm{CWT}({}^{\omega>}\omega, \triangleleft)$ and $\mathbb{Q}$ be an ${}^{\omega}\omega$-bounding proper forcing notion that preserves some P-point. Then $(\mathrm{psb}(B))^{\mathbf{V}}$ is big in $\mathbf{V}^{\mathbb{Q}}$; cf. Definition 4.3.

[5] The condition of being strongly bounded can be replaced by "the player NU has no winning strategy"; the condition of preserving an ultrafilter can be replaced by "$([\mathbb{N}]^{\aleph_0})^{\mathbf{V}}$ is big in $\mathbf{V}^{\mathbb{Q}}$".

Proof. Let D be a P-point ultrafilter such that $\Vdash_{\mathbb{Q}}$ "D generates an ultrafilter" and $p \in \mathbb{Q}$. Suppose that $p \Vdash$ "$\underset{\sim}{c} : \max(\underset{\sim}{B}) \longrightarrow \{0,1\}$". Let χ be a large enough regular cardinal and $N \prec (\mathscr{H}_\chi, \in)$ be a countable model with $B, \mathbb{Q}, p, \underset{\sim}{c}, \ldots \in N$. Let $q \in \mathbb{Q}$ be such that $p \leq_{\mathbb{Q}} q$, q is $(N, \mathbb{Q})$-generic, for some $g \in ({}^\omega\omega)^{\mathbf{V}}$ we have $q \Vdash$ "if $\underset{\sim}{f} \in {}^\omega\omega \cap N$, then $\underset{\sim}{f} <_{J^{\rm bd}_\omega} g$", and for some $A \in D$ we have $q \Vdash$ "if $\underset{\sim}{B} \in D \cap N$, then $A \subseteq^* \underset{\sim}{B}$". From (g, A) we can compute $\mathbf{c}$ and $B' \in (\mathrm{psb}(B))^{\mathbf{V}}$ such that $q \Vdash$ "$\underset{\sim}{c} \restriction B'$ is constantly $\mathbf{c}$", so we are done. □

Theorem 4.6. Let $\mathbb{Q}$ be a proper forcing notion and D_* is a Ramsey ultrafilter in $\mathbf{V}$ such that

$$\Vdash_{\mathbb{Q}} \text{"fil}(D_*) \text{ is a Ramsey ultrafilter."} \qquad (\dagger)$$

Assume that $\mathbf{x} \in \mathbf{K}$ and $B \in \mathscr{B}_{\mathbf{x}}$. Then $(\mathrm{psb}(B))^{\mathbf{V}}$ is large in $\mathbf{V}^{\mathbb{Q}}$ (cf. Definition 4.3).

Proof. We prove this by induction on $\mathrm{Dp}(B)$ for $B \in \mathscr{B}_{\mathbf{x}}$. Let $\mathbf{c} : \max(B) \longrightarrow \mathbb{N}$ be from $\mathbf{V}^{\mathbb{Q}}$ and we should find (B', Y) as promised. We shall work in $\mathbf{V}^{\mathbb{Q}}$. If $\mathrm{Dp}(B) = 0$, i.e., $|B| = 1$, this is trivial. If $\mathrm{Dp}(B) = 1$ let $\langle \eta_n : \eta \in \mathbb{N}\rangle \in \mathbf{V}$ list $\mathrm{suc}_B(\mathrm{rt}_{\mathbf{x}})$: by assumption † in $\mathbf{V}^{\mathbb{Q}}$, for some $A \in \mathrm{fil}(D_*)$ the sequence $\langle \mathbf{c}(\eta_n) : n \in A\rangle$ is constant or without repetitions. Without loss of generality $A \in D_* \subseteq \mathbf{V}$ and then $\{\mathrm{rt}_{\mathbf{x}}\} \cup \{\eta_n : n \in A\}$ is as required.

