CATEGORICITY IN ABSTRACT ELEMENTARY CLASSES: GOING UP INDUCTIVE STEP SH600 - PART 1 AND 2

SAHARON SHELAH

Abstract. We deal with beginning stability theory for "reasonable" non-elementary classes without any remnants of compactness like dealing with models above Hanf number or by the class being definable by $\mathbb{L}_{\omega_1,\omega}$. We introduce and investigate good λ-frame, show that they can be found under reasonable assumptions and prove we can advance from λ to λ^+ when non-structure fail. That is, assume $2^{\lambda^{+n}} < 2^{\lambda^{+n+1}}$ for $n < \omega$. So if an AEC is cateogorical in λ, λ^+ and has intermediate number of models in λ^{++} and $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$, $\mathrm{LS}(\mathfrak{k}) \leq \lambda$). Then there is a good λ-frame $\mathfrak{s}$ and if $\mathfrak{s}$ fails non-structure in λ^{++} then $\mathfrak{s}$ has a successor $\mathfrak{s}^+$, a good λ^+-frame hence $K^{\mathfrak{s}}_{\lambda^{+3}} \neq \emptyset$, and we can continue.

Date: 2023-06-17.

2020 *Mathematics Subject Classification.* 03C45, 03C75, 03C95, 03C50 .

Key words and phrases. model theory, abstract elementary classes, classification theory, categoricity, non-structure theory.

I thank Alice Leonhardt for the beautiful typing This research was supported by the United States-Israel Binational Science Foundation and in its final stages also by the Israel Science Foundation (Grant no. 242/03). Hopefully, final work 8/99 First version - Spring '95 Latest Revision - 08/Apr/2.

§ 0. Introduction

The paper's main explicit result is proving Theorem 0.1 below. It is done axiomatically, in a "superstable" abstract framework with the set of "axioms" of the frame, verified by applying earlier works, so it suggests this frame as the, or at least a major, non-elementary parallel of superstable.

A major case to which this is applied, is the one from [She01] represented in [She09c]; we continue this work in several ways but the use of [She01] is only in verifying the basic framework; we refer the reader to the book's introduction or [She01, §0] for background and some further claims but all the definitions and basic properties appear here. Otherwise, the heavy use of earlier works is in proving that our abstract framework applies in those contexts. If $\lambda = \aleph_0$ is O.K. for you, you may use [She09a] or [She75] instead of [She01] as a starting point.

Naturally, our deeper aim is to develop stability theory (actually a parallel of the theory of superstable elementary classes) for non-elementary classes. We use the number of non-isomorphic models as test problem. Our main conclusion is 0.1 below. As a concession to supposedly general opinion, we restrict ourselves here to the λ-good framework and delay dealing with weak relatives (see [She09d], Jarden-Shelah [JS13], hopefully [S$^+$]. Also, we assume that the (normal) weak-diamond ideal on the $\lambda^{+\ell}$ is not saturated (for $\ell = 1, \dots, n-1$). We had intended to rely on [She01, §3], but actually in the end we prefer to rely on the lean version of [She09d], see "reading plan A" in [She09d, §0]. Relying on the full version of [She09d], we can eliminate this extra assumption "not $\lambda^{+\ell+1}$-saturated[1] (ideal)". On $\mu_{\rm unif}(\lambda^{+\ell+1}, 2^{\lambda^{+\ell}})$, see, e.g. [She09a, 88r-0.wD](3)).

theorem 0.1. *Assume $2^\lambda < 2^{\lambda^{+1}} < \dots < 2^{\lambda^{+n+1}}$ and the (so called weak diamond) normal [1] ideal* $\mathrm{WDmId}(\lambda^{+\ell})$ *is not $\lambda^{+\ell+1}$-saturated [2] for $\ell = 1, \dots, n$.*

1) Let $\mathfrak{k}$ be an abstract elementary class (see §1 below) categorical in λ and λ^+ with $\mathrm{LS}(\mathfrak{k}) \le \lambda$ (e.g. the class of models of $\psi \in \mathbb{L}_{\lambda^+,\omega}$ with $\le_{\mathfrak{k}}$ defined naturally). If $1 \le \dot{I}(\lambda^{+2}, \mathfrak{k})$ and $2 \le \ell \le n \Rightarrow \dot{I}(\lambda^{+\ell}, \mathfrak{k}) < \mu_{\rm unif}(\lambda^{+\ell}, 2^{\lambda^{+\ell-1}})$, then $\mathfrak{k}$ has a model of cardinality λ^{+n+1}.

2) Assume $\lambda = \aleph_0$, and $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$. If $1 \le \dot{I}(\lambda^{+\ell}, \psi) < \mu_{\rm unif}(\lambda^{+\ell}, 2^{\lambda^{+\ell-1}})$ for $\ell = 1, \dots, n-1$ then ψ has a model in λ^{+n} (see [She75]*).*

Note that if $n = 3$, then 0.1(1) is already proved in [She01] $\approx$ [She09c]. If $\mathfrak{k}$ is the class of models of some $\psi \in \mathbb{L}_{\omega_1,\omega}$ this is proved in [She83a], [She83b], but the proof here does not generalize the proofs there. It is a different one (of course, they are related). There, for proving the theorem for n, we have to consider a few statements on $(\aleph_m, \mathcal{P}^-(n-m))$-systems for all $m \le n$, (going up and down). A major point (there) is that for $n = 0$, as $\lambda = \aleph_0$ we have the omitting type theorem and the types are "classical", that is, are sets of formulas. This helps in proving strong dichotomies; so the analysis of what occurs in $\lambda^{+n} = \aleph_n$ is helped by those dichotomies. Whereas here we deal with $\lambda, \lambda^+, \lambda^{+2}, \lambda^{+3}$ and then "forget" λ and deal with $\lambda^+, \lambda^{+2}, \lambda^{+3}, \lambda^{+4}$, etc. So having started with poor assumptions there is less reason to go back from λ^{+n} to λ. However, there are some further theorems proved in [She83a], [She83b], whose parallels are not proved here, mainly that if for every n, in λ^{+n} we get the "structure" side, then the class has models in every $\mu \ge \lambda$, and theorems about categoricity. We shall deal with them in subsequent

[1] Recall that as $2^{\lambda_{\ell-1}} < 2^{\lambda_\ell}$ this ideal is not trivial (i.e. $\lambda^{+\ell}$ is not in the ideal).

[2] Actually, the statement "some normal ideal on μ^+ is μ^{++}-saturated" is "expensive" (i.e. of large consistency strength, etc.), so it is "hard" for this assumption to fail.

works, mainly [She09e]. Also in [She75], [She87a] = [She09a] we started to deal with $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ dealing with $\aleph_1, \aleph_2$. Of course, we integrate them too into our present context. In the axiomatic framework (introduced in §2) we are able to present a lemma, speaking only on 4 cardinals, and which implies the theorem 0.1. (Why? Because in §3 by [She01] $\approx$ [She09c] we can get a so-called good λ^+-frame $\mathfrak{s}$ with $K^{\mathfrak{s}} \subseteq \mathfrak{k}$, and then we prove a similar theorem on good frames by induction on n, with the induction step done by the lemma mentioned above). For this, parts of the proof are a generalization of the proof of [She01, §8,§9,§10]. A major theme

here (and even more so in [She09e]) is:

Thesis 0.2. *It is worthwhile to develop model theory (and superstability in particular) in the context of $\mathfrak{k}_\lambda$ or $K_{\lambda^{+\ell}}, \ell \in \{0,\dots,n\}$, i.e., restrict ourselves to one, few, or an interval of cardinals. We may have good understanding of the class in this context, while in general cardinals we are lost.*

As in [She90] for first order classes

Thesis 0.3. *It is reasonable first to develop the theory for the class of (quite) saturated enough models as it is smoother and even if you prefer to investigate the non-restricted case, the saturated case will clarify it and you will e able to rely on it. In our case this will mean investigating $\mathfrak{s}^{+n}$ for each n and then $\bigcap\{\mathfrak{k}^{\mathfrak{s}^{+n}} : n < \omega\}$.*

Thesis 0.4. *[The Better to be poor Thesis] Better to know what is essential. e.g., you may have better closure properties (here a major point of poverty is having no formulas, this is even more noticeable in* [She09e]*).*

I thank John Baldwin, Alex Usvyatsov, Andres Villaveces and Adi Yarden for many complaints and corrections.

§1 gives a self-contained introduction to AEC (abstract elementary classes), including definitions of types, "M_2 is (λ,κ)-brimmed over M_1," and saturativity = universality + model homogeneity. An interesting point is observing that any λ-AEC $\mathfrak{k}_\lambda$ can be lifted to $\mathfrak{k}_{\geq\lambda}$, uniquely; so it does not matter if we deal with $\mathfrak{k}_\lambda$ or $\mathfrak{k}_{\geq\lambda}$ (unlike the situation for good λ-frames, which if we lift, we in general, lose some essential properties).

The good λ-frames introduced in §2 are a very central notion here. It concentrates on one cardinal λ, in $\mathfrak{k}_\lambda$ we have amalgamation and more, hence types, in the orbital sense, not in the classical sense of set of formulas, for models of cardinality λ can be reasonably defined and "behave" reasonably (we concentrate on so-called basic types) and we axiomatically have a non-forking relation for them.

In §3 we show that starting with classes belonging to reasonably large families, from assumptions on categoricity (or few models), good λ-frames arise. In §4 we deduce some things on good λ-frames; mainly: stability in λ, existence and (full) uniqueness of $(\lambda,*)$-brimmed extensions of $M \in K_\lambda$.

Concerning §5 we know that if $M \in K_\lambda$ and $p \in \mathcal{S}^{\rm bs}(M)$ then there is $(M,N,a) \in K^{3,\rm bs}_\lambda$ such that $\mathrm{ortp}(a,M,N) = p$. But can we find a special ("minimal" or "prime") triple in some sense? Note that if $(M_1,N_1,a) \leq_{\rm bs} (M_2,N_2,a)$ then N_2 is an amalgamation of N_1, M_2 over M_1 (restricting ourselves to the case "$\mathrm{ortp}(a,M_2,N_2)$ does not fork over M_1") and we may wonder is this amalgamation unique (i.e., allowing to increase or decrease N_2). If this holds for any such (M_2,N_2,a) we say (M_1,N_1,a) has uniqueness (= belongs to $K^{3,\rm uq}_\lambda = K^{3,\rm uq}_{\mathfrak{s}}$). Specifically we ask: is $K^{3,\rm uq}_\lambda$ dense in $(K^{3,\rm bs}_\lambda, \leq_{\rm bs})$? If no, we get a non-structure result; if yes, we shall (assuming categoricity) deduce the "existence for $K^{3,\rm uq}_{\mathfrak{s}}$" and this is used later as a building block for non-forking amalgamation of models.

So our next aim is to find "non-forking" amalgamation of models (in §6). We first note that there is at most one such notion which fulfills our expectations (and "respect" $\mathfrak{s}$). Now if $\bigcup(M_0, M_1, a, M_3)$, $M_0 \le_{\mathfrak{k}} M_2 \le_{\mathfrak{k}} M_3$ (equivalently, $(M_0, M_2, a) \le_{\rm bs} (M_1, M_3, a)$ and $(M_0, M_2, a) \in K^{3,\rm uq}_\lambda$), by our demands we have to say that M_1, M_2 are in non-forking amalgamation over M_0 inside M_3. Closing this family under the closure demands we expect to arrive to a notion $\mathrm{NF}_\lambda = \mathrm{NF}_{\mathfrak{s}}$ which should be the right one (if a solution exists at all). But then we have to work on proving that it has all the properties it hopefully has.

A major aim in advancing to λ^+ is having a superlimit model in $\mathfrak{k}_{\lambda^+}$. So in §7 we find out who it should be: the saturated model of $\mathfrak{k}_{\lambda^+}$, but is it superlimit? We use our NF_λ to define a "nice" order $\le^*_{\lambda^+}$ on $\mathfrak{k}_{\lambda^+}$, investigate it and prove the existence of a superlimit model under this partial order. To advance the move to λ^+ we would like to have that the class of λ^+-saturated model with the partial order $\le^*_{\lambda^+}$ is a λ^+-AEC Well, we do not prove it but rather use it as a dividing line: if it fails we eventually get many models in $\mathfrak{k}_{\lambda^{++}}$ (coding a stationary subset of[3] λ^{++}); see §8.

Lastly, we pay our debts: prove the theorems which were the motivation of this work, in §9.

* * *

Reading Plans: As usual, these are instructions on what you can avoid reading.

Note that §3 contains the examples, i.e., it shows how "good λ-frame", our main object of study here, arise in previous works. This, on the one hand, may help the reader to understand what is a good frame and, on the other hand, helps us in the end to draw conclusions continuing those works. However, it is <u>not</u> necessary here otherwise, so you may ignore it.

Note that we treat the subject axiomatically, in a general enough way to treat the cases which exist without trying too much to eliminate axioms as long as the cases are covered (and probably most potential readers will feel they are more than general enough).

We shall assume

$(*)_0$ $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{+2}} < \ldots < 2^{\lambda^{+n}}$ and $n \ge 2$.

In the beginning of §1 there are some basic definitions.

Reading Plan 0: We accept the good frames as interesting *per se*, so ignore §3 (which gives "examples") and: §1 tells you all you need to know on abstract elementary classes; §2 presents frames, etc.

Reading Plan 1: The reader decides to understand why we reprove the main theorem of [She83a], [She83b] so

$(*)_1$ K is the class of models of some $\psi \in \mathbb{L}_{\lambda^+,\omega}$ (with a natural notion of elementary embedding $\prec_{\mathscr{L}}$ for $\mathscr{L}$ a fragment of $\mathbb{L}_{\lambda^+,\omega}$ of cardinality $\le \lambda$ to which ψ belongs).

So in fact (as we can replace, for this result, K by any class with fewer models still satisfying the assumptions) without loss of generality

$(*)'_1$ if $\lambda = \aleph_0$ then K is the class of atomic models of some complete first order theory, $\le_{\mathfrak{k}}$ is being elementary submodel.

The theorems we are seeking are of the form

[3]Really, any $S \subseteq \{\delta < \lambda^{++} : \mathrm{cf}(\delta) = \lambda^+\}$

$(*)_2$ if K has few models in $\lambda + \aleph_1, \lambda^+, \dots, \lambda^{+n}$ then it has a model in λ^{+n+1}.
[Why "$\lambda + \aleph_1$"? If $\lambda > \aleph_0$ this means λ whereas if $\lambda = \aleph_0$ this means that we do not require "few model in $\lambda = \aleph_0$". The reason is that for the class or models of $\psi \in \mathbb{L}_{\omega_1,\omega}$ (or $\in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ or an AEC which is $\mathrm{PC}_{\aleph_0}$, see Definition 3.4) we have considerable knowledge of general methods of building models of cardinality $\aleph_1$, for general λ we are very poor in such knowedge (probably as there is much less).]

But, of course, what we would really like to have are rudiments of stability theory (non-forking amalgamation, superlimit models, etc.). Now reading plan 1 is to follow reading plan 2 below but replacing the use of Claim 3.10 and [She01] by the use of a simplified version of 3.5 and [She83a]. Reading Plan 2: The reader would

like to understand the proof of $(*)_2$ for arbitrary $\mathfrak{k}$ and λ. The reader

(a) knows at least the main definitions and results of [She01] $\approx$ [She09c], or just

(b) reads the main definitions of §1 here (in 1.1 - 1.7) and is willing to believe some quotations of results of [She01] $\approx$ [She09c].

We start assuming $\mathfrak{k}$ is an abstract elementary class, $\mathrm{LS}(\mathfrak{k}) \le \lambda$ (or read §1 here until 1.17) and $\mathfrak{k}$ is categorical in λ, λ^+ and $1 \le \dot{I}(\lambda^{++}, K) < \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$ and moreover, $1 \le \dot{I}(\lambda^{++}, K) < \mu_{\mathrm{unif}}(\lambda^{++}, 2^{\lambda^+})$. As an appetizer and to understand types and the definition of types and saturated (in the present context) and brimmed, read from §1 until 1.18.

He should read in §2 Definition 2.1 of λ-good frame, an axiomatic framework and then read the following two Definitions 2.4, 2.5 and Claim 2.6. In §3, 3.10 show how by [She01] $\approx$ [She09c] the context there gives a λ^+-good frame; of course the reader may just believe instead of reading proofs, and he may remember that our basic types are minimal in this case.

In §4 he should read some consequences of the axioms.

Then in §5 we show some amount of unique amalgamation. Then §6,§7,§8 do a parallel to [She01, §8,§9,§10] in our context; still there are differences, in particular our context is not necessarily uni-dimensional which complicates matters. But if we restrict ourselves to continuing [She01] $\approx$ [She09c], our frame is "uni-dimensional", we could have simplified the proofs by using $\mathcal{S}^{\mathrm{bs}}(M)$ as the set of minimal types.

Reading Plan 3: $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$, so $\lambda = \aleph_0$ and $1 \le \dot{I}(\aleph_1, \psi) < 2^{\aleph_1}$, recalling $\mathbf{Q}$ denotes the quantifier "there are uncountably many".

For this, [She01] $\approx$ [She09c] is irrelevant (except if we quote the "*black box*" use of the combinatorial section §3 of [She01] when using the weak diamond to get many non-isomorphic models in §5, but we prefer to use [She09d]).

Now reading plan 3 is to follow reading plan 2 but 3.10 is replaced by 3.8 which relies on [She75], i.e., it proves that we get an $\aleph_1$-good frame investigating $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$.

Note that our class may well be such that $\mathfrak{k}$ is the parallel of "superstable non-multidimensional complete first order theory"; e.g.

$$\psi_1 = (\mathbf{Q}x)[P(x)] \wedge (\mathbf{Q}x)[\neg P(x)],$$

$\tau_\psi = \{P\}$, P a unary predicate; this is categorical in $\aleph_1$ and has no model in $\aleph_0$ and ψ_1 has 3 models in $\aleph_2$. But if we use $\psi_0 = (\forall x)[P(x) \equiv P(x)]$ we have $\dot{I}(\aleph_1, \psi_0) = \aleph_0$; however, even starting with ψ_1, the derived AEC $\mathfrak{k}$ has exactly three non-isomorphic models in $\aleph_1$. In general we derived an AEC $\mathfrak{k}$ from ψ such that: $\mathfrak{k}$ is an AEC with LS number $\aleph_0$, categorical in $\aleph_0$, and the number of somewhat "saturated" models of $\mathfrak{k}$ in λ is $\le \dot{I}(\lambda, \psi)$ for $\lambda \ge \aleph_1$. The relationship of ψ and

$\mathfrak{k}$ is not comfortable; as it means that, for general results to be applied, they have to be somewhat stronger, e.g. "$\mathfrak{k}$ has $2^{\lambda^{++}}$ non-isomorphic $\underline{\lambda^+\text{-saturated}}$ models of cardinality λ^{++}". The reason is that $\mathrm{LS}(\mathfrak{k}) = \lambda = \aleph_0$; we have to find many somewhat λ^+-saturated models as we have first in a sense eliminate the quantifier $\mathbf{Q} = \exists^{\geq \aleph_1}$, (i.e., the choice of the class of models and of the order guaranteed that what has to be countable is countable, and λ^+-saturation guarantees that what should be uncountable is uncountable). This is the role of $K^{\mathbf{F}}_{\aleph_1}$ in [She09a, §3].

Reading Plan 4: $\mathfrak{k}$ an abstract elementary class which is PC_ω (= $\aleph_0$-presentable, see Definition 3.4); see [She09a] or [Mak85] which includes a friendly presentation of [She87a, §1-§3] so of [She09a, §1-§3]).

Like plan 3 but we have to use 3.5 instead of 3.8 and fortunately the reader is encouraged to read [She09a, §4,§5] to understand why we get a λ-good quadruple.

§ 1. Abstract elementary classes

First we present the basic material on AEC $\mathfrak{k}$, that is types, saturativity and (λ, κ)-brimmness (so most is repeating some things from [She09a, §1] and from [She09f]).

Second we show that the situation in $\lambda = \mathrm{LS}(\mathfrak{k})$ determine the situation above λ, moreover such lifting always exists; so a λ-AEC can be lifted to a $(\geq \lambda)$-AEC in one and only one way.

Convention 1.1. Here $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$, where K is a class of τ-models for a fixed vocabulary $\tau = \tau_K = \tau_{\mathfrak{k}}$ and $\leq_{\mathfrak{k}}$ is a two-place relation on the models in K. We do not always strictly distinguish between $\mathfrak{k}, K$ and $(K, \leq_{\mathfrak{k}})$. We shall assume that $K, \leq_{\mathfrak{k}}$ are fixed, and $M \leq_{\mathfrak{k}} N \Rightarrow M, N \in K$; and we assume that it is an abstract elementary class, see Definition 1.4 below. When we use $\leq_{\mathfrak{k}}$ in the $\prec$ sense (elementary submodel for first order logic), we write $\prec_{\mathbb{L}}$ as $\mathbb{L}$ is first order logic.

Definition 1.2. For a class of τ_K-models we let

$$\dot{I}(\lambda, K) = \big|\{M/\cong : M \in K,\ \|M\| = \lambda\}\big|.$$

Definition 1.3. 1) We say $\overline{M} = \langle M_i : i < \mu \rangle$ is a representation or filtration of a model M of cardinality μ <u>if</u> $\tau_{M_i} = \tau_M$, M_i is $\subseteq$-increasing continuous, $\|M_i\| < \|M\|$ and $M = \bigcup\{M_i : i < \mu\}$, and $\mu = \chi^+ \Rightarrow \|M_i\| = \chi$.

2) We say $\overline{M}$ is a $\leq_{\mathfrak{k}}$-representation or $\leq_{\mathfrak{k}}$-filtration of M <u>if</u> in addition $M_i \leq_{\mathfrak{k}} M$ for $i < \|M\|$ (hence $M_i, M \in K$ and $\langle M_i : i < \mu \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous, by Ax.V from Definition 1.4).

Definition 1.4. We say $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$ is an abstract elementary class, AEC in short, if (τ is as in 1.1, $Ax0$ holds and) AxI-VI hold, where:

$Ax0$: The holding of $M \in K, N \leq_{\mathfrak{k}} M$ depends on N, M only up to isomorphism, i.e., $[M \in K, M \cong N \Rightarrow N \in K]$, and [if $N \leq_{\mathfrak{k}} M$ and f is an isomorphism from M onto the τ-model M' mapping N onto N' <u>then</u> $N' \leq_{\mathfrak{k}} M'$], and of course 1.1.

AxI: If $M \leq_{\mathfrak{k}} N$ then $M \subseteq N$ (i.e. M is a submodel of N).

$AxII$: $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_2$ implies $M_0 \leq_{\mathfrak{k}} M_2$ and $M \leq_{\mathfrak{k}} M$ for $M \in K$.

$AxIII$: If λ is a regular cardinal, M_i (for $i < \lambda$) is $\leq_{\mathfrak{k}}$-increasing (i.e. $i < j < \lambda$ implies $M_i \leq_{\mathfrak{k}} M_j$) and continuous (i.e. for limit ordinal $\delta < \lambda$ we have $M_\delta = \bigcup\limits_{i<\delta} M_i$) <u>then</u> $M_0 \leq_{\mathfrak{k}} \bigcup\limits_{i<\lambda} M_i$.

$AxIV$: If λ is a regular cardinal, M_i (for $i < \lambda$) is $\leq_{\mathfrak{k}}$-increasing continuous and $M_i \leq_{\mathfrak{k}} N$ for $i < \lambda$ <u>then</u> $\bigcup\limits_{i<\lambda} M_i \leq_{\mathfrak{k}} N$.

AxV: If $M_0 \subseteq M_1$ and $M_\ell \leq_{\mathfrak{k}} N$ for $\ell = 0, 1$, <u>then</u> $M_0 \leq_{\mathfrak{k}} M_1$.

$AxVI$: $\mathrm{LS}(\mathfrak{k})$ exists [4], where $\mathrm{LS}(\mathfrak{k})$ is the minimal cardinal λ such that: if $A \subseteq N$ and $|A| \leq \lambda$ <u>then</u> for some $M \leq_{\mathfrak{k}} N$ we have $A \subseteq |M|$ and $\|M\| \leq \lambda$.

Notation 1.5. : 1) $K_\lambda = \{M \in K : \|M\| = \lambda\}$ and $K_{<\lambda} = \bigcup\limits_{\mu<\lambda} K_\mu$, etc.

[4] We normally assume $M \in \mathfrak{k} \Rightarrow \|M\| \geq \mathrm{LS}(\mathfrak{k})$ so may forget to write $\|M\|$"$+\mathrm{LS}(\mathfrak{k})$" instead $\|M\|$, here there is no loss in it. It is also natural to assume $|\tau(\mathfrak{k})| \leq \mathrm{LS}(\mathfrak{k})$ which means just increasing $\mathrm{LS}(\mathfrak{k})$, but no real need here; dealing with Hanf numbers it is natural.

Definition 1.6. 1) The function $f : N \to M$ is $\leq_{\mathfrak{k}}$-embedding <u>when</u> f is an isomorphism from N onto N' where $N' \leq_{\mathfrak{k}} M$, (so $f : N \to N'$ is an isomorphism onto).

2) We say f is a $\leq_{\mathfrak{k}}$-embedding of M_1 into M_2 over M_0 when for some M_1' we have: $M_0 \leq_{\mathfrak{k}} M_1, M_0 \leq_{\mathfrak{k}} M_1' \leq_{\mathfrak{k}} M_2$ and f is an isomorphism from M_1 onto M_1' extending the mapping id_{M_0}.

Recall

Observation 1.7. *Let I be a directed set (i.e., I is partially ordered by $\leq = \leq^I$ such that any two elements have a common upper bound).*
1) If M_t is defined for $t \in I$, and $t \leq s \in I$ implies $M_t \leq_{\mathfrak{k}} M_s$ <u>then</u> for every $t \in I$ we have $M_t \leq_{\mathfrak{k}} \bigcup\limits_{s \in I} M_s$.
2) If in addition $t \in I$ implies $M_t \leq_{\mathfrak{k}} N$ <u>then</u> $\bigcup\limits_{s \in I} M_s \leq_{\mathfrak{k}} N$.

Proof. Easy; or see [She09a, 88r-1.6], which does not rely on anything else. $\square_{1.7}$

Claim 1.8. *1) For every $N \in K$ there is a directed partial order I of cardinality $\leq \|N\|$ and sequence $\overline{M} = \langle M_t : t \in I \rangle$ such that $t \in I \Rightarrow M_t \leq_{\mathfrak{k}} N, \|M_t\| \leq \mathrm{LS}(\mathfrak{k}), I \models s < t \Rightarrow M_s \leq_{\mathfrak{k}} M_t$ and $N = \bigcup\limits_{t \in I} M_t$. If $\|N\| \geq \mathrm{LS}(\mathfrak{k})$ we can add $\|M_t\| = \mathrm{LS}(\mathfrak{k})$ for $t \in I$.*
2) For every $N_1 \leq_{\mathfrak{k}} N_2$ we can find $\langle M_t^\ell : t \in I_\ell \rangle$ as in part (1) for $\ell = 1, 2$ such that $I_1 \subseteq I_2$ and $t \in I_1 \Rightarrow M_t^2 = M_t^1$.
3) Any $\lambda \geq \mathrm{LS}(\mathfrak{k})$ satisfies the requirement in the definition of $\mathrm{LS}(\mathfrak{k})$.

Proof. Easy or see [She09a, 88r-1.7] which does not require anything else. $\square_{1.8}$

We now (in 1.9) recall the (non-classical) definition of type (note that it is natural to look at types only over models which are amalgamation bases, see part (4) of 1.9 below and consider only extensions of the models of the same cardinality). Note that though the choice of the name indicates that they are supposed to behave like complete types over models as in classical model theory (on which we are not relying), this does not guarantee most of the basic properties. E.g., when $\mathrm{cf}(\delta) = \aleph_0$, uniqueness of $p_\delta \in \mathcal{S}(M_\delta)$ such that $i < \delta \Rightarrow p_\delta \restriction M_i = p_i$ is not guaranteed even if $p_i \in \mathcal{S}(M_i), M_i$ is $\leq_{\mathfrak{k}}$-increasing continuous for $i \leq \delta$ and $i < j < \delta \Rightarrow p_i = p_j \restriction M_i$. Still we have existence: if for $i < \delta, p_i \in \mathcal{S}(M_i)$ increasing with i, then there is $p_\delta \in \mathcal{S}(\bigcup\{M_i : i < \delta\})$ such that $i < \delta \Rightarrow p_i = p_\delta \restriction M_i$. But when $\mathrm{cf}(\delta) > \aleph_0$ even existence is not guaranteed.

Definition 1.9. 1) For $M \in K_\mu$, $M \leq_{\mathfrak{k}} N \in K_\mu$, and $a \in N$, let $\mathrm{ortp}(a, M, N) = \mathrm{ortp}_{\mathfrak{k}}(a, M, N) = (M, N, a)/\mathcal{E}_M$, where $\mathcal{E}_M$ is the transitive closure of $\mathcal{E}_M^{\mathrm{at}}$, and the two-place relation $\mathcal{E}_M^{\mathrm{at}}$ is defined by:

$$\begin{aligned}(M, N_1, a_1)\ \mathcal{E}_M^{\mathrm{at}}\ (M, N_2, a_2) \text{ \underline{iff} } & M \leq_{\mathfrak{k}} N_\ell,\ a_\ell \in N_\ell,\ \|N_\ell\| = \mu = \|M\| \text{ for } \ell = 1, 2 \\ & \text{and there is } N \in K_\mu \text{ and } \leq_{\mathfrak{k}}\text{-embeddings} \\ & f_\ell : N_\ell \to N \text{ for } \ell = 1, 2 \text{ such that} \\ & f_1 \restriction M = \mathrm{id}_M = f_2 \restriction M \text{ and } f_1(a_1) = f_2(a_2).\end{aligned}$$

We may say $p = \mathrm{ortp}(a, M, N)$ is the type which a realizes over M in N. Of course, all those notions depend on $\mathfrak{k}$ so we may write $\mathrm{ortp}_{\mathfrak{k}}(a, M, N)$ and $\mathcal{E}_M[\mathfrak{k}], \mathcal{E}^{\mathrm{at}}_M[\mathfrak{k}]$.

(If in Definition 1.4 we do not require $M \in K \Rightarrow \|M\| \geq \mathrm{LS}(\mathfrak{k})$, here we should allow any N such that $\|M\| \leq \|N\| \leq M + \mathrm{LS}(\mathfrak{k})$.) The restriction to $N \in K_\mu$ is essential, and pedantically $(M, N, a)/\mathcal{E}_M$ should be replaced by $((M, N, a)/\mathcal{E}_\mu) \cap \mathcal{H}(\chi_{(M,N,a)})$ where $\chi_{(M,N,a)} = \min\{\chi : ((M, N, a)/\mathcal{E}_M) \cap \mathcal{H}(\chi) \neq \varnothing\}$ so that the equivalence class is a set.

1A) For $M \in \mathfrak{k}_\mu$ let[5] $\mathcal{S}_{\mathfrak{k}}(M) = \{\mathrm{ortp}(a, M, N) : M \leq_{\mathfrak{k}} N$ and $N \in K_\mu$ (or just $N \in K_{\leq(\mu+\mathrm{LS}(\mathfrak{k}))}$) and $a \in N\}$ and $\mathcal{S}^{\mathrm{na}}_{\mathfrak{k}}(M) = \{\mathrm{ortp}(a, M, N) : M \leq_{\mathfrak{k}} N$ and $N \in K_{\leq(\mu+\mathrm{LS}(\mathfrak{k}))}$ and $a \in N \setminus M\}$ (na stands for *non-algebraic*). We may write $\mathcal{S}^{\mathrm{na}}(M)$ omitting $\mathfrak{k}$ when $\mathfrak{k}$ is clear from the context; so omitting na means $a \in N$ rather than $a \in N \setminus M$.

2) Let $M \in K_\mu$ and $M \leq_{\mathfrak{k}} N$. We say "a realizes p in N" and "$p = \mathrm{ortp}(a, M, N)$" when: if $a \in N, p \in \mathcal{S}(M)$ and $N' \in K_{\leq(\mu+\mathrm{LS}(\mathfrak{k}))}$ satisfies $M \leq_{\mathfrak{k}} N' \leq_{\mathfrak{k}} N$ and $a \in N'$ then $p = \mathrm{ortp}(a, M, N')$ and there is at least one such N'; so $M, N' \in K_\mu$ (or just $M \leq \|N'\| \leq \mu + \mathrm{LS}(\mathfrak{k})$) but possibly $N \notin K_\mu$.

3) We say "a_2 strongly [6] realizes $(M, N^1, a_1)/\mathcal{E}^{\mathrm{at}}_M$ in N" <u>when</u> for some N^2 of cardinality $\leq \|M\| + \mathrm{LS}(\mathfrak{k})$ we have $M \leq_{\mathfrak{k}} N^2 \leq_{\mathfrak{k}} N$, $a_2 \in N^2$, and

$$(M, N^1, a_1)\, \mathcal{E}^{\mathrm{at}}_M\, (M, N^2, a_2)$$

hence $\mu = \|N^1\|$.

4) We say $M_0 \in K_\lambda$ is an amalgamation base (in $\mathfrak{k}$, but normally $\mathfrak{k}$ is understood from the context) <u>if</u>: for every $M_1, M_2 \in K_\lambda$ and $\leq_{\mathfrak{k}}$-embeddings $f_\ell : M_0 \to M_\ell$ (for $\ell = 1, 2$) there is $M_3 \in K_\lambda$ and $\leq_{\mathfrak{k}}$-embeddings $g_\ell : M_\ell \to M_3$ (for $\ell = 1, 2$) such that $g_1 \circ f_1 = g_2 \circ f_2$. Similarly for $\mathfrak{k}_{\leq\lambda}$.

4A) $\mathfrak{k}$ has amalgamation in λ (or λ-amalgamation or $\mathfrak{k}_\lambda$ has amalgamation) <u>when</u> every $M \in K_\lambda$ is an amalgamation base.

4B) $\mathfrak{k}$ has the λ-JEP or JEP_λ (or $\mathfrak{k}_\lambda$ has the JEP) <u>when</u> any $M_1, M_2 \in K_\lambda$ can be $\leq_{\mathfrak{k}}$-embedded into some $M \in K_\lambda$.

5) We say $\mathfrak{k}$ is stable in λ <u>if</u> ($\mathrm{LS}(\mathfrak{k}) \leq \lambda$ and) $M \in K_\lambda \Rightarrow |\mathcal{S}(M)| \leq \lambda$, and moreover there are no λ^+ pairwise non-$\mathcal{E}^{\mathrm{at}}_\mu$-equivalent triples (M, N, a), $M \leq_{\mathfrak{k}} N \in K_\lambda$, $a \in N$.

6) We say $p = q \restriction M$ if $p \in \mathcal{S}(M)$, $q \in \mathcal{S}(N)$, $M \leq_{\mathfrak{k}} N$, and for some N^+, $N \leq_{\mathfrak{k}} N^+$, and $a \in N^+$ we have $p = \mathrm{ortp}(a, M, N^+)$ and $q = \mathrm{ortp}(a, N, N^+)$; see 1.11(1),(2). We may express this also as "q extends p" or "p is the restriction of q to M".

7) For finite m, for $M \leq_{\mathfrak{k}} N$, $\bar{a} \in {}^m N$ we can define $\mathrm{ortp}(\bar{a}, M, N)$ and $\mathcal{S}^m_{\mathfrak{k}}(M)$ similarly and $\mathcal{S}^{<\omega}_{\mathfrak{k}}(M) = \bigcup\limits_{m<\omega} \mathcal{S}^m_{\mathfrak{k}}(M)$; similarly for $\mathcal{S}^\alpha(M)$ (but we shall not use this in any essential way, so we agree $\mathcal{S}(M) = \mathcal{S}^1(M)$.) Again we may omit $\mathfrak{k}$ when clear from the context.

8) We say that $p \in \mathcal{S}_{\mathfrak{k}}(M)$ is algebraic when some $a \in M$ realizes it.

9) We say that $p \in \mathcal{S}_{\mathfrak{k}}(M)$ is minimal <u>when</u> it is not algebraic and, for every $N \in K$ of cardinality $\leq \|M\| + \mathrm{LS}(\mathfrak{k})$ which $\leq_{\mathfrak{k}}$-extends M, the type p has at most one non-algebraic extension in $\mathcal{S}_{\mathfrak{k}}(M)$.

Remark 1.10. 1) Note that here "amalgamation base" means only for extensions of the same cardinality!

---

[5]If we omit $M \in K \Rightarrow \|M\| \geq \mathrm{LS}(\mathfrak{k})$ in 1.4, still we can insist that $N \in K_\mu$, the difference is not serious.

[6]Note that $\mathcal{E}^{\mathrm{at}}_M$ is not necessarily an equivalence relation, and hence in general it is not $\mathcal{E}_M$.

2) The notion "minimal type" is important (for categoricity) but not used much in this chapter.

Observation 1.11. *0) Assume $M \in K_\mu$ and $M \leq_{\mathfrak{k}} N$, $a \in N$ then* $\text{ortp}(a, M, N)$ *is well defined and is p if for some $M' \in K_\mu$ we have $M \cup \{a\} \subseteq M' \leq_{\mathfrak{k}} N$ and $p = \text{ortp}(a, M, M')$.*

1) If $M \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} N_2$, $M \in K_\mu$, and $a \in N_1$ then $\text{ortp}(a, M, N_1)$ is well defined and equal to $\text{ortp}(a, M, N_2)$. (More transparent if $\mathfrak{k}$ has the μ-amalgamation, which is the real case anyhow.)

2) If $M \leq_{\mathfrak{k}} N$ and $q \in \mathcal{S}(N)$ then for one and only one p we have $p = q \restriction M$.

3) If $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_2$ and $p \in \mathcal{S}(M_2)$ then $p \restriction M_0 = (p \restriction M_1) \restriction M_0$.

4) If $M \in \mathfrak{k}_\mu$ is an amalgamation base then $\mathcal{E}^{\text{at}}_M$ is a transitive relation hence is equal to $\mathcal{E}_M$.

5) If $M \leq_{\mathfrak{k}} N$ are from $\mathfrak{k}_\lambda$, M is an amalgamation base and $p \in \mathcal{S}(M)$ then there is $q \in \mathcal{S}(N)$ extending p, so the mapping $q \mapsto q \restriction M$ is a function from $\mathcal{S}(N)$ onto $\mathcal{S}(M)$.

Proof. Easy. $\square_{1.11}$

Definition 1.12. 1) We say N is λ-*universal over* M when $\lambda \geq \|N\|$ and for every M' with $M \leq_{\mathfrak{k}} M' \in K_\lambda$, there is a $\leq_{\mathfrak{k}}$-embedding of M' into N over M. If we omit λ we mean $\|N\|$; clearly if N is universal over M and both are from K_λ then M is an amalgamation base.

2) $K^{3,\text{na}}_\lambda = \{(M, N, a) : M \leq_{\mathfrak{k}} N, a \in N \setminus M$ and $M, N \in \mathfrak{k}_\lambda\}$, with the partial order $\leq$ defined by $(M, N, a) \leq (M', N', a')$ iff $a = a', M \leq_{\mathfrak{k}} M'$ and $N \leq_{\mathfrak{k}} N'$.

3) We say $(M, N, a) \in K^{3,\text{na}}_\lambda$ is minimal when: if $(M, N, a) \leq (M', N_\ell, a) \in K^{3,\text{na}}_\lambda$ for $\ell = 1, 2$ implies $\text{ortp}(a, M', N_1) = \text{ortp}(a, M', N_2)$. Moreover, $(M', N_1, a) \, \mathcal{E}^{\text{at}}_\lambda \, (M', N_2, a)$ (this strengthening is not needed if every $M' \in K_\lambda$ is an amalgamation bases).

4) $N \in \mathfrak{k}$ is λ-universal if every $M \in \mathfrak{k}_\lambda$ can be $\leq_{\mathfrak{k}}$-embedded into it.

5) We say $N \in \mathfrak{k}$ is universal for $K' \subseteq \mathfrak{k}$ when every $M \in K'$ can be $\leq_{\mathfrak{k}}$-embedded into N.

Remark 1.13. Why do we use $\leq$ on $K^{3,\text{na}}_\lambda$? Because those triples serve us as a representation of types for which direct limit exists.

Definition 1.14. 1) $M^* \in K_\lambda$ is *superlimit* if clauses (a) + (b) + (c) below hold, *locally superlimit* if clauses $(a)^- + (b) + (c)$ below hold, and is *pseudo superlimit* if clauses (b) + (c) below hold, where:

- (a) It is universal (i.e. every $M \in K_\lambda$ can be $\leq_{\mathfrak{k}}$-embedded into M^*).
- (b) If $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous, $\delta < \lambda^+$ and $i < \delta \Rightarrow M_i \cong M^*$ then $M_\delta \cong M^*$.
- $(a)^-$ If $M^* \leq_{\mathfrak{k}} M_1 \in K_\lambda$ then there is $M_2 \in K_2$ which $\leq_{\mathfrak{k}}$-extends M_1 and is isomorphic to M^*.
- (c) There is M^{**} isomorphic to M^* such that $M^* <_{\mathfrak{k}} M^{**}$.

2) M is λ-saturated above μ <u>when</u> $\|M\| \geq \lambda > \mu \geq \mathrm{LS}(\mathfrak{k})$ and if $N \leq_{\mathfrak{k}} M$, $\mu \leq \|N\| < \lambda$, $N \leq_{\mathfrak{k}} N_1$, $\|N_1\| \leq \|N\|+\mathrm{LS}(\mathfrak{k})$, and $a \in N_1$ then some $b \in M$ strongly realizes $(N, N_1, a)/\mathcal{E}_N^{\mathrm{at}}$ in M (see Definition 1.9(3)). Omitting "above μ" means "for some $\mu < \lambda$," hence "M is λ^+-saturated" means that "M is λ^+-saturated above λ" and $K(\lambda^+\text{-saturated}) = \{M \in K : M \text{ is } \lambda^+\text{-saturated}\}$ and "M is saturated" means "M is $\|M\|$-saturated".

In the following lemma note that amalgamation in $\mathfrak{k}_{<\lambda}$ is not assumed: it is even deduced. For variety we allow $K_{<\mathrm{LS}(\mathfrak{k})} \neq \varnothing$.

Lemma 1.15. *[The Model-homogeneity = Saturativity Lemma] Let $\lambda > \mu + \mathrm{LS}(\mathfrak{k})$ and $M \in K$.*

1) M is λ-saturated above μ <u>iff</u> M is $(\mathbb{D}_{\mathfrak{k}_{\geq\mu}}, \lambda)$-homogeneous above μ, which means: for every $N_1 \leq_{\mathfrak{k}} N_2 \in K$ such that $\mu \leq \|N_1\| \leq \|N_2\| < \lambda$ and $N_1 \leq_{\mathfrak{k}} M$, there is a $\leq_{\mathfrak{k}}$-embedding f of N_2 into M over N_1.

2) If $M_1, M_2 \in K_\lambda$ are λ-saturated above $\mu < \lambda$ and for some $N_1 \leq_{\mathfrak{k}} M_1, N_2 \leq_{\mathfrak{k}} M_2$, both of cardinality $\in [\mu, \lambda)$, we have $N_1 \cong N_2$ <u>then</u> $M_1 \cong M_2$; in fact, any isomorphism f from N_1 onto N_2 can be extended to an isomorphism from M_1 onto M_2.

3) If in (2) we demand only "M_2 is λ-saturated" and $M_1 \in K_{\leq\lambda}$ <u>then</u> f can be extended to a $\leq_{\mathfrak{k}}$-embedding from M_1 into M_2.

4) In part (2) instead of $N_1 \cong N_2$ it suffices to assume that N_1 and N_2 can be $\leq_{\mathfrak{k}}$-embedded into some $N \in K$, which holds if $\mathfrak{k}$ has the JEP or just θ-JEP for some $\theta < \lambda, \theta \geq \mu$. Similarly for part (3).

5) If N is λ-universal over $M \in K_\mu$ and $\mathfrak{k}$ has μ-JEP then N is λ-universal (where $\lambda \geq \mathrm{LS}(\mathfrak{k})$ for simplicity).

6) Assume M is λ-saturated above μ. If $N \leq_{\mathfrak{k}} M$ and $\mu \leq \|N\| < \lambda$ <u>then</u> N is an amalgamation base (in $K_{\leq(\|N\|+\mathrm{LS}(\mathfrak{k}))}$ and even in $\mathfrak{k}_{\leq\lambda}$) and $|\mathcal{S}(N)| \leq \|M\|$. So if every $N \in K_\mu$ can be $\leq_{\mathfrak{k}}$-embedded into M then $\mathfrak{k}$ has μ-amalgamation.

Proof. 1) The "if" direction is easy as $\lambda > \mu + \mathrm{LS}(\mathfrak{k})$. Let us prove the other direction.

We prove this by induction on $\|N_2\|$. Now first consider the case $\|N_2\| > \|N_1\| + \mathrm{LS}(\mathfrak{k})$ then we can find a $\leq_{\mathfrak{k}}$-increasing continuous sequence $\langle N_{1,\varepsilon} : \varepsilon < \|N_2\|\rangle$ with union N_2 with $N_{1,0} = N_1$ and $\|N_{1,\varepsilon}\| \leq \|N_1\| + |\varepsilon|$. Now we choose f_ε, a $\leq_{\mathfrak{k}}$-embedding of $N_{1,\varepsilon}$ into M, increasing continuous with ε such that $f_0 = \mathrm{id}_{N_1}$. For $\varepsilon = 0$ this is trivial for ε limit take unions and for ε successor use the induction hypothesis. So without loss of generality $\|N_2\| \leq \|N_1\| + \mathrm{LS}(\mathfrak{k})$.

Let $|N_2| = \{a_i : i < \kappa\}$, and we know $\mu \leq \kappa'' := \|N_1\| \leq \kappa := \|N_2\| \leq \kappa' := \|N_1\|+\mathrm{LS}(\mathfrak{k}) < \lambda$; so if, as usual, $\|N_1\| \geq \mathrm{LS}(\mathfrak{k})$ then $\kappa' = \kappa$. We define by induction on $i \leq \kappa, N_1^i, N_2^i, f_i$ such that:

(a) $N_1^i \leq_{\mathfrak{k}} N_2^i$ and $\|N_1^i\| \leq \|N_2^i\| \leq \kappa'$
(b) N_1^i is $\leq_{\mathfrak{k}}$-increasing continuous with i
(c) N_2^i is $\leq_{\mathfrak{k}}$-increasing continuous with i
(d) f_i is a $\leq_{\mathfrak{k}}$-embedding of N_1^i into M
(e) f_i is increasing continuous with i
(f) $a_i \in f_i(N_1^{i+1})$
(g) $N_1^0 = N_1, N_2^0 = N_2, f_0 = \mathrm{id}_{N_1}$.

For $i = 0$, clause (g) gives the definition. For i limit let:
$N_1^i = \bigcup_{j<i} N_1^j$ and $N_2^i = \bigcup_{j<i} N_2^j$ and $f_i = \bigcup_{j<i} f_j$.

Now (a)-(f) continues to hold by continuity (and $\|N_2^i\| \le \kappa'$ easily).

For i successor we use our assumption; more elaborately, let $M_1^{i-1} \le_{\mathfrak{k}} M$ be $f_{i-1}(N_1^{i-1})$ and let M_2^{i-1}, g_{i-1} be such that g_{i-1} is an isomorphism from N_2^{i-1} onto M_2^{i-1} extending f_{i-1}, so $M_1^{i-1} \le_{\mathfrak{k}} M_2^{i-1}$ (but without loss of generality $M_2^{i-1} \cap M = M_1^{i-1}$). Now apply the saturation assumption[7] with $M, (M_1^{i-1}, M_2^{i-1}), g_{i-1}(a))$ here standing for $M, (N, N_1, a)$ there (note: $a_{i-1} \in N_2 = N_2^0 \subseteq N_2^{i-1}$ and

$$\lambda > \kappa' \ge \|N_2^{i-1}\| = \|M_2^{i-1}\| \ge \|M_1^{i-1}\| = \|N_1^{i-1}\| \ge \|N_1^0\| = \|N_1\| = \kappa'' \ge \mu,$$

so the requirements — including the requirements on the cardinalities in Definition 1.14(2) — hold). So there is $b \in M$ such that

$$\text{ortp}(b, M_1^{i-1}, M) = \text{ortp}(g_{i-1}(a_{i-1}), M_1^{i-1}, M_2^{i-1}).$$

Moreover,[8] remembering the end of the first sentence in 1.14(2) which speaks about "strongly realizes", b strongly realizes $(M_1^{i-1}, M_3^{i-1}, g_{i-1}(a_{i-1}))/\mathcal{E}^{\text{at}}_{M_1^{i-1}}$ in M. This means (see Definition 1.9(3)) that for some $M_1^{i,*}$ we have $b \in M_1^{i,*}$ and $M_1^{i-1} \le_{\mathfrak{k}} M_1^{i,*} \le_{\mathfrak{k}} M$ and

$$(M_1^{i-1}, M_2^{i-1}, g_{i-1}(a_{i-1}))\ \mathcal{E}^{\text{at}}_{M_1^{i-1}}\ (M_1^{i-1}, M_1^{i,*}, b).$$

This means (see Definition 1.9(1)) that $M_1^{i,*}$ also has cardinality $\le \kappa'$ and there is $M_2^{i,*} \in K_{\le\kappa'}$ such that $M_1^{i-1} \le_{\mathfrak{k}} M_2^{i,*}$ and there are $\le_{\mathfrak{k}}$-embeddings h_2^i, h_1^i of $M_2^{i-1}, M_1^{i,*}$ into $M_2^{i,*}$ over M_1^{i-1} respectively, such that $h_2^i(g_{i-1}(a_{i-1})) = h_1^i(b)$.

Now changing names, without loss of generality h_1^i is the identity.

Let N_2^i, h_i be such that $N_2^{i-1} \le_{\mathfrak{k}} N_2^i$ and h_i an isomorphism from N_2^i onto $M_2^{i,*}$ extending g_{i-1}. Let $N_1^i = h_i^{-1}(M_1^{i,*})$ and $f_i = (h_i \restriction N_1^i)$.

We have carried the induction. Now f_κ is a $\le_{\mathfrak{k}}$-embedding of N_1^κ into M over N_1, but $|N_2| = \{a_i : i < \kappa\} \subseteq N_1^\kappa$. Hence by Ax.V of Definition 1.4, $N_2 \le_{\mathfrak{k}} N_1^\kappa$, so $f_\kappa \restriction N_2 : N_2 \to M$ is as required.

2), 3) By the hence and forth argument (or see [She09a, 88r-2.3], [She09a, 88r-2.4] or see [She87b, II,§3] = [She09f, §3]).

4),5),6) Easy, too. $\square_{1.15}$

Definition 1.16. 1) For $\partial = \text{cf}(\partial) \le \lambda^+$, we say <u>$N$ is (λ, ∂)-brimmed over M</u> if ($M \le_{\mathfrak{k}} N$ are in K_λ and) we can find a sequence $\langle M_i : i < \partial\rangle$ which is $\le_{\mathfrak{k}}$-increasing,[9] $M_i \in K_\lambda, M_0 = M, M_{i+1}$ is $\le_{\mathfrak{k}}$-universal[10] over M_i and $\bigcup_{i<\partial} M_i = N$. We say N is (λ, ∂)-brimmed over A if $A \subseteq N \in K_\lambda$ and we can find $\langle M_i : i < \partial\rangle$ as above such that $A \subseteq M_0$ but $M_0 \restriction A \le_{\mathfrak{k}} M_0 \Rightarrow M_0 = A$; if $A = \varnothing$ we may omit "over A". We say continuously (λ, ∂)-brimmed (over M) <u>when</u> the sequence $\langle M_i : i < \partial\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous; if $\mathfrak{k}_\lambda$ has amalgamation, the two notions coincide.

[7]See Definition 1.14(21).

[8]If $\mathfrak{k}$ has amalgamation in μ the proof is slightly shorter

[9]We have not asked continuity; because in the direction we are going, it makes no difference if we add "continuous". Then we have in general fewer cases of existence, uniqueness (of being (λ, ∂)-brimmed over $M \in K_\lambda$) does not need extra assumptions, and existence is harder.

[10]Hence M_i is an amalgamation base.

2) We say N is $(\lambda, *)$-brimmed over M <u>if</u> for some $\partial \le \lambda, N$ is (λ, ∂)-brimmed over M. We say N is $(\lambda, *)$-brimmed if for some M, N is $(\lambda, *)$-brimmed over M.

3) If $\alpha < \lambda^+$ let "N is (λ, α)-brimmed over M" mean $M \le_{\mathfrak{k}} N$ are from K_λ and $\mathrm{cf}(\alpha) \ge \aleph_0 \Rightarrow N$ is $(\lambda, \mathrm{cf}(\alpha))$-brimmed over M.

On the meaning of (λ, ∂)-brimmed for elementary classes, see 3.1(2) below. Recall

Claim 1.17. *Assume $\lambda \ge \mathrm{LS}(\mathfrak{k})$.*

1) If $\mathfrak{k}$ has amalgamation in λ, is stable in λ and $\partial = \mathrm{cf}(\partial) \le \lambda$, <u>then</u>

- *(a) for every $M \in \mathfrak{k}_\lambda$ there is $N, M \le_{\mathfrak{k}} N \in K_\lambda$, universal over M*
- *(b) for every $M \in \mathfrak{k}_\lambda$ there is $N \in \mathfrak{k}_\lambda$ which is (λ, ∂)-brimmed over M*
- *(c) if N is (λ, ∂)-brimmed over M <u>then</u> N is universal over M.*

2) If N_ℓ is $(\lambda, \aleph_0)$-brimmed over M for $\ell = 1, 2$, <u>then</u> N_1, N_2 are isomorphic over M.

3) Assume $\partial = \mathrm{cf}(\partial) \le \lambda^+$, and for every $\aleph_0 \le \theta = \mathrm{cf}(\theta) < \partial$ any (λ, θ)-brimmed model is an amalgamation base (in $\mathfrak{k}$). <u>Then</u>:

- *(a) if N_ℓ is (λ, ∂)-brimmed over M for $\ell = 1, 2$ <u>then</u> N_1, N_2 are isomorphic over M*
- *(b) if $\mathfrak{k}$ has λ-JEP (i.e., the joint embedding property in λ) and N_1, N_2 are (λ, ∂)-brimmed <u>then</u> N_1, N_2 are isomorphic.*

3A) There is a (λ, ∂)-brimmed model N over $M \in K_\lambda$ <u>when</u>: M is an amalgamation base, and for every $\le_{\mathfrak{k}_\lambda}$-extension M_1 of M there is a $\le_{\mathfrak{k}_\lambda}$-extension M_2 of M_1 which is an amalgamation base and there is a λ-universal extension $M_3 \in K_\lambda$ of M_2.

4) Assume $\mathfrak{k}$ has λ-amalgamation and the λ-JEP and $\overline{M} = \langle M_i : i \le \lambda \rangle$ is $\le_{\mathfrak{k}}$-increasing continuous and $M_i \in K_\lambda$ for $i \le \lambda$.

- *(a) If λ is regular and for every $i < \lambda, p \in \mathcal{S}(M_i)$ for some $j \in (i, \lambda)$, some $a \in M_j$ realizes p, <u>then</u> M_λ is universal over M_0 and is (λ, λ)-brimmed over M_0*
- *(b) if for every $i < \lambda$ every $p \in \mathcal{S}(M_i)$ is realized in M_{i+1} <u>then</u> M_λ is $(\lambda, \mathrm{cf}(\lambda))$-brimmed over M_0.*

5) Assume $\partial = \mathrm{cf}(\partial) \le \lambda$ and $M \in \mathfrak{k}$ is continuous (λ, ∂)-brimmed. <u>Then</u> M is a locally $(\lambda, \{\partial\})$-strongly limit model in $\mathfrak{k}_\lambda$ (see Definition [She09a, 88r-3.1]*(2),(7), not used).*

6) If N is (λ, ∂)-brimmed over M and $A \subseteq N, |A| < \partial$, e.g. $A = \{a\}$ <u>then</u> for some M' we have $M \cup A \subseteq M' <_{\mathfrak{k}} M$ and M is (λ, ∂)-brimmed over M'.

Proof. 1) Clause (c) holds by Definition 1.16.

As for clause (a), for any given $M \in K_\lambda$, easily there is an $\le_{\mathfrak{k}}$-increasing continuous sequence $\langle M_i : i \le \lambda \rangle$ of models from $K_\lambda, M_0 = M$ such that $p \in \mathcal{S}(M_i) \Rightarrow p$ is realized in M_{i+1}, this by stability + amalgamation. So $\langle M_i : i \le \lambda \rangle$ is as in part (4) below hence by clause (b) of part (4) below, we get that M_δ is $\le_{\mathfrak{k}}$-universal over $M_0 = M$ so we are done. Clause (b) follows by (a).

2) By (3)(a) because the extra assumption in part (3) is empty when $\partial = \aleph_0$.

3) **Clause (a)** holds by the hence and forth argument, that is assume $\langle N_{\ell,i} : i < \partial \rangle$ is $\le_{\mathfrak{k}}$-increasing with union $N_{\ell,\partial}, N_{\ell,0} = M, N_{\ell,i+1}$ is universal over $N_{\ell,i}$ and $N_\ell = N_{\ell,\partial}$ so $N_{\ell,i} \in \mathfrak{k}_\lambda$.

Now for each limit $\delta < \partial$ the model $N'_{\ell,\delta} := \cup\{N_{\ell,i} : i < \delta\}$ is an amalgamation base (and is $\le_{\mathfrak{k}} N_{\ell,\delta+1}$) hence without loss of generality $\langle N_{\ell,i} : i \le \partial \rangle$ is $\le_{\mathfrak{k}}$-increasing continuous. We now choose f_i by induction on $i \le \partial$ such that:

(*i*) if i is odd, f_i is a $\leq_{\mathfrak{k}}$-embedding of $N_{1,i}$ into $N_{2,i}$.

