

ATOMIC SATURATION OF REDUCED POWERS SH1064

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ABSTRACT. Our aim was to try to generalize some theorems about the saturation of ultra-powers to reduced powers. Naturally, we deal with saturation for types consisting of atomic formulas. We succeed to generalize “the theory of dense linear order (or T with the strict order property) is maximal and so is any pair (T, Δ) which is SOP_3 ”, (where Δ consists of atomic or conjunction of atomic formulas). However, the theorem on “it is enough to deal with symmetric pre-cuts” (so the $p = t$ theorem) cannot be generalized in this case. Similarly the uniqueness of the dual cofinality fails in this context.

After publication, a full proof of 2.10m3as added (and minor proofs done).

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References like [Shea, Th0.2=L_y5] means the label of Theorem 0.2 is y_5 . The reader should note that the version in my website is usually more updated than the one in the mathematical archive. Was proof read with proofs for the Journal.

Annotated Content

§0 Introduction, pg.3

§1 Axiomatizing [She90, Ch.VI,2.6], pg. 8

[We phrase and prove a theorem which axiomatizes [She90, Ch.VI,2.6]. The theorem there says that if D is a regular ultra-filter on I and for every model M of the theory of dense linear orders (or T with the strict order property), the model M^I/D is λ^+ -saturated, then D is λ^+ -good and λ -regular.]

§2 Applying the axiomatized frame, pg.12

[The axiomatization in §1 can be phrased as a set of sentences, surprisingly moreover Horn ones (first order if $\theta_{\mathfrak{r}} = \aleph_0$). Now in this case we can straightforwardly derive [She78, Ch.VI,2.6]. However we can get more, because the axiomatization being Horn, we can now deal also with the $(\lambda^+, \text{atomic})$ -saturation of reduced powers. We then deal with infinitary logics and comment on models of Bounded Peano Arithmetic. After an inquiring of G. Cherlin the proof of 2.10 was given.]

§3 Criterion for atomic saturation of reduced powers, pg.19

[For a complete first order T we characterize when a filter D on I is such that M^I/D is (λ, atomic) -saturated for every model M of T .]

§4 The counterexample, pg.25

[We prove that for reduced powers, the parallel of $\mathfrak{t} \leq \mathfrak{p}$ in general fails. Also, similarly the uniqueness of the dual cofinality. More specifically, for $\lambda \geq \aleph_1$, for some regular filter D on λ , the partial order $(\mathbb{Q}, <)^\lambda/D$ has no symmetric pre-cut of cofinality $\geq \lambda$ but has such an asymmetric pre-cut.]

§ 0. INTRODUCTION

§ 0(A). **Background, Questions and Answers.**

We know much on saturation of ultrapowers, see Keisler [Kei67], [She90, Ch.IV] and later mainly works of Malliaris and the author, e.g. ([MS16b], [MS16c]). But we know considerably less on reduced powers. For transparency, let T denote a first-order complete countable theory with elimination of quantifiers and M will denote a model of T . For D a regular filter on $\lambda > \aleph_0$ we may ask: when is M^λ/D λ^+ -saturated? For D an ultra-filter, Keisler [Kei64] proves that this holds for every T iff D is λ^+ -good iff this holds for $T =$ theory of Boolean algebras, such T is called \leq_λ -maximal.

By [She78, Ch.VI,2.6] the maximality holds for $T =$ theory of dense linear orders or just any T with the strict order property and by [She96], any T with the 3-strong order property, SOP_3 is \leq_λ -maximal.

What about reduced powers for λ -regular filter D on λ ? By [She72a], M^λ/D is λ^+ -saturated for every T (of cardinality $\leq \lambda$) iff D is λ^+ -good and $\mathcal{P}(\lambda)/D$ is a λ^+ -saturated Boolean algebra. Parallel results hold when we replace λ^+ -saturated by $(\lambda^+, \Sigma_{1+n}(\mathbb{L}_r(T)))$ -saturated. We shall concentrate on $(\lambda^+, \text{atomic})$ -saturated and introduce the related partial order \leq_λ^{rp} , see definitions below.

Concerning ultrapowers, lately, Malliaris-Shelah [MS16b] proved that a regular ultra-filter D on a cardinal λ is λ^+ -good iff for any linear order M we have M^λ/D has no symmetric pre-cut with cofinality $\leq \lambda$. This was proved together with the theorem $\mathfrak{p} = \mathfrak{t}$ and “for a f.o. complete countable T , being SOP_2 suffices for \triangleleft_λ -maximality”. In a later work [MS17a], it is proved that at least for a relative \triangleleft_λ^* (see [She96]) this is “iff” assuming a case of G.C.H., relying also on works with Dzamonja [DS04], and with Usvyastov [SU08]. Part of the proof is axiomatized by Malliaris-Shelah [MS16a].

Note also that [Sheb] deals with saturation but only for ultrapowers by θ -complete ultra-filters for θ a compact cardinal; and also with ω -ultra-limits.

Now what do we accomplish here?

First, in §1 we axiomatize the proof of [She78, Ch.VI,2.6], i.e. we define when $\mathbf{r} = (M, \Delta)$ is a so-called RSP and for it to prove that the relevant model $N_{\mathbf{r}}$ is $(\min\{\mathfrak{p}_{\mathbf{r}}, \mathfrak{t}_{\mathbf{r}}\}, \Delta)$ -saturated. Second, in §2 we prove, of course, that [She78, Ch.VI,2.6] follows, but also we show that the axiomatization of RSP is by Horn sentences. Hence we can apply it to reduced powers. So T is \leq_λ^{rp} -maximal if $T = \text{Th}(\mathbb{Q}, <)$ and moreover for every T having the SOP_3 ; lastly, we comment on models of Peano Arithmetic.

In §3 we try to sort out when for models of T we get the relevant atomic saturation.

Can we generalize also results [MS16b] to reduced powers?

The main result of §4 says that no. We also sort out the parallel of goodness, excellence, and morality for filters and atomic saturation for reduced powers. In a hopeful continuation [S⁺], we shall try to sort out the order \leq_λ^{rp} , and in particular consider non-maximality and parallel statements for infinitary logics (see [Sheb]).

The reader can ignore Boolean ultra-powers (that is 0.12, 0.13, 0.14 for sections 1,2 and can in first reading deal only with first-order logic (so $\theta = \aleph_0$, and the

assumptions concerning the completeness of filter disappear. We thank the referee for many helpful comments.

Note that by 2.10

Conclusion 0.1. *If (T, Δ) has the SOP_3 , then it is $\leq_{\lambda}^{\text{rp}}$ -maximal.*

Question 0.2. Do we have: if D is (λ_2, T) -good and regular then D is (λ_1, T) -good when $\lambda_1 < \lambda_2$ (or more)?

§ 0(B). **Further Questions.**

Convention 0.3. 1) Let T be a theory with elimination of quantifiers if not said otherwise. Let Mod_T be the class of models of T .

2) The main case is for T a countable complete first-order theory with elimination of quantifiers, moreover, with every formula equivalent to an atomic one.

So it is natural to ask

Conjecture 0.4. The pair (T, Δ) is \leq_{rp} -maximal iff (T, Δ) has the SOP_3 .

So which T (with elimination of quantifiers) are maximal under $\leq_{\lambda}^{\text{rp}}$? That is, when for every regular filter D on λ , M^{λ}/D is $(\lambda^+, \text{atomic})$ -saturated iff D is λ^+ -good? Is T_{feq} maximal? (see [She93], it is a proto-typical non-simple T , but see more in [She23]) As we have not proved this even for ultra-filters, the reasonable hope is that it will be easier to show non-maximality for $\leq_{\lambda}^{\text{rp}}$. Also in light of [MS16c] for simple theories we like to prove non-maximality with no large cardinals. We may hope to use just $NSOP_2$, but still it would not settle the problem of characterizing the maximal ones as e.g. $SOP_2 \equiv SOP_3$ is open for such T ; for pairs $(T, \varphi(\bar{x}, \bar{y}))$ they are different.

Note that for first order T , it makes sense to use μ^+ -saturated models and D is μ^+ -complete.

Also, the “ T stable” case should be resolved.

Conjecture 0.5. M^{λ}/D is $(\aleph_0^{\lambda}/D, \text{atomic})$ -saturated when :

- (a) T a theory as in 0.3(2),
- (b) T is stable without the fcp,
- (c) D is a regular filter on λ .

Remark 0.6. Maybe given a $1 - \varphi$ -type $p \subseteq \{\varphi(x, \bar{a}) : \bar{a} \in {}^m(M^I/D)\}$ of cardinality $\leq \lambda$ in M^I/D , we try just to find a dense set of $A \in D^+$ such that in $M^I/(D + A)$ the $1 - \varphi$ -type is realized. Then continue; opaque.

§ 0(C). **Preliminaries.**

Notation 0.7. 1) T denote a f.o. theory, usually complete.

2) Let τ denote a vocabulary, $\tau_T = \tau(T)$ denote the vocabulary of the theory T

3) We use M, N to denote models, $\tau_M = \tau(M)$ is the vocabulary of M and P^M, F^M denote the interpretation of P, F respectively.

4) let $\mathbb{L}(\tau)$ denote the f.o. language for the vocabulary τ .

5) We allow function symbol $F \in \tau$ to be interpreted in a τ -model M as a partial function, but with domain P_F^M , with $P_F \in \tau$ a predicate with the same arity.

Notation 0.8. 1) Let \mathfrak{B} denote a Boolean algebra, $\text{comp}(\mathfrak{B})$ its completion, $\mathfrak{B}^+ = \mathfrak{B} \setminus \{0_{\mathfrak{B}}\}$, $\text{uf}(\mathfrak{B})$ the set of ultra-filters on \mathfrak{B} , $\text{fil}(\mathfrak{B})$ the set of filters on \mathfrak{B} . For $\mathbf{a} \in \mathfrak{B}$ let $\mathbf{a}^{\text{if(true)}} = \mathbf{a}^{\text{if(1)}}$ be \mathbf{a} and let $\mathbf{a}^{\text{if(false)}} = \mathbf{a}^{\text{if(0)}}$ be $1_{\mathfrak{B}} - \mathbf{a}$.

1A) Let $\mathfrak{B}_1 \triangleleft \mathfrak{B}_2$ mean that \mathfrak{B}_1 is a sub-algebra of \mathfrak{B}_2 , and moreover, a complete one, which means that every maximal anti-chain of \mathfrak{B}_1 is a maximal anti-chain of \mathfrak{B}_2 .

2) For a model M let $\tau_M = \tau(M)$ be its vocabulary.

3) For a filter D on a set I let $D^+ = \{B \subseteq I : I \setminus B \notin D\}$.

Now about cuts (they are closed to but different than gaps, see [MS17b]).

Definition 0.9. 1) For a partial order $\mathcal{T} = (\mathcal{T}, \leq_{\mathcal{T}})$, we say (C_1, C_2) is pre-cut when (but we may in this paper omit the “pre”):

- (a) $C_1 \cup C_2$ is a subset of \mathcal{T} linearly ordered by $\leq_{\mathcal{T}}$,
- (b) if $a_1 \in C_1, a_2 \in C_2$ then $a_1 \leq_{\mathcal{T}} a_2$,
- (c) for no $c \in \mathcal{T}$ do we have $a_1 \in C_1 \Rightarrow a_1 \leq_{\mathcal{T}} c$ and $a_2 \in C_2 \Rightarrow c \leq_{\mathcal{T}} a_2$.

2) Above we say (C_1, C_2) is a (κ_1, κ_2) -pre-cut when in addition:

- (d) C_1 has cofinality κ_1 ,
- (e) C_2^* , the inverse of C_2 , has cofinality κ_2 ,
- (f) so κ_1, κ_2 are regular cardinals, (here we ignore the case 0,1 if not said otherwise).

2A) Above we call κ_1, κ_2 the cofinalities of the pre-cut (C_1, C_2) . We say that the pre-cut is symmetric when $\kappa_1 = \kappa_2$ and then we may say κ_1 is its cofinality.

3) We may replace C_ℓ by a sequence \bar{a}_ℓ , if not said otherwise such that \bar{a}_1 is $\leq_{\mathcal{T}}$ -increasing and \bar{a}_2 is $\leq_{\mathcal{T}}$ -decreasing.

4) We say (C_1, C_2) is a (κ_1, κ_2) -linear-cut of \mathcal{T} when it is a (κ_1, κ_2) -pre-cut and $C_1 \cup C_2$ is downward closed, so natural for \mathcal{T} a tree.

5) We say (C_1, C_2) is a weak pre-cut when (b),(c) of part (1) holds.

Remark 0.10. 1) If \mathcal{T} is a (model-theoretic) tree, $\kappa_2 > 0$ and (C_1, C_2) is a (κ_1, κ_2) -pre-cut then it induces one and only one (κ_1, κ_2) -linear-cut (C'_1, C'_2) , i.e. one satisfying $C_1 \subseteq C'_1, C_2 \subseteq C'_2$ such that $C_1 \cup C_2$ is cofinal in $C'_1 \cup C'_2$.

2) In 0.11 below, if $L = \mathbb{L}(\tau)$ then $\theta = \aleph_0, \sigma = 1$ suffice, but not so in more general cases.

Definition 0.11. 1) We say M is fully $(\lambda, \theta, \sigma, L)$ -saturated (may omit the fully); where $L \subseteq \mathcal{L}(\tau_M)$ and \mathcal{L} is a logic; we may write \mathcal{L} if $L = \mathcal{L}(\tau_M)$, when :

- if Γ is a set of $< \lambda$ formulas from L with parameters from M with $< 1 + \sigma$ free variables, and Γ is $(< \theta)$ -satisfiable in M , then Γ is realized in M .

2) We say “locally” when using one $\varphi = \varphi(\bar{x}, \bar{y}) \in \mathcal{L}$ with $\ell g(\bar{x}) < 1 + \sigma$, i.e. all members of Γ have the form¹ $\varphi(\bar{x}, \bar{b})$.

¹In [Sheb] we use a $L \subseteq \mathbb{L}_{\theta, \theta}$, θ a compact cardinal and if $\sigma > \theta$ we use a slightly different version of the definition of local and of the default value of σ was θ .

- 3) Saying “locally/fully (λ, \mathcal{L}) -saturated” the default values (i.e. we may omit) of σ is $\sigma = \theta$, of (σ, θ) is $\theta = \aleph_0 \wedge \sigma = \aleph_0$ and of \mathcal{L} is \mathbb{L} (first order logic) and of L is \mathcal{L} . Omitting λ means $\lambda = \|M\|$.
- 4) If $\varphi(\bar{x}, \bar{y}) \in \mathcal{L}(\tau_M)$ and $\bar{a} \in {}^{\ell g(\bar{y})}M$ then $\varphi(M, \bar{a}) := \{\bar{b} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{b}, \bar{a}]\}$.
- 5) Let $\bar{x}_{[u]} = \langle x_s : s \in u \rangle$.

In 0.12, 0.13, 0.14 we shall deal with complete Boolean algebras and ultrapowers, and then we define an order between theories.

Definition 0.12. Assume we are given a Boolean algebra \mathfrak{B} usually complete and a model or a set M and D a filter on $\text{comp}(\mathfrak{B})$, the completion of \mathfrak{B} .

1) Let $M^{\mathfrak{B}}$ be the set of partial functions f from \mathfrak{B}^+ into M such that for some maximal anti-chain $\langle a_i : i < i(*) \rangle$ of \mathfrak{B} , $\text{dom}(f)$ includes $\{a_i : i < i(*)\}$ and is included in ${}^2 \{a \in \mathfrak{B}^+ : (\exists i)(a \leq a_i)\}$ and f is a function into M and $f \upharpoonright \{a \in \text{dom}(f) : a \leq a_i\}$ is constant for each i .

1A) Naturally for $f_1, f_2 \in M^{\mathfrak{B}}$ we say f_1, f_2 are D -equivalent, or $f_1 = f_2 \pmod D$ when for some $b \in D$ we have $a_1 \in \text{dom}(f_1) \wedge a_2 \in \text{dom}(f_2) \wedge a_1 \cap a_2 \cap b > 0_{\mathfrak{B}} \Rightarrow f_1(a_1) = f_2(a_2)$.

2) We define $M^{\mathfrak{B}}/D$ naturally, as well as $\text{TV}_M(\varphi(f_0, \dots, f_{n-1})) \in \text{comp}(\mathfrak{B})$ when $\varphi(x_0, \dots, x_{n-1}) \in \mathbb{L}(\tau_M)$ and $f_0, \dots, f_{n-1} \in M^{\mathfrak{B}}$ where:

- (a) TV stands for truth value,
- (b) $\text{TV}_M(\varphi(f_0, \dots, f_{n-1}))$ is equal to

$$\sup\{a \in \mathfrak{B}^+ : a \cap \bigcap_{\ell < n} \text{Dom}(f_\ell) : M \models (\varphi(f_0(a), \dots, f_{n-1}(a)))\},$$

- (c) M is defined by letting, for φ an atomic formulas

$$M^{\mathfrak{B}}/D \models \varphi[f_0/D, \dots, f_{n-1}/D] \text{ iff } \text{TV}_M(\varphi(f_0, \dots, f_{n-1})) \in D.$$

2A) Abusing notation, not only $M^{\mathfrak{B}_1} \subseteq M^{\mathfrak{B}_2}$ but $M^{\mathfrak{B}_1}/D_1 \subseteq M^{\mathfrak{B}_2}/D_2$ when $\mathfrak{B}_1 < \mathfrak{B}_2$, $D_\ell \in \text{fil}(\mathfrak{B}_\ell)$ for $\ell = 1, 2$ and $D_1 = \mathfrak{B}_1 \cap D_2$. Also $[f_1, f_2 \in M^{\mathfrak{B}_1} \Rightarrow f_1 = f_2 \pmod{D_1} \Leftrightarrow f_1 = f_2 \pmod{D_2}]$. So for $f \in M^{\mathfrak{B}_1}$ we identify f/D_1 and f/D_2 .

3) For complete \mathfrak{B} , we say $\langle a_n : n < \omega \rangle$ represents $f \in \mathbb{N}^{\mathfrak{B}}$ when $\langle a_n : n < \omega \rangle$ is a maximal anti-chain of \mathfrak{B} (so $a_n = 0_{\mathfrak{B}}$ is allowed) and for some $f' \in \mathbb{N}^{\mathfrak{B}}$ which is D -equivalent to f (see 0.12(1A)) we have $f'(a_n) = n$.

4) We say $\langle (a_n, k_n) : n < \omega \rangle$ represents $f \in \mathbb{N}^{\mathfrak{B}}$ when :

- (a) the k_n are natural numbers with no repetition,
- (b) $\langle a_n : n < \omega \rangle$ is a maximal anti-chain,
- (c) $f(a_n) = k_n$.

5) If \mathcal{I} is a maximal anti-chain of \mathfrak{B} and $\bar{M} = \langle M_a : a \in \mathcal{I} \rangle$ is a sequence of τ -models, then we define $\bar{M}^{\mathfrak{B}}$ as the set of partial functions f from \mathfrak{B}^+ to $\cup\{M_a : a \in \mathcal{I}\}$ such that for some maximal anti-chain $\langle a_i : i < i(*) \rangle$ of \mathfrak{B} refining \mathcal{I} (i.e. $(\forall i < i(*))(\exists b \in \mathcal{I})(a_i \leq_{\mathfrak{B}} b)$) we have:

- (a) $\{a_i : i < i(*)\} \subseteq \text{dom}(f) \subseteq \{b \in \mathfrak{B}^+ : b \leq_{\mathfrak{B}} a_i \text{ for some } i < i(*)\}$

²for the $D_\ell \in \text{uf}(\mathfrak{B}_\ell)$ ultra-product, without loss of generality \mathfrak{B} is complete, then without loss of generality $f \upharpoonright \{a_i : i < i(*)\}$ is one to one. But in general we allow $a_i = 0_{\mathfrak{B}}$, those are redundant but natural in 0.12(3).