So assume $\mathrm{Dp}(B) > 1$. Without loss of generality $0 \notin \mathrm{Rang}(\mathbf{c})$. For $\nu \in B \setminus \max(B)$ let $\langle \eta_{\nu,n} : n \in \mathbb{N}\rangle$ list $\mathrm{suc}_B(\nu)$ so that the function $(\nu, n) \mapsto \eta_{\nu,n}$ belongs to $\mathbf{V}$. In $\mathbf{V}^{\mathbb{Q}}$, by downward induction on $\nu \in B$, we choose $k_\nu = k(\nu)$, $A_\nu, A_{\nu,\varrho}$ and $\mathbf{t}_{\nu,\varrho}$ so that the following requirements are satisfied: (a) $k_\nu \in \mathbb{N}$, $A_\nu \in D_*$, (b) if $\nu \in \max(B)$, then $k_n = \mathbf{c}(\nu)$, so > 0, (c) if $\nu \notin \max(B)$, then either $k_\nu = 0$ and $\langle k(\eta_{\nu,n}) : n \in A_\nu\rangle$ is with no repetitions, all non-zero, or $\langle k(\eta_{\nu,n}) : n \in A_\nu\rangle$ is constantly k_ν, (d) for $\nu, \varrho \in B \setminus \max(B)$ we have $A_{\nu,\varrho} \in D_*$ and $\mathbf{t}_{\nu,\varrho} \in \{0,1\}$ and either $\mathbf{t}_{\nu,\varrho} = 1$ and $n \in A_{\nu,\varrho} \Rightarrow k(\eta_{\varrho,n}) = k(\eta_{\nu,n})$ or $\mathbf{t}_{\nu,\varrho} = 0$ and $\{k(\eta_{\varrho,n}) : n \in A_{\nu,\ell}\}$ is disjoint to $\{k(\eta_{\nu,n}) : n \in A_{\nu,\varrho}\}$. This is possible by assumption †. By the same assumption, there is $A_* \in D_*$ such that if $\nu \in B \setminus \max(B)$, then $A_* \subseteq^* A_\nu$ and if $\nu, \varrho \in B \setminus \max(B)$, then $A_* \subseteq^* A_{\nu,\varrho}$.

Let $\langle \nu_n : n \in \mathbb{N}\rangle$ list $B \setminus \max(B)$ and let f_1 be the function with domain $B \setminus \max(B)$ such that

$$f_1(\nu) = \{\eta_{\nu,n} : n \in A_* \backslash A_\nu \text{ or for some } k < \ell \text{ we have } \nu = \nu_\ell \ \wedge \ n \in A_* \backslash A_{\nu_k, \nu_\ell}\}$$

(so $f_1(\nu) \in [\mathrm{suc}_B(\nu)]^{<\aleph_0}$).

As the forcing $\mathbb{Q}$ satisfies †, it is bounding, so there is a function $f_2 \in \mathbf{V}$ with domain $B \setminus \max(B)$ such that $f_1(\nu) \subseteq f_2(\nu) \in [\mathrm{suc}_B(\nu)]^{<\aleph_0}$. Clearly, letting $B_1 := A_{B,f} := \{\nu \in B :$ if $\varrho \in B$ satisfies $\mathrm{rt}_{\mathbf{x}} \leq_B \varrho <_B \nu$ and n is such that $\eta_{\varrho,n} \leq_B \nu$, then $n \in A_*$ but $\eta_{\varrho,n} \notin f_2(\nu)\}$, we have $B_1 \in \mathrm{psb}(B)^{\mathbf{V}}$.

Define $Y := \{\nu \in B_1 :$ if $k_\nu \neq 0$ and $\varrho <_B \nu$, then $k_\varrho = 0\}$. Plainly, the set Y is a front of B_1, and if $\nu \in Y$, then $\mathbf{c} \restriction (B_1)_{\geq \nu}$ is constantly k_ν. Note that if $\nu \in B_1$ and $k_\nu = 0$, then either $k_\eta = 0$ for all $\eta \in \mathrm{suc}_{B_1}(\nu)$, or $k_\eta > 0$ for all $\eta \in \mathrm{suc}_{B_1}(\nu)$. Hence, if $\nu \in B_1 \setminus \max(B_1)$ and $\mathrm{suc}_{B_1}(\nu)$ is not disjoint to Y, then $\mathrm{suc}_{B_1}(\nu) \subseteq Y$. If $Y = \{\mathrm{rt}_{\mathbf{x}}\}$, we are done, so assume not. Let $Z = \{\eta \in B_1 : \eta \notin \max(B_1)$ and $\mathrm{suc}_{B_1}(\eta) \subseteq Y\}$. So both Z and Y are fronts of B_1, both Z and Y belong to $\mathbf{V}$, and if $\nu \in Y$, then $\langle k_\varrho : \varrho \in \max((B_1)_{\geq \nu})\rangle$ is constantly k_ν. Also if $Z = \{\mathrm{rt}_{\mathbf{x}}\}$ we are done, so assume not. Let $\langle \nu_n : n \in \mathbb{N}\rangle$ list Z. As $\mathrm{fil}(D_*)$ is a Ramsey ultrafilter we can find $\bar{n}$ such that $\bar{n} = \langle n(i) : i \in \mathbb{N}\rangle$ is an increasing enumeration of a member of D_*, hence $\bar{n} \in \mathbf{V}$, if $\ell \leq i$, then $\eta_{\nu_\ell, n(i)} \in B_1$, if $\ell < i$, $\mathbf{t}_{\nu_\ell, \nu_i} = 0$ and $\nu_\ell, \nu_i \in B_1[\leq Z]$, then $\{k(\eta_{\nu_i, n(j)}) : i \leq j\}$ is disjoint from $\{k(\eta_{\nu_\ell, n(j)}) : i \leq j\}$, moreover it is disjoint from $\{k(\eta_{\nu_\ell, n(j)} : j \in \mathbb{N}\}$. Lastly, as $\bar{n} \in \mathbf{V}$ we can find in $\mathbf{V}$ a partition $\langle C_\ell : \ell \in \mathbb{N}\rangle$ of $\mathbb{N}$ to (pairwise disjoint) infinite sets and let $B_2 := \{\sigma \in B_1 :$ if $\nu_\ell <_{B_1} \sigma$ and $\nu_\ell \in B_1[\leq Z]$, then for some $i \in C_\ell$ we have $i > \ell$ and $\eta_{\nu_\ell, n(i)} \leq_{B_2} \sigma\}$. Easily $B_2 \in \mathbf{V}$, $B_2 \in \mathrm{psb}(B_1)$ and it is as required. □