(*ii*) if i is even, f_i^{-1} is a $\leq_{\mathfrak{k}}$-embedding of $N_{2,i}$ into $N_{1,i}$.

(*iii*) if i is limit then f_i is an isomorphism from $N_{1,i}$ onto $N_{2,i}$.

(*iv*) f_i is increasing continuous with i.

(*v*) if $i = 0$ then $f_0 = \mathrm{id}_M$.

For $i = 0$ let $f_0 = \mathrm{id}_M$. If $i = 2j + 2$ use "$N_{1,i}$ is a universal extension of $N_{1,2j+1}$ (in $\mathfrak{k}_\lambda$) and f_{2j+1} is a $\leq_{\mathfrak{k}}$-embedding of $N_{1,2j+1}$ into $N_{2,2j+1}$ (by clause (i) applied to $2j + 1$) and $N_{1,2j+1}$ is an amalgamation base". That is, $N_{2,i}$ is a $\leq_{\mathfrak{k}}$-extension of $f_{2j+1}(N_{2j+1})$ which is an amalgamation base so f_{2j+1}^{-1} can be extended to a $\leq_{\mathfrak{k}}$-embedding of f_i^{-1} of $N_{2,i}$ into $N_{1,i}$. For $i = 2j+1$ use "$N_{2,i}$ is a universal extension (in $\mathfrak{k}_\lambda$) of $N_{2,2j}$ and f_{2j}^{-1} is a $\leq_{\mathfrak{k}}$-embedding of $N_{2,2j}$ into $N_{1,2j}$ and $N_{2,2j}$ is an amalgamation base (in $\mathfrak{k}_\lambda$)".

For i limit let $f_i = \cup\{f_j : j < i\}$. Clearly f_∂ is an isomorphism from $N_1 = N_{1,\partial}$ onto $N_{2,\partial} = N_2$ so we are done, i.e. clause (a) holds.

As for clause (b), for $\ell = 1, 2$ we can assume that $\langle N_{\ell,i} : i \leq \partial\rangle$ exemplifies "N_ℓ is (λ, ∂)-brimmed" so $N_\ell = N_{\ell,\partial}$ and without loss of generality as above $\langle N_{\ell,i} : i \leq \partial\rangle$ is $\leq_{\mathfrak{k}_\lambda}$-increasing continuous. By the λ-JEP there is a pair (g_1, N) such that $N_{1,0} \leq_{\mathfrak{k}} N \in K_\lambda$ and g_1 is a $\leq_{\mathfrak{k}}$-embedding of $N_{2,0}$ into N. As above there is a $\leq_{\mathfrak{k}}$-embedding g_2 of N into $N_{1,1}$ over $N_{1,0}$. Let $f_0 = (g_2 \circ g_1)^{-1}$ and continue as in the proof of clause (a).

3A) Easy, too.

4) We first proved weaker version of (a) and of (b) called $(a)^-$,$(b)^-$ respectively.

Clause (a)$^-$: Like (a) but we conclude only: M_λ is universal over M_0.

So let N satisfy $M_0 \leq_{\mathfrak{k}} N \in K_\lambda$ and we shall prove that N is $\leq_{\mathfrak{k}}$-embeddable into M_λ over M_0. Let $\langle S_i : i < \lambda\rangle$ be a partition of λ such that $|S_i| = \lambda$, $\min(S_i) \geq i$ for $i < \lambda$. We choose a quadruple $(N_i, f_i, \bar{\mathbf{a}}_i, j_i)$ by induction on $i < \lambda$ such that:

⊛ (a) $N_i \in K_\lambda$ is $\leq_{\mathfrak{k}}$-increasing continuous.
(b) $N_0 = N$
(c) $\bar{\mathbf{a}}_i = \langle a_\alpha : \alpha \in S_i\rangle$ lists the members of N_i.
(d) $j_i < \lambda$ is increasing continuous.
(e) f_i is a $\leq_{\mathfrak{k}}$-embedding of M_i into M_i.
(f) $f_0 = \mathrm{id}_{M_0}$
(g) f_i is $\subseteq$-increasing continuous.
(h) If $i = \alpha + 1$ then $a_\alpha \in \mathrm{rang}(f_i)$.

There is no problem to carry the definition (below, proving (a) we give more details) and necessarily $f = \cup\{f_i : i < \lambda\}$ is an isomorphism from M_λ onto $N_\lambda := \cup\{N_i : i < \lambda\}$, so $f^{-1} \restriction N$ is a $\leq_{\mathfrak{k}}$-embedding of N into M_λ over M_0 (as $f^{-1} \restriction N \supseteq \mathrm{id}_{M_0}$), so we are done.

Clause (b)$^-$: Like clause (b) but we conclude only: M_λ is universal over M_0.

Similar to the proof of $(a)^-$ except that we demand $j_i = i$.

Clause (a): Let $M_0 \leq_{\mathfrak{k}} N \in K_\lambda$ and we let $\langle S_i : i < \lambda\rangle$ be a partition of λ to λ sets each with λ members, $i \leq \min(S_i)$. Let $M_{1,i} = M_i$ for $i \leq \lambda$ and we choose $\langle M_{2,i} : i \leq \delta\rangle$ which is $\leq_{\mathfrak{k}}$-increasing such that $M_{2,i} \in \mathfrak{k}, M_{2,0} = M_{1,0}, N \leq_{\mathfrak{k}} M_{2,1}$ and $M_{2,i+1} \in K_\lambda$ is $\leq_{\mathfrak{k}}$-universal over $M_{2,i}$, possible as we have already proved clause $(a)^-$ recalling $\mathfrak{k}$ has λ-amalgamation and the λ-JEP.

We shall prove that $M_{1,\lambda}, M_{2,\lambda}$ are isomorphic over $M_0 = M_{1,0}$, this clearly suffices. We choose a quintuple $(j_i, M_{3,i}, f_{1,i}, f_{2,i}, \bar{\mathbf{a}}_i)$ by induction on $i < \lambda$ such that

⊛ (a) $j_i < \lambda$ is increasing continuous.

(b) $M_{3,i} \in K_\lambda$ is $\leq_{\mathfrak{k}}$-increasing continuous.
(c) $f_{\ell,i}$ is a $\leq_{\mathfrak{k}}$-embedding of M_{ℓ,j_i} into M, for $\ell = 1,2$.
(d) $f_{\ell,i}$ is increasing continuous with i, for $\ell = 1,2$.
(e) $\bar{\mathbf{a}}_i = \langle a^i_\varepsilon : \varepsilon \in S_i\rangle$ lists the members of $M_{3,i}$.
(f) If $\varepsilon \in S_i$ then $a^i_\varepsilon \in \operatorname{rang}(f_{1,2\varepsilon+1})$ and $a^i_\varepsilon \in \operatorname{rang}(f_{2,2\varepsilon+2})$.

If we succeed then $f_\ell := \cup\{f_{\ell,i} : i < \lambda\}$ is a $\leq_{\mathfrak{k}}$-embedding of $M_{\ell,\lambda}$ into $M_{3,\lambda} := M_3 := \cup\{M_{3,i} : i < \lambda\}$ and this embedding is onto because $a \in M_3 \Rightarrow$ for some $i < \lambda, a \in M_{3,i} \Rightarrow$ for some $i < \lambda$ and $\varepsilon \in S_i, a = a^i_\varepsilon \Rightarrow a = a^i_\varepsilon \in \operatorname{rang}(f_{\ell,\varepsilon+1}) \Rightarrow a \in \operatorname{rang}(f_\ell)$. So $f_1^{-1} \circ f_2$ is an isomorphism from $M_{2,\lambda}$ onto $M_{1,\lambda} = M_\lambda$ so as said above we are done.

Carrying the induction; for $i = 0$ use "$\mathfrak{k}$ has the λ-JEP" for $M_{1,0}, M_{2,0}$.

For i limit take unions.

For $i = 2\varepsilon+1$ let $j_i = \min\{j < \lambda_i : j > j_{2\varepsilon}$ and $(f^1_{2\varepsilon})^{-1}(\operatorname{ortp}(a^i_\varepsilon, f^1_{2\varepsilon}(M_{1,i}), M_{3,i})) \in \mathcal{S}_{\mathfrak{k}}(M_{1,i})$ is realized in M_j and continue as in the proof of 1.15(1), so can avoid using "$(f^1_i)^{-2}$ of a type.

For $i = 2\varepsilon+2$, the proof is similar. So $M_{2,\lambda}$ is $(\lambda, \mathrm{cf}(\lambda))$-brimmed over $M_{2,0} = M_0$ hence also M_λ being isomorphic to $M_{2,\lambda}$ over M_0 is $(\lambda, \mathrm{cf}(\lambda))$-brimmed over M_0, as required.

Clause (b): As in the proof of clause (a) but now $j_i = i$.

5) Easy and not used. (Let $\langle M_i : i \leq \partial\rangle$ witness "M is (λ, ∂)-brimmed", so M can be $\leq_{\mathfrak{k}}$-embedded into M_i, hence without loss of generality $M_0 \cong M_1$. Now use $\mathbf{F}$ such that $\mathbf{F}(M')$ is a $\leq_{\mathfrak{k}_\lambda}$-extension of M' which is $\leq_{\mathfrak{k}_\lambda}$-universal over it and is an amalgamation base.)

6) Easy. $\square_{1.17}$

Claim 1.18. *1) Assume that $\mathfrak{k}$ is an AEC, $\mathrm{LS}(\mathfrak{k}) \leq \lambda$, $\mathfrak{k}$ has λ-amalgamation and is stable in λ, and no $M \in K_\lambda$ is $\leq_{\mathfrak{k}}$-maximal. Then there is a saturated $N \in K_{\lambda^+}$. Also for every saturated $N \in K_{\lambda^+}$ (in $\mathfrak{k}$, above λ of course) we can find a $\leq_{\mathfrak{k}}$-representation $\bar{N} = \langle N_i : i < \lambda^+\rangle$, with N_{i+1} being $(\lambda, \mathrm{cf}(\lambda))$-brimmed over N_i and N_0 being (λ,λ)-brimmed.*

2) If for $\ell = 1,2$ we have $\bar{N}^\ell = \langle N^\ell_i : i < \lambda^+\rangle$ as in part (1), then there is an isomorphism f from N^1 onto N^2 mapping N^1_i onto N^2_i for each $i < \lambda^+$. Moreover, for any $i < \lambda^+$ and isomorphism g from N^1_i onto N^2_i we can find an isomorphism f from N^1 onto N^2 extending g and mapping N^1_j onto N^2_j for each $j \in [i, \lambda^+)$.

3) If $N^0 \leq_{\mathfrak{k}} N^1$ are both saturated (above λ) and are in K_{λ^+} (hence $\mathrm{LS}(\mathfrak{k}) \leq \lambda$), then we can find $\leq_{\mathfrak{k}}$-representation $\bar{N}^\ell$ of N^ℓ as in (1) for $\ell = 1,2$ with $N^0_i = N^0 \cap N^1_i$, (so $N^0_i \leq_{\mathfrak{k}} N^1_i$) for $i < \lambda^+$.

4) If $M \in K_{\lambda^+}$ and $\mathfrak{k}$ has λ-amalgamation and is stable in λ (and $\mathrm{LS}(\mathfrak{k}) \leq \lambda$), then for some $N \in K_{\lambda^+}$ saturated (above λ) we have $M \leq_{\mathfrak{k}} N$.

Proof. Easy (for (2),(3) using 1.15(6)), e.g.

4) There is a $\leq_{\mathfrak{k}}$-increasing continuous sequence $\langle M_i : i < \lambda^+\rangle$ with union M such that $M_i \in K_\lambda$. Now we choose N_i by induction on $i < \lambda$

$(*)$ (a) $N_i \in K_\lambda$ is $\leq_{\mathfrak{k}}$-increasing continuous
(b) N_{i+1} is $(\lambda, \mathrm{cf}(\lambda))$-brimmed over N_i
(c) $N_0 = M_0$.

This is possible by 1.17(1). Then by induction on $i \le \lambda^+$ we choose a $\le_{\mathfrak{k}}$-embedding f_i of M_i into N_i, increasing continuous with i. For $i = 0$ let $f_i = \mathrm{id}_{M_0}$. For i limit use union.

Lastly, for $i = j + 1$ use "$\mathfrak{k}$ has λ-amalgamation" and "N_j is universal over N_i". Now by renaming without loss of generality $f_{\lambda^+} = \mathrm{id}_{N_{\lambda^+}}$ and we are done. (Of course, we have assumed less). $\square_{1.18}$

You may wonder why in this work we have not restricted ourselves $\mathfrak{k}$ to "abstract elementary class in λ" say in §2 below (or in [She01]); by the following facts (mainly 1.24) this is immaterial.

Definition 1.19. 1) We say that $\mathfrak{k}_\lambda$ is a λ-abstract elementary class or λ-AEC in short, when:

- (a) $\mathfrak{k}_\lambda = (K_\lambda, \le_{\mathfrak{k}_\lambda})$,
- (b) K_λ is a class of τ-models of cardinality λ closed under isomorphism for some vocabulary $\tau = \tau_{\mathfrak{k}_\lambda}$,
- (c) $\le_{\mathfrak{k}_\lambda}$ a partial order of K_λ, closed under isomorphisms
- (d) axioms (0 and) I,II,III,IV,V of abstract elementary classes (see 1.4) hold except that in Ax.III we demand $\delta < \lambda^+$ (you can demand this also in Ax.IV).

2) For an abstract elementary class $\mathfrak{k}$ let $\mathfrak{k}_\lambda = (K_\lambda, \le_{\mathfrak{k}} \restriction K_\lambda)$ and similarly $\mathfrak{k}_{\ge\lambda}, \mathfrak{k}_{\le\lambda}, \mathfrak{k}_{[\lambda,\mu]}$ and define $(\le \lambda)$-AEC and $[\lambda, \mu]$-AEC, etc.

3) Definitions 1.9, 1.12, 1.14, 1.16 apply to λ-AEC $\mathfrak{k}_\lambda$.

Observation 1.20. *If $\mathfrak{k}^1$ is an AEC with $K^1_\lambda \ne \varnothing$ then*

(A) $\mathfrak{k}^1_\lambda$ is a λ-AEC.

(B) if $\mathfrak{k}^2_\lambda$ is a λ-AEC and $\mathfrak{k}^1_\lambda = \mathfrak{k}^2_\lambda$ then Definitions 1.9, 1.12, 1.14, 1.16, when applied to $\mathfrak{k}^1$ (but restricting ourselves to models of cardinality λ) and when applied to $\mathfrak{k}^2_\lambda$, are equivalent.

Proof. Just read the definitions. $\square_{1.20}$

We may wonder

Problem 1.21. : Suppose $\mathfrak{k}^1, \mathfrak{k}^2$ are AEC such that for some $\lambda > \mu \ge \mathrm{LS}(\mathfrak{k}^1)$, $\mathrm{LS}(\mathfrak{k}^2)$ and $\mathfrak{k}^1_\lambda = \mathfrak{k}^2_\lambda$. Can we bound the first such λ above μ? (Well, better bound than the Lowenheim number of $\mathbb{L}_{\mu^+,\mu^+}$(second order)).

Observation 1.22. *1) Let $\mathfrak{k}$ be an AEC with $\lambda = \mathrm{LS}(\mathfrak{k})$ and $\mu \ge \lambda$ and we define $\mathfrak{k}_{\ge\mu}$ by: $M \in \mathfrak{k}_{\ge\mu}$ iff $M \in K$ and $\|M\| \ge \mu$ and $M \le_{\mathfrak{k}_{\ge\mu}} N$ if $M \le_{\mathfrak{k}} N$ and $\|M\|, \|N\| \ge \mu$. Then $\mathfrak{k}_{\ge\mu}$ is an AEC with $\mathrm{LS}(\mathfrak{k}_{\ge\mu}) = \mu$.*

2) If $\mathfrak{k}_\lambda$ is a λ-AEC then Observation 1.7 holds when $|I| \le \lambda$.

3) Claims 1.11, 1.17 apply to λ-AEC.

Proof. Easy. $\square_{1.22}$

Remark 1.23. Recall if $\mathfrak{k}$ is an AEC with Lowenheim-Skolem number λ, then every model of $\mathfrak{k}$ can be written as a direct limit (by $\leq_{\mathfrak{k}}$) of members of $\mathfrak{k}_\lambda$ (see 1.8(1)). Alternating we prove below that given a λ-abstract elementary class $\mathfrak{k}_\lambda$, the class of direct limits of members of $\mathfrak{k}_\lambda$ is an AEC $\mathfrak{k}^{\rm up}$. We show below $(\mathfrak{k}_\lambda)^{\rm up} = \mathfrak{k}$, hence $\mathfrak{k}_\lambda$ determines $\mathfrak{k}_{\geq\lambda}$.

Lemma 1.24. *Suppose $\mathfrak{k}_\lambda$ is a λ-abstract elementary class.*

1) The pair $(K', \leq_{\mathfrak{k}'})$ is an abstract elementary class with Lowenheim-Skolem number λ which we denote also by $\mathfrak{k}^{\rm up}$ where we define

$$K' = \Big\{ M : M \text{ is a } \tau_{\mathfrak{k}_\lambda}\text{-model, and for some directed partial order } I \text{ and } \overline{M} = \langle M_s : s \in I\rangle \text{ we have } M = \bigcup_{s\in I} M_s,\ s \in I \Rightarrow M_s \in K_\lambda,\ I \models s < t \Rightarrow M_s \leq_{\mathfrak{k}_\lambda} M_t \Big\}.$$

We call such $\langle M_s : s \in I\rangle$ a witness for $M \in K'$, we call it reasonable if $|I| \leq \|M\|$

$$M \leq_{\mathfrak{k}'} N \text{ \underline{iff} for some directed partial order } J, \text{ and directed } I \subseteq J \text{ and } \langle M_s : s \in J\rangle \text{ we have } M = \bigcup_{s\in I} M_s, N = \bigcup_{t\in J} M_t, M_s \in K_\lambda \text{ and } J \models s < t \Rightarrow M_s \leq_{\mathfrak{k}_\lambda} M_t.$$

We call such $I, \langle M_s : s \in J\rangle$ witnesses for $M \leq_{\mathfrak{k}'} N$ or say $(I, J, \langle M_s : s \in J\rangle)$ witness $M \leq_{\mathfrak{k}'} N$.

2) Moreover, $K'_\lambda = K_\lambda$ and $\leq_{\mathfrak{k}'_\lambda}$ (which means $\leq_{\mathfrak{k}'} \restriction K'_\lambda$) is equal to $\leq_{\mathfrak{k}_\lambda}$ so $(\mathfrak{k}')_\lambda = \mathfrak{k}_\lambda$.

3) If $\mathfrak{k}''$ is an abstract elementary class satisfying (see 1.22) $K''_\lambda = K_\lambda, <_{\mathfrak{k}''} \restriction K_\lambda = \leq_{\mathfrak{k}_\lambda}$ and ${\rm LS}(\mathfrak{k}'') \leq \lambda$ <u>then</u>[11] $\mathfrak{k}''_{\geq\lambda} = \mathfrak{k}'$.

4) If $\mathfrak{k}''$ is an AEC, $K_\lambda \subseteq K''_\lambda$ and $\leq_{\mathfrak{k}_\lambda} = \leq_{\mathfrak{k}''} \restriction K_\lambda$, <u>then</u> $K' \subseteq K''$ and $\leq_{\mathfrak{k}'} \subseteq \leq_{\mathfrak{k}''} \restriction K'$ and if ${\rm LS}(\mathfrak{k}'') \leq \lambda$ then equality holds..

Proof. The proof of part (2) is straightforward (recalling 1.7) and part (3) follows from 1.8 and part (4) is also straightforward hence we concentrate on part (1). So let us check the axioms one by one.

<u>Ax 0</u>: K' is a class of τ-models, $\leq_{\mathfrak{k}'}$ a two-place relation on K', both closed under isomorphisms.
[Why? Trivially by their definitions.]

<u>Ax I</u>: If $M \leq_{\mathfrak{k}'} N$ then $M \subseteq N$.
[Why? trivial.] <u>Ax II</u>: $M_0 \leq_{\mathfrak{k}'} M_1 \leq_{\mathfrak{k}'} M_2$ implies $M_0 \leq_{\mathfrak{k}'} M_2$ and $M \in K' \Rightarrow M \leq_{\mathfrak{k}'} M$.
[Why? The second phrase is trivial (as if $\overline{M} = \langle M_t : t \in I\rangle$ witness $M \in K'$

[11] If we assume in addition that $M \in \mathfrak{k}'' \Rightarrow \|M\| \geq \lambda$ then we can show that equality holds.

then $(I, I, \overline{M})$ witness $M \leq_{\mathfrak{k}'} M$ above). For the first phrase let for $\ell \in \{1,2\}$ the directed partial orders $I_\ell \subseteq J_\ell$ and $\overline{M}^\ell = \langle M^\ell_s : s \in J_\ell \rangle$ witness $M_{\ell-1} \leq_{\mathfrak{k}'} M_\ell$ and let $\overline{M}^0 = \langle M^0_s : s \in I_0 \rangle$ witness $M_0 \in K'$. Now without loss of generality $\overline{M}^0$ is reasonable, i.e. $|I_0| \leq \|M_0\|$, why? by

$\boxtimes_1$ every $M \in K'$ has a reasonable witness, in fact, if $\overline{M} = \langle M_t : t \in I \rangle$ is a witness for M then for some $I' \subseteq I$ of cardinality $\leq \|M\|$ we have $\overline{M} \restriction I'$ is a reasonable witness for M.

[Why? If $\overline{M} = \langle M_t : t \in I \rangle$ is a witness, for each $a \in M$ choose $t_a \in I$ such that $a \in M_{t_a}$ and let $F : [I]^{<\aleph_0} \to I$ be such that $F(\{t_1, \dots, t_n\})$ is an upper bound of $\{t_1, \dots, t_n\}$ and let J be the closure of $\{t_a : a \in M\}$ under F; now $\overline{M} \restriction J$ is a reasonable witness of $M \in K'$.]

Similarly

$\boxtimes_2$ if $(I, J, \langle M_s : s \in J \rangle$ witness $M \leq_{\mathfrak{k}'} N$ then for some directed $I' \subseteq I, |I'| \leq \|M\|$ we have $(I', J, \langle M_s : s \in J \rangle)$ witness $M \leq_{K'} N$

$\boxtimes_3$ if $I, \overline{M} = \langle M_t : t \in J \rangle$ witness $M \leq_{\mathfrak{k}'} N$ <u>then</u> for some directed $J' \subseteq J$ we have $\|J'\| \leq |I| + \|N\|, I \subseteq J'$ and $I, \overline{M} \restriction J'$ witness $M \leq_{\mathfrak{k}'} N$.

Clearly $\boxtimes_1$ (and $\boxtimes_2, \boxtimes_3$) are cases of the LS-argument. We shall find a witness $(I, J, \langle M_s : s \in J \rangle)$ for $M_0 \leq_{\mathfrak{k}'} M_2$ such that $\langle M_s : s \in I \rangle = \langle M^0_s : s \in I_0 \rangle$ so $I = I_0$ and $|J| \leq \|M_2\|$. This is needed for the proof of Ax III below. Without loss of generality I_1, I_2 has cardinality $\leq \|M_0\|, \|M_1\|$ respectively, by $\boxtimes_2$. Also without loss of generality $\overline{M}^1, \overline{M}^1 \restriction I_1, \overline{M}^2, \overline{M}^2 \restriction I_2$ are reasonable as by the same argument we can have $|J_1| \leq \|M_1\|, |J_2| \leq \|M_2\|$ by $\boxtimes_3$.

As $\langle M^0_s : s \in I_0 \rangle$ is reasonable, there is a one-to-one function h from I_0 into M_2 (and even M_0); the function h will be used to get that J defined below is directed. We choose by induction on $m < \omega$, for every $\bar{c} \in {}^m(M_2)$, sets $I_{0,\bar{c}}, I_{1,\bar{c}}, I_{2,\bar{c}}, J_{1,\bar{c}}, J_{2,\bar{c}}$ such that:

$\otimes_1(a)$ $I_{\ell,\bar{c}}$ is a directed subset of I_ℓ of cardinality $\leq \lambda$ for $\ell \in \{0,1,2\}$

(b) $J_{\ell,\bar{c}}$ is a directed subset of J_ℓ of cardinality $\leq \lambda$ for $\ell \in \{1,2\}$

(c) $\bigcup_{s \in I_{\ell+1,\bar{c}}} M^{\ell+1}_s = \big(\bigcup_{s \in J_{\ell+1,\bar{c}}} M^{\ell+1}_s\big) \cap M_\ell$ for $\ell = 0,1$

(d) $\bigcup_{s \in I_{0,\bar{c}}} M^0_s = (\bigcup_{s \in I_{1,\bar{c}}} M^1_s) \cap M_0$

(e) $\bigcup_{s \in J_{1,\bar{c}}} M^1_s = \bigcup_{s \in I_{2,\bar{c}}} M^2_s$

(f) $\bar{c} \subseteq \bigcup_{s \in J_{2,\bar{c}}} M^2_s$

(g) if $\bar{d}$ is a permutation of $\bar{c}$ (i.e., letting $m = \ell g(\bar{c})$ for some one to one $g : \{0, \dots, m-1\} \to \{0, \dots, m-1\}$ we have $d_\ell = c_{g(\ell)}$) <u>then</u> $I_{\ell,\bar{c}} = I_{\ell,\bar{d}}, J_{m,\bar{c}} = J_{m,\bar{d}}$
(for $\ell \in \{0,1,2\}, m \in \{1,2\}$)

(h) if $\bar{d}$ is a subsequence of $\bar{c}$ (equivalently: an initial segment of some permutation of $\bar{c}$) <u>then</u> $I_{\ell,\bar{d}} \subseteq I_{\ell,\bar{c}}, J_{m,\bar{d}} \subseteq J_{m,\bar{c}}$ for $\ell \in \{0,1,2\}, m \in \{1,2\}$

(i) if $h(s) = c$ so $s \in I_0$ then $s \in I_{0,<c>}$.

There is no problem to carry the definition by LS-argument recalling clauses (a) + (b) and $\|M^\ell_s\| = \lambda$ when $\ell = 0 \wedge s \in I_0$ or $\ell = 1 \wedge s \in J_1$ or $\ell = 2 \wedge s \in J_2$. Without loss of generality $I_\ell \cap {}^{\omega>}(M_2) = \varnothing$.

Now let J have as set of elements $I_0 \cup \{\bar{c} : \bar{c}$ a finite sequence from $M_2\}$ ordered by: $J \models x \leq y$ iff $I_0 \models x \leq y$ <u>or</u> $x \in I_0, y \in J \setminus I_0, \exists z \in I_{0,y}[x \leq_{I_0} z]$ <u>or</u> $x, y \in J \setminus I_0$ and x is an initial segment of a permutation of y (or you may identify $\bar{c}$ with its set of permutations). Let $I = I_0$.

Let M_x be M^0_x if $x \in I_0$ and $\bigcup\limits_{s \in J_{2,x}} M^2_s$ if $x \in J \setminus I_0$.

Now

$(*)_1$ J is a partial order

[Clearly $x \le_J y \le_J x \Rightarrow x = y$, hence it is enough to prove transitivity. Assume $x \le_J y \le_J z$; if all three are in I_0 use "I_0 is a partial order", if all three are not in $J \setminus I_0$, use the definition of the order. As $x' \le_J y' \in I_0 \Rightarrow x' \in I_0$ without loss of generality $x \in I_0, z \in J \setminus I_0$. If $y \in I_0$ then (as $y \le_J z$) for some $y', y \le_{I_0} y' \in I_{0,z}$ but $x \le_{I_0} y$ (as $x, y \in I_0, x \le_J y$) hence $x \le_{I_0} y' \in I_{0,z}$ so $x \le_J z$. If $y \notin I_0$ then $I_{0,y} \subseteq I_{0,z}$ (by clause (h)) so we can finish similarly. So we have covered all cases.]

$(*)_2$ J is directed and $I \subseteq J$ is directed

[Let $x, y \in J$ and we shall find a common upper bound. If $x, y \notin I_0$ their concatenation $x\hat{\ }y$ can serve. If $x, y \in I_0$ use "I_0 is directed". If $x \in I_0, y \in J \setminus I_0$, then $\langle h(x) \rangle \in J \setminus I_0$ and $z = y\hat{\ }\langle h(x) \rangle \in J \setminus I_0$ is $<_J$ above y (by the choice of $\le_J$) and is $\le_J$-above x as $x \in I_{0,\langle h(x) \rangle} \subseteq I_{0,z}$ by clause (i) of $\otimes_1$ so we are done. If $x \in J \setminus I_0, y \in J_0$ the dual proof works. Lastly, $I \subseteq J$ as a partial order by the definition of I, J, and I is directed as I_0 is and $I = I_0$.]

$(*)_3$ if $x \in J \setminus I_0$ then $M_x \cap M_\ell \le_{\mathfrak{k}_x} M_x$ for $\ell = 0, 1$

[Why? Clearly $M_x \cap M_0 = (\cup\{M^2_t : t \in J_{1,x}\}) \cap M_0 = ((\cup\{M^2_t : t \in J_{2,x}) \cap M_1) \cap M_0 = (\cup\{M^2_t : t \in I_{2,x}\}) \cap M_0 = (\cup\{M^1_t : t \in J_{1,x}\}) \cap M_0 = \cup\{M^1_t : t \in I_{1,x}\}$ by the choice of M^2_x, as $M_0 \subseteq M_1$, by clause (c) for $\ell = 1$, by clause (e) and by clause (c) for $\ell = 0$, respectively. Similarly $M_x \cap M_1 = \cup\{M^1_t : t \in J_{1,x}\}$. Now the sets $I_{1,x} \subseteq J_{1,x} (\subseteq J_1)$ are directed by $\le_{J_1}$ so by the assumption on $\langle M^1_t : t \in J_1 \rangle$ and Lemma 1.7 we have $M_x \cap M_0 \le_{\mathfrak{k}_\lambda} M_x \cap M_1$. Using J_2 we can similarly prove $M_x \cap M_1 \le_{\mathfrak{k}_\lambda} M_x \cap M_2$ and trivially $M_x \cap M_2 = M_x$. As $\le_{\mathfrak{k}_\lambda}$ is transitive we are done.]

$(*)_4$ if $x \le_J y$ then $M_x \le_{\mathfrak{k}_\lambda} M_y$

[Why? If $x, y \in I_0$ use the choice of $\langle M^0_s : s \in I_0 \rangle$. If $x, y \in J \setminus I_0$ the proof is similar to that of $(*)_3$ using J_2. If $x \in I_0, y \in J \setminus I_0$ there is $s \in I_{0,y}$ such that $x \le_{I_0} s$, hence $M_x = M^0_x \le_{\mathfrak{k}_\lambda} M^0_s$ and as $\langle M^0_t : t \in I_{0,y} \rangle$ is $\le_{\mathfrak{k}_\lambda}$-directed clearly $M^0_s \le_{\mathfrak{k}_\lambda} \cup\{M^0_t : t \in I_{0,y}\} = M_y \cap M_0$ and $M_y \cap M_0 \le_{\mathfrak{k}_\lambda} M_y$ by $(*)_3$. By the transitivity of $\le_{\mathfrak{k}_\lambda}$ we are done.]

$(*)_5$ $\bigcup\{M_x : x \in I\} = \cup\{M^0_x : x \in I_0\} = M_0$

[Why? Trivially recalling $I_0 = I$ and $x \in I \Rightarrow M_x = M^0_x$.]

$(*)_6$ $M_2 = \cup\{M_x : x \in J\}$

[Why? Trivially as $\bar{c} \subseteq M^2_{\bar{c}} \subseteq M_2$ for $\bar{c} \in {}^{\omega>}(M_2)$ and $t \in I_0 \Rightarrow M^0_t \subseteq M_0 \subseteq M_1 \subseteq M_2$.]

By $(*)_1 + (*)_2 + (*)_4 + (*)_5 + (*)_6$ we have checked that $I, \langle M_x : x \in J \rangle$ witness $M_0 \le_{\mathfrak{k}'} M_2$. This completes the proof of AxII, but we also have proved

$\otimes_2$ if $\overline{M} = \langle M_t : t \in I \rangle$ is a reasonable witness to $M \in K'$ and $M \le_{\mathfrak{k}'} N \in K'$, then there is a witness $I', \overline{M}' = \langle M'_t : t \in J' \rangle$ to $M \le_{\mathfrak{k}'} N$ such that $I' = I, \overline{M}' \restriction I = \overline{M}$ and $\overline{M}'$ is reasonable and $x \le_{J'} y \wedge y \in I' \Rightarrow x \in I'$; can add $M = N \Rightarrow I' = I$.]

Ax III: If θ is a regular cardinal, M_i (for $i < \theta$) is $\le_{\mathfrak{k}'}$-increasing and continuous, then $M_0 \le_{\mathfrak{k}'} \bigcup\limits_{i<\theta} M_i$ (in particular $\bigcup\limits_{i<\theta} M_i \in \mathfrak{k}'$).

[Why? Let $M_\theta = \bigcup\limits_{i<\theta} M_i$, without loss of generality $\langle M_i : i < \theta \rangle$ is not eventually constant and so without loss of generality $i < \theta \Rightarrow M_i \neq M_{i+1}$ hence $\|M_i\| \geq |i|$; (this helps below to get "reasonable", i.e. $|I_\ell| = \|M_i\|$ for limit i). We choose by induction on $i \leq \theta$, a directed partial order I_i and M_s for $s \in I_i$ such that:

$\otimes_3(a)$ $\langle M_s : s \in I_i \rangle$ witness $M_i \in K'$

(b) for $j < i, I_j \subseteq I_i$ and $(I_j, I_i, \langle M_s : s \in I_i \rangle)$ witness $M_j \leq_{\mathfrak{k}'} M_i$

(c) I_i is of cardinality $\leq \|M_i\|$

(d) if $I_i \models s \leq t$ and $j < i, t \in I_j$ then $s \in I_j$

For $i = 0$ use the definition of $M_0 \in K'$.
For i limit let $I_i := \bigcup_{j<i} I_j$ (and the already defined M_s's) are as required because $M_i = \bigcup_{j<i} M_j$ and the induction hypothesis (and $|I_i| \leq \|M_i\|$ as we have assumed above that $j < i \Rightarrow M_j \neq M_{j+1}$) .
For $i = j + 1$ use the proof of Ax.II above with $M_j, M_i, M_i, \langle M_s : s \in I_j \rangle$ here serving as $M_0, M_1, M_2, \langle M^0_j : s \in I_0 \rangle$ there, that is, we use $\otimes_2$ from there. Now for $i = \theta, \langle M_s : s \in I_\theta \rangle$ witness $M_\theta \in K'$ and $(I_i, I_\theta, \langle M_s : s \in I_\theta \rangle)$ witness $M_i \leq_{\mathfrak{k}'} M_\theta$ for each $i < \theta$.] Axiom IV: Assume θ is regular and $\langle M_i : i < \theta \rangle$ is $\leq_{\mathfrak{k}}$-increasingly continuous, $M \in K'$ and $i < \theta \Rightarrow M_i \leq_{\mathfrak{k}'} M$ and $M_\theta = \bigcup_{i<\theta} M_i$ (so $M_\theta \subseteq M$). Then $M_\theta \leq_{\mathfrak{k}'} M$.

[Why? By the proof of Ax.III there are $\langle M_s : s \in I_i \rangle$ for $i < \theta$ satisfying clauses (a),(b),(c) and (d) of $\otimes_3$ there and without loss of generality $I_i \cap \theta = \varnothing$. For each $i < \theta$ as $M_i \leq_{\mathfrak{k}'} M$ there are J_i and M_s for $s \in J_i \setminus I_i$ such that $(I_i, J_i, \langle M_s : s \in J_i \rangle)$ witnesses it; without loss of generality with $\langle \bigcup_{i<\theta} I_i \rangle \hat{} \langle J_i \setminus I_i : i < \theta \rangle$ a sequence of pairwise disjoint sets; exist by $\otimes_2$ above. Let $I := \bigcup_{i<\theta} I_i$, let $\mathbf{i} : I \to \theta$ be $\mathbf{i}(s) = \min\{i : s \in I_i\}$ and recall $|I| \leq \|M_\theta\|$ hence by clause (d) of $\otimes_3$ we have $s \leq_I t \Rightarrow \mathbf{i}(s) \leq \mathbf{i}(t)$ and let h be a one-to-one function from I into M_θ. Without loss of generality the union below is disjoint and let

$(*)_7$ $J := I \cup \big\{(A, S) : A$ a finite subset of M and S a finite subset of I with a maximal element$\big\}$.

ordered by: $J \models x \leq y$ iff $x, y \in I, I \models x \leq y$ or $x \in I, y = (A, S) \in J \setminus I$ and $x \in S$ or $x = (A^1, S^1) \in J \setminus I, y = (A^2, S^2) \in J \setminus I, A^1 \subseteq A^2, S^1 \subseteq S^2$. We choose N_y for $y \in J$ as follows: If $y \in I$ we let $N_y = M_y$. By induction on $n < \omega$, if $y = (A, S) \in J \setminus I$ satisfies $n = |A| + |S|$, we choose the objects $N_y, I_{y,s}, J_{y,s}$ for $s \in S$ such that:

$\otimes_4(a)$ $I_{y,s}$ is a directed subset of $I_{\mathbf{i}(s)}$ of cardinality $\leq \lambda$ and $s \in I_{y,s}$

(b) $J_{y,s}$ is a directed subset of $J_{\mathbf{i}(s)}$ of cardinality $\leq \lambda$

(c) $s \in I_{\mathbf{i}(s)}$ for $s \in S$ (follows from the definition of $\mathbf{i}(s)$)

(d) $I_{y,s} \subseteq J_{y,s}$ for $s \in S$ and for $s <_I t$ from S we have $I_{y,s} \subseteq I_{y,t}$ and $J_{y,s} \subseteq J_{y,t}$

(e) if $y_1 = (A_1, S_1) \in J \setminus I, (A_1, S_1) <_J (A, S)$ and $s \in S_1$ then $I_{y_1,s} \subseteq I_{y,s}, J_{y_1,s} \subseteq J_{y,s}$

(f) $N_y = \bigcup_{t \in J_{y,s}} M_t$ for any $s \in S$

(g) $A \subseteq M_t$ for some $t \in J_{y,s}$ for any $s \in S$, hence $A \subseteq N_y$.

No problem to carry the induction and check that $(I, J, \langle N_y : y \in J \rangle)$ witness $M_\theta \leq_{\mathfrak{k}'} M$. Axiom V: Assume $N_0 \leq_{\mathfrak{k}'} M$ and $N_1 \leq_{\mathfrak{k}'} M$.

If $N_0 \subseteq N_1$, then $N_0 \leq_{\mathfrak{k}'} N_1$.

[Why? Let $(I_0, J_0, \langle M^0_s : s \in J_0\rangle)$ witness $N_0 \le_{\mathfrak{k}'} M$ and without loss of generality $|I_0| \le \|N_0\|$ and $h_0 : I_0 \to N_0$ be one-to-one. Let $\langle M^1_s : s \in I_1\rangle$ witness $N_1 \in \mathfrak{k}'$ and without loss of generality I_1 is isomorphic to $([N_1]^{<\aleph_0}, \subseteq)$ and let h_1 be an isomorphism from I_1 onto $([N_1]^{<\aleph_0}, \subseteq)$. Now by induction on n, for $s \in I_1$ satisfying $n = |\{t : t <_{I_1} s\}|$ we choose directed subsets $F_0(s), F_1(s)$ of I_0, I_1 respectively, each of cardinality $\le \lambda$ such that:

- (i) $s \in I_1 \Rightarrow s \in F_1(s)$ and $t <_{I_1} s \Rightarrow F_0(t) \subseteq F_0(s)$ and $F_1(t) \subseteq F_1(s)$
- (ii) if $s \in I_1$ then
  - (α) $\bigcup\{M^0_t : t \in F_0(s)\} = \bigcup\{M^1_t : t \in F_1(s)\} \cap N_0$
  - (β) $r \in I_0$ and $t \in I_1$ and $h_0(r) \in M^1_t \Rightarrow r \in F_0(s)$.

Now letting $M^2_s = \cup\{M^1_t : t \in F_1(s)\}$ and letting $F = F_0$ we get:

- (iii) $t \in I_1 \wedge s \in F(t)(\subseteq I_0) \Rightarrow M^0_s \subseteq M^2_t$
- (iv) F is a function from I_1 to $[I_0]^{\le\lambda}$
- (v) for $s \in I_1, F(s)$ is a directed subset of I_0 of cardinality $\le \lambda$
- (vi) for $s \in I_1, M^2_s \cap N_0 = \cup\{M^0_t : t \in F(s)\}$
- (vii) $I_1 \models s \le t \Rightarrow F(s) \subseteq F(t)$
- $(viii)$ $\langle M^2_s : s \in I_1\rangle$ witness $N_1 \in K'$.

As $N_1 \le_{\mathfrak{k}'} M$ by the proof of Ax.II, i.e., by $\otimes_2$ above we can find J_1 extending I_1 and M^2_s for $s \in J_1 \setminus I_1$ such that $(I_1, J_1, \langle M^2_s : s \in J_1\rangle)$ witnesses $N_1 \le_{\mathfrak{k}'} M$. We now prove

$\boxtimes_4$ if $r \in I_1, s \in I_0$ and $s \in F(r)$ then $M^0_s \le_{\mathfrak{k}_\lambda} M^2_r$.

[Why? As $\langle M^0_t : t \in J_0\rangle, \langle M^2_t : t \in J_1\rangle$ are both witnesses for $M \in K'$, clearly for $r \in I_1(\subseteq J_1)$ we can find directed $J'_0(r) \subseteq J_0$ of cardinality $\le \lambda$ and directed $J'_1(r) \subseteq J_1$ of cardinality $\le \lambda$ such that $r \in J'_1(r), F(r) \subseteq J'_0(r)$ and $\bigcup\limits_{t\in J'_0(r)} M^0_t = \bigcup\limits_{t\in J'_1(r)} M^2_t$, call it M^*_r.

Now $M^*_r \in K'_\lambda = K_\lambda$ (by part (2) and 1.7) and $t \in J'_1(r) \Rightarrow M^2_t \le_{\mathfrak{k}_\lambda} M^*_r$ (as $\mathfrak{k}_\lambda$ is a λ-abstract elementary class applying the parallel to Observation 1.7, i.e., 1.22(2)) and similarly $t \in J'_0(r) \Rightarrow M^0_t \le_{\mathfrak{k}_\lambda} M^*_r$. Now the s from $\boxtimes_4$ satisfied $s \in F(r) \subseteq J'_0(r)$ hence $M^0_s \subseteq M^1_r$ (why? by clause (iii) above $s \in F(r)$ is as required in $\boxtimes_4$). But above we got $M^0_s \le_{\mathfrak{k}} M^*_r, M^2_r \le_{\mathfrak{k}} M^*_r$, so by Ax.V for $\mathfrak{k}_\lambda$ we have $M^0_s \le_{\mathfrak{k}} M^1_r$ as required in $\boxtimes_4$.]

Without loss of generality $I_0 \cap I_1 = \varnothing$ and define the partial order J with set of elements $I_0 \cup I_1$ by $J \models x \le y$ iff $x, y \in I_0, I_0 \models x \le y$ or $x \in I_0, y \in I_1$ and $x \in F(y)$ or $x, y \in I_1, I_1 \models x \le y$.

$\boxtimes_5$ J is a partial order and $x \le_J y in I_0 \Rightarrow x \in I_0$ (hence $x \le_J y$ and $x \in I_1 \Rightarrow y \in I_1$).

[Why? The second phrase holds by the definition of $\le_J$. For J being a partial order obviously $x \le_J y \le_J x \Rightarrow x = y$, so assume $x \le_J y \le_J z$ and we shall prove $x \le_J z$. If $x \in I_1$ then $y, z \in I_1$ and we use "I_1 is a partial order", and if $z \in I_0$ then $x, y \in I_0$ and we can use "I_0 is a partial order". So assume $x \in I_0, z \in I_1$. If $y \in I_0$ use "$F(z) = F_1(z)$ satisfies clause (i) above. If $y \in I_1$, use clause (vii) above with (y, z) here standing for (s, t) there.]

$\boxtimes_6$ J is directed.

[Why? Note that I_0, I_1 are directed, $x \le_J y \in I_0 \Rightarrow x \in I_0$ and $(\forall x \in I_0)(\exists y \in I_1)[x \le_J y]$ because given $r \in I_0, h_0(r) \in N_0$ hence $h_0(r)$ belongs to M^1_t for some $t \in I_1$, and so by clause (i) we have $t \in F_1(t)$ hence by clause $(ii)(\beta)$ above $r \in F_0(t)$. Together this is easy.]

Define M_s for $s \in J$ as M^0_s if $s \in I_0$ and as M^2_s if $s \in I_1$

$\boxtimes_7$ $M_s \in K_\lambda$ for $s \in J$.

[Why? Obvious.]

$\boxtimes_8$ if $x \leq_J y$ then $M_x \leq_x M_y$.

[Why? If $y \in I_0$ (hence $x \in I_0$) use $\langle M^0_t : t \in I_0\rangle$ is a witness for $N_0 \in K'$. If $x \in I_1$ (hence $y \in I_1$) use clause (viii) above; i.e. $\langle M^2_s : s \in I_1\rangle$ is a witness for $N_1 \in K'$.]

$\boxtimes_9$ $\bigcup\{M_x : x \in J\} = N_1$.

[Why? As $(\forall x \in I_0)(\exists y \in I_1)[x \leq_J y]$, see the proof of $\boxtimes_6$ recalling $\boxtimes$ we have $\bigcup\{M_x : x \in J\} = \cup\{M_x : x \in I_1\}$ but the latter is $\bigcup\{M^2_x : x \in I_1\}$ which is equal to N_2.]

$\boxtimes_{10}$ $I_0 \subseteq J$ is directed and $\bigcup\{M_x : x \in J\} = N_1$.

[Why? Obvious.]

Together $(I_0, J, \langle M_s : s \in J\rangle)$ witnesses $N_0 \leq_{\mathfrak{k}'} N_1$ are as required.]

<u>Axiom VI</u>: $\mathrm{LS}(\mathfrak{k}') = \lambda$.

[Why? Let $M \in K'$ and $A \subseteq M$ with $|A| + \lambda \leq \mu < \|M\|$, and let $\langle M_s : s \in J\rangle$ witness $M \in K'$. As $\|M\| > \mu$ we can choose a directed $I \subseteq J$ of cardinality $\leq \mu$ such that $A \subseteq M' := \bigcup\limits_{s\in I} M_s$ and so $(I, J, \langle M_s : s \in J\rangle)$ witnesses $M' \leq_{\mathfrak{k}'} M$, so as $A \subseteq M'$ and $\|M'\| \leq |A| + \mu$; this is more than enough.] $\square_{1.24}$

We may like to use $\mathfrak{k}_{\leq\lambda}$ instead of $\mathfrak{k}_\lambda$; no need as essentially $\mathfrak{k}$ consists of two parts $\mathfrak{k}_{\leq\lambda}$ and $\mathfrak{k}_{\geq\lambda}$ which have just to agree in λ. That is,

Claim 1.25. *Assume*

(a) $\mathfrak{k}^1$ *is an abstract elementary class with* $\lambda = \mathrm{LS}(\mathfrak{k}^1), K^1 = K^1_{\geq\lambda}$

(b) $\mathfrak{k}^2_{\leq\lambda}$ *is a* $(\leq\lambda)$*-abstract elementary class (defined as in 1.19(1) with the obvious changes so* $M \in \mathfrak{k}^2_{\leq\lambda} \Rightarrow \|M\| \leq \lambda$ *and in Axiom III,* $\|\bigcup\limits_i M_i\| \leq \lambda$ *is required)*

(c) $K^2_\lambda = K^1_\lambda$ *and* $\leq_{\mathfrak{k}^2} \restriction K^2_\lambda = \leq_{\mathfrak{k}^1} \restriction K^1_\lambda$

(d) *we define* $\mathfrak{k}$ *as follows:* $K = K^1 \cup K^2, M \leq_{\mathfrak{k}} N$ *iff* $M \leq_{\mathfrak{k}^1} N$ *or* $M \leq_{\mathfrak{k}^2} N$ *or for some* $M', M \leq_{\mathfrak{k}^2} M' \leq_{\mathfrak{k}^1} N$.

<u>*Then*</u> $\mathfrak{k}$ *is an abstract elementary class and* $\mathrm{LS}(\mathfrak{k}) = \mathrm{LS}(\mathfrak{k}^2)$ *which trivially is* $\leq \lambda$.

Proof. Straight. E.g.

Axiom V: We shall use freely

$(*)$ $\mathfrak{k}_{\leq\lambda} = \mathfrak{k}^2$ and $\mathfrak{k}_{\geq\lambda} = \mathfrak{k}^1$.

So assume $N_0 \leq_{\mathfrak{k}} M, N_1 \leq_{\mathfrak{k}} M, N_0 \subseteq N_1$.

Now if $\|N_0\| \geq \lambda$ use assumption (a), so we can assume $\|N_0\| < \lambda$. If $\|M\| \leq \lambda$ we can use assumption (b) so we can assume $\|M\| > \lambda$ and by the definition of $\leq_{\mathfrak{k}}$ there is $M'_0 \in K^1_\lambda = K^2_\lambda$ such that $N_0 \leq_{\mathfrak{k}^2} M'_0 \leq_{\mathfrak{k}^1} M$. First assume $\|N_1\| \leq \lambda$, so we can find $M'_1 \in K^1_\lambda$ such that $N_1 \leq_{\mathfrak{k}^2} M'_1 \leq_{\mathfrak{k}^1} M$ (why? if $N_1 \in K_{<\lambda}$, by the definition of $\leq_{\mathfrak{k}}$ and if $N_1 \in K_\lambda$ just choose $M'_1 = N_1$). Now by assumption (a) we can find $M'' \in K^1_\lambda$ such that $M'_0 \cup M'_1 \subseteq M'' \leq_{\mathfrak{k}^1} M$, hence by assumption (a) (i.e. Ax.V for $\mathfrak{k}^1$) we have $M'_0 \leq_{\mathfrak{k}^1} M'', M'_1 \leq_{\mathfrak{k}^1} M''$, so by assumption (c) we have $M'_0 \leq_{\mathfrak{k}^2} M'', M'_1 \leq_{\mathfrak{k}^2} M''$. As $N_0 \leq_{\mathfrak{k}^2} M'_0 \leq_{\mathfrak{k}^2} M'' \in K_{\leq\lambda}$ by assumption (b) we have $N_0 \leq_{\mathfrak{k}^2} M''$, and similarly we have $N_1 \leq_{\mathfrak{k}^2} M''$. So $N_0 \subseteq N_1, N_0 \leq_{\mathfrak{k}^2} M'', N_1 \leq_{\mathfrak{k}^2} M'$ so by assumption (b) we have $N_0 \leq_{\mathfrak{k}^2} N_1$ hence $N_0 \leq_{\mathfrak{k}} N_1$.

We are left with the case $\|N_1\| > \lambda$; by assumption (a) there is $N'_1 \in K_\lambda$ such that $N_0 \subseteq N'_1 \leq_{\mathfrak{k}^1} N_1$. By assumption (a) we have $N'_1 \leq_{\mathfrak{k}^1} M$, so by the previous paragraph we get $N_0 \le_{\mathfrak{k}^2} N'_1$, together with the previous sentence we have $N_0 \le_{\mathfrak{k}^2} N'_1 \le_{\mathfrak{k}^1} N_1$ so by the definition of $\le_{\mathfrak{k}}$ we are done. $\square_{1.25}$

Recall

Definition 1.26. If $M \in K_\lambda$ is locally superlimit or just pseudo superlimit let $K_{[M]} = K^{[M]}_\lambda = \{N \in K_\lambda : N \cong M\}, \mathfrak{k}_{[M]} = \mathfrak{k}^{[M]}_\lambda = (K_{[M]}, \le_{\mathfrak{k}} \restriction K^{[M]}_\lambda)$ and let $\mathfrak{k}^{[M]}$ be the $\mathfrak{k}'$ we get in 1.24(1) for $\mathfrak{k} = \mathfrak{k}_{[M]} = \mathfrak{k}^{[M]}_\lambda$. We may write $\mathfrak{k}_\lambda[M], \mathfrak{k}[M]$.

Trivially but still important is showing that assuming categoricity in one λ is a not so strong assumption.

Claim 1.27. *1) If $\mathfrak{k}$ is an λ-AEC, $M \in K_\lambda$ is locally superlimit or just pseudo superlimit <u>then</u> $\mathfrak{k}_{[M]}$ is a λ-AEC which is categorical (i.e. categorical in λ).*
2) Assume $\mathfrak{k}$ is an AEC and $M \in \mathfrak{k}_\lambda$ is not $\le_{\mathfrak{k}}$- maximal. M is pseudo superlimit (in $\mathfrak{k}$, i.e., in $\mathfrak{k}_\lambda$) <u>iff</u> $\mathfrak{k}_{[M]}$ is a λ-AEC which is categorical <u>iff</u> $\mathfrak{k}^{[M]}$ is an AEC, categorical in λ and $\le_{\mathfrak{k}^{[M]}} = \le_{\mathfrak{k}} \restriction K^{[M]}$.
3) In (1) and (2), $\mathrm{LS}(\mathfrak{k}^{[M]}) = \lambda = \min\{\|N\| : N \in \mathfrak{k}^{[M]}\}$.

Proof. Straightforward. $\square_{1.27}$

Exercise 1.28. Assume $\mathfrak{k}$ is a λ-AEC with amalgamation and stability in λ. <u>Then</u> for every $M_1 \in K_\lambda$, $p_1 \in \mathcal{S}_{\mathfrak{k}}(M_1)$ we can find $M_2 \in K$ and minimal $p_2 \in \mathcal{S}_{\mathfrak{k}}(M_2)$ such that $M_1 \le_{\mathfrak{k}} M_2$, $p_1 = p_2 \restriction M_1$.
[Hint: See [She09c, 2b.4](2).]

Exercise 1.29. 1) Any $\le_{\mathfrak{k}_\lambda}$-embedding f_0 of M^1_0 into M^2_0 can be extended to an isomorphism f from M^1_δ onto M^2_δ such that $f(M^1_{2\alpha}) \le_{\mathfrak{k}_\lambda} M^2_{2\alpha}, f^{-1}(M^2_{2\alpha+1}) \le_{\mathfrak{k}_\lambda} M^1_{2\alpha+1}$ for every $\alpha < \delta$, <u>provided that</u>

⊛ (a) $\mathfrak{k}_\lambda$ is a λ-AEC with amalgamation and δ is a limit ordinal $\le \lambda^+$.
(b) $\langle M^\ell_\alpha : \alpha \le \delta\rangle$ is $\le_{\mathfrak{k}_\lambda}$-increasing continuous for $\ell = 1, 2$.
(c) M^ℓ_α is an amalgamation base in $\mathfrak{k}_\lambda$ (for $\alpha < \delta$ and $\ell = 1, 2$).
(d) $M^\ell_{\alpha+1}$ is $\le_{\mathfrak{k}_\lambda}$-universal extension of M^ℓ_α for $\alpha < \delta$, $\ell = 1, 2$.

2) Write the axioms of "a λ-AEC" which are used.
3) For $\mathfrak{k}_\lambda, \delta$ as in (a) above, for any $M \in K_\lambda$ there is $N \in K_\lambda$ which is $(\lambda, \mathrm{cf}(\delta))$-brimmed over it.
[Hint: Should be easy; is similar to 1.17 (or 1.18).]

§ 2. Good Frames

We first present our central definition: good λ-frame (in Definition 2.1). We are given the relation "$p \in \mathcal{S}(N)$ does not fork over $M \leq_{\mathfrak{k}} N$ when p is basic" (by the basic relations and axioms) so it is natural to look at how well we can "lift" the definition of non-forking to models of cardinality λ and later to non-forking of models (and types over them) in cardinalities $> \lambda$. Unlike the lifting of λ-AEC in Lemma 1.24, life is not so easy. We define in 2.4, 2.5, 2.7 and we prove basic properties in 2.6, 2.8, 2.10 and less obvious ones in 2.9, 2.11, 2.12. This should serve as a reasonable exercise in the meaning of good frames; however, the lifting, in general, does not give good μ-frames for $\mu > \lambda$. There may be no $M \in K_\mu$ at all and/or amalgamation may fail. Also the existence and uniqueness of non-forking types is problematic. We do not give up and will return to the lifting problem, under additional assumptions in [She09e, §12] and [SV].

In 2.16 (recalling 1.27) we show that the case "$\mathfrak{k}^{\mathfrak{s}}$ categorical in λ" is not so rare among good λ-frames; in fact if there is a superlimit model in λ we can restrict $\mathfrak{k}_\lambda$ to it. So in a sense superstability and categoricity are close, a point which does not appear in first order model theory, <u>but</u> if T is a complete first order superstable theory and $\lambda \geq 2^{|T|}$, <u>then</u> the class $\mathfrak{k} = \mathfrak{k}_{T,\lambda}$ of λ-saturated models of T is in general not an elementary class (though is a PC_λ class) but is an AEC categorical in λ though in general not in λ^+ and for some good λ-frame $\mathfrak{s}, K_{\mathfrak{s}} = \mathfrak{k}_{T,\lambda}$. How justified is our restriction here to something like "the λ-saturated model"? It is O.K. for our test problems but more so it is justified as our approach is to first analyze the quite saturated models.