- (b) if $a \in \text{dom}(f)$ and $a \leq a_i$ then $f(a) = f(a_i)$
- (c) if $a_i \leq_{\mathfrak{B}} b, b \in \mathcal{I}$ then $f(a_i) \in M_b$.

6) For $\bar{M}, \mathfrak{B}, \mathcal{I}$ as above and a filter D on \mathfrak{B} we define $\bar{M}^{\mathfrak{B}}/D$ as in part (2) replacing $M^{\mathfrak{B}}$ there by $\bar{M}^{\mathfrak{B}}$ here, see part (7).

7) For $\bar{M}, \mathfrak{B}, \mathcal{I}$ as above, $\varphi = \varphi(\bar{x}) = \varphi(x_0, \dots, x_{n-1}) \in \mathbb{L}(\tau_M)$ and $\bar{f} = \langle f_\ell : \ell < n \rangle$ where $f_0, \dots, f_{n-1} \in \bar{M}^{\mathfrak{B}}$, let $\text{TV}(\varphi[\bar{f}]) = \text{TV}(\varphi[\bar{f}], \bar{M}^{\mathfrak{B}})$ be $\sup\{a \in \mathfrak{B}^+ : \text{if } \ell < n \text{ then } a \in \text{dom}(f_\ell) \text{ and } a \leq b \in \mathcal{I} \text{ then } M_b \models \varphi[f_0(b), \dots, f_{n-1}(b)]\}$.

8) We say \mathfrak{B} is $(< \sigma)$ -distributive when it is θ -distributive for every $\theta < \sigma$, where

8A) \mathfrak{B} is θ -distributive when: if for $\alpha < \theta$, \mathcal{I}_α is a maximal antichain of \mathfrak{B} then there is a maximal antichain of \mathfrak{B} refining every \mathcal{I}_α ($\alpha < \theta$); this holds, e.g. when $\mathfrak{B} = \mathcal{P}(\lambda)$ or just there is a dense $Y \subseteq \mathfrak{B}^+$ closed under intersection of θ .

Definition 0.13. 1) Let \mathfrak{B} be a complete Boolean algebra and D a filter on \mathfrak{B} . We say that D is (μ, θ) -regular when for some (\bar{c}, \bar{u}) we have:

- (a) $\bar{c} = \langle c_\alpha : \alpha < \alpha_* \rangle$ is a maximal anti-chain of \mathfrak{B} ,
- (b) $\bar{u} = \langle u_\alpha : \alpha < \alpha_* \rangle$ with $u_\alpha \in [\mu]^{< \theta}$,
- (c) if $i < \mu$ then $\sup\{c_\alpha : \alpha \text{ satisfies } i \in u_\alpha\} \in D$.

2) A filter D is called λ -regular when it is (λ, \aleph_0) -regular; the filter D on a set I (that is the Boolean algebra $\mathcal{P}(I)$) is called regular when it is a filter on a set I and it is $|I|$ -regular.

Claim 0.14. Assume \mathfrak{B} is a complete Boolean algebra which is $(< \lambda)$ -distributive and D a filter on \mathfrak{B} and $\theta = \text{cf}(\theta) \leq \lambda$.

1) Assume D is a θ -complete ultra-filter. The parallel of Loś theorem holds for $\mathbb{L}_{\theta, \theta}$ and if D is λ -complete even for $\mathbb{L}_{\lambda, \theta}$ which means: if $\bar{M} = \langle M_b : b \in \mathcal{I} \rangle$ is a sequence of τ -models, \mathcal{I} is a maximal antichain of the complete Boolean algebra \mathfrak{B} and $\varepsilon < \theta$, $\varphi = \varphi(\bar{x}_{[\varepsilon]}) \in \mathbb{L}_{\lambda, \theta}(\tau)$ and $f_\zeta \in \bar{M}^{\mathfrak{B}}$ for $\zeta < \varepsilon$ then $M^{\mathfrak{B}}/D \models \varphi[\langle f_\zeta/D : \zeta < \varepsilon \rangle]$ iff $\text{TV}_M(\varphi[\langle f_\zeta/D : \zeta < \varepsilon \rangle])$ belongs to D .

2) If in addition D is (λ, θ) -regular and M, N are $\mathbb{L}_{\theta, \theta}$ -equivalent then $M^{\mathfrak{B}}/D, N^{\mathfrak{B}}/D$ are $\mathbb{L}_{\lambda^+, \theta}$ -equivalent.

Definition 0.15. 1) Assume Δ_ℓ is a set of atomic formulas in $\mathbb{L}(\tau(T_\ell))$. Then we say $(T_1, \Delta_1) \leq_{\lambda, \theta}^{\text{rp}} (T_2, \Delta_2)$ when: if D is a (λ, θ) -regular filter on λ and M_ℓ is a λ^+ -saturated model of T_ℓ for $\ell = 1, 2$ and M_2^λ/D is $(\lambda^+, \theta, \Delta_2)$ -saturated then M_1^λ/D is $(\lambda^+, \theta, \Delta_1)$ -saturated.

2) For general Δ_1, Δ_2 we define $(T_1, \Delta_1) \leq_{\lambda, \theta}^{\text{rp}} (T_2, \Delta_2)$ as meaning $(T_1^+, \Delta_1^+) \leq_{\lambda, \theta}^{\text{rp}} (T_2^+, \Delta_2^+)$ where (as Morley [Mor65] does):

- $T_\ell^+ = T_\ell \cup \{(\forall \bar{x})(\varphi(\bar{x}) \equiv P_\varphi(\bar{x})) : \varphi(\bar{x}) \in \Delta_\ell\}$ with $\langle P_\varphi^\ell : \varphi \in \Delta_\ell \rangle$ new pairwise distinct predicates with suitable number of places
- $\Delta_\ell^+ = \{P_\varphi^\ell(\bar{x}_\varphi) : \varphi \in \Delta_\ell\}$.

3) In (2), $T_1 \leq_{\lambda, \theta}^{\text{rp}} T_2$ means $\Delta_\ell =$ the set of atomic $\mathbb{L}_{\theta, \theta}(\tau_{T_\ell})$ -formulas.

Observation 0.16. Assume $\Delta \subseteq \mathbb{L}(\tau_T)$ is closed under \exists and \wedge . A model M of T is (μ^+, μ^+, Δ) -saturated iff it is $(\mu^+, 1, \Delta)$ -saturated.

Question 0.17. 1) Under \leq_{rp} characterize the minimal/maximal pairs (T, Δ) .

2) What about the parallel of \leq^{**} (see [She93], [MS17a])?

§ 1. AXIOMATIZING [She90, Ch.VI,2.6]

Note that while the notation $\mathfrak{t}(\mathcal{T})$ is obviously natural the notation $\mathfrak{p}(\mathcal{T})$ is really justified just by the results here.

Definition 1.1. 1) For a partial order $\mathcal{T} = (\mathcal{T}, \leq_{\mathcal{T}})$ let $\mathfrak{p}_{\mathcal{T}} = \mathfrak{p}(\mathcal{T})$ be $\min\{\kappa_1 + \kappa_2 : (\kappa_1, \kappa_2) \in \mathcal{C}_{\mathcal{T}}\}$ and $\mathfrak{p}_{\theta}(\mathcal{T}) = \min\{\kappa_1 + \kappa_2 : (\kappa_1, \kappa_2) \in \mathcal{C}_{\mathcal{T}, \theta}\}$; where:

2) $\mathcal{C}_{\theta}(\mathcal{T}) = \{(\kappa_1, \kappa_2) : \text{the partial order } \mathcal{T} \text{ has a } (\kappa_1, \kappa_2)\text{-cut and } \kappa_1 \geq \theta, \kappa_2 \geq \aleph_0\}$. If $\theta = \aleph_0$ then we may omit θ , (yes, when $\theta > \aleph_0$ this is not symmetric).

3) For a partial order \mathcal{T} let $\mathfrak{t}_{\mathcal{T}} = \mathfrak{t}(\mathcal{T})$ be the minimal $\kappa \geq \aleph_0$ such that there is a $<_{\mathcal{T}}$ -increasing sequence of length κ with no $<_{\mathcal{T}}$ -upper bound.

4) Let $\mathfrak{p}_{\mathcal{T}}^* = \mathfrak{p}^*(\mathcal{T})$ be $\min\{\mathfrak{t}_{\mathcal{T}}, \mathfrak{p}_{\mathcal{T}}\}$.

5) $\mathfrak{p}_{\theta\text{-sym}}(\mathcal{T}) = \min\{\kappa : (\kappa, \kappa) \in \mathcal{C}_{\theta}(\mathcal{T})\}$ and if $\theta = \aleph_0$ we may write $\mathfrak{p}_{\text{sym}}^*(\mathcal{T})$.

6) In Definition 1.2 below let $\mathfrak{t}_{\mathbf{r}} = \mathfrak{t}_{\mathcal{T}_{\mathbf{r}}}$, $\mathfrak{p}_{\mathbf{r}} = \mathfrak{p}_{\theta_{\mathbf{r}}}(\mathcal{T}_{\mathbf{r}})$.

Definition 1.2. 1) For $\iota = 1, 2$ (the difference is only in closed (i)), we say \mathbf{r} or (M, Δ) is a (θ, ι) -realization³ spectrum problem, in short (θ, ι) -RSP or (θ, ι) -1-RSP when \mathbf{r} consists of (if $\iota = 2$ we may omit it, similarly if $\theta = \aleph_0$; we may omit Δ and write M when Δ is the set of atomic formulas in $\mathbb{L}(\tau_{N_M})$, see below, so M below = $M_{\mathbf{r}}$, etc.):

(a) M a model,

(b) for the relations $\mathcal{T} = \mathcal{T}^M, \leq_{\mathcal{T}} = \leq_{\mathcal{T}}^M$ of M (i.e. $\mathcal{T}, \leq_{\mathcal{T}}$ are predicates from τ_M) we have $\mathcal{T} = (\mathcal{T}, \leq_{\mathcal{T}})$ a partial order (so definable in M) with root $c^M = \text{rt}(\mathcal{T})$, so $c \in \tau_M$ is an individual constant and $t \in \mathcal{T} \Rightarrow \text{rt}(\mathcal{T}) \leq_{\mathcal{T}} t$; as in other cases we may write $\mathcal{T}_{\mathbf{r}}, \leq_{\mathbf{r}}$ for $\mathcal{T}, \leq_{\mathcal{T}}$; we do not require \mathcal{T} to be a tree; but do require $t \in \mathcal{T} \Rightarrow t \leq_{\mathcal{T}} t$,

(c) a model $N = N_{\mathbf{r}} = N_M$ with universe $P^M, \tau(N) \subseteq \tau(M)$ such that

- $Q \in \tau_N \Rightarrow Q^M = Q^N$,
- $F \in \tau_N \Rightarrow F^N = F^M$, (we understand F^M, F^N to be partial functions),
so every $\varphi \in \mathbb{L}(\tau_N)$ can be interpreted as $\varphi^{[*]} \in \mathbb{L}(\tau_M)$, all variables varying on P (include quantification); we may forget the superscript $[*]$.

(d) the cardinal θ and $\Delta \subseteq \{\varphi : \varphi = \varphi(x, \bar{y}) \in \mathbb{L}_{\theta, \theta}(\tau_N)\}$ which is closed under conjunctions meaning: if $\varphi_{\ell}(x, \bar{y}_{\ell}) \in \Delta$ for $\ell = 1, 2$ then $\varphi(x, \bar{y}'_1, \bar{y}''_2) = \varphi_1(x, \bar{y}'_1) \wedge \varphi_2(x, \bar{y}''_2) \in \Delta$,

(e) $R^M \subseteq |N| \times \mathcal{T}^M$ so a two-place relation; and let $R_t^M = \{b : bR^M t\}$ for $t \in \mathcal{T}^M$,

(f) $|N| \times \{\text{rt}_{\mathcal{T}}\} \subseteq R^M$, i.e. $R_{\text{rt}(\mathcal{T})}^M = |N|$,

(g) if $s \leq_{\mathcal{T}} t$ then $a \in N \wedge aRt \Rightarrow aRs$, i.e. $R_s^M \supseteq R_t^M$,

(h) $t \in \mathcal{T} \Rightarrow R_t^M \neq \emptyset$,

(i) if $s \in \mathcal{T}, \varphi(x, \bar{a}) \in \Delta(N) := \{\varphi(x, \bar{y}) : \varphi(x, \bar{y}) \in \Delta \text{ and } \bar{a} \in {}^{\ell g(\bar{y})}N\}$ and for some $b \in R_s^M, N \models \varphi[b, \bar{a}]$ then there is $t \in \mathcal{T}$ such that $s \leq_{\mathcal{T}} t$ and $R_t^M = \{b \in R_s^M : N \models \varphi[b, \bar{a}]\}$,

³When P and τ_N (hence N) are understood from the context we may omit them.

- (i)⁺ if $\iota = 1$ like clause (i) but⁴ moreover $t = F_{\varphi,1}^M(s, \bar{a})$ where $F_{\varphi,1}^M : \mathcal{T}_{\mathbf{r}} \times {}^{\ell g(\bar{y})}(P^M) \rightarrow \mathcal{T}_{\mathbf{r}}$,
- (j) if $t \in \mathcal{T}_{\mathbf{r}}$ and $\varphi(x, \bar{a}) \in \Delta(N)$ and $\varphi(N, \bar{a}) \neq \emptyset$ then
 - (α) $s = F_{\varphi,2}^M(t, \bar{a})$ is such that $R_s^M \cap \varphi(N, \bar{a}) \neq \emptyset$ and $s \leq_{\mathcal{T}} t$,
 - (β) if $s = F_{\varphi,2}^M(t, \bar{a})$, $s_1 \leq_{\mathcal{T}} t$ and $R_{s_1}^M \cap \varphi(N, \bar{a}) \neq \emptyset$ then $s_1 \leq_{\mathcal{T}} s$.
- (k) if $\theta > \aleph_0$ then in $(\mathcal{T}, \leq_{\mathcal{T}})$ any increasing chain of length $< \theta$ which has an upper bound has a $\leq_{\mathcal{T}}$ -lub.

Remark 1.3. We may consider adding: S^M a being successor, (but this is not Horn), i.e.:

- (l) if $\iota = 1$ we also have S^M is the set of pairs (a, b) such that b is a $\leq_{\mathcal{T}}$ -successor of a which means:
 - (α) if $a \leq b \wedge a \neq b$ then for some c , $S(a, c) \wedge c \leq b$,
 - (β) if $b \in \mathcal{T} \setminus \{\text{rt}_{\mathcal{T}}\}$ then for some unique a we have $S^M(a, b)$,
 - (γ) $S(a, b) \Rightarrow a \leq b$,
 - (δ) $S(a, b_1) \wedge S(a, b_2) \wedge b_1 \neq b_2 \Rightarrow \neg(b_1 \leq b_2)$,
 - (ε) in clause (j) we can add $S^M(s, t)$.

Remark 1.4. Presently, it may be that $a \leq_{\mathcal{T}} b \leq_{\mathcal{T}} a$ but $a \neq b$. Not a disaster to forbid but no reason.

How does this axiomatize realizations of types?

Claim/Definition 1.5. *Let $\iota = \{1, 2\}$, θ is \aleph_0 or just a regular cardinal.*

1) *For any model N and $\Delta \subseteq \{\varphi : \varphi = \varphi(x, \bar{y}) \in \mathbb{L}_{\theta, \theta}(\tau_T)\}$ closed under conjunctions of $< \theta$, the canonical (θ, ι) -RSP, $\mathbf{r} = \mathbf{r}_{N, \Delta}^{\theta}$ defined below is indeed a θ -RSP.*

2) $\mathbf{r} = \mathbf{r}_{N, \Delta}^{\theta}$ (if $\theta = \aleph_0$ we may omit it) is defined by:

- (a) $\Delta_{\mathbf{r}} = \Delta$, $N_{\mathbf{r}} = N$ and $\theta_{\mathbf{r}} = \theta$,
- (b) $\mathcal{T}_{\mathbf{r}} = \{\langle \varphi_{\varepsilon}(x, \bar{a}_{\varepsilon}) : \varepsilon < \zeta \rangle : \zeta < \theta \text{ and for every } \varepsilon < \zeta \text{ we have } \varphi_{\varepsilon}(x, \bar{a}_{\varepsilon}) \in \Delta(N) \text{ and } N \models (\exists x)(\bigwedge_{\varepsilon < \zeta} \varphi_{\varepsilon}(x, \bar{a}_{\varepsilon}))\}$,
- (c) $\leq_{\mathbf{r}} =$ being the initial segment relation on $\mathcal{T}_{\mathbf{r}}$,
- (d) $M = M_{\mathbf{r}}$ is the model with universe $\mathcal{T}_{\mathbf{r}} \cup |N|$; without loss of generality $\mathcal{T}_{\mathbf{r}} \cap |N| = \emptyset$, with the relations and functions of N , $\mathcal{T}_{\mathbf{r}}$, $\leq_{\mathbf{r}}$ and,
 - $P^M = |N|$,
 - $c^M = \langle \rangle \in \mathcal{T}_{\mathbf{r}}$,
 - $R^M = \{(b, t) : b \in N, t = \langle \varphi_{t, \varepsilon}(x, \bar{a}_{t, \varepsilon}) : \varepsilon < \zeta_t \rangle \in \mathcal{T} \text{ and } N \models \varphi_{t, \ell}(b, \bar{a}_{t, \varepsilon}) \text{ for every } \varepsilon < \zeta_t\}$,
 - $F_{\varphi, 2}^M$ as in Definition 1.2(j),
 - if $\iota = 1$ then $F_{\varphi, 1}^M$ is as in Definition 1.2(i)⁺.

Remark 1.6. If we adopt 1.3 it is natural to add:

⁴We may not add a function, maybe it matters when we try to build \mathbf{r} with $\text{Th}(M_{\mathbf{r}})$ nice first order.

(e) for $\iota = 1, S^M = \{(\bar{\varphi}_1, \bar{\varphi}_2) : \bar{\varphi}_2 = \bar{\varphi}_1 \wedge \langle \varphi(x, \bar{a}) \rangle \in \mathcal{T}_{\mathbf{r}} \text{ for some } \varphi(x, \bar{a}) \in \Delta(N)\}$.

Proof. Obvious. □_{1.5}

Main Claim 1.7. 1) Assume \mathbf{r} is an RSP. If $\kappa = \min\{\mathfrak{t}_{\mathbf{r}}, \mathfrak{p}_{\mathbf{r}}\}$ then the model N is $(\kappa, 1, \Delta_{\mathbf{r}})$ -saturated, i.e.

⊕ if $p(x) \subseteq \Delta_{\mathbf{r}}(N_{\mathbf{r}})$ is finitely satisfiable in $N_{\mathbf{r}}$ (= is a type in $N_{\mathbf{r}}$) of cardinality $< \kappa$ then p is realized in $N_{\mathbf{r}}$.