Motivated by Definition 4.1 we introduce the following bounding games for a forcing notion $\mathbb{Q}$.

Definition 4.7. Let $\mathbb{Q}$ be a forcing notion and $p \in \mathbb{Q}$. We shall define three games: $\eth^{\rm bd}_p = \eth^{\rm bd}_{\mathbb{Q},p}$, $\eth^{\rm ufbd}_p = \eth^{\rm ufbd}_{\mathbb{Q},p}$, and $\eth^{\rm vfbd}_p = \eth^{\rm vfbd}_{\mathbb{Q},p}$. Each of the games lasts ω rounds, and in each round player NU moves first, and player BND second. The games $\eth^{\rm bd}, \eth^{\rm ufbd}, \eth^{\rm vfbd}$ are defined analogously, but here the condition p will be chosen by player NU in his first move.

In the nth round of the game $\eth^{\rm bd}_p$, first the player NU gives a $\mathbb{Q}$-name $\underset{\sim}{\tau}_n$ of a member of $\mathbf{V}$ and then the player BND gives a finite set $w_n \subseteq \mathbf{V}$. After ω rounds, the player BND wins the play if and only if there is $q \in \mathbb{Q}$ above p forcing "$\underset{\sim}{\tau}_n \in w_n$" for every n.

In the nth round of the game $\eth^{\rm ufbd}_p$, first the player NU chooses an ultrafilter E_n on some set I_n from $\mathbf{V}$ and a $\mathbb{Q}$-name $\underset{\sim}{E}^+_n$ of an ultrafilter on I_n extending E_n and a $\mathbb{Q}$-name $\underset{\sim}{X}_n$ of a member of $\underset{\sim}{E}^+_n$; then the player BND chooses $t_n \in I_n$. In the end of the play the player BND wins the play if and only if there is $q \in \mathbb{Q}$ above p forcing "$t_n \in \underset{\sim}{X}_n$" for every n.

The game $\Game_p^{\rm vfbd}$ is similar to $\Game_p^{\rm ufbd}$, but now we demand $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{X}_n \in E_n$ or just includes a member of E_n", so $\underset{\sim}{E}_n^+$ is redundant.

Basic relations between the games introduced above are given by the following result.

Proposition 4.8. Let $\mathbb{Q}$ be a forcing notion.

1. If BND wins in $\Game^{\rm sb}_{\mathbb{Q},p}$, then BND wins in $\Game^{\rm bd}_{\mathbb{Q},p}$ which implies that $\mathbb{Q}$ is a bounding forcing.
2. The player BND wins in $\Game^{\rm bd}_{\mathbb{Q},p}$ iff BND wins in $\Game^{\rm vfbd}_{\mathbb{Q},p}$.
3. If the player BND wins in $\Game^{\rm vfbd}_{\mathbb{Q},p}$, then BND wins in $\Game^{\rm ufbd}_{\mathbb{Q},p}$.
4. We can replace in 1–3 above "wins" by "does not lose".