Last but not least in 2.18 we show that one of the axioms from 2.1, i.e., (E)(i), follows from the rest in our present definition; additional implications are in Claims 2.19, 2.21. Later "Ax(X)(y)" will mean (X)(y) from Definition 2.1.

Recall that good λ-frame is intended to be a parallel to (bare bones) superstable elementary class stable in λ; here we restrict ourselves to models of cardinality λ.

Definition 2.1. We say $\mathfrak{s} = (\mathfrak{k}, \bigcup_\lambda, \mathcal{S}^{\mathrm{bs}}_\lambda) = (\mathfrak{k}^{\mathfrak{s}}, \bigcup_{\mathfrak{s}}, \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}})$ is a good frame in λ or a good λ-frame (λ may be omitted when its value is clear, note that $\lambda = \lambda_{\mathfrak{s}} = \lambda(\mathfrak{s})$ is determined by $\mathfrak{s}$ and we may write $\mathcal{S}_{\mathfrak{s}}(M)$ instead of $\mathcal{S}_{\mathfrak{k}^{\mathfrak{s}}}(M)$ and $\mathrm{ortp}_{\mathfrak{s}}(a, M, N)$ instead of $\mathrm{ortp}_{\mathfrak{k}^{\mathfrak{s}}}(a, M, N)$ when $M \in K^{\mathfrak{s}}_\lambda, N \in K^{\mathfrak{s}}$; we may write $\mathrm{ortp}(a, M, N)$ for $\mathrm{ortp}_{\mathfrak{k}^{\mathfrak{s}}}(a, M, N)$) <u>when</u> the following conditions hold:

- (A) $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$ is an abstract elementary class also denoted by $\mathfrak{k}[\mathfrak{s}]$, the Löwenheim Skolem number of $\mathfrak{k}$, being $\leq \lambda$ (see Definition 1.4); there is no harm in assuming $M \in K \Rightarrow \|M\| \geq \lambda$; let $\mathfrak{k}_{\mathfrak{s}} = \mathfrak{k}^{\mathfrak{s}}_\lambda$ and $\leq_{\mathfrak{s}} = \leq_{\mathfrak{k}} \restriction K_\lambda$, and let $\mathfrak{k}_{\mathfrak{s}} = (K_\lambda, \leq_{\mathfrak{s}})$ and $\mathfrak{k}[\mathfrak{s}] = \mathfrak{k}^{\mathfrak{s}}$ so we may write $\mathfrak{s} = (\mathfrak{k}_{\mathfrak{s}}, \bigcup_{\mathfrak{s}}, \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}})$
- (B) $\mathfrak{k}$ has a superlimit model in λ which [12] is not $<_{\mathfrak{k}}$-maximal.
- (C) $\mathfrak{k}_\lambda$ has the amalgamation property, the JEP (joint embedding property), and has no $\leq_{\mathfrak{k}}$-maximal member.
- $(D)(a)$ $\mathcal{S}^{\mathrm{bs}} = \mathcal{S}^{\mathrm{bs}}_\lambda$ (the class of basic types for $\mathfrak{k}_\lambda$) is included in $\bigcup\{\mathcal{S}(M) : M \in K_\lambda\}$ and is closed under isomorphisms including automorphisms; for $M \in K_\lambda$ let $\mathcal{S}^{\mathrm{bs}}(M) = \mathcal{S}^{\mathrm{bs}} \cap \mathcal{S}(M)$; no harm in allowing types of finite sequences, i.e., replacing $\mathcal{S}(M)$ by $\mathcal{S}^{<\omega}(M)$, ($\mathcal{S}^\omega(M)$) is different as being new (= non-algebraic) is not preserved under increasing unions).
  - (b) if $p \in \mathcal{S}^{\mathrm{bs}}(M)$, <u>then</u> p is non-algebraic (i.e. not realized by any $a \in M$).

[12] in fact, the "is not $<_{\mathfrak{k}}$-maximal" follows by (C)

(c) (density)
if $M \leq_{\mathfrak{k}} N$ are from K_λ and $M \neq N$, then for some $a \in N \setminus M$ we have $\text{ortp}(a, M, N) \in \mathcal{S}^{\text{bs}}$
[intention: examples are: minimal types in [She01], i.e. [She09c], regular types for superstable first order (= elementary) classes].

(d) bs-stability
$\mathcal{S}^{\text{bs}}(M)$ has cardinality $\leq \lambda$ for $M \in K_\lambda$.

$(E)(a)$ $\bigcup\limits_{\lambda}$ denoted also by $\bigcup\limits_{\mathfrak{s}}$ or just $\bigcup$, is a four place relation [13] called non-forking with $\bigcup(M_0, M_1, a, M_3)$ implying $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_3$ are from $K_\lambda, a \in M_3 \setminus M_1$ and $\text{ortp}(a, M_0, M_3) \in \mathcal{S}^{\text{bs}}(M_0)$ and $\text{ortp}(a, M_1, M_3) \in \mathcal{S}^{\text{bs}}(M_1)$. Also $\bigcup$ is preserved under isomorphisms and we demand: if $M_0 = M_1 \leq_{\mathfrak{k}} M_3$ both in K_λ and $a \in M_3$, then: $\bigcup(M_0, M_1, a, M_3)$ is equivalent to "$\text{ortp}(a, M_0, M_3) \in \mathcal{S}^{\text{bs}}(M_0)$". The assertion $\bigcup(M_0, M_1, a, M_3)$ is also written as $M_1 \overset{M_3}{\underset{M_0}{\bigcup}} a$ and also as "$\text{ortp}(a, M_1, M_3)$ does not fork over M_0 (inside M_3)" (this is justified by clause (b) below). So $\text{ortp}(a, M_1, M_3)$ forks over M_0 (where $M_0 \leq_{\mathfrak{s}} M_1 \leq_{\mathfrak{s}} M_3, a \in M_3$) is just the negation

[Explanation: The intention is to axiomatize non-forking of types, but we already commit ourselves to dealing with basic types only. Note that in [She01], i.e. [She09c] we know something on minimal types but other types are something else.]

(b) (monotonicity):
if $M_0 \leq_{\mathfrak{k}} M_0' \leq_{\mathfrak{k}} M_1' \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_3 \leq_{\mathfrak{k}} M_3', M_1' \cup \{a\} \subseteq M_3'' \leq_{\mathfrak{k}} M_3'$ all of them in K_λ, then $\bigcup(M_0, M_1, a, M_3) \Rightarrow \bigcup(M_0', M_1', a, M_3')$ and $\bigcup(M_0', M_1', a, M_3') \Rightarrow \bigcup(M_0', M_1', a, M_3'')$, so it is legitimate to just say "$\text{ortp}(a, M_1, M_3)$ does not fork over M_0".

[Explanation: non-forking is preserved by decreasing the type, increasing the basis (= the set over which it does not fork) and increasing or decreasing the model inside which all this occurs, i.e. where the type is computed. The same holds for stable theories only here we restrict ourselves to "legitimate", i.e., basic types. But note that here the "restriction of $\text{ortp}(a, M_1, M_3)$ to M_1' is basic" is a worthwhile information.]

(c) (local character):
if $\langle M_i : i \leq \delta + 1 \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous in $\mathfrak{k}_\lambda, a \in M_{\delta+1}$ and $\text{ortp}(a, M_\delta, M_{\delta+1}) \in \mathcal{S}^{\text{bs}}(M_\delta)$ then for every $i < \delta$ large enough $\text{ortp}(a, M_\delta, M_{\delta+1})$ does not fork over M_i.

[Explanation: This is a replacement for superstability which says that: if $p \in \mathcal{S}(A)$ then there is a finite $B \subseteq A$ such that p does not fork over B.]

(d) (transitivity):
if $M_0 \leq_{\mathfrak{s}} M_0' \leq_{\mathfrak{s}} M_0'' \leq_{\mathfrak{s}} M_3$ are from K_λ and $a \in M_3$ and $\text{ortp}(a, M_0'', M_3)$ does not fork over M_0' and $\text{ortp}(a, M_0', M_3)$ does not fork over M_0 (all models are in K_λ, of course, and necessarily the three relevant types are in $\mathcal{S}^{\text{bs}}$), then $\text{ortp}(a, M_0'', M_3)$ does not fork over M_0

[13] we tend to forget to write the λ, this is justified by 2.6(2), and see Definition 2.5

(e) uniqueness:
if $p, q \in \mathcal{S}^{\rm bs}(M_1)$ do not fork over $M_0 \leq_{\mathfrak{k}} M_1$ (all in K_λ) and $p \restriction M_0 = q \restriction M_0$ then $p = q$

(f) symmetry:
if $M_0 \leq_{\mathfrak{k}} M_3$ are in $\mathfrak{k}_\lambda$ and for $\ell = 1, 2$ we have $a_\ell \in M_3$ and $\text{ortp}(a_\ell, M_0, M_3) \in \mathcal{S}^{\rm bs}(M_0)$, then the following are equivalent:

(α) there are M_1, M_3' in K_λ such that $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_3'$, $a_1 \in M_1, M_3 \leq_{\mathfrak{k}} M_3'$ and $\text{ortp}(a_2, M_1, M_3')$ does not fork over M_0

(β) there are M_2, M_3' in K_λ such that $M_0 \leq_{\mathfrak{k}} M_2 \leq_{\mathfrak{k}} M_3'$, $a_2 \in M_2, M_3 \leq_{\mathfrak{k}} M_3'$ and $\text{ortp}(a_1, M_2, M_3')$ does not fork over M_0.

[Explanation: this is a replacement to "$\text{ortp}(a_1, M_0 \cup \{a_2\}, M_3)$ forks over M_0 iff $\text{ortp}(a_2, M_0 \cup \{a_1\}, M_3)$ forks over M_0" which is not well defined in our context.]

(g) extension existence:
if $M \leq_{\mathfrak{k}} N$ are from K_λ and $p \in \mathcal{S}^{\rm bs}(M)$ then some $q \in \mathcal{S}^{\rm bs}(N)$ does not fork over M and extends p

(h) continuity:
if $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous, all in K_λ (recall δ is always a limit ordinal), $p \in \mathcal{S}(M_\delta)$ and $i < \delta \Rightarrow p \restriction M_i \in \mathcal{S}^{\rm bs}(M_i)$ does not fork over M_0 then $p \in \mathcal{S}^{\rm bs}(M_\delta)$ and moreover p does not fork over M_0.

[Explanation: This is a replacement to: for an increasing sequence of types which do not fork over A, the union does not fork over A; equivalently if p forks over A then some finite subtype does.]

(i) non-forking amalgamation:
if for $\ell = 1, 2, M_0 \leq_{\mathfrak{k}} M_\ell$ are from $K_\lambda, a_\ell \in M_\ell \setminus M_0$, $\text{ortp}(a_\ell, M_0, M_\ell) \in \mathcal{S}^{bs}(M_0)$, then we can find f_1, f_2, M_3 satisfying $M_0 \leq_{\mathfrak{k}} M_3 \in K_\lambda$ such that for $\ell = 1, 2$ we have f_ℓ is a $\leq_{\mathfrak{k}}$-embedding of M_ℓ into M_3 over M_0 and $\text{ortp}(f_\ell(a_\ell), f_{3-\ell}(M_{3-\ell}), M_3)$ does not fork over M_0 for $\ell = 1, 2$.

[Explanation: This strengthens clause (g), (existence) saying we can do it twice so close to (f), symmetry, but see 2.18.]

* * *

Discussion 2.2. : 0) On connections between the axioms see 2.18, 2.19, 2.21.

1) What justifies the choice of the good λ-frame as a parallel to (bare bones) superstability? Mostly starting from assumptions on few models around λ in the AEC $\mathfrak{k}$ and reasonable, "semi ZFC" set theoretic assumptions (e.g. involving categoricity and weak cases of G.C.H., see §3) we can prove that, essentially, for some $\bigcup, \mathcal{S}$ the demands in Definition 2.1 hold. So here we shall get (i.e., applying our general theorem to the case of 3.5) an alternative proof of the main theorem of [She83a], [She83b] in a local version, i.e., dealing with few cardinals rather than having to deal with all the cardinals $\lambda, \lambda^{+1}, \lambda^{+2}, \ldots, \lambda^{+n}$ as in [She83a], [She83b] in an inductive proof. That is, in [She83b], we get dichotomies by the omitting type theorem for countable models (and theories). So problems on $\aleph_n$ are "translated" down to $\aleph_{n-1}$ (increasing the complexity) till we arrive to $\aleph_0$ and then "translated" back. Hence it is important there to deal with $\aleph_0, \ldots, \aleph_n$ together. Here our λ may not have special helpful properties, so if we succeed to prove the relevant claims then they apply to λ^+, too. There are advantages to being poor.

2) Of course, we may just point out that the axioms seem reasonable and that eventually we can say much more.

3) We may consider weakening bs-stability (i.e., $\mathrm{Ax}(D)(d)$ in Definition 2.1) to $M \in K_\lambda \Rightarrow |\mathcal{S}^{\mathrm{bs}}(M)| \le \lambda^+$, we have not looked into it here; Jarden-Shelah [JS13] will; actually [She09a] deals in a limited way with this in a considerably more restricted framework.

4) On stability in λ and existence of (λ, ∂)-brimmed extensions see 4.2.

From the rest of this section we shall use mainly the defintion of $K^{3,\mathrm{bs}}_\lambda$ in Definition 2.4(3), also 2.23 (restricting ourselves to a superlimit). We sometimes use implications among the axioms (in 2.18 - 2.21). The rest is, for now an exercise to familiarize the reader with λ-frames, in particular (2.3-2.16) to see what occurs to non-forking and basic types in cardinals $> \lambda$. This is easy (but see below). For this we first present the basic definitions.

Convention 2.3. 1) We fix $\mathfrak{s}$, a good λ-frame so $K = K^{\mathfrak{s}}, \mathcal{S}^{\mathrm{bs}} = \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}$.

2) By $M \in K$ we mean $M \in K_{\ge\lambda}$ if not said otherwise.

We lift the properties to $\mathfrak{k}_{\ge\lambda}$ by reflecting to the situation in K_λ. But do not be too excited: the good properties do not lift automatically, we shall be working on that later (under additional assumptions). Of course, from the definition below later we shall use mainly $K^{3,\mathrm{bs}}_{\mathfrak{s}} = K^{3,\mathrm{bs}}_\lambda$.

Definition 2.4. 1)

$$K^{3,\mathrm{bs}} = K^{3,\mathrm{bs}}_{\ge\mathfrak{s}} := \Big\{(M,N,a) : M \le_{\mathfrak{k}} N, a \in N \setminus M \text{ and there is } M' \le_{\mathfrak{k}} M \text{ satisfying } M' \in K_\lambda, \text{ such that for every } M'' \in K_\lambda \text{ we have: } [M' \le_{\mathfrak{k}} M'' \le_{\mathfrak{k}} M \Rightarrow \mathrm{ortp}(a, M'', N) \in \mathcal{S}^{\mathrm{bs}}(M'') \text{ does not fork over } M']; \text{ equivalently } [M' \le_{\mathfrak{k}} M'' \le_{\mathfrak{k}} M \text{ and } M'' \le_{\mathfrak{k}} N'' \le_{\mathfrak{k}} N \text{ and } N'' \in K_\lambda \text{ and } a \in N'' \Rightarrow \bigcup_\lambda(M', M'', a, N'')]\Big\}.$$

2) $K^{3,\mathrm{bs}}_{=\mu} = K^{3,\mathrm{bs}}_{\mathfrak{s},\mu} := \{(M,N,a) \in K^{3,\mathrm{bs}}_{\ge\mathfrak{s}} : M, N \in \mathfrak{k}^{\mathfrak{s}}_\mu\}$.

3) $K^{3,\mathrm{bs}}_{\mathfrak{s}} := K^{3,\mathrm{bs}}_{=\lambda,\mathfrak{s}}$; and let $K^{3,\mathrm{bs}}_\mu = K^{3,\mathrm{bs}}_{=\mu}$, used mainly for $\mu = \lambda_{\mathfrak{s}}$ and $K^{3,\mathrm{bs}}_{\mathfrak{s},\ge\mu}$ is defined naturally.

Definition 2.5. We define $\bigcup_{<\infty}(M_0, M_1, a, M_3)$ (rather than $\bigcup_\lambda$) as follows: it holds <u>iff</u> $M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_3$ are from K (not necessarily K_λ), $a \in M_3 \setminus M_1$ and there is $M'_0 \le_{\mathfrak{k}} M_0$ which belongs to K_λ satisfying: if $M'_0 \le_{\mathfrak{k}} M'_1 \le_{\mathfrak{k}} M_1, M'_1 \in K_\lambda$, $M'_1 \cup \{a\} \subseteq M'_3 \le_{\mathfrak{k}} M_3$ and $M'_3 \in K_\lambda$ <u>then</u> $\bigcup_\lambda(M'_0, M'_1, a, M'_3)$.

We now check that $\bigcup_{<\infty}$ behaves correctly when restricted to K_λ.

Claim 2.6. *1) Assume $M \le_{\mathfrak{k}} N$ are from K_λ and $a \in N$. <u>Then</u> $(M,N,a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ <u>iff</u> $\mathrm{ortp}(a, M, N) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$.*

2) Assume $M_0, M_1, M_3 \in K_\lambda$ and $a \in M_3$. <u>Then</u> $\bigcup_{<\infty}(M_0, M_1, a, M_3)$ <u>iff</u>

$\bigcup\limits_{\lambda}(M_0, M_1, a, M_3)$.

3) Assume $M \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} N_2$ and $a \in N_1$. Then $(M, N_1, a) \in K^{3,\mathrm{bs}}_{\geq \mathfrak{s}} \Leftrightarrow (M, N_2, a) \in K^{3,\mathrm{bs}}_{\geq \mathfrak{s}}$.

4) Assume $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_3 \leq_{\mathfrak{k}} M_3^*$ and $a \in M_3$ then: $\bigcup\limits_{<\infty}(M_0, M_1, a, M_3)$ iff $\bigcup\limits_{<\infty}(M_0, M_1, a, M_3^*)$.

Proof. 1) First assume $\mathrm{ortp}(a, M, N) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ and check the definition of $(M, N, a) \in K^{3,\mathrm{bs}}$. Clearly $M \leq_{\mathfrak{k}} N, a \in N$ and $a \in N \setminus M$; we have to find M' as required in Definition 2.4(1); we let $M' = M$, so $M' \leq_{\mathfrak{k}} M, M' \in K_\lambda$ and

$$M' \leq_{\mathfrak{k}} M'' \leq_{\mathfrak{k}} M \text{ and } M'' \in K_\lambda \Rightarrow M'' = M$$
$$\Rightarrow \mathrm{ortp}_{\mathfrak{k}_\lambda}(a, M'', N) = \mathrm{ortp}_{\mathfrak{k}_\lambda}(a, M, N) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M) = \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M'')$$

so we are done.

Second, assume $(M, N, a) \in K^{3,\mathrm{bs}}$ so there is $M' \leq_{\mathfrak{k}} M$ as asserted in the definition 2.4(1) of $K^{3,\mathrm{bs}}$ so $(\forall M'')[M' \leq_{\mathfrak{k}} M'' \leq_{\mathfrak{k}} M$ and $M'' \in K_\lambda \Rightarrow \mathrm{ortp}(a, M'', N) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M'')]$ in particular this holds for $M'' = M$ and we get $\mathrm{ortp}(a, M, N) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ as required.

2) First assume $\bigcup\limits_{<\infty}(M_0, M_1, a, M_3)$.

So there is M_0' as required in Definition 2.5; this means

$$M_0' \in K_\lambda, M_0' \leq_{\mathfrak{k}} M_0 \text{ and}$$
$$(\forall M_1' \in K_\lambda)(\forall M_3' \in K_\lambda)[M_0' \leq_{\mathfrak{k}} M_1' \leq M_1 \text{ and } M_1' \cup \{a\} \subseteq M_3' \leq_{\mathfrak{k}} M_3$$
$$\rightarrow \bigcup\limits_{\lambda}(M_0', M_1', a, M_3')].$$

In particular, we can choose $M_1' = M_1, M_3' = M_3$ so the antecedent holds hence $\bigcup\limits_{\lambda}(M_0', M_1', a, M_3')$ which means $\bigcup\limits_{\lambda}(M_0', M_1, a, M_3)$ and by clause $(E)(b)$ of Definition 2.1, $\bigcup\limits_{\lambda}(M_0, M_1, a, M_3)$ holds, as required.

Second assume $\bigcup\limits_{\lambda}(M_0, M_1, a, M_3)$. So in Definition 2.5 the demands $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_3, a \in M_3 \setminus M_1$ hold by clause $(E)(a)$ of Definition 2.1; and we choose M_0' as M_0; clearly $M_0' \in K_\lambda$ and $M_0' \leq_{\mathfrak{k}} M_0$. Now suppose $M_0' \leq_{\mathfrak{k}} M_1' \leq_{\mathfrak{k}} M_1$ and $M_1' \in K_\lambda, M_1' \cup \{a\} \leq_{\mathfrak{k}} M_3' \leq M_3$; by clause $(E)(b)$ of Definition 2.1 we have $\bigcup\limits_{\lambda}(M_0', M_1', a, M_3')$; so M_0' is as required so really $\bigcup\limits_{<\infty}(M_0, M_1, a, M_3)$.

3) We prove something stronger: for any $M' \in \mathfrak{k}_{\mathfrak{s}}$ which is $\leq_{\mathfrak{k}[\mathfrak{s}]} M, M'$ witnesses $(M, N_1, a) \in K^{3,\mathrm{bs}}$ iff M' witnesses $(M, N_2, a) \in K^{3,\mathrm{bs}}$ (of course, witness means: as required in Definition 2.4). So we have to check the statement there for every $M'' \in K_\lambda$ such that $M' \leq_{\mathfrak{s}} M'' \leq_{\mathfrak{k}} M$. The equivalence holds because for every $M'' \leq_{\mathfrak{k}} M, M'' \in K_\lambda$ we have $\mathrm{ortp}(a, M'', N_1) = \mathrm{ortp}(a, M'', N_2)$, by 1.11(2), more transparent as $\mathfrak{k}_\lambda$ has the amalgamation property (by clause (C) of Definition 2.1) and so one is "basic" iff the other is by clause $(E)(b)$ of Definition 2.1.

4) The direction $\Leftarrow$ is because if M_0' witness $\bigcup\limits_{<\infty}(M_0, M_1, a, M_3^*)$ (see Definition 2.5), then it witnesses $\bigcup\limits_{<\infty}(M_0, M_1, a, M_3)$ as there are just fewer pairs (M_1', M_3') to consider. For the direction $\Rightarrow$ the demands $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_3, a \in M_3 \setminus M_1$, of course, hold and let M_0' be as required in the definition of $\bigcup\limits_{<\infty}(M_0, M_1, a, M_3)$; let

$M'_0 \le_{\mathfrak{k}} M'_1 \le_{\mathfrak{k}} M_1, M'_1 \cup \{a\} \subseteq M'_3 \le_{\mathfrak{k}} M^*_3, M'_3 \in K_\lambda$. As $\lambda \ge \mathrm{LS}(\mathfrak{k})$ we can find $M''_3 \le_{\mathfrak{k}} M_3$ such that $M'_1 \cup \{a\} \subseteq M''_3 \in K_\lambda$ and then find $M'''_3 \le_{\mathfrak{s}} M^*_3$ such that $M'_3 \cup M''_3 \subseteq M'''_3 \in K_\lambda$. So by the choice of M'_0 and M''_3 clearly $\bigcup\limits_{\lambda}(M'_0, M'_1, a, M''_3)$ and by clause $(E)(b)$ of Definition 2.1 we have

$$\bigcup_{\lambda}(M'_0, M'_1, a, M''_3) \Leftrightarrow \bigcup_{\lambda}(M'_0, M'_1, a, M'''_3) \Leftrightarrow \bigcup_{\lambda}(M'_0, M'_1, a, M'_3)$$

(note that we know the left statement and need the right statement) so M'_1 is as required to complete the checking of $\bigcup\limits_{<\infty}(M_0, M_1, a, M^*_3)$. $\square_{2.6}$

We extend the definition of $\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$ from $M \in K_\lambda$ to arbitrary $M \in K$.

Definition 2.7. 1) For $M \in K$ we let

$$\mathcal{S}^{\mathrm{bs}}(M) = \mathcal{S}^{\mathrm{bs}}_{\ge \mathfrak{s}}(M) = \Big\{ p \in \mathcal{S}(M) : \text{for some } N \text{ and } a, \quad p = \mathrm{ortp}(a, M, N) \text{ and } (M, N, a) \in K^{3,\mathrm{bs}}_{\ge \mathfrak{s}} \Big\}$$

(for $M \in K_\lambda$ we get the old definition by 2.6(1); note that as we do not have amalgamation (in general) the meaning of types is more delicate. Not so in $\mathfrak{k}_\lambda$ as in a good λ-frame we have amalgamation in $\mathfrak{k}_\lambda$ but not necessarily in $\mathfrak{k}_{\ge\lambda}$).

2) We say that $p \in \mathcal{S}^{\mathrm{bs}}_{\ge \mathfrak{s}}(M_1)$ does not fork over $M_0 \le_{\mathfrak{k}} M_1$ <u>if</u> for some M_3, a we have $p = \mathrm{ortp}_{\mathfrak{k}[\mathfrak{s}]}(a, M_1, M_3)$ and $\bigcup\limits_{<\infty}(M_0, M_1, a, M_3)$. (Again, for $M \in K_\lambda$ this is equivalent to the old definition by 2.6).

3) For $M \in K$ let $\mathcal{E}^\lambda_M$ be the following two-place relation on $\mathcal{S}(M)$: $p_1 \mathcal{E}^\lambda_M p_2$ iff $p_1, p_2 \in \mathcal{S}^{\mathrm{bs}}(M)$ and if $p_\ell = \mathrm{ortp}(a_\ell, M, M^*), N \le_{\mathfrak{k}} M, N \in K_\lambda$ then $p_1 \restriction N = p_2 \restriction N$. Let $\mathcal{E}^{\mathfrak{s}}_M = \mathcal{E}^{\lambda(\mathfrak{s})}_M \restriction \mathcal{S}^{\mathrm{bs}}(M)$.

4) $\mathfrak{k}$ is (λ, μ)-local if every $M \in \mathfrak{k}_\mu$ is λ-local which means that $\mathcal{E}^\lambda_M$ is equality; let $(\mathfrak{s}, \mu)$-local means $(\lambda_{\mathfrak{s}}, \mu)$-local. Though we will prove below some nice things, having the extension property is more problematic. We may define "the extension" in a formal way, for $M \in K_{>\lambda}$ but then it is not clear if it is realized in any $\le_{\mathfrak{k}}$-extension of M. Similarly for the uniqueness property. That is, assume $M_0 \le_{\mathfrak{k}} M \le_{\mathfrak{k}} N_\ell$ and $a_\ell \in N_\ell \setminus M$, and $M_0 \in \mathfrak{k}_{\mathfrak{s}}$ and $\mathrm{ortp}(a_\ell, M, N_\ell)$ does not fork over M_0 for $\ell = 1, 2$ and $\mathrm{ortp}(a_1, M_0, N_1) = \mathrm{ortp}(a_2, M_0, N_1)$. Now does it follow that $\mathrm{ortp}(a_1, M, N_1) = \mathrm{ortp}(a_2, M, N_2)$? This requires the existence of some form of amalgamation in $\mathfrak{k}$, which we are not justified in assuming. So we may prefer to define $\mathcal{S}^{\mathrm{bs}}(M)$ "formally", the set of stationarization of $p \in \mathcal{S}^{\mathrm{bs}}(M_0), M_0 \in \mathfrak{k}_{\mathfrak{s}}$, see [SV]. We now note that in definition 2.7 "some" can be replaced by "every".

Fact 2.8. 1) For $M \in K$

$$\mathcal{S}^{\mathrm{bs}}_{\ge \mathfrak{s}}(M) = \Big\{ p \in \mathcal{S}_{\mathfrak{k}[\mathfrak{s}]}(M) : \text{for every } N, a \text{ we have: if } M \le_{\mathfrak{k}} N, a \in N \setminus M \text{ and } p = \mathrm{ortp}_{\mathfrak{k}}(a, M, N) \text{ then } (M, N, a) \in K^{3,\mathrm{bs}}_{\ge \mathfrak{s}} \Big\}.$$

2) The type $p \in \mathcal{S}_{\mathfrak{k}[\mathfrak{s}]}(M_1)$ does not fork over $M_0 \le_{\mathfrak{k}} M_1$ iff for every a, M_3 satisfying $M_1 \le_{\mathfrak{k}} M_3 \in K, a \in M_3 \setminus M_1$ and $p = \mathrm{ortp}_{\mathfrak{k}[\mathfrak{s}]}(a, M_1, M_3)$ we have $\underset{<\infty}{\bigcup}(M_0, M_1, a, M_3)$.

3) $(M, N, a) \in K^{3,\mathrm{bs}}_{\ge \mathfrak{s}}$ is preserved by isomorphisms.

4) If $M \le_{\mathfrak{k}} N_\ell, a_\ell \in N_\ell \setminus M$ for $\ell = 1,2$ and $\mathrm{ortp}(a_1, M, N_1)\mathcal{E}^{\mathfrak{s}}_M \mathrm{ortp}(a_2, M, N_2)$ then $(M, N_1, a_1) \in K^{3,\mathrm{bs}}_{\ge \mathfrak{s}} \Leftrightarrow (M, N_2, a_2) \in K^{3,\mathrm{bs}}_{\ge \mathfrak{s}}$.

5) $\mathcal{E}^{\mathfrak{s}}_M$ is an equivalence relation on $\mathcal{S}^{\mathrm{bs}}_{\ge \mathfrak{s}}(M)$ and if $p, q \in \mathcal{S}^{\mathrm{bs}}_{\ge \mathfrak{s}}(M)$ do not fork over $N \in K_\lambda$ so $N \le_{\mathfrak{k}} M$ then $p\mathcal{E}^{\mathfrak{s}}_M q \Leftrightarrow (p \restriction N = q \restriction N)$.

Proof. 1) By 2.6(3) and the definition of type.

2) By 2.6(4) and the definition of type.

3) Easy.

4) Enough to deal with the case $(M, N_1, a_1)E^{\mathrm{at}}_M, (M, N_2, a_2)$ or (by (3)) even $a_1 = a_2, N_1 \le_{\mathfrak{k}} N_2$. This is easy.

5) Easy, too. $\square_{2.8}$

We can also get that there are enough basic types, as follows:

Claim 2.9. *If $M \le_{\mathfrak{k}} N$ and $M \ne N$, <u>then</u> for some $a \in N \setminus M$ we have $\mathrm{ortp}_{\mathfrak{k}}(a, M, N) \in \mathcal{S}^{\mathrm{bs}}(M)$.*

Proof. Suppose not. So as we are assuming $K = K_{\ge \lambda}$, by clause (D)(c) of Definition 2.1, necessarily $\|N\| > \lambda$. If $\|M\| = \lambda < \|N\|$ choose N' satisfying $M <_{\mathfrak{k}} N' \le_{\mathfrak{k}} N$, $N' \in K_\lambda$ and by clause $(D)(c)$ of Definition 2.1 choose $a^* \in N' \setminus M$ such that $\mathrm{ortp}_{\mathfrak{s}}(a^*, M, N') \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M)$. So we can assume $\|M\| > \lambda$; choose $a^* \in N \setminus M$. We choose $M_i, N_i, M_{i,c}$ by induction on $i < \omega$ (for $c \in N_i \setminus M_i$) such that:

(a) $M_i \le_{\mathfrak{k}} M$ is $\le_{\mathfrak{k}}$-increasing.
(b) $M_i \in K_\lambda$
(c) $N_i \le_{\mathfrak{k}} N$ is $\le_{\mathfrak{k}}$-increasing.
(d) $N_i \in K_\lambda$
(e) $a^* \in N_0$
(f) $M_i \le_{\mathfrak{k}} N_i$
(g) If $c \in N_i \setminus M$, $\mathrm{ortp}_{\mathfrak{s}}(c, M_i, N) \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_i)$ and there is $M' \in K_\lambda$ such that $M_i \le_{\mathfrak{k}} M' \le_{\mathfrak{k}} M$ and $\mathrm{ortp}_{\mathfrak{s}}(c, M', N)$ forks over M_i then $M_{i,c}$ satisfies this, otherwise $M_{i,c} = M_i$
(h) M_{i+1} includes the set $\bigcup\limits_{c \in N_i \setminus M} M_{i,c} \cup (N_i \cap M)$.

There is no problem to carry the definition; in stage $i+1$ first choose $M_{i,c}$ for $c \in N_i \setminus M$ then choose M_{i+1} and lastly choose N_{i+1}. Let $M^* = \bigcup\limits_{i<\omega} M_i$ and $N^* = \bigcup\limits_{i<\omega} N_i$. It is easy to check that:

(i) $M_i \le_{\mathfrak{k}} M^* \le_{\mathfrak{k}} M$ for $i < \omega$
(by clause (a))

(ii) $M^* \in K_\lambda$
(by clause (i) we have $M^* \in K$ and $\|M^*\| = \lambda$ by the choice of M^* and clause (b))

(iii) $N_i \leq_{\mathfrak{k}} N^* \leq_{\mathfrak{k}} N$
(by clause (c))

(iv) $N^* \in K_\lambda$
(by clause (iii) we have $N^* \in K$ and $\|N^*\| = \lambda$ by the choice of N^* and clause (d))

(v) $M_i \leq_{\mathfrak{k}} M^* \leq_{\mathfrak{k}} N^* \leq_{\mathfrak{k}} N$
(by clauses (a) + (f) + (iii) we have $M_i \leq_{\mathfrak{k}} N^*$ hence by clause (a) and the choice of M^* we have $M^* \leq_{\mathfrak{k}} N^*$, and $N^* \leq_{\mathfrak{k}} N$ by clause (iii))

(vi) $M^* = N^* \cap M$
(by clauses (f) + (h) and the choices of M^*, N^*)

(vii) $M^* \neq N^*$
(as $a^* \in N \setminus M$ and $a^* \in N_0 \leq_{\mathfrak{k}} N^* \leq_{\mathfrak{k}} N$ and $M^* = N^* \cap M$;
they hold by the choice of a^*, clause (e), clause (iii), clause (iii) and clause (vi) respectively)

(viii) there is $b^* \in N^* \setminus M^*$ such that $\text{ortp}(b^*, M^*, N^*) \in \mathcal{S}^{\text{bs}}(M^*)$
[why? by clause (v) and (viii) recalling Definition 2.1 clause (D)(c) (density)]

(ix) for some $i < \omega$ we have $\bigcup(M_i, M^*, b^*, N^*)$, so

$\text{ortp}(b^*, M^*, N^*) \in \mathcal{S}^{\text{bs}}_{\mathfrak{s}}(M^*)$ and $\text{ortp}_{\mathfrak{s}}(b^*, M_j, N^*) \in \mathcal{S}^{\text{bs}}_{\mathfrak{s}}(M_j)$ for $j \in [i, \omega)$

[why? by Definition 2.1 clause $(E)(c)$ (local character) applied to the sequence $\langle M_n : n < \omega\rangle\char`^\langle M^*, N^*\rangle$ and the element b^*, using of course (E)(a) of Definition 2.1 and clause (viii)]

(x) $\bigcup(M_i, M_{i,b^*}, b^*, N^*)$

[why? by clause (ix) and Definition 2.1$(E)(b)$ (monotonicity) as
$M_i \leq_{\mathfrak{k}} M_{i,b^*} \leq_{\mathfrak{k}} M_{i+1} \leq_{\mathfrak{k}} M^*$ by clause (g) in the construction]

(xi) if $M_i \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} M$ and $M' \cup \{b^*\} \subseteq N' \leq_{\mathfrak{k}} N$ and $M' \in K_\lambda, N' \in K_\lambda$ then $\bigcup(M_i, M', b^*, N')$

[why? by clause (x) and clause (g) in the construction.]

So we are done. $\square_{2.9}$

Claim 2.10. *If $M \leq_{\mathfrak{k}} N, a \in N \setminus M$, and $\text{ortp}(a, M, N) \in \mathcal{S}^{\text{bs}}_{\geq \mathfrak{s}}(M)$ <u>then</u> for some $M_0 \leq_{\mathfrak{k}} M$ we have*

(A) $M_0 \in K_\lambda$

(B) $\text{ortp}(a, M_0, N) \in \mathcal{S}^{\text{bs}}_{\mathfrak{s}}(M_0)$

(C) If $M_0 \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} M$, <u>then</u> $\text{ortp}(a, M', N) \in \mathcal{S}^{\text{bs}}_{\mathfrak{s}}(M')$ does not fork over M_0.

Proof. Easy by now. $\square_{2.10}$

Claim 2.11. *1) Assume $M_1 \leq_{\mathfrak{k}} M_2$ and $p \in \mathcal{S}_{\mathfrak{k}}(M_2)$. <u>Then</u> $p \in \mathcal{S}^{\text{bs}}_{\geq \mathfrak{s}}(M_2)$ and p does not fork over M_1 <u>iff</u> for some $N_1 \leq_{\mathfrak{k}} M_1, N_1 \in K_\lambda$ and p does not fork over N_1 <u>iff</u> for some $N_1 \leq_{\mathfrak{k}} M_1, N_1 \in K_\lambda$ and we have $(\forall N)[N_1 \leq_{\mathfrak{k}} N \leq_{\mathfrak{k}} M_2$ and $N \in K_\lambda \Rightarrow p \restriction N \in \mathcal{S}^{\text{bs}}_{\mathfrak{s}}(N)$ and $(p \restriction N$ does not fork over $N_1)]$; we call such N_1 a witness, so every $N'_1 \in K_\lambda, N_1 \leq_{\mathfrak{k}} N'_1 \leq M_1$ is a witness, too.*
2) Assume $M^* \in K$ and $p \in \mathcal{S}_{\mathfrak{k}}(M^*)$.
<u>Then</u>: $p \in \mathcal{S}^{\text{bs}}_{\geq \mathfrak{s}}(M^*)$ <u>iff</u> for some $N^* \leq_{\mathfrak{k}} M^*$ we have $N^* \in K_\lambda, p \restriction N^* \in \mathcal{S}^{\text{bs}}(N^*)$

and $(\forall N \in K_\lambda)(N^* \le_{\mathfrak{k}} N \le_{\mathfrak{k}} M^* \Rightarrow p \restriction N \in \mathcal{S}^{\rm bs}(N)$ *and does not fork over* $N^*)$ *(we say such* N^* *is a witness, so any* $N' \in K_\lambda, N^* \le_{\mathfrak{k}} N' \le_{\mathfrak{k}} M$ *is a witness, too).*

3) (Monotonicity)
If $M_1 \le_{\mathfrak{k}} M_1' \le_{\mathfrak{k}} M_2' \le_{\mathfrak{k}} M_2$ *and* $p \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(M_2)$ *does not fork over* M_1, *then* $p \restriction M_2' \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(M_2')$ *and it does not fork over* M_1'.

4) (Transitivity)
If $M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_2$ *and* $p \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(M_2)$ *does not fork over* M_1 *and* $p \restriction M_1$ *does not fork over* M_0, *then* p *does not fork over* M_0.

5) (Local character) If $\langle M_i : i \le \delta + 1\rangle$ *is* $\le_{\mathfrak{k}}$*-increasing continuous and* $a \in M_{\delta+1}$ *and* $\mathrm{ortp}_{\mathfrak{k}}(a, M_\delta, M_{\delta+1}) \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(M_\delta)$ *then for some* $i < \delta$ *we have* $\mathrm{ortp}_{\mathfrak{k}}(a, M_\delta, M_{\delta+1})$ *does not fork over* M_i.

6) Assume that $\langle M_i : i \le \delta + 1\rangle$ *is* $\le_{\mathfrak{k}}$*-increasing,* $p \in \mathcal{S}(M_\delta)$, *and for every* $i < \delta$ *we have that* $p \restriction M_i \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(M_i)$ *does not fork over* M_0. *Then* $p \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(M_\delta)$ *and* p *does not fork over* M_0.

Proof. 1), 2) Check the definitions.

3) As $p \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(M_2)$ does not fork over M_1, there is $N_1 \in K_\lambda$ which witnesses it. This same N_1 witnesses that $p \restriction M_2'$ does not fork over M_1'.

4) Let $N_0 \le_{\mathfrak{k}} M_0$ witness that $p \restriction M_1$ does not fork over M_0 (in particular $N_0 \in K_\lambda$); let $N_1 \le_{\mathfrak{k}} M_1$ witness that p does not fork over M_1 (so in particular $N_1 \in K_\lambda$). Let us show that N_0 witnesses p does not fork over M_0, so let $N \in K_\lambda$ be such that $N_0 \le_{\mathfrak{k}} N \le_{\mathfrak{k}} M_2$ and we should just prove that $p \restriction N$ does not fork over N_0. We can find $N' \le_{\mathfrak{k}} M_1, N' \in K_\lambda$ such that $N_0 \cup N_1 \subseteq N'$, we can also find $N'' \le_{\mathfrak{k}} M_2$ satisfying $N'' \in K_\lambda$ such that $N' \cup N \subseteq N''$. As N_1 witnesses that p does not fork over M_1, clearly $p \restriction N'' \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N'')$ does not fork over N_1, hence by monotonicity does not fork over N'. As N_0 witnesses $p \restriction M_1$ does not fork over M_0, clearly $p \restriction N'$ belongs to $\mathcal{S}^{\rm bs}(N')$ and does not fork over N_0, so by transitivity (in $\mathfrak{k}_{\mathfrak{s}}$) we know that $p \restriction N''$ does not fork over N_0; hence by monotonicity $p \restriction N$ does not fork over N_0.

5) Let $p = \mathrm{ortp}_{\mathfrak{k}}(a, M_\delta, M_{\delta+1})$ and let $N^* \le_{\mathfrak{k}} M_\delta$ witness $p \in \mathcal{S}^{\rm bs}(M_\delta)$. Assume toward contradiction that the conclusion fails. Without loss of generality $\mathrm{cf}(\delta) = \delta$.

Case 0: $\|M_\delta\| \le \lambda (= \lambda_{\mathfrak{s}})$.
Trivial.

Case 1: $\delta < \lambda^+, \|M_\delta\| > \lambda$.
As $\|M_\delta\| > \lambda$, for some i, $\|M_i\| > \lambda$ so without loss of generality $i < \delta \Rightarrow \|M_i\| > \lambda$. We choose by induction on $i < \delta$, models N_i, N_i' such that:

- (α) $N_i \in K_\lambda$
- (β) $N_i \le_{\mathfrak{k}} M_i$ (hence $N_i \le_{\mathfrak{k}} M_j$ for $j \in [i, \delta)$)
- (γ) N_i is $\le_{\mathfrak{k}}$-increasing continuous
- (δ) $N_i' \in K_\lambda, N^* \le_{\mathfrak{k}} N_0'$
- (ε) $N_i \le_{\mathfrak{k}} N_i' \le_{\mathfrak{k}} M_\delta$,
- (ζ) N_i' is $\le_{\mathfrak{k}}$-increasing continuous
- (η) $p \restriction N_i'$ forks over N_i when $i \ne 0$ for simplicity
- (θ) $N_i \cup \bigcup_{j \le i}(N_j' \cap M_{i+1}) \subseteq N_{i+1}$.

No problem to carry the induction, but we give details.

First, if $i = 0$ trivial. Second let i be a limit ordinal.

Let $N_i = \cup\{N_j : j < i\}$, now $N_i \le_{\mathfrak{k}} M_i$ by clauses $(\beta) + (\gamma)$ and $\mathfrak{k}$ being AEC and $\|N_i\| = \lambda$ by clause (α), as $i \le \delta < \lambda^+$; so clauses $(\alpha), (\beta), (\gamma)$ hold. Next, let $N'_i = \cup\{N'_j : j < i\}$ and similarly clauses $(\delta), (\varepsilon), (\zeta)$ hold. Lastly, we shall prove clause (η) and assume toward contradiction that it fails; so $p \restriction N'_i$ does not fork over N_i in particular $p \restriction N_i \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N_i)$ hence for some $j < i$ the type $p \restriction N'_i$ does not fork over $N_j \le_{\mathfrak{k}} N_i$, (by (E)(c) of Definition 2.1) hence by transitivity (for $\mathfrak{k}_{\mathfrak{s}}$), $p \restriction N'_i$ does not fork over N_j hence by monotonicity $p \restriction N'_j$ does not fork over N_j (see (E)(b) of Definition 2.1) contradicting the induction hypothesis.

Lastly, clause (θ) is vacuous.

Third assume $i = j + 1$, so first choose N_i satisfying clause (θ) (with j, i here standing for $i, i + 1$ there), and $(\alpha), (\beta), (\gamma)$; this is possible by the L.S. property. Now N_i cannot witness "p does not fork over M_i" hence for some $N^*_i \in K_\lambda$ we have $N_i \le_{\mathfrak{k}} N^*_i \le_{\mathfrak{k}} M_\delta$ and $p \restriction N^*_i$ forks over N_i; again by L.S. choose $N'_i \in K_\lambda$ such that $N'_i \le_{\mathfrak{k}} M_\delta$ and $N^* \cup N_i \cup N'_j \cup N^*_i \subseteq N'_i$, easily (N_i, N'_i) are as required.

Let $N_\delta = \bigcup\limits_{i<\delta} N_i$, so by clause $(\beta), (\gamma)$ we have $N_\delta \le_{\mathfrak{k}} M_\delta$ and by clause (α), as $\delta < \lambda^+$ we have $N_\delta \in K_\lambda$ and by clauses $(\delta) + (\theta)$ in the construction we have $i < \delta \Rightarrow N'_i = \cup\{N'_i \cap M_{j+1} : j \in [i, \delta)\} \subseteq N$ so by clause $(\delta), N^* \le_{\mathfrak{k}} N'_0 \le_{\mathfrak{k}} N_\delta$. Hence by the choice of $N^*, p \restriction N_\delta \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N_\delta)$ and it does not fork over N^*. Now as $p \restriction N_\delta \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N_\delta)$ by local character, i.e., clause $(E)(c)$ of Definition 2.1, for some $i < \delta, p \restriction N_\delta$ does not fork over N_i (so $p \restriction N_i \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N_i)$). Now $N_i \le_{\mathfrak{k}} N'_i \le_{\mathfrak{k}} M_\delta$ and by clause (θ) of the construction $N'_i \subseteq N_\delta$ hence $N_i \le_{\mathfrak{k}} N'_i \le_{\mathfrak{k}} N_\delta$ hence by monotonicity of non-forking (i.e. clause (E)(b) of Definition 2.1), $p \restriction N'_i \in \mathcal{S}^{\rm bs}(N_i)$ does not fork over N_i. But this contradicts the choice of N'_i (i.e., clause (η) of the construction).

Case 2: $\delta = {\rm cf}(\delta) > \lambda$.

Recall that $N^* \le_{\mathfrak{k}} M_\delta, N^*$ is from K_λ and $N^* \le_{\mathfrak{k}} N \le_{\mathfrak{k}} M_\delta$ and $N \in K_\lambda \Rightarrow$ $p \restriction N \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(N)$. Now as $\delta = {\rm cf}(\delta) > \lambda \ge \|N^*\|$ clearly for some $i < \delta$ we have $N^* \subseteq M_i$ hence $N^* \le_{\mathfrak{k}} M_i$ (hence $i \le j < \delta \Rightarrow p \restriction M_j \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(M_j)$), and N^* witnesses that $p \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(M_\delta)$ does not fork over M_i so we are clearly done.

6) Let $N_0 \in K_\lambda$, $N_0 \le_{\mathfrak{k}} M_0$ witness $p \restriction M_0 \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(M_0)$. By the proof of part (4) clearly $i < \delta$ and $N_0 \le_{\mathfrak{k}} N \in K_\lambda$ and $N \le_{\mathfrak{k}} M_i \Rightarrow p \restriction N$ does not fork over N_0. If ${\rm cf}(\delta) > \lambda$ we are done, so assume ${\rm cf}(\delta) \le \lambda$. Let $N_0 \le_{\mathfrak{k}} N^* \in K_\lambda$ and $N^* \le_{\mathfrak{k}} M_\delta$, and we shall prove that $p \restriction N^*$ does not fork over N_0, this clearly suffices. As in Case 1 in the proof of part (5) we can find $N_i \le_{\mathfrak{k}} M_i$ for $i \in (0, \delta)$ such that $\langle N_i : i \le \delta\rangle$ is $\le_{\mathfrak{k}}$-increasing with i, each N_i belongs to $\mathfrak{k}_\lambda$ and $N^* \cap M_i \subseteq N_{i+1}$, hence $N^* \subseteq N_\delta := \bigcup\limits_{i<\delta} N_i$. Now $N_\delta \le_{\mathfrak{k}} M_\delta$ and as said as $i < \delta \Rightarrow p \restriction N_i \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(N_i)$ does not fork over N_0 hence $p \restriction N_\delta$ does not fork over N_0 and by monotonicity $p \restriction N^*$ does not fork over N_0, as required. $\square_{2.11}$

Lemma 2.12. *If $\mu = {\rm cf}(\mu) > \lambda$ and $M \le_{\mathfrak{k}} N$ are in K_μ, <u>then</u> we can find $\le_{\mathfrak{k}}$-representations $\overline{M}, \overline{N}$ of M, N respectively such that:*

(i) $N_i \cap M = M_i$ for $i < \mu$

(ii) if $i < j < \mu$ and $a \in N_i$ <u>then</u>

$$(a) \qquad {\rm ortp}(a, M_i, N) \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(M_i) \Leftrightarrow {\rm ortp}(a, M_j, N) \in \mathcal{S}^{\rm bs}_{\ge \mathfrak{s}}(M_j)$$
$$\Leftrightarrow {\rm ortp}(a, M, N) \text{ does not fork over } M_i$$
$$\Leftrightarrow {\rm ortp}(a, M_j, N) \text{ is a non-forking extension of } {\rm ortp}(a, M_i, N)$$

(b) *$M_i \leq_{\mathfrak{k}} N_i \leq_{\mathfrak{k}} N_j$ and $M_i \leq_{\mathfrak{k}} M_j \leq_{\mathfrak{k}} N_j$ (and obviously $M_i \leq_{\mathfrak{k}} N_j$ and $M_i \leq_{\mathfrak{k}} M, M_i \leq_{\mathfrak{k}} N, N_i \leq_{\mathfrak{k}} N$).*

Remark 2.13. In fact for any representations $\bar{M}, \overline{N}$ of M, N respectively, for some club E of μ the sequences $\overline{M} \restriction E, \overline{N} \restriction E$ are as above.

Proof. Let $\overline{M}$ be a $\leq_{\mathfrak{k}}$-representation of M. For $a \in N$ we define $S_a = \{\alpha < \mu : \mathrm{ortp}(a, M_\alpha, N) \in \mathcal{S}^{\mathrm{bs}}_{\geq \mathfrak{s}}(M_\alpha)\}$. Clearly if $\delta \in S_a$ is a limit ordinal then for some $i(a,\delta) < \delta$ we have $i(a,\delta) \leq i < \delta \Rightarrow i \in S_a$ and $(\mathrm{ortp}(a, M_i, N)$ does not fork over $M_{i_{(a,\delta)}})$ by 2.11(5). So if S_a is stationary, then for some $i(a) < \mu$ the set $S'_a = \{\delta \in S_a : i(a,\delta) = i(a)\}$ is a stationary subset of λ hence by monotonicity we have $i(a) \leq i \leq \mu \Rightarrow \mathrm{ortp}(a, M_i, N)$ does not fork over $M_{i(a)}$. Let E_a be a club of μ such that: if S_a is not stationary (subset of μ) then $E_a \cap S_a = \varnothing$ and if S_a is not stationary then $S_a \cap E_a = \varnothing$.

Let $\overline{N}$ be a representation of N, and let

$$E^* = \{\delta < \mu : N_\delta \cap M = M_\delta \text{ and } M_\delta \leq_{\mathfrak{k}} M, N_\delta \leq_{\mathfrak{k}} N \text{ and for every } a \in N_\delta \text{ we have } \delta \in E_a\}.$$

Clearly it is a club of μ and $\overline{M} \restriction E^*$, $\overline{N} \restriction E^*$ are as required. $\square_{2.12}$

* * *

We may treat the lifting of $K^{3,\mathrm{bs}}_\lambda$ as a special case of the "lifting" of $\mathfrak{k}_\lambda$ to $\mathfrak{k}_{\geq \lambda} = (\mathfrak{k}_\lambda)^{\mathrm{up}}$ in Claim 1.24; this may be considered a good exercise.

Claim 2.14. *1) $(K^{3,\mathrm{bs}}_\lambda, \leq_{\mathrm{bs}})$ is a λ-AEC.*
2) $(K^{3,\mathrm{bs}}_{\geq \mathfrak{s}}, \leq_{\mathrm{bs}})$ is $(K^{3,\mathrm{bs}}_\lambda, \leq_{\mathrm{bs}})^{\mathrm{up}}$.

Remark 2.15. What is the class in 2.14(1)? Formally, let $\tau^+ = \{R_{[\ell]} : R$ a predicate of τ_K, $\ell = 1,2\} \cup \{F_{[\ell]} : F$ a function symbol from τ_K and $\ell = 1,2\} \cup \{c\}$ where $R_{[\ell]}$ is an n-place predicate when $R \in \tau$ is an n-place predicate and similarly $F_{[\ell]}$ and c is an individual constant. A triple (M, N, a) is identified with the following τ^+-model N^+ defined as follows:

(A) Its universe is the universe of N.
(B) $c^{N^+} = a$
(C) $R^{N^+}_{[2]} = R^N$
(D) $F^{N^+}_{[2]} = F^N$
(E) $R^{N^+}_{[1]} = R^M$
(F) $F^{N^+}_{[1]} = F^M$

(if you do not like partial functions, extend them to functions with full domain by $F(a_0, \ldots) = a_0$ when not defined if F has arity > 0, if F has arity zero it is an individual constant, $F^{N^+} = F^N$ so no problem).

Proof. Left to the reader (in particular, this means that $K^{3,\mathrm{bs}}_\lambda$ is closed under $\leq_{\mathrm{bs}}$-increasing chains of length $< \lambda^+$). $\square_{2.14}$

Continuing 1.24, 1.27 (and see more in 2.23), note that:

Lemma 2.16. *Assume*

(a) $\mathfrak{k}$ *is an abstract elementary class with* $\mathrm{LS}(\mathfrak{k}) \leq \mu$.

(b) $K'_{\leq\mu}$ *is a class of* τ_K*-model,* $K'_{\leq\mu} \subseteq K_{\leq\mu}$ *is non-empty and closed under* $\leq_{\mathfrak{k}}$*-increasing unions of length* $< \mu^+$ *and isomorphisms (e.g. the class of* μ*-superlimit models of* $\mathfrak{k}_\mu$*, if there is one).*

(c) $K' := \{M \in K : M$ *is a* $\leq_{\mathfrak{k}}$*-directed union of members of* $K'_\mu\} \cup K'_{\leq\mu}$

(d) Let $\mathfrak{k}' = (K', \leq_{\mathfrak{k}} \restriction K')$ *so* $\leq_{\mathfrak{k}'}$ *is* $\leq_{\mathfrak{k}} \restriction K'$*, so* $\mathfrak{k}'_{\leq\mu} := (K'_{\leq\mu}, \leq_{\mathfrak{k}} \restriction K'_{\leq\mu})$*; or* $\leq_{\mathfrak{k}}$ *is as in 1.24(1) (see 1.24(4)).*

Then

(A) $\mathfrak{k}'$ *is an abstract elementary class,* $\mathrm{LS}(\mathfrak{k}) \leq \mathrm{LS}(\mathfrak{k}') \leq \mu$.

(B) If $\mu \leq \lambda$ *and* $(\mathfrak{k}, \bigcup, \mathcal{S}^{\mathrm{bs}})$ *is a good* λ*-frame,* $\mathfrak{k}'_\lambda$ *has amalgamation and JEP, and* $M \in \mathfrak{k}'_\lambda \Rightarrow \mathcal{S}_{\mathfrak{k}'}(M) = \mathcal{S}_{\mathfrak{k}}(M)$*, then* $(\mathfrak{k}', \bigcup, \mathcal{S}^{\mathrm{bs}})$ *(with* $\bigcup$*,* $\mathcal{S}^{\mathrm{bs}}$ *restricted to* $\mathfrak{k}'$*) is a good* λ*-frame.*

(C) In clause (B), instead of "$M \in \mathfrak{k}'_\lambda \Rightarrow \mathcal{S}_{\mathfrak{k}'}(M) = \mathcal{S}_{\mathfrak{k}}(M)$*," it suffices to require: if* $M \in \mathfrak{k}'_\lambda$*,* $M \leq_{\mathfrak{k}} N \in \mathfrak{k}'_\lambda$*,* $p \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N)$*, p does not fork over M, and* $p \restriction M$ *is realized in some* M' *with* $M \leq_{\mathfrak{k}'} M'$ *then p is realized in some* N' *with* $N \leq_{\mathfrak{k}} N' \in \mathfrak{k}'_\lambda$.

Remark 2.17. If in 2.16, K'_μ is not closed under $\leq_{\mathfrak{k}}$-increasing unions, we can close it but then the "so $\mathfrak{k}'_{\leq\mu} = \ldots$" in clause (d) may fail.

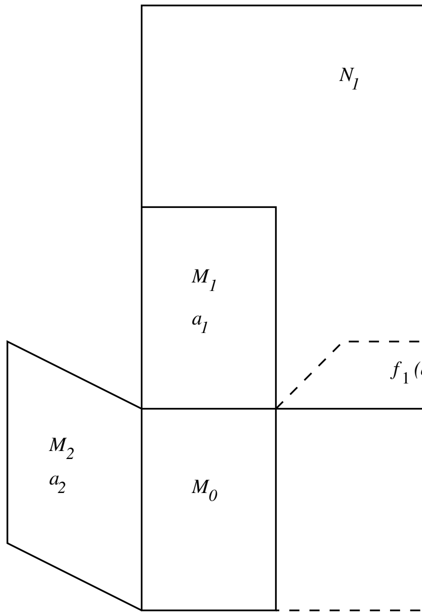
Proof. **Clause (A)**: As in 1.24. Clauses (B),(C): Check. $\square_{2.16}$

* * *

Next we deal with some implications between the axioms in 2.1.