2) If $\theta > \aleph_0$ and \mathbf{r} is a θ -RSP, then $N_{\mathbf{r}}$ is $(\kappa, 1, \Delta_{\mathbf{r}})$ -saturated where $\kappa = \min\{\mathfrak{t}_{\mathbf{r}}, \mathfrak{p}_{\mathbf{r}}\}$ recalling 1.1(6), i.e. $\mathfrak{p}_{\mathbf{r}} = \mathfrak{p}_{\mathcal{T}_{\mathbf{r}}, \theta}$.

3) If $\theta > \aleph_0, \mathbf{r}$ is a θ -RSP satisfying $(k)^+$ below then $N_{\mathbf{r}}$ is $(\mathfrak{t}_{\mathbf{r}}, 1, \Delta_{\mathbf{r}})$ -saturated when:

$(k)^+$ in $(\mathcal{T}, \leq_{\mathcal{T}})$ any increasing chain which has an upper bound, has a $\leq_{\mathcal{T}}$ -lub.

Proof. This is an abstract version of [She78, Ch.VI,2.6] = [She90, Ch.VI,2.6]; recall that [She78, Ch.VI,2.7] translates trees to linear orders.

1) Let $N = N_{\mathbf{r}}, \Delta = \Delta_{\mathbf{r}}$, etc.

Let p be a $(\Delta, 1)$ -type in N of cardinality $< \kappa$. Without loss of generality p is infinite and closed under conjunctions.

So let

(*)₁ $\alpha_* < \kappa, p = \{\varphi_{\alpha}(x, \bar{a}_{\alpha}) : \alpha < \alpha_*\} \subseteq \Delta(N), p$ is finitely satisfiable in N .

We shall try to choose t_{α} by induction on $\alpha \leq \alpha_*$ such that:

- (*)₂ (a) $t_{\alpha} \in \mathcal{T}$ and $\beta < \alpha \Rightarrow t_{\beta} \leq_{\mathcal{T}} t_{\alpha}$,
- (b) if $\beta < \alpha_*$ then there is $b \in R_{t_{\alpha}}^M$ such that $N \models \varphi_{\beta}[b, \bar{a}_{\beta}]$,
- (c) if $\beta < \alpha$ then $b \in R_{t_{\alpha}}^M \Rightarrow N \models \varphi_{\beta}[b, \bar{a}_{\alpha}]$.

If we succeed, this is enough because if $t = t_{\alpha_*}$ is well defined then $R_t^M \neq \emptyset$ by Definition 1.2(h) and any $b \in R_t^M$ realizes the type by (*)₂(c) and Definition 1.2(h). Why can we carry the definition?

Case 1: $\alpha = 0$.

Let $t_{\alpha} = \text{rt}_{\mathcal{T}}$, hence $R_{t_{\alpha}}^M = |N|$ by Definition 1.2(f). Now clause (a) of (*)₂ holds as $t_{\alpha} \in \mathcal{T}_{\mathbf{r}}$ and there is no $\beta < \alpha$. Also clause (b) of (*)₂ holds because p is a type and $R_{\text{rt}(\mathcal{T})}^M = |N_{\mathbf{r}}|$ by Definition 1.2(h).

Lastly, clause (c) of (*)₂ holds trivially.

Case 2: $\alpha = \beta + 1$.

If $\iota = 1$ let $t = F_{\varphi_{\beta, 1}}^M(t_{\beta}, \bar{a}_{\beta})$ and see clause (i)⁺ of Definition 1.2. If $\iota = 2$ use clause (i) of the definition recalling p is closed under conjunctions.

Case 3: α a limit ordinal

As $\mathfrak{t}_{\mathcal{T}_{\mathbf{r}}} \geq \kappa > \alpha_*$ by the claim's assumption (on $\mathfrak{t}_{\mathcal{T}_{\mathbf{r}}}$, see Definition 1.1(2)) necessarily there is $s \in \mathcal{T}$ such that $\beta < \alpha \Rightarrow t_{\alpha} \leq_{\mathcal{T}} s$. We now try to choose s_i by induction on $i \leq \alpha_*$ such that

(*)_{2.1} (a) $s_i \in \mathcal{T}$,

- (b) $\beta < \alpha \Rightarrow t_\beta \leq_{\mathcal{I}} s_i$,
- (c) $j < i \Rightarrow s_i \leq_{\mathcal{I}} s_j$,
- (d) if $i = j + 1$ then $R_{s_i}^M$ is not disjoint to $\varphi_j(N, \bar{a}_j)$.

If we succeed, then s_{α_*} satisfies all the demands on t_α (e.g. $(*)_2(b)$ holds by Definition 1.2(g) and $(*)_{2.1}(d)$), so we have just to carry the induction for α . Now if $i = 0$ clearly $s_0 = s$ it as required. If $i = j + 1$ let $s_i = F_{\varphi,2}^M(s_j, \bar{a}_j)$, by Definition 1.2(j) it is as required. For i a limit ordinal use $\kappa \leq \mathfrak{p}_{\mathcal{I}}$ hence carry the induction on i so finish case 3.

So we succeed to carry the induction on α hence (as said after $(*)_2$) get the desired conclusion.

2) Similar, except concerning case 3. Note that without loss of generality $\theta > \aleph_0$ by part (1).

Case 3A: α is a limit ordinal of cofinality $\geq \theta$.

As in the proof of part (1).

Case 3B: α is a limit ordinal of cofinality $< \theta$.

Again there is an upper bound s of $\{t_\beta : \beta < \alpha\}$. Now by clause (k) of Definition 1.2, without loss of generality s is a $<_{\mathcal{I}}$ -lub of $\{t_\beta : \beta < \alpha\}$. So easily for every $i < \alpha_*$, $F_{\varphi_i,2}^N(s, \bar{a}_i)$ is $\geq t_\beta$ for $\beta < \alpha$ hence is equal to s , so $s_\alpha := s$ is as required.

3) Similarly. □_{1.7}

Discussion 1.8. 1) What about “ (λ^+, n, Δ) -saturation”? We can repeat the same analysis or we can change the models to code n -tuples. More generally, replacing $\varphi(\bar{x}_{[\varepsilon]}, \bar{y})$ by $\varphi(\langle F_\zeta(x) : \zeta < \varepsilon \rangle, \bar{y})$, using $F_\zeta \in \tau_M$ (though not necessarily $F_\zeta \in \tau_{N_r}$), so we can allow infinite ε .

2) Hence the same is true for $(\lambda^+, \aleph_0, \Delta)$ -saturation, e.g. λ^+ -saturated by an assumption.

§ 2. APPLYING THE AXIOMATIZED FRAME

Consider a filter D on a set I and cardinals $\lambda \geq \mu$. We may ask for a model M of cardinality $\geq \mu$, whether M^I/D is $(\lambda^+, \text{atomic})$ -saturated, varying M .

We here apply §1 to show that: when D is an ultra-filter, the model $(^{\omega >} \mu, \triangleleft)$ is the hardest, this is 2.1, We then (in 2.2) show that §1 has axiomatization which is a Horn theory. Hence we can prove results like 2.1 below for filters D (not just for ultra-filters),

Conclusion 2.1. 1) *If D is an ultra-filter on a set I, N a model, $\mu = \|N\| + \|\tau_N\|$ and $(^{\omega >} \mu, \triangleleft)^I/D$ is $(\lambda^+, \text{atomic})$ -saturated then N^I/D is λ^+ -saturated.*

2) *Instead of “ $(^{\omega >} \mu, \trianglelefteq)^I/D$ is $(\lambda^+, 1, \text{atomic})$ -saturated” we can demand “ J^I/D is $(\lambda^+, 1, \text{atomic})$ -saturated” where J is the linear order with set of elements $\{-1, 1\} \times ^{\omega >} \mu$ ordered by $(\iota_1, \eta_1) < (\iota_2, \eta_2)$ iff $\iota_1 < \iota_2$ or $\iota_1 = -1 = \iota_2 \wedge \eta_1 <_{\text{lex}} \eta_2$ or $\iota_1 = -1 = \iota_2 \wedge \eta_2 <_{\text{lex}} \eta_1$.*

Proof. 1) Let $N_1 = N$. As D is an ultra-filter without loss of generality $\text{Th}(N_1)$ has elimination of quantifiers and even every formula is equivalent to an atomic formula. Let $\Delta = \mathbb{L}(\tau_N)$, by 1.5 $\mathbf{r}_1 := \mathbf{r}_{N_1, \Delta}$ is an RSP. Let $N_2 = N_1^I/D$ and let $M_1 = M_{\mathbf{r}_1}, M_2 = M_1^I/D$ and let \mathbf{r}_2 be the RSP(M_2, Δ). Clearly, \mathbf{r}_2 is an RSP as the demands in 1.2 are first order (see more in 2.2).

Now,

$$(*)_1 \quad \mathcal{T}_{\mathbf{r}_1} \cong (^{\omega >} \mu, \triangleleft).$$

[Why? See 1.5(2).]

$$(*)_2 \quad \mathcal{T}_{\mathbf{r}_2} = (\mathcal{T}_{\mathbf{r}_1})^I/D \text{ is } (\lambda^+, \text{atomic})\text{-saturated.}$$

[Why? By an assumption.]

$$(*)_3 \quad \mathfrak{t}(\mathcal{T}_{\mathbf{r}_1}), \mathfrak{p}(\mathcal{T}_{\mathbf{r}_2}) \geq \lambda^+.$$

[Why? Follows by $(*)_2$.]

Hence by 1.7, N_2 is $(\lambda^+, 1, 1, \Delta)$ -saturated which means $N_2 = (N_1)^I/D$ is λ^+ -saturated.

2) Easy (or see [She78, Ch.VI,2.7] or see [She72b]). □_{2.1}

To apply the criterion of the Main Claim 1.7 to reduced products we need:

Claim 2.2. *If Δ is the set of conjunctions of atomic formulas (no negation!) in $\mathbb{L}(\tau_0)$ and $\tau = \{\mathcal{F}, \leq_{\mathcal{F}}, R, P, c\} \cup \{F_{\varphi, \ell} : \varphi \in \Delta \text{ and } \ell = 2 \text{ or } \ell = 1 \text{ if relevant}\} \cup \tau_0$, (disjoint union, recall c is $\text{rt}_{\mathcal{F}}$), then there is a set T of Horn sentences from $\mathbb{L}(\tau)$ such that for every τ -model M*

- (M, Δ) is a RSP (i.e. 2-RSP) iff $M \models T$.

Proof. Consider Definition 1.2. For each clause we consider the sentences expressing the demands there.

Clause (a): Obvious.

Clause (b): Clearly the following are Horn:

- $x \leq_{\mathcal{F}} y \rightarrow \mathcal{F}(x)$ and $x \leq_{\mathcal{F}} y \rightarrow \mathcal{F}(y)$,

- $x \leq_{\mathcal{T}} y \wedge y \leq_{\mathcal{T}} z \rightarrow x \leq_{\mathcal{T}} z$,
- $\mathcal{T}(\text{rt}_{\mathcal{T}})$ and $\mathcal{T}(s) \rightarrow \text{rt}_{\mathcal{T}} \leq s$,
- $\mathcal{T}(x) \rightarrow x \leq_{\mathcal{T}} x$.

Note that $(\mathcal{T}, \leq_{\mathcal{T}})$ being a tree is not a Horn sentence but this is not required.

Clause (c):

- $Q(x_0, \dots, x_{n(Q)-1}) \rightarrow P(x_{\ell})$ when Q is an $n(Q)$ -place predicate from $\tau(N)$ and $\ell < n(Q)$; clearly it is Horn,
- for any n -place function symbol $F \in \tau_0$ the sentence: $P(x_0) \wedge \dots \wedge P(x_{n-1}) \rightarrow P(F(x_0, \dots, x_{n-1}))$ and $y = F(x_0, \dots, x_{n-1}) \rightarrow P(x_{\ell})$.

Clause (d): nothing to prove - see the present claim assumption on Δ .

Recall that for $F \in \tau_N$, F stand for a partial function symbol with domain P_F .

Clause (e): $yRs \rightarrow \mathcal{T}(s)$, $yRs \rightarrow P(y)$ are Horn.

Clause (f): $P(x) \rightarrow xR(\text{rt}_{\mathcal{T}})$ is Horn.

Clause (g): $s \leq_{\mathcal{T}} t \wedge xRt \rightarrow xRs$ is Horn.

Clause (h): $(\forall t)(\exists x)(\mathcal{T}(t) \rightarrow xRt)$ is Horn.

Clause (i): Let $\varphi(x, \bar{y}) \in \Delta$.

First assume $\iota = 1$. Note the following are Horn: for any $\varphi(x, \bar{y}) \in \Delta$,

- $\mathcal{T}(s) \wedge xRs \wedge \varphi(x, \bar{y}) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_{\ell}) \wedge t = F_{\varphi,1}(s, \bar{y}) \rightarrow \mathcal{T}(t) \wedge s \leq_{\mathcal{T}} t$,
- $\mathcal{T}(s) \wedge xRs \wedge \varphi(x, \bar{y}) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_{\ell}) \wedge t = F_{\varphi,1}(s, \bar{y}) \rightarrow xRt$,
- $\mathcal{T}(s) \wedge x'Rs \wedge x'R F_{\varphi,1}(s, \bar{y}) \rightarrow \varphi(x', \bar{y})$.

This suffices. The proof when $\iota = 2$ is similar.

Clause (j): Similarly but we give details.

Let $\varphi = \varphi(x, \bar{y}) \in \Delta$, so the following are Horn:

- $\varphi(x_1, \bar{y}) \wedge P(x_1) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_{\ell}) \wedge s = F_{\varphi,2}(t, \bar{y}) \rightarrow s \leq_{\mathcal{T}} t$,
- $\varphi(x_1, \bar{y}) \wedge P(x_1) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_{\ell}) \wedge s = F_{\varphi,2}(t, \bar{y}) \rightarrow (\exists x)(xRs \wedge \varphi(x, \bar{y}))$,
- $P(x) \wedge \bigwedge_{\ell < \ell g(\bar{y})} P(y_{\ell}) \wedge s = F_{\varphi,2}(t, \bar{y}) \wedge z \leq_{\mathcal{T}} t \wedge xRz \wedge \varphi(x, \bar{y}) \rightarrow z \leq_{\mathcal{T}} s$.

Clause (k): As $\theta = \aleph_0$ this is empty.

This suffices. □_{2.2}

Claim 2.3. *Also for $\theta > \aleph_0$ (see 1.2(2)) Claim 2.2 holds but some of the formulas are in $\mathbb{L}_{\theta, \theta}$.*

Proof. Clause (k): When $\theta > \aleph_0$.

Should be clear because for each limit ordinal $\delta < \kappa$, we have that the sentences $\psi_{\delta} = (\forall x_0, \dots, x_{\alpha}, \dots, x_{\delta})(\exists y)(\forall z)((\bigwedge_{\alpha < \beta < \delta} x_{\alpha} \leq_{\mathcal{T}} x_{\beta} \leq_{\mathcal{T}} y \leq_{\mathcal{T}} x_{\delta}) \wedge$

($\bigwedge_{\alpha < \beta < \delta} x_\alpha \leq_{\mathcal{F}} x_\beta \leq_{\mathcal{F}} z \leq_{\mathcal{F}} y \leq_{\mathcal{F}} x_\delta \rightarrow y = z$) is a Horn sentence and it expresses “any $\leq_{\mathcal{F}}$ -increasing chain of length δ has a \leq -lub”. $\square_{2.3}$

Conclusion 2.4. 1) Assume

- (a) D be a filter on I ,
- (b) N a model, $\lambda = \|N\| + |\tau_N|$, Δ the set of atomic formulas (in $\mathbb{L}(\tau_N)$),
- (c) $\mathcal{F} = (\mathcal{F}, \leq_{\mathcal{F}}) := (\omega > \lambda, \trianglelefteq)^I / D$,
- (d) $\kappa = \mathfrak{p}_{\mathcal{F}}^* = \min\{\mathfrak{t}_{\mathcal{F}}, \mathfrak{p}_{\theta}(\mathcal{F}_1)\}$ see Definition 1.1(6).

Then the reduced power N^I / D is $(\kappa, 1, \Delta)$ -saturated.

2) Assume⁵

- (a) D is a θ -complete filter on I , $\theta = \text{cf}(\theta) > \aleph_0$,
- (b) N is (θ, Δ) -saturated, Δ a set of atomic formulas,
- (c) $\mathcal{F}_1 := (\theta > \lambda, \trianglelefteq)^I / D$
- (d) $\kappa = \min\{\mathfrak{t}_{\mathcal{F}_1}, \mathfrak{p}_{\theta}(\mathcal{F}_1)\}$.

Then N^I / D is $(\kappa, \theta, 1, \Delta)$ -saturated.

3) We can above replace N^I / D by $N^{\mathfrak{B}} / D$ where D is a filter on the complete Boolean algebra \mathfrak{B} which has $(< \theta)$ -distributivity when $\theta > \aleph_0$.

Proof. 1) Let $\theta = \aleph_0$ and $\mathbf{r}_0 = (M_0, \Delta)$ be $\mathbf{r}_{N, \Delta}^{\theta}$ from 1.5, so $\theta_{\mathbf{r}_0} = \theta$.

By Claim 1.5, M_0 is an RSP hence by Claim 2.2 also $M = M_0^I / D$ is an RSP. Now apply the Main Claim 1.7(1).

2) Similarly using 1.7(2).

3) Similarly. $\square_{2.4}$

Remark 2.5. 1) No harm in assuming $\Delta = \{Q(\bar{y}) : Q \text{ a predicate}\}$. Note that allowing bigger Δ is problematic except in trivial cases (φ and $\neg\varphi$ are equivalent to Horn formulas), see proof of clauses (i),(j) of Definition 1.2.

2) Using 2.4(1) above, if D is an ultra-filter, not surprisingly we get [She90, Ch.VI,2.6], i.e. the theory of dense linear orders is \trianglelefteq -maximal (well, using the translation from dense linear orders to trees in 2.1(2) equivalently [She90, Ch.VI,2.7]). The new point here is that 2.4 does this also for reduced powers, i.e. for D a filter.

3) So a natural question is: can we replace the strict property by SOP_2 ? We shall show that for reduced power we also have non-peculiar cuts, see §4.

4) Why is the reduced power of a tree not necessarily a tree? Let M be the tree $(\omega > \omega, \triangleleft)$. Let $\eta_1 \triangleleft \eta_2 \triangleleft \eta_3 \in \omega > \omega$ and let $A_1, A_2 \in D^+$ be disjoint and define $f_\ell : I \rightarrow \omega > \omega$ for $\ell = 1, 2, 3$ by:

- $f_3(s) = \eta_3$ for $s \in I$,
- $f_2(s)$ is η_2 if $s \in A_2$ and η_0 otherwise,
- $f_1(s)$ is η_2 if $s \in A_1$ and η_0 otherwise.

Clearly if $N = M^I / D$ then in N we have:

- $f_1 / D \triangleleft f_3 / D$,

⁵Note that κ here may be bigger than in part (1).

- $f_2/D \triangleleft f_3/D$,
- $\neg(f_1/D \triangleleft f_2/D)$,
- $\neg(f_2/D \triangleleft f_1/D)$,
- $\neg(f_1/D = f_2/D)$.