Proof. We start by observing that (3) is obvious and that our proofs all work with both "wins" and "does not lose" (which shows (4)). Let us therefore start by showing (1). The second implication is obvious, so we concentrate on the first. For every $\underset{\sim}{\tau}$, a $\mathbb{Q}$-name of an ordinal we define a pair $(T_{\underset{\sim}{\tau}}, \underset{\sim}{F}_{\underset{\sim}{\tau}})$ as follows: let $u = \{\alpha : \nVdash_{\mathbb{Q}} \text{“}\underset{\sim}{\tau} \neq \alpha\text{”}\}$, it is a non-empty set of $\leq |\mathbb{Q}|$ ordinals; let $T_{\underset{\sim}{\tau}}$ be the tree $\{\eta : \eta \in {}^{\omega>}u\}$, i.e., ordered by $\lhd$ (being an initial segment), and let $\underset{\sim}{F}_{\underset{\sim}{\tau}}(\eta) = \eta^\frown\langle\underset{\sim}{\tau}\rangle$ for $\eta \in T_{\underset{\sim}{\tau}}$. Clearly, $T_{\underset{\sim}{\tau}}$ is a tree with ω levels in $\mathbf{V}$, $\underset{\sim}{F}_{\underset{\sim}{\tau}}$ is a $\mathbb{Q}$-name of a function with domain $T_{\underset{\sim}{\tau}}$ such that $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{F}_{\underset{\sim}{\tau}}(\eta) \in \mathrm{suc}_{T_{\underset{\sim}{\tau}}}(\eta)$". Furthermore, if $q \in \mathbb{Q}$ and $\eta \in T_{\underset{\sim}{\tau}}$ (so $\mathrm{Rang}(\eta)$ is a finite subset of u), then we have that $q \Vdash$ "$\underset{\sim}{\tau} \in \mathrm{Rang}(\eta)$" if and only if $q \Vdash$ "for some $\nu \lhd \eta$ we have $\nu^\frown\langle\underset{\sim}{F}_{\underset{\sim}{\tau}}(\nu)\rangle \unlhd \eta$".

So playing the game $\Game^{\rm bd}_{\mathbb{Q},p}$ we can translate it to a play of $\Game^{\rm sb}_{\mathbb{Q},p}$ replacing the NU choice of $\underset{\sim}{\tau}_n$ by the choice of $(T_{\underset{\sim}{\tau}}, \underset{\sim}{F}_{\underset{\sim}{\tau}})$. Thus every strategy $\mathbf{st}_1$ of BND in $\Game^{\rm sb}_{\mathbb{Q},p}$ translates it to a strategy $\mathbf{st}_2$ of the player BND in $\Game^{\rm bd}_{\mathbb{Q},p}$.

For (2), we need two translations as follows.

Translating $\Game^{\rm vfbd}_{\mathbb{Q},p}$ to $\Game^{\rm bd}_{\mathbb{Q},p}$:
We are given a move $y = (I, E, \underset{\sim}{X})$ of NU in a play of $\Game^{\rm vfbd}_{\mathbb{Q},p}$ as in Definition 4.7, i.e., $I \in \mathbf{V}$, E is an ultrafilter on I, in $\mathbf{V}$, and $\Vdash_{\mathbb{Q}}$ "$\underset{\sim}{X} \in E$ or just includes a member $\underset{\sim}{X}'$ of E". Now we have that if $q \Vdash$ "$\underset{\sim}{X}' \in \mathscr{W}$" where $\mathscr{W} \subseteq E$ is finite ($\mathscr{W}$ an object in $\mathbf{V}$ not a name), then $\bigcap\{A : A \in \mathscr{W}\}$ is non-empty and $t \in \bigcap\{A : A \in \mathscr{W}\}$ implies $q \Vdash$ "$t \in \underset{\sim}{X}' \subseteq \underset{\sim}{X}$".

Translating $\Game^{\rm bd}_{\mathbb{Q},p}$ to $\Game^{\rm vfbd}_{\mathbb{Q},p}$:
Given $y = (I, \underset{\sim}{\tau})$, $\underset{\sim}{\tau}$ a $\mathbb{Q}$-name of a member I of $\mathbf{V}$ we define $I_y = [I]^{<\aleph_0} \in \mathbf{V}$ and choose $E_y \in \mathbf{V}$ an ultrafilter on I_y such that $u_* \in [I]^{<\aleph_0}$ implies $\{u \in [I]^{<\aleph_0} : u_* \subseteq u\} \in E$; lastly we choose $\underset{\sim}{X}_y = \{u \in [I]^{<\aleph_0} : \underset{\sim}{\tau} \in u\}$. So, $(I_y, E_y, \underset{\sim}{X}_y)$ is a legal move in $\Game^{\rm vfbd}_{\mathbb{Q},p}$ and for a finite subset t of I, we have that if $q \Vdash$ "$t \in \underset{\sim}{X}_y$", then $q \Vdash$ "$\underset{\sim}{\tau} \in t$". □

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CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

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