Claim 2.18. *1) In Definition 2.1 clause (E)(i) is redundant, i.e., follows from the rest, recalling*

(E)(i) non-forking amalgamation:
if for $\ell = 1, 2, M_0 \leq_{\mathfrak{k}} M_\ell$ *are in* $K_\lambda, a_\ell \in M_\ell \setminus M_0, \mathrm{ortp}(a_\ell, M_0, M_\ell) \in \mathcal{S}^{\mathrm{bs}}(M_0)$*, then we can find* f_1, f_2, M_3 *satisfying* $M_0 \leq_{\mathfrak{k}} M_3 \in K_\lambda$ *such that for* $\ell = 1, 2$ *we have* f_ℓ *is a* $\leq_{\mathfrak{k}}$*-embedding of* M_ℓ *into* M_3 *over* M_0 *and* $\mathrm{ortp}(f_\ell(a_\ell), f_{3-\ell}(M_{3-\ell}), M_3)$ *does not fork over* M_0.



2) In fact, proving part (1) we use Axioms (A),(C),(E)(b),(d),(f),(g) only.

Proof. By Axiom (E)(g) (existence) applied with $\mathrm{ortp}(a_2, M_0, M_2), M_0, M_1$ here standing for p, M, N there; there is q_1 such that:

(a) $q_1 \in \mathcal{S}^{\mathrm{bs}}(M_1)$

(b) q_1 does not fork over M_0

(c) $q_1 \restriction M_0 = \mathrm{ortp}(a_2, M_0, M_2)$.

By the definition of types and as $\mathfrak{k}_\lambda$ has amalgamation (by Axiom (C)) there are N_1, f_1 such that

(d) $M_1 \le_{\mathfrak{k}} N_1 \in K_\lambda$

(e) f_1 is a $\le_{\mathfrak{k}}$-embedding of M_2 into N_1 over M_0

(f) $f_1(a_2)$ realizes q_1 inside N_1.

Now consider Axiom (E)(f) (symmetry) applied with $M_0, N_1, a_1, f_1(a_2)$ here standing for M_0, M_3, a_1, a_2 there; now as clause (α) of (E)(f) holds (use M_1, N_1 for M_1, M_3') we get that clause (β) of (E)(f) holds which means that there are N_2, N_2^* (standing for M_3', M_2 in clause (β) of (E)(f)) such that:

(g) $N_1 \le_{\mathfrak{k}} N_2 \in K_\lambda$

(h) $M_0 \cup \{f_1(a_2)\} \subseteq N_2^* \le_{\mathfrak{k}} N_2$

(i) $\mathrm{ortp}(a_1, N_2^*, N_2) \in \mathcal{S}^{\mathrm{bs}}(N_2^*)$ does not fork over M_0.

As $\mathfrak{k}_\lambda$ has amalgamation (see Axiom (C)) and the definition of type and as
$\mathrm{ortp}(f_1(a_2), M_0, f_1(M_2)) = \mathrm{ortp}(f_1(a_2), M_0, N_2) = \mathrm{ortp}(f_1(a_2), M_0, N_2^*)$, we can find N_3^*, f_2 such that

(j) $N_2^* \le_{\mathfrak{k}} N_3^* \in K_\lambda$

(k) f_2 is a $\leq_{\mathfrak{k}}$-embedding [14] of $f_1(M_2)$ into N_3^* over $M_0 \cup \{f_1(a_2)\}$.

As by clause (i) above $\text{ortp}(a_1, N_2^*, N_2) \in \mathcal{S}^{\text{bs}}(N_2^*)$, so by Axiom (E)(g) (extension existence) there are N_3, f_3 such that

(l) $N_2 \leq_{\mathfrak{k}} N_3 \in K_\lambda$

(m) f_3 is a $\leq_{\mathfrak{k}}$-embedding of N_3^* into N_3 over N_2^*

(n) $\text{ortp}(a_1, f_3(N_3^*), N_3) \in \mathcal{S}^{\text{bs}}(N_3^*)$ does not fork over N_2^*.

By Axiom (E)(d) (transitivity) using clauses (i) + (n) above we have

(o) $\text{ortp}(a_1, f_3(N_3^*), N_3) \in \mathcal{S}^{\text{bs}}(N_3^*)$ does not fork over M_0.

Letting $f = f_3 \circ f_2 \circ f_1$ as $f(M_2) \subseteq f_3(N_3^*)$ by clauses $(e), (k), (m)$ we have

(p) f is a $\leq_{\mathfrak{k}}$-embedding of M_2 into N_3 over M_0.

By (E)(b) (monotonicity) and clause (o) and clause (p)

(q) $\text{ortp}(a_1, f(M_2), N_3) \in \mathcal{S}^{\text{bs}}(f(M_2))$ does not fork over M_0.

As $\text{ortp}(f_1(a_2), M_1, N_3) = \text{ortp}(f_1(a_2), M_1, N_1) = q_1$ does not fork over M_0 by clauses (b) + (f), and $f_2(f_1(a_2)) = f_1(a_2)$ by clause (k) and $f_3(f_1(a_2)) = f_1(a_2)$ by clauses (m) + (h), we get

(r) $\text{ortp}(f(a_2), M_1, N_3) \in \mathcal{S}^{\text{bs}}(M_1)$ does not fork over M_0.

So by clauses (o) and (r) we have id_{M_1}, f, N_3 are as required on f_1, f_2, M_3 in our desired conclusion. $\square_{2.18}$

Claim 2.19. *1) In the local character Axiom (E)(c) of Definition 2.1 if $\mathcal{S}^{\text{bs}}_{\mathfrak{s}} = \mathcal{S}^{\text{na}}_{\mathfrak{k}_{\mathfrak{s}}}$ recalling $\mathcal{S}^{\text{na}}_{\mathfrak{k}_{\mathfrak{s}}}(M) = \{\text{ortp}(a, M, N) : M \leq_{\mathfrak{s}} N$ and $a \in N \setminus M\}$ <u>then</u> it suffices to restrict ourselves to the case that δ has cofinality $\aleph_0$ (i.e., the general case follows from this special case and the other axioms).*

2) In fact in part (1) we need only Axioms (E)(b),(h) and you may say (A),(D)(a),(E)(a).

3) If $\mathcal{S}^{\text{bs}} = \mathcal{S}^{\text{na}}$ <u>then</u> the continuity Axiom (E)(h) follows from the rest.

4) In (3) actually we need only Axioms (E)(c), (local character) (d), (transitivity) and you may say (A),(D)(a),(E)(a).

Proof. 1), 2) Let $\langle M_i : i \leq \delta + 1 \rangle$ be $\leq_{\mathfrak{k}_\lambda}$-increasing, $a \in M_{\delta+1} \setminus M_\delta$ and without loss of generality $\aleph_0 < \delta = \text{cf}(\delta)$, so for every $\alpha \in S := \{\alpha < \delta : \text{cf}(\alpha) = \aleph_0\}$, $\text{ortp}(a, M_\alpha, M_{\delta+1}) \in \mathcal{S}^{\text{bs}}(M_\alpha)$ by the assumption "$\mathcal{S}^{\text{bs}}_{\mathfrak{s}} = \mathcal{S}^{\text{na}}_{\mathfrak{k}_{\mathfrak{s}}}$ hence there is $\beta_\alpha < \alpha$ such that $\text{ortp}(a, M_\alpha, M_{\delta+1})$ does not fork over M_{β_α}, so for some $\beta < \delta$ the set $S_1 = \{\alpha \in S : \beta_\alpha = \beta\}$ is a stationary subset of δ. By Axiom $(E)(b)$ (monotonicity) it follows that for any $\gamma_1 \leq \gamma_2$ from $[\beta, \delta)$ the type $\text{ortp}(a, M_{\gamma_2}, M_{\delta+1}) \in \mathcal{S}^{\text{bs}}(M_{\gamma_2})$ does not fork over M_{γ_1}. Now for any $\gamma \in [\beta, \delta)$ the type $\text{ortp}(a, M_\delta, M_{\delta+1})$ does not fork over M_γ by applying $(E)(h)$ (continuity) to $\langle M_\alpha : \alpha \in [\gamma, \delta + 1]$ so we have finished.

3),4) So assume $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous, all in K_λ and δ is a limit ordinal, $p \in \mathcal{S}(M_\delta)$ and $p_i := p \restriction M_i \in \mathcal{S}^{\text{bs}}(M_i)$ does not fork over M_0 for each $i < \delta$; we should prove that $p \in \mathcal{S}^{\text{bs}}(M_\delta)$ and p does not fork over M_0.

First, for each $i < \delta, p_i \in \mathcal{S}^{\text{bs}}(M_i)$ hence p_i is not realized in M_i. As $M_\delta = \cup\{M_i : i < \delta\}$ clearly p is not realized in M_δ so $p \in \mathcal{S}^{\text{na}}(M_\delta) = \mathcal{S}^{\text{bs}}(M_\delta)$.

Second, by Ax(E)(c) the type p does not fork over M_j for some $j < \delta$. As $p_j = p \restriction M_j$ does not fork over M_0 (by assumption) by the transitivity Axiom $(E)(d)$, we get that p does not fork over M_0, as required. $\square_{2.19}$

[14] we could have chosen $N_3^* = N_2, f_2 = \text{id}_{f_1(M_2)}$

Remark 2.20. So in some sense by 2.19 we can omit in 2.1, the local character Axiom $(E)(c)$ or the continuity Axiom $(E)(h)$ but not both. In fact (under reasonable assumptions) they are equivalent.

Claim 2.21. *In Definition 2.1, Clause (E)(d), i.e., transitivity of non-forking follows from (A),(C),(D)(a),(b),(E)(a),(b),(e),(g).*

Proof. As $\mathfrak{k}_\lambda$ is an λ-AEC with amalgamation, types as well as restriction of types are not only well defined but are "reasonable".

So assume $M_0 \le_{\mathfrak{s}} M_0' \le_{\mathfrak{s}} M_0'' \le_{\mathfrak{s}} M_3, a \in M_3$ and $p'' := \mathrm{ortp}_{\mathfrak{s}}(a, M_0'', M_3)$ does not fork over M_0' and $p' := \mathrm{ortp}_{\mathfrak{s}}(a, M_0', M_3)$ does not fork over M_0. Let $p = p' \restriction M_0$. As p' does not fork over M_0, by Axiom $(E)(a)$ we have $p' \in \mathcal{S}^{\mathrm{bs}}(M_0')$ and $p = \mathrm{ortp}(a, M_0, M_3) = p' \restriction M_0$ belongs to $\mathcal{S}^{\mathrm{bs}}(M_0)$. As p'' does not fork over M_0' clearly $p'' \in \mathcal{S}^{\mathrm{bs}}(M_0'')$ and recall $p'' \restriction M_0' = p'$. By the existence axiom $(E)(g)$ the type p has an extension $q'' \in \mathcal{S}^{\mathrm{bs}}(M_0'')$ which does not fork over M_0. By the monotonicity Axiom (E)(b) the type q'' does not fork over M_0' and $q' = q'' \restriction M_0'$ does not fork over M_0. As $p', q' \in \mathcal{S}^{\mathrm{bs}}(M_0')$ do not fork over M_0 and $p' \restriction M_0 = p = q'' \restriction M_0 = q' \restriction M_0$, by the uniqueness Axiom Ax(E)(e), we have $p' = q'$. Similarly $p'' = q''$, but q'' does not fork over M_0 hence p'' does not fork over M_0 as required. $\square_{2.21}$

Claim 2.22. *1) The symmetry axiom (E)(f) is equivalent to (E)(f)$'$ below if we assume (A),(B),(C),(D)(a),(b),(E)(a),(b),(g) in Definition 2.1*

(E)(f)$'$ there are no $M_\ell (\ell \le 3)$ and $a_\ell (\ell \le 2)$ such that

- *(a)* $M_0 \le_{\mathfrak{s}} M_1 \le_{\mathfrak{s}} M_2 \le_{\mathfrak{s}} M_3$
- *(b)* $\mathrm{ortp}(a_\ell, M_\ell, M_{\ell+1})$ *does not fork over* M_0 *for* $\ell = 0, 1, 2$
- *(c)* $\mathrm{ortp}_{\mathfrak{s}}(a_0, M_0, M_1) = \mathrm{ortp}_{\mathfrak{s}}(a_2, M_0, M_3)$
- *(d)* $\mathrm{ortp}_{\mathfrak{s}}(\langle a_0, a_1 \rangle, M_0, M_1) \ne \mathrm{ortp}_{\mathfrak{s}}(\langle a_2, a_1 \rangle, M_0, M_1)$.

Proof. Easy. $\square_{2.22}$

* * *

A most interesting case of 2.16 is the following. In particular it tells us that the categoricity assumption is not so rare and it will have essential uses here.

Claim 2.23. *If $\mathfrak{s} = (\mathfrak{k}, \bigcup_\lambda, \mathcal{S}^{\mathrm{bs}})$ is a good λ-frame and $M \in K_\lambda$ is a superlimit model in $\mathfrak{k}_\lambda$ and we define $\mathfrak{s}' = \mathfrak{s}^{[M]} = \mathfrak{s}[M] = (\mathfrak{k}[\mathfrak{s}^{[M]}], \bigcup_\lambda[\mathfrak{s}^{[M]}], \mathcal{S}^{\mathrm{bs}}[\mathfrak{s}^{[M]}])$ by*

$$\mathfrak{k}[\mathfrak{s}^{[M]}] = \mathfrak{k}^{[M]}, \text{ see Definition 1.26 so } \mathfrak{k}_{\mathfrak{s}[M]} = \mathfrak{k} \restriction \{N : N \cong M\}$$

$$\bigcup_\lambda[\mathfrak{s}^{[M]}] = \{(M_0, M_1, a, M_3) \in \bigcup_\lambda : M_0, M_1, M_3 \in K_\lambda^{[M]}\}$$

$$\mathcal{S}^{\mathrm{bs}}[\mathfrak{s}^{[M]}] = \big\{\mathrm{ortp}_{\mathfrak{k}[M]}(a, M_0, M_1) : M_0 \le_{\mathfrak{k}} M_1, M_0 \in K_\lambda^{[M]}, N \in K_\lambda^{[M]} \text{ and } \mathrm{ortp}_{\mathfrak{k}}(a, M_0, M_1) \in \mathcal{S}^{\mathrm{bs}}(M_0)\big\}.$$

Then

(A) $\mathfrak{s}'$ is a good λ-frame

(B) $\mathfrak{k}[\mathfrak{s}'] \subseteq \mathfrak{k}_{\geq\lambda}[\mathfrak{s}]$

(C) $\leq_{\mathfrak{k}[\mathfrak{s}']} = \leq_{\mathfrak{k}} \restriction K[\mathfrak{s}']$

(D) $K_\lambda[\mathfrak{s}']$ *is categorical.*

Proof. Straight by 1.24, 1.27, 2.16. $\square_{2.23}$

---

§ 3. Examples

We show here that the context from §2 occurs in earlier investigation: in [She87a] = [She09a], [She01] that is [She09c], [She75] (and [She83a], [She83b]). Of course, also the class K of models of a superstable (first order) theory T (working in $\mathfrak{C}^{\rm eq}$), with $\le_{\mathfrak{k}} = \prec$ and $\mathcal{S}^{\rm bs}(M)$ being the set of regular types (when we work in $\mathfrak{C}^{\rm eq}$) or just "the set non-algebraic types" works, with $\bigcup(M_0, M_1, a, M_3)$ iff $M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_3$ are in $K_\lambda, a \in M_3$ and ${\rm ortp}(a, M_1, M_3) \in \mathcal{S}^{\rm bs}(M_1)$ does not fork over M_0, (in the sense of [She90, III], of course). The reader may concentrate on 3.10 (or 3.5) below for easy life.

Note that 3.5 (or 3.8) will be used to continue [She87a] = [She09a] and also to give an alternative proof to the theorem of [She83a], [She83b] + (deducing "there is a model in $\aleph_n$" if there are not too many models in $\aleph_\ell$ for $\ell < n$) and note that 3.8 will be used to continue [She75], i.e., on $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ and 3.10 will be used to continue [She01]. Many of the axioms from 2.1 are easy.

§ 3(A). The superstable prototype.

Claim 3.1. *Assume T is a first order complete theory and λ be a cardinal $\ge |T| + \aleph_0$; let $\mathfrak{k} = \mathfrak{k}_{T,\lambda} = (K_{T,\lambda} \le_{\mathfrak{k}_{T,\lambda}})$ be defined by:*

- *(a) $K_{T,\lambda}$ is the class of models of T of cardinality $\ge \lambda$*
- *(b) $\le_{\mathfrak{k}_{T,\lambda}}$ is "being an elementary submodel".*

0) $\mathfrak{k}$ is an AEC with ${\rm LS}(\mathfrak{k}) = \lambda$.

1) If T is superstable, stable in λ, then $\mathfrak{s} = \mathfrak{s}_{T,\lambda}$ is a good λ-frame when $\mathfrak{s} = (\mathfrak{k}_{T,\lambda}\mathcal{S}^{\rm bs}, \bigcup)$ is defined by:

- *(c) $p \in \mathcal{S}^{\rm bs}(M)$ iff $p = {\rm ortp}_{\mathfrak{k}_{t,\lambda}}(a, M, N)$ for some a, N such that ${\rm tp}_{\mathbb{L}(\tau_T)}(a, M, N)$, see Definition 3.3 is a non-algebraic complete 1-type over M, so $M \prec N, a \in N \setminus M$*
- *(d) $\bigcup(M_0, M_1, a, M_3)$ iff $M_0 \prec M_1 \prec M_3$ are in $K_{T,\lambda}$ and $a \in M_3$ and ${\rm tp}_{\mathbb{L}(\tau_T)}(a, M_1, M_3)$ is a type that does not fork over M_0 in the sense of [She90, III].*

2) Let $\kappa = {\rm cf}(\kappa) \le \lambda$. The model M is a (λ, κ)-brimmed model for $\mathfrak{k}_{T,\lambda}$ iff (i)+(ii) or (i)+(iii) where

- *(i) T is stable in λ*
- *(ii) $\kappa = {\rm cf}(\kappa) \ge \kappa(T)$ and M is a saturated model of T of cardinality λ*
- *(iii) $\kappa = {\rm cf}(\kappa) < \kappa(T)$ and there is a $\prec$-increasing continuous sequence $\langle M_i : i \le \kappa\rangle$ (by $\prec$, equivalently by $\le_{\mathfrak{s}}$) such that $M = M_\kappa$ and $(M_{i+1}, c)_{c \in M_i}$ is saturated for $i < \kappa$.*

2A) So there is a (λ, κ)-brimmed model for $\mathfrak{k}_{T,\lambda}$ iff T is stable in λ.

3) M is (λ, κ)-brimmed over M_0 in $\mathfrak{k}_{T,\lambda}$ iff $(M, c)_{c \in M_0}$ is (λ, κ)-brimmed.

4) Assume T is superstable first order complete theory stable in λ and we define $\mathfrak{s}^{reg}_{T,\lambda}$ as above only $\mathcal{S}^{\rm bs}(M)$ is the set of regular types $p \in \mathcal{S}_{\mathfrak{k}_T}(M)$ and we work in $T^{\rm eq}$. Then $\mathfrak{s}^{reg}_{T,\lambda}$ is a good λ-frame.

5) For $\kappa \le \lambda$ or $\kappa = \aleph_\varepsilon$ (abusing notation), $\mathfrak{s}^\kappa_{T,\lambda}$ is defined similarly restricting ourselves to $\mathbf{F}^a_\kappa$-saturated models. (Let $\mathfrak{s}^0_{t,\lambda} = \mathfrak{s}_{T,\lambda}$.) If T is superstable, stable in λ then $\mathfrak{s}^\kappa_{T,\lambda}$ is a good λ frame.

Remark 3.2. We can replace (c) of 3.1 by:

(c)′ $p \in \mathcal{S}^{\rm bs}(M)$ iff $p = \text{ortp}_{\mathfrak{k}_{T,\lambda}}(a, M, N)$ for some a, N such that $\text{tp}_{\mathbb{L}(\tau_T)}(a, M, N)$ is a complete 1-type over M

except that clause (D)(b) of Definition 2.1 fail. In fact the proofs are easier in this case; of course, the two meaning of types essentially agree.

Proof. 0),1),2),2A),3) Obvious (see [She90]).
4) As in (1), except density of regular types which holds by [HS89].
5) Also by [She90]. $\square_{3.1}$

Recall

Definition 3.3. 1) For a logic $\mathscr{L}$ and vocabulary $\tau, \mathscr{L}(\tau)$ is the set of $\mathscr{L}$-formulas in this vocabulary.
2) $\mathbb{L} = \mathbb{L}_{\omega,\omega}$ is first order logic.
3) A theory in $\mathscr{L}(\tau)$ is a set of sentences from $\mathscr{L}(\tau)$ which we assume has a model if not said otherwise. Similarly in a language $L(\subseteq \mathscr{L}(\tau))$

Very central in [She09a] (and [She09b]) but peripheral here (except when in (parts of) §3 we continue [She09a] in our framework) is:

Definition 3.4. Let T_1 be a theory in $\mathbb{L}(\tau_1), \tau \subseteq \tau_1$ vocabularies, Γ a set of types in $\mathbb{L}(\tau_1)$; (i.e. for some m, a set of formulas $\varphi(x_0, \dots, x_{m-1}) \in \mathbb{L}(\tau_1)$).
1) $\text{EC}(T_1, \Gamma) = \{M : M$ a τ_1-model of T_1 which omits every $p \in \Gamma\}$.
(So without loss of generality τ_1 is reconstructible from T_1, Γ) and

$$\text{PC}_\tau(T_1, \Gamma) = \text{PC}(T_1, \Gamma, \tau) = \{M : M \text{ is a } \tau\text{-reduct of some } M_1 \in \text{EC}(T_1, \Gamma)\}.$$

2) We say that $\mathfrak{k}$ is PC^μ_λ or $\text{PC}_{\lambda,\mu}$ <u>if</u> for some $T_1, T_2, \Gamma_1, \Gamma_2$ and τ_1 and τ_2 we have: (T_ℓ a first order theory in the vocabulary τ_ℓ, Γ_ℓ a set of types in $\mathbb{L}(\tau_\ell)$ and) $K = \text{PC}(T_1, \Gamma_1, \tau_{\mathfrak{k}})$ and $\{(M, N) : M \leq_{\mathfrak{k}} N$ and $M, N \in K\} = \text{PC}(T_2, \Gamma_2, \tau')$ where $\tau' = \tau_{\mathfrak{k}} \cup \{P\}$, ($P$ a new one place predicate and (M, N) means the τ'-model N^+ expanding N where $P^{N^+} = |M|$) and $|T_\ell| \leq \lambda, |\Gamma_\ell| \leq \mu$ for $\ell = 1, 2$.
3) If $\mu = \lambda$, we may omit μ.

§ 3(B). An abstract elementary class which is $\text{PC}_{\aleph_0}$.

theorem 3.5. *Assume $2^{\aleph_0} < 2^{\aleph_1}$ and consider the statements*

(α) *$\mathfrak{k}$ is an abstract elementary class with $\text{LS}(\mathfrak{k}) = \aleph_0$ (the last phrase follows by clause (β)) and $\tau = \tau(\mathfrak{k})$ is countable*

(β) *$\mathfrak{k}$ is $\text{PC}_{\aleph_0}$, equivalently for some sentences $\psi_1, \psi_2 \in \mathbb{L}_{\omega_1,\omega}(\tau_1)$ where τ_1 is a countable vocabulary extending τ we have*

$$K = \{M_1 \restriction \tau : M_1 \text{ a model of } \psi_1\}$$
$$\{(N, M) : M \leq_{\mathfrak{k}} N\} = \{(N_1 \restriction \tau, M_1 \restriction \tau) : (N_1, M_1) \text{ a model of } \psi_2\}$$

(γ) $1 \leq \dot{I}(\aleph_1, \mathfrak{k}) < 2^{\aleph_1}$

(δ) *$\mathfrak{k}$ is categorical in $\aleph_0$, has the amalgamation property in $\aleph_0$ and is stable in $\aleph_0$*

(δ)$^-$ *like (δ) but "stable in $\aleph_0$" is weakened to: $M \in \mathfrak{k}_{\aleph_0} \Rightarrow |\mathcal{S}(M)| \leq \aleph_1$*

(ε) *all models of $\mathfrak{k}$ are $\mathbb{L}_{\infty,\omega}$-equivalent and $M \leq_{\mathfrak{k}} N \Rightarrow M \prec_{\mathbb{L}_{\infty,\omega}} N$.*

For $M \in \mathfrak{k}_{\aleph_0}$ we define $\mathfrak{k}'_M$ as follows: the class of members is $\{N \in K : N \equiv_{\mathbb{L}_{\infty,\omega}} M\}$ *and* $N_1 \le_{\mathfrak{k}'_M} N_2$ *iff* $N_1 \le_{\mathfrak{k}} N_2$ *and* $N_1 \prec_{\mathbb{L}_{\infty,\omega}} N_2$.

1) Assume $(\alpha)+(\beta)+(\gamma)$, *then for some* $M \in \mathfrak{k}_{\aleph_0}$ *the class* $\mathfrak{k}'_M$ *satisfies* $(\alpha)+(\beta)+(\gamma)+(\delta)^-+(\varepsilon)$*; in fact any* $M \in \mathfrak{k}_{\aleph_0}$ *such that* $(\mathfrak{k}'_M)_{\aleph_1} \neq \varnothing$ *will do and there are such* $M \in K_{\aleph_0}$*. Moreover, if* $\mathfrak{k}$ *satisfies* (δ) *then also* $\mathfrak{k}'_M$ *satisfies it; also trivially* $K'_M \subseteq K$ *and* $\le_{\mathfrak{k}'_M} \subseteq \le_{\mathfrak{k}}$.

1A) Also there is $\mathfrak{k}'$ *such that:* $\mathfrak{k}'$ *satisfies* $(\alpha)+(\beta)+(\gamma)+(\delta)+(\varepsilon)$*, and for every* μ *we have* $K'_\mu \subseteq K_\mu$*. In fact, in the notation of* [She09a, 88r-5.6] *for every* $\alpha < \omega_1$ *we can choose* $\mathfrak{k}' = \mathfrak{k}_{\mathbf{D}_\alpha}$.

2) Assume $(\alpha)+(\beta)+(\gamma)+(\delta)$*. Then* $(\mathfrak{k}, \bigcup, \mathcal{S}^{\rm bs})$ *is a good* $\aleph_0$*-frame for some* $\bigcup$ *and* $\mathcal{S}^{\rm bs}$.

3) In fact, in part (2) we can choose $\mathcal{S}^{\rm bs}(M) = \{p \in \mathcal{S}(M) : p \text{ not algebraic}\}$ *and* $\bigcup$ *is defined by* [She09a, 88r-5.11] *(the definable extensions).*

Remark 3.6. 1) In [She09a, 88r-5.23] we use the additional assumption $\dot I(\aleph_2, K) < \mu_{\rm unif}(\aleph_2, 2^{\aleph_1})$. But this Theorem is not used here!

2) Note that $\mathfrak{k}'_M$ is related to $K^{[M]}$ from Definition 1.26 but is different.

3) In the proof we relate the types in the sense of $\mathcal{S}_{\mathfrak{s}}(M)$, and those in [She09a, §5]. Now in [She09a, §5] we have lift types, from $\mathfrak{k}_{\aleph_0}$ to any $\mathfrak{k}_\mu$, i.e., define $\mathbf{D}(N)$ for $N \in \mathfrak{k}_\mu$. In $\mu > \aleph_0$, in general we do not know how to relate them to types $\mathcal{S}_{\mathfrak{k}_{\mathfrak{s}}}(N)$. But when $\mathfrak{s}^+$ is defined (in the "successful" cases, see §8 here and [She09e, §1]) we can get the parallel claim.

Discussion 3.7. 1) What occurs if we do not pass in 3.5 to the case "$\mathbf{D}(N)$ countable for every $N \in K_{\aleph_0}$"? If we still assume "$\mathfrak{k}$ categorical in $\aleph_0$" then as $|\mathbf{D}(N_0)| \le \aleph_1$, if we assume "there is a superlimit model in $\mathfrak{k}_{\aleph_1}$" we can find a good $\aleph_1$-frame $\mathfrak{s}$; this assumption is justified by [She09a, 88r-5.23], [She09a, 88r-5.24].

Proof. 1) Note that for any $M \in K_{\aleph_0}$, the class $\mathfrak{k}'_M$ satisfies $(\alpha), (\beta), (\varepsilon)$ and it is categorical in $\aleph_0$ and $(K'_M)_\mu \subseteq K_\mu$ hence $\dot I(\mu, K'_M) \le \dot I(\mu, K)$. By Theorem [She09a, 88r-3.6], (note: if you use the original version (i.e., [She87a]) by its proof or use it and get a less specified class with the desired properties) for some $M \in K_{\aleph_0}$ we have $(\mathfrak{k}'_M)_{\aleph_1} \neq \varnothing$. By [She09a, 88r-3.5] we get that $\mathfrak{k}'_M$ has amalgamation in $\aleph_0$ and by [She09a] almost we get that in $\mathfrak{k}'_M$ the set $\mathcal{S}(M)$ is of small cardinality ($\le \aleph_1$); be careful - the types there are defined differently than here, but by the amalgamation (in $\aleph_0$) and the omitting types theorem in this case they are the same, see more in the proof of part (3) below. So by [She09a, 88r-5.1], [She09a, 88r-5.2] we have $M \in (\mathfrak{k}'_\mu)_{\aleph_0} \Rightarrow |\mathcal{S}_{\mathfrak{k}'_\mu}(M)| \le \aleph_1$.

Also the second sentence in (1) is easy.

1A) Use [She09a, 88r-5.18], [She09a, 88r-5.19].

In more detail, (but not much point in reading without some understanding of [She09a, §5], however we should not use [She09a, 88r-5.23] as long as we do not strengthen our assumptions) by part (1) we can assume that clauses $(\delta)^- + (\varepsilon)$ hold. (Looking at the old version [She87a] of [She09a] remember that there $\prec$ means $\le_{\mathfrak{k}}$.) We can find $\mathbf{D}_* = \mathbf{D}^*_\alpha, \alpha < \omega_1$, which is a good countable diagram (see Definition [She09a, 88r-5.6.1] and Fact [She09a, 88r-5.6] or [She09a, 88r-5.16], [She09a, 88r-5.17]. So in particular (give the non-maximality of models below) such that for some countable $M_0 <_{\mathfrak{k}} M_1 <_{\mathfrak{k}} M_2$ we have M_m is $(\mathbf{D}^*(M_\ell), \aleph_0)$-homogeneous for $\ell < m \le 2$. In [She09a, 88r-5.18] we define $(K_{\mathbf{D}_*}, \le_{\mathbf{D}_*})$. By [She09a, 88r-5.19] the pair $(K_{\mathbf{D}_*}, \le_{\mathbf{D}_*})$ is an abstract elementary class (the choice of $\mathbf{D}_*$ a part, e.g. transitivity = Axiom II which holds by the existence of the M_ℓ's above and [She09a, 88r-5.16]) categorical in $\aleph_0$ and no maximal countable model (by $\le_{\mathbf{D}_*}$, see [She09a, 88r-5.6](2). Now $\aleph_0$-stability holds by [She09a, 88r-5.19](2) and the equality of the three definitions of types in the proof of parts (2),(3) and $K_{\mathbf{D}_*} \subseteq K$ so we are done by part 3) below.

2),3) The first part of the proof serves also part (1) of the theorem so we assume $(\delta)^-$ instead of (δ). We should be careful: the notion of type has three relevant meanings here. For $N \in K_{\aleph_0}$ the three definitions for $\mathcal{S}^{<\omega}(N)$ and of $\text{tp}(\bar a, N, M)$ when $\bar a \in {}^{\omega>}M, N \le_{\mathfrak{k}} M \in K_{\aleph_0}$ (of course we can use just 1-types) are:

(α) the one we use here (recall 1.9) which uses elementary mappings; for the present proof we call them $\mathcal{S}_0^{<\omega}(M), \text{tp}_0(\bar a, M, N)$

(β) $\mathbf{S}_1(N)$ which is (recall that materialize is close to but different from realize)

$$\begin{aligned}\mathbf{D}(N) = \big\{p : \ & p \text{ a complete } \mathbb{L}^0_{\aleph_1,\aleph_0}(N)\text{-type over } N \\ & \text{(so in each formula only finitely many parameters from } N \text{ appear)} \\ & \text{such that for some } M, \bar a \in {}^{\omega>}M, \bar a \text{ materializes } p \text{ in } (M,N)\big\}\end{aligned}$$

("materializing a type" is defined in [She09a, 88r-4.2](2)) so

$$\mathbf{S}_1(N) = \{\text{tp}_1(\bar a, N, M) : \bar a \in {}^{\omega>}M \text{ and } N \le_{\mathfrak{k}} M \in K_{\aleph_0}\}$$

where

$$\text{tp}_1(\bar a, N, M) = \{\varphi(\bar x) \in \mathbb{L}^0_{\aleph_1,\aleph_0}(N) : M \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi(\bar a)\}$$

(see [She09a, 88r-4.2](1) on the meaning of this forcing relation).

(γ) $\mathbf{S}_2(N)$ which is

$$\begin{aligned}\mathbf{D}^*(N) = \big\{p : \ & p \text{ a complete } \mathbb{L}^0_{\aleph_1,\aleph_0}(N;N)\text{-type over } N \\ & \text{(so in each formula all members of } N \text{ may appear)} \\ & \text{such that for some } M \in K_{\aleph_0} \text{ and} \\ & \bar a \in {}^{\omega>}M \text{ satisfying } N \le_{\mathfrak{k}} M \text{ the sequence} \\ & \bar a \text{ materializes } p \text{ in } (M,N)\big\}\end{aligned}$$

so

$$\mathbf{S}_2(N) = \{\text{tp}_2(\bar a, N, M) : \bar a \in {}^{\omega>}M \text{ and } N \le_{\mathfrak{k}} M \in K_{\aleph_0}\}$$

$$\text{tp}_2(\bar a, N, M) = \{\varphi(\bar x) \in \mathbb{L}^0_{\aleph_1,\aleph_0}(N,N) : M \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi(\bar a)\}.$$

As we have amalgamation in $K_{\aleph_0}$ it is enough to prove for $\ell, m < 3$ that

$(*)_{\ell,m}$ if $k < \omega, N \le_{\mathfrak{k}} M \in K_{\aleph_0}$ and $\bar a, \bar b \in {}^kM$, then
$\text{tp}_\ell(\bar a, N, M) = \text{tp}_\ell(\bar b, N, M) \Rightarrow \text{tp}_m(\bar a, N, M) = \text{tp}_m(\bar b, N, M)$.

Now $(*)_{2,1}$ holds trivially (more formulas) and $(*)_{1,2}$ holds by [She09a, 88r-5.5]. By amalgamation in $\mathfrak{k}_{\aleph_0}$, if $\text{tp}_0(\bar a, N, M) = \text{tp}_0(\bar b, N, M)$, then for some $M', M \le_{\mathfrak{k}} M' \in K_{\aleph_0}$ there is an automorphism f of M' over N such that $f(\bar a) = \bar b$, so trivially $(*)_{0,1}, (*)_{0,2}$ hold (we use the facts that $\text{tp}_\ell(\bar a, N, M)$ is preserved by isomorphism and by replacing M by M_1 if $M \le_{\mathfrak{k}} M_2 \in K_{\aleph_0}$ and $N \cup \bar a \subseteq M_1 \le_{\mathfrak{k}} M_2$). Lastly we prove $(*)_{2,0}$.

So $N \le_{\mathfrak{k}} M \in K_{\aleph_0}$, hence $\text{tp}_2(\bar c, N, M) : \bar c \in {}^{\omega>}M\} \subseteq \mathbf{D}^*(N)$ is countable so by [She09a, 88r-5.6](b),(c) for some countable $\alpha < \omega_1$ we have $\{\text{tp}_2(\bar c, N, M) : \bar c \in {}^{\omega>}M\} \subseteq \mathbf{D}^*_\alpha(N)$. Now there is $M' \in K_{\aleph_0}$ such that $M \le_{\mathfrak{k}} M', M'$ is $(\mathbf{D}^*_\alpha, \aleph_0)^*$-homogeneous (by [She09a, 88r-5.6](e) see Definition [She09a, 88r-5.7]) hence M' is

$(\mathbf{D}^*_\alpha(N), \aleph_0)^*$- homogeneous (by [She09a, 88r-5.6](f)), and $\text{tp}_2(\bar{a}, N, M') = \text{tp}_2(\bar{b}, N, M')$ by [She09a, 88r-5.4.1](3), (N here means N_0 there, that is increasing the model preserve the type).

Lastly by Definition [She09a, 88r-5.7] there is an automorphism f of M' over N mapping $\bar{a}$ to $\bar{b}$, so we have proved $(*)_{2,0}$, so the three definitions of type are equivalent.

Now we define for $M \in K_{\aleph_0}$:

(a) $\mathcal{S}^{\text{bs}}(M) = \{p \in \mathcal{S}_{\mathfrak{k}}(M) : p \text{ not algebraic}\}$

(b) for $M_0, M_1, M_3 \in K_{\aleph_0}$ and an element $a \in M_3$ we define:
$\bigcup(M_0, M_1, a, M_3)$ iff $M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_3$ and $a \in M_3 \setminus M_1$ and

$\text{tp}_1(a, M_1, M_3)(= \text{gtp}(a, M_1, M_3)$ in [She09a]'s notation)
is definable over some finite $\bar{b} \in {}^{\omega>}M_0$ (equivalently is preserved by every automorphism of M_1 over $\bar{b}$ (see [She09a, 88r-5.11])
equivalently $\text{gtp}(a, M_1, M_3)$ is the stationarization of $\text{gtp}(a, M_0, M_3)$.

Now we should check the axioms from Definition 2.1.

Clause (A): By clause (α) of the assumption.

Clauses (B),(C): By clause (δ) or $(\delta)^-$ of the assumption except "the superlimit $M \in K_{\aleph_0}$ is not $\le_{\mathfrak{k}}$-maximal" which holds by clause $(\gamma) + (\delta)$ or $(\gamma) + (\delta)^-$.

Clause (D): By the definition (note that about clause (d), bs-stability, that it holds by assumption (δ), and about clause (c), i.e., the density is trivial by the way we have defined $\mathcal{S}^{\text{bs}}$).

Subclause (E)(a): By the definition.

Subclause (E)(b)(monotonicity):

Let $M_0 \le_{\mathfrak{k}} M'_0 \le_{\mathfrak{k}} M'_1 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_3 \le M'_3$ be all in $\mathfrak{k}_{\aleph_0}$ and assume $\bigcup(M_0, M_1, a, M_3)$.

So $M'_0 \le_{\mathfrak{k}} M'_1 \le_{\mathfrak{k}} M_3 \le M'_3$ and $a \in M_3 \setminus M_1 \subseteq M'_3 \setminus M'_1$. Now by the assumption and the definition of $\bigcup$, for some $\bar{b} \in {}^{\omega>}(M_0)$, $\text{gtp}(a, M_1, M_3)$ is definable over $\bar{b}$. So the same holds for $\text{gtp}(a, M'_1, M_3)$ by [She09a, 88r-5.13], in fact (with the same definition) and hence for $\text{gtp}(a, M'_1, M'_3) = \text{gtp}(a, M'_1, M_3)$ by [She09a, 88r-5.4.1](3), so as $\bar{b} \in {}^{\omega>}(M_0) \subseteq {}^{\omega>}(M'_0)$ we have gotten $\bigcup(M'_0, M'_1, a, M'_3)$.

For the additional clause in the monotoncity Axiom, assume in addition $M'_1 \cup \{a\} \subseteq M''_3 \le_{\mathfrak{k}} M'_3$ again by [She09a, 88r-5.4.1](3) clearly $\text{gtp}(a, M'_1, M''_3) = \text{gtp}(a, M'_1, M'_3)$, so (recalling the beginning of the proof) we are done.

Sublcause (E)(c)(local character):

So let $\langle M_i : i \le \delta + 1\rangle$ be $\le_{\mathfrak{k}}$-increasing continuous in $K_{\aleph_0}$ and $a \in M_{\delta+1}$ and $\text{ortp}(a, M_\delta, M_{\delta+1}) \in \mathcal{S}^{\text{bs}}(M_\delta)$, so $a \notin M_\delta$ and $\text{gtp}(a, M_\delta, M_{\delta+1})$ is definable over some $\bar{b} \in {}^{\omega>}(M_\delta)$ by [She09a, 88r-5.4].

As $\bar{b}$ is finite, for some $\alpha < \delta$ we have $\bar{b} \subseteq M_\alpha$, hence we have ($\text{ortp}(a, M_\beta, M_{\delta+1}) \in \mathcal{S}^{\text{bs}}(M_\beta)$ trivially and) $\text{ortp}(a, M_\delta, M_{\delta+1})$ does not fork over M_β.

Sublcause (E)(d)(transitivity):

By [She09a, 88r-5.13](2) or even better [She09a, 88r-5.16].

Subclause (E)(e)(uniqueness):

Holds by the Definition [She09a, 88r-5.11].

Subclause (E)(f)(symmetry):

By [She09a, 88r-5.20] + uniqueness we get (E)(f). Actually [She09a, 88r-5.20] gives this more directly.

Subclause (E)(g)(extension existence):

By [She09a, 88r-5.11] (i.e., by [She09a, 88r-5.4] + all $M \in K_{\aleph_0}$ are $\aleph_0$-homogeneous).
Alternatively, see [She09a, 88r-5.15].

Subclause (E)(h)(continuity):

Suppose $\langle M_\alpha : \alpha \leq \delta\rangle$ is $\leq_{\mathfrak{k}}$- increasingly continuous, $M_\alpha \in K_{\aleph_0}, \delta < \omega_1, p \in \mathcal{S}(M_\delta)$ and $\alpha < \delta \Rightarrow p \restriction M_\alpha$ does not fork over M_0. Now we shall use (E)(c)+(E)(d). As $p \restriction M_\alpha \in \mathcal{S}^{\rm bs}(M_\alpha)$ clearly $p \restriction M_\alpha$ is not realized in M_α hence p is not realized in M_α; as $M_\delta = \bigcup\limits_{\alpha<\delta} M_\alpha$ necessarily p is not realized in M_δ, hence p is not algebraic.

So $p \in \mathcal{S}^{\rm bs}(M_\delta)$. For some finite $\bar b \in {}^{\omega>}(M_\delta), p$ is definable over $\bar b$, let $\alpha < \delta$ be such that $\bar b \in {}^{\omega>}(M_\alpha)$, so as in the proof of (E)(c), (or use it directly) the type p does not fork over M_α. As $p \restriction M_\alpha$ does not fork over M_0, by (E)(d) we get that p does not fork over M_0 as required. Actually we can derive (E)(h) by 2.19.

Subclause (E)(i)(non-forking amalgamation):

One way is by [She09a, 88r-5.20]; (note that in [She09a, 88r-5.23] we get more, but assuming, by our present notation $\dot I(\aleph_2, K) < \mu_{\rm wd}(\aleph_2)$); but another way is just to use 2.18. $\square_{3.5}$

§ 3(C). **The uncountable cardinality quantifier case, $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$.** Now we turn to sentences in $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$.

Conclusion 3.8. *Assume $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ and $1 \leq \dot I(\aleph_1, \psi) < 2^{\aleph_1}$ and $2^{\aleph_0} < 2^{\aleph_1}$.*
Then for some abstract elementary classes $\mathfrak{k}, \mathfrak{k}^+$ (note $\tau_\psi \subset \tau_{\mathfrak{k}} = \tau_{\mathfrak{k}^+}$) we have:

- (a) *$\mathfrak{k}$ satisfies $(\alpha), (\beta), (\delta), (\varepsilon)$ from 3.5 with $\tau_{\mathfrak{k}} \supseteq \tau_\psi$ countable (for $(\gamma), (b)$ is a replacement)*
- (b) *for every $\mu > \aleph_0, \dot I(\mu, \mathfrak{k}(\aleph_1\text{-saturated})) \leq \dot I(\mu, \psi)$, where [15] "$\aleph_1$-saturated" is well defined as $\mathfrak{k}_{\aleph_0}$ has amalgamation, see 1.15*
- (c) *for some $\bigcup, \mathcal{S}^{\rm bs}$ (and $\lambda = \aleph_0$), the triple $(\mathfrak{k}, \bigcup, S^{\rm bs})$ is as in 3.5(2) so is a good $\aleph_0$-frame*
- (d) *every $\aleph_1$-saturated member of $\mathfrak{k}$ belongs to $\mathfrak{k}^+$ and there is an $\aleph_1$-saturated member of $\mathfrak{k}$ (and naturally it is uncountable, even of cardinality $\aleph_1$)*
- (e) *$\mathfrak{k}^+$ is an AEC, has LS number $\aleph_1$ and $\{M \restriction \tau_\psi : M \in \mathfrak{k}^+\} \subseteq \{M : M \models \psi\}$ and every τ-model M of ψ has a unique expansion in $\mathfrak{k}^+$ hence $\mu \geq \aleph_1 \Rightarrow \dot I(\mu, \psi) = \dot I(\mu, \mathfrak{k}^+)$ and $\mathfrak{k}^+$ is the class of models of some complete $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$.*

Proof. Essentially by [She75] and 3.5.

I feel that upon reading [She75] the proof should not be inherently difficult, much more so having read 3.5, but will give full details.

Recall Mod(ψ) is the class of τ_ψ-models of ψ. We can find a countable fragment $\mathscr{L}$ of $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau_\psi)$ to which ψ belongs and a sentence $\psi_1 \in \mathscr{L} \subseteq \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau_\psi)$ such that ψ_1 is "nice" for [She75, Definition 3.1,3.2], [She75, Lemma 3.1]

$\circledast_1$ (a) ψ_1 has uncountable models

(b) $\psi_1 \vdash \psi$, i.e., every model of ψ_1 is a model of ψ

(c) ψ_1 is $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$-complete

(d) every model $M \models \psi_1$ realizes just countably many complete $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau_\psi)$-types (of any finite arity, over the empty set), each isolated by a formula in $\mathscr{L}$.

The proof of $\circledast_1(d)$ is sketched in Theorem 2.5 of [She75]. The reference to Keisler [Kei71] is to the generalization of theorems 12 and 28 of Keisler's book from $\mathbb{L}_{\omega_1,\omega}$ to $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$, see [She09a, 88r-0.1].

Let

[15] much less than saturation suffice, like "obeying" $<^{**}$

$\circledast_2$ (i) $\mathfrak{k}_0 = (\text{Mod}(\psi), \prec_{\mathscr{L}})$,

(ii) $\mathfrak{k}_1 = (\text{Mod}(\psi_1), \prec_{\mathscr{L}})$

$\circledast_3$ $\mathfrak{k}_\ell$ is an AEC with L.S. number $\aleph_1$ for $\ell = 0, 1$.

Toward defining $\mathfrak{k}$, let $\tau_{\mathfrak{k}} = \tau_\psi \cup \{R_{\varphi(\bar{x})} : \varphi(\bar{x}) \in \mathscr{L}\}, R_{\varphi(\bar{x})}$ a new $\ell g(\bar{x})$-predicate and let $\psi_2 = \psi_1 \wedge \{(\forall \bar{y})(R_{\varphi(\bar{x})}(\bar{y}) = \varphi(\bar{y}) : \varphi(\bar{x}) \in \mathbb{L}\}$. For every $M \in \text{Mod}(\psi)$ we define M^+ by

$\circledast_4$ M^+ is M expanded to a $\tau_{\mathfrak{k}}$-model by letting $R^{M^+}_{\varphi(\bar{x})} = \{\bar{a} \in {}^{\ell g(\bar{x})}M : M \models \varphi[\bar{a}]\}$

$\circledast_5$ (a) $\mathfrak{k}^+_0 = (\{M^+ : M \in \text{Mod}(\psi)\}, \prec_{\mathbb{L}})$ is an AEC with $\text{LS}(\mathfrak{k}^+_0) = \aleph_1$

(b) $\mathfrak{k}^+_1 = (\{M^+ : M \in \text{Mod}(\psi_1)\}, \prec_{\mathbb{L}})$ is an AEC with $\text{LS}(\mathfrak{k}^+) = \aleph_1$.

Clearly

$\circledast_6$ if $M \models \psi_1$ then M^+ is an atomic model of the complete first-order theory T_{ψ_1} where T_{ψ_1} is the set of first order consequences in $\mathbb{L}(\tau_{\mathfrak{k}})$ of ψ_2.

So it is natural to define $\mathfrak{k}$:

$\circledast_7$ (a) $N \in \mathfrak{k}$ iff

(α) N is a $\tau_{\mathfrak{k}}$-model which is an atomic model of T_{ψ_1}

(β) if $\psi_1 \vdash (\forall \bar{x})[\varphi_1(\bar{x}) = (\mathbf{Q}y)\varphi_2(y, \bar{x})]$ and $\varphi_1, \varphi_2 \in \mathscr{L}$ and $N \models \neg R_{\varphi_1(\bar{x})}[\bar{a}]$ then $\{b \in N : N \models R_{\varphi_2(y,\bar{x})}(b, \bar{a})\}$ is countable

(b) $N_1 \le_{\mathfrak{k}} N_2$ <u>iff</u> $(N_1, N_2 \in K, N_1 \prec_{\mathbb{L}} N_2$ equivalently $N_1 \subseteq N_2$ and) for $\varphi_1(\bar{x}), \varphi_2(y, \bar{x})$ as in subclause (β) of clause (a) above, if $\bar{a} \in {}^{\ell g(\bar{x})}(N_1)$,
$N_1 \models \neg R_{\varphi_1(\bar{x})}[\bar{a}]$ and $b \in N_2 \setminus N_1$ then $N_2 \models \neg R_{\varphi_2(y,\bar{x})}[b, \bar{a}]$.

Observe

$\circledast_8$ $N \in \mathfrak{k}$ iff N is an atomic $\tau_{\mathfrak{k}}$-model of the first order $\mathbb{L}(\tau_{\mathfrak{k}})$-consequences ψ_2 (i.e. of ψ and every $\tau_{\mathfrak{k}}$ sentence of the form $\forall \bar{x}[R_\varphi(\bar{x}) \equiv \varphi(\bar{x})]$) and clause (β) of $\circledast_7(a)$ holds

$\circledast_9$ $\mathfrak{k}$ is an AEC with $\text{LS}(\mathfrak{k}) = \aleph_0$ and is $\text{PC}_{\aleph_0}, \mathfrak{k}$ is categorical in $\aleph_0$ (and $\le_{\mathfrak{k}}$ is called $\le^*$ in [She75, Definition 3.3]).

Note that $\mathfrak{k}_1, \mathfrak{k}^+_1$ has the same number of models, but $\mathfrak{k}$ has "more models" than $\mathfrak{k}^+_1$, in particular, it has countable members and $\mathfrak{k}_0$ has at least as many models as $\mathfrak{k}_1$. For $N \in \mathfrak{k}$ to be in $\mathfrak{k}^+_1 = \{M^+ : M \in \text{Mod}(\psi_1)\}$ what is missing is the other implications in $\circledast_7(a)(\beta)$.

This is very close to 3.5, but $\mathfrak{k}$ may have many models in $\aleph_1$ (as $\mathbf{Q}$ is not necessarily interpreted as expected). However,

$\circledast_{10}$ constructing $M \in K_{\aleph_1}$ by the union as $\le_{\mathfrak{k}}$-increasing continuous chain $\langle M_i : i < \omega_1 \rangle$, to make sure $M \in \mathfrak{k}^+_1$ it is enough that for unboundedly many $\alpha < \omega_1, M_\alpha <^{**} M_{\alpha+1}$ and $(\forall M \in \mathfrak{k}_{\aleph_0})(\exists N \in \mathfrak{k}_{\aleph_0})(M <^{**} N)$ where

$\circledast_{11}$ for $M, N \in \mathfrak{k}, M <^{**} N$ iff

(i) $M \le_{\mathfrak{k}} N$

(ii) in $\circledast_7(b)$ also the inverse direction holds.

Does $\mathfrak{k}$ have amalgamation in $\aleph_0$? Now [She75, Lemma 3.4], almost says this but it assumed $\diamondsuit_{\aleph_1}$ instead of $2^{\aleph_0} < 2^{\aleph_1}$; and [She09a, 88r-3.5] almost says this, but the models are from $\mathfrak{k}_{\aleph_1}$ rather than $\mathfrak{k}^+_{\aleph_1}$ but [She09a, 88r-3.8.4] fully says this using the so called $K^{\mathbf{F}}_{\aleph_1}$, see Definition [She09a, 88r-3.8.1] and using $\mathbf{F}$ such that $M \in \mathfrak{k}_{\aleph_0} \Rightarrow M <^{**} \mathbf{F}(N) \in \mathfrak{k}_{\aleph_0}$; or pedantically $\mathbf{F} = \{(M, N) : M <^{**} N$ are from $\mathfrak{k}\}$. So

$\circledast_{12}$ $\mathfrak{k}$ has the amalgamation property in $\aleph_0$.

It should be clear by now that we have proved clauses (a),(b),(d),(e) of 3.8 using $\mathfrak{k}$. We have to prove clause (c); we cannot quote 3.5 as clause (γ) there is only almost true. The proof is similar to (but simpler than) that of 3.5 quoting [She75] instead of [She09a]; a marked difference is that in the present case the number of types over a countable model is countable (in $\mathfrak{k}$) whereas in [She09a] it seemingly could be $\aleph_1$, generally [She75] situation is more similar to the first order logic case.

Recall that all models from $\mathfrak{k}$ are atomic (in the first order sense) and we shall use below $\mathrm{tp}_{\mathbb{L}}$.

As $\mathfrak{k}$ has $\aleph_0$-amalgamation (by $\circledast_{12}$), clearly [She75, §4] applies; now by [She75, Lemma 2.1](B) + Definition 3.5, being $(\aleph_0, 1)$-stable as defined in [She75, Definition 3.5](A) holds. Hence all clauses of [She75, Lemma 4.2] hold, in particular $((D)(\beta)$ there and clause (A), i.e., [She75, Def.3.5](B)), so

$\circledast_{13}$ (i) if $M \le_{\mathfrak{k}} N$ and $\bar{a} \in N$ then $\mathrm{tp}_{\mathbb{L}}(\bar{a}, M, N)$ is definable over a finite subset of M

(ii) if $M \in \mathfrak{k}_{\aleph_0}$ then $\{\mathrm{tp}_{\mathbb{L}}(\bar{a}, M, N) : \bar{a} \in {}^{\omega>}N$ and $M \le_{\mathfrak{k}} N\}$ is countable.

By [She75, Lemma 4.4] it follows that

$\circledast_{14}$ if $M \le_{\mathfrak{k}} N$ are countable and $\bar{a} \in M$ then $\mathrm{tp}_{\mathbb{L}}(\bar{a}, M, N)$ determine $\mathrm{tp}_{\mathfrak{k}}(\bar{a}, M, N)$.

Now we define $\mathfrak{s} = (\mathfrak{k}_{\aleph_0}, \mathcal{S}^{\mathrm{bs}}, \bigcup)$ by

$\circledast_{15}$ $\mathcal{S}^{\mathrm{bs}}(M) = \{\mathrm{ortp}_{\mathfrak{k}}(\bar{a}, M, N) : M \le_{\mathfrak{k}} N$ are countable and $\bar{a} \in {}^{\omega>}N$ but $\bar{a} \notin {}^{\omega>}M\}$

$\circledast_{16}$ $\mathrm{ortp}_{\mathfrak{k}}(\bar{a}, M_1, M_3)$ does not fork over M_0 where $M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_3 \in \mathfrak{k}_{\aleph_0}$ iff $\mathrm{tp}_{\mathbb{L}}(\bar{a}, M_1, M_3)$ is definable over some finite subset of M_0.

Now we check "$\mathfrak{s}$ is a good frame", i.e., all clauses of Definition 2.1.

Clause (A): By $\circledast_9$ above.

Clause (B): As $\mathfrak{k}$ is categorical in $\aleph_0$, has an uncountable model and $\mathrm{LS}(\mathfrak{k}) = \aleph_0$ this should be clear.

Clause (C): $\mathfrak{k}_{\aleph_0}$ has amalgamation by $\circledast_{12}$ and has the JEP by categoricity in $\aleph_0$ and $\mathfrak{k}_{\aleph_0}$ has no maximal model by (categoricity and) having uncountable models (and $\mathrm{LS}(\mathfrak{k}) = \aleph_0$).

Clause (D): Obvious; stability, i.e., (D)(d) holds by $\circledast_{13}(ii) + \circledast_{14}$.

Subclause (E)(a),(b): By the definition.

Subclause (E)(c): (Local character).

If $\langle M_i : i \le \delta + 1\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous $M_i \in K_{\aleph_0}, \bar{a} \in {}^{\omega>}(M_{\delta+1})$ and $\bar{a} \in {}^{\omega>}(M_\delta)$ then for some finite $A \subseteq M_\delta, \mathrm{tp}_{\mathbb{L}}(\bar{a}, M_\delta, M_{\delta+1})$ is definable over A, so for some $i < \delta, A \subseteq M_\delta$ hence $j \in [i, \delta) \Rightarrow \mathrm{tp}_{\mathbb{L}}(\bar{a}, M_i, M_{\delta+1})$ is definable over $A \Rightarrow \bigcup(M_i, M_\delta, \bar{a}, M_{\delta+i})$.