Conclusion 2.6. N^I/D is $(\kappa, 1, \Delta_1)$ -saturated and $\kappa \geq \theta$ when :

- (*) (a) D is a θ -complete filter on I ,
- (b) $\Delta \subseteq \{\varphi : \varphi(x, \bar{y}) \in \mathbb{L}_{\theta, \theta}(\tau_N) \text{ is atomic (hence } \in \mathbb{L}(\tau_N))\}$,
- (c) $\Delta_1 = \text{cl}_{< \theta}(\Delta) = \text{the closure of } \Delta \text{ under conjunction of } < \theta \text{ formulas}$,
- (d) N is (θ, Δ) -saturated, i.e. if $p(x) \subseteq \Delta(N) = \{\varphi(x, \bar{a}) : \varphi(x, \bar{y}) \in \Delta, \bar{a} \in {}^{\text{lg}(\bar{y})}M\}$ has cardinality $< \theta$ and is finitely satisfiable in N then p is realized in N ,
- (e) $\kappa = \min\{\mathfrak{p}_{\mathcal{F}}, \mathfrak{t}_{\theta}(\mathcal{F})\}$ where $\mathcal{F} = (\theta^{>} \lambda, \trianglelefteq)^I/D$ and $\lambda = \theta^{>}(\|N\| + |\Delta|)$.

Proof. Let $\mathbf{r} = \mathbf{r}_{N, \Delta_1}^{\theta}$ recalling Definition 1.5 and $M_0 = M_{\mathbf{r}}$.

Now apply 1.7(2) noting that:

- (*)₁ $N_1 = N_0^I/D$ satisfies: every set of $< \theta$ formulas from $\Delta(N)$ which is finitely satisfiable in N_1 is realized in N_1 .

[Why? Let $\langle \varphi_{\alpha}(x, f_{\alpha,0}/D, \dots, f_{\alpha, n(\alpha)-1}/D) : \alpha < \alpha_* \rangle$ be finitely satisfiable in N_1 and $\alpha_* < \theta, \alpha < \alpha_* \Rightarrow \varphi_{\alpha} \in \Delta$. For every finite $u \subseteq \alpha_*$ we have $N_1 \models (\exists x)(\bigwedge_{\alpha \in u} \varphi_{\alpha}(x, f_{\alpha,0}/D, \dots, f_{\alpha, n(\alpha)-1}/D))$ hence the set

$$I_u := \{s \in I : N_1 \models (\exists x) \bigwedge_{\alpha \in u} \varphi_{\alpha}(x, f_{\alpha,0}(s), \dots, f_{\alpha, n(\alpha)-1}(s))\}$$

belongs to D . But D is θ -complete, hence $I_* = \bigcap \{I_u : u \subseteq \alpha_* \text{ is finite}\}$ belongs to D . Now for each $s \in I_*$, the set $p_s := \{\varphi_{\alpha}(x, f_{\alpha,0}(s), \dots, f_{\alpha, n(\alpha)-1}(s)) : \alpha < \alpha_*\}$ is finitely satisfiable in N , hence is realized by some $a_s \in N$. Let $g \in {}^I N$ be such that $s \in I_* \Rightarrow g(s) = a_s$; clearly g/D realizes p , so we are done.]

Similarly

- (*)₂ in $\mathcal{F} = (\theta^{>} \lambda, \trianglelefteq)^I/D$ we have,
 - (a) every increasing sequence of length $< \theta$ has an upper bound,
 - (b) any increasing sequence of length $< \theta$ with an upper bound has a lub,
 - (c) there is no infinite decreasing sequence so $(\kappa_1, \kappa_2) \in \mathcal{C}(\mathcal{F}) \Rightarrow \kappa_2 = 1$.

[Why? For clause (a) note that $(\forall x_0, \dots, x_{\alpha}, \dots)_{\alpha < \delta} (\exists y) (\bigwedge_{\alpha < \beta < \delta} x_{\alpha} \leq_{\mathcal{F}} x_{\beta} \rightarrow \bigwedge_{\alpha < \delta} x_{\alpha} \leq_{\mathcal{F}} y)$ is a Horn sentence. For clause (b) see 2.3, i.e. proof of clause (k) in 2.3.]

- (*)₃ $M_1 = M_{\mathbf{r}}^I/D$ is a θ -RSP.

[Why? See above recalling 2.2, 2.3.]

- (*)₄ if $\theta \geq \aleph_1$ then \mathbf{r} satisfies $(k)^+$ from 1.7(3).

[Why? Easily as D is a \aleph_1 -complete ultra-filter.]

So we are done by 1.7(3). □_{2.6}

It is natural to wonder

Question 2.7. Assume $\lambda \geq \theta = \text{cf}(\theta) > \aleph_0$.

- 1) Is there a θ -complete (λ, θ) -regular ultra-filter D on λ such that $\lambda < \mathfrak{t}((^{\theta}>\theta, \trianglelefteq)^\lambda / D)$?
- 2) Similarly for filters.
- 3) Use $\leq_{\mathcal{F}} = \trianglelefteq$ or $<_{\mathcal{F}} = \triangleleft$?
- 4) If $\lambda = \lambda^{<\theta}$, D a fine normal ultra-filter on $I = [\lambda]^{<\theta}$, we get $\lambda \leq \mathfrak{t}((^{\theta}>\theta, \trianglelefteq) / D)$.

Remark 2.8. Now [MS16c, §5] answers 2.7(1) positively for θ a super-compact cardinal.

Conclusion 2.9. Let \mathfrak{B} be a complete Boolean algebra and D a filter on \mathfrak{B} .

- 1) For every model N , letting $\lambda = \|N\| + |\tau_N|$, we have $N^{\mathfrak{B}}/D$ is (μ^+, atomic) -saturated if $\mu^+ \leq \min\{\mathfrak{p}((^{\omega}>\lambda, \trianglelefteq)^{\mathfrak{B}}/D), \mathfrak{t}((^{\omega}>\lambda, \trianglelefteq)^{\mathfrak{B}}/D)\}$.
- 2) Assume \mathfrak{B} is $(< \theta)$ -distributive (e.g. for some dense $Y \subseteq \mathfrak{B}^+$, for every decreasing sequence⁶ in \mathfrak{B} of elements from Y of length $< \theta$ has a positive lower bound), and D is a θ -complete filter on \mathfrak{B} . If N is (μ^+, atomic) -saturated then $N^{\mathfrak{B}}/D$ is $\mathfrak{t}((^{\theta}>\lambda, \trianglelefteq)^{\mathfrak{B}}/D)$ -atomic saturated.

Proof. As, e.g. in 2.6 above or 2.13 below. □_{2.9}

Conclusion 2.10. Assume $(T, \varphi(\bar{x}, \bar{y}))$ has SOP_3 .

Then, recalling 0.15, T is $\trianglelefteq_{\lambda}^{\text{FP}}$ -maximal for every λ and even $(T, \{\varphi(\bar{x}, \bar{y})\})$ is.

Proof. By [She78, Ch. VI.2.7] = [She90], but we elaborate.

Without loss of generality assume φ is an atomic formula.

Let $(T_1, \Delta_1) = (T_1, \{\varphi(x, y)\})$ and let T_2 be a first-order theory, Δ_2 a set of atomic formulas in $\mathbb{L}(\tau(T_2))$, $|T_2| \leq \lambda$, $\mathcal{B} = \mathcal{P}(\lambda)$, D a regular filter on λ , or, equivalently, on the Boolean algebra \mathcal{B} . Finally, let M_ℓ be a λ^+ -saturated model of T_ℓ for $\ell = 1, 2$. We assume M_1^λ/D is (λ^+, Δ_1) -saturated and we should prove that also M_2^λ/D is (λ^+, Δ_2) -saturated. By Conclusion 2.9,

$$(*)_1 \text{ It suffice to prove } \mu^+ \leq \min\{\mathfrak{p}((^{\omega}>\lambda, \trianglelefteq)^{\mathfrak{B}}/D), \mathfrak{t}((^{\omega}>\lambda, \trianglelefteq)^{\mathfrak{B}}/D)\}.$$

Next,

- (*)₂ Let I be the following linear order:
 - the set of elements is $\{(\eta, \iota) : \eta \in {}^{\omega}>\lambda, \iota \in \{1, -1\}\}$,
 - the linear order $<_I$ is: $(\eta_1, \iota_1) < (\eta_2, \iota_2)$ iff one of the following occurs:
 - (a) $\eta_1 \triangleleft \eta_2$ and $\iota_1 = -1$,
 - (b) $\eta_2 \triangleleft \eta_1$ and $\iota_1 = 1$,
 - (c) $\eta_1 = \eta_2$ and $\iota_1 = -1, \iota_2 = 1$,
 - (d) for some $k < \text{lg}(\eta_1), \text{lg}(\eta_2)$, we have $\eta_1 \upharpoonright k = \eta_2 \upharpoonright k$ and $\eta_1(k) < \eta_2(k)$,

⁶can weaken the demand.

Next, we can find $a_{(\eta,\iota)} \in M_1$ for $(\eta, \iota) \in I$ such that $(\eta_1, \iota_1) <_I (\eta_2, \iota_2) \Rightarrow M_1 \models \varphi[a_{(\eta_1, \iota_1)}, a_{(\eta_2, \iota_2)}]$.

Let $N = (|M_1|, P^N, R^N)$, where:

$$P^N = \{a_{(\eta,\iota)} : (\eta, \iota) \in I\} \text{ and } R^N = \{(a_{(\eta_1, \iota_1)}, a_{(\eta_2, \iota_2)}) : (\eta_1, \iota_1) <_I (\eta_2, \iota_2)\}.$$

Let $M_1^+ = (M_1, P^N, R^N)$. Now,

(*)₃ in $(M_1^+)^{\lambda/D}$ every set of formulas $\subseteq \{\varphi(x, a), \varphi(a, x) : a \in (M_1^+)^{\lambda/D}\}$ of cardinality $\leq \lambda$, which is finitely satisfiable is realized.

[Why? Think of $M_1^{\lambda/D}$.]

(*)₄ In $N^{\lambda/D}$ every set of formulas $\subseteq \{xRa, aRx : a \in N^{\lambda/D}\}$ of cardinality $\leq \lambda$, which is finitely satisfiable is realized.

So easily we are done. □_{2.10}

* * *

On the connection to Peano arithmetic and to Pabion [Pab82], see Malliaris-Shelah [MS17a]. We repeat some results of [MS17b] in the present context; but first recalling:

Definition 2.11. 1) PA, Peano arithmetic, is the f.o. theory consisting of:

- (a) the obvious axioms on $0, 1, x < y, x + y, xy$
- (b) all the cases of the induction scheme, i.e. for every f.o. φ :
“if $\{x : \varphi(x, \bar{y})\}$ is not empty then it has a first member”,

2) BPA, the bounded Peano arithmetic, is defined similarly, but in clause (b), the formulas φ is bounded, i.e. all the quantifications inside it are of the form $(\forall x < y)$ or $(\exists x < y)$.

Definition 2.12. 1) $N \models$ BPA is boundedly κ -saturated up to (c_1, c_2) where $c_1, c_2 \in N$ when: if $p(x) \cup \{x < c_1\}$ is a type in N (= finitely satisfiable) of cardinality $< \kappa$ consisting of bounded formulas but with parameters $\leq c_2$, then $p(x) \cup \{x < c_1\}$ is realized in N .

2) If above $c_1 = c = c_2$ we may write c instead of (c_1, c_2) . We say N is strongly boundedly κ -saturated up to c when it holds for $(c, c_2), c_2 = \infty$, i.e. we do not bound the parameters.

3) Omitting “up to c ” in part (3) means for every $c \in N$.

Conclusion 2.13. Assume N is a model of BPA.

1) Assume $a_* \in N$ is non-standard and the power in the N -sense c^{a_*} exists for every $c \in N$.

For any uncountable cardinal κ the following conditions are equivalent:

- (a) N is boundedly κ -saturated up to c for any $c \in N$,
- (b) if (C_1, C_2) is a cut of N of cofinality (κ_1, κ_2) and κ_1, κ_2 are infinite (so $C_1, C_2 \neq 0$) then $\kappa_1 + \kappa_2 \geq \kappa$,
- (c) like clause (b) but $\kappa_1 = \kappa_2$, that is restricting ourselves to symmetric cuts.

2) We can weaken the assumption of part (1) by fixing c , as well as N, a_* . That is, assume $N \models$ “ $n < a_*$ and $c_n = c^{(a_*)^n}$ exist” for every standard n from N . For every uncountable cardinal κ the following are equivalent:

- (a)' N is boundedly κ -saturated up to c_n for each n
- (b)' if (C_1, C_2) is a cut of N of cofinality (κ_1, κ_2) with κ_1, κ_2 infinite such that $c_n \in C_2$ for some n then $\kappa_1 + \kappa_2 \geq \kappa$
- (c)' like clause (b)' but $\kappa_1 = \kappa_2$.

3) Moreover we can add in part (2):

- (c) N is strongly boundedly κ -saturated up to c .

Proof. 1) By (2).

2) (a)' \Rightarrow (b)':

Trivial.

(b)' \Rightarrow (a)':

Without loss of generality c is not standard (in N) and $n = 0$. Let $N^+ = (N, c, a_*)$ and $\tau^+ = \tau(N^+) = \tau(N) \cup \{c, a_*\}$ and $\Delta = \{\varphi(x, \bar{y}) \wedge x < c \wedge \bigwedge_{\ell} y_\ell < c : \varphi(x, \bar{y}) \in \mathbb{L}(\tau_N) \text{ is a bounded formula}\}$. We define \mathbf{r} naturally - the tree of sequences of length $< a_*$ of members of $\Delta(N_{\leq c})$ possibly non-standard but of length $< a_*$. Now apply 1.7.

(b)' \Rightarrow (c)':

Obvious.

(c)' \Rightarrow (b)':

By [MS16b].

3) We just repeat the proof of 1.7 or see 2.16 below. □_{2.13}

Question 2.14. Is a_* necessary in 2.13(1)? We conjecture that yes.

A partial answer:

Fact 2.15. If N is a model of PA, then N is κ -saturated iff $\text{cf}(|N|, <^N) \geq \kappa$ and N is boundedly κ -saturated.

Claim 2.16. If (A) then (B) where:

- (A) (a) \mathbf{r}_α is an RSP for $\alpha < \delta$,
- (b) $\Delta_{\mathbf{r}_\alpha} = \Delta$ is a set of quantifier-free formulas,
- (c) $\mathcal{T}_{\mathbf{r}_\alpha} = \mathcal{T}_{\mathbf{r}_0}$ and $N_{\mathbf{r}_\alpha}$ is increasing with α ,
- (d) $Q \in \tau(N_{\mathbf{r}_\alpha})$ and $Q^{N_{\mathbf{r}_\alpha}} = Q^{N_{\mathbf{r}_0}}$,
- (e) if $\varphi(x, \bar{y}) \in \Delta_{\mathbf{r}_\alpha}$ and $\bar{b} \in {}^{\ell g(\bar{y})}(N_{\mathbf{r}_\alpha})$ then $\varphi(N_{\mathbf{r}_\alpha}, \bar{b}) \subseteq Q^{N_{\mathbf{r}_\alpha}}$,
- (f) $\kappa = \min\{\mathfrak{p}_\alpha(\mathcal{T}_{\mathbf{r}_0}), \mathfrak{t}(\mathcal{T}_{\mathbf{r}_0})\}$.
- (B) the model $\cup\{N_{\mathbf{r}_\alpha} : \alpha < \delta\}$ is $(\kappa, 1, \Delta)$ -saturated.

Proof. As in 1.7. □_{2.16}

§ 3. CRITERION FOR ATOMIC SATURATION OF REDUCED POWERS

Malliaris-Shelah [MS16b] have dealt with such problems for ultra-filters (on sets). The main case here is $\theta = \aleph_0$.

Definition 3.1. Assume D is a filter on the complete Boolean algebra \mathfrak{B} , T an $\mathbb{L}_{\theta, \theta}(\tau_T)$ -theory, $\Delta \subseteq \mathbb{L}(\tau_T)$ and $\mu \geq |\Delta|$. We say D is a $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral filter on \mathfrak{B} (writing ε instead $\varepsilon!$ means for every $\varepsilon' < 1 + \varepsilon$; if $\mathfrak{B} = \mathcal{P}(\lambda)$ we may say good instead of moral): when for every $D - (\mu, \theta, \varepsilon!, \Delta, T)$ -problem there is a $D - (\mu, \theta)$ -solution where:

- (a) $\bar{\mathbf{a}}$ is a $D - (\mu, \theta, \varepsilon!, \Delta, T)$ -(moral)-problem when :
- (α) $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\theta} \rangle$,
 - (β) $\mathbf{a}_u \in D$ (hence $\in \mathfrak{B}^+$),
 - (γ) $\bar{\mathbf{a}}$ is \subseteq -decreasing, that is $u \subseteq v \in [\mu]^{<\theta} \Rightarrow \mathbf{a}_v \leq \mathbf{a}_u$ and $\mathbf{a}_\emptyset = 1_{\mathfrak{B}}$,
 - (δ) for some sequence $\langle \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_\alpha) : \alpha < \mu \rangle$ of formulas from Δ for every $\mathbf{a} \in \mathfrak{B}^+$ and $u \subseteq \mu$ of cardinality $< \theta$ we can find $M \models T$ and $\bar{b}_\alpha \in {}^{\ell g(\bar{y}_\alpha)} M$ for $\alpha \in u$ such that:
 - for every $v \subseteq u$ we have:

$$\mathbf{a} \leq \mathbf{a}_v \Rightarrow M \models \left(\exists \bar{x}_{[\varepsilon]} \bigwedge_{\alpha \in v} \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_\alpha) \right)$$
 and $\mathbf{a} \leq 1 - \mathbf{a}_v \Rightarrow M \models \left(\neg(\exists \bar{x}_{[\varepsilon]} \bigwedge_{\alpha \in v} \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_\alpha)) \right)$
- (b) $\bar{\mathbf{b}}$ is a $D - (\mu, \theta)$ -(moral)-solution of the $D - (\mu, \theta, \varepsilon!, \Delta, T)$ -(moral)-problem $\bar{\mathbf{a}}$ when :
- (α) $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\mu]^{<\theta} \rangle$,
 - (β) $\mathbf{b}_u \in D$ and $\mathbf{b}_\emptyset = 1_{\mathfrak{B}}$,
 - (γ) $\mathbf{b}_u \leq \mathbf{a}_u$,
 - (δ) $\bar{\mathbf{b}}$ is multiplicative, i.e. $\bar{\mathbf{b}}_u = \bigcap \{ \mathbf{b}_{\{\alpha\}} : \alpha \in u \}$ and $\mathbf{b}_\emptyset = 1_{\mathfrak{B}}$.

Remark 3.2. 1) The θ here means “a type is $(< \theta)$ -satisfiable”.

2) The use of “ $\varepsilon!$ ” is to conform with Definition 0.11.

Recall (from 0.11):

Definition 3.3. 1) Let τ be a vocabulary and $\Delta \subseteq \{ \varphi \in \mathbb{L}(\tau) : \varphi = \varphi(\bar{x}, \bar{y}) \}$ but $\varphi(\bar{x}, \bar{y}) \in \Delta$ means we can add to \bar{x} dummy variables. Let $\lambda > \theta$ (dull otherwise).

A τ -model M is $(\lambda, \theta, \varepsilon!, \Delta)$ -saturated when: if $p \subseteq \{ \varphi(\bar{x}_{[\varepsilon]}, \bar{a}) : \varphi(\bar{x}_{[\varepsilon]}, \bar{y}) \in \Delta, \bar{a} \in {}^{\ell g(\bar{y})} M \}$ has cardinality $< \lambda$ and is $(< \theta)$ satisfiable in M then p is realized in M .

Claim 3.4. 1) For a (μ, θ) -regular θ -complete ultra-filter D on a set I and θ -saturated or just $(\theta, \aleph_0, \varepsilon!, \Delta)$ -saturated model M , a cardinal μ and $\Delta = \mathbb{L}_{\theta, \theta}(\tau_M)$, the following conditions are equivalent:

- (a) D is $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral ultra-filter on the Boolean algebra $\mathcal{P}(I)$,
- (b) if $M \in \text{Mod}_T$ then M^I/D is $(\mu, \theta, \varepsilon!, \Delta)$ -saturated.

2) Similarly for D a ultra-filter on a $(< \theta)$ -distributive (see 0.12(8)) complete Boolean algebra \mathfrak{B} .

Proof. Similar to 3.5, it actually follows from it because as D is an ultra-filter, we can start with $M \models T$, expand it to M^+ by adding a predicate to any definable relation and applying 3.5 to $T^+ = \text{Th}(M^+)$. $\square_{3.4}$

Claim 3.5. 1) If (A) then (B) \Leftrightarrow (C) where:

- (A) (a) $\mathfrak{B} = \mathcal{P}(I)$,
- (b) D is a θ -complete (μ, θ) -regular filter on \mathfrak{B} ,
- (c) $\theta > \varepsilon$ or just $\mu^+ > \varepsilon$,
- (d) T is an $\mathbb{L}_{\theta, \theta}(\tau)$ -theory,
- (e) Δ is a set of conjunctions of $< \theta$ atomic formulas from $\mathbb{L}_{\theta, \theta}(\tau)$.
- (B) D is a $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral filter on \mathfrak{B} .
- (C) if M_s is a model of T for $s \in I$ then $\prod_{s \in I} M_s / D$ is $(\mu^+, \theta, \varepsilon!, \Delta)$ -saturated.

2) If (A)' then (B)' \Leftrightarrow (C)' where:

- (A)' (a) \mathfrak{B} is a $(< \theta)$ -distributive (see 0.12(8)) complete Boolean algebra,
- (b) – (e) as above (on regularity see Definition, 0.13)
- (d)⁺ T is a complete $\mathbb{L}_{\theta, \theta}(\tau)$ -theory.
- (B)' as (B) above.
- (C)' (a) if M is a model of T then $M^{\mathfrak{B}} / D$ is $(\mu^+, \theta, \varepsilon!, \Delta)$ -saturated,
- (b) if \mathcal{S} is a maximal anti-chain of \mathfrak{B} and $\bar{M} = \langle M_b : b \in \mathcal{S} \rangle$ is a sequence of τ -models then $\bar{M}^{\mathfrak{B}} / D$ is $(\mu^+, \theta, \varepsilon!, \Delta)$ -saturated.

Proof. 1) Proving (B) \Rightarrow (C): Let $N = \prod_{s \in I} M_s / D$ let $\bar{x} = \bar{x}_{[\varepsilon]}$, $\varphi_\alpha = \varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ and assume that $p(\bar{x}) = \{\varphi_\alpha(\bar{x}, \bar{b}_\alpha) : \alpha < \alpha_*\}$ is $(< \theta)$ -satisfiable in N and $|\alpha_*| \leq \mu$, so without loss of generality $\alpha_* = \mu$; without loss of generality let $\varphi_\alpha = \varphi_\alpha(\bar{x}, \bar{y}_{[\xi_\alpha]})$ so $\bar{b}_\alpha \in \xi_\alpha(\prod_{s \in I} M_s)$.

Let $\bar{b}_\alpha = \langle f_{\alpha, \xi} / D : \xi < \xi_\alpha \rangle$ where $f_{\alpha, \xi} \in \prod_{s \in I} M_s$ and for $s \in I$ let $\bar{b}_{\alpha, s} = \langle f_{\alpha, \xi}(s) : \xi < \xi_\alpha \rangle$; now for $u \in [\mu]^{< \theta}$ we let

$$(*)_0 \mathbf{a}_u := \{s \in I : M_s \models (\exists \bar{x}) \bigwedge_{\alpha \in u} \varphi_\alpha(\bar{x}, \bar{b}_{\alpha, s})\}.$$

Now

$$(*)_1 \bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{< \theta} \rangle \text{ is a } D - (\mu, \theta, \varepsilon!, \Delta, T)\text{-problem.}$$

[Why? We should check Definition 3.1, clause (a): now (a)(α) is trivial; also $\mathbf{a}_u \subseteq I$ holds by the choice of \mathbf{a}_α . Toward clause (a)(β) fix a set $u \in [\mu]^{< \theta}$; some $\bar{c} \in {}^\varepsilon N$ realizes the type $p_u(\bar{x}_{[\varepsilon]}) = \{\varphi_\alpha(\bar{x}, \bar{b}_\alpha) : \alpha \in u\}$ in N because $p(\bar{x})$ is $(< \theta)$ -satisfiable in N , see Definition 3.3, so let $\bar{c} = \langle g_\zeta / D : \zeta < \varepsilon \rangle$ for some $g_\zeta \in \prod_{s \in I} M_s$ for $\zeta < \varepsilon$

and let $\bar{c}_s = \langle g_\zeta(s) : \zeta < \varepsilon \rangle \in {}^\varepsilon(M_s)$. So $\mathbf{a}'_{\{\alpha\}} = \{s \in I : M \models \varphi_\alpha[\bar{c}_s, \bar{b}_s]\}$ belong to D because $N \models \varphi_\alpha[\bar{c}, \bar{b}_\alpha]$ by the definition of N if φ_α is atomic, but recalling D is θ -complete also for our φ_α , remembering clause (A)(e) of 3.5(1). As D is θ -complete clearly, $\mathbf{a}'_u = \cap \{\mathbf{a}'_{\{\alpha\}} : \alpha \in u\}$ belongs to D and by our choices, $\mathbf{a}'_u \subseteq_{\mathfrak{B}} \mathbf{a}_u$, hence $\mathbf{a}_u \in D$ so subclause (a)(β) of Def 3.1 holds indeed.

By the choice of \mathbf{a}_u , $\bar{\mathbf{a}}$ is \subseteq -decreasing so subclause (a)(γ) of Def 3.1 holds.

Lastly, subclause (a)(δ) of Def 3.1 holds by the definition of \mathbf{a}_u 's recalling $p(\bar{x})$ is ($< \theta$)-satisfiable (and $\emptyset \notin D$).

(*)₂ there is $\bar{\mathbf{b}}$, a $D - (\mu, \theta)$ -solution of \mathbf{a} in \mathfrak{B} .

[Why? Because we are presently assuming clause (B) of 3.5 which says that D is $(\mu, \theta, \varepsilon!, \Delta, T)$ -good, see Definition 3.1.]

(*)₃ without loss of generality $s \in I \Rightarrow \{\alpha < \mu : s \in \mathbf{b}_{\{\alpha\}}\}$ has cardinality $< \theta$.

[Why? As D is (μ, θ) -regular.]

Next for $s \in I$ let $u_s = \{\alpha < \mu : s \in \mathbf{b}_{\{\alpha\}}\}$ but $\bar{\mathbf{b}}$ is multiplicative (see 3.1(b)(δ)) so $\mathbf{b}_{u_s} = \bigcap \{\mathbf{b}_{\{\alpha\}} : \alpha \in u_s\} = \bigcap \{\mathbf{b}_\alpha : \text{the ordinal } \alpha \text{ satisfies } s \in \mathbf{b}_{\{\alpha\}}\}$ hence $s \in \mathbf{b}_{u_s}$ hence (see 3.1(b) recalling that $|u_\alpha| < \theta$ by (*)₂) we have $s \in \mathbf{a}_{u_s}$ hence (by the choice of \mathbf{a}_{u_s}) there is $\bar{a}_s \in {}^\varepsilon(M_s)$ realizing $\{\varphi(\bar{x}_{[\varepsilon]}, \langle f_{\alpha, \varepsilon_\zeta}(s) : \zeta < \varepsilon \rangle) : \alpha \in u_s\}$.

Let $\bar{a}_s = \langle a_{s, \zeta} : \zeta < \varepsilon \rangle$. Now for $\zeta < \varepsilon = \ell g(\bar{x})$ let $g_\zeta \in \prod_{s \in I} M_s$ be defined by $g_\zeta(s) = a_{s, \zeta} \in M_s$ and let $\bar{a} = \langle g_\zeta/D : \zeta < \varepsilon \rangle$ noting $g_\zeta/D \in \prod_{s \in I} M_s/D = N$.

Hence for every $\alpha < \mu$, $\{s \in I : M_s \models \varphi_\alpha(\langle g_\zeta(s) : \zeta < \varepsilon \rangle, \bar{b}_{\alpha, s})\} \supseteq \mathbf{b}_{\{\alpha\}} \in D$ so $N \models \varphi[\bar{a}, \bar{b}_\alpha]$.

Hence \bar{a} realizes $p(\bar{x})$ in N as promised.

Proving (C) \Rightarrow (B):

To prove clause (B), let $\bar{\mathbf{a}}$ be a $D - (\mu, \theta, \varepsilon!, \Delta, T)$ -problem and let $\bar{\varphi} = \langle \varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_\alpha) : \alpha < \mu \rangle$ be a sequence of formulas from Δ as in clause (a)(δ) of Definition 3.1.

As D is (λ, θ) -regular, we can choose $\bar{w} = \langle w_s : s \in I \rangle$ a sequence of subsets of μ each of cardinality $< \theta$ such that $\alpha < \mu \Rightarrow \{s \in I : \alpha \in w_s\} \in D$. For $u \in [\mu]^{< \theta}$ let $\mathbf{c}_u = \{s \in I : u \subseteq w_s\}$, so clearly $\mathbf{c}_u \in D$ and $\langle \mathbf{c}_u : u \in [\lambda]^{< \theta} \rangle$ is multiplicative.

For each $s \in I$ applying Definition 3.1(a)(δ) to $\mathbf{a} = \{s\}$ and $u = w_s$ we can find a model M_s of T and $\bar{b}_{s, \alpha} \in {}^{\ell g(\bar{y}_\alpha)}(M_s)$ for $\alpha \in w_s$ satisfying \bullet there.

Now choose $\bar{b}_{s, \alpha}$ also for $s \in I, \alpha \in \mu \setminus w_s$, as any sequence of members of M_s of length $\ell g(\bar{y}_\alpha)$. Now for every $\alpha < \mu$ and $j < \ell g(\bar{y}_\alpha)$ we define $g_{\alpha, j} \in \prod_{s \in I} M_s$ by

$$g_{\alpha, j}(s) = (\bar{b}_{s, \alpha})_j.$$

Hence $g_{\alpha, \zeta}/D \in \prod_{s \in I} M_s/D = N$ and $\bar{b}_\alpha = \langle g_{\alpha, \zeta}/D : \zeta < \ell g(\bar{y}_\alpha) \rangle \in {}^{\ell g(\bar{y}_\alpha)}N$ and

consider the set $p = \{\varphi_\alpha(\bar{x}, \bar{b}_\alpha) : \alpha < \mu\}$. Is p a ($< \theta$)-satisfiable type in N ? We shall prove that Yes, so let $u \in [\mu]^{< \theta}$, then recall $\mathbf{c}_u = \{s \in I : u \subseteq w_s\} \in D$ and $s \in \mathbf{c}_u \cap \mathbf{a}_u \Rightarrow \{\varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_{s, \alpha}) : \alpha \in u\}$ is realized in M_s , [why? by the choice of $\langle \bar{b}_{s, \alpha} : \alpha \in w_s \rangle$.]

So let the type $\{\varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_{s, \alpha}) : \alpha \in w_s\}$ be realized $\bar{a}_s = \langle a_{s, \zeta} : \zeta < \varepsilon \rangle$; for $s \in I$ and let $f_{\alpha, \zeta} \in \prod_{s \in I} M_s$ be $f_{\alpha, \zeta}(s) = a_{s, \zeta}$. Easily $\langle f_{\alpha, j}/D : \zeta < \varepsilon \rangle$ realizes

$\{\varphi_\alpha(\bar{x}_{[\varepsilon]}) : \alpha \in u\}$ because $\mathbf{a}_u \cap \mathbf{c}_u \in D$. Hence $p(\bar{x}_{[\varepsilon]})$ is ($< \theta$)-satisfiable indeed.

Next, we apply clause (C) we are assuming hence $p(\bar{x}_{[\varepsilon]})$ is realized in N . So let $\bar{a} = \langle a_\zeta : \zeta < \varepsilon \rangle \in {}^\varepsilon N$ realize p and let $a_\zeta = h_\zeta/D$ where $h_\zeta \in \prod_{s \in I} M_s$ and lastly

let:

$$\mathbf{b}_u = \{s \in I : M_s \models \varphi_\alpha[\langle h_\zeta(s) : \zeta < \varepsilon \rangle, \bar{b}_{s, \alpha}]\} \text{ for every } \alpha \in u \text{ and } s \in \mathbf{c}_u\}.$$

Now check that $\langle \mathbf{b}_u : u \in [\lambda]^{<\theta} \rangle$ is as required, recalling $\langle \mathbf{c}_u : u \in [\lambda]^{<\theta} \rangle$ is multiplicative. So the desired conclusion of 3.1(B) holds indeed so we are done proving $(C) \Rightarrow (B)$.

2) Similarly; e.g. for clause (a) let $p(\bar{x})$ be as there but

- $f_{\alpha,\xi} \in M^{\mathfrak{B}}$ is supported by the maximal anti-chain $\langle \mathbf{c}_{\alpha,\xi,i} : i < i(\alpha,\xi) \rangle$
- (*)₀ $\mathbf{a}_u = \sup\{\mathbf{c} : \text{we have } \alpha \in u \wedge \xi < \xi_\alpha \Rightarrow (\exists \mathbf{d})(\mathbf{d} \in \text{dom}(f_{\alpha,\xi}) \wedge \mathbf{c} \leq \mathbf{d}) \text{ and } M \models (\exists \bar{x}_{[\varepsilon]}) \bigwedge_{\alpha \in u} \varphi(\bar{x}_{[\varepsilon]}, \langle f_{\alpha,\xi}(\mathbf{c}) : \xi < \xi_\alpha \rangle)\}$
- (*)₁ $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\theta} \rangle$ is a $D - (\mu, \theta, \varepsilon!, \Delta, T)$ -problem.

[Why? As there.]

- (*)₂ let $\bar{\mathbf{b}}$ be a $D - (\mu, \theta)$ -solution.

[Why does $\bar{\mathbf{b}}$ exist? By $(B)'$ recalling Definition 3.1.]

Also the rest is as above. □_{3.5}

Remark 3.6. If $\mathcal{S} \subseteq [\mu]^{<\theta}$ is cofinal, $u \in [\mu]^{<\theta} \Rightarrow |\mathcal{P}(u) \cap \mathcal{S}| < \theta_1$ we may consistently replace $[\mu]^{<\theta}$ by \mathcal{S} and 2^{θ_1} by θ_1 .

Definition 3.7. 1) A filter D on a complete Boolean algebra \mathfrak{B} is (μ, θ) -excellent when: if $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\theta} \rangle$ is a sequence of members of \mathfrak{B} , (yes! not necessarily from D) then we can find $\bar{\mathbf{b}}$ which is a multiplicative refinement of $\bar{\mathbf{a}}$ for D , meaning:

- (a) $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\mu]^{<\theta} \rangle$,
- (b) $\mathbf{b}_u \leq \mathbf{a}_u$ and $\mathbf{b}_u = \mathbf{a}_u \pmod{D}$,
- (c) if $\mathbf{a}_{u_1} \cap \mathbf{a}_{u_2} = \mathbf{a}_{u_1 \cap u_2} \pmod{D}$ then $\mathbf{b}_{u_1} \cap \mathbf{b}_{u_2} = \mathbf{b}_{u_1 \cap u_2}$.

2) For a Boolean algebra \mathfrak{B} and filter D on \mathfrak{B} we say $\bar{\mathbf{a}}$ is a $D - (\mu, \theta)$ -problem (or a $D - (\mu, \theta)$ -moral problem) when clauses (a)(α), (β), (γ) of Definition 3.1 holds.

3) A filter D on a complete Boolean algebra \mathfrak{B} is (μ, θ) -good when every $D - (\mu, \theta)$ -problem has a $D - (\mu, \theta)$ -solution

Claim 3.8. 1) Assuming (*) below, the filter D on I (i.e. on the Boolean algebra $\mathcal{P}(I)$) is $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral iff the filter D_1 on \mathfrak{B}_1 is $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral where:

- (*) (a) \mathfrak{B}_1 is a complete Boolean algebra,
- (b) \mathbf{j} is a homomorphism from $\mathcal{P}(I)$ onto \mathfrak{B}_1 ,
- (c) $D_0 = \{A \subseteq I : \mathbf{j}(A) = 1_{\mathfrak{B}_1}\}$ is a (μ, θ) -excellent filter on I ,
- (d) D_1 is a filter on \mathfrak{B}_1 ,
- (e) $D = \{A \subseteq I : \mathbf{j}(A) \in D_1\}$. is a filter on I

2) We can replace $\mathcal{P}(I)$ by a complete Boolean algebra \mathfrak{B}_2 .

Proof. The “if” direction:

We assume D_1 is $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral and should prove it for D . So let $\bar{A} = \langle A_u : u \in [\mu]^{<\theta} \rangle$ be a $D - (\mu, \theta, \varepsilon!, \Delta, T)$ -problem and we should find a $D - (\mu, \theta)$ -solution \bar{B} of it.

Clearly $\mathbf{a}_u := \mathbf{j}(A_u) \in \mathfrak{B}_1^+$ and $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\theta} \rangle = \langle \mathbf{j}(A_u) : u \in [\mu]^{<\theta} \rangle$ is a $D_1 - (\mu, \theta, \varepsilon!, \Delta, T)$ -problem.

Hence by our present assumption (D_1 is $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral) there is a $D_1 - (\mu, \theta)$ -solution $\bar{\mathbf{b}}$ of $\bar{\mathbf{a}}$, let $\bar{\mathbf{b}} = \langle \mathbf{b}_u : u \in [\mu]^{<\theta} \rangle$ so in particular $u \in [\mu]^{<\theta} \Rightarrow \mathbf{b}_u \in D_1$. For $u \in [\mu]^{<\theta}$ choose $B_u^1 \subseteq I$ such that $\mathbf{j}(B_u^1) = \mathbf{b}_u$, possible because \mathbf{j} is a homomorphism from $\mathcal{P}(I)$ onto \mathfrak{B}_1 . So $\bar{B}^1 = \langle B_u^1 : u \in [\mu]^{<\theta} \rangle$ is a multiplicative modulo D_0 , i.e. $\langle B_u^1/D_0 : u \in [\mu]^{<\theta} \rangle$ is a multiplicative sequence of members of $\mathcal{P}(I)/D_0$.