Subclause (E)(d): (Transitivity).

As if $M' \le_{\mathfrak{k}} M'' \in \mathfrak{k}_{\aleph_0}$, two definitions in M' of complete types, which give the same result in M' give the same result in M''.

Sublause (E)(e)(uniqueness): By $\circledast_{14}$ and the justification of transitivity.

Subclause (E)(f)(symmetry): By [She75, Theorem 5.4], we have the symmetry property see [She75, Definition 5.2]. By [She75, 5.5] + the uniqueness proved above we can finish easily.

Subclause (E)(g): Extension existence.

Easy, included in [She75, 5.5].

Subclause (E)(h): Continuity.

As $\mathcal{S}^{\rm bs}_{\mathfrak{s}}(M)$ is the set of non-algebraic types this follows from "finite character", that is by 2.19(3)(4).

Subclause (E)(i): non-forking amalgamation
By 2.18. $\square_{3.8}$

Remark 3.9. So if $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ and $1 \le \dot{I}(\aleph_1, \psi) < 2^{\aleph_1}$, we essentially can apply Theorem 0.1, exactly see 9.5.

§ 3(D). **Starting at $\lambda > \aleph_0$.** The next theorem puts the results of [She01] in our context hence rely on it heavily.

(Alternatively, even eliminating "WDmId(λ^+) is λ^{++}-saturated" we can deduce 3.10 by [She09c], [She09d], i.e. by [She09c, 0z.1](2) there is a so called almost good λ-frame $\mathfrak{s}$ and by [She09d, e.6A] it is even a good λ-frame, and by §9 here, also $\mathfrak{s}^+$ is a good λ^+-frame and easily it is the frame described in 3.10(2).)

We use $K^{3,\rm na}_\lambda$ as in [She09c] called K^3_λ is [She01]. Note that while the material does not [She01, §1,§2,§4,§7] appears in [She09c], the material in [She01, §8,§9,§10] similar to §6 - §9 here, so we still need some parts of [She01], though as said above we can avoid it.

theorem 3.10. *Assume $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ and*

- (α) *$\mathfrak{k}$ is an abstract elementary class with* $\mathrm{LS}(\mathfrak{k}) \le \lambda$
- (β) *$\mathfrak{k}$ is categorical in λ and in λ^+*
- (γ) *$\mathfrak{k}$ has a model in λ^{++}*
- (δ) $\dot{I}(\lambda^{+2}, K) < \mu_{\rm unif}(\lambda^{+2}, 2^{\lambda^+})$ *and* $\mathrm{WDmId}(\lambda^+)$ *is not λ^{++}-saturated* <u>*or*</u> *just some consequences: density of minimal types (see by* [She09c, 4d.19,4d.23]*) and $\otimes$, i.e. $K^{3,\rm uq}_\lambda \ne \varnothing$ of* [She01, 6.4,pg.99] $=$ [She09c, 6f.5] *proved by the conclusion of* [She01, Th.6.7]*(pg.101) or* [She09c, 6f.13].

<u>*Then*</u> *1) Letting $\mu = \lambda^+$ we can choose $\bigcup\limits_{\mu}, \mathcal{S}^{\rm bs}$ such that $(\mathfrak{k}_{\ge\mu}, \bigcup\limits_{\mu}, \mathcal{S}^{\rm bs})$ is a μ-good frame.*

2) Moreover, we can let

(a) $\mathcal{S}^{\rm bs}(M) := \{\mathrm{ortp}_{\mathfrak{k}}(a, M, N) :$ *for some M, N, a we have $(M, N, a) \in K^{3,\rm na}_{\lambda^+}$ and for some $M' \le_{\mathfrak{k}} M$ we have $M' \in K_\lambda$ and $\mathrm{ortp}_{\mathfrak{k}}(a, M', N) \in \mathcal{S}_{\mathfrak{k}}(M')$ is minimal*$\}$

(see Definition [She01, 2.3]*(4),pg.56 and* [She01, 2.5]*(1),(13),pg.57-58* <u>*or*</u> *(*[She09c, 1a.19,1a.34]*)*

(b) $\bigcup = \bigcup\limits_{\mu}$ *be defined by: $\bigcup(M_0, M_1, a, M_3)$* <u>*iff*</u> *$M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_3$ are from K_μ, $a \in M_3 \backslash M_1$ and for some $N \le_{\mathfrak{k}} M_0$ of cardinality λ, the type $\mathrm{ortp}_{\mathfrak{k}}(a, N, M_3) \in \mathcal{S}_{\mathfrak{k}}(N)$ is minimal.*

Proof. 1), 2). Note that $\mathfrak{k}$ has amalgamation in λ and in λ^+, see [She09a, 88r-3.5]. By clause (δ) of the assumption, we can use the "positive" results of [She01] in particular [She01] freely. Now (see Definition 1.12(2))

- $(*)$ if $(M, N, a) \in K^{3,\rm na}_{\lambda^+}$ and $M' \le_{\mathfrak{k}} M, M' \in K_\lambda$ and $p = \mathrm{ortp}_{\mathfrak{k}}(a, M', N)$ is minimal (see Definition 1.9(0)) <u>then</u>
  - (a) if $q \in \mathcal{S}_{\mathfrak{k}}(M)$ is not algebraic and $q \restriction M' = p$ then $q = \mathrm{ortp}_{\mathfrak{k}}(a, M, N)$

(b) if $\langle M_\alpha : \alpha < \mu\rangle, \langle N_\alpha : \alpha < \mu\rangle$ are $\leq_{\mathfrak{k}}$-representations of M, N respectively then for a club of $\delta < \mu$ we have $\mathrm{ortp}_{\mathfrak{k}}(a, M_\delta, N_\delta) \in \mathcal{S}_{\mathfrak{k}}(M_\delta)$ is minimal and reduced

[Why? For clause (b) let $\alpha^* = \min\{\alpha : M' \leq_{\mathfrak{k}} M_\alpha\}$, so α^* is well defined and as M is saturated (for $\mathfrak{k}$), for a club of $\delta < \mu = \lambda^+$, the model M_δ is $(\lambda, \mathrm{cf}(\delta))$-brimmed over M' hence by [She01, 7.5](2)(pg.106) we are done.

For clause (a) let $M^0 = M, M^1 = N$ and $a^1 = a$ and $M^2, a^2 = a$ be such that $(M^0, M^2, a^2) \in K^{3,\mathrm{na}}_\mu = K^{3,\mathrm{na}}_{\lambda^+}$ and $q = \mathrm{ortp}_{\mathfrak{k}}(a^2, M^0, M)$. Now we repeat the proof of [She01, 9.5](pg.120) but instead $f(a^2) \notin M^1$ we require $f(a^2) = a^1$; we are using [She01, 10.5](1)(pg.125) which says $<^*_{\lambda^+} = <_{\mathfrak{k}} \restriction K_{\lambda^+}$.]

In particular we have used

(∗∗) if $M_0 \leq_{\mathfrak{k}_\lambda} M_1, M_1$ is (λ, κ)-brimmed over $M_0, p \in \mathcal{S}_{\mathfrak{k}}(M_1)$ is not algebraic and $p \restriction M_0$ is minimal, then p is minimal and reduced.

Clause (A):

This is by assumption (α).

Clause (B):

As K is categorical in $\mu = \lambda^+$, the existence of superlimit $M \in K_\mu$ follows; the superlimit is not maximal as $\mathrm{LS}(\mathfrak{k}) \leq \lambda$ and $K_{\mu^+} = K_{\lambda^{++}} \neq \varnothing$ by assumption (γ).

Clause (C):

K_{λ^+} has the amalgamation property by [She09a, 88r-3.5] or [She01, 1.4](pg.46),1.6(pg.48) and $\mathfrak{k}_\lambda$ has the JEP in λ^+ by categoricity in λ^+.

Clause (D):

Subclause (D)(a),(b):

By the definition of $\mathcal{S}^{\mathrm{bs}}(M)$ and of minimal types (in $\mathcal{S}_{\mathfrak{k}}(N), N \in K_\lambda$, [She01, 2.5](1)+(3)(pg.57),2.3(4)+(6)](pg.56)), this is clear.

Subclause (D)(c):

Suppose $M \leq_{\mathfrak{k}} N$ are from K_μ and $M \neq N$; let $\langle M_i : i < \lambda^+\rangle, \langle N_i : i < \lambda^+\rangle$ be a $\leq_{\mathfrak{k}}$-representation of M, N respectively, choose $b \in N \setminus M$ so $E = \{\delta < \lambda^+ : N_\delta \cap M = M_\delta$ and $b \in N_\delta\}$ is a club of λ^+. Now for $\delta = \min(E)$ we have $M_\delta \neq N_\delta, M_\delta \leq_{\mathfrak{k}} N_\delta$ and there is a minimal inevitable $p \in \mathcal{S}_{\mathfrak{k}}(M_\delta)$ by [She01, 5.3,pg.94] and categoricity of K in λ; so for some $a \in N_\delta \setminus M_\delta$ we have $p = \mathrm{ortp}_{\mathfrak{k}}(a, M_\delta, N_\delta)$. So $\mathrm{ortp}_{\mathfrak{k}}(a, M, N)$ is non-algebraic as $a \in M \Rightarrow a \in M \cap N_\delta = M_\delta$, a contradiction, so $\mathrm{ortp}_{\mathfrak{k}}(a, M, N) \in \mathcal{S}^{\mathrm{bs}}(M)$ as required.

Subclause (D)(d): If $M \in K_\mu$ let $\langle M_i : i < \lambda^+\rangle$ be a $\leq_{\mathfrak{k}}$-representation of M, so by $(*)(a)$ above $p \in \mathcal{S}^{\mathrm{bs}}(M)$ is determined by $p \restriction M_\alpha$ if $p \restriction M_\alpha$ is minimal and reduced. But for every such p there is such $\alpha(p) < \lambda^+$ by the definition of $\mathcal{S}^{\mathrm{bs}}(M)$ and for each $\alpha < \lambda^+$ there are $\leq \lambda$ possible such $p \restriction M_\alpha$ as $\mathfrak{k}$ is stable in λ by [She01, 5.7](a)(pg.97), so the conclusion follows. Alternatively, $M \in K_\mu \Rightarrow |\mathcal{S}^{\mathrm{bs}}(M)| \leq \mu$ as by [She01, 10.5](pg.125), we have $\leq^*_{\lambda^+} = \leq_{\mathfrak{k}} \restriction K_{\lambda^+}$, so we can apply [She01, 9.7](pg.121); or use $(*)$ above.

Clause (E):

Subclause (E)(a):

Follows by the definition.

Subclause (E)(b): (Monotonicity)

Obvious properties of minimal types in $\mathcal{S}(M)$ for $M \in K_\lambda$.

Subclause (E)(c): (Local character)

Let $\delta < \mu^+ = \lambda^{++}$ and $M_i \in K_\mu$ be $\le_{\mathfrak{k}}$-increasing continuous for $i \le \delta$ and $p \in \mathcal{S}^{\rm bs}(M_\delta)$, so for some $N \le_{\mathfrak{k}} M_\delta$ we have $N \in K_\lambda$ and $p \restriction N \in \mathcal{S}_{\mathfrak{k}}(N)$ is minimal. Without loss of generality $\delta = {\rm cf}(\delta)$ and if $\delta = \lambda^+$, there is $i < \delta$ such that $N \subseteq M_i$ and easily we are done. So assume $\delta = {\rm cf}(\delta) < \lambda^+$.

Let $\langle M^i_\zeta : \zeta < \lambda^+\rangle$ be a $\le_{\mathfrak{k}}$-representation of M_i for $i \le \delta$, hence E is a club of λ^+ where:

$$\begin{aligned} E := \big\{\zeta < \lambda^+ :&\zeta \text{ a limit ordinal and for } j < i \le \delta \text{ we have} \\ &M^i_\zeta \cap M_j = M^j_\zeta \text{ and for } \xi < \zeta, i \le \delta \text{ we have :} \\ &M^i_\zeta \text{ is } (\lambda, {\rm cf}(\zeta))\text{-brimmed over } M^i_\xi \text{ and } N \le_{\mathfrak{k}} M^\delta_\zeta\big\}. \end{aligned}$$

Let ζ_i be the i-th member of E for $i \le \delta$, so $\langle \zeta_i : i \le \delta\rangle$ is increasing continuous, $\langle M^i_{\zeta_i} : i \le \delta\rangle$ is $\le_{\mathfrak{k}}$-increasingly continuous in K_λ and $M^{i+1}_{\zeta_{i+1}}$ is $(\lambda, {\rm cf}(\zeta_{i+1}))$-brimmed over $M^{i+1}_{\zeta_i}$ hence also over $M^i_{\zeta_i}$. Also $p \restriction M^\delta_{\zeta_\delta}$ is non-algebraic (as p is) and extends $p \restriction N$ (as $N \le_{\mathfrak{k}} M^\delta_{\zeta_\delta}$ as $\zeta_\delta \in E$) hence $p \restriction M^\delta_{\zeta_\delta}$ is minimal.

Also $M^\delta_{\zeta_\delta}$ is $(\lambda, {\rm cf}(\zeta_\delta))$-brimmed over $M^\delta_{\zeta_0}$ hence over N, hence by $(**)$ above we get that $p \restriction M^\delta_{\zeta_\delta}$ is not only minimal but also reduced. Hence by [She01, 7.3](2)(pg.103) applied to $\langle M^i_{\zeta_i} : i \le \delta\rangle, p \restriction M^\delta_{\zeta_\delta}$ we know that for some $i < \delta$ the type $p \restriction M^i_{\zeta_i} = (p \restriction M^\delta_{\zeta_\delta}) \restriction M^i_{\zeta_i}$ is minimal and reduced, so it witnesses that $p \restriction M_j \in \mathcal{S}^{\rm bs}(M_j)$ for every $j \in [i, \delta)$, as required.

Subclause $(E)(d)$: (Transitivity)

Easy by the definition of minimal.

Subclause $(E)(e)$: (Uniqueness)

By $(*)(a)$ above.

Subclause $(E)(f)$: (Symmetry)

By the symmetry in the situation assume $M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_3$ are from K_μ, $a_1 \in M_1 \backslash M_0, a_2 \in M_3 \backslash M_1$ and ${\rm ortp}_{\mathfrak{k}}(a_1, M_0, M_3) \in \mathcal{S}^{\rm bs}(M_0)$ and ${\rm ortp}_{\mathfrak{k}}(a_2, M_1, M_3) \in \mathcal{S}^{\rm bs}(M_1)$ does not fork over M_0; hence for $\ell = 1, 2$ we have ${\rm ortp}_{\mathfrak{k}}(a_\ell, M_0, M_3) \in \mathcal{S}^{\rm bs}(M_0)$. By the existence of disjoint amalgamation (by [She01, 9.11](pg.122),10.5(1)(pg.125)) there are M_2, M'_3, f such that $M_0 \le_{\mathfrak{k}} M_2 \le_{\mathfrak{k}} M'_3 \in K_\mu$, $M_3 \le_{\mathfrak{k}} M'_3, f$ is an isomorphism from M_3 onto M_2 over M_0, and $M_3 \cap M_2 = M_0$. By ${\rm ortp}_{\mathfrak{k}}(a_2, M_0, M_3) \in \mathcal{S}^{\rm bs}(M_1)$ and as $f(a_2) \notin M_1$ being in $M_2 \setminus M_0 = M_2 \setminus M_3$ and $a_2 \notin M_1$ by assumption and as $a_2, f(a_2)$ realize the same type from $\mathcal{S}_{\mathfrak{k}}(M_0)$ clearly by $(*)(a)$ we have ${\rm ortp}_{\mathfrak{k}}(a_2, M_1, M'_3) = {\rm ortp}_{\mathfrak{k}}(f(a_2), M_1, M'_3)$.

Using amalgamation in $\mathfrak{k}_\mu$ (and equality of types) there is M''_3 such that: $M'_3 \le_{\mathfrak{k}} M''_3 \in K_\mu$, and there is an $\le_{\mathfrak{k}}$-embedding g of M'_3 into M''_3 such that $g \restriction M_1 = {\rm id}_{M_1}$ and $g(f(a_2)) = a_2$. Note that as $a_1 \notin g(M_2), M_1 \le_{\mathfrak{k}} g(M_2) \in K_\mu$ and ${\rm ortp}_{\mathfrak{k}}(a_1, M_1, M''_3)$ is minimal then necessarily ${\rm ortp}_{\mathfrak{k}}(a_1, g(M_2), M''_3)$ is its non-forking extension. So $g(M_2), M''_3$ are models as required.

Subclause $(E)(g)$: (Extension existence)

Claims [She01, 9.11](pg.122),10.5(1)(pg.125) do even more.

Subclause $(E)(h)$: (Continuity)

Easy.

Subclause $(E)(i)$: (Non-forking amalgamation)

Like $(E)(f)$ or use 2.18. $\square_{3.10}$

Question 3.11. If $\mathfrak{k}$ is categorical in λ and in μ and $\mu > \lambda \geq \mathrm{LS}(\mathfrak{k})$, can we conclude categoricity in $\chi \in (\mu, \lambda)$?

Fact 3.12. In 3.10:

1) If $p \in \mathcal{S}^{\mathrm{bs}}(M)$ and $M \in K_\mu$, then for some $N \leq_{\mathfrak{k}} M, N \in K_\lambda$ and $p \restriction N$ is minimal and reduced.

2) If $M <_{\mathfrak{k}} N, M \in K_\mu$ and $p \in \mathcal{S}^{\mathrm{bs}}(M)$, then some $a \in N \setminus M$ realizes p, (i.e., "a strong version of uni-dimensionality" holds).

Proof. The proof is included in the proof of 3.10.

□

* * *

(E) An Example:

A trivial example (of an approximation to good λ-frame) is:

Definition/Claim 3.13. 1) Assume that $\mathfrak{k}$ is an AEC and $\lambda \geq \mathrm{LS}(\mathfrak{k})$ or $\mathfrak{k}$ is a λ-AEC We define $\mathfrak{s} = \mathfrak{s}_\lambda[\mathfrak{k}]$ as the triple $\mathfrak{s} = (\mathfrak{k}_\lambda, \mathcal{S}^{\mathrm{na}}, \underset{\mathrm{na}}{\bigcup})$ where:

- (a) $\mathcal{S}^{\mathrm{na}}(M) = \{\mathrm{ortp}_{\mathfrak{k}}(a, M, N), M \leq_{\mathfrak{k}} N \text{ and } a \in N \setminus M\}$
- (b) $\bigcup(M_0, M_1, a, M_3)$ iff $M_0 \leq_{\mathfrak{k}_\lambda} M_1 \leq_{\mathfrak{k}_\lambda} M_3$ and $a \in M_3 \setminus M_1$.

2) Then $\mathfrak{s}$ satisfies Definition 2.1 of good λ-frame except possibly: (B), existence of superlimits, (C) amalgamation and JEP, (D)(d) stability and (E)(e),(f),(g),(i) uniqueness, symmetry, extension existence and non-forking amalgamation.

§ 4. Inside the frame

We investigate good λ-frames. We prove stability in λ (we have assumed in Definition 2.1 only stability for basic types), hence the existence of a (λ, ∂)-brimmed $\le_{\mathfrak{k}}$-extension in K_λ over $M_0 \in K_\lambda$ (see 4.2), and we give a sufficient condition for "M_δ is $(\lambda, \mathrm{cf}(\delta))$-brimmed over M_0" (in 4.3). We define again $K^{3,\mathrm{bs}}_\lambda$ (like K^3_λ from 1.12(2) but the type is basic) and the natural order $\le_{\mathrm{bs}}$ on them as well as "reduced" (Definition 4.5), and indicate their basic properties (4.7).

We may like to construct sometimes pairs $N_i \le_{\mathfrak{k}_\lambda} M_i$ such that M_i, N_i are increasing continuous with i and we would like to guarantee that M_γ is $(\lambda, \mathrm{cf}(\gamma))$-brimmed over N_γ, of course we need to carry more inductive assumptions. Toward this we may give a sufficient condition for building a $(\lambda, \mathrm{cf}(\gamma))$-brimmed extension over N_γ where $\langle N_i : i \le \gamma\rangle$ is $\le_{\mathfrak{k}_\lambda}$-increasing continuous, by a triangle of extensions of the N_i's, with non-forking demands of course (see 4.8). We also give conditions on a rectangle of models to get such pairs in both directions (4.12), for this we use nice extensions of chains (4.10, 4.11).

Then we can deduce that if "M_1 is (λ, ∂)-brimmed over M_0" then the isomorphism type of M_1 over M_0 does not depend on ∂ (see 4.9), so the brimmed N over M_0 is unique up to isomorphism (i.e. being (λ, ∂)-brimmed over M_0 does not depend on ∂). We finish giving conclusion about $K_{\lambda^+}, K_{\lambda^{++}}$.

Hypothesis 4.1. $\mathfrak{s} = (\mathfrak{k}, \bigcup, \mathcal{S}^{\mathrm{bs}})$ is a good λ-frame.

Claim 4.2. *1) $\mathfrak{k}$ is stable in λ, i.e., $M \in \mathfrak{k}_\lambda \Rightarrow |\mathcal{S}(M)| \le \lambda$.*

2) For every $M_0 \in K_\lambda$ and $\partial \le \lambda$ there is M_1 such that $M_0 \le_{\mathfrak{k}} M_1 \in K_\lambda$ and M_1 is (λ, ∂)-brimmed over M_0 (see Definition 1.16) and it is universal[16] *over M_0.*

Proof. 1) Let $M_0 \in K_\lambda$ and we choose by induction on $\alpha \in [1, \lambda], M_\alpha \in K_\lambda$ such that:

- (i) M_α is $\le_{\mathfrak{k}}$-increasing continuous
- (ii) if $p \in \mathcal{S}^{\mathrm{bs}}(M_\alpha)$ then this type is realized in $M_{\alpha+1}$.

No problem to carry this: for clause (i) use Axiom(A), for clause (ii) use Axiom $(D)(d)$ and amalgamation in $\mathfrak{k}_\lambda$, i.e., Axiom (C). If every $q \in \mathcal{S}(M_0)$ is realized in M_λ we are done. So let q be a counterexample, so let $M_0 \le_{\mathfrak{k}} N \in K_\lambda$ be such that q is realized in N. We now try to choose by induction on $\alpha < \lambda$ a triple $(N_\alpha, f_\alpha, \bar{\mathbf{a}}_\alpha)$ such that:

- (A) $N_\alpha \in K_\lambda$ is $\le_{\mathfrak{k}}$-increasingly continuous
- (B) f_α is a $\le_{\mathfrak{k}}$-embedding of M_α into N_α
- (C) f_α is increasing continuous
- (D) $f_0 = \mathrm{id}_{M_0}$ and $N_0 = N$
- (E) $\bar{\mathbf{a}}_\alpha = \langle a_{\alpha,i} : i < \lambda\rangle$ lists the elements of N_α
- (F) if there are $\beta \le \alpha, i < \lambda$ such that $\mathrm{ortp}(a_{\beta,i}, f_\alpha(M_\alpha), N_\alpha) \in \mathcal{S}^{\mathrm{bs}}(f_\alpha(M_\alpha))$ <u>then</u> for some such pair (β_α, i_α) we have:
  - (i) the pair (β_α, i_α) is minimal in an appropriate sense, that is: if (β, i) is another such pair then $\beta + i > \beta_\alpha + i_\alpha$ or $\beta + i = \beta_\alpha + i_\alpha$ and $\beta > \beta_\alpha$ or $\beta + i = \beta_\alpha + i_\alpha$ and $\beta = \beta_\alpha$ and $i \ge i_\alpha$
  - (ii) $a_{\beta_\alpha, i_\alpha} \in \mathrm{rang}(f_{\alpha+1})$.

[16] in fact, this follows

This is easy: for successor α we use the definition of type and let $N_\lambda := \cup\{N_\alpha : \alpha < \lambda\}$. Clearly $f_\lambda := \cup\{f_\alpha : \alpha < \lambda\}$ is a $\leq_{\mathfrak{s}}$-embedding of M_λ into N_λ over M_0.

As in N, the type q is realized and it is not realized in M_λ necessarily $N \nsubseteq f_\lambda(M_\lambda)$ hence $N_\lambda \neq f_\lambda(M_\lambda)$ but easily $f_\lambda(M_\lambda) \leq_{\mathfrak{k}} N_\lambda$. So by Axiom $(D)(c)$ for some $c \in N_\lambda \setminus f_\lambda(M_\lambda)$ we have $p = \text{ortp}(c, f_\lambda(M_\lambda), N_\lambda) \in \mathcal{S}^{\text{bs}}(f_\lambda(M_\lambda))$. As $\langle f_\gamma(M_\gamma) : \gamma \leq \lambda\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous, by Axiom (E)(c) for some $\gamma < \lambda$ we have $\text{ortp}(c, f_\lambda(M_\lambda), N_\lambda)$ does not fork over $f_\gamma(M_\gamma)$, also as $c \in N_\lambda = \bigcup\limits_{\beta<\lambda} N_\beta$ clearly $c \in N_\beta$ for some $\beta < \lambda$ and let $i < \lambda$ be such that $c = a_{\beta,i}$. Now if $\alpha \in [\max\{\gamma,\beta\},\lambda)$ then (β,i) is a legitimate candidate for (β_α, i_α) that is $\text{ortp}(a_{\beta,i}, f_\alpha(M_\alpha), N_\alpha) \in \mathcal{S}^{\text{bs}}(f_\alpha(M_\alpha))$ by monotonicity of non-forking, i.e., Axiom (E)(b). So (β_α, i_α) is well defined for any such α and $\beta_\alpha + i_\alpha \leq \beta + i$ by clause (F)(i). But $\alpha_1 < \alpha_2 \Rightarrow a_{\beta_{\alpha_1}, i_{\alpha_1}} \neq a_{\beta_{\alpha_2}, i_{\alpha_2}}$ (as one belongs to $f_{\alpha_1+1}(M_{\alpha_1})$ and the other not), contradiction by cardinality consideration.

2) So $\mathfrak{k}_\lambda$ is stable in λ and has amalgamation, hence (see 1.17) the conclusion holds; alternatively use 4.3 below. $\square_{4.2}$

Claim 4.3. *Assume*

- (a) *$\delta < \lambda^+$ is a limit ordinal divisible by λ*
- (b) *$\overline{M} = \langle M_\alpha : \alpha \leq \delta\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous sequence in $\mathfrak{k}_\lambda$*
- (c) *if $i < \delta$ and $p \in \mathcal{S}^{\text{bs}}(M_i)$, <u>then</u> for λ ordinals $j \in (i,\delta)$ there is $c' \in M_{j+1}$ realizing the non-forking extension of p in $\mathcal{S}^{\text{bs}}(M_j)$.*

<u>Then</u> *M_δ is $(\lambda, \text{cf}(\delta))$-brimmed over M_0 and universal over it.*

Remark 4.4. 1) See end of proof of 6.31.

2) Of course, by renaming, M_δ is $(\lambda, \text{cf}(\delta))$-brimmed over M_α for any $\alpha < \delta$.

3) Why in clause (c) of 4.3 we ask for "λ ordinals $j \in (i,\delta)$" rather than "for unboundedly many $j \in (i,\delta)$"? For λ regular there is no difference but for λ singular not so. Think of $\mathfrak{k}$ the class of $(A,R), R$ an equivalence relation on A; (so it is not categorical) but for some λ-good frames $\mathfrak{s}, \mathfrak{k}_{\mathfrak{s}} = \mathfrak{k}_\lambda$ and exemplifies a problem; some equivalence class of M_δ may be of cardinality $< \lambda$.

Proof. Like 4.2, but we give details.

Let $g : \delta \to \lambda$ be a one to one and choose by induction on $\alpha \leq \delta$ a triple $(N_\alpha, f_\alpha, \bar{\mathbf{a}}_\alpha)$ such that

- (A) $N_\alpha \in K_\lambda$ is $\leq_{\mathfrak{k}}$-increasing continuous
- (B) f_α is a $\leq_{\mathfrak{k}}$-embedding of M_α into N_α
- (C) f_α is increasing continuous
- (D) $f_0 = \text{id}_{M_0}, N_0 = M_0$
- (E) $\bar{\mathbf{a}}_\alpha = \langle a_{\alpha,i} : i < \lambda\rangle$ list the elements of N_α
- (F) $N_{\alpha+1}$ is universal over N_α
- (G) if $\alpha < \delta$ and there is a pair $(\beta,i) = (\beta_\alpha, i_\alpha)$ satisfying the condition $(*)^{\beta,i}_{f_\alpha,N_\alpha}$ stated below and it is minimal in the sense that
  $(*)^{\beta',i'}_{f_\alpha,N_\alpha} \Rightarrow (**)^{\beta',i',\beta,i}_g$, see below, <u>then</u> $a_{\beta,i} \in \text{rang}(f_{\alpha+1})$,
  where

$(*)^{\beta,i}_{f_\alpha,N_\alpha}$ (a) $\beta \leq \alpha$ and $i < \lambda$

(a) (b) $\text{ortp}(a_{\beta,i}, f_\alpha(M_\alpha), N_\alpha) \in \mathcal{S}^{\text{bs}}(f_\alpha(M_\alpha))$

(b) (c) some $c \in M_{\alpha+1}$ realizes $f_\alpha^{-1}(\text{ortp}(a_{\beta,i}, f_\alpha(M_\alpha), N_\alpha)$, so by clause (b) it follows that $c \in M_{\alpha+1} \setminus M_\alpha$

$(**)_g^{\beta',i',\beta,i}$ $[g(\beta) + i < g(\beta') + i'] \vee$ $[g(\beta)+i = g(\beta')+i'$ and $g(\beta) < g(\beta')] \vee [g(\beta)+i = g(\beta')+i'$ and $g(\beta) = g(\beta')$ and $i \le i']$.

There is no problem to choose f_α, N_α. Now in the end, by clauses (A),(F) clearly N_δ is $(\lambda, \text{cf}(\delta))$-brimmed over N_0, i.e., over M_0, so it suffices to prove that f_δ is onto N_δ. If not, then by Axiom (D)(c), the density, there is $d \in N_\delta \setminus f_\delta(M_\delta)$ such that $p := \text{ortp}(d, f_\delta(M_\delta), N_\delta) \in \mathcal{S}^{\text{bs}}(f_\delta(M_\delta))$ hence for some $\beta(*) < \delta$ we have $d \in N_{\beta(*)}$ so for some $i(*) < \lambda, d = a_{\beta(*),i(*)}$. Also by Axiom (E)(c), (the local character) for every $\beta < \delta$ large enough say $\ge \beta_d$ the type p does not fork over $f_\delta(M_\beta)$, without loss of generality $\beta_d = \beta(*)$. Let $q = f_\delta^{-1}(\text{ortp}(d, f_\delta(M_\delta), N_\delta)$, so it $\in \mathcal{S}^{\text{bs}}(M_\delta)$.

Let $u = \{\alpha : \beta(*) \le \alpha < \delta$ and $q \restriction M_\alpha \in \mathcal{S}^{\text{bs}}(M_\alpha)$ (note $\beta(*) \le \alpha$) is realized in $M_{\alpha+1}\}$. By clause (c) of the assumption clearly $|u| = \lambda$. Also by the definition of v for every $\alpha \in u$ the condition $(*)_{N_\alpha,f_\alpha}^{\beta(*),i(*)}$ holds, hence in clause (F) the pair (β_α, i_α) is well defined and is "below" $(\beta(*), i(*))$ in the sense of clause (G). But there are only $\le |g(\beta(*)) \times i(*)| < \lambda$ such pairs hence for some $\alpha_1 < \alpha_2$ in u we have $(\beta_{\alpha_1}, i_{\alpha_1}) = (\beta_{\alpha_2}, i_{\alpha_2})$, a contradiction: $a_{\beta_{\alpha_1},i_{\alpha_1}} \in \text{rang}(f_{\alpha_1+1}) \subseteq \text{rang}(f_{\alpha_2}) = f_{\alpha_2}(M_{\alpha_2})$ hence $\text{ortp}(a_{\beta_{\alpha_1},i_{\alpha_1}}, f_{\alpha_2}(M_{\alpha_2}), N_{\alpha_2}) \notin \mathcal{S}^{\text{bs}}(f_{\alpha_2}(M_{\alpha_2}))$, contradiction. So we are done. $\square_{4.3}$

* * *

The following is helpful for constructions so that we can amalgamate disjointly preserving non-forking of a type; we first repeat the definition of $K_\lambda^{3,\text{bs}}, <_{\text{bs}}$.

Definition 4.5. 1) Let $(M, N, a) \in K_\lambda^{3,\text{bs}}$ if $M \le_{\mathfrak{k}} N$ are models from $K_\lambda, a \in N \backslash M$ and $\text{ortp}(a, M, N) \in \mathcal{S}^{\text{bs}}(M)$. Let $(M_1, N_1, a) \le_{\text{bs}} (M_2, N_2, a)$ or write $\le_{\text{bs}}^{\mathfrak{s}}$, when: both triples are in $K_\lambda^{3,\text{bs}}, M_1 \le_{\mathfrak{k}} M_2, N_1 \le_{\mathfrak{k}} N_2$ and $\text{ortp}(a, M_2, N_2)$ does not fork over M_1.

2) We say (M, N, a) is bs-reduced <u>when</u> if it belongs to $K_\lambda^{3,\text{bs}}$ and $(M, N, a) \le_{\text{bs}} (M', N', a) \in K_\lambda^{3,\text{bs}} \Rightarrow N \cap M' = M$.

3) We say $p \in \mathcal{S}^{\text{bs}}(N)$ is a (really the) stationarization of $q \in \mathcal{S}^{\text{bs}}(M)$ if $M \le_{\mathfrak{k}} N$ and p is an extension of q which does not fork over M.

Remark 4.6. 1) The definition of $K_\lambda^{3,\text{bs}}$ is compatible with the one in 2.4 by 2.6(1).

2) We could have strengthened the definition of bs-reduced (4.5), e.g., add: for no $b \in N' \setminus M'$, do we have $\text{ortp}(b, M', N') \in \mathcal{S}^{\text{bs}}(M')$ and there are M'', N'' such that $(M', N', a) \le_{\text{bs}} (M'', N'', a)$ and $\text{ortp}(b, M'', N'')$ forks over M'.

Claim 4.7. *For parts (3),(4),(5) assume $\mathfrak{s}$ is categorical (in λ).*

1) If $\kappa \le \lambda, (M, N, a) \in K_\lambda^{3,\text{bs}}$, <u>then</u> we can find M', N' such that: $(M, N, a) \le_{\text{bs}} (M', N', a) \in K_\lambda^{3,\text{bs}}, M'$ is (λ, κ)-brimmed over M, N' is (λ, κ)-brimmed over N and (M', N', a) is bs*-reduced.*

1A) If $(M, N_\ell, a_\ell) \in K_\lambda^{3,\text{bs}}$ for $\ell = 1, 2$, <u>then</u> we can find M^+, f_1, f_2 such that: $M \le_{\mathfrak{k}} M^+ \in K_\lambda$ and for $\ell \in \{1, 2\}, f_\ell$ is a $\le_{\mathfrak{k}}$-embedding of N_ℓ into M^+ over M and $(M, f_\ell(N_\ell), f_\ell(a_\ell)) \le_{\text{bs}} (f_{3-\ell}(N_{3-\ell}), M^+, f_\ell(a_\ell))$, equivalently $\text{ortp}(f_\ell(a_\ell), f_{3-\ell}(N_{3-\ell}), M^+)$ *does not fork over M.*

2) If $(M_\alpha, N_\alpha, a) \in K^{3,\mathrm{bs}}_\lambda$ is $\le_{\mathrm{bs}}$-increasing for $\alpha < \delta$ and $\delta < \lambda^+$ is a limit ordinal <u>*then*</u> *their union $(\bigcup_{\alpha<\delta} M_\alpha, \bigcup_{\alpha<\delta} N_\alpha, a)$ is a $\le_{\mathrm{bs}}$-lub. If each (M_α, N_α, a) is bs-reduced then so is their union.*

3) Let λ divide $\delta, \delta < \lambda^+$. We can find $\langle N_j, a_i : j \le \delta, i < \delta\rangle$ such that: $N_j \in K_\lambda$ is $\le_{\mathfrak{k}}$-increasing continuous, $(N_j, N_{j+1}, a_j) \in K^{3,\mathrm{bs}}_\lambda$ is bs-reduced and if $i < \delta, p \in \mathcal{S}^{\mathrm{bs}}(N_i)$ <u>*then*</u> *for λ ordinals $j \in (i, i+\lambda)$ the type $\mathrm{ortp}(a_j, N_j, N_{j+1})$ is a non-forking extension of p; so N_δ is $(\lambda, \mathrm{cf}(\delta))$-brimmed over each $N_i, i < \delta$. We can add "N_0 is brimmed".*

4) For any $(M_0, M_1, a) \in K^{3,\mathrm{bs}}_\lambda$ and $M_2 \in K_\lambda$ such that $M_0 \le_{\mathfrak{k}} M_2$ there are N_0, N_1 such that $(M_0, M_1, a) \le_{\mathrm{bs}} (N_0, N_1, a), M_0 = M_1 \cap N_0$ and M_2, N_0 are isomorphic over M_0. (In fact, if $(M_0, M_2, b) \in K^{3,\mathrm{bs}}_\lambda$ we can add that for some isomorphism f from M_2 onto N_0 over M_0 we have $(M_0, N_0, f(a)) \le_{\mathrm{bs}} (M_1, N_1, f(a))$.)

5) If $M_0 \in K_\lambda$ is brimmed and $M_0 \le_{\mathfrak{s}} M_\ell$ for $\ell = 1, 2$ and there is a disjoint $\le_{\mathfrak{s}}$-amalgamation of M_1, M_2 over M_0.

Proof. 1) We choose $M_i, N_i, b^\ell_i (\ell = 1, 2), \bar{\mathbf{c}}_i$ by induction on $i < \delta := \lambda$ such that

- (a) $(M_i, N_i, a) \in K^{3,\mathrm{bs}}_{\mathfrak{s}}$ is $\le_{\mathrm{bs}}$-increasing continuous
- (b) $(M_0, N_0) = (M, N)$
- $(c)_1$ $b^1_i \in M_{i+1} \setminus M_i$ and $\mathrm{ortp}(b^1_i, M_i, M_{i+1}) \in \mathcal{S}^{\mathrm{bs}}(M_i)$,
- $(c)_2$ $b^2_i \in N_{i+1} \setminus N_i$ and $\mathrm{ortp}(b^2_i, N_i, N_{i+1}) \in \mathcal{S}^{\mathrm{bs}}(N_i)$
- $(d)_1$ if $i < \lambda$ and $p \in \mathcal{S}^{\mathrm{bs}}(M_i)$ <u>then</u> the set $\{j : i \le j < \lambda$ and $\mathrm{ortp}(b^1_j, M_j, M_{j+1})$ is a non-forking extension of $p\}$ has order type λ
- $(d)_2$ if $i < \lambda$ and $p \in \mathcal{S}^{\mathrm{bs}}(N_i)$ then the set $\{j : i \le j < \lambda$ and $\mathrm{ortp}(b^2_j, N_j, N_{j+1})$ is the non-forking extension of $p\}$ has order type λ
- (e) $\bar{\mathbf{c}}_i = \langle c_{i,j} : j < \lambda\rangle$ list N_i
- (f) if $\alpha < \lambda, i \le \alpha, j < \lambda, c_{i,j} \notin M_\alpha$ but for some (M'', N'') we have $(M_{\alpha+1}, N_{\alpha+1}, a) \le_{\mathrm{bs}} (M'', N'', a)$ and $c_{i,j} \in M''$ <u>then</u> for some $i_1, j_1 \le \max\{i, j\}$ we have $c_{i_1,j_1} \in M_{\alpha+1} \setminus M_\alpha$.

Lastly, let $M' = \cup\{M_i : i < \lambda\}, N' = \cup\{N_i : i < \lambda\}$, by 4.3 M' is $(\lambda, \mathrm{cf}(\lambda))$-brimmed over M (using $(d)_1$), and N' is $(\lambda, \mathrm{cf}(\lambda))$-brimmed over N (using $(d)_2$).

Lastly, being bs-reduced holds by clauses (e)+(f).

1A) Easy.

2) Recall Ax(E)(h).

3) For proving part (3) use part (1) and the "so" is by using 4.3.

4) For proving part (4), without loss of generality M_2 is $(\lambda, \mathrm{cf}(\lambda))$-brimmed over M_0, as we can replace M_2 by M'_2 if $M_2 \le_{\mathfrak{k}} M'_2 \in K_\lambda$. By part (3) there is a sequence $\langle a_i : i < \delta\rangle$ and an $\le_{\mathfrak{k}}$-increasing continuous $\langle N_i : i \le \delta\rangle$ with $N_0 = M_0, N_\delta = M_2$ and $(N_i, N_{i+1}, a_i) \in K^{3,\mathrm{bs}}_\lambda$ is reduced. Then use (1A) successively.

5) By part (3) as in the proof of part (4). $\square_{4.7}$

Claim 4.8. *Assume*

- (a) *$\gamma < \lambda^+$ is a limit ordinal*
- (b) *$\delta_i < \lambda^+$ is divisible by λ for $i \le \gamma, \langle \delta_i : i \le \gamma\rangle$ is increasing continuous*
- (c) *$\langle N_i : i < \gamma\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous in K_λ*
- (d) *$\langle M_i : i < \gamma\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous in K_λ*
- (e) *$N_i \le_{\mathfrak{k}} M_i$ for $i < \gamma$*
- (f) *$\langle M_{i,j} : j \le \delta_i\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous in K_λ for each $i < \gamma$*

(g) $M_{i,0} = N_i, M_{i,\delta_i} = M_i, a_j \in M_{i,j+1} \setminus M_{i,j}$ *and* $\mathrm{ortp}(a_j, M_{i,j}, M_{i,j+1}) \in \mathcal{S}^{\mathrm{bs}}(M_{i,j})$ *when* $i < \gamma, j < \delta_i$

(h) *if* $j \leq \delta_{i(*)}, i(*) < \gamma$ *then* $\langle M_{i,j} : i \in [i(*), \gamma) \rangle$ *is* $\leq_{\mathfrak{k}}$*-increasing continuous*

(i) $\mathrm{ortp}(a_j, M_{\beta,j}, M_{\beta,j+1})$ *does not fork over* $M_{i,j}$ *when* $i < \gamma, j < \delta_i, i \leq \beta < \gamma$

(j) *if* $i < \gamma, j < \delta_i, p \in \mathcal{S}^{\mathrm{bs}}(M_{i,j})$ *then for* λ *ordinals* $j_1 \in [j, \delta_i)$ *we have* $\mathrm{ortp}(a_{j_1}, M_{i,j_1}, M_{i,j_1+1}) \in \mathcal{S}^{\mathrm{bs}}(M_{i,j_1})$ *is a non-forking extension of* p
or we can ask less

$(j)^-$ *if* $i < \gamma, j < \delta_i$ *and* $p \in \mathcal{S}^{\mathrm{bs}}(M_{i,j})$ *then for* λ *ordinals* $j_1 \in [j, \delta_\gamma)$ *for some* $i_1 \in [i, \gamma)$ *we have* $\mathrm{ortp}(a_{j_1}, M_{i_1,j_1}, M_{i_1,j_1+1}) \in \mathcal{S}^{\mathrm{bs}}(M_{i_1,j_1})$ *is a non-forking extension of* p.

Then $M_\gamma := \cup\{M_{i,j} : i < \gamma, j < \delta_i\} = \{M_i : i < \gamma\}$ *is* $(\lambda, \mathrm{cf}(\gamma))$*-brimmed over* $N_\gamma := \cup\{N_i : i < \gamma\}$.

Proof. For $j < \delta_\gamma$ let $M_{\gamma,j} = \cup\{M_{i,j} : i < \gamma\}$, and let $M_{\gamma,\delta_\gamma} = M_\gamma$ be $\bigcup\{M_{\gamma,j} : j < \delta_\gamma\}$. Easily $\langle M_{\gamma,j} : j \leq \delta_\gamma \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous, $M_{\gamma,j} \in K_\lambda$ and $i \leq \gamma \wedge j < \delta_i \Rightarrow M_{i,j} \leq_{\mathfrak{k}} M_{\gamma,j}$. Also if $i < \gamma, j < \delta_i$ then $\mathrm{ortp}(a_j, M_{\gamma,j}, M_{\gamma,j+1}) \in \mathcal{S}^{\mathrm{bs}}(M_{\gamma,j})$ does not fork over $M_{i,j}$ by Axiom (E)(h), continuity.

Now if $j < \delta_\gamma$ and $p \in \mathcal{S}^{\mathrm{bs}}(M_{\gamma,j})$ then for some $i < \gamma, p$ does not fork over $M_{i,j}$ (by Ax(E)(c)) and without loss of generality $j < \delta_i$.

Hence if clause (j) holds we have $u := \{\varepsilon : j < \varepsilon < \delta_i$ and $\mathrm{ortp}(a_\varepsilon, M_{i,\varepsilon}, M_{i,\varepsilon+1})$ is a non-forking extension of $p \restriction M_{i,j}\}$ has λ members. But for $\varepsilon \in u$, $\mathrm{ortp}(a_\varepsilon, M_{\gamma,\varepsilon}, M_{\gamma,\varepsilon+1})$ does not fork over $M_{i,\varepsilon}$ (by clause (i) of the assumption) hence does not fork over $M_{i,j}$ and by monotonicity it does not fork over $M_{\gamma,i}$ and by uniqueness it extends p. If clause $(j)^-$ holds the proof is similar. By 4.3 the model M_γ is $(\lambda, \mathrm{cf}(\gamma))$-brimmed over N_γ. $\square_{4.8}$

Lemma 4.9. *1) If* $M \in K_\lambda$ *and the models* $M_1, M_2 \in K_\lambda$ *are* $(\lambda, *)$*-brimmed over* M *(see Definition 1.16(2)), then* M_1, M_2 *are isomorphic over* M.

2) If $M_1, M_2 \in K_\lambda$ *are* $(\lambda, *)$*-brimmed then they are isomorphic.*

We prove some claims before proving 4.9; we will not much use the lemma, but it is of obvious interest and its proof is crucial in one point of §6.

Claim 4.10. *1)*

$(E)(i)^+$ *long non-forking amalgamation for* $\alpha < \lambda^+$:

if $\langle N_i : i \leq \alpha \rangle$ *is* $\leq_{\mathfrak{k}}$*-increasing continuous sequence of members of* $K_\lambda, a_i \in N_{i+1} \setminus N_i$ *for* $i < \alpha, p_i = \mathrm{ortp}(a_i, N_i, N_{i+1}) \in \mathcal{S}^{\mathrm{bs}}(N_i)$ *and* $q \in \mathcal{S}^{\mathrm{bs}}(N_0)$, *then we can find a* $\leq_{\mathfrak{k}}$*-increasing continuous sequence* $\langle N'_i : i \leq \alpha \rangle$ *of members of* K_λ *such that:* $i \leq \alpha \Rightarrow N_i \leq_{\mathfrak{k}} N'_i$; *some* $b \in N'_0 \setminus N_0$ *realizes* $q, \mathrm{ortp}(b, N_\alpha, N'_\alpha)$ *does not fork over* N_0 *and* $\mathrm{ortp}(a_i, N'_i, N'_{i+1})$ *does not fork over* N_i *for* $i < \alpha$.

2) Above assume in addition that there are M, b^* *such that* $N_0 \leq_{\mathfrak{k}} M \in K_\lambda, b^* \in M$ *and* $\mathrm{ortp}(b^*, N_0, M) = q$. *Then we can add: there is a* $\leq_{\mathfrak{k}}$*-embedding of* M *into* N'_0 *over* N_0 *mapping* b^* *to* b.

Proof. Straight (remembering Axiom (E)(i) on non-forking amalgamation of Definition 2.1). In details

1) Let M_0, b^* be such that $N_0 \leq_{\mathfrak{k}[\mathfrak{s}]} M_0$ and $q = \mathrm{ortp}(b^*, N_0, M_0)$ and apply part (2).

2) We choose (M_i, f_i) by induction on $i \leq \alpha$ such that

$\circledast$ (a) $M_i \in \mathfrak{k}_\mathfrak{s}$ is $\leq_\mathfrak{k}$-increasing continuous.
(b) f_i is a $\leq_\mathfrak{k}$-embedding of N_i into M_i.
(c) f_i is increasing continuous with $i \leq \alpha$.
(d) $M_0 = M$ and $f_0 = \mathrm{id}_{N_0}$.
(e) $\mathrm{ortp}(b^*, f_i(N_i), M_i)$ does not fork over N_0.
(f) $\mathrm{ortp}(f_{i+1}(a_i), M_i, M_{i+1})$ does not fork over $f_i(N_i)$.

For $i = 0$ there is nothing to do. For i limit take unions; clause (e) holds by Ax(E)(h). Lastly, for $i = j + 1$, we can find (M'_i, f'_i) such that $f_j \subseteq f'_i$ and f'_i is an isomorphism from N_i onto M. Hence $f_j(N_j) \leq_{\mathfrak{k}[\mathfrak{s}]} N'_i$. Now use Ax(E)(i) for $f_j(N_j), M'_i, N_i, f'_i(a_j), b^*$.

Having carried the induction, we rename to finish. $\square_{4.10}$

In the claim below, we are given a $\leq_{\mathfrak{k}_\lambda}$-increasing continuous $\langle M_i : i \leq \delta\rangle$ and $u_0, u_1, u_2 \subseteq \delta$ such that: u_0 is where we are already given $a_i \in M_{i+1} \setminus M_i, u_1 \subseteq \delta$ is where we shall choose $a_i (\in M'_{i+1} \setminus M'_i)$ and $u_2 \subseteq \delta$ is the place which we "leave for future use"; main case is $u_1 = \delta; u_0 = u_2 = \emptyset$.

Claim 4.11. *1) Assume*

(a) $\delta < \lambda^+$ *is divisible by* λ
(b) u_0, u_1, u_2 *are disjoint subsets of* δ
(c) $\delta = \sup(u_1)$ *and* $\mathrm{otp}(u_1)$ *is divisible by* λ
(d) $\langle M_i : i \leq \delta\rangle$ *is* $\leq_\mathfrak{k}$*-increasing continuous in* $\mathfrak{k}_\lambda$
(e) $\bar{\mathbf{a}} = \langle a_i : i \in u_0\rangle, a_i \in M_{i+1} \setminus M_i, \mathrm{ortp}(a_i, M_i, M_{i+1}) \in \mathcal{S}^{\mathrm{bs}}(M_i)$.

<u>*Then*</u> *we can find* $\overline{M}' = \langle M'_i : i \leq \delta\rangle$ *and* $\bar{\mathbf{a}}' = \langle a_i : i \in u_1\rangle$ *such that*

(α) $\overline{M}'$ *is* $\leq_\mathfrak{k}$*-increasing continuous in* K_λ
(β) $M_i \leq_\mathfrak{k} M'_i$
(γ) *if* $i \in u_0$ *then* $\mathrm{ortp}(a_i, M'_i, M'_{i+1})$ *is a non-forking extension of* $\mathrm{ortp}(a_i, M_i, M_{i+1})$
(δ) *if* $i \in u_2$ *then* $M_i = M_{i+1} \Rightarrow M'_i = M'_{i+1}$
(ε) *if* $i \in u_1$ *then* $\mathrm{ortp}(a_i, M'_i, M'_{i+1}) \in \mathcal{S}^{\mathrm{bs}}(M'_i)$
(ζ) *if* $i < \delta, p \in \mathcal{S}^{\mathrm{bs}}(M'_i)$ *then for* λ *ordinals* $j \in u_1 \cap (i, \delta)$ *the type* $\mathrm{ortp}(a_j, M'_j, M'_{j+1})$ *is a non-forking extension of* p.

2) If we add in part (1) the assumption

(g) $M_0 \leq_\mathfrak{k} N \in K_\lambda$

<u>*then*</u> *we can add to the conclusion*

(η) *there is an* $\leq_\mathfrak{k}$*-embedding* f *of* N *into* M'_0 *over* M_0 *and moreover* f *is onto.*

3) If we add in part (1) the assumption

$(h)^+$ $M_0 \leq_\mathfrak{k} N \in K_\lambda$ *and* $b \in N \setminus M_0, \mathrm{ortp}(b, M_0, N) \in \mathcal{S}^{\mathrm{bs}}(M_0)$

<u>*then*</u> *we can add to the conclusion*

$(\eta)^+$ *as in* (η) *and* $\mathrm{ortp}(f(b), M_\delta, M'_\delta)$ *does not fork over* M_0.

4) We can strengthen clause (ζ) *in part (1) to*

$(\zeta)^+$ *if* $i < \delta$ *and* $p \in \mathcal{S}^{\rm bs}(M'_i)$ *then for* λ *ordinals* j *we have* $j \in [i, \delta) \cap u_1$ *and* $\operatorname{ortp}(a_j, M'_j, M'_{j+1})$ *is a non-forking extension of* p *and* $\operatorname{otp}(u_1 \cap j \setminus i) < \lambda$.

Proof. Straight like 4.10(2). Note that we can find a sequence $\langle u_{1,i,\varepsilon} : i < \delta, \varepsilon < \lambda\rangle$ such that: this is a sequence of pairwise disjoint subsets of u_1 each of cardinality λ satisfying $u_{1,i,\varepsilon} \subseteq \{j : i < j, j \in u_1 \text{ and } |u_1 \cap (i,j)| < \lambda\}$ (or we can demand that $i \le i_1 < i_2 \le \delta \wedge |u_1 \cap (i_1, i_2)| = \lambda \Rightarrow |u_{1,i,\varepsilon} \cap (i_1, i_2)| = \lambda$). $\square_{4.11}$

Toward building our rectangles of models with sides of difference lengths (and then we shall use 4.8) we show (to understand the aim of the clauses in the conclusion of 4.12 see the proof of 4.9 below):

Claim 4.12. *Assume*

(a) $\delta_\ell < \lambda^+$ *is divisible by* λ *for* $\ell = 1, 2$

(b) $\overline{M}^\ell = \langle M^\ell_\alpha : \alpha \le \delta_\ell\rangle$ *is* $\le_{\mathfrak{k}}$*-increasing continuous for* $\ell = 1, 2$

(c) $u^\ell_0, u^\ell_1, u^\ell_2$ *are disjoint subsets of* $\delta_\ell, \operatorname{otp}(u^\ell_1)$ *is divisible by* λ *and* $\delta_\ell = \sup(u^\ell_1)$ *for* $\ell = 1, 2$

(d) $\bar{\mathbf{a}}^\ell \equiv \langle a^\ell_\alpha : \alpha \in u^\ell_0\rangle$ *and* $\operatorname{ortp}(a^\ell_\alpha, M^\ell_\alpha, M^\ell_{\alpha+1}) \in \mathcal{S}^{\rm bs}(M^\ell_\alpha)$ *for* $\ell = 1, 2, \alpha \in u^\ell_0$

(e) $M^1_0 = M^2_0$

(f) $\alpha \in u^\ell_1 \cup u^\ell_2 \Rightarrow M^\ell_\alpha = M^\ell_{\alpha+1}$ *for* $\ell = 1, 2$.

Then we can find $\bar{f}^\ell = \langle f^\ell_\alpha : \alpha \le \delta_\ell\rangle, \bar{\mathbf{b}}^\ell = \langle b^\ell_\alpha : \alpha \in u^\ell_0 \cup u^\ell_1\rangle$ *for* $\ell = 1, 2$ *and* $\overline{M} = \langle M_{\alpha,\beta} : \alpha \le \delta_1, \beta \le \delta_2\rangle$ *and functions* $\zeta : u^1_1 \to \delta_2$ *and* $\varepsilon : u^2_1 \to \delta_1$ *such that*

$(\alpha)_1$ *for each* $\alpha \le \delta_1, \langle M_{\alpha,\beta} : \beta \le \delta_2\rangle$ *is* $\le_{\mathfrak{k}}$*-increasing continuous*

$(\alpha)_2$ *for each* $\beta \le \delta_2, \langle M_{\alpha,\beta} : \alpha \le \delta_1\rangle$ *is* $\le_{\mathfrak{k}}$*-increasing continuous*

$(\beta)_1$ *for* $\alpha \in u^1_0, b^1_\alpha$ *belongs to* $M_{\alpha+1,0}$ *and* $\operatorname{ortp}(b^1_\alpha, M_{\alpha,\delta_2}, M_{\alpha+1,\delta_2}) \in \mathcal{S}^{\rm bs}(M_{\alpha,\delta_2})$ *does not fork over* $M_{\alpha,0}$

$(\beta)_2$ *for* $\beta \in u^2_0, b^2_\beta$ *belongs to* $M_{0,\beta+1}$ *and* $\operatorname{ortp}(b^2_\beta, M_{\delta_1,\beta}, M_{\delta_1,\beta+1}) \in \mathcal{S}^{\rm bs}(M_{\delta_1,\beta})$ *does not fork over* $M_{0,\beta}$

$(\gamma)_1$ *for* $\alpha \in u^1_1, \zeta(\alpha) < \delta_2$ *and we have* $b^1_\alpha \in M_{\alpha+1,\zeta(\alpha)+1}$ *and* $\operatorname{ortp}(b^1_\alpha, M_{\alpha,\delta_2}, M_{\alpha+1,\delta_2})$ *does not fork over* $M_{\alpha,\zeta(\alpha)+1}$

$(\gamma)_2$ *for* $\beta \in u^2_1, \varepsilon(\beta) < \delta_1$ *and we have* $b^2_\beta \in M_{\varepsilon(\beta)+1,\beta+1}$ *and* $\operatorname{ortp}(b^2_\beta, M_{\delta_1,\beta}, M_{\delta_1,\beta+1})$ *does not fork over* $M_{\varepsilon(\beta)+1,\beta}$

$(\delta)_1$ *if* $\alpha < \delta_1, \beta < \delta_2$ *and* $p \in \mathcal{S}^{\rm bs}(M_{\alpha,\beta})$ *or just* $p \in \mathcal{S}^{\rm bs}(M_{\alpha,\beta+1})$ *<u>then</u> for* λ *ordinals* [17] $\alpha' \in [\alpha, \delta_1) \cap u^1_1$*, the type* $\operatorname{ortp}(b^1_{\alpha'}, M_{\alpha',\beta+1}, M_{\alpha+1,\beta+1})$ *is a (well defined) non-forking extension of* p *and* $\beta = \zeta(\alpha')$

$(\delta)_2$ *if* $\alpha < \delta_1, \beta < \delta_2$ *and* $p \in \mathcal{S}^{\rm bs}(M_{\alpha,\beta})$ *or just* $p \in \mathcal{S}^{\rm bs}(M_{\alpha+1,\beta})$ *<u>then</u> for* λ *ordinals* [18] $\beta' \in [\beta, \delta_2) \cap u^2_1$*, the type* $\operatorname{ortp}(b^2_{\beta'}, M_{\alpha+1,\beta'}, M_{\alpha+1,\beta'+1})$ *is a non-forking extension of* p *and* $\alpha = \varepsilon(\beta')$

(ε) $M_{0,0} = M^1_0 = M^2_0$

$(\zeta)_1$ f^1_α *is an isomorphism from* M^1_α *onto* $M_{\alpha,0}$ *such that* $\alpha \in u^1_0 \Rightarrow f^1_\alpha(a^1_\alpha) = b^1_\alpha$ $f^1_0 = \operatorname{id}_{M^1_0}$ *and* f^1_α *is increasing continuous with* α

$(\zeta)_2$ f^2_β *is an isomorphism from* M^2_β *onto* $M_{0,\beta}$ *such that* $\beta \in u^2_0 \Rightarrow f^2_\beta(a^2_\beta) = b^2_\beta$ $f^2_0 = \operatorname{id}_{M^2_0}$ *and* f^2_α *is increasing continuous with* α

$(\eta)_1$ *if* $\alpha \in u^1_2$ *then* $M_{\alpha,\beta} = M_{\alpha+1,\beta}$ *for every* $\beta \le \delta_2$

[17] we can add "and $\operatorname{otp}(\alpha' \cap u^1_1 \setminus \alpha_2) < \lambda$"
[18] we can add "and $\operatorname{otp}(\beta' \cap u^2_1 \setminus \beta_2) < \lambda$"

$(\eta)_2$ *if* $\beta \in u^2_2$ *then* $M_{\alpha,\beta} = M_{\alpha,\beta+1}$ *for every* $\alpha \le \delta_1$.