Let $B_u^2 = B_u^1 \cap A_u$, let

- $B_u^1 \subseteq A_u \pmod{D_0}$.

[Note that we have written B_u^1 and not B_u^2 . So why this statement holds? As $\mathbf{j}(B_u^1) = \mathbf{b}_u \leq \mathbf{a}_u = \mathbf{j}(A_u)$.]

- $B_u^2 \subseteq B_u^1$ and $B_u^2 \subseteq A_u \pmod{D_0}$,
- $B_u^2 \in D$,
- $\langle B_u^2 : u \in [\mu]^{<\theta} \rangle$ is multiplicative modulo D_0 (see 3.7).

By Definition 3.7(1) applied to $\langle B_u^2 : u \in [\mu]^{<\theta} \rangle$ recalling clause (c) of the assumption of the claim, we can find $\bar{B} = \langle B_u : u \in [\mu]^{<\theta} \rangle$ which is a multiplicative refinement of \bar{B}^2 and is multiplicative, and $B_u \in D$ because $B_n = B_n^2$ modulo $D_0 \subseteq D$ and $B_u^2 \in D$.

So we are done for the “if” direction.

The “only if” direction:

So we are assuming D is a $(\mu, \theta, \varepsilon!, \Delta, T)$ -good filter on I and we have to prove D_1 is $(\mu, \theta, \varepsilon!, \Delta, T)$ -moral.

So let $\bar{\mathbf{a}}$ be a $D_1 - (\mu, \theta, \varepsilon!, \Delta, T)$ -moral problem (on \mathfrak{B}_1), we have to find a solution. For $u \in [\mu]^{<\theta}$ choose $A_u^1 \subseteq I$ such that $\mathbf{j}(A_u^1) = \mathbf{a}_u$, so $A_u^1 \in D$ (by clause (e)) and $u \subseteq v \in [\mu]^{<\theta} \rightarrow A_u^1 \subseteq A_v^1$ modulo D_0 . Now by 3.7, i.e. clause (b) of the assumption of the claim there is $\bar{A}^2 = \langle A_u^2 : u \in [\mu]^{<\theta} \rangle$ such that $A_u^2 \subseteq A_u^1, A_u^2 = A_u^1 \pmod{D_0}$ hence $A_u^2 \in D$ and \bar{A}^2 is \subseteq -decreasing [Why? Because \bar{A}^1 is \subseteq -decreasing modulo D_0 as $\bar{\mathbf{a}}$ is decreasing hence \bar{A}^2 is \subseteq -decreasing.]

As D is $(\mu, \theta, \varepsilon!, \Delta, T)$ -good filter on I there is a D -multiplicative refinement $\langle B_u^2 : u \in [u]^{<\theta} \rangle$ of $\langle A_u^2 : u \in [\mu]^{<\theta} \rangle$. Let $\mathbf{b}_u = \mathbf{j}(B_u^2)$, now $\langle \mathbf{b}_u : u \in [\mu]^{<\theta} \rangle$ is as required.

2) Similarly. □_{3.8}

Claim 3.9. *Let D be a filter on I .*

- 1) D is (μ, θ) -excellent implies D is (μ, θ) -good, see 3.7(3).
- 2) D is (μ, θ) -good implies D is $(\mu, \theta, \varepsilon, \Delta, T)$ -moral.

Proof. 1) So let $\bar{\mathbf{a}} = \langle \mathbf{a}_u : u \in [\mu]^{<\theta} \rangle$ be a D -problem and we should find a $D - (\mu, \theta)$ -solution $\bar{\mathbf{b}}$ below $\bar{\mathbf{a}}$. As D is (μ, θ) -excellent we apply this to $\bar{\mathbf{a}}$ and $\bar{\mathbf{b}}$ as in 3.7(2). Easily it is as required.

2) Just read the definitions: there are fewer problems. □_{3.9}

Remark 3.10. We may wonder, e.g. in 3.5(1): can we remove the regularity demand on the filter D from clause (A) to clause (B)? The answer is yes for most T 's.

Claim 3.11. *The filter D is (μ, θ) -regular when:*

- (A) (a) $\mathfrak{B} = \mathcal{P}(I)$

- (b) D is a θ -complete ultra-filter on \mathfrak{B} ,
- (c) the demand $\theta > \varepsilon$, is natural but not actually required,
- (d) T is a complete $\mathbb{L}_{\theta,\theta}(\tau)$ -theory, e.g. $T = \text{Th}_{\mathbb{L}_{\theta,\theta}}(M)$, M a θ -saturated model (note that $T = T_0^{[\theta]}$ where $T_0 = \text{Th}_{\mathbb{L}_{\aleph_0,\aleph_0}}(M)$, i.e. T is determined by T_0 and θ),
- (B) T has a model M and $p = \{\varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{b}_\alpha) : \alpha < \mu\}$, $\varphi_\alpha(\bar{x}_{[\varepsilon]}, \bar{y}_\alpha) \in \mathbb{L}_{\theta,\theta}$, $\bar{b}_\alpha \in {}^{\ell g(\bar{y}_\alpha)}M$ satisfying:
 - for every $q \subseteq p$,
 - q is realized in M iff $|q| < \theta$.
- (C) if M_s is a model of T for $s \in I$ then $\prod_s M_s/D$ is $(\mu^+, \theta, \varepsilon!, \Delta)$ -saturated.

Proof. Should be clear.

□_{3.11}

§ 4. A COUNTEREXAMPLE

In §2 we generalize [She90, Ch.VI, 2.6] to filters, using the class of relevant RSP's \mathbf{r} being closed under reduced powers (being a Horn class, see 2.2). Can we generalize the result of Malliaris-Shelah [MS16b]? Here we give a counter-example.

For this, we have to find:

- (*)₁ D a filter of λ such that the partial order $N_1 = (\mathbb{Q}, <)^{\lambda}/D$ satisfies $\mathfrak{p}^*(N_1) = \kappa_1 + \kappa_2 < \mu^+ \leq \mathfrak{p}_{\text{sym}}^*(N_1)$, $\kappa_1 \neq \kappa_2$, $(\kappa_1, \kappa_2) \in \mathcal{C}(N_1)$, so in fact N_1 has no (θ_1, θ_2) -cut when $\theta_1 = \text{cf}(\theta_1) = \theta_2 \leq \mu$ and when $\theta_\ell \geq \mu^+ \wedge \theta_{3-\ell} \in \{0, 1\}$,
- (*)₂ preferably: $\lambda = \mu$
- (*)₃ or at least for some dense linear order M_0 there is a complete Boolean algebra \mathfrak{B} and a filter D on \mathfrak{B} such that $N_0 = M_0^{\mathfrak{B}}/D$ is as above.

We presently deal with the (main) case $\theta = \aleph_0$ and carry this out. It seems reasonable that we can prove, e.g. $T_{\text{ceq}} \not\leq_{\text{rp}} T_{\text{ord}}$ but we have not arrived to it; see [She23] on T_{ceq} and [She93] on the closely related T_{feq} . Later we hope to say more. Clearly, we can control the set of non-symmetric pre-cuts.

Convention 4.1. T_{ord} is the first order theory of (\mathbb{Q}, \leq) , see 4.4(1)(d).

Definition 4.2. Let κ be a regular uncountable cardinal.

1) Let K_{κ}^{ba} be the class of \mathbf{m} such that:

- (a) $\mathbf{m} = (\mathfrak{B}, D) = (\mathfrak{B}_{\mathbf{m}}, D_{\mathbf{m}})$,
- (b) \mathfrak{B} is a complete Boolean algebra satisfying the κ -c.c.,
- (c) D is a filter on \mathfrak{B} .

2) Let $\leq_{\kappa}^{\text{ba}}$ be the following two-place relation on K_{κ}^{ba} : $\mathbf{m} \leq_{\kappa}^{\text{ba}} \mathbf{n}$ iff:

- (a) $\mathbf{m}, \mathbf{n} \in K_{\kappa}^{\text{ba}}$,
- (b) $\mathfrak{B}_{\mathbf{m}} \leq \mathfrak{B}_{\mathbf{n}}$,
- (c) $D_{\mathbf{m}} = D_{\mathbf{n}} \cap \mathfrak{B}_{\mathbf{m}}$.

3) Let S_{κ}^{ba} be the class of $\leq_{\kappa}^{\text{ba}}$ -increasing continuous sequences $\bar{\mathbf{m}}$ which means:

- (a) $\bar{\mathbf{m}} = \langle \mathbf{m}_{\alpha} : \alpha < \text{lg}(\bar{\mathbf{m}}) \rangle$,
- (b) $\mathbf{m}_{\alpha} \in K_{\kappa}^{\text{ba}}$,
- (c) if $\alpha < \beta < \text{lg}(\bar{\mathbf{m}})$ then $\mathbf{m}_{\alpha} \leq_{\kappa}^{\text{ba}} \mathbf{m}_{\beta}$,
- (d) if $\beta < \text{lg}(\bar{\mathbf{m}})$ is a limit ordinal then:
 - (α) $\mathfrak{B}_{\mathbf{m}_{\beta}}$ is the completion of $\cup\{\mathfrak{B}_{\mathbf{m}_{\alpha}} : \alpha < \beta\}$,
 - (β) $D_{\mathbf{m}_{\beta}}$ is generated (as a filter) by $\cup\{D_{\mathbf{m}_{\alpha}} : \alpha < \beta\}$.

4) If $\kappa = \aleph_1$ we may write $K_{\text{ba}}^1, \leq_{\text{ba}}^1, S_{\text{ba}}^1$, and if $\kappa = \infty$ we may write $K_{\text{ba}}^2, \leq_{\text{ba}}^2, S_{\text{ba}}^2$ or $K_{\infty}^{\text{ba}}, \leq_{\infty}^{\text{ba}}, S_{\infty}^{\text{ba}}$.

5) We say \mathbf{m} is of cardinality λ when $\mathfrak{B}_{\mathbf{m}}$ is of cardinality λ .

Claim 4.3. 1) For every λ there is $\mathbf{m} \in K_{\kappa}^{\text{ba}}$ of cardinality $\lambda^{<\kappa}$.

2) $\leq_{\kappa}^{\text{ba}}$ is a partial order on K_{κ}^{ba} .

3) If $\bar{\mathbf{m}} = \langle \mathbf{m}_{\alpha} : \alpha < \delta \rangle$ is a $\leq_{\kappa}^{\text{ba}}$ -increasing continuous sequence, then for some \mathbf{m}_{δ} , the sequence $\bar{\mathbf{m}} \hat{\ } \langle \mathbf{m}_{\delta} \rangle$ is $\leq_{\kappa}^{\text{ba}}$ -increasing continuous.

Proof. 1) E.g. \mathfrak{B}_m is the completion of a free Boolean algebra generated by $\lambda^{<\kappa}$ elements.

2) Easy.

3) If $\text{cf}(\delta) \geq \kappa$, then $\mathfrak{B}_{m_\delta} = \bigcup_{\alpha < \delta} \mathfrak{B}_{m_\alpha}$, if $\text{cf}(\delta) < \kappa$ it is the (pendantically a) completion of the union. D_{m_δ} is the filter generated by $\bigcup \{D_{m_\alpha} : \alpha < \delta\}$. Classically κ -c.c. is preserved. $\square_{4.3}$

Definition 4.4. Let $m \in K_{\text{ba}}^2$ and κ_1, κ_2 are (infinite) regular cardinals.

1) We say \bar{a} is a $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral problem in m when:

- (a) $m \in K_{\text{ba}}^2$, (actually already assumed),
- (b) $I = I(\kappa_1, \kappa_2)$ is the linear order $I_1 + I_2$ where:
 - $I_1 = I_1(\kappa_1) = (\{1\} \times \kappa_1)$,
 - $I_2 = I_2(\kappa_2) = (\{2\} \times \kappa_2^*)$,
- (c) $\bar{a} = \langle \mathbf{a}_{s,t} : s <_{I(\kappa_1, \kappa_2)} t \rangle$ is a sequence of members of D_m ,
- (d) if $u \subseteq I$ is finite, $\mathbf{t} : u \times u \rightarrow \{0, 1\}$ and $\bigcap \{\mathbf{a}_{s,t}^{\text{if}(t(s,t))} : s, t \in u\} > 0_m$, then there is a function $f : u \rightarrow \{0, \dots, |u| - 1\}$ such that:
 - if $s, t \in u$ then $\mathbf{t}(s, t) = 1$ iff $f(s) \leq f(t)$.
- (e) hence $s_1 <_I s_2 <_I s_3 \Rightarrow \mathbf{a}_{s_1, s_2} \cap \mathbf{a}_{s_2, s_3} \leq \mathbf{a}_{s_1, s_3}$ and we stipulate $\mathbf{a}_{s,s} = 1_{\mathfrak{B}_m}$, $\mathbf{a}_{t,s} = \mathbf{a}_{s,t}$ when $s <_I t$.

2) We say \bar{b} is a solution of \bar{a} in m where \bar{a} is as above when:

- (a) $\bar{b} = \langle \mathbf{b}_s : s \in I \rangle$,
- (b) $\mathbf{b}_s \in D_m$,
- (c) if $s_1 \in I_1, s_2 \in I_2$ then $\mathbf{b}_{s_1} \cap \mathbf{b}_{s_2} \leq \mathbf{a}_{s_1, s_2}$.

Definition 4.5. 1) For $\iota = 1, 2$ let \mathbf{S}_ι be the class of tuples $\mathbf{s} = (I, D_0, \mathbf{j}, \mathfrak{B}, D_1, D)$ such that:

- (a) \mathbf{j} is a homomorphism from $\mathcal{P}(I)$ onto the complete Boolean algebra \mathfrak{B} ,
- (b) D_1 is a filter on \mathfrak{B} ,
- (c) $D_0 = \{A \subseteq I : \mathbf{j}(A) = 1_{\mathfrak{B}}\}$ (or see §3),
- (d) $D = \{A \subseteq I : \mathbf{j}(A) \in D_1\}$,
- (e) the pair (\mathfrak{B}, D) belongs to K_{ba}^ι .

2) For $\mathbf{s} \in \mathbf{S}$ let $m_s = (\mathfrak{B}_s, D_s)$.

3) We say $\mathbf{s} \in \mathbf{S}$ is (μ, θ) -excellent (if $\theta = \aleph_0$ may omit it) when D_0 is an excellent filter on I , see Definition 3.7(2).

4) We say $\mathbf{s} \in \mathbf{S}$ is (μ, θ) -regular (if $\theta = \aleph_0$ we may omit θ) when D_0 is a (μ, θ) -regular filter.

5) Let $\mathbf{S}_{\mu, \theta}^\iota$ be the class of (μ, θ) -excellent (μ, θ) -regular $\mathbf{s} \in \mathbf{S}_\iota$; we may omit θ if $\theta = \aleph_0$.

6) Let $\mathbf{S}_{\mu, \theta, \kappa}$ be the class of $\mathbf{s} \in \mathbf{S}_{\mu, \theta}^2$ such that \mathfrak{B}_s satisfies the κ -c.c.

Claim 4.6. 1) Assume $m = (\mathfrak{B}, D) \in K_{\text{ba}}$ and κ_1, κ_2 are infinite and regular cardinals. Then for some $M \in \text{Mod}_{T_{\text{ord}}}$, $M^{\mathfrak{B}}/D$ has a (κ_1, κ_2) -pre-cut iff some $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral problem in m has no solution.

2) Let $\mu \geq \aleph_0 = \theta$. If $\mathbf{s} \in \mathbf{S}_{\mu, \theta}$ so is μ -excellent and μ -regular and $\kappa_1, \kappa_2 \geq \aleph_0$ are regular and $\kappa_1 + \kappa_2 \leq \mu$ then the following conditions are equivalent:

- (a) for some linear order M , $M^{I(\mathbf{s})}/D_{\mathbf{s}}$ has a (κ_1, κ_2) -pre-cut,
- (b) for every infinite linear order, $M^{I(\mathbf{s})}/D_{\mathbf{s}}$ has a (κ_1, κ_2) -pre-cut,
- (c) not every $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral problem in $\mathbf{m}_{\mathbf{s}}$ has a solution.

Proof. As in the proof of 3.5(1), relying on Def 4.4 instead of Def 3.1; recalling

- ▮ if M_s^t for $s \in I, t \in \{1, 2\}$ are τ -models, $|\tau| \leq \mu$, D a μ -regular filter on I and M_s^1, M_s^2 are elementarily equivalent, then $N_1 = \prod_{s \in I} M_s^1/D, N_2 = \prod_{s \in I} M_s^2/D$ are $\mathbb{L}_{\mu^+, \mu^+}$ -equivalent (and more, see Kennedy-Shelah [KS02], [KS04] and Kennedy-Shelah-Vaananen [KSV08] on the subject).

□_{4.6}

Observation 4.7. Assume $\mathbf{m} \in K_{\text{ba}}^2$ and $\bar{\mathbf{a}}$ is a $T_{\text{ord}} - (\kappa_1, \kappa_1)$ -moral problem for \mathbf{m} so (see 4.5(5)) $I_\ell = I_\ell(\kappa_\ell)$ for $\ell = 1, 2$.

1) If $I'_1 \subseteq I_1$ is cofinal in I_1 and $I'_2 \subseteq I_2$ is co-initial in I_2 then $\bar{\mathbf{a}}$ has a solution in \mathbf{m} iff $\bar{\mathbf{a}}' = \bar{\mathbf{a}} \upharpoonright (I'_1 + I'_2) = \langle \mathbf{a}_{s,t} : s <_I t \text{ and } s, t \in I'_1 + I'_2 \rangle$ has a solution in \mathbf{m} .

1A) Also, above, if $\bar{\mathbf{b}}$ is a solution of $\bar{\mathbf{a}}$ in \mathbf{m} , then $\bar{\mathbf{b}} \upharpoonright (I'_1 + I'_2)$ is a solution of $\bar{\mathbf{a}}'$.

1B) Also above, if $\bar{\mathbf{b}}'$ is a solution of $\bar{\mathbf{a}}'$, then $\bar{\mathbf{b}}$ is a solution of $\bar{\mathbf{a}}$ when:

- (a) if $s \in I_1$ and $t \in I'_1$ is minimal such that $s \leq_I t$ then $\mathbf{b}_s = \mathbf{b}'_t \cap \mathbf{a}_{s,t}$ if $s <_I t$ and $\mathbf{b}_s = \mathbf{b}'_t$ if $s = t$,
- (b) like (a) replacing $I_1, I'_1, s <_I t, \mathbf{a}_{s,t}$ by $I_2, I'_2, t \leq_I s, \mathbf{a}_{t,s}$.

2) If $\bar{\mathbf{b}}$ is a solution of $\bar{\mathbf{a}}$ in \mathbf{m} and $\mathbf{b}'_s \in D \wedge \mathbf{b}'_s \leq \mathbf{b}_s$ for $s \in I_1 + I_2$ then $\langle \mathbf{b}'_s : s \in I \rangle$ is a solution of $\bar{\mathbf{a}}$ for \mathbf{m} .

Proof. 1) Easy using the proofs of 3.5, 4.6 or using (1A), (1B).

1A), 1B), 2) Check.