Proof. Straight, divide u^ℓ_1 to $\delta_{3-\ell}$ subsets large enough), in fact, we can first choose the function $\zeta(-), \varepsilon(-)$. Now choose $\langle M_{\alpha,\beta} : \alpha \le \delta_1, \beta \le \beta^* \rangle, \langle f^1_\alpha : \alpha \le \delta_1 \rangle, \langle f^2_\beta : \beta \le \beta^* \rangle$ and $\langle b^1_\alpha : \zeta(\alpha) \in \beta^* \rangle, \langle b^2_\beta : \beta < \beta^* \rangle$ by induction on β^* using 4.11. $\square_{4.12}$

Proof. [Proof of 4.9] By 1.17(3), i.e., uniqueness of the (λ, θ_ℓ)-brimmed model over M, it is enough to show for any regular $\theta_1, \theta_2 \le \lambda$ that there is a model $N \in K_\lambda$ which is (λ, θ_ℓ)-brimmed over M for $\ell = 1, 2$. Let $\delta_1 = \lambda \times \theta_1, \delta_2 = \lambda \times \theta_2$ (ordinal multiplication, of course), $M^1_\alpha = M^2_\beta = M$ for $\alpha \le \delta_1, \beta \le \delta_2, u^1_0 = u^2_0 = \varnothing, u^1_1 = \delta_1, u^2_1 = \delta_2, u^1_2 = u^2_2 = \varnothing$. So there are $\langle M_{\alpha,\beta} : \alpha \le \delta_1, \beta \le \delta_2 \rangle, \langle b^1_\alpha : \alpha < \delta_1 \rangle, \langle b^2_\beta : \beta < \delta_2 \rangle$ and $\langle f^1_\alpha : \alpha \le \delta_1 \rangle, \langle f^2_\beta : \beta \le \delta_2 \rangle$ as in Claim 4.12. Without loss of generality $f^1_\alpha = f^2_\alpha = \mathrm{id}_M$. Now

- $(*)_1$ $\langle M_{\alpha,\delta_2} : \alpha \le \delta_1 \rangle$ is $\le_{\mathfrak{k}}$-increasing continuous in K_λ (by clause $(\alpha)_1$, of 4.12). Also
- $(*)_2$ if $\alpha < \delta_1$ and $p \in \mathcal{S}(M_{\alpha,\delta_2})$ <u>then</u> for λ ordinals $\alpha' \in (\alpha, \delta_1) \cap u^1_1$ the type $\mathrm{ortp}(b^1_{\alpha',\delta_2}, M_{\alpha',\delta_2}, M_{\alpha'+1,\delta_2})$ is a non-forking extension of p.

(Easy, by Axiom (E)(c) for some $\beta < \delta_2, p$ does not fork over $M_{\alpha,\beta+1}$ and use clause $(\delta)_1$ of 4.12).

So by 4.8, M_{δ_1,δ_2} is $(\lambda, \mathrm{cf}(\delta_1))$-brimmed over M_{0,δ_2} which is M.

Similarly M_{δ_1,δ_2} is $(\lambda, \mathrm{cf}(\delta_2))$-brimmed over $M_{\delta_1,0}$ which is M; so together we are done. $\square_{4.9}$

Claim 4.13. *1) If $M \in K_{\lambda^+}$ and $p \in \mathcal{S}^{\mathrm{bs}}(M_0), M_0 \le_{\mathfrak{k}} M$ (so $M_0 \in K_\lambda$), <u>then</u> we can find $b, \langle N^0_\alpha : \alpha \le \lambda^+ \rangle$ and $\langle N^1_\alpha : \alpha \le \lambda^+ \rangle$ such that*

- *(a) $\langle N^0_\alpha : \alpha < \lambda^+ \rangle$ is a $\le_{\mathfrak{k}}$-representation of $N^0_{\lambda^+} = M$*
- *(b) $\langle N^1_\alpha : \alpha < \lambda^+ \rangle$ is a $\le_{\mathfrak{k}}$-representation of $N^1_{\lambda^+} \in K_{\lambda^+}$*
- *(c) $N^1_{\alpha+1}$ is (λ, λ)-brimmed over N^1_α (hence $N^1_{\lambda^+}$ is saturated over λ in $\mathfrak{k}$)*
- *(d) $M_0 \le N^0_0$ and $N^0_\alpha \le_{\mathfrak{k}} N^1_\alpha$*
- *(e) $\mathrm{ortp}_{\mathfrak{s}}(b, N^0_\alpha, N^1_\alpha)$ is a non-forking extension of p for every $\alpha < \lambda^+$.*

2) We can add

- *(f) for $\alpha < \beta < \lambda^+, N^1_\beta$ is $(\lambda, *)$-brimmed over $N^0_\beta \cup N^1_\alpha$.*

Proof. 1) Easy by long non-forking amalgamation 4.10 (see 1.18).
2) Use 4.8. $\square_{4.13}$

Conclusion 4.14. *1) $K_{\lambda^{++}} \ne \varnothing$.*
2) $K_{\lambda^+} \ne \varnothing$.
3) No $M \in K_{\lambda^+}$ is $\le_{\mathfrak{k}}$-maximal.

Proof. 1) By (2) + (3).
2) By (B) of 2.1.
3) By 4.13. $\square_{4.14}$

Exercise 4.15. : 1) Let $M \in K_{\mathfrak{s}}$ be superlimit and $\mathfrak{t} = \mathfrak{s}_{[M]}$, so $K_{\mathfrak{t}}$ is categorical. If $(M, N, a) \in K^{\rm bs}_{\mathfrak{t}}$ is reduced for $\mathfrak{t}$, then it is reduced for $\mathfrak{s}$.

2) In 4.7(3),(4),(5), we can omit the assumption "$\mathfrak{s}$ is categorical" if:

- (a) we add in part (3), each N_i is superlimit (equivalently brimmed)
- (b) in parts (4),(5) add the assumption "M_0 is superlimit".

2) Some extra assumption in 4.7(5) is needed.

§ 5. Non-structure or some unique amalgamation

We shall assuming $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ get from essentially $\dot I(\lambda^{++}, K) < 2^{\lambda^{++}}$ pedantically $< \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$ or just $\dot I(\lambda^{++}, K(\lambda^+\text{-saturated})) < \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$, many cases of uniqueness of amalgamation assuming in addition $\mathrm{WDmId}(\lambda^+)$ is not λ^{++}-saturated, a weak assumption. The proof is similar to [She], [She01, §3] but now we rely on [She09d], the "lean" version; and by the "full version" without we can eliminate the additional assumption.

We define $K^{3,\rm bt}_\lambda$, it is a brimmed relative of $K^{3,\rm bs}_\lambda$ hence the choice of bt; it guarantees much brimness (see Definition 5.2) hence it guarantees some uniqueness, that is, if $(M, N, a) \in K^{3,\rm bt}_\lambda$, M is unique (recalling the uniqueness of the brimmed model) and more crucially, we consider $K^{3,\rm uq}_\lambda$, (the family of members of $K^{3,\rm bs}_\lambda$ for which we have uniqueness in relevant extensions). Having enough such triples is the main conclusion of this section (in 5.9 under "not too many non-isomorphic models" assumptions). In 5.4 we give some properties of $K^{3,\rm bt}_\lambda, K^{3,\rm uq}_\lambda$.

To construct models in λ^{++} we use approximations of cardianlity in λ^+ with "obligation" on the further construction, which are presented as pairs $(\overline{M}, \bar{\mathbf a}) \in K^{\rm sq}_\lambda$ ordered by $\le_{\rm ct}$, see Definition 5.5, Claims 5.6, 5.7. We need more: the triples $(\overline{M}, \bar{\mathbf a}, \mathbf f) \in K^{\rm mqr}_S, K^{\rm nqr}_S$ in Definition 5.12, Claim 5.13. All this enables us to quote results of [She01, §3] or better [She09d, §2], but apart from believing the reader do not need to know non of them.

Hypothesis 5.1. (a) $\mathfrak s = (\mathfrak k, \bigcup, \mathcal S^{\rm bs})$ is a good λ-frame.

Definition 5.2. 1) Let $K^{3,\rm bt}_\lambda = K^{3,\rm bt}_{\mathfrak s}$ be the set of triples (M, N, a) such that for some $\partial = \mathrm{cf}(\partial) \le \lambda$, $M \le_{\mathfrak k} N$ are both (λ, ∂)-brimmed members of K_λ, $a \in N \setminus M$ and $\mathrm{ortp}(a, M, N) \in \mathcal S^{\rm bs}(M)$.

2) For $(M_\ell, N_\ell, a_\ell) \in K^{3,\rm bt}_\lambda$ for $\ell = 1, 2$ let $(M_1, N_1, a_1) <_{\rm bt} (M_2, N_2, a_2)$ mean $a_1 = a_2$, $\mathrm{ortp}(a_1, M_2, N_2)$ does not fork over M_1 and for some $\partial_2 = \mathrm{cf}(\partial_2) \le \lambda$, the model M_2 is (λ, ∂_2)-brimmed over M_1 and the model N_2 is (λ, ∂_2)-brimmed over N_1. Finally $(M_1, N_1, a_2) \le_{\rm bt} (M_2, N_2, a_2)$ means $(M_1, N_1, a_1) <_{\rm bt} (M_2, N_2, a_2)$ or $(M_1, N_1, a_1) = (M_2, N_2, a_2)$.

Definition 5.3. 1) Let "$(M_0, M_2, a) \in K^{3,\rm uq}_\lambda$" mean: $(M_0, M_2, a) \in K^{3,\rm bs}_\lambda$ and: for every M_1 satisfying $M_0 \le_{\mathfrak k} M_1 \in K_\lambda$, the amalgamation M of M_1, M_2 over M_0, with $\mathrm{ortp}(a, M_1, M)$ not forking over M_0, is unique, that is:

(∗) <u>if</u> for $\ell = 1, 2$ we have $M_0 \le_{\mathfrak k} M_1 \le_{\mathfrak k} M^\ell \in K_\lambda$ and f_ℓ is a $\le_{\mathfrak k}$-embedding of M_2 into M^ℓ over M_0 (so $f_1 \restriction M_0 = f_2 \restriction M_0 = \mathrm{id}_{M_0}$) such that $\mathrm{ortp}(f_\ell(a), M_1, M^\ell)$ does not fork over M_0, <u>then</u>
- (a) [uniqueness]: for some M', g_1, g_2 we have: $M_1 \le_{\mathfrak k} M' \in K_\lambda$ and g_ℓ is a $\le_{\mathfrak k}$-embedding of M^ℓ into M' over M_1 for $\ell = 1, 2$ such that $g_1 \circ f_1 \restriction M_2 = g_2 \circ f_2 \restriction M_2$
- (b) [being reduced] $f_\ell(M_2) \cap M_1 = M_0$ [this is "for free" in the proofs; and is not really necessary so the decision if to include it is not important but simplify notation, but see 5.4(3)].

2) $K^{3,\rm uq}_\lambda$ is dense (or $\mathfrak s$ has density for $K^{3,\rm uq}_\lambda$) when $K^{3,\rm uq}_\lambda$ is dense in $(K^{3,\rm bs}_\lambda, \le_{\rm bs})$, i.e., for every $(M_1, M_2, a) \in K^{3,\rm bs}_\lambda$ there is $(M_1, N_2, a) \in K^{3,\rm uq}_\lambda$ such that $(M_1, M_2, a) \le_{\rm bs} (N_1, N_2, a) \in K^{3,\rm uq}_\lambda$.

3) $K^{3,\mathrm{uq}}_\lambda$ has existence or $\mathfrak{s}$ has existence for $K^{3,\mathrm{uq}}_\lambda$ when for every $M_0 \in K_\lambda$ and $p \in \mathcal{S}^{\mathrm{bs}}(M_0)$ for some M_1, a we have $(M_0, M_1, a) \in K^{3,\mathrm{uq}}_\lambda$ and $p = \mathrm{ortp}(a, M_0, M_1)$.
4) $K^{3,\mathrm{uq}}_{\mathfrak{s}} = K^{3,\mathrm{uq}}_\lambda$.

Claim 5.4. *1) The relation $\le_{\mathrm{bt}}$ is a partial order on $K^{3,\mathrm{bt}}_\lambda$ that is transitive and reflexive (but not necessarily satisfying the parallel of Ax V of AEC see Definition 1.4).*
2) If $(M_\alpha, N_\alpha, a) \in K^{3,\mathrm{bt}}_\lambda$ is $\le_{\mathrm{bt}}$-increasing continuous for $\alpha < \delta$ where δ is a limit ordinal $< \lambda^+$ then $(M, N, a) = (\bigcup_{\alpha<\delta} M_\alpha, \bigcup_{\alpha<\delta} N_\alpha, a)$ belongs to $K^{3,\mathrm{bt}}_\lambda$ and $\alpha < \delta \Rightarrow (M_\alpha, N_\alpha, a) \le_{\mathrm{bt}} (M, N, a)$ and so (M, N, a) is a $\le_{\mathrm{bt}}$-upper bound of $\langle (M_\alpha, N_\alpha, a) : \alpha < \delta \rangle$.
3) In $(*)$ of 5.3(1), clause (b) follows from (a).

Proof. Easy; e.g. (3) by the uniqueness (i.e., clause (a)) and 4.7(4). $\square_{5.4}$

We now define $K^{\mathrm{sq}}_{\lambda^+}$, a family of $\le_{\mathfrak{k}}$-increasing continuous sequences (the reason for sq) in K_λ of length λ^+, will be used to approximate stages in constructing models in $K_{\lambda^{++}}$.

Definition 5.5. 1) Let $K^{\mathrm{sq}}_{\lambda^+} = K^{\mathrm{sq}}_{\mathfrak{s}}$ be the set of pairs $(\overline{M}, \bar{\mathbf{a}})$ such that (sq stands for *sequence*):

- (a) $\overline{M} = \langle M_\alpha : \alpha < \lambda^+ \rangle$ is a $\le_{\mathfrak{k}}$-increasing continuous sequence of models from K_λ
- (b) $\bar{\mathbf{a}} = \langle a_\alpha : \alpha \in S \rangle$, where $S \subseteq \lambda^+$ is stationary in λ^+ and $a_\alpha \in M_{\alpha+1} \setminus M_\alpha$
- (c) for some club E of λ^+ for every $\alpha \in S \cap E$ we have $\mathrm{ortp}(a_\alpha, M_\alpha, M_{\alpha+1}) \in \mathcal{S}^{\mathrm{bs}}(M_\alpha)$
- (d) if $p \in \mathcal{S}^{\mathrm{bs}}(M_\alpha)$ then for stationarily many $\delta \in S$ we have: $\mathrm{ortp}(a_\delta, M_\delta, M_{\delta+1}) \in \mathcal{S}^{\mathrm{bs}}(M_\delta)$ does not fork over M_α and extends p.

In such cases we let $M = \bigcup_{\alpha<\lambda^+} M_\alpha$.

2) When for $\ell = 1, 2$ we are given $(\overline{M}^\ell, \bar{\mathbf{a}}^\ell) \in K^{\mathrm{sq}}_{\lambda^+}$ we say $(\overline{M}^1, \bar{\mathbf{a}}^1) \le_{\mathrm{ct}} (\overline{M}^2, \bar{\mathbf{a}}^2)$ if for some club E of λ^+, letting $\bar{\mathbf{a}}^\ell = \langle a^\ell_\delta : \delta \in S^\ell \rangle$ for $\ell = 1, 2$, of course, we have

- (a) $S^1 \cap E \subseteq S^2 \cap E$
- (b) if $\delta \in S^1 \cap E$ then
  - (α) $M^1_\delta \le_{\mathfrak{k}} M^2_\delta$,
  - (β) $M^1_{\delta+1} \le_{\mathfrak{k}} M^2_{\delta+1}$
  - (γ) $a^2_\delta = a^1_\delta$
  - (δ) $\mathrm{ortp}(a^1_\delta, M^2_\delta, M^2_{\delta+1})$ does not fork over M^1_δ, so in particular $a^1_\delta \notin M^2_\delta$.

Observation 5.6. *1) If $(\overline{M}, \bar{\mathbf{a}}) \in K^{\mathrm{sq}}_{\lambda^+}$ then $M := \bigcup_{\alpha<\lambda^+} M_\alpha \in K_{\lambda^+}$ is saturated.*

2) $K^{\mathrm{sq}}_{\lambda^+}$ is partially ordered by $\le_{\mathrm{ct}}$. $\square_{5.6}$

Claim 5.7. *Assume $\langle (\overline{M}^\zeta, \bar{\mathbf{a}}^\zeta) : \zeta < \zeta^* \rangle$ is $\le_{\mathrm{ct}}$-increasing in $K^{\mathrm{sq}}_{\lambda^+}$, and ζ^* is a limit ordinal $< \lambda^{++}$, then the sequence has a $\le_{\mathrm{ct}}$-l.u.b. $(\overline{M}, \bar{\mathbf{a}})$.*

Proof. Let $\bar{\mathbf{a}}^\zeta = \langle a^\zeta_\delta : \delta \in S_\zeta \rangle$ for $\zeta < \zeta^*$ and without loss of generality $\zeta^* = \mathrm{cf}(\zeta^*)$ and for $\zeta < \xi < \zeta^*$ let $E_{\zeta,\xi}$ be a club of λ^+ consisting of limit ordinals witnessing $(\overline{M}^\zeta, \bar{\mathbf{a}}^\zeta) \le_{\mathrm{ct}} (\overline{M}^\xi, \bar{\mathbf{a}}^\xi)$, i.e. as in 5.5(2).

Case 1: $\zeta^* < \lambda^+$.

Let $E = \cap\{E_{\zeta,\xi} : \zeta < \xi < \zeta^*\}$ and for $\delta \in E$ let $M_\delta = \cup\{M^\zeta_\delta : \zeta < \zeta^*\}$ and $M_{\delta+1} = \cup\{M^\zeta_{\delta+1} : \zeta < \zeta^*\}$ and for any other $\alpha, M_\alpha = M_{\min(E \setminus \alpha)}$. Let $S = \bigcup\limits_{\zeta<\zeta^*} S_\zeta \cap E$ and for $\delta \in S$ let $a_\delta = a^\zeta_\delta$ for every ζ for which $\delta \in S_\zeta$. Clearly $M_\alpha \in K_\lambda$ is $\le_{\mathfrak{k}}$-increasing continuous and $\zeta < \zeta^* \wedge \delta \in E \Rightarrow M^\zeta_\delta \le_{\mathfrak{k}} M_\delta$ and $M^\zeta_{\delta+1} \le_{\mathfrak{k}} M_{\delta+1}$.

Now if $\delta \in E \cap S_\zeta$ then $\xi \in [\zeta, \zeta^*)$ implies $\mathrm{ortp}(a_\delta, M^\xi_\delta, M_{\delta+1}) = \mathrm{ortp}(a^\zeta_\delta, M^\xi_\delta, M^\xi_{\delta+1})$ does not fork over M^ζ_δ (and $\langle M^\xi_\delta : \xi \in [\zeta, \delta) \rangle, \langle M^\xi_{\delta+1} : \xi \in [\zeta, \delta) \rangle$ are $\le_{\mathfrak{k}}$-increasing continuous); hence by Axiom (E)(h) we know that $\mathrm{ortp}(a_\delta, M_\delta, M_{\delta+1})$ does not fork over M^ζ_δ and in particular $\in \mathcal{S}^{\mathrm{bs}}(M_\delta)$. Also if $N \le_{\mathfrak{k}} M := \bigcup\limits_{\alpha<\lambda^+} M_\alpha, N \in K_\lambda$ and $p \in \mathcal{S}^{\mathrm{bs}}(N)$ then for some $\delta(*) \in E, N \le_{\mathfrak{k}} M_{\delta(*)}$, let $p_1 \in \mathcal{S}^{\mathrm{bs}}(M_{\delta(*)})$ be a non-forking extension of p, so for some $\zeta < \zeta^*, p$ does not fork over $M^\zeta_{\delta(*)}$ hence for stationarily many $\delta \in S_\zeta, q^0_\delta = \mathrm{ortp}(a_\delta, M^\zeta_\delta, M^\zeta_{\delta+1})$ is a non-forking extension of $p_1 \restriction M^\zeta_{\delta(*)}$, hence this holds for stationarily many $\delta \in S \cap E$ and for each such $\delta, q_\delta = \mathrm{ortp}(a_\delta, M_\delta, M_{\delta+1})$ is a non-forking extension of $p_1 \restriction M^\zeta_{\delta(*)}$, hence of p_1 hence of p. Looking at the definitions, clearly $(\overline{M}, \bar{\mathbf{a}}) \in K^{\mathrm{sq}}_{\lambda^+}$ and $\zeta < \zeta^* \Rightarrow (\bar{M}^\zeta, \bar{\mathbf{a}}^\zeta) \le_{\mathrm{ct}} (\overline{M}, \bar{\mathbf{a}})$.

Lastly, it is easy to check the $\le_{\mathrm{ct}}$-l.u.b.

Case 2: $\zeta^* = \lambda^+$.

Similarly using diagonal union, i.e., $E = \{\delta < \lambda^+ : \delta$ is a limit ordinal such that $\zeta < \xi < \delta \Rightarrow \delta \in E_{\zeta,\varepsilon}\}$ and we choose $M_\alpha = \cup\{M^\zeta_\alpha : \zeta < \alpha\}$ when $\alpha \in E$ and $M_\alpha = M_{\min(E\setminus(\alpha+1))}$ otherwise. $\square_{5.7}$

Observation 5.8. *Assume $K^{3,\mathrm{uq}}_\lambda$ is dense in $K^{3,\mathrm{bs}}_\lambda$, i.e., in $(K^{3,\mathrm{bs}}_\lambda, \le_{\mathrm{bs}})$ and even in $(K^{3,\mathrm{bt}}_\lambda, <_{\mathrm{bt}})$. Then*

(a) *if $M \in K_\lambda$ is superlimit and $p \in \mathcal{S}^{\mathrm{bs}}(M)$ then there are N, a such that $(M, N, a) \in K^{3,\mathrm{uq}}_\lambda$ and $p = \mathrm{ortp}(a, M, N)$*

(b) *if in addition $K_{\mathfrak{s}}$ is categorical (in λ) then $\mathfrak{s}$ has existence for $K^{3,\mathrm{uq}}_\lambda$ (recall that this means that for every $M \in K_{\mathfrak{s}}$ and $p \in \mathcal{S}^{\mathrm{bs}}(M)$ for some pair (N, a) we have $(M, N, a) \in K^{3,\mathrm{uq}}_\lambda$ and $p = \mathrm{ortp}(a, M, N)$).*

Proof. Should be clear. $\square_{5.8}$

Now the assumptions of 5.8 are justified by the following theorem (and the categoricity in (b) is justified by Claim 1.27).

Claim 5.9. *[First Main Claim] Assume that*

(a) *as in 5.1*

(b) $\mathrm{WDmId}(\lambda^+)$ *is not λ^{++}-saturated and*[19] $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$.

[19] alternatively the parallel versions for the definitional weak diamond, but not here

If $\dot{I}(\lambda^{++}, K) < \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+}$ or just $\dot{I}(\lambda^{++}, K(\lambda^+\text{-saturated})) < \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$, then for every $(M,N,a) \in K^{3,{\rm bs}}_\lambda$ there is $(M^*,N^*,a) \in K^{3,{\rm bt}}_\lambda$ such that $(M,N,a) <_{\rm bt} (M^*,N^*,a)$ and $(M^*,N^*,a) \in K^{3,{\rm uq}}_\lambda$.

Explanation 5.10. The reader who agrees to believe in 5.9 can ignore the rest of this section (though it can still serve as a good exercise).

Let $\langle S_\alpha : \alpha < \lambda^{++}\rangle$ be a sequence of subsets of λ^+ such that $\alpha < \beta \Rightarrow |S_\alpha \setminus S_\beta| \le \lambda$ and $S_{\alpha+1} \setminus S_\alpha \neq \varnothing$ mod $\text{WDmId}(\lambda^+)$, exists by assumption.

Why having (M,N,a) failing the conclusion of 5.9 helps us to construct many models in $K_{\lambda^{++}}$? The point is that we can choose $(\overline{M}^\alpha, \bar{\mathbf{a}}^\alpha) \in K^{\rm sq}_{\lambda^+}$ with $\text{Dom}(\bar{\mathbf{a}}^\alpha) = S_\alpha$ for $\alpha < \lambda^{++}$, $<_{\rm ct}$-increasing continuous (see 5.7).

Now for $\alpha = \beta+1$, having $(\overline{M}^\beta, \bar{\mathbf{a}}^\beta)$, without loss of generality M^β_{i+1} is brimmed over M^β_i and we shall choose M^α_i by induction on $i < \lambda^+$ (for simplicity we assume $M^\alpha_i \cap \cup\{M^\beta_j : j < \lambda^+\} = M^\beta_i$) and $M^\beta_i \le_{\mathfrak{k}} M^\alpha_i \in K_\lambda$ and $\text{ortp}(a^\beta_i, M^\alpha_i, M^\alpha_{i+1})$ does not fork over M^β_i and M^α_{i+1} is brimmed over M^α_i).

Given $(\overline{M}^\beta, \bar{\mathbf{a}}^\beta), \overline{M}^\beta = \langle M^\beta_i : i < \lambda^+\rangle, \bar{\mathbf{a}}^\beta = \langle a^\beta_i : i \in S_\beta\rangle$ we work toward building $(\overline{M}^\alpha, \bar{\mathbf{a}}^\alpha), \alpha_{\beta+1}$.

We start with choosing (M^α_0, b) such that no member of $K^{3,{\rm bs}}_\lambda$ which is $\le_{\rm bs}$-above $(M^\beta_0, M^\alpha_0, b) \in K^{3,{\rm bs}}_\lambda$ belongs to $K^{3,{\rm uq}}_\lambda$ and will choose M^β_i by induction on i such that $(M^\beta_i, M^\alpha_i, b) \in K^{3,{\rm bs}}_\lambda$ is $\le_{\rm bs}$-increasing continuous and even $<_{\rm bt}$-increasing hence in particular that $\text{ortp}(b, M^\beta_i, M^\alpha_i)$ does not fork over M^α_0. Now in each stage $i = j+1$, as M^β_i is universal over M^β_j, and the choice of M^α_0, b we have some freedom. So it makes sense that we will have many possible outcomes, i.e., models $M = \cup\{M^\alpha_i : \alpha < \lambda^{++}, i < \lambda^+\}$ which are in $K_{\lambda^{++}}$. The combination of what we have above and [She01, §3] better [She09d, §2] gives that $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ is enough to materialize this intuition. If in addition $2^\lambda = \lambda^+$ and moreover $\diamondsuit_{\lambda^+}$ it is considerably easier. In the end we still have to define $\bar{\mathbf{a}}^\alpha \restriction (S_\alpha \setminus S_\beta)$ as required in Definition 5.5, [GSar]. An alternative is to force a model in λ^{++}. Now below we replace $K^{3,{\rm sq}}_{\lambda^+}$ by $K^{\rm mqr}_{\lambda^+}, K^{\rm nqr}_S$ but actually $K^{3,{\rm sq}}_{\lambda^+}$ is enough. So we need a somewhat more complicated relative as elaborated below which anyhow seems to me more natural.

Claim 5.11. *[Second Main Claim] Assume $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}}$ (or the parallel versions for the definitional weak diamond). If $\dot{I}(\lambda^{++}, K(\lambda^+\text{-saturated})) < \mu_{\rm unif}(\lambda^{++}, 2^{\lambda^+})$, then for every $(M,N,a) \in K^{3,{\rm bt}}_\lambda$ there is $(M^*,N^*,a) \in K^{3,{\rm bt}}_\lambda$ such that $(M,N,a) <_{\rm bt} (M^*,N^*,a)$ and $(M^*,N^*,a) \in K^{3,{\rm uq}}_\lambda$.*

We shall not prove here 5.11 and shall not use it, it is proved in the full version of [She09d]; toward proving 5.9 (by quoting) let

Definition 5.12. Let $S \subseteq \lambda^+$ be a stationary subset of λ^+.

1) Let $K^{\rm mqr}_S$ or $K^{\rm mqr}_{\lambda^+}[S]$ be the set of triples $(\overline{M}, \bar{\mathbf{a}}, \mathbf{f})$ such that:

- (a) $\overline{M} = \langle M_\alpha : \alpha < \lambda^+\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous, $M_\alpha \in K_\lambda$ (we denote $\bigcup\limits_{\alpha<\lambda^+} M_\alpha$ by M) and demand $M \in K_{\lambda^+}$
- (b) $\bar{\mathbf{a}} = \langle a_\alpha : \alpha < \lambda\rangle$ with $a_\alpha \in M_{\alpha+1}$
- (c) $\mathbf{f}$ is a function from λ^+ to λ^+ such that for some club E of λ^+ for every $\delta \in E \cap S$ and ordinal $i < \mathbf{f}(\delta)$ we have $\text{ortp}(a_{\delta+i}, M_{\delta+i}, M_{\delta+i+1}) \in \mathcal{S}^{\rm bs}(M_{\delta+i})$

(*d*) for every $\alpha < \lambda^+$ and $p \in \mathcal{S}^{\rm bs}(M_\alpha)$, stationarily many $\delta \in S$ satisfies: for some $\varepsilon < \mathbf{f}(\delta)$ we have $\text{ortp}(a_{\delta+\varepsilon}, M_{\delta+\varepsilon}, M_{\delta+\varepsilon+1})$ is a non-forking extension of p.

1A) $K^{\rm nqr}_{\lambda^+}[S] = K^{\rm nqr}_S$ is the set of triples $(\overline{M}, \bar{\mathbf{a}}, \mathbf{f}) \in K^{\rm mqr}_S$ such that:

(*e*) for a club of $\delta < \lambda^+$, if $\delta \in S$ then $\mathbf{f}(\delta)$ is divisible by λ and [20] for every $i < \mathbf{f}(\delta)$ if $q \in \mathcal{S}^{\rm bs}(M_{\delta+i})$ then for λ ordinals $\varepsilon \in [i, \mathbf{f}(\delta))$ the type $\text{ortp}(a_{\delta+\varepsilon}, M_{\delta+\varepsilon}, M_{\delta+\varepsilon+1}) \in \mathcal{S}^{\rm bs}(M_{\delta+\varepsilon})$ is a stationarization of q (= non-forking extension of q, see Definition 4.5).

2) Assume $(\overline{M}^\ell, \bar{\mathbf{a}}^\ell, \mathbf{f}^\ell) \in K^{\rm mqr}_S$ for $\ell = 1, 2$; we say $(\overline{M}^1, \bar{\mathbf{a}}^1, \mathbf{f}^1) \le^0_S (\overline{M}^2, \bar{\mathbf{a}}^2, \mathbf{f}^2)$ <u>iff</u> for some club E of λ^+, for every $\delta \in E \cap S$ we have:

(*a*) $M^1_{\delta+i} \le_{\mathfrak{k}} M^2_{\delta+i}$ for [21] $i \le \mathbf{f}^1(\delta)$

(*b*) $\mathbf{f}^1(\delta) \le \mathbf{f}^2(\delta)$

(*c*) for $i < \mathbf{f}^1(\delta)$ we have $a^1_{\delta+i} = a^2_{\delta+i}$ and $\text{ortp}(a^1_{\delta+i}, M^2_{\delta+i}, M^2_{\delta+i+1})$ does not fork over $M^1_{\delta+i}$.

3) We define the relation $<^1_S$ on $K^{\rm mqr}_S$ as in part (2) adding

(*d*) if $\delta \in E$ and $i < \mathbf{f}^1(\delta)$ then $M^2_{\delta+i+1}$ is $(\lambda, *)$-brimmed over $M^1_{\delta+i+1} \cup M^2_{\delta+i}$.

Claim 5.13. *0) If $(\overline{M}, \bar{\mathbf{a}}, \mathbf{f}) \in K^{\rm mqr}_S$ <u>then</u> $\bigcup_{\alpha<\lambda^+} M_\alpha \in K_{\lambda^+}$ is saturated.*

1) The relation $\le^0_S$ is a quasi-order [22] on $K^{\rm mqr}_\lambda$; also $<^1_S$ is.

2) $K^{\rm mqr}_S \supseteq K^{\rm nqr}_S \neq \emptyset$ for any stationary $S \subseteq \lambda^+$.

3) For every $(\overline{M}, \bar{\mathbf{a}}, \mathbf{f}) \in K^{\rm mqr}_\lambda[S]$ for some $(\overline{M}', \bar{\mathbf{a}}, \mathbf{f}') \in K^{\rm nqr}_\lambda[S]$ we have $(\overline{M}, \bar{\mathbf{a}}, \mathbf{f}) <^1_S (\overline{M}', \bar{\mathbf{a}}, \mathbf{f}')$.

4) For every $(\overline{M}^1, \bar{\mathbf{a}}^1, \mathbf{f}^1) \in K^{\rm mqr}_S$ and $q \in \mathcal{S}^{\rm bs}(M^1_\alpha), \alpha < \lambda^+$, <u>there is</u> $(M^2, \bar{\mathbf{a}}^2, \mathbf{f}^2) \in K^{\rm mqr}_S$ such that $(\overline{M}^1, \bar{\mathbf{a}}^1, \mathbf{f}^1) <^1_S (\overline{M}^2, \bar{\mathbf{a}}^2, \mathbf{f}^2) \in K^{\rm nqr}_S$ and $b \in M^2_\alpha$ realizing q such that for every $\beta \in [\alpha, \lambda^+)$ we have $\text{ortp}(b, M^1_\beta, M^2_\beta) \in \mathcal{S}^{\rm bs}(M^1_\beta)$ does not fork over M^1_α.

5) If $\langle (\overline{M}^\zeta, \bar{\mathbf{a}}^\zeta, \mathbf{f}^\zeta) : \zeta < \xi(*) \rangle$ is $\le^0_S$-increasing continuous in $K^{\rm mqr}_S$ and $\xi(*) < \lambda^{++}$ a limit ordering, <u>then</u> the sequence has a $\le^0_S$-l.u.b..

Proof. 0, 1) Easy.

2) The inclusion $K^{\rm mqr}_S \supseteq K^{\rm nqr}_S$ is obvious, so let us prove $K^{\rm nqr}_S \neq \emptyset$. We choose by induction on $\alpha < \lambda^+, a_\alpha, M_\alpha, p_\alpha$ such that

(*a*) $M_\alpha \in K_\lambda$ is a super limit model,

(*b*) M_α is $\le_{\mathfrak{k}}$-increasingly continuous,

(*c*) if $\alpha = \beta + 1$, then $a_\beta \in M_\alpha \setminus M_\beta$ realizes $p_\beta \in \mathcal{S}^{\rm bs}(M_\beta)$,

(*d*) if $p \in \mathcal{S}^{\rm bs}(M_\alpha)$, then for some $i < \lambda$, for every $j \in [i, \lambda)$ for at least one ordinal $\varepsilon \in [j, j+i), p_{\alpha+\varepsilon} \restriction M_\alpha = p$ and $p_{\alpha+\varepsilon}$ does not fork over M_α.

For $\alpha = 0$ choose $M_0 \in K_\lambda$. For α limit, $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ is as required. Then use Axiom(E)(g) to take care of clause (d) (with careful bookkeeping). Lastly, let $\mathbf{f} : \lambda^+ \to \lambda^+$ be constantly $\lambda, \overline{M} = \langle M_\alpha : \alpha < \lambda \rangle, \bar{\mathbf{a}} = \langle a_\alpha : \alpha < \lambda \rangle$; now for any stationary $S \subseteq \lambda^+$, the triple $(\overline{M}, \bar{\mathbf{a}} \restriction S, \mathbf{f} \restriction S)$ belong to $K^{\rm nqr}_S$.

---

[20] if we have an a priori bound $\mathbf{f}^* : \lambda^+ \to \lambda^+$ which is a $<_{\mathcal{D}_{\lambda^+}}$-upper bound of the "first" λ^{++} functions in ${}^{\lambda^+}(\lambda^+)/D$, we can use bookkeeping for u_i's as in the proof of 4.11

[21] could have used (systematically) $i < \mathbf{f}^1(\delta)$

[22] quasi order $\le$ is a transitive relation, so we waive $x \le y \le x \Rightarrow x = y$

3) Let E be a club witnessing $(\overline{M}^1, \bar{\mathbf{a}}^1, \mathbf{f}^1) \in K_S^{\rm mqr}$ such that $\delta \in E \Rightarrow \delta + \mathbf{f}^1(\delta) < \min(E \backslash (\delta+1))$. Choose $\mathbf{f}^2 : \lambda^+ \to \lambda^+$ such that $\alpha < \lambda^+$ implies $\mathbf{f}^1(\alpha) < \mathbf{f}^2(\alpha) < \lambda^+$ and $\mathbf{f}^2(\alpha)$ is divisible by λ. We choose by induction on $\alpha < \lambda^+, f_\alpha, M^2_\alpha, p_\alpha, a^2_\alpha$ such that:

$(a),(b),(c)$ as in the proof of part (2)

(d) f_α is a $\le_{\mathfrak{k}}$-embedding of M^1_α into M^2_α

(e) f_α is increasing continuous

(f) if $\delta \in E \cap S$ and $i < \mathbf{f}^1(\delta)$ hence $\text{ortp}(a^1_{\delta+i}, M^1_{\delta+i}, M^1_{\delta+i+1}) \in \mathcal{S}^{\rm bs}(M^1_{\delta+i})$, <u>then</u> $f_{\delta+i+1}(a^1_{\delta+i}) = a^2_{\delta+i}$ and $p_{\varepsilon+i} = \text{ortp}(a^2_{\delta+i}, M^2_{\delta+i}, M^2_{\delta+i+1}) \in \mathcal{S}^{\rm bs}(M^2_{\delta+i})$ is a stationarization of $\text{ortp}\big(f_{\delta+i+1}(a^1_{\delta+i}), f_{\delta+i}(M^1_{\delta+i}), f_{\delta+i+1}(M^1_{\delta+i+1})\big) = \text{ortp}(a^2_{\delta+i}, f_{\delta+i}(M^1_{\delta+i}), M^2_{\delta+i+1})$

(g) if $\delta \in E$ and $i < \mathbf{f}^2(\delta), q \in \mathcal{S}^{\rm bs}(M^2_{\delta+i})$ then for some λ ordinals $\varepsilon \in (i, \mathbf{f}^2(\delta))$ the type $p_{\delta+\varepsilon}$ is a stationarization of q

(h) if $\delta \in E, i < \mathbf{f}^2(\delta)$ then $M_{\delta+i+1}$ is $(\lambda,*)$-brimmed over $M_{\delta+i} \cup f_{\delta+i+1}(M^1_{\delta+i+1})$.

The proof is as in part (2) only the bookkeeping is different. At the end without loss of generality $\bigcup\limits_{\alpha<\lambda^*} f_\alpha$ is the identity and we are done. 4) Similar proof but in some cases we have to use Axiom (E)(i), the non-forking amalgamation of Definition 2.1, in the appropriate cases.

5) Without loss of generality $\text{cf}(\xi(*)) = \xi(*)$. First assume that $\xi(*) \le \lambda$. For $\varepsilon < \zeta < \xi(*)$ let $E_{\varepsilon,\zeta}$ be a club of λ^+ witnessing $\overline{M}^\varepsilon <^0_S \overline{M}^\zeta$. Let $E^* = \bigcap\limits_{\varepsilon<\zeta<\xi(*)} E_{\varepsilon,\zeta} \cap \{\delta < \lambda^+ : \text{for every } \alpha < \delta \text{ we have } \underset{\varepsilon<\xi(*)}{\to} \sup \mathbf{f}^\varepsilon(\alpha) < \delta\}$, it is a club of λ^+. Let $\mathbf{f}^{\xi(*)} : \lambda^+ \to \lambda^+$ be $\mathbf{f}^{\xi(*)}(i) = \underset{\varepsilon<\xi(*)}{\to} \sup \mathbf{f}^\varepsilon(i)$ now define $M^{\xi(*)}_i$ as follows: **Case 1**: If $\delta \in E^*$ and $\varepsilon < \xi(*)$ and $i \le \mathbf{f}^\varepsilon(\delta)$ and $i \ge \bigcup\limits_{\zeta<\varepsilon} \mathbf{f}^\zeta(\delta)$ then

(α) $M^{\xi(*)}_{\delta+i} = \bigcup\{M^\zeta_{\delta+i} : \zeta \in [\varepsilon, \xi(*))\}$

(β) $i < \mathbf{f}^\varepsilon(\delta) \Rightarrow a^{\xi(*)}_{\delta+i} = a^\varepsilon_{\delta+i}$.

(Note: we may define $M^{\xi(*)}_{\delta+i}$ twice if $i = \mathbf{f}^\varepsilon(\delta)$, but the two values are the same).
Case 2: If $\delta \in E^*, i = \mathbf{f}^{\xi(*)}(\delta)$ is a limit ordinal let

$$M^{\xi(*)}_{\delta+i} = \bigcup_{j<i} M^{\xi(*)}_{\delta+i}.$$

Case 3: If $M^{\xi(*)}_i$ has not been defined yet, let it be $M^{\xi(*)}_{\min(E^*\backslash i)}$. **Case 4**: If $a^{\xi(*)}_i$ has not been defined yet, let $a^{\xi(*)}_i \in M^{\xi(*)}_{i+1}$ be arbitrary.

Note that Case 3,4 deal with the "unimportant" cases.

Let $\varepsilon < \xi(*)$, why $(\overline{M}^\varepsilon, \bar{\mathbf{a}}^\varepsilon, \mathbf{f}^\varepsilon) \le^0_S (\overline{M}^{\xi(*)}, \bar{\mathbf{a}}^{\xi(*)}, \mathbf{f}^{\xi(*)}) \in K^{\rm mqr}_S$? Enough to check that the club E^* witnesses it.

Why $\text{ortp}(a_{\delta+i}, M^{\xi(*)}_{\delta+i}, M^{\xi(*)}_{\delta+i+1}) \in \mathcal{S}^{\rm bs}(M^{\xi(*)}_{\delta+i})$ and when $\delta \in E^*, i < \mathbf{f}^{\xi(*)}(i)$, and does not fork over $M^\varepsilon_{\delta+i}$ when $i < \mathbf{f}^\varepsilon(\delta)$? by Axiom (E)(h) of Definition 2.1.

Why clause (e) of Definition 5.12(1A)? By Axiom (E)(c), local character of non-forking.

The case $\xi(*) = \lambda^+$ is similar using diagonal intersections. $\square_{5.13}$

Remark 5.14. If we use weaker versions of "good λ-frames", we should systematically concentrate on successor $i < \mathbf{f}(\delta)$.

Proof. [Proof of 5.9] We can use [She09d, 2b.3] or more explicitly [She09d, e.4]: the older version runs as follows. The use of $\lambda^{++} \notin \text{WDmId}(\lambda^{++})$ is as in the proof of [She01, 3.19](pg.79)=3.12t. But now we need to preserve saturation in limit stages $\delta < \lambda^{++}$ of cofinality $< \lambda^+$, we use $<^1_S$, otherwise we act as in [She01, §3]. $\square_{5.9}$

Let us elaborate.

Definition 5.15. We define $\mathbf{C} = (\mathfrak{k}^+, \mathbf{S}eq, \leq^*)$ as follows:

(a) $\tau^+ = \tau \cup \{P, <\}, \mathfrak{k}^+$ is the set of $(M, P^M, <^M)$ where $M \in \mathfrak{k}_{<\lambda}, P^M \subseteq M, <^M$ a linear ordering of P^M (but $=^M$ may be as in [She01, 3.1](2) and $M_1 \leq_{\mathfrak{k}^+} M_2$ iff $(M_1 \restriction \tau) \leq_{\mathfrak{k}} (M_2 \restriction \tau)$ and $M_1 \subseteq M_2$

(b) $\mathbf{Seq}_\alpha = \{\overline{M} : \overline{M} = \langle M_i : i \leq \alpha \rangle$ is an increasing continuous sequence of members of $\mathfrak{k}^+$ and $\langle M_i \restriction \tau : i \leq \alpha\rangle$ is $\leq_{\mathfrak{k}}$-increasing, and for $i < j < \alpha : P^{M_i}$ is a proper initial segment of $(P^{M_j}, <^{M_j})$ and there is a first element in the difference$\}$

we denote the $<^{M_{i+1}}$-first element of $P^{M_{i+1}} \setminus P^{M_i}$, by $a_i[\overline{M}]$ and we demand $\text{ortp}(a_i(\overline{M}), M_i\tau \restriction, M_{i+1} \restriction \tau) \in \mathcal{S}^{\text{bs}}(M_i \restriction \tau)$ and if $\alpha = \lambda, M = \cup\{M_i \restriction \tau : i < \lambda^+\}$ is saturated

(c) $\overline{M} <^*_t \overline{N}$ <u>iff</u>
$\overline{M} = \langle M_i : i < \alpha^* \rangle, \overline{N} = \langle N_i : i < \alpha^{**}\rangle$ are from $\mathbf{Seq}, t$ is a set of pairwise disjoint closed intervals of α^* and for any $[\alpha, \beta] \in t$ we have ($\beta < \alpha^*$ and): $\gamma \in [\alpha, \beta) \Rightarrow M_\gamma \leq_{\mathfrak{k}} N_\gamma$ and $a_\gamma[\overline{M}] \notin N_\gamma$, moreover
$a_\gamma[\overline{M}] = a_\gamma[\overline{N}]$ and $\text{ortp}(a_j[\overline{M}], N_\gamma \restriction \tau, N_{\gamma+1}, \tau)$ does not fork over $M_\gamma \restriction \tau$.

Claim 5.16. *1)* $\mathbf{C}$ *is a* λ^+*-construction framework (see* [She01, 3.3]*(pg.68).*
2) $\mathbf{C}$ *is weakly nice (see Definition* [She01, 3.14]*(2)(pg.76).*
4) $\mathbf{C}$ *has the weakening* λ^+*-coding property.*

Discussion 5.17. Is it better to use (see [She01, 3.14](1)(pg.75)) stronger axiomatization in [She01, §3] to cover this?
But at present this will be the only case.

Proof. Straight. $\square_{5.16}$

Now 5.11 follows by [She01, 3.19](pg.79).

§ 6. Non-forking amalgamation in $\mathfrak{k}_\lambda$

We deal in this section only with $\mathfrak{k}_\lambda$.
We would like to, at least, approximate "non-forking amalgamation of models" using as a starting point the conclusion of 5.9, i.e., $K^{3,\mathrm{uq}}_\lambda$ is dense. We use what looks like a stronger hypothesis: the existence for $K^{3,\mathrm{uq}}_\lambda$ (also called "weakly successful"); but in our application we can assume categoricity in λ; the point being that as we have a superlimit $M \in K_\lambda$, this assumption is reasonable when we restrict ourselves to $\mathfrak{k}^{[M]}$, recalling that we believe in first analyzing the saturated enough models; see 5.8. By 4.9, the "$(\lambda, \mathrm{cf}(\delta))$-brimmed over" is the same for all limit ordinals $\delta < \lambda^+$, (but not for $\delta = 1$ or just δ non-limit); nevertheless for possible generalizations we do not use this.

It may help the reader to note, that (assuming 6.9 below, of course), if there is a 4-place relation $\mathrm{NF}_\lambda(M_0, M_1, M_2, M_3)$ on K_λ, satisfying the expected properties of "M_1, M_2 are amalgamated in a non-forking = free way over M_0 inside M_3", i.e., is a $\mathfrak{k}_\lambda$-non-forking relation from Definition 6.1 below then Definition 6.13 below (of NF_λ) gives it (provably!). So we have "a definition" of NF_λ satisfying that: if desirable non-forking relation exists, our definition gives it (assuming the hypothesis 6.9). So during this section we are trying to get better and better approximations to the desirable properties; have the feeling of going up on a spiral, as usual.

For the readers who know on non-forking in stable first order theory we note that in such context $\mathrm{NF}_\lambda(M_0, M_1, M_2, M_3)$ says that $\mathrm{ortp}(M_2, M_1, M_3)$, the type of M_2 over M_1 inside M_3, does not fork over M_0. It is natural to say that there are $\langle N_{1,\alpha}, N_{2,\alpha} : \alpha \le \alpha^* \rangle, N_{\ell,\alpha}$ is increasing continuous. $N_{1,0} = M_0, N_{2,0} = M_2, M_1 \subseteq M_{1,\alpha}, M_3 \subseteq M'_3, N_{2,\alpha} \subseteq M'_3, N_{\ell,\alpha+2}$ is prime over $N_{\ell,\alpha} + a_\alpha$ for $\ell = 1, 2$ and $\mathrm{ortp}(a_\alpha, N_{2,\alpha})$ does not fork over $N_{1,\alpha}$ but this is not available. The $K^{3,\mathrm{uq}}_\lambda$ is a substitute.

Definition 6.1. 1) Assume that $\mathfrak{k} = \mathfrak{k}_\lambda$ is a λ-AEC We say NF is a non-forking relation on ${}^4(\mathfrak{k}_\lambda)$ or just a $\mathfrak{k}_\lambda$-non-forking relation <u>when</u>:

$\boxtimes_{\mathrm{NF}}(a)$ NF is a 4-place relation on K_λ and NF is preserved under isomorphisms

- (b) $\mathrm{NF}(M_0, M_1, M_2, M_3)$ implies $M_0 \le_{\mathfrak{k}} M_\ell \le_{\mathfrak{k}} M_3$ for $\ell = 1, 2$
- $(c)_1$ (monotonicity): if $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and $M_0 \le_{\mathfrak{k}} M'_\ell \le_{\mathfrak{k}} M_\ell$ for $\ell = 1, 2$ then $\mathrm{NF}(M_0, M'_1, M'_2, M_3)$
- $(c)_2$ (monotonicity): if $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and $M_3 \le_{\mathfrak{k}} M'_3 \in K_\lambda, M_1 \cup M_2 \subseteq M''_3 \le_{\mathfrak{k}} M'_3$ <u>then</u> $\mathrm{NF}(M_0, M_1, M_2, M''_3)$
- (d) (symmetry) $\mathrm{NF}(M_0, M_1, M_2, M_3)$ iff $\mathrm{NF}(M_0, M_2, M_1, M_3)$
- (e) ((long) transitivity) if $\mathrm{NF}(M_i, N_i, M_{i+1}, N_{i+1})$ for $i < \alpha, \langle M_i : i \le \alpha \rangle$ is $\le_{\mathfrak{k}}$-increasing continuous and $\langle N_i : i \le \alpha \rangle$ is $\le_{\mathfrak{k}}$-increasing continuous <u>then</u> $\mathrm{NF}(M_0, N_0, M_\alpha, N_\alpha)$
- (f) (existence) if $M_0 \le_{\mathfrak{k}} M_\ell$ for $\ell = 1, 2$ (all in K_λ) <u>then</u> for some $M_3 \in K_\lambda, f_1, f_2$ we have $M_0 \le_{\mathfrak{k}} M_3, f_\ell$ is a $\le_{\mathfrak{k}}$-embedding of M_ℓ into M_3 over M_0 for $\ell = 1, 2$ and $\mathrm{NF}(M_0, f_1(M_1), f_2(M_2), M_3)$
- (g) (uniqueness) if $\mathrm{NF}(M^\ell_0, M^\ell_1, M^\ell_2, M^\ell_3)$ and for $\ell = 1, 2$ and f_i is an isomorphism from M^1_i onto M^2_i for $i = 0, 1, 2$ and $f_0 \subseteq f_1, f_0 \subseteq f_2$ <u>then</u> $f_1 \cup f_2$ can be extended to an embedding f_3 of M^1_3 into some $M^2_4, M^2_3 \le_{\mathfrak{k}_\lambda} M^2_4$.

2) We say that NF is a pseudo non-forking relation on ${}^4(K_\lambda)$ or a weak $\mathfrak{k}_\lambda$-non-forking relation <u>if</u> clauses (a)-(f) of $\boxtimes_{\mathrm{NF}}$ above holds but not necessarily clause (g).

3) Assume $\mathfrak{s}$ is a good λ-frame and NF is a non-forking relation on $\mathfrak{k}$ or just a weak one. We say that NF respects $\mathfrak{s}$ or NF is an $\mathfrak{s}$-non-forking relation <u>when</u>:

- (h) if $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and $a \in M_2 \setminus M_0$, $\mathrm{ortp}_{\mathfrak{s}}(a, M_0, M_2) \in \mathcal{S}^{\mathrm{bs}}(M_0)$ then $\mathrm{ortp}_{\mathfrak{s}}(a, M_1, M_3)$ does not fork over M_0 in the sense of $\mathfrak{s}$.

Observation 6.2. *Assume $\mathfrak{k}_\lambda$ is a λ-AEC and NF is a non-forking relation on ${}^4(\mathfrak{k}_\lambda)$.*

1) Assume $\mathfrak{k}$ is stable in λ. If in clause (g) of 6.1(1) above we assume in addition that M_3^ℓ is (λ, ∂)-brimmed over $M_1^\ell \cup M_2^\ell$, <u>then</u> in the conclusion of (g) we can add $M_3^2 = M_4^2$, i.e., $f_1 \cup f_2$ can be extended to an isomorphism from M_3^1 onto M_3^2. This version of (g) is equivalent to it (assuming stability in λ; note that "$\mathfrak{k}_\lambda$ has amalgamation" follows by clause (f) of Definition 6.1).

2) If $M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_3$ are from K_λ then $\mathrm{NF}(M_0, M_0, M_1, M_3)$.

3) In Definition 6.1(1), clause (d), symmetry, it is enough to demand "if".

Proof. 1) Chase arrows and the uniqueness from 1.17.

2) By clause (f) of $\boxtimes_{\mathrm{NF}}$ of 6.1(1) and clause $(c)_2$, i.e., first apply existence with (M_0, M_0, M_3) here standing for (M_0, M_1, M_2) there, then chase arrows and use the monotonicity as in $(c)_2$.

3) Easy. $\square_{6.2}$

The main point of the following claim shows that there is at most one non-forking relation respecting $\mathfrak{s}$; so it justifies the definition of $\mathrm{NF}_{\mathfrak{s}}$ later. The assumption "NF respects $\mathfrak{s}$" is not so strong by 6.7.

Claim 6.3. *1) If $\mathfrak{s}$ is a good λ-frame and* NF *is a non-forking relation on ${}^4(\mathfrak{k}_{\mathfrak{s}})$ respecting $\mathfrak{s}$ and $(M_0, N_0, a) \in K_\lambda^{3,\mathrm{uq}}$ and $(M_0, N_0, a) \le_{\mathrm{bs}} (M_1, N_1, a)$ then $\mathrm{NF}(M_0, N_0, M_1, N_1)$.*

2) If $\mathfrak{s}$ is a good λ-frame, weakly successful (which means $K_{\mathfrak{s}}^{3,\mathrm{uq}}$ has existence in $K_{\mathfrak{s}}^{3,\mathrm{uq}}$, i.e., $\mathfrak{s}$ satisfies hypothesis 6.9 below) and NF *is a non-forking relation on ${}^4(\mathfrak{k}_{\mathfrak{s}})$ respecting $\mathfrak{s}$ <u>then</u> the relation $\mathrm{NF}_\lambda = \mathrm{NF}_{\mathfrak{s}}$, i.e., $N_1 \underset{N_0}{\overset{N_3}{\bigcup}} N_2$ defined in Definition 6.13 below is equivalent to $\mathrm{NF}(N_0, N_1, N_2, N_3)$. [Recalling 6.36, but see 6.37(2), 6.38.]*

3) If $\mathfrak{s}$ is a weakly successful good λ-frame and for $\ell = 1, 2$, the relation NF_ℓ is a non-forking relation on ${}^4(\mathfrak{k}_{\mathfrak{s}})$ respecting $\mathfrak{s}$, <u>then</u> $\mathrm{NF}_1 = \mathrm{NF}_2$.