□_{4.7}

A key point in the inductive construction is:

Claim 4.8. There is no solution to $\bar{\mathbf{a}}$ in \mathbf{m}_δ when:

- (a) $\bar{\mathbf{m}} = \langle \mathbf{m}_\alpha : \alpha \leq \delta \rangle \in S_{\text{ba}}^2$,
- (b) $\bar{\mathbf{a}}$ is a $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral problem in \mathbf{m}_0 ,
- (c) if $\alpha < \delta$ then $\bar{\mathbf{a}}$ has no solution in \mathbf{m}_α ,
- (d) $\text{cf}(\delta) \neq \kappa_1$ or $\text{cf}(\delta) \neq \kappa_2$.

Proof. Let $\mathbf{m}_\gamma = (\mathfrak{B}_\gamma, D_\gamma)$ for $\gamma \leq \delta$; by symmetry without loss of generality $\text{cf}(\delta) \neq \kappa_1$ and toward contradiction assume $\bar{\mathbf{b}} = \langle \mathbf{b}_s : s \in I_1 + I_2 \rangle$ is a solution of $\bar{\mathbf{a}}$ in \mathbf{m}_δ .

Hence $\mathbf{b}_s \in D$. Now D_δ is not necessarily equal to $\bigcup_{\gamma < \delta} D_\gamma$ but recalling 4.2(3)(d)(β) and $\langle D_\gamma : \gamma < \delta \rangle$ being increasing, clearly, every member of D_δ is above some member of $\bigcup_{\gamma < \delta} D_\gamma$.

So by Observation 4.7(2), without loss of generality $s \in I_1 + I_2 \Rightarrow \mathbf{b}_s \in \bigcup_{\gamma < \delta} D_\gamma \subseteq$

$$\bigcup_{\gamma < \delta} \mathfrak{B}_\gamma.$$

As $\text{cf}(\delta) \neq \kappa_1$, for some $\gamma < \delta$ we have $\kappa_1 = \sup\{\alpha < \kappa_1 : \mathbf{b}_{(1,\alpha)} \in \mathfrak{B}_\gamma\}$, i.e. $\{s \in I_1 : \mathbf{b}_s \in \mathfrak{B}_\gamma\}$ is co-final in I_1 . So by 4.7(1) without loss of generality

$$(a) \ s \in I_1 \Rightarrow \mathbf{b}_s \in \mathfrak{B}_\gamma.$$

As $D_\gamma = D_\delta \cap \mathfrak{B}_\gamma$ by 4.2(2)(c) clearly

$$(b) \ s \in I_1 \Rightarrow \mathbf{b}_s \in D_\gamma.$$

For $t \in I_2$ let $\mathbf{b}'_t = \min\{\mathbf{b} \in \mathfrak{B}_\gamma : \mathfrak{B}_\delta \models \mathbf{b}_t \leq \mathbf{b}\}$, well defined because \mathfrak{B}_γ is complete.

Now

$$(c) \ \mathbf{b}'_t \in D_\gamma \text{ for } t \in I_2.$$

[Why? Clearly $\mathbf{b}_t \in \mathfrak{B}_\delta$ as $\bar{\mathbf{b}}$ is a solution of $\bar{\mathbf{a}}$ in \mathbf{m}_δ and $\mathbf{b}_t \leq \mathbf{b}'_t, \mathbf{b}'_t \in \mathfrak{B}_\gamma$ by its choice. Also $\mathbf{b}'_t \in D_\delta$ because $\mathbf{b}_t \leq \mathbf{b}'_t \wedge \mathbf{b}_t \in D_\delta$ and D_δ is a filter on \mathfrak{B}_δ and lastly $\mathbf{b}'_t \in D_\gamma$ as $D_\gamma = D_\delta \cap \mathfrak{B}_\gamma$.]

$$(d) \ \text{if } s \in I_1, t \in I_2 \text{ then } \mathbf{b}_s \cap \mathbf{b}'_t \leq \mathbf{a}_{s,t}.$$

[Why? Note $\mathfrak{B}_\delta \models \mathbf{b}_s \cap \mathbf{b}_t \leq \mathbf{a}_{s,t}$ because $\bar{\mathbf{b}}$ is a solution of $\bar{\mathbf{a}}$ in \mathfrak{B}_δ hence $\mathbf{b}_t \leq \mathbf{a}_{s,t} \cup (1 - \mathbf{b}_s)$ and the later $\in \mathfrak{B}_\gamma$. So by the choice of $\mathbf{b}'_t, \mathbf{b}'_t \leq \mathbf{a}_{s,t} \cup (1 - \mathbf{b}_s)$ hence $\mathbf{b}_s \cap \mathbf{b}'_t \leq \mathbf{a}_{s,t}$.]

$$(e) \ \langle \mathbf{b}_s : s \in I_1 \rangle \wedge \langle \mathbf{b}'_t : t \in I_2 \rangle \text{ solves } \bar{\mathbf{a}} \text{ in } \mathfrak{B}_\gamma.$$

[Why? By (a) + (b) + (c) + (d).]

But this contradicts an assumption. □_{4.8}

Definition 4.9. Assume $\mathbf{m} \in K_{\text{ba}}^2$ and $\bar{\mathbf{a}}$ is a $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral problem in \mathbf{m} . We say \mathbf{n} is a simple $\bar{\mathbf{a}}$ -solving extension of \mathbf{m} when:

- (a) $\mathfrak{B}_\mathbf{n}$ is the completion of $\mathfrak{B}_\mathbf{n}^o$ where:
- (b) $\mathfrak{B}_\mathbf{n}^o$ is the Boolean algebra generated by $\mathfrak{B}_\mathbf{m} \cup \{y_s : s \in I(\kappa_1, \kappa_2)\}$ freely except the equations which holds in $\mathfrak{B}_\mathbf{m}$ and $\Gamma_{\bar{\mathbf{a}}} = \{y_{s_1} \cap y_{s_2} \leq \mathbf{a}_{s_1, s_2} : s_1 \in I_1(\kappa_1) \text{ and } s_2 \in I_2(\kappa_2)\}$,
- (c) $D_\mathbf{n}$ is the filter on $\mathfrak{B}_\mathbf{n}$ generated by $D_\mathbf{m} \cup \{y_s : s \in I(\kappa_1, \kappa_2)\}$.

Claim 4.10. Assume $\bar{\mathbf{a}}$ is a $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -moral problem in $\mathbf{m} \in K_\kappa^{\text{ba}}$ and⁷ $\kappa = \text{cf}(\kappa) > \kappa_1 + \kappa_2$.

1) There is $\mathbf{n} \in K_\kappa^{\text{ba}}$ which is a simple $\bar{\mathbf{a}}$ -solving extension of \mathbf{m} , unique up to isomorphism over $\mathfrak{B}_\mathbf{m}$.

2) Above $\mathbf{m} \leq_\kappa^{\text{ba}} \mathbf{n}$ (so $\mathbf{n} \in K_\kappa^{\text{ba}}$).

3) If $\bar{\mathbf{a}}^*$ is a $T_{\text{ord}} - (\theta_1, \theta_2)$ -moral problem of \mathbf{m} with no solution in \mathbf{m} and $\theta_1 \notin \{\kappa_1, \kappa_2\}$ or $\theta_2 \notin \{\kappa_1, \kappa_2\}$ then $\bar{\mathbf{a}}^*$ has no solution in \mathbf{n} .

Proof. 1) As above let $I_\ell = I_\ell(\kappa_\ell)$ for $\ell = 1, 2$ and $I = I_1 + I_2$.

First,

⁷It seems that $\min\{\kappa_1, \kappa_2\} < \kappa$ suffice; the only difference in the proof is in proving (*).₅

(*)₁ the set of equations $\Gamma_{\bar{a}}$ is finitely satisfiable in $\mathfrak{B}_{\mathbf{m}}$.

Why? We shall prove two stronger statements (each implying (*))₁).

(*)_{1.1} if $t_1 \in I_1$ then we can find $\langle \mathbf{b}'_s : s \in I \rangle \in {}^I \mathfrak{B}$ such that:

- (a) $\mathbf{b}'_s \in D_{\mathbf{m}} \subseteq \mathfrak{B}_{\mathbf{m}}$ if $(s \leq_{I_1} t_1) \vee (s \in I_2)$,
- (b) if $s_1 \in I_1, s_2 \in I_2$ then $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} \leq \mathbf{a}_{s_1, s_2}$.

[Why? Let \mathbf{b}'_s be:

- \mathbf{a}_{s, t_1} if $s \leq_I t_1$ (so $s \in I_1$),
- $\mathbf{a}_{t_1, s}$ if $s \in I_2$,
- $0_{\mathfrak{B}}$ if $t_1 <_I s \in I_1$.

Now clause (a) is obvious (recalling $\mathbf{a}_{t_1, t_1} = 1_{\mathfrak{B}_{\mathbf{m}}}$ and as for clause (b), let $s_1 \in I_1, s_2 \in I_2$, now if $t_1 \leq_I s_1 \in I_1$ then $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} = 0_{\mathfrak{B}_{\mathbf{m}}} \cap \mathbf{b}'_{s_2} = 0_{\mathfrak{B}_{\mathbf{m}}} \leq \mathbf{a}_{s_1, s_2}$ and if $s_1 <_I t_1$ then $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} = \mathbf{a}_{s_1, t_1} \cap \mathbf{a}_{t_1, s_2}$ which is $\leq \mathbf{a}_{s_1, s_2}$ by 4.4(1)(d),(e).]

(*)_{1.2} if $t_2 \in I_2$ then we can find $\langle \mathbf{b}'_s : s \in I \rangle \in {}^I \mathfrak{B}$ such that:

- (a) $\mathbf{b}'_s \in D_{\mathbf{m}} \subseteq \mathfrak{B}_{\mathbf{m}}$ if $s \in I_1$ or $t_2 \leq_{I_2} s$,
- (b) if $s_1 \in I_2, s_2 \in I_2$ then $\mathbf{b}'_{s_1} \cap \mathbf{b}'_{s_2} \leq \mathbf{a}_{s_1, s_2}$.

[Why? Similarly.]

Now (*))₁ is easy: if $\Gamma' \subseteq \Gamma_{\bar{a}}$ is finite let $t_* \in I_1$ be such that: if $t \in I_1$ and y_t appears in Γ' then $t \leq_I t_*$. Choose $\langle b'_s : s \in I \rangle$ as in (*))_{1.1} for t_* and let h be the function $y_s \mapsto \mathbf{b}'_s$ for $s \in I$. Now think, so (*))₁ holds indeed.

Clearly it follows by (*))₁ that

- (*)₂ (a) there is a Boolean algebra $\mathfrak{B}_{\mathbf{n}}^o$ extending $\mathfrak{B}_{\mathbf{m}}$ as described in clause (b) of Definition 4.9,
- (b) there is a Boolean algebra $\mathfrak{B}_{\mathbf{n}}$ as described in (a) of Definition 4.9: the completion of $\mathfrak{B}_{\mathbf{n}}^o$,
- (c) $D_{\mathbf{n}}$ is chosen as the filter on $\mathfrak{B}_{\mathbf{n}}$ generated by $D_{\mathbf{m}} \cup \{y_s : s \in I\}$ it satisfies $D_{\mathbf{m}} = D_{\mathbf{n}} \cap \mathfrak{B}_{\mathbf{m}}$, in particular $0_{\mathfrak{B}_{\mathbf{m}}} \notin D_{\mathbf{n}}$,
- (d) $\mathfrak{B}_{\mathbf{n}}$ satisfies the κ -c.c.,
- (e) $D_{\mathbf{n}}$ is generated (as a filter) by $D_{\mathbf{n}} \cap \mathfrak{B}_{\mathbf{n}}^o$.

[Why? Clauses (a), (b) follows by (*))₁ and for clauses (c),(d) see (*))₄ and (*))₅ in the proof of (2), respectively; in particular $0_{\mathfrak{B}_{\mathbf{m}}} \notin D_{\mathbf{n}}$ and clause (e) holds by clause (c).]

Together we have $\mathbf{n} = (\mathfrak{B}_{\mathbf{n}}, D_{\mathbf{n}}) \in K_{\text{ba}}^2$, as for $\mathbf{m} \leq_{\text{ba}} \mathbf{n}$, see part (2).

2) Now (by part (1) we have $\mathfrak{B}_{\mathbf{m}} \subseteq \mathfrak{B}_{\mathbf{n}}$, but we shall show that moreover)

(*)₃ $\mathfrak{B}_{\mathbf{m}} \triangleleft \mathfrak{B}_{\mathbf{n}}$.

[Why? If not, then some $\mathbf{d} \in \mathfrak{B}_{\mathbf{n}}^+$ is disjoint to \mathbf{b} for a dense subset of $\mathbf{b} \in \mathfrak{B}_{\mathbf{m}}^+$. Let $\mathbf{d} = \sigma(y_{s_0}, \dots, y_{s_{n-1}}, \bar{c})$ where σ is a Boolean term, $s_0 <_I \dots <_I s_{n-1}$ and \bar{c} is from $\mathfrak{B}_{\mathbf{m}}$. We may replace \mathbf{d} by any $\mathbf{d}' \in \mathfrak{B}_{\mathbf{n}}^+$ satisfying $\mathbf{d}' \leq_{\mathfrak{B}} \mathbf{d}$. Hence without loss of generality $\mathbf{d} = \bigcap \{y_{s_\ell}^{\text{if}(\eta(\ell))} : \ell < n\} \cap c > 0_{\mathbf{n}}$ where $c \in \mathfrak{B}_{\mathbf{m}}, \eta(\ell) \in \{0, 1\}$ for $\ell < n$; also without loss of generality for every $\ell, k < n$ we have $s_\ell \in I_1 \wedge s_k \in I_2 \Rightarrow (c \leq \mathbf{a}_{s_\ell, s_k}) \vee (c \cap \mathbf{a}_{s_\ell, s_k} = 0_{\mathfrak{B}_{\mathbf{n}}})$.

We now define a function h from $\{y_s : s \in I\}$ into \mathfrak{B}_m as follows: $h(y_s)$ is:

- ₁ c , if $s = s_\ell \wedge \eta(\ell) = 1$,
- ₂ $0_{\mathfrak{B}_m}$, if otherwise.

Now

- ₃ if $t_1 \in I_1, t_2 \in I_2$ then $\mathfrak{B}_m \models "h(y_{t_1}) \cap h(y_{t_2}) \leq \mathbf{a}_{t_1, t_2}"$.

[Why? If $h(t_1) = 0_{\mathfrak{B}_m} \vee h(t_2) = 0_{\mathfrak{B}_m}$ this is obvious, otherwise for some $\ell(1) < \ell(2) < n$ we have $t_1 = s_{\ell(1)}, t_2 = s_{\ell(2)}$ and $\eta(\ell(1)) = 1 = \eta(\ell(2))$. So it suffice to prove $c = c \cap c \leq \mathbf{a}_{t_1, t_2}$ but otherwise by the choice of $c, c \cap \mathbf{a}_{t_1, t_2} = 0$, hence recalling 4.9(b) we have $\mathfrak{B}_n \models "y_{s_1} \cap y_{s_2} \cap c = 0"$ contradiction to our current assumption $\mathfrak{B}_n \models "d > 0"$; so •₃ holds indeed.]

By the choice of $\Gamma_{\bar{\mathbf{a}}}$ and of \mathfrak{B}_n recalling \mathfrak{B}_m is complete, by the choice of h and •₃ there is a projection \hat{h} from \mathfrak{B}_n onto \mathfrak{B}_m extending h , so clearly $\hat{h}(d) = c$ and this implies $c_1 \in \mathfrak{B}_m \wedge 0 < c_1 \leq c \Rightarrow \mathfrak{B}_n \models "c_1 \cap \mathbf{d} \geq 0_{\mathfrak{B}_n}"$ contradicting the choice of \mathbf{d} . So indeed (*)₃ holds.]

- (*)₄ $D_m = D_n \cap \mathfrak{B}_m$.

[Why? Otherwise there are $c_1 \in D_m, c_2 \in \mathfrak{B}_m \setminus D_m$ and $s_0 <_I \dots <_I s_{n-1}$ such that $\mathfrak{B}_n \models " \bigcap_{\ell < n} y_{s_\ell} \cap c_1 \leq c_2 "$. As $\mathbf{a}_{t_1, t_2} \in D_m$ for $t_1 <_I t_2$, without loss of generality $c_1 \leq \mathbf{a}_{s_\ell, s_k}$ for $\ell < k < n, s_\ell \in I_1, s_k \in I_2$.

Now letting $c = c_1 - c_2$ we continue as in the proof of (*)₃ defining h, \hat{h} and apply the projection \hat{h} to " $\bigcap_{\ell < n} y_{s_\ell} \cap c_1 \leq c_2$ ".]

- (*)₅ \mathfrak{B}_n satisfies the κ -c.c..

[Why? If not, then there are pairwise disjoint, positive $d_i \in \mathfrak{B}_n$ for $i < \kappa$. So as in the proof of (*)₃, without loss of generality $d_i = \bigcap \{y_{s(i, \ell)}^{\text{if}(\eta(i, \ell))} : \ell < n(i)\} \cap c_i$ where $c_i \in \mathfrak{B}_m, \eta(i, \ell) \in \{0, 1\}$ and $s(i, 0) <_I s(i, 1) <_I \dots <_I s(i, n(i) - 1)$. Let $m(i) \leq n(i)$ be such that for every $\ell < n(i)$ we have $s_\ell \in I_1$ iff $\ell < m(i)$.

Again as there, without loss of generality for every $\ell < m(i) \leq k < n(i)$ we have $(\mathbf{a}_{s(i, \ell), s(i, k)} \leq c_i) \vee (\mathbf{a}_{s(i, \ell), s(i, k)} \cap c_i = 0)$ so $\eta(i, \ell) = 1 = \eta(i, k) \wedge \ell < m(i) \leq k < n(i) \Rightarrow c_i \leq \mathbf{a}_{s(i, \ell), s(i, k)}$.

As $\kappa = \text{cf}(\kappa) > \kappa_1 + \kappa_2$ by an assumption of 4.10 without loss of generality $n(i) = n, m(i) = m, \eta(i, \ell) = \eta(\ell)$ and $s(i, \ell) = s_\ell$ for $i < \kappa, \ell < n$ and as \mathfrak{B}_m satisfies the κ -c.c. we can find $i < j < \kappa$ such that $\mathfrak{B}_m \models "0 < c_i \cap c_j"$ and let $c = c_i \cap c_j$ so we can continue as before.]

So together by (*)₃, (*)₄, (*)₅ we have $\mathbf{m} \leq_{\kappa}^{\text{ba}} \mathbf{n} \in K_{\kappa}^{\text{ba}}$ as promised.

3) Let $I^* = I(\theta_1, \theta_2), I_1^* = I_1(\theta_1), I_2^* = I_2(\theta_2)$ and recall $\bar{\mathbf{a}}^* = \langle \mathbf{a}_{s, t}^* : s <_{I^*} t \rangle$ is a $T_{\text{ord}} - (\theta_1, \theta_2)$ -moral problem in \mathbf{m} . Toward a contradiction assume that the sequence $\bar{\mathbf{b}} = \langle \mathbf{b}_t : t \in I^* \rangle$ solve the problem $\bar{\mathbf{a}}^*$ in \mathbf{n} so $\mathbf{b}_t \in D_n$ and let $\mathbf{b}_t = \sigma_t(y_{s(t, 0)} \dots, y_{s(t, n(t) - 1)}, c_{t, 0}, \dots, c_{t, m(t) - 1})$ with $c_{t, k} \in \mathfrak{B}_m, s(t, \ell) \in I$ and without loss of generality $s(t, \ell) <_I s(t, \ell + 1)$ for $\ell < n(t) - 1$ so $s(t, k) \in I$ for $k < n(t)$.

The reader may wonder: we have to prove that there is no solution in \mathfrak{B}_n , not just in \mathfrak{B}_n^o , so how can we use finitary terms? The point is that though \mathfrak{B}_n is the completion of \mathfrak{B}_n^o , the filter D_n is generated (as a filter) by $\mathfrak{B}_n^o \cap D_n$.

By symmetry without loss of generality,

$$(*)_6 \theta_1 \notin \{\kappa_1, \kappa_2\}.$$

Recalling 4.7, we can replace \mathbf{b}_t by any $\mathbf{b}'_t \leq \mathbf{b}_t$ which is from $D_{\mathbf{n}}$, so as $\bigwedge_{\ell} y_{s(t,\ell)} \in D_{\mathbf{n}}$, without loss of generality $\ell < n(t) \Rightarrow \mathbf{b}_t \leq y_{s(t,\ell)}$, so without loss of generality

$$(*)_7 \mathbf{b}_t = \cap \{y_{s(t,\ell)} : \ell < n(t)\} \cap c_t \text{ for some } c_t \in D_{\mathbf{m}} \text{ recalling } D_{\mathbf{m}} = D_{\mathbf{m}} \cap \mathfrak{B}_{\mathbf{m}}.$$

By the Δ -system lemma (recalling 4.7(1)) without loss of generality,

\oplus if $\theta_1 > \aleph_0$ then,

$$(a) t \in I_1^* \Rightarrow n(t) = n(*),$$

$$(b) \text{ if } t \in I_1^* \text{ then } s(t,\ell) \in I_1^* \Leftrightarrow \ell < \ell(*),$$

$$(c) \langle \langle s(t,\ell) : \ell < n(*) \rangle : t \in I_1^* \rangle \text{ is an indiscernible sequence in the linear order } I = I(\kappa_1, \kappa_2), \text{ for quantifier free formulas.}$$

But we shall not use \oplus . As $\theta_1 \neq \kappa_1, \kappa_2$, by 4.7(1),(1A) it follows that without loss of generality for some s_1°, s_2° we have:

$$(*)_8 s_1^\circ \in I_1, s_2^\circ \in I_2 \text{ and } s(t,\ell) \notin [s_1^\circ, s_2^\circ]_I \text{ for every } t \in I_1^*, \ell < n(t).$$

Again by 4.7(2) without loss of generality,

$$(*)_9 \text{ if } t \in I_2^* \text{ then } \mathbf{b}_t \leq y_{s_1^\circ} \cap y_{s_2^\circ}.$$

We now define a function h from $\{y_s : s \in I\}$ into $\mathfrak{B}_{\mathbf{n}}$, (yes! not $\mathfrak{B}_{\mathbf{m}}$) by:

(*)₁₀ $h(y_s)$ is:

- $\mathbf{a}_{s,s_1^\circ} \cap \mathbf{a}_{s_1^\circ,s_2^\circ}$ if $s <_I s_1^\circ$,
- $\mathbf{a}_{s_1^\circ,s} \cap y_s \cap \mathbf{a}_{s,s_2^\circ}$ if $s \in I, s_1^\circ \leq_I s \leq_I s_2^\circ$,
- $\mathbf{a}_{s_1^\circ,s_2^\circ} \cap \mathbf{a}_{s_2^\circ,s}$ if $s_2^\circ <_I s$.

Note

$$(*)_{11} h(y_s) \in D_{\mathbf{n}} \text{ for } s \in I.$$

[Why? Because $\mathbf{a}_{s,t} \in D_{\mathbf{n}}$ for $S \in I_1, t \in I$ and $y_s \in D_{\mathbf{n}}$ for $s \in I$.]

$$(*)_{12} h(y_{s_1}) \cap h(y_{s_2}) \leq \mathbf{a}_{s_1,s_2} \text{ for } s_1 \in I_1, s_2 \in I_2.$$

[Why? If $s_1, s_2 \in [s_1^\circ, s_2^\circ]_I$ this holds by the definition of $\mathfrak{B}_{\mathbf{n}}$, i.e. as $h(y_{s_1}) \leq y_{s_1}, h(y_{s_2}) \leq y_{s_2}$ and $\mathfrak{B}_{\mathbf{n}} \models "y_{s_1} \cap y_{s_2} \leq \mathbf{a}_{s_1,s_2}"$.

If $s_1 <_{I^*} s_1^\circ \wedge s_2^\circ <_{I^*} s_2$ then (*)₁₁ says: $\mathbf{a}_{s_1,s_1^\circ} \cap \mathbf{a}_{s_1^\circ,s_2^\circ} \cap \mathbf{a}_{s_2^\circ,s_2} \leq \mathbf{a}_{s_1,s_2}$ which obviously holds (as $\bar{\mathbf{a}}$ is a $T_{\text{ord}} - (\kappa_1, \kappa_2)$ -problem in \mathbf{m}).

If $s_1 <_{I^*} s_1^\circ \wedge s_2 \in [s_1^\circ, s_2^\circ]_{I^*}$ then this means: $(\mathbf{a}_{s_1,s_1^\circ} \cap \mathbf{a}_{s_1^\circ,s_2^\circ}) \cap (\mathbf{a}_{s_1^\circ,s_2} \cap y_{s_2} \cap \mathbf{a}_{s,s_2^\circ}) \leq \mathbf{a}_{s_1,s_2}$; but as we have $\mathbf{a}_{s_1,s_1^\circ} \cap \mathbf{a}_{s_1^\circ,s_2} \leq \mathbf{a}_{s_1,s_2}$ this holds.

If $s_1 \in [s_1^\circ, s_2^\circ]_{I^*}$ and $s_2^\circ <_{I^*} s_2$ this means $(\mathbf{a}_{s_1^\circ,s_1} \cap y_{s_1} \cap \mathbf{a}_{s_1,s_2}) \cap (\mathbf{a}_{s_1^\circ,s_2^\circ} \cap \mathbf{a}_{s_2^\circ,s}) \leq \mathbf{a}_{s_1,s_2}$ which holds for similar reasons. So (*)₁₂ holds indeed.]

By the choice of $\mathfrak{B}_{\mathbf{n}}^\circ$ and $\mathfrak{B}_{\mathbf{n}}$ there is a homomorphism \hat{h} from $\mathfrak{B}_{\mathbf{n}}^\circ$ into $\mathfrak{B}_{\mathbf{n}}$, extending $\text{id}_{\mathfrak{B}_{\mathbf{m}}}$ and extending h . Now easily $\hat{h}(\mathbf{b}_t) \in D$ for $t \in I^*$ because $\mathbf{b}_t = \cap \{y_{s(t,\ell)} : \ell < n(t)\} \cap c_t, c_t \in D_{\mathbf{m}}$ hence $\hat{h}(c_t) = c_t \in D_{\mathbf{m}}$ and by (*)₁₀ we have $\hat{h}(y_{s(t,\ell)}) \in D_{\mathbf{m}}$.

Now $\langle \hat{h}(\mathbf{b}_t) : t \in I^* \rangle$ still form a solution of $\bar{\mathbf{a}}^*$ and by (*)₇ + (*)₈ + (*)₁₀ we have $t \in I_1^* \Rightarrow h(\mathbf{b}_t) \in \mathfrak{B}_{\mathbf{m}}$ hence without loss of generality:

(*)₁₃ $t \in I_1^* \Rightarrow \mathbf{b}_t \in \mathfrak{B}_m$.

Now define \mathbf{b}'_t for $t \in I^*$ by: \mathbf{b}'_t is:

- \mathbf{b}_t if $t \in I_1^*$,
- c_t if $t \in I_2^*$.

It suffices to prove that $\langle \mathbf{b}'_t : t \in I^* \rangle$ solves $\bar{\mathbf{a}}^*$ in \mathbf{m} . Clearly $t \in I^* \Rightarrow \mathbf{b}'_t \in D_m$, so let $t_1 \in I_1^*, t_2 \in I_2^*$. We have to prove that $\mathbf{b}'_{t_1} \cap \mathbf{b}'_{t_2} \leq \mathbf{a}_{t_1, t_2}$ but we know only that $\mathbf{b}_{t_1} \cap \mathbf{b}_{t_2} \leq \mathbf{a}_{t_1, t_2}$ which means $\mathbf{a}_{t_1, t_2} \geq \mathbf{b}'_{t_1} \cap \left(\bigcap_{\ell < n(t_2)} y_{s(t_2, \ell)} \cap c_{t_2} \right) =$

$(\mathbf{b}'_{t_1} \cap \mathbf{b}'_{t_2}) \cap \bigcap \{y_{s(t_2, \ell)} : \ell < n(t_2)\}$.

Let h_{t_2} be a projection from \mathfrak{B}_n onto \mathfrak{B}_m such that $h_{t_2}(y_{s(t_2, \ell)}) = c_t$ if $\ell < n(t)$ and $h_{t_2}(y_s) = 0_{\mathfrak{B}_m}$ if $s \in I \setminus \{s(t_2, \ell) : \ell < n(t_2)\}$, as earlier it exists and applying it we get the desired inequality. $\square_{4.10}$

Theorem 4.11. *For any λ and regular $\theta_1, \theta_2 \leq \lambda$ such that $\theta_1 + \theta_2 > \aleph_0$ there is a regular filter D on λ such that:*

- (a) *for every dense linear order M , in M^λ/D there is a (θ_1, θ_2) -pre-cut but no (κ_1, κ_2) -pre-cut when κ_1, κ_2 are regular $\leq \lambda$ and $\{\theta_1, \theta_2\} \not\subseteq \{\kappa_1, \kappa_2\}$*
- (b) *if M is $(\omega^{>2}, \triangleleft)^\lambda/D$ then $t(M) \geq \lambda^+$.*

Remark 4.12. 1) Why do we need $\theta_1 + \theta_2 > \aleph_0$? To prove (*)₁.

2) In fact, this demand is necessary, see 4.14 below.

Proof. We prove clause (a), which is the main result, clause (b) holds by 4.15. Let $\kappa = \lambda^+$.

(*)₁ there are \mathbf{m}_0, \mathbf{a} such that:

- (a) $\mathbf{m}_0 \in K_\kappa^{\text{ba}}$,
- (b) \mathbf{a} is a $T_{\text{ord}} - (\theta_1, \theta_2)$ -moral problem in \mathbf{m}_0 not solved in it.

[Why? By [She90, Ch.VI, §3] there is an ultra-filter D on λ such that in $(\mathbb{Q} <)^\lambda/D$ there is a (θ_1, θ_2) -cut. Define \mathbf{m} by $\mathfrak{B}_m = \mathcal{P}(\lambda), D_m = D$, now check. E.g. as $\kappa = \lambda^+$, easily the Boolean algebra \mathfrak{B}_m satisfies the κ -c.c.; alternatively let \mathfrak{B}_n be generated by $\{\mathbf{a}_{s,t} : s \in I_1, t \in I_2\}$ freely; and let D_n be the ultra-filter on \mathfrak{B}_n generated by $\{\mathbf{a}_{s,t} : s \in I_1, t \in I_2\}$. Now check.]

Let $\langle W_\alpha : \alpha < 2^\lambda \rangle$ be a partition of 2^λ to sets each of cardinality 2^λ such that $W_\alpha \cap \alpha = \emptyset$.

(*)₂ we can choose \mathbf{m}_α and $\langle \bar{\mathbf{a}}_\gamma : \gamma \in W_\alpha \rangle$ by induction on $\alpha \leq 2^\lambda$ such that:

- (a) $\mathbf{m}_\alpha \in K_\kappa^{\text{ba}}$ has cardinality $\leq 2^\lambda$,
- (b) $\langle \mathbf{m}_\beta : \beta \leq \alpha \rangle \in S_\kappa^{\text{ba}}$,
- (c) \mathbf{m}_0 is as in (*)₁,
- (d) $\langle \bar{\mathbf{a}}_\gamma : \gamma \in W_\alpha \rangle$ be such that $\bar{\mathbf{a}}_\gamma$ is a $T_{\text{ord}} - (\kappa_{\gamma,1}, \kappa_{\gamma,2})$ problem in \mathbf{m}_α and $\kappa_{\gamma,1}, \kappa_{\gamma,2}$ are regular $\leq \lambda$ and $\{\theta_1, \theta_2\} \not\subseteq \{\kappa_{\gamma,1}, \kappa_{\gamma,2}\}$ and any such $\bar{\mathbf{a}}$ appears in the sequence,
- (e) if $\alpha = \gamma + 1$ then necessarily $\gamma \in W_\beta$ for some $\beta \leq \alpha$ and in \mathbf{m}_α there is a solution for $\bar{\mathbf{a}}_\gamma$,
- (f) in \mathbf{m}_α there is no solution to $\bar{\mathbf{a}}^*$.

[Why can we carry the induction?

Now for $\alpha = 0$ use $(*)_1$, for α limit use 4.8 and for α successor use 4.10.]

$(*)_3$ letting $\mathbf{m} = \mathbf{m}_{2^\lambda}$ we have $\mathfrak{B}_{\mathbf{m}} = \cup\{\mathfrak{B}_{\mathbf{m}_\alpha} : \alpha < 2^\lambda\}$ and $D_{\mathbf{m}} = \cup\{D_{\mathbf{m}_\alpha} : \alpha < 2^\lambda\}$.

[Why? Because $\langle \mathbf{m}_\alpha : \alpha \leq 2^\lambda \rangle \in S_\kappa^{\text{ba}}$ and $\text{cf}(2^\lambda) \geq \kappa$.]

$(*)_4$ there is a regular excellent filter D_0 on λ and homomorphism \mathbf{j} from $\mathcal{P}(\lambda)$ onto $\mathfrak{B}_{\mathbf{m}}$.

[Why? See [MS14].]

$(*)_5$ let $D = \mathbf{j}^{-1}(D_{\mathbf{m}})$.

So D is a filter on λ , and by 3.8 for $\theta = \aleph_0$ (or Malliaris-Shelah [MS14]) we are done. $\square_{4.11}$

Conclusion 4.13. *If $\lambda \geq \aleph_2$ then the results of Malliaris-Shelah [MS16b] cannot be generalized to reduced powers (atomic types, of course), that is (clause (A) is in contrast to [MS16b, Th.10.25(b)] \Rightarrow (d)); clause (B) is in contrast to [MS16b, Th.10.1], and clause (C) is in contrast to [MS16b, Th.3.1]*

- (A) *If $\lambda \geq \aleph_1$ then for some regular filter D on λ we have: in ultra-powers of infinite linear orders we have a pre-cut with small cofinalities, but no symmetric pre-cut, that is:*
 - (a) *in the ultra-power $(\mathbb{Q}, <)^{\lambda}/D$ there is a (\aleph_1, \aleph_0) -pre-cut,*
 - (b) *in this ultra-power, there is no symmetric pre-cut of cofinality σ for $\sigma \leq \lambda$.*
- (B) *treetops: we can add above that in $(^{\omega}>\omega, \triangleleft)^{\lambda}/D$ every increasing sequence of length $\leq \lambda$ has an upper bound;*
- (C) *if $\lambda \leq \aleph_2$ then we can add in part (A), there are two pre-cuts with the same small left cofinality but a different small right cofinalities, e.g. \aleph_1 from the left and \aleph_2, \aleph_0 from the right.*

Proof. For clause (A) we apply clause (a) of 4.11 choosing the pair (θ_1, θ_2) as (\aleph_1, \aleph_0) .

For clause (B) apply clause (b) of 4.11.

For clause (C) we repeat the proof of 4.11 but starting (with $\kappa = \lambda^+$ as there) and choose as there $\mathbf{m}_0 \in K_\kappa$ of cardinality $\leq 2^\lambda$ such that some (\aleph_1, \aleph_0) -moral problem and (\aleph_1, \aleph_2) -moral problem in \mathbf{m}_0 are not solve. Then continue as there. $\square_{4.13}$

Observation 4.14. *If $\mathbf{m} \in K_\kappa^{\text{ba}}$ then any $T_{\text{ord}} - (\aleph_0, \aleph_0)$ -problem $\bar{\mathbf{a}}$ in \mathbf{m} has a solution.*

Proof. Let $\mathbf{b}_{(1,n)} = \mathbf{b}_{(2,n)} = \mathbf{b}_n := \cap\{\mathbf{a}_{(1,\ell),(2,k)} : \ell, k \leq n\}$, clearly $s \in I(\aleph_0, \aleph_0) \Rightarrow \mathbf{b}_s \in D$ and $(s, t) \in I(1, \aleph_0) \times I(2, \aleph_0) \Rightarrow \mathbf{b}_s \cap \mathbf{b}_t \leq \mathbf{a}_{s,t}$. $\square_{4.14}$

Claim 4.15. *In $M_*^{\mathfrak{B}}/D$, any increasing sequence of length $< \kappa^+$ has an upper bound when (A) or (B) holds, where:*

- (A) (a) $M_* = (^{\omega}>\mu, \trianglelefteq)$,
- (b) \mathfrak{B} is a complete Boolean algebra which is $(< \theta)$ -distributive,

- (c) D is a (μ, θ) -regular, θ -complete filter on \mathfrak{B} ,
 - (d) $(\mathbb{Q}, <)^{\mathfrak{B}}/D$ has no (σ, σ) -pre-cut for any regular $\sigma \leq \kappa$,
 - (e) $\mathbf{m} = (\mathfrak{B}, D)$.
- (B) (a) – (c) as above,
- (d) every $T_{\text{tr}} - (\sigma, \sigma)$ -moral problem in \mathbf{m} has a $T_{\text{tr}} - (\sigma, \sigma)$ -moral, solution in \mathbf{m} where:
 - (α) $\bar{\mathbf{a}}$ is a T_{tr} -moral problem when:
 - $\bar{\mathbf{a}} = \langle \mathbf{a}_{\alpha, \beta} : \alpha < \beta < \sigma \rangle$,
 - $\mathbf{a}_{\alpha, \beta} \in D$,
 - if $u \subseteq \sigma$ is finite and $\mathbf{c} \in \mathfrak{B}^+$ then for some $\bar{\eta} = \langle \eta_\alpha : \alpha \in u \rangle$ we have $\eta_\alpha \in {}^{|u|}>|u|$ for $\alpha \in u$ and $\mathbf{c} \leq \mathbf{a}_{\alpha, \beta} \Rightarrow \eta_\alpha \trianglelefteq \eta_\beta$ and $\mathbf{c} \cap \mathbf{a}_{\alpha, \beta} = \mathbf{0}_{\mathfrak{B}} \Rightarrow \neg(\eta_\alpha \trianglelefteq \eta_\beta)$ for $\alpha < \beta$ from u .
 - (β) $\bar{\mathbf{b}} = \langle \mathbf{b}_\alpha : \alpha < \sigma \rangle$ is a $T_{\text{tr}} - (\sigma, \sigma)$ -solution of $\bar{\mathbf{a}}$ when $\mathbf{b}_\alpha \in D$ and $\mathbf{b}_\alpha \cap \mathbf{b}_\beta \leq \mathbf{a}_{\alpha, \beta}$ for $\alpha < \beta < \sigma$.

Proof. If clause (A), as in [She78, Ch.VI,2.7] or [MS16b].

If clause (B), as above.

□_{4.15}

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