Proof. Straightforward, but we elaborate.

1) We can find (M_1', N_1') such that $\mathrm{NF}(M_0, N_0, M_1', N_1')$ and M_1, M_1' are isomorphic over M_0, say f_1 is such an isomorphism from M_1 onto M_1' over M_0; why such (M_1', N_1', f_1) exists? by clause (f) of $\boxtimes_{\mathrm{NF}}$ of Definition 6.1.

As NF respects $\mathfrak{s}$, see Definition 6.1(2), recalling $\mathrm{ortp}(a, M_0, N_0) \in \mathcal{S}^{\mathrm{bs}}(M_0)$ we know that $\mathrm{ortp}(a, M_1', N_1')$ does not fork over M_0, so by the definition of $\le_{\mathrm{bs}}$ we have $(M_0, N_0, a) \le_{\mathrm{bs}} (M_1', N_1', a)$.

As $(M_0, N_0, a) \in K_\lambda^{3,\mathrm{uq}}$, by the definition of $K_\lambda^{3,\mathrm{uq}}$ (and chasing arrows) we conclude that there are N_2, f_2 such that:

- $(*)$ $N_1 \le_{\mathfrak{k}[\mathfrak{s}]} N_2 \in K_\lambda$ and f_2 is a $\le_{\mathfrak{k}}$-embedding of N_1' into N_2 extending f_1^{-1} and id_{N_0}.

As $\mathrm{NF}(M_0, N_0, M_1', N_1')$ and NF is preserved under isomorphisms (see clause (a) in 6.1(1)) it follows that $\mathrm{NF}(M_0, N_0, M_1, f_2(N_1'))$. By the monotonicity of NF (see clause $(c)_2$ of Definition 6.1) it follows that $\mathrm{NF}(M_0, N_0, M_1, N_2)$. Again by the same monotonicity we have $\mathrm{NF}(M_0, N_0, M_1, N_1)$, as required.

2) First we prove that $\mathrm{NF}_{\lambda,\bar{\delta}}(N_0, N_1, N_2, N_3)$, which is defined in Definition 6.12 below implies $\mathrm{NF}(N_0, N_1, N_2, N_3)$. By definition 6.12, clause (f) there are $\langle (N_{1,i}, N_{2,i} : i \le \lambda \times \delta_1\rangle), \langle c_i : i < \lambda \times \delta_1\rangle$ as there. Now we prove by induction on $j \le \lambda \times \delta_1$ that $i \le j \Rightarrow \mathrm{NF}(N_{1,i}, N_{2,i}, N_{1,j}, N_{2,j})$. For $j = 0$ or more generally when $i = j$ this is trivial by 6.2(2). For j a limit ordinal use the induction hypothesis and transitivity of NF (see clause (e) of 6.1(1)).

Lastly, for j successor by the demands in Definition 6.12 we know that $N_{1,j-1} \le_{\mathfrak{k}} N_{1,j} \le_{\mathfrak{k}} N_{2,j}, N_{1,j-1} \le_{\mathfrak{k}} N_{2,j-1} \le_{\mathfrak{k}} N_{2,j}$ are all in K_λ, $\mathrm{ortp}(c_{j-1}, N_{2,j-1}, N_{2,j})$ does not fork over $N_{1,j-1}$ and $(N_{1,j-1}, N_{1,j}, c_{j-1}) \in K^{3,\mathrm{uq}}_\lambda$. By part (1) of this claim we deduce that $\mathrm{NF}(N_{1,j-1}, N_{1,j}, N_{2,j-1}, N_{2,j})$ hence by symmetry (i.e., clause (d) of Definition 6.1(1)) we deduce $\mathrm{NF}(N_{1,j-1}, N_{2,j-1}, N_{1,j}, N_{2,j})$.

So we have gotten $i < j \Rightarrow \mathrm{NF}(N_{1,i}, N_{2,i}, N_{1,j}, N_{2,j})$.

[Why? If $i = j - 1$ by the previous sentence and for $i < j - 1$ note that by the induction hypothesis $\mathrm{NF}(N_{1,i}, N_{2,i}, N_{1,j-1}, N_{1,j-1})$ so by transitivity (clause (e) of 6.1(1) of Definition 6.1) we get $\mathrm{NF}(N_{1,i}, N_{2,i}, N_{1,j}, N_{2,j})$].

We have carried the induction so in particular for $i = 0, j = \alpha$ we get $\mathrm{NF}(N_{1,0}, N_{2,0}, N_{1,\alpha}, N_{2,\alpha})$, which means $\mathrm{NF}(N_0, N_1, N_2, N_3)$ as promised. So we have proved $\mathrm{NF}_{\lambda,\bar{\delta}}(N_0, N_1, N_2, N_3) \Rightarrow \mathrm{NF}(N_0, N_1, N_2, N_3)$.

Second, if $\mathrm{NF}_\lambda(N_0, N_1, N_2, N_3)$ as defined in Definition 6.13 then there are $M_0, M_1, M_2, M_3 \in K_\lambda$ such that $\mathrm{NF}_{\lambda,\langle\lambda,\lambda\rangle}(M_0, M_1, M_2, M_3), N_\ell \le_{\mathfrak{k}} M_\ell$ for $\ell < 4$ and $N_0 = M_0$. By what we have proved above we can conclude $\mathrm{NF}(M_0, M_1, M_2, M_3)$. As $N_0 = M_0 \le_{\mathfrak{k}} N_\ell \le_{\mathfrak{k}} M_\ell$ for $\ell = 1, 2$ by clause $(c)_1$ of Definition 6.1(1) we get $\mathrm{NF}(M_0, N_1, N_2, M_3)$ and by clause $(c)_2$ of Definition 6.1(1) we get $\mathrm{NF}(N_0, N_1, N_2, N_3)$. So we have proved the implication $\mathrm{NF}_\lambda(N_0, N_1, N_2, N_3) \Rightarrow \mathrm{NF}(N_0, N_1, N_2, N_3)$.

For the other implication assume $\mathrm{NF}(N_0, N_1, N_2, M_3)$. Now as we have existence for NF_λ (as proved below, see 6.23), we can find N_ℓ' for $\ell = 0, 1, 2, 3$ and f_ℓ for $\ell = 0, 1, 2$ such that $\mathrm{NF}_\lambda(N_0', N_1', N_2', N_3'), f_\ell$ is an isomorphism from N_ℓ onto N_ℓ' for $\ell = 0, 1, 2$ and $f_0 \subseteq f_1, f_0 \subseteq f_2$. But what we have already proved it follows that $\mathrm{NF}(N_0', N_1', N_2', N_3')$. As we have uniqueness for NF by clause (g) of Definition 6.1 we can find (f_3, N_3'') such that $N_3' \le_{\mathfrak{k}_\lambda} N_3''$ and f_3 is a $\le_{\mathfrak{k}}$-embedding of N_3 into N_3'' extending $f_1 \cup f_2$. As NF_λ satisfies clause $(c)_2$ of 6.1, recalling $\mathrm{NF}_\lambda(N_0', N_1', N_2', N_3')$ it follows that $\mathrm{NF}_\lambda(N_0', N_1', N_2', f_3(N_3))$ holds. As NF_λ is preserved by isomorphisms, it follows that $\mathrm{NF}_\lambda(N_0, N_1, N_2, N_3)$ holds as required.

3) By the rest of this section, i.e., the main conclusion 6.36, the relation NF_λ defined in 6.13 is a non-forking relation on ${}^4(K_{\mathfrak{s}})$ respecting $\mathfrak{s}$. Hence by part (2) of the present claim we have $\mathrm{NF}_1 = \mathrm{NF}_\lambda = \mathrm{NF}_2$. $\square_{6.3}$

Example 6.4. : Do we need $\mathfrak{s}$ in 6.3(3)? Yes.

Let $\mathfrak{k}$ be the class of graphs and $M \le_{\mathfrak{k}} N$ iff $M \subseteq N$; so $\mathfrak{k}$ is an AEC with $\mathrm{LS}(\mathfrak{k}) = \aleph_0$. For cardinal λ and $\ell = 1, 2$ we define $\mathrm{NF}^\ell = \{(M_0, M_1, M_2, M_3) : M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_3$ and $M_0 \le_{\mathfrak{k}} M_2 \le_{\mathfrak{k}} M_3$ and $M_1 \cap M_2 = M_0$ and if $a \in M_1 \setminus M_0, b \in M_2 \setminus M_0$ then $\{a, b\}$ is an edge of M_3 iff $\ell = 2\}$ and $\mathrm{NF}^\ell_\lambda := \{(M_0, M_1, M_2, M_3) \in \mathrm{NF} : M_0, M_1, M_2, M_3 \in K_\lambda\}$. Then NF^ℓ_λ is a non-forking relation on ${}^4(\mathfrak{k}_\lambda)$ but $\mathrm{NF}^1_\lambda \neq \mathrm{NF}^2_\lambda$.

Remark 6.5. 1) So the assumption on $\mathfrak{k}_\lambda$ that for some good λ-frame $\mathfrak{s}$ we have $\mathfrak{k}_{\mathfrak{s}} = \mathfrak{k}_\lambda$ is quite a strong demand on $\mathfrak{k}_\lambda$.

2) However, the assumption "respect" essentially is not necessary as it can be deduced when $\mathfrak{s}$ is good enough.

3) Below on "good$^+$" see [She09e, §1] in particular Definition [She09e, 705-stg.1].

Exercise 6.6. : 1) Assume NF_1,NF_2 are non-forking relations on ${}^4(\mathfrak{k}_\lambda)$.
If $\mathrm{NF}_1 \subseteq \mathrm{NF}_2$ <u>then</u> $\mathrm{NF}_1 = \mathrm{NF}_2$.

2) In part (1) write down the clauses from 6.1. We need to assume on NF_1, and those we need assume on NF_2. [Hint: Read the last paragraph of the proof of 6.3(3).]

Claim 6.7. *Assume that $\mathfrak{s}$ is a good$^+$ λ-frame and* NF *is a non-forking relation on* ${}^4(\mathfrak{k}_\mathfrak{s})$. <u>*Then*</u> NF *respects* $\mathfrak{s}$.

Remark 6.8. The construction in the proof is similar to the ones in 4.10, 6.15.

Proof. Assume $\mathrm{NF}(M_0, M_1, M_2, M_3)$ and $a \in M_2 \backslash M_0$, $\mathrm{ortp}(a, M_0, M_2) \in \mathcal{S}^{\mathrm{bs}}(M_0)$. We define $(N_{0,i}, N_{1,i}, f_i)$ for $i < \lambda_\mathfrak{s}^+$ as follows:

$\otimes_1$ (a) $N_{0,i}$ is $\leq_\mathfrak{s}$-increasing continuous and $N_{0,0} = M_0$.
(b) $N_{1,i}$ is $\leq_\mathfrak{s}$-increasing continuous and $N_{1,0} = M_1$.
(c) $\mathrm{NF}(N_{0,i}, N_{1,i}, N_{0,i+1}, N_{1,i+1})$
(d) f_i is a $\leq_\mathfrak{k}$-embedding of M_2 into $N_{0,i+1}$ over $M_0 = N_{0,0}$ such that $\mathrm{ortp}(f_i(a), N_{0,i}, N_{0,i+1})$ does not fork over $M_0 = N_{0,0}$.

We shall choose f_i together with $N_{0,i+1}, N_{1,i+1}$.

Why can we define? For $i = 0$ there is nothing to do. For i limit take unions. For $i = j + 1$ choose $f_j, N_{0,i}$ satisfying clause (d) and $N_{0,j} \leq_\mathfrak{s} N_{0,i}$, this is possible for $\mathfrak{s}$ as we have the existence of non-forking extensions of $\mathrm{ortp}(a, M_0, M_2)$ (and amalgamation).

Lastly, we take care of the rest (mainly clause (c) of $\otimes_1$ by clause (f) of Definition 6.1(1), existence). Now

$\circledast_2$ For $i < j < \lambda^+$ we have $\mathrm{NF}(N_{0,i}, N_{1,i}, N_{0,j}, N_{1,j})$.
[why? by transitivity for NF, i.e., clause (e) of Definition 6.1(1), transitivity]

$\circledast_3$ For some i, $\mathrm{ortp}(f_i(a), N_{1,i}, N_{1,i+1})$ does not fork over M_0.
[why? by the definition of good$^+$].

So for this i, $M_0 \leq_\mathfrak{s} f_i(M_2) \leq_\mathfrak{s} N_{0,i+1}$ by clause (d) of $\otimes_1$, hence by clause $(c)_1$ of Definition 6.1, monotonicity we have $\mathrm{NF}(M_0, M_1, f_i(M_2), N_{1,i+1})$. Now again by the choice of i, i.e., by $\circledast_3$ we have $\mathrm{ortp}(f_i(a), M_1, N_{1,i+1})$ does not fork over M_0. By clause (g) of Definition 6.1(1), i.e., uniqueness of NF (and preservation by isomorphisms) we get $\mathrm{ortp}(a, M_1, M_3)$ does not fork over M_0 as required. $\square_{6.3}$

We turn to our main task in this section proving that such NF exist; till 6.36 we assume:

Hypothesis 6.9. 1) $\mathfrak{s} = (\mathfrak{k}, \bigcup, \mathcal{S}^{\mathrm{bs}})$ is a good λ-frame.

2) $\mathfrak{s}$ is weakly successful which just means that it has existence for $K_\lambda^{3,\mathrm{uq}}$: for every $M \in K_\lambda$ and $p \in \mathcal{S}^{\mathrm{bs}}(M)$ there are N, a such that $(M, N, a) \in K_\lambda^{3,\mathrm{uq}}$ (see Definition 5.3) and $p = \mathrm{ortp}(a, M, N)$. (This follows by $K_\mathfrak{s}^{3,\mathrm{uq}}$ is dense in $K_\mathfrak{s}^{3,\mathrm{bs}}$; when $\mathfrak{s}$ is categorical, see 5.8.)

In this section we deal with models from K_λ only.

Claim 6.10. *If $M \in K_\lambda$ and N is (λ,κ)-brimmed over M, then we can find $\overline{M} = \langle M_i : i \le \delta\rangle, \le_{\mathfrak{k}}$-increasing continuous, $(M_i, M_{i+1}, c_i) \in K^{3,\mathrm{uq}}_\lambda$, $M_0 = M$, and $M_\delta = N$ for δ any pre-given limit ordinal $< \lambda^+$ of cofinality κ divisible by λ.*

Proof. Let δ be given (e.g. $\delta = \lambda \times \kappa$). By 6.9(2) we can find a $\le_{\mathfrak{k}}$-increasing sequence $\langle M_i : i \le \delta\rangle$ of members of K_λ and $\langle a_i : i < \delta\rangle$ such that $M_0 = M$ and $i < \delta \Rightarrow (M_i, M_{i+1}, a_i) \in K^{3,\mathrm{uq}}_\lambda$ and for every $i < \delta$ and $p \in \mathcal{S}^{\mathrm{bs}}(M_i)$ for λ ordinals $j \in (i, i+\lambda)$ we have that $\mathrm{ortp}(a_j, M_j, M_{j+1})$ is a non-forking extension of p. So the demands in 4.3 hold, hence M_δ is (λ,κ)-brimmed over $M_0 = M$. Now we are done by the uniqueness of N being (λ,κ)-brimmed over M_0, see 1.17(3). $\square_{6.10}$

Claim 6.11. *If $M^\ell_0 \le_{\mathfrak{k}} M^\ell_1 \le_{\mathfrak{k}} M^\ell_3$ and $M^\ell_0 \le_{\mathfrak{k}} M^\ell_2 \le_{\mathfrak{k}} M^\ell_3, c_\ell \in M^\ell_1$ and $(M^\ell_0, M^\ell_1, c_\ell) \in K^{3,\mathrm{uq}}_\lambda$ and $\mathrm{ortp}(c_\ell, M^\ell_2, M^\ell_3) \in \mathcal{S}^{\mathrm{bs}}(M^\ell_2)$ does not fork over M^ℓ_0 and M^ℓ_3 is (λ,∂)-brimmed over $M^\ell_1 \cup M^\ell_2$ all this for $\ell = 1,2$ and f_i is an isomorphism from M^1_i onto M^2_i for $i = 0,1,2$ such that $f_0 \subseteq f_1, f_0 \subseteq f_2$ and $f_1(c_1) = c_2$, then $f_1 \cup f_2$ can be extended to an isomorphism from M^1_3 onto M^2_3.*

Proof. Chase arrows (and recall definition of $K^{3,\mathrm{uq}}_\lambda$), that is by 6.1(1) and Definition 6.2(1) and 1.17(3). $\square_{6.11}$

Definition 6.12. Assume $\bar\delta = \langle \delta_1, \delta_2, \delta_3\rangle, \delta_1, \delta_2, \delta_3$ are ordinals $< \lambda^+$, maybe 1. We say that $\mathrm{NF}_{\lambda,\bar\delta}(N_0, N_1, N_2, N_3)$ or, in other words N_1, N_2 are brimmedly smoothly amalgamated in N_3 over N_0 for $\bar\delta$ when:

- (a) $N_\ell \in K_\lambda$ for $\ell \in \{0,1,2,3\}$
- (b) $N_0 \le_{\mathfrak{k}} N_\ell \le_{\mathfrak{k}} N_3$ for $\ell = 1,2$
- (c) $N_1 \cap N_2 = N_0$ (i.e. in disjoint amalgamation, actually follows by clause (f))
- (d) N_1 is (λ,cf(δ_1))-brimmed over N_0; recall that if $\mathrm{cf}(\delta_1) = 1$ this just means $N_0 \le_{\mathfrak{k}} N_1$
- (e) N_2 is (λ,cf(δ_2))-brimmed over N_0; so that if $\mathrm{cf}(\delta_2) = 1$ this just means $N_0 \le_{\mathfrak{k}} N_2$ and
- (f) there are $N_{1,i}, N_{2,i}$ for $i \le \lambda \times \delta_1$ and c_i for $i < \lambda \times \delta_1$ (called witnesses and $\langle N_{1,i}, N_{2,i}, c_j : i \le \lambda \times \delta_1, j < \lambda \times \delta_1\rangle$ is called a witness sequence as well as $\langle N_{1,i} : i \le \lambda \times \delta_1\rangle, \langle N_{2,i} : i \le \lambda \times \delta_1\rangle$) such that:
  - (α) $N_{1,0} = N_0, N_{1,\lambda\times\delta_1} = N_1$
  - (β) $N_{2,0} = N_2$
  - (γ) $\langle N_{\ell,i} : i \le \lambda \times \delta_1\rangle$ is a $\le_{\mathfrak{k}}$-increasing continuous sequence of models for $\ell = 1,2$
  - (δ) $(N_{1,i}, N_{1,i+1}, c_i) \in K^{3,\mathrm{uq}}_\lambda$
  - (ε) $\mathrm{ortp}(c_i, N_{2,i}, N_{2,i+1}) \in \mathcal{S}^{\mathrm{bs}}(N_{2,i})$ does not fork over $N_{1,i}$ and $N_{2,i} \cap N_1 = N_{1,i}$, for $i < \lambda \times \delta_1$ (follows by Definition 5.3)
  - (ζ) N_3 is (λ,cf(δ_3))-brimmed over $N_{2,\lambda\times\delta_1}$; so for $\mathrm{cf}(\delta_3) = 1$ this means just $N_{2,\lambda\times\delta_1} \le_{\mathfrak{k}} N_3$

Definition 6.13. 1) We say $N_1 \bigcup_{N_0}^{N_3} N_2$ (or N_1, N_2 are smoothly amalgamated over N_0 inside N_3 or $\mathrm{NF}_\lambda(N_0, N_1, N_2, N_3)$ or $\mathrm{NF}_{\mathfrak{s}}(N_0, N_1, N_2, N_3)$) when we can find $M_\ell \in K_\lambda$ (for $\ell < 4$) such that:

- (a) $\mathrm{NF}_{\lambda,\langle\lambda,\lambda,\lambda\rangle}(M_0, M_1, M_2, M_3)$
- (b) $N_\ell \le_{\mathfrak{k}} M_\ell$ for $\ell < 4$
- (c) $N_0 = M_0$
- (d) M_1, M_2 are $(\lambda, \mathrm{cf}(\lambda))$-brimmed over N_0 (follows by (a) see clauses (d), (e) of 6.12).

2) We call (M, N, a) strongly bs-reduced if $(M, N, a) \in K^{3,\mathrm{bs}}_\lambda$ and $(M, N, a) \le_{\mathrm{bs}} (M', N', a) \in K^{3,\mathrm{bs}}_\lambda \Rightarrow \mathrm{NF}_\lambda(M, N, M', N')$; not used.

Clearly we expect "strongly bs-reduced" to be equivalent to "$\in K^{3,\mathrm{uq}}_\lambda$", e.g. as this occurs in the first order case. We start by proving existence for $\mathrm{NF}_{\lambda,\bar\delta}$ from Definition 6.12.

Claim 6.14. *1) Assume $\bar\delta = \langle \delta_1, \delta_2, \delta_3\rangle, \delta_\ell$ an ordinal $< \lambda^+$ and $N_\ell \in K_\lambda$ for $\ell < 3$ and N_1 is $(\lambda, \mathrm{cf}(\delta_1))$-brimmed over N_0 and N_2 is $(\lambda, \mathrm{cf}(\delta_2))$-brimmed over N_0 and $N_0 \le_{\mathfrak{k}} N_1$ and $N_0 \le_{\mathfrak{k}} N_2$ and for simplicity $N_1 \cap N_2 = N_0$. Then we can find N_3 such that $\mathrm{NF}_{\lambda,\bar\delta}(N_0, N_1, N_2, N_3)$.*
2) Moreover, we can choose any $\langle N_{1,i} : i \le \lambda \times \delta_1\rangle, \langle c_i : i < \lambda \times \delta_1\rangle$ as in 6.12 subclauses (f)$(\alpha), (\gamma), (\delta)$ as part of the witness.
3) If $\mathrm{NF}_\lambda(N_0, N_1, N_2, N_3)$ then $N_1 \cap N_2 = N_0$.

Proof. 1) We can find $\langle N_{1,i} : i \le \lambda \times \delta_1\rangle$ and $\langle c_i : i < \lambda \times \delta_1\rangle$ as required in part (2) by Claim 6.10, the $(\lambda, \mathrm{cf}(\lambda \times \delta_1))$-brimmedness holds by 4.3 and apply part (2).
2) We choose the $N_{2,i}$ (by induction on i) by 4.10 preserving $N_{2,i} \cap N_{1,\lambda\times\delta_2} = N_{1,i}$; in the successor case use Definition 5.3 + Claim 5.4(3). We then choose N_3 using 4.2(2).
3) By the definitions of NF_λ, $\mathrm{NF}_{\lambda,\bar\delta}$. $\square_{6.14}$

The following claim tells us that if we have "$(\lambda, \mathrm{cf}(\delta_3))$-brimmed" in the end, then we can have it in all successor stages.

Claim 6.15. *In Definition 6.12, if δ_3 is a limit ordinal and $\kappa = \mathrm{cf}(\kappa) \ge \aleph_0$, then without loss of generality (even without changing $\langle N_{1,i} : i \le \lambda\times\delta_1\rangle, \langle c_i : i < \lambda\times\delta_1\rangle$)*

- *(g) $N_{2,i+1}$ is (λ, κ)-brimmed over $N_{1,i+1} \cup N_{2,i}$ (which means that it is (λ, κ)-brimmed over some N, where $N_{1,i+1} \cup N_{2,i} \subseteq N \le_{\mathfrak{k}} N_{2,i+1}$).*

Proof. So assume $\mathrm{NF}_{\lambda,\bar\delta}(N_0, N_1, N_2, N_3)$ holds as being witnessed by $\langle N_{\ell,i} : i \le \lambda \times \delta_1\rangle, \langle c_i : i < \lambda \times \delta_1\rangle$ for $\ell = 1, 2$. Now we choose by induction on $i \le \lambda \times \delta_1$ a model $M_{2,i} \in K_\lambda$ and f_i such that:

- (i) f_i is a $\le_{\mathfrak{k}}$-embedding of $N_{2,i}$ into $M_{2,i}$
- (ii) $M_{2,0} = f_i(N_2)$
- (iii) $M_{2,i}$ is $\le_{\mathfrak{k}}$-increasing continuous and also f_i is increasing continuous
- (iv) $M_{2,j} \cap f_i(N_{1,i}) = f_i(N_{1,j})$ for $j \le i$
- (v) $M_{2,i+1}$ is (λ, κ)-brimmed over $M_{2,i} \cup f_i(N_{2,i+1})$
- (vi) $\mathrm{ortp}(f_{i+1}(c_i), M_{2,i}, M_{2,i+1}) \in \mathcal{S}^{\mathrm{bs}}(M_{2,i})$ does not fork over $f_i(N_{1,i})$.

There is no problem to carry the induction. Using in the successor case $i = j+1$ the existence Axiom (E)(g) of Definition 2.1, there is a model $M'_{2,i} \in K_{\mathfrak{s}}$ such that $M_{2,j} \leq_{\mathfrak{k}} M'_{2,i}$ and $f_i \supseteq f_j$ as required in clauses (i), (iv), (vi) and then use Claim 4.2 to find a model $M_{2,i} \in K_\lambda$ which is (λ,κ)-brimmed over $M_{2,j} \cup f_i(N_{2,i})$.

Having carried the induction, without loss of generality $f_i = \mathrm{id}_{N_{2,i}}$. Let M_3 be such that $M_{2,\lambda\times\delta_1} \leq_{\mathfrak{k}} M_3 \in K_\lambda$ and M_3 is $(\lambda,\mathrm{cf}(\delta_3))$-brimmed over $M_{2,\lambda\times\delta_1}$, it exists by 4.2(2) but $N_{2,\lambda\times\delta_1} \leq_{\mathfrak{k}} M_{2,\lambda\times\delta_1}$, hence it follows that M_3 is (λ,κ)-brimmed over $N_{1,\lambda\times\delta_1}$. So both M_3 and N_3 are $(\lambda,\mathrm{cf}(\delta_3))$-brimmed over $N_{2,\lambda\times\delta_1}$, hence they are isomorphic over $N_{2,\lambda\times\delta_1}$ (by 1.17(1)) so let f be an isomorphism from M_3 onto N_3 which is the identity over $N_{2,\lambda\times\delta_1}$.
Clearly $\langle N_{1,i} : i \leq \lambda\times\delta_1\rangle, \langle f(M_{2,i}) : i \leq \lambda\times\delta_1\rangle$ are also witnesses for $\mathrm{NF}_{\lambda,\bar\delta}(N_0,N_1,N_2,N_3)$ satisfying the extra demand (g) from 6.15. $\square_{6.15}$

The point of the following claim is that having uniqueness in every atomic step we have uniqueness in the end (using the same "ladder" $N_{1,i}$ for now).

Claim 6.16. *(Weak Uniqueness).*
Assume that for $x \in \{a,b\}$, we have $\mathrm{NF}_{\lambda,\bar\delta^x}(N^x_0,N^x_1,N^x_2,N^x_3)$ holds as witnessed by $\langle N^x_{1,i} : i \leq \lambda\times\delta^x_1\rangle, \langle c^x_i : i < \lambda\times\delta^x_1\rangle, \langle N^x_{2,i} : i \leq \lambda\times\delta^x_1\rangle$ and $\delta_1 := \delta^a_1 = \delta^b_1, \mathrm{cf}(\delta^a_2) = \mathrm{cf}(\delta^b_2)$ and $\mathrm{cf}(\delta^a_3) = \mathrm{cf}(\delta^b_3) \geq \aleph_0$.
(Note that $\mathrm{cf}(\lambda\times\delta^a_1) \geq \aleph_0$ by the definition of NF).

Suppose further that f_ℓ is an isomorphism from N^a_ℓ onto N^b_ℓ for $\ell = 0,1,2$, moreover: $f_0 \subseteq f_1, f_0 \subseteq f_2$ and $f_1(N^a_{1,i}) = N^b_{1,i}, f_1(c^a_i) = c^b_i$.
<u>Then</u> we can find an isomorphism f from N^a_3 onto N^b_3 extending $f_1 \cup f_2$.

Proof. Without loss of generality for each $i < \lambda\times\delta_1$, the model $N^x_{2,i+1}$ is (λ,λ)-brimmed over $N^x_{1,i+1} \cup N^x_{2,i}$ (by 6.15, note there the statement "without changing the $N_{1,i}$'s"). Now we choose by induction on $i \leq \lambda\times\delta_1$ an isomorphism g_i from $N^a_{2,i}$ onto $N^b_{2,i}$ such that: g_i is increasing with i and g_i extends $(f_1 \restriction N^a_{1,i}) \cup f_2$.

For $i = 0$ choose $g_0 = f_2$ and for i limit let g_i be $\bigcup_{j<i} g_j$ and for $i = j+1$ it exists by 6.11, whose assumptions hold by $(N^x_{1,i}, N^x_{1,i+1}, c^x_i) \in K^{3,\mathrm{uq}}_\lambda$ (see 6.12, clause (f)(δ)) and the extra brimmedness clause from 6.15. Now by 1.17(3) we can extend $g_{\lambda\times\delta_1}$ to an isomorphism from N^a_3 onto N^b_3 as N^x_3 is $(\lambda,\mathrm{cf}(\delta_3))$-brimmed over $N^x_{2,\lambda\times\delta_1}$ (for $x \in \{a,b\}$). $\square_{6.16}$

Note that even knowing 6.16 the choice of $\langle N_{1,i} : i \leq \lambda\times\delta_1\rangle, \langle c_i : i < \lambda\times\delta_1\rangle$ still possibly matters. Now we prove an "inverted" uniqueness, using our ability to construct a "rectangle" of models which is a witness for $\mathrm{NF}_{\lambda,\bar\delta}$ in two ways.

Claim 6.17. *Suppose that*

(*a*) *for $x \in \{a,b\}$ we have $\mathrm{NF}_{\lambda,\bar\delta^x}(N^x_0,N^x_1,N^x_2,N^x_3)$*
(*b*) *$\bar\delta^x = \langle\delta^x_1,\delta^x_2,\delta^x_3\rangle, \delta^a_1 = \delta^b_2,\ \delta^a_2 = \delta^b_1, \mathrm{cf}(\delta^a_3) = \mathrm{cf}(\delta^b_3)$, all limit ordinals*
(*c*) *f_0 is an isomorphism from N^a_0 onto N^b_0*
(*d*) *f_1 is an isomorphism from N^a_1 onto N^b_2*
(*e*) *f_2 is an isomorphism from N^a_2 onto N^b_1*
(*f*) *$f_0 \subseteq f_1$ and $f_0 \subseteq f_2$.*

<u>Then</u> there is an isomorphism from N^a_3 onto N^b_3 extending $f_1 \cup f_2$.

Before proving we shall construct a third "rectangle" of models such that we shall be able to construct appropriate isomorphisms each of N^a_3, N^b_3

Subclaim 6.18. *Assume*

(a) $\delta^a_1, \delta^a_2, \delta^a_3 < \lambda^+$ *are limit ordinals*

$(b)_1$ $\overline{M}^1 = \langle M^1_\alpha : \alpha \le \lambda \times \delta^a_1 \rangle$ *is* $\le_{\mathfrak{k}}$*-increasing continuous in* K_λ *and* $(M^1_\alpha, M^1_{\alpha+1}, c_\alpha) \in K^{3,\text{bs}}_\lambda$

$(b)_2$ $\overline{M}^2 = \langle M^2_\alpha : \alpha \le \lambda \times \delta^a_2 \rangle$ *is* $\le_{\mathfrak{k}}$*-increasing continuous in* K_λ *and* $(M^2_\alpha, M^2_{\alpha+1}, d_\alpha) \in K^{3,\text{bs}}_\lambda$

(c) $M^1_0 = M^2_0$ *we call it* M *and* $M^1_\alpha \cap M^2_\beta = M$ *for* $\alpha \le \lambda \times \delta^a_1, \beta \le \lambda \times \delta^a_2$.

Then we can find $M_{i,j}$ *(for* $i \le \lambda \times \delta^a_1$ *and* $j \le \lambda \times \delta^a_2$*) and* M_3 *such that:*

(A) $M_{i,j} \in K_\lambda$ *and* $M_{0,0} = M$ *and* $M_{i,0} = M^1_i, M_{0,j} = M^2_j$

(B) $i_1 \le i_2$ *and* $j_1 \le j_2 \Rightarrow M_{i_1,j_1} \le_{\mathfrak{k}} M_{i_2,j_2}$

(C) *if* $i \le \lambda \times \delta^a_1$ *is a limit ordinal and* $j \le \lambda \times \delta^a_2$ *then* $M_{i,j} = \bigcup\limits_{\zeta < i} M_{\zeta,j}$

(D) *if* $i \le \lambda \times \delta^a_1$ *and* $j \le \lambda \times \delta^a_2$ *is a limit ordinal then* $M_{i,j} = \bigcup\limits_{\xi < j} M_{i,\xi}$

(E) $M_{\lambda \times \delta^a_1, j+1}$ *is* $(\lambda, \text{cf}(\delta^a_1))$*-brimmed over* $M^a_{\lambda \times \delta^a_1, j}$ *for* $j < \lambda \times \delta^a_2$

(F) $M_{i+1, \lambda \times \delta^a_2}$ *is* $(\lambda, \text{cf}(\delta^a_2))$*-brimmed over* $M_{i, \lambda \times \delta^a_2}$ *for* $i < \lambda \times \delta^a_1$

(G) $M_{\lambda \times \delta^a_1, \lambda \times \delta^a_2} \le_{\mathfrak{k}} M_3 \in K_\lambda$ *moreover* M_3 *is* $(\lambda, \text{cf}(\delta^a_3))$*-brimmed over* $M_{\lambda \times \delta^a_1, \lambda \times \delta^a_2}$

(H) *for* $i < \lambda \times \delta^a_1, j \le \lambda \times \delta^a_2$ *we have* $\text{ortp}(c_i, M_{i,j}, M_{i+1,j})$ *does not fork over* $M_{i,0}$

(I) *for* $j < \lambda \times \delta^a_2, i \le \lambda \times \delta^a_1$ *we have* $\text{ortp}(d_j, M_{i,j}, M_{i,j+1})$ *does not fork over* $M_{0,j}$.

We can add

(J) *for* $i < \lambda \times \delta^a_1, j < \lambda \times \delta^b_2$ *the model* $M_{i+1,j+1}$ *is* $(\lambda, *)$*-brimmed over* $M_{i,j+1} \cup M_{i+1,j}$.

Remark 6.19. 1) We can replace in 6.18 the ordinals $\lambda \times \delta^a_\ell$ ($\ell = 1,2,3$) by any ordinal $\alpha^a_\ell < \lambda^+$ (for $\ell = 1,2,3$) we use the present notation just to conform with its use in the proof of 6.17.

2) Why do we need u^ℓ_1 in the proof below? This is used to get the brimmedness demands in 6.18.

Proof. We first change our towers, repeating models to give space for bookkeeping. That is we define ${}^*M^1_\alpha$ for $\alpha \le \lambda \times \lambda \times \delta^a_1$ as follows:

if $\lambda \times \beta < \alpha \le \lambda \times \beta + \lambda$ and $\beta < \lambda \times \delta^a_1$ then ${}^*M^1_\alpha = M^1_{\beta+1}$

if $\alpha = \lambda \times \beta$, then ${}^*M^1_\alpha = M^1_\beta$. Let $u^1_0 = \{\lambda\beta : \beta < \delta^a_1\}, u^1_1 = \lambda \times \lambda \times \delta^a_1 \backslash u^1_0, u^1_2 = \varnothing$

and for $\alpha = \lambda\beta \in u^1_0$ let $a^1_\alpha = c_\beta$.

Similarly let us define ${}^*M^2_\alpha$ (for $\alpha \le \lambda \times \lambda \times \delta^a_2$),$u^2_0, u^2_1, u^2_2$ and $\langle a^2_\alpha : \alpha \in u^2_0 \rangle$.

Now apply 4.12 (check) and get ${}^*M_{i,j}, (i \le \lambda \times \lambda \times \delta^a_1, j \le \lambda \times \lambda \times \delta^a_2)$. Lastly, for $i \le \delta^a_1, j \le \delta^a_2$ let $M_{i,j} = {}^*M_{\lambda \times i, \lambda \times j}$. By 4.3 clearly ${}^*M_{\lambda \times i+\lambda, \lambda \times j+\lambda}$ is $(\lambda, \text{cf}(\lambda))$-brimmed over ${}^*M_{\lambda \times i+1, \lambda \times j+1}$ hence $M_{i+1,j+1}$ is $(\lambda, \text{cf}(\lambda))$-brimmed over $M_{i+1,j} \cup M_{i,j+1}$. And, by 4.2(1) choose $M_3 \in K_\lambda$ which is $(\lambda, \text{cf}(\delta^a_3))$-brimmed over $M_{\lambda \times \delta^a_1, \lambda \times \delta^a_2}$. $\square_{6.18}$

Proof. [Proof of 6.17] We shall let $M_{i,j}, M_3$ be as in 6.18 for $\bar{\delta}^a$ and $\overline{M}^1, \overline{M}^2$ determined below. For $x \in \{a,b\}$ as $\mathrm{NF}_{\lambda,\bar{\delta}^x}(N^x_0, N^x_1, N^x_2, N^x_3)$, we know that there are witnesses $\langle N^x_{1,i} : i \le \lambda \times \delta^x_1\rangle, \langle c^x_i : i < \lambda \times \delta^x_1\rangle, \langle N^x_{2,i} : i \le \lambda \times \delta^x_1\rangle$ for this. So $\langle N^x_{1,i} : i \le \lambda \times \delta^x_1\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous and $(N^x_{1,i}, N^x_{1,i+1}, c^x_i) \in K^{3,\mathrm{uq}}_\lambda$ for $i < \lambda \times \delta^x_1$. Hence by the freedom we have in choosing $\overline{M}^1$ and $\langle c_i : i < \lambda \times \delta_1\rangle$ without loss of generality there is an isomorphism g_1 from $N^a_{1,\lambda\times\delta^a_1}$ onto $M_{\lambda\times\delta^a_1}$ mapping $N^a_{1,i}$ onto $M^1_i = M_{i,0}$ and c^a_i to c_i; remember that $N^a_{1,\lambda\times\delta^a_1} = N^a_1$. Let $g_0 = g_1 \restriction N^a_0 = g_1 \restriction N^a_{1,0}$ so $g_0 \circ f^{-1}_0$ is an isomorphism from N^b_0 onto $M_{0,0}$.

Similarly as $\delta^b_1 = \delta^a_2$, and using the freedom we have in choosing $\overline{M}^2$ and $\langle d_i : i < \lambda \times \delta^b_1\rangle$ without loss of generality there is an isomorphism g_2 from $N^b_{1,\lambda\times\delta^a_2}$ onto $M^2_j = M_{0,\lambda\times\delta^a_2}$ mapping $N^b_{1,j}$ onto $M_{0,j}$ (for $j \le \lambda \times \delta^a_2$) and mapping c^b_i to d_i and g_2 extends $g_0 \circ f^{-1}_0$.
Now would like to use the weak uniqueness 6.16 and for this note:

(α) $\mathrm{NF}_{\lambda,\bar{\delta}^a}(N^a_0, N^a_1, N^a_2, N^a_3)$ is witnessed by the sequences $\langle N^a_{1,i} : i \le \lambda \times \delta^a_1\rangle$, and $\langle N^a_{2,i} : i \le \lambda \times \delta^a_1\rangle$
[why? an assumption]

(β) $\mathrm{NF}_{\lambda,\bar{\delta}^a}(M_{0,0}, M_{\lambda\times\delta^a_1,0}, M_{0,\lambda\times\delta^a_2}, M_3)$ is witnessed by the sequences $\langle M_{i,0} : i \le \lambda \times \delta^a_1\rangle, \langle M_{i,\lambda\times\delta^a_2} : i \le \lambda \times \delta^a_1\rangle$
[why? check]

(γ) g_0 is an isomorphism from N^a_0 onto $M_{0,0}$
[why? see its choice]

(δ) g_1 is an isomorphism from N^a_1 onto $M_{\lambda\times\delta^a_1,0}$ mapping $N^a_{1,i}$ onto $M_{i,0}$ for $i < \lambda \times \delta^a_1$ and c^a_i to c_i for $i < \lambda \times \delta^a_1$ and extending g_0
[why? see the choice of g_1 and of g_0]

(ε) $g_2 \circ f_2$ is an isomorphism from N^a_2 onto $M_{0,\lambda\times\delta^a_2}$ extending g_0
[why? f_2 is an isomorphism from N^a_2 onto N^b_1 and g_2 is an isomorphism from N^b_1 onto $M_{0,\lambda\times\delta^a_1}$ extending $g_0 \circ f^{-1}_0$ and $f_0 \subseteq f_2$].

So there is by 6.16 an isomorphism g^a_3 from N^a_3 onto M_3 extending both g_1 and $g_2 \circ f_2$.

We next would like to apply 6.16 to the N^b_i's, so note:

$(\alpha)'$ $\mathrm{NF}_{\lambda,\bar{\delta}^b}(N^b_0, N^b_1, N^b_2, N^b_3)$ is witnessed by the sequences $\langle N^b_{1,i} : i \le \lambda \times \delta^a_2\rangle$, $\langle N^b_{2,i} : i \le \lambda \times \delta^a_2\rangle$

$(\beta)'$ $\mathrm{NF}_{\lambda,\bar{\delta}^b}(M_{0,0}, M_{0,\lambda\times\delta^a_2}, M_{\lambda\times\delta^a_1,0}, M_3)$ is witnessed by the sequences $\langle M_{0,j} : j \le \lambda \times \delta^a_2\rangle, \langle M_{\lambda\times\delta^a_1,j} : j \le \lambda \times \delta^a_2\rangle$

$(\gamma)'$ $g_0 \circ (f_0)^{-1}$ is an isomorphism from N^b_0 onto $M_{0,0}$
[why? Check.]

$(\delta)'$ g_2 is an isomorphism from N^b_1 onto $M_{0,\lambda\times\delta^a_2}$ mapping $N^b_{1,j}$ onto $M_{0,j}$ and c^a_j to d_j for $j \le \lambda \times \delta^a_2$ and extending $g_0 \circ (f_2)^{-1}$
[why? see the choice of g_2: it maps $N^b_{1,j}$ onto $M_{0,j}$]

$(\varepsilon)'$ $g_1 \circ (f_1)^{-1}$ is an isomorphism from N^b_2 onto $M_{\lambda\times\delta^a_0}$ extending g_0
[why? remember f_1 is an isomorphism from N^a_1 onto N^b_2 extending f_0 and the choice of g_1: it maps N^a_1 onto $M_{\lambda\times\delta^a_1,0}$].

So there is an isomorphism g^b_3 form N^b_3 onto M_3 extending g_2 and $g_1 \circ (f_1)^{-1}$.
Lastly, $(g^b_3)^{-1} \circ g^a_3$ is an isomorphism from N^a_3 onto N^b_3 (chase arrows). Also

$$\begin{aligned}((g_3^b)^{-1} \circ g_3^a) \restriction N_1^a &= (g_3^b)^{-1}(g_3^a \restriction N_1^a) \\ &= (g_3^b)^{-1} g_1 = ((g_3^b)^{-1} \restriction M_{\lambda \times \delta_1^a, 0}) \circ g_1 \\ &= (g_3^b \restriction N_2^b)^{-1} \circ g_1 = ((g_1 \circ (f_1)^{-1})^{-1}) \circ g_1 \\ &= (f_1 \circ (g_1)^{-1}) \circ g_1 = f_1.\end{aligned}$$

Similarly $((g_3^b)^{-1} \circ g_3^a) \restriction N_2^a = f_2$.

So we have finished. $\square_{6.17}$

But if we invert twice we get straight; so

Claim 6.20. *[Uniqueness]. Assume for $x \in \{a, b\}$ we have $\mathrm{NF}_{\lambda, \bar\delta^x}(N_0^x, N_1^x, N_2^x, N_3^x)$ and $\mathrm{cf}(\delta_1^a) = \mathrm{cf}(\delta_1^b), \mathrm{cf}(\delta_2^a) = \mathrm{cf}(\delta_2^b), \mathrm{cf}(\delta_3^a) = \mathrm{cf}(\delta_3^b)$, all δ_ℓ^x limit ordinals $< \lambda^+$.*

If f_ℓ is an isomorphism from N_ℓ^a onto N_ℓ^b for $\ell < 3$ and $f_0 \subseteq f_1, f_0 \subseteq f_2$ then there is an isomorphism f from N_3^a onto N_3^b extending f_1, f_2.

Proof. Let $\bar\delta^c = \langle \delta_1^c, \delta_2^c, \delta_3^c \rangle = \langle \delta_2^a, \delta_1^a, \delta_3^a \rangle$; by 6.14(1) there are N_ℓ^c (for $\ell \le 3$) such that $\mathrm{NF}_{\lambda, \bar\delta^c}(N_0^c, N_1^c, N_2^c, N_3^c)$ and $N_0^c \cong N_0^a$. There is for $x \in \{a, b\}$ an isomorphism g_0^x from N_0^x onto N_0^c and without loss of generality $g_0^a = g_0^b \circ f_0$. Similarly for $x \in \{a, b\}$ there is an isomorphism g_1^x from N_1^x onto N_2^c extending g_0^x (as N_1^x is $(\lambda, \mathrm{cf}(\delta_1^x))$-brimmed over N_0^x and also N_2^c is $(\lambda, \mathrm{cf}(\delta_2^c))$-brimmed over N_0^c and $\mathrm{cf}(\delta_2^c) = \mathrm{cf}(\delta_1^a) = \mathrm{cf}(\delta_1^x)$) and without loss of generality $g_1^b = g_1^a \circ f_1$. Similarly for $x \in \{a, b\}$ there is an isomorphism g_2^x from N_2^x onto N_1^c extending g_0^x (as N_2^x is $(\lambda, \mathrm{cf}(\delta_2^x))$-brimmed over N_0^x and also N_1^c is $(\lambda, \mathrm{cf}(\delta_1^c))$-brimmed over N_0^c and $\mathrm{cf}(\delta_1^c) = \mathrm{cf}(\delta_2^a) = \mathrm{cf}(\delta_2^x)$) and without loss of generality $g_2^a = g_2^b \circ f_2$.
So by 6.17 for $x \in \{a, b\}$ there is an isomorphism g_3^x from N_3^x onto N_3^c extending g_1^x and g_2^x. Now $(g_3^b)^{-1} \circ g_3^a$ is an isomorphism from N_3^a onto N_3^b extending f_1, f_2 as required. $\square_{6.20}$

So we have proved the uniqueness for $\mathrm{NF}_{\lambda, \bar\delta}$ when all δ_ℓ are limit ordinals; this means that the arbitrary choice of $\langle N_{1,i} : i \le \lambda \times \delta_1 \rangle$ and $\langle c_i : i < \lambda \times \delta_1 \rangle$ is immaterial; it figures in the definition and, e.g. existence proof but does not influence the net result. The power of this result is illustrated in the following conclusion.

Conclusion 6.21. *[Symmetry].*

If $\mathrm{NF}_{\lambda, \langle \delta_1, \delta_2, \delta_3 \rangle}(N_0, N_1, N_2, N_3)$ where $\delta_1, \delta_2, \delta_3$ are limit ordinals $< \lambda^+$ then $\mathrm{NF}_{\lambda, \langle \delta_2, \delta_1, \delta_3 \rangle}(N_0, N_2, N_1, N_3)$.

Proof. By 6.18 we can find $N'_\ell (\ell \le 3)$ such that: $N'_0 = N_0, N'_1$ is $(\lambda, \mathrm{cf}(\delta_1))$-brimmed over N'_0, N'_2 is $(\lambda, \mathrm{cf}(\delta_2))$-brimmed over N'_0 and N'_3 is $(\lambda, \mathrm{cf}(\delta_3))$-brimmed over $N'_1 \cup N'_2$ and $\mathrm{NF}_{\lambda, \langle \delta_1, \delta_2, \delta_3 \rangle}(N'_0, N'_1, N'_2, N'_3)$ and $\mathrm{NF}_{\lambda, \langle \delta_2, \delta_1, \delta_3 \rangle}(N'_0, N'_2, N'_1, N'_3)$. Let f_1, f_2 be an isomorphism from N_1, N_2 onto N'_1, N'_2 over N_0, respectively. By 6.20 (or 6.17) there is an isomorphism f'_3 form N_3 onto N'_3 extending $f_1 \cup f_2$. As isomorphisms preserve NF, we are done. $\square_{6.21}$

Now we turn to smooth amalgamation (not necessarily brimmed, see Definition 6.13). If we use Lemma 4.9, of course, we do not really need 6.22.

Claim 6.22. *1) If* $\mathrm{NF}_{\lambda,\bar\delta}(N_0,N_1,N_2,N_3)$ *and* $\delta_1,\delta_2,\delta_3$ *are limit ordinals,* *then* $\mathrm{NF}_\lambda(N_0,N_1,N_2,N_3)$ *(see Definition 6.13).*
2) In Definition 6.13(1) we can add:

$(d)^+$ M_ℓ *is* $(\lambda,\mathrm{cf}(\lambda))$*-brimmed over* N_0 *and moreover over* N_ℓ,

(e) M_3 *is* $(\lambda,\mathrm{cf}(\lambda))$*-brimmed over* $M_1\cup M_2$ *(actually this is given by clause* $(f)(\zeta)$ *of Definition 6.12).*

3) If $N_0\le_{\mathfrak{k}} N_\ell$ *for* $\ell=1,2$ *and* $N_1\cap N_2=N_0$, *then* *we can find* N_3 *such that* $\mathrm{NF}_\lambda(N_0,N_1,N_2,N_3)$.

Proof. 1) Note that even if every δ_ℓ is limit and we waive the "moreover" in clause $(d)^+$, the problem is in the case that e.g. $(\mathrm{cf}(\delta^a),\mathrm{cf}(\delta^b),\mathrm{cf}(\delta^c))\neq(\mathrm{cf}(\lambda),\mathrm{cf}(\lambda),\mathrm{cf}(\lambda))$. For $\ell=1,2$ we can find $\overline{M}^\ell=\langle M^\ell_i:i\le\lambda\times(\delta_\ell+\lambda)\rangle$ and $\langle c^\ell_i:i<\lambda\times(\delta_i+\lambda)\rangle$ such that $M^\ell_0=N_0,\overline{M}^1$ is $\le_{\mathfrak{k}}$-increasing continuous $(M^\ell_i,M^\ell_{i+1},c_i)\in K^{3,\mathrm{uq}}_{\mathfrak{s}}$ and if $p\in\mathcal{S}^{\mathrm{bs}}(M^\ell_i)$ and $i<\lambda\times(\delta_\ell+\lambda)$ then for λ ordinals $j<\lambda$, $\mathrm{ortp}(c_i,M^\ell_{i+j},M^\ell_{i+j+1})$ is a non-forking extension of p. So $M^\ell_{\lambda\times\delta_\ell}$ is $(\lambda,\mathrm{cf}(\delta_\ell))$-brimmed over $M^\ell_0=N_0$ and $M^\ell_{\lambda\times(\delta_\ell+\lambda)}$ is $(\lambda,\mathrm{cf}(\lambda))$-brimmed over $M^\ell_{\lambda\times\delta_\ell}$; so without loss of generality $M^\ell_{\lambda\times\delta_\ell}=N_\ell$ for $\ell=1,2$.

By 6.18 we can find $M_{i,j}$ for $i\le\lambda\times(\delta_1+\lambda), j\le\lambda\times(\delta_2+\lambda)$ for $\bar\delta':=\langle\delta_1+\lambda,\delta_2+\lambda,\delta_3\rangle$ such that they are as in 6.18 for $\overline{M}^1,\overline{M}^2$ so $M_{0,0}=N_0$; then choose $M'_3\in K_\lambda$ which is $(\lambda,\mathrm{cf}(\delta_3))$-brimmed over $M_{\lambda\times\delta_1,\lambda\times\delta_2}$. So $\mathrm{NF}_{\lambda,\bar\delta}(M_{0,0},M_{\lambda\times\delta_1,0},M_{0,\lambda\times\delta_2},M'_3)$, hence by 6.20 without loss of generality $M_{0,0}=N_0, M_{\lambda\times\delta_1,0}=N_1$, $M_{0,\lambda\times\delta_2}=N_2$, and $N_3=M'_3$. Lastly, let M_3 be $(\lambda,\mathrm{cf}(\lambda))$-brimmed over M'_3. Now clearly also $\mathrm{NF}_{\lambda,\langle\delta_1+\lambda,\delta_2+\lambda,\delta_3+\lambda\rangle}(M_{0,0},M_{\lambda\times(\delta_1+\lambda),0},M_{0,\lambda\times(\delta_2+\lambda)},M_3)$ and $N_0=M_{0,0},N_1=M_{\lambda\times\delta_2,0}\le_{\mathfrak{k}}M_{\lambda\times(\delta_2+\lambda),0},N_2=M_{0,\lambda\times\delta_2}\le_{\mathfrak{k}}M_{0,\lambda\times(\delta_2+\lambda)}$ and $M_{\lambda\times(\delta_1+\lambda),0}$ is $(\lambda,\mathrm{cf}(\lambda))$-brimmed over $M_{\lambda\times\delta_1,0}$ and $M_{0,\lambda\times(\delta_2+\lambda)}$ is $(\lambda,\mathrm{cf}(\lambda))$-brimmed over $M_{0,\lambda\times\delta_2}$ and $N_3=M'_3\le_{\mathfrak{k}}M_3$. So we get all the requirements for $\mathrm{NF}_\lambda(N_0,N_1,N_2,N_3)$ (as witnessed by $\langle M_{0,0},M_{\lambda\times(\delta_1+\lambda),0},M_{0,\lambda\times(\delta_2+\lambda)},M_3\rangle$).
2) Similar proof.
3) By 6.14 and the proof above. $\square_{6.22}$

Now we turn to NF_λ; existence is easy.

Claim 6.23. NF_λ *has existence, i.e., clause (f) of 6.1(1).*

Proof. By 6.22(3). $\square_{6.23}$

Next we deal with real uniqueness

Claim 6.24. *[Uniqueness of smooth amalgamation]:*
1) If $\mathrm{NF}_\lambda(N^x_0,N^x_1,N^x_2,N^x_3)$ *for* $x\in\{a,b\}$, f_ℓ *an isomorphism from* N^a_ℓ *onto* N^b_ℓ *for* $\ell<3$ *and* $f_0\subseteq f_1, f_0\subseteq f_2$ *then* $f_1\cup f_2$ *can be extended to a* $\le_{\mathfrak{k}}$*-embedding of* N^a_3 *into some* $\le_{\mathfrak{k}}$*-extension of* N^b_3.
2) So if above N^x_3 *is* (λ,κ)*-brimmed over* $N^x_1\cup N^x_2$ *for* $x=a,b$, *we can extend* $f_1\cup f_2$ *to an isomorphism from* N^a_3 *onto* N^b_3.

Proof. 1) For $x\in\{a,b\}$ let the sequence $\langle M^x_\ell:\ell<4\rangle$ be a witness to $\mathrm{NF}_\lambda(N^x_0,N^x_1,N^x_2,N^x_3)$ as in 6.13, 6.22(2), so in particular $\mathrm{NF}_{\lambda,\langle\lambda,\lambda,\lambda\rangle}(M^x_0,M^x_1,M^x_2,M^x_3)$. By chasing arrows (disjointness) and uniqueness, i.e. 6.20 without loss of generality $M^a_\ell=M^b_\ell$ for $\ell<4$ and $f_0=\mathrm{id}_{N^a_0}$. As M^a_1 is $(\lambda,\mathrm{cf}(\lambda))$-brimmed over N^a_1 and also over N^b_1 (by clause $(d)^+$ of 6.22(2)) and f_1

is an isomorphism from N_1^a onto N_1^b, clearly by 1.17 there is an automorphism g_1 of M_1^a such that $f_1 \subseteq g_1$, hence also $\mathrm{id}_{N_0^a} = f_0 \subseteq f_1 \subseteq g_1$. Similarly there is an automorphism g_2 of M_2^a extending f_2 hence f_0. So $g_\ell \in \mathrm{Aut}(M_\ell^a)$ for $\ell = 1,2$ and $g_1 \restriction M_0^a = f_0 = g_2 \restriction M_0^a$. By the uniqueness of $\mathrm{NF}_{\lambda,\langle\lambda,\lambda,\lambda\rangle}$ (i.e. Claim 6.20) there is an automorphism g_3 of M_3^a extending $g_1 \cup g_2$. This proves the desired conclusion.
2) Should be clear. $\square_{6.24}$

We now show that in the cases the two notions of non-forking amalgamations are meaningful then they coincide, one implication already is a case of 6.22.

Claim 6.25. *Assume*

(a) $\bar\delta = \langle\delta_1,\delta_2,\delta_3\rangle, \delta_\ell < \lambda^+$ *is a limit ordinal for* $\ell = 1,2,3$;
$N_0 \le_{\mathfrak{k}} N_\ell \le_{\mathfrak{k}} N_3$ *are in* K_λ *for* $\ell = 1,2$

(b) N_ℓ *is* $(\lambda, \mathrm{cf}(\delta_\ell))$*-brimmed over* N_0 *for* $\ell = 1,2$

(c) N_3 *is* $\mathrm{cf}(\delta_3)$*-brimmed over* $N_1 \cup N_2$.

Then $\mathrm{NF}_\lambda(N_0,N_1,N_2,N_3)$ *iff* $\mathrm{NF}_{\lambda,\bar\delta}(N_0,N_1,N_2,N_3)$.

Proof. The "if" direction holds by 6.22(1). As for the "only if" direction, basically it follows from the existence for $\mathrm{NF}_{\lambda,\bar\delta}$ and uniqueness for NF_λ; in details by the proof of 6.22(1) (and Definition 6.12, 6.13) we can find $M_\ell (\ell \le 3)$ such that $M_0 = N_0$ and $\mathrm{NF}_{\lambda,\bar\delta}(M_0,M_1,M_2,M_3)$ and clauses (b), (c), (d) of Definition 6.13 and $(d)^+$ of 6.22(2) hold so by 6.22 also $\mathrm{NF}_\lambda(M_0,M_1,M_2,M_3)$. Easily there are for $\ell < 3$, isomorphisms f_ℓ from M_ℓ onto N_ℓ such that $f_0 = f_\ell \restriction M_\ell$ where $f_0 = \mathrm{id}_{N_0}$. By the uniqueness of smooth amalgamations (i.e., 6.24(2)) we can find an isomorphism f_3 from M_3 onto N_3 extending $f_1 \cup f_2$. So as $\mathrm{NF}_{\lambda,\bar\delta}(M_0,M_1,M_2,M_3)$ holds also $\mathrm{NF}_{\lambda,\bar\delta},(f_0(M_0),f_3(M_1),f_3(M_2),f_3(M_3))$; that is $\mathrm{NF}_{\lambda,\bar\delta}(N_0,N_1,N_2,N_3)$ is as required. $\square_{6.25}$

Claim 6.26. *[Monotonicity]: If* $\mathrm{NF}_\lambda(N_0,N_1,N_2,N_3)$ *and* $N_0 \le_{\mathfrak{k}} N_1' \le_{\mathfrak{k}} N_1$ *and* $N_0 \le_{\mathfrak{k}} N_2' \le_{\mathfrak{k}} N_2$ *and* $N_1' \cup N_2' \subseteq N_3' \le_{\mathfrak{k}} N_3'', N_3 \le_{\mathfrak{k}} N_3''$ *then* $\mathrm{NF}_\lambda(N_0,N_1',N_2',N_3')$.

Proof. Read Definition 6.13(1). $\square_{6.26}$

Claim 6.27. *[Symmetry]:* $\mathrm{NF}_\lambda(N_0,N_1,N_2,N_3)$ *holds if and only if* $\mathrm{NF}_\lambda(N_0,N_2,N_1,N_3)$ *holds.*

Proof. By Claim 6.21 (and Definition 6.13). $\square_{6.27}$

We observe

Conclusion 6.28. *If* $\mathrm{NF}_\lambda(N_0,N_1,N_2,N_3), N_3$ *is* (λ,∂)*-brimmed over* $N_1 \cup N_2$ *and* $\lambda \ge \partial, \kappa \ge \aleph_0$, *then there is* N_2^+ *such that*

(A) $\mathrm{NF}_\lambda(N_0,N_1,N_2^+,N_3)$

(B) $N_2 \le_{\mathfrak{k}} N_2^+$

(C) N_2^+ *is* (λ,κ)*-brimmed over* N_0 *and even over* N_2.

(D) N_3 *is* (λ,∂)*-brimmed over* $N_1 \cup N_2^+$.

Proof. Let N_2^+ be (λ,κ)-brimmed over N_2 be such that $N_2^+ \cap N_3 = N_2$. So by existence 6.23 there is N_3^+ such that $\mathrm{NF}_\lambda(N_0,N_1,N_2^+,N_3^+)$ and N_3^+ is (λ,∂)-brimmed over $N_1 \cup N_2^+$. By monotonicity 6.26 we have $\mathrm{NF}_\lambda(N_0,N_1,N_2,N_3^+)$. So by uniqueness (i.e., 6.24(2)) without loss of generality $N_3 = N_3^+$, so we are done. $\square_{6.28}$

The following claim is a step toward proving transitivity for NF_λ; so we first deal with $\mathrm{NF}_{\lambda,\bar\delta}$. Note below: if we ignore N_i^c we have problem showing $\mathrm{NF}_{\lambda,\bar\delta}(N_0^a,N_\alpha^a,N_0^b,N_\alpha^b)$. Note that it is not clear at this stage whether, e.g. N_ω^b is even universal over N_ω^a, but N_ω^c is; note that the N_i^c are $\le_{\mathfrak{k}}$-increasing with i but not necessarily continuous. However once we finish proving that NF_λ is a non-forking relation on $\mathfrak{k}_{\mathfrak{s}}$ respecting $\mathfrak{s}$ this claim will lose its relevance.

Claim 6.29. *Assume $\alpha < \lambda^+$ is an ordinal and for $x \in \{a,b,c\}$ the sequence $\overline{N}^x = \langle N_i^x : i \le \alpha\rangle$ is a $\le_{\mathfrak{k}}$-increasing sequence of members of K_λ, and for $x = a,b$ the sequence $\overline{N}^x$ is $\le_{\mathfrak{k}}$-increasing continuous, $N_i^b \cap N_\alpha^a = N_i^a, N_i^c \cap N_\alpha^a = N_i^a, N_i^a \le_{\mathfrak{k}} N_i^b \le_{\mathfrak{k}} N_i^c$ and N_0^b is (λ,δ_2)-brimmed over N_0^a and $\mathrm{NF}_{\lambda,\bar\delta^i}(N_i^a,N_{i+1}^a,N_i^c,N_{i+1}^b)$ (so necessarily $i < \alpha \Rightarrow N_i^c \le_{\mathfrak{k}} N_{i+1}^b$) where*
$\bar\delta^i = \langle \delta_1^i,\delta_2^i,\delta_3^i\rangle$ with $\delta_1^i,\delta_2^i,\delta_3^i$ are ordinals $< \lambda^+$ and $\delta_3 < \lambda^+$ is limit, N_α^c is $(\lambda,\mathrm{cf}(\delta_3))$-brimmed over N_α^b, $\delta_1 = \sum_{\beta<\alpha} \delta_1^\beta$ and $\delta_3 = \delta_3^\alpha$ and $\delta_2 = \delta_2^0, \bar\delta = \langle \delta_1,\delta_2,\delta_3\rangle$.

<u>Then</u> $\mathrm{NF}_{\lambda,\bar\delta}(N_0^a,N_\alpha^a,N_0^b,N_\alpha^c)$.

Proof. For $i < \alpha$ let $\langle N_{1,\varepsilon}^i, N_{2,\varepsilon}^i, d_\zeta^i : \varepsilon \le \lambda \times \delta_1^i, \zeta < \lambda \times \delta_1^i\rangle$ be a witness to $\mathrm{NF}_{\lambda,\bar\delta^i}(N_i^a,N_{i+1}^a,N_i^c,N_{i+1}^b)$. Now we define a sequence $\langle N_{1,\varepsilon}, N_{2,\varepsilon}, d_\zeta^i : \varepsilon \le \lambda \times \delta_1$ and $\zeta < \lambda \times \delta_1\rangle$ where

- (a) $N_{1,0} = N_0^a, N_{2,0} = N_0^b$ and
- (b) if $\lambda \times (\sum_{j<i} \delta_1^j) < \zeta \le \lambda \times (\sum_{j\le i} \delta_1^j)$ then we let $N_{1,\zeta} = N_{1,\varepsilon_\zeta}^i, N_{2,\zeta} = N_{2,\varepsilon_\zeta}^i$ where $\varepsilon_\zeta = \zeta - \lambda \times (\sum_{j<i} \delta_1^j)$ and
- (c) if $0 < \zeta = \lambda \times \sum_{j<\alpha} \delta_1^j$ we let $N_{1,\zeta} = N_i^a, N_{2,\zeta} = N_i^b = \alpha$ (if i is non-limit we should note that this is compatible with clause (b), note that by this if $i = \alpha$ then $N_{1,\zeta} = N_\alpha^a, N_{2,\zeta} = \cup\{N_{2,\lambda\times\delta_1}^i : i < \alpha\}$
- (d) if $\lambda \times (\sum_{j<i} \delta_1^j) \le \zeta < \lambda \times (\sum_{j\le i} \delta_1^j)$ then we let $d_\zeta = d_{\varepsilon_\zeta}^i$ where $\varepsilon_\zeta = \zeta - \lambda \times (\sum_{j<i} \delta_j^j) = \cup\{N_{2,\zeta}^* : \zeta < \lambda \times (\sum_{j<\alpha} \delta_1^j)$.

Clearly $\langle N_{1,\zeta} : \zeta \le \lambda\times\delta_1\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous, and also $\langle N_{2,\zeta} : \zeta \le \lambda\times\delta_1\rangle$ is. Obviously $(N_{1,\zeta}, N_{1,\zeta+1}, d_\zeta) \in K_\lambda^{3,\mathrm{uq}}$ as this just means $(N_{1,\varepsilon_\zeta}^i, N_{1,\varepsilon_\zeta+1}^i, d_\zeta^i) \in K_\lambda^{3,\mathrm{uq}}$ when $\lambda \times \sum_{j<i} \delta_1^j : j \le \zeta < \lambda \times \sum_{j\le i} \delta_1^j$ and ε_ζ as above.

Why $\mathrm{ortp}(d_\zeta, N_{2,\zeta}, N_{2,\zeta+1})$ does not fork over $N_{1,\zeta}$ for ζ, i such that $\lambda\times(\sum_{j<i} \delta_1^j)\zeta < \lambda \times (\sum_{j\le i} \delta_j^j)$? If $\lambda \times \sum_{j<i} \delta_1^j < \zeta$ this holds as it means $\mathrm{ortp}(d_{\varepsilon_\zeta}^i, N_{2,\varepsilon_\zeta}^i, N_{2,\varepsilon_\zeta+1}^i)$ does not fork over $N_{1,\zeta}^i$. If $\lambda \times \sum_{j<i} \delta_1^j = \zeta$ this is not the case but $N_{1,0}^i = N_{1,\zeta} \le_{\mathfrak{k}} N_{2,\zeta} \le_{\mathfrak{k}} N_i^c = N_{2,0}^i$ and we know that $\mathrm{ortp}(d_\zeta, N_{2,0}^i, N_{2,1}^i)$ does not fork over $N_{1,0}^i = N_{1,\zeta}$ hence by monotonicity of non-forking $\mathrm{ortp}(d_\zeta, N_{2,\zeta}, N_{2,\zeta+1})$ does not fork over $N_{1,\zeta}$ is as required.

Note that we have not demanded or used "$\overline{N}^c$ continuous"; the N_i^c is really needed for i limit as we do not know that N_i^b is brimmed over N_i^a. $\square_{6.29}$

Claim 6.30. *[transitivity] 1) Assume that $\alpha < \lambda^+$ and for $x \in \{a,b\}$ we have $\langle N^x_i : i \le \alpha \rangle$ is a $\le_{\mathfrak{k}}$-increasing continuous sequence of members of K_λ. If $\mathrm{NF}_\lambda(N^a_i, N^a_{i+1}, N^b_i, N^b_{i+1})$ for each $i < \alpha$ then $\mathrm{NF}_\lambda(N^a_0, N^a_\alpha, N^b_0, N^b_\alpha)$.*

2) Assume that $\alpha_1 < \lambda^+, \alpha_2 < \lambda^+$ and $M_{i,j} \in K_\lambda$ (for $i \le \alpha_1, j \le \alpha_2$) satisfy clauses (B), (C), (D), from 6.18, and for each $i < \alpha_1, j < \alpha_2$ we have:

$$M_{i,j+1} \underset{M_{i,j}}{\overset{M_{i+1,j+1}}{\bigcup}} M_{i+1,j}.$$

$$\underline{\text{Then}}\ M_{i,0} \underset{M_{0,0}}{\overset{M_{\alpha_1,\alpha_2}}{\bigcup}} M_{0,j} \text{ for } i \le \alpha_1, j \le \alpha_2.$$

Proof. 1) We first prove special cases and use them to prove more general cases.

Case A: N^a_{i+1} is (λ,κ_i)-brimmed over N^a_i and N^b_{i+1} is (λ,∂_i)-brimmed over $N^a_{i+1} \cup N^b_i$ for $i < \alpha$ (∂_i infinite, of course).

In essence the problem is that we do not know "N^b_i is brimmed over N^a_i" (i limit) so we shall use 6.29; for this we introduce appropriate N^c_i.

Let $\delta^i_1 = \kappa_i, \delta^i_2 = \kappa_i, \delta^i_3 = \partial_i$ where we stipulate $\partial_\alpha = \lambda$. For $i \le \alpha$ we can choose $N^c_i \in K_\lambda$ such that

- (a) $N^b_i \le_{\mathfrak{k}} N^c_i \le_{\mathfrak{k}} N^b_{i+1}, N^c_i$ is (λ,κ_i)-brimmed over N^b_i, and $\mathrm{NF}_{\lambda,\langle \delta^i_1,\delta^i_2,\delta^i_3\rangle}(N^a_i, N^a_{i+1}, N^c_i, N^b_{i+1})$
- (b) $N^c_\alpha \in K_\lambda$ is $(\lambda,\delta^\alpha_3)$-brimmed over N^b_α
- (c) $\langle N^c_i : i < \alpha\rangle$ is $\le_{\mathfrak{k}}$-increasing (in fact follows)

(Possible by 6.28). Now we can use 6.29. **Case B**: For each $i < \alpha$ we have: N^a_{i+1} is (λ,κ_i)-brimmed over N^a_i.

In essence our problem is that we do not know anything about brimmedness of the N^b_i, so we shall "correct it".

Let $\bar\delta^i = (\kappa_i,\lambda,\lambda)$.

We can find a $\le_{\mathfrak{k}}$-increasing sequence $\langle M^x_i : i \le \alpha\rangle$ of models in K_λ for $x \in \{a,b,c\}$, continuous for $x = a,b$ such that $i < \alpha \Rightarrow M^a_i \le_{\mathfrak{k}} M^b_i \le_{\mathfrak{k}} M^c_i \le_{\mathfrak{k}} M^b_{i+1}$ and $M^b_\alpha \le_{\mathfrak{k}} M^c_\alpha$ and M^c_i is (λ,κ_i)-brimmed over M^b_i (hence over M^a_i) and $\mathrm{NF}_{\lambda,\bar\delta^i}(M^a_i, M^a_{i+1}, M^c_i, M^b_{i+1})$ by choosing M^a_i, M^b_i, M^c_i by induction on $i, M^a_0 = N^a_0$ and M^b_0 is universal over M^a_0 recalling that the $\mathrm{NF}_{\lambda,\bar\delta^i}$ implies some brimmedness condition, e.g. M^b_{i+1} is $(\lambda,\mathrm{cf}(\delta^i_3))$-brimmed over $M^a_{i+1} \cup M^b_i$. By Case A we know that $\mathrm{NF}_\lambda(M^a_0, M^a_\alpha, M^b_0, M^c_\alpha)$ holds.

We can now choose an isomorphism f^a_0 from N^a_0 onto M^a_0, as the identity (exists as $M^a_0 = N^a_0$) and then a $\le_{\mathfrak{k}}$-embedding f^b_0 of N^b_0 into M^b_0 extending f^a_0. Next we choose by induction on $i \le \alpha, f^a_i$ an isomorphism from N^a_i onto M^a_i such that: $j < i \Rightarrow f^a_j \subseteq f^a_i$, possible by "uniqueness of the (λ,κ_i)-brimmed model over M^a_i" so here we are using the assumption of this case.

Now we choose by induction on $i \le \alpha$, a $\le_{\mathfrak{k}}$-embedding f^b_i of N^b_i into M^b_i extending f^a_i and f^b_j for $j < i$. For $i = 0$ we have done it, for i limit use $\bigcup\limits_{j<i} f^b_j$, lastly for i a successor ordinal let $i = j+1$, now we have

$(*)_2$ $\mathrm{NF}_\lambda(M^a_j, M^a_{j+1}, f^b_j(N^b_j), M^b_{j+1})$
[why? because $\mathrm{NF}_{\lambda,\bar\delta^j}(M^a_j, M^a_{j+1}, M^c_j, M^b_{j+1})$ by the choice of the M^x_ζ's hence by 6.25 we have $\mathrm{NF}_\lambda(M^a_j, M^a_{j+1}, M^c_j, M^b_{j+1})$ and as $M^a_j = f^a_j(N^a_j) \le_{\mathfrak{k}} f^b_j(N^b_j) \le M^b_j \le_{\mathfrak{k}} M^c_j$ by 6.26 we get $(*)_2$.]

By $(*)_2$ and the uniqueness of smooth amalgamation 6.24 and as M^b_{j+1} is $(\lambda, \mathrm{cf}(\delta^3_j))$-brimmed over $M^a_{j+1} \cup M^b_j$ hence over $M^a_{j+1} \cup f^b_j(N^b_j)$ clearly there is f^b_i as required. So without loss of generality f^a_α is the identity, so we have $N^a_0 = M^a_0, N^a_\alpha = M^a_\alpha, N^b_0 \leq_{\mathfrak{k}} M^b_0, N^b_\alpha \leq_{\mathfrak{k}} M^b_\alpha$; also as said above $\mathrm{NF}_\lambda(M^a_0, M^a_\alpha, M^b_0, M^b_\alpha)$ holds (using Case A) so by monotonicity, i.e., 6.26 we get $\mathrm{NF}_\lambda(N^a_0, N^a_\alpha, N^b_0, N^b_\alpha)$ as required.
Case C: General case.

We can find M^ℓ_i for $\ell < 3, i \leq \alpha$ such that (note that $M^1_0 = M^0_0$):

- (a) $M^\ell_i \in K_\lambda$
- (b) for each $\ell < 3, M^\ell_i$ is $\leq_{\mathfrak{k}}$-increasing in i (but for $\ell = 1, 2$ they are not required to be continuous)
- (c) $M^0_i = N^a_i$
- (d) $M^{\ell+1}_{i+1}$ is (λ, λ)-brimmed over $M^\ell_{i+1} \cup M^{\ell+1}_i$ for $\ell < 2, i < \alpha$
- (e) $\mathrm{NF}_\lambda(M^\ell_i, M^\ell_{i+1}, M^{\ell+1}_i, M^{\ell+1}_{i+1})$ for $\ell < 2, i < \alpha$
- (f) $M^1_0 = M^0_0$ and M^2_0 is $(\lambda, \mathrm{cf}(\lambda))$-brimmed over M^1_0
- (g) for $\ell < 2$ and $i < \alpha$ limit we have
$$M^{\ell+1}_i \text{ is } (\lambda, \lambda)\text{-brimmed over } \bigcup_{j<i} M^{\ell+1}_j \cup M^\ell_i$$
- (h) for $i < \alpha$ limit we have
$$\mathrm{NF}_\lambda(\bigcup_{j<i} M^1_j, M^1_i, \bigcup_{j<i} M^2_j, M^2_i).$$

  [How? As in the proof of 6.18 or just do by hand.]

Now note:

- $(*)_3$ $M^{\ell+1}_i$ is $(\lambda, \mathrm{cf}(\lambda \times (1+i)))$-brimmed over M^ℓ_i if $\ell = 1 \vee i \neq 0$
  [why? If $i = 0$ by clause (f), if i a successor ordinal by clause (d) and if i is a limit ordinal then by clause (g)]
- $(*)_4$ for $i < \alpha, \mathrm{NF}_\lambda(M^0_i, M^0_{i+1}, M^2_i, M^2_{i+1})$.
  [Why? If $i = 0$ by clause (e) for $\ell = 1, i = 0$ we get $\mathrm{NF}_\lambda(M^1_0, M^1_1, M^2_0, M^2_1)$ so by clause (f) (i.e., $M^1_0 = M^0_0$) and monotonicity (i.e., Claim 6.26) we have $\mathrm{NF}_\lambda(M^0_0, M^1_0, M^2_0, M^2_1)$ as required. If $i > 0$ we use Case B for $\alpha = 2$ with $M^0_i, M^0_{i+1}, M^1_i, M^1_{i+1}, M^2_i, M^2_{i+1}$ here standing for $N^a_0, N^b_0, N^a_1, N^b_1, N^a_2, N^b_2$ there (and symmetry).]

Let us define N^ℓ_i for $\ell < 3, i \leq \alpha$ by: N^ℓ_i is M^ℓ_i if i is non-limit and $N^\ell_i = \cup\{N^\ell_j : j < i\}$ if i is limit.

- $(*)_5(i)$ $\langle N^\ell_i : i \leq \alpha\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous, $N^0_i = N^a_i$ and $N^\ell_i \leq_{\mathfrak{k}} M^\ell_i$
  - (ii) for $i < \alpha$, $\mathrm{NF}_\lambda(N^0_i, N^0_{i+1}, N^2_i, N^2_{i+1})$
    [why? by $(*)_4$+ monotonicity of NF_λ]
  - (iii) for $i < \alpha, N^2_{i+1}$ is $(\lambda, \mathrm{cf}(\lambda))$-brimmed over $N^0_{i+1} \cup N^2_i$ and even over $N^1_{i+1} \cup N^2_i$
    [why? by clause (d)]
- $(*)_6$ $\mathrm{NF}_{\lambda,\langle\lambda,\lambda,1\rangle}(N^1_0, N^1_\alpha, N^2_0, N^2_\alpha)$.
  [Why? As we have proved case A (or, if you prefer, by 6.29; easily the assumption there holds).]

Choose $f^a_i = \mathrm{id}_{N^a_i}$ for $i \leq \alpha$ and let f^b_0 be a $\leq_{\mathfrak{k}}$-embedding of N^b_0 into N^2_0.

Now we continue as in Case B defining by induction on i a $\leq_{\mathfrak{k}}$-embedding f^b_i of N^b_i into N^2_i, the successor case is possible by $(*)_5(ii) + (*)_5(iii)$. In the end by $(*)_6$ and monotonicity of NF_λ (i.e., Claim 6.26) we are done.
2) Apply for each $i < \alpha_2$ part (1) to the sequences $\langle M_{\beta,i} : \beta \leq \alpha_1\rangle, \langle M_{\beta,i+1} :$

$\beta \le \alpha_1\rangle$ so we get $M_{\alpha_1,i} \underset{M_{0,i}}{\overset{M_{\alpha_1,i+1}}{\bigcup}} M_{0,i+1}$ hence by symmetry (i.e., 6.24) we have $M_{0,i+1} \underset{M_{0,i}}{\overset{M_{\alpha_1,i+1}}{\bigcup}} M_{\alpha_1,i}$. Applying part (1) to the sequences $\langle M_{0,j} : j \le \alpha_2\rangle, \langle M_{\alpha_1,j} : j \le \alpha_2\rangle$ we get $M_{0,\alpha_2} \underset{M_{0,0}}{\overset{M_{\alpha_1,\alpha_2}}{\bigcup}} M_{\alpha_1,0}$ hence by symmetry (i.e. 6.24) we have $M_{\alpha_1,0} \underset{M_{0,0}}{\overset{M_{\alpha_1,\alpha_2}}{\bigcup}} M_{0,\alpha_2}$; so we get the desired conclusion. $\square_{6.30}$

Claim 6.31. *Assume $\alpha < \lambda^+, \langle N^\ell_i : i \le \alpha\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous sequence of models for $\ell = 0, 1$ where $N^\ell_i \in K_\lambda$ and N^1_{i+1} is (λ, κ_i)-brimmed over $N^0_{i+1} \cup N^1_i$ and $\mathrm{NF}_\lambda(N^0_i, N^1_i, N^0_{i+1}, N^1_{i+1})$.*

Then N^1_α is $(\lambda, \mathrm{cf}(\sum\limits_{i<\alpha} \kappa_i))$-brimmed over $N^0_\alpha \cup N^1_0$.

Remark 6.32. 1) If our framework is uni-dimensional (see [She09e, §2]; as for example when it comes from [She01]) we can simplify the proof.

2) Assuming only "N^1_{i+1} is universal over $N^0_{i+1} \cup N^1_i$" suffices when α is a limit ordinal, i.e., we get N^1_α is $(\lambda, \mathrm{cf}(\alpha))$-brimmed over N^0_α. Why? We choose N^2_j for $j \le i$ such that $N^2_j = N^1_j$ if $j = 0$ or j a limit ordinal and N^2_j is a model $\le_{\mathfrak{s}} N^1_j$ and (λ, κ_1)-brimmed over $N^0_j \cup N^1_i$ when $j = i+1$. Now $\langle N^2_j : j \le \alpha\rangle$ satisfies all the requirements in $\langle N^1_j : j \le \alpha\rangle$ in 6.31.

3) We could have proved this earlier and used it, e.g. in 6.30.

Proof. The case α not a limit ordinal is trivial so assume α is a limit ordinal. We choose by induction on $i \le \alpha$, an ordinal $\varepsilon(i)$ and a sequence $\langle M_{i,\varepsilon} : \varepsilon \le \varepsilon(i)\rangle$ and $\langle c_\varepsilon : \varepsilon < \varepsilon(i)$ non-limit$\rangle$ such that:

- (a) $\langle M_{i,\varepsilon} : \varepsilon \le \varepsilon(i)\rangle$ is (strictly) $<_{\mathfrak{k}}$-increasing continuous in K_λ.
- (b) $N^0_i \le_{\mathfrak{k}} M_{i,\varepsilon} \le_{\mathfrak{k}} N^1_i$
- (c) $N^0_i = M_{i,0}$ and $N^1_i = M_{i,\varepsilon(i)}$.
- (d) $\varepsilon(i)$ is (strictly) increasing continuous in i and $\varepsilon(i)$ is divisible by λ.
- (e) $j < i$ and $\varepsilon \le \varepsilon(j) \Rightarrow M_{i,\varepsilon} \cap N^1_j = M_{j,\varepsilon}$.
- (f) For $j < i$ and $\varepsilon \le \varepsilon(j+1)$, the sequence $\langle M_{\beta,\varepsilon} : \beta \in (j,i]\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous.
- (g) For $j < i$ and $\varepsilon < \varepsilon(j)$ non-limit, the type $\mathrm{ortp}(c_\varepsilon, M_{i,\varepsilon}, M_{i,\varepsilon+1}) \in \mathcal{S}^{\mathrm{bs}}(M_{i,\varepsilon})$ does not fork over $M_{j,\varepsilon}$. (Actually, allowing all ε here is OK as well.)
- (h) $M_{i+1,\varepsilon+1}$ is $(\lambda, \mathrm{cf}(\lambda))$-brimmed over $M_{i+1,\varepsilon} \cup M_{i,\varepsilon+1}$.
- (i) If $\varepsilon < \varepsilon(i)$ and $p \in \mathcal{S}^{\mathrm{bs}}(M_{i,\varepsilon})$ then, for λ successor ordinals $\xi \in [\varepsilon, \varepsilon(i))$, the type $\mathrm{ortp}(c_\xi, M_{i,\xi}, M_{i,\xi+1})$ is a non-forking extension of p.

If we succeed, then $\langle M_{\alpha,\varepsilon} : \varepsilon \le \varepsilon(\alpha)\rangle$ is a (strictly) $<_{\mathfrak{k}}$-increasing continuous sequence of models from K_λ, $M_{\alpha,0} = N^0_\alpha$, and $M_{\alpha,\varepsilon(\alpha)} = N^1_\alpha$. We can apply 4.3 and we conclude that $N^1_\alpha = M_{\alpha,\varepsilon(\alpha)}$ is $(\lambda, \mathrm{cf}(\alpha))$-brimmed over $M_{\alpha,\varepsilon(j)}$ hence over $N^0_\alpha \cup N^1_0$ (both $\le_{\mathfrak{k}} M_{\alpha,1}$).

Carrying the induction is easy. For $i = 0$, there is not much to do. For i successor we use "N^j_{i+1} is brimmed over $N^0_{i+1} \cup N^1_i$" the existence of non-forking amalgamations and 4.2, bookkeeping and the extension property $(E)(g)$. For i limit we have no problem. $\square_{6.31}$

Conclusion 6.33. *1) If* $\mathrm{NF}_\lambda(N_0, N_1, N_2, N_3)$ *and* $\langle M_{0,\varepsilon} : \varepsilon \le \varepsilon(*)\rangle$ *is an* $\le_{\mathfrak{k}}$*-increasing continuous sequence of models from* K_λ*,* $N_0 \le_{\mathfrak{k}} M_{0,\varepsilon} \le_{\mathfrak{k}} N_2$ *then we can find* $\langle M_{1,\varepsilon} : \varepsilon \le \varepsilon(*)\rangle$ *and* N'_3 *such that:*

- *(a)* $N_3 \le_{\mathfrak{k}} N'_3 \in K_\lambda$
- *(b)* $\langle M_{1,\varepsilon} : \varepsilon \le \varepsilon(*)\rangle$ *is* $\le_{\mathfrak{k}}$*-increasing continuous*
- *(c)* $M_{1,\varepsilon} \cap N_2 = M_{0,\varepsilon}$
- *(d)* $N_1 \le_{\mathfrak{k}} M_{1,\varepsilon} \le_{\mathfrak{k}} N'_3$
- *(e) if* $M_{0,0} = N_0$ *then* $M_{1,0} = N_1$
- *(f)* $\mathrm{NF}_\lambda(M_{0,\varepsilon}, M_{1,\varepsilon}, N_2, N'_3)$*, for every* $\varepsilon \le \varepsilon(*)$*.*

2) If N_3 *is universal over* $N_1 \cup N_2$*, then without loss of generality* $N'_3 = N_3$*.*

3) In part (1) we can add

- *(g)* $M_{1,\varepsilon+1}$ *is brimmed over* $M_{0,\varepsilon+1} \cup M_{1,\varepsilon}$*.*

Proof. 1) Define $M'_{0,i}$ for $i \le \varepsilon^* := 1 + \varepsilon(*) + 1$ by $M'_{0,0} = N_0, M'_{0,1+\varepsilon} = M_{0,\varepsilon}$ for $\varepsilon \le \varepsilon(*)$ and $M'_{0,1+\varepsilon(*)+1} = N_2$. By existence (6.23) we can find an $\le_{\mathfrak{k}}$-increasing continuous sequence $\langle M'_{1,\varepsilon} : \varepsilon \le \varepsilon^*\rangle$ with $M'_{1,0} = N_1$ and $\le_{\mathfrak{k}}$-embedding f of N_2 into M'_{1,ε^*} such that $\varepsilon < \varepsilon^* \Rightarrow \mathrm{NF}_\lambda(f(M'_{0,\varepsilon}), M'_{1,0}, f(M'_{0,\varepsilon+1}), M'_{1,\varepsilon+1})$. By transitivity we have $\mathrm{NF}_\lambda(f(M'_{0,0}), M'_{1,0}, f(M'_{0,\varepsilon^*}), M'_{1,\varepsilon^*})$. By disjointness (i.e., $f(M'_{0,\varepsilon^*}) \cap M'_{1,0} = M'_{0,0}$, see 6.14(3)) without loss of generality f is the identity. By uniqueness for NF there are $N'_3, N_3 \le_{\mathfrak{k}} N'_3 \in K_\lambda$ and $\le_{\mathfrak{k}}$-embedding of M'_{1,ε^*} onto N'_3 over $N_1 \cup N_2 = M'_{0,\varepsilon^*} \cup M'_{1,0}$ so we are done.

2) Follows by (1).

3) Similar to (1). $\square_{6.33}$

Claim 6.34. NF_λ *respects* $\mathfrak{s}$*; that is assume* $\mathrm{NF}_\lambda(M_0, M_1, M_2, M_3)$ *and* $a \in M_1 \setminus M_0$ *satisfies* $\mathrm{ortp}(a, M_0, M_3) \in \mathcal{S}^{\mathrm{bs}}(M_0)$*, then* $\mathrm{ortp}(a, M_2, M_3) \in \mathcal{S}^{\mathrm{bs}}(M_2)$ *does not fork over* M_0*.*

Proof. Without loss of generality M_1 is $(\lambda, *)$-brimmed over M_0. [Why? By the existence we can find M_1^+ which is a $(\lambda, *)$-brimmed extension of M_1. By the existence for NF_λ without loss of generality we can find M_3^+ such that $\mathrm{NF}_\lambda(M_1, M_1^+, M_3, M_3^+)$, hence by transitivity for NF_λ we have $\mathrm{NF}_\lambda(M_0, M_1^+, M_2, M_3^+)$.] By the hypothesis of the section there are M'_1, a' such that $M_0 \cup \{a'\} \subseteq M'_1$ and $\mathrm{ortp}(a', M_0, M'_1) = \mathrm{ortp}(a, M_0, M_1)$ and $(M_0, M'_1, a) \in K^{3,\mathrm{uq}}_\lambda$; as M_1^+ is $(\lambda, *)$-brimmed over M_0 without loss of generality $M' \le_{\mathfrak{k}} M_1^+$ and $a' = a$ and M_1 is $(\lambda, *)$-brimmed over M'_1. We can apply 6.10 to M'_1, M_1^+ getting $\langle M^*_i, a_i : i \le \delta < \lambda^+\rangle$ as there. Let M'_i be: M_0 if $i = 0, M^*_j$ if $1 + j = i$ so $M'_1 = M^*_0 = M'_1$ and let a_i be a if $i = 0, a_j$ if $1+j = i$. So we can find M'_3 and f such that $M_2 \le_{\mathfrak{k}} M'_3, f$ is a $\le_{\mathfrak{k}}$-embedding of M_1^+ into M'_3 extending id_{M_0} such that $\mathrm{NF}_{\lambda,\langle\delta,\lambda,\lambda\rangle}(M_0, f(M_1^+), M_2, M'_3)$ and M'_3, this is witnessed by $\langle f(M'_i) : i \le \delta\rangle, \langle M''_i : i \le \delta\rangle, \langle f(a_i) : i < \delta\rangle$ and $M''_0 = M_2$; this is possible by 6.14(2). Hence $\mathrm{NF}_\lambda(M_0, f(M_1^+), M_2, N) = \mathrm{NF}_\lambda(f(M'_0), f(M'_\delta), M''_0, N)$ hence by the uniqueness for NF_λ without loss of generality $f = \mathrm{id}_{M_1^+}$ and $M_3 \le_{\mathfrak{k}} N$. By the choice of f, N we have that $\mathrm{ortp}(a, M_2, M_3) = \mathrm{ortp}(a_0, M_2, N) = \mathrm{ortp}(a_0, M''_0, M'_1) \in \mathcal{S}^{\mathrm{bs}}(M''_0) = \mathcal{S}^{\mathrm{bs}}(M_2)$ does not fork over $M'_0 = M_0$ as required. $\square_{6.34}$

Conclusion 6.35. *If $M_0 \leq_{\mathfrak{k}} M_\ell \leq_{\mathfrak{k}} M_3$ for $\ell = 1, 2$ and $(M_0, M_1, a) \in K^{3,\rm uq}_\lambda$ and $\mathrm{ortp}(a, M_2, M_3) \in \mathcal{S}^{\rm bs}(M_2)$ does not fork over M_0 then $\mathrm{NF}(M_0, M_1, M_2, M_3)$.*

Proof. By the definition of $K^{3,\rm uq}_\lambda$ and existence for NF_λ and 6.34 (or use 6.3 + 6.36. $\square_{6.35}$

We can sum up our work by

Conclusion 6.36. *[Main Conclusion] NF_λ is a non-forking relation on ${}^4(\mathfrak{k}_\lambda)$ which respects $\mathfrak{s}$.*

Proof. We have to check clauses (a)-(g)+(h) from 6.1. Clauses (a),(b) hold by the Definition 6.13 of NF_λ. Clauses $(c)_1, (c)_2$, i.e., monotonicity hold by 6.26. Clause (d), i.e., symmetry holds by 6.27. Clause (e), i.e., transitivity holds by 6.30. Clause (f), i.e., existence hold by 6.23. Clause (g), i.e., uniqueness holds by 6.24.

Lastly, clause (h), i.e., NF_λ respecting $\mathfrak{s}$ by 6.34. $\square_{6.36}$

The following definition is not needed for now but is natural (of course, we can omit "there is superlimit" from the assumption and the conclusion). For the rest of the section we stop assuming Hypothesis 6.9.

Definition 6.37. 1) A good λ-frame $\mathfrak{s}$ is type-full when for $M \in \mathfrak{k}_{\mathfrak{s}}, \mathcal{S}^{\rm bs}(M) = \mathcal{S}^{\rm na}_{\mathfrak{k}_\lambda}(M)$.
2) Assume $\mathfrak{k}_\lambda$ is a λ-AEC and NF is a 4-place relation on K_λ. We define $\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}_\lambda,\mathrm{NF}} = (K_{\mathfrak{t}}, \bigcup_{\mathfrak{t}}, \mathcal{S}^{\rm bs}_{\mathfrak{t}})$ as follows:

- (A) $\mathfrak{k}_{\mathfrak{t}}$ is the λ-AEC $\mathfrak{k}_\lambda$
- (B) $\mathcal{S}^{\rm bs}_{\mathfrak{t}}(M)$ is $\mathcal{S}^{\rm na}_{\mathfrak{k}_\lambda}(M)$ for $M \in \mathfrak{k}_\lambda$
- (C) $\bigcup_{\mathfrak{t}}$ is defined by: $(M_0, M_1, a, M_3) \in \bigcup_{\mathfrak{t}}$ when we can find M_2, M_3' such that $M_0 \leq_{\mathfrak{k}_\lambda} M_2 \leq_{\mathfrak{k}_\lambda} M_3', M_3 \leq_{\mathfrak{k}_\lambda} M_3', a \in M_2 \setminus M_0$ and $\mathrm{NF}(M_0, M_1, M_2, M_3')$.

Claim 6.38. *1) Assume that*

- *(A) $\mathfrak{k}_\lambda$ is a λ-AEC with amalgamation (actually follows by (c)) and a superlimit model*
- *(B) $\mathfrak{k}_\lambda$ is stable*
- *(C) NF is a $\mathfrak{k}_\lambda$-non-forking relation, see Definition 6.1(1).*

Then $\mathfrak{t} = \mathfrak{t}_{\mathfrak{k}_\lambda,\mathrm{NF}}$ is a type-full good λ-frame.

2) Assume that $\mathfrak{s}$ is a good λ-frame which has existence for $K^{3,\rm uq}_\lambda$ (see 6.9(2)) and $\mathrm{NF} = \mathrm{NF}_\lambda$. Then $\mathfrak{t}$ is very close to $\mathfrak{s}$, i.e.:

- *(A) $\mathfrak{k}_{\mathfrak{s}} = \mathfrak{k}_{\mathfrak{t}}$*
- *(B) if $p \in \mathcal{S}^{\rm bs}_{\mathfrak{s}}(M_1)$ and $M_0 \leq_{\mathfrak{k}_\lambda} M_1$ then $p \in \mathcal{S}^{\rm bs}_{\mathfrak{t}}(M_1)$ and p forks over M_0 for $\mathfrak{s}$ iff p forks over M_0 for $\mathfrak{t}$.*

Proof. For the time being, left to the reader (but before it is really used, it is proved in [She09e, 705-9.11A]). $\square_{6.38}$

Remark 6.39. Note that this actually says that from now on we could have used type-full $\mathfrak{s}$, but it is not necessary for a long time.

Definition 6.40. 1) Let $\mathfrak{s}$ be a good λ-frame. We say that NF is a weak $\mathfrak{s}$-non-forking relation when

- (a) NF is a pseudo $\mathfrak{k}_{\mathfrak{s}}$-non-forking relation, see Definition 6.1(2), i.e., uniqueness is omitted
- (b) NF respects $\mathfrak{s}$, see Definition 6.1(3)
- (c) NF satisfies 6.33, (NF-lifting of an $\le_{\mathfrak{k}}$-increasing sequence).

1A) If in part (1) we replace "$\mathfrak{s}$-non-forking" by "non-forking", we mean that we omit clause (c).

1B) In part (1) we omit "weak" when we omit the "pseudo" in clause (a), so clause (c) becomes redundant.

2) We say $\mathfrak{s}$ is pseudo-successful if some NF is a weak $\mathfrak{s}$-non-forking relation witnesses it.

Observation 6.41. *1) If $\mathfrak{s}$ is a good λ-frame which is weakly successful (i.e., has existence for $K^{3,\mathrm{uq}}_\lambda$, i.e., 6.9) <u>then</u> $\mathrm{NF}_\lambda = \mathrm{NF}_{\mathfrak{s}}$ is a $\mathfrak{s}$-non-forking relation.*

2) If $\mathfrak{s}$ is a good λ-frame and NF *is a weak $\mathfrak{s}$-non-forking relation then 6.35 holds.*

3) If $\mathfrak{s}$ is a good λ-frame and NF is an $\mathfrak{s}$-non-forking relation <u>then</u> NF is a weak $\mathfrak{s}$-non-forking relation which implies NF is a pseudo non-forking relation.

Proof. Straight.

1) Follows by 6.36, NF_λ satisfies clauses (a)+(b) and by 6.33 it satisfies also clause (c) of Definition 6.1(1).

2) Also easy.

3) We have just to check the proof of 6.33 still works. $\square_{6.41}$

Remark 6.42. 1) In [She09e, §1-§11] we can use "$\mathfrak{s}$ is pseudo successful as witnessed by NF" so has lifting of decompositions instead of "$\mathfrak{s}$ is weakly successful". We shall return to this elsewhere: see [She09d], [SV].

§ 7. Nice extensions in K_{λ^+}

Hypothesis 7.1. Assume the hypothesis 6.9.

So by §6 we have reasonable control on smooth amalgamation in K_λ. We use this to define "nice" extensions in K_{λ^+} and prove some basic properties. This will be treated again in §8.

Definition 7.2. 1) $K^{\rm nice}_{\lambda^+}$ is the class of saturated $M \in K_{\lambda^+}$.
2) Let $M_0 \le^*_{\lambda^+} M_1$ mean:

> $M_0 \le_{\mathfrak{k}} M_1$ and they are from K_{λ^+} and we can find $\overline{M}^\ell = \langle M^\ell_i : i < \lambda^+\rangle$, a $\le_{\mathfrak{k}}$-representation of M_ℓ for $\ell = 0, 1$ such that:
> ${\rm NF}_\lambda(M^0_i, M^0_{i+1}, M^1_i, M^1_{i+1})$ for $i < \lambda^+$.

3) Let $M_0 <^+_{\lambda^+,\kappa} M_1$ mean [23] that ($M_0, M_1 \in K_{\lambda^+}$ and) $M_0 \le^*_{\lambda^+} M_1$ by some witnesses M^ℓ_i (for $i < \lambda^+, \ell < 2$) such that ${\rm NF}_{\lambda,\langle 1,1,\kappa\rangle}(M^0_i, M^0_{i+1}, M^1_i, M^1_{i+1})$ for $i < \lambda^+$; of course $M_0 \le_{\mathfrak{k}} M_1$ in this case. Let $M_0 \le^+_{\lambda^+,\kappa} M_1$ mean ($M_0 = M_1 \in K_{\lambda^+}$) $\vee$ ($M_0 <^+_{\lambda^+,\kappa} M_1$). If $\kappa = \lambda$, we may omit it.
4) Let $K^{3,\rm bs}_{\lambda^+} = \{(M, N, a) : M \le^*_{\lambda^+} N$ are from K_{λ^+} and $a \in N \setminus M$ and for some $M_0 \le_{\mathfrak{k}} M, M_0 \in K_\lambda$ we have $[M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M$ and $M_1 \in K_\lambda$ implies ${\rm ortp}(a, M_1, N) \in \mathcal{S}^{\rm bs}(M_1)$ and does not fork over $M_0]\}$. We call M_0 or ${\rm ortp}(a, M_0, N)$ a witness for $(M, N, a) \in K^{3,\rm bs}_{\lambda^+}$. (In fact this definition on $K^{3,\rm bs}_{\lambda^+}$ is compatible with the definition in §2 for triples such that $M \le^*_{\lambda^+} N$ but we do not know now whether even $(K^{\rm nice}_{\lambda^+}, \le^*_{\lambda^+})$ is a λ^+-AEC.)

Claim 7.3. *0) $K^{\rm nice}_{\lambda^+}$ has one and only one model up to isomorphism and $M \in K^{\rm nice}_{\lambda^+}$ implies $M \le^*_{\lambda^+} M$ and $M \le^+_{\lambda^+} M$; moreover, $M \in K_{\lambda^+} \Rightarrow M \le^*_{\lambda^+} M$. Also $\le^*_{\lambda^+}$ is a partial order and if $M_\ell \in K_{\lambda^+}$ for $\ell = 0, 1, 2$ and $M_0 \le_{\mathfrak{k}} M_1 \le_{\mathfrak{k}} M_2$ and $M_0 \le^*_{\lambda^+} M_2$ then $M_0 \le^*_{\lambda^+} M_1$.*

1) If $M_0 \le^*_{\lambda^+} M_1$ and $\overline{M}^\ell = \langle M^\ell_i : i < \lambda^+\rangle$ is a representation of M_ℓ for $\ell = 0, 1$ then

- $(*)$ *For some club E of λ^+,*
  - (a) *for every $\alpha < \beta$ from E we have ${\rm NF}_\lambda(M^0_\alpha, M^0_\beta, M^1_\alpha, M^1_\beta)$.*
  - (b) *if $\ell < 2$ and $M_\ell \in K^{\rm nice}_{\lambda^+}$ then for $\alpha < \beta$ from E the model M^ℓ_β is $(\lambda, *)$-brimmed over M^ℓ_α.*

2) Similarly for $<^+_{\lambda^+,\kappa}$: if $M_0 <^+_{\lambda^+,\kappa} M_1, \overline{M}^\ell = \langle \overline{M}^\ell_i : i < \lambda^+\rangle$ a representation of M_ℓ for $\ell = 0, 1$ then for some club E of λ^+ for every $\alpha < \beta$ from E we have ${\rm NF}_{\lambda,\langle 1,1,\kappa\rangle}(M^0_\alpha, M^0_\beta, M^1_\alpha, M^1_\beta)$, moreover ${\rm NF}_{\lambda,\langle 1,{\rm cf}(\lambda\times(1+\beta)),\kappa\rangle}(M^0_\alpha, M^0_\beta, M^1_\alpha, M^1_\beta)$ and if $(M_\alpha, \overline{M}^0_\beta, M^1_\alpha, M^1_\beta), M_0 \in K^{\rm nice}_{\lambda^+}$ then we can add ${\rm NF}_{\lambda,\langle\lambda,{\rm cf}(\lambda\times(1+\beta)),\kappa\rangle}(M^0_\alpha, M^0_\beta, M'_\alpha, M'_\beta)$.

3) The κ in Definition 7.2(3) does not matter.

4) If $M_0 <^+_{\lambda^+,\kappa} M_1$, then $M_1 \in K^{\rm nice}_{\lambda^+}$.

5) If $M \in K_{\lambda^+}$ is saturated, equivalently $M \in K^{\rm nice}_{\lambda^+}$ then M has a $\le_{\mathfrak{k}}$-representation $\overline{M} = \langle M_\alpha : \alpha < \lambda^+\rangle$ such that M_{i+1} is (λ, λ)-brimmed over M_i for $i < \lambda^+$ and also the inverse is true.

[23] Note that $M_0 <^+_{\lambda^+,\kappa} M_1$ implies $M_1 \in K^{\rm nice}_{\lambda^+}$ but in general $M_0 \in K^{\rm nice}_{\lambda^+}$ does not follow.

6) If $M \leq^*_{\lambda^+} N$ and $N_0 \leq_{\mathfrak{k}} N, N_0 \in K_\lambda$ then we can find $M_1 \leq_{\mathfrak{k}} N_1$ from K_λ such that $M_1 \leq_{\mathfrak{k}} M, N_0 \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} N$ and: for every $M_2 \in K_\lambda$ satisfying $M_1 \leq_{\mathfrak{k}} M_2 \leq_{\mathfrak{k}} M$ there is $N_2 \leq_{\mathfrak{k}} N$ such that $\mathrm{NF}_{\mathfrak{s}}(M_1, M_2, N_1, N_2)$.

Proof. 0) Obvious by now (for the second sentence use part (1) and $\mathrm{NF}_{\mathfrak{s}}$ being a non-forking relation on $\mathfrak{k}_{\mathfrak{s}}$); in particular transitivity and monotonicity.

1) Straight by 6.30 as any two representations agree on a club.

2) Up to "moreover" quite straight. For the "moreover" use 6.31 to show that M^1_β is $(\lambda, \mathrm{cf}(\beta))$-brimmed over M^0_β. Lastly, for the "we can add" just use part (5), choosing thin enough club E of λ^+ then use $\{\alpha \in E : \mathrm{otp}(\alpha \cap E)$ is divisible by $\lambda\}$.

3) By 6.31.

4) By 6.31.

5) Trivial.

6) Easy. $\square_{7.3}$

Claim 7.4. *0) For every $M_0 \in K_{\lambda^+}$ for some $M_1 \in K^{\mathrm{nice}}_{\lambda^+}$ we have $M_0 \leq_{\mathfrak{k}} M_1$.*

1) For every $M_0 \in K_{\lambda^+}$ and $\kappa = \mathrm{cf}(\kappa) \leq \lambda$ for some $M_1 \in K_{\lambda^+}$ we have $M_0 <^+_{\lambda^+,\kappa} M_1$ so $M_1 \in K^{\mathrm{nice}}_{\lambda^+}$.

1A) Moreover, if $N_0 \leq_{\mathfrak{k}} M_0 \in K_{\lambda^+}, N_0 \in K_\lambda, p \in \mathcal{S}^{\mathrm{bs}}(N_0)$ then in (1) we can add that for some $a, (M_0, M_1, a) \in K^{3,\mathrm{bs}}_\lambda$ as witnessed by p.

2) $\leq^*_{\lambda^+}$ and $<^+_{\lambda^+,\kappa}$ are transitive.

3) If $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_2$ are in K_{λ^+} and $M_0 \leq^*_{\lambda^+} M_2$, then $M_0 \leq^*_{\lambda^+} M_1$.

4) If $M_1 <^+_{\lambda^+,\kappa} M_2$, then $M_1 <^*_{\lambda^+} M_2$.

5) If $M_0 <^*_{\lambda^+} M_1 <^+_{\lambda,\kappa} M_2$ then $M_0 <^+_{\lambda,\kappa} M_2$.

Proof. 0) Easy, and follows from the proof of part (1) below.

1), 1A) Let $\langle M^0_i : i < \lambda^+\rangle$ be a $\leq_{\mathfrak{k}}$-representation of M_0 with M^0_i brimmed and brimmed over M^0_j for $j < i$ and for part (1A) we have $M^0_0 = N_0$, and for part (1) let p be any member of $\mathcal{S}^{\mathrm{bs}}(M^0_0)$. We choose by induction on i a model $M^1_i \in K_\lambda$ and $a \in M^1_0$ such that M^1_i is $(\lambda, \mathrm{cf}(\lambda \times (1+i)))$-brimmed over $M^0_i, \langle M^1_i : i < \lambda^+\rangle$ is $<_{\mathfrak{k}}$-increasing continuous, $M^1_i \cap M_0 = M^0_i$ and $\mathrm{ortp}(a, M^0_0, M^1_0) = p$ and M^1_{i+1} is (λ, κ)-brimmed over $M^0_{i+1} \cup M^1_i$ and $\mathrm{NF}_{\lambda,\langle 1,\mathrm{cf}(\lambda\times(1+i)),\kappa\rangle}(M^0_i, M^0_{i+1}, M^1_i, M^1_{i+1})$ for $i < \lambda^+$. Note that for limit i, by 6.31, M^1_i is $(\lambda, \mathrm{cf}(i))$-brimmed over $M^0_i \cup M^1_j$ for any $j < i$.

Note that for $i < \lambda^+$, the type $\mathrm{ortp}(a, M^0_i, M^1_i)$ does not fork over $M^0_0 = N_0$ and extends p by 6.34 (saying NF_λ respects $\mathfrak{s}$) 6.27 (symmetry) and 6.25. So clearly we are done.

2) Concerning $<^+_{\lambda^+,\kappa}$ use 7.3 and 6.30 (i.e. transitivity for smooth amalgamations). The proof for $<^*_{\lambda^+}$ is the same.

3) By monotonicity for smooth amalgamations in $\mathfrak{k}_\lambda$; i.e., 6.26.

4), 5) Check. $\square_{7.4}$

Claim 7.5. *1) If $(M_0, M_1, a) \in K^{3,\mathrm{bs}}_{\lambda^+}$ and $M_1 \leq^*_{\lambda^+} M_2 \in K_{\lambda^+}$ then $(M_0, M_2, a) \in K^{3,\mathrm{bs}}_{\lambda^+}$.*

2) If $M_0 <^*_{\lambda^+} M_1$, then for some $a, (M_0, M_1, a) \in K^{3,\mathrm{bs}}_{\lambda^+}$.

Proof. 1) By the transitivity of $\le^*_{\lambda^+}$ which holds by 7.4(2).
2) As in the proof of 2.9; in fact, it follows from it. $\square_{7.5}$

Remark 7.6. Note that the parallel to 7.4(1A) is problematic in §2 as, e.g. locality may fail; i.e. $(M, N_i, a_i) \in K^{3,\text{bs}}_{\lambda^+}$ and $M' \le_{\mathfrak{k}} M \wedge M' \in K_\lambda \Rightarrow \text{ortp}_{\mathfrak{s}}(a_1, M', N_1) = \text{ortp}_{\mathfrak{s}}(a_2, M', N_2)$ but $\text{ortp}_{K^{\mathfrak{s}}_{\lambda^+}}(a_1, M, N_1) \ne \text{ortp}_{K^{\mathfrak{s}}_{\lambda^+}}(\bar{a}_2, M, N_2)$.

Claim 7.7. *1) [Amalgamation of $\le^*_{\lambda^+}$ and toward extending types] If $M_0 \le^*_{\lambda^+} M_\ell$ for $\ell = 1, 2, \kappa = \text{cf}(\kappa) \le \lambda$ and $a \in M_2 \setminus M_0$ is such that $(M_0, M_2, a) \in K^{3,\text{bs}}_{\lambda^+}$ is witnessed by p, then for some M_3 and f we have: $M_1 <^+_{\lambda^+,\kappa} M_3$ and f is an $\le_{\mathfrak{k}}$-embedding of M_2 into M_3 over M_0 with $f(a) \notin M_1$, moreover, $f(M_2) \le^*_{\lambda^+} M_3$ and $(M_1, M_3, f(a)) \in K^{3,\text{bs}}_{\lambda^+}$ is witnessed by p.*
2) [uniqueness] Assume $M_0 <^+_{\lambda^+,\kappa} M_\ell$ for $\ell = 1, 2$ then there is an isomorphism f from M_1 onto M_2 over M_0.
3) [locality] Moreover,[24] *in (2), if $a_\ell \in M_\ell \setminus M_0$ for $\ell = 1, 2$ and $[N \le_{\mathfrak{k}} M_0$ and $N \in K_\lambda \Rightarrow \text{ortp}(a_1, N, M_1) = \text{ortp}(a_2, N, M_2)]$, then we can demand $f(a_1) = a_2$ (so in particular $\text{ortp}(a_1, M_0, M_1) = \text{ortp}(a_2, M_0, M_2)$ where the types are as defined in $\mathfrak{k}_{\lambda^+}$ and even in $(K_{\lambda^+}, \le^*_{\lambda^+})$.*
4) Moreover in (2), assume further that for $\ell = 1, 2$, the following hold: $N_0 \le_{\mathfrak{k}} N_\ell \le_{\mathfrak{k}} M_\ell, N_0 \in K_\lambda, N_0 \le_{\mathfrak{k}} N_\ell, N_\ell \in K_\lambda$ and $(\forall N \in K_\lambda)[N_0 \le_{\mathfrak{k}} N \le_{\mathfrak{k}} M_0 \rightarrow (\exists N' \in K_\lambda)(N \cup N_\ell \subseteq N' \le_{\mathfrak{k}} M_\ell \wedge \text{NF}_\lambda(N_0, N_\ell, N, N')]$. If f_0 is an isomorphism from N_1 onto N_2 over N_0 then we can add $f \supseteq f_0$.

Proof. We first prove part (2).
2) By 7.3(1) + (2) there are representations $\overline{M}^\ell = \langle M^\ell_i : i < \lambda^+ \rangle$ of M_ℓ for $\ell < 3$ such that for $\ell = 1, 2$ we have: $M^\ell_i \cap M_0 = M^\ell_0$ and $\text{NF}_{\lambda,\langle 1,1,\kappa \rangle}(M^0_i, M^0_{i+1}, M^\ell_i, M^\ell_{i+1})$ and without loss of generality M^ℓ_0 is (λ, κ)-brimmed over M^0_0 for $\ell = 1, 2$.

Now we choose by induction on $i < \lambda^+$ an isomorphism f_i from M^1_i onto M^2_i, increasing with i and being the identity over M^0_i. For $i = 0$ use "M^ℓ_0 is (λ, κ)-brimmed over M^0_0 for $\ell = 1, 2$" which we assume above. For i limit take unions, for i successor ordinal use uniqueness (Claim 6.20).

[Proof of part (1)] By 7.4(1) there are for $\ell = 1, 2$ models $N^*_\ell \in K_{\lambda^+}$ such that $M_\ell <^+_{\lambda^+,\kappa} N^*_\ell$. Now let $\overline{M}^\ell = \langle M^\ell_i : i < \lambda^+ \rangle$ be a representation of M_ℓ for $\ell = 0, 1, 2$ and let $\overline{N}^\ell = \langle N^\ell_i : i < \lambda^+ \rangle$ be a representation of N^*_ℓ for $\ell = 1, 2$. By 7.4(4) and 7.3(2) without loss of generality N^ℓ_0 is (λ, κ)-brimmed over M^ℓ_0 and $\text{NF}_\lambda(M^0_i, M^0_{i+1}, M^\ell_i, M^\ell_{i+1})$ and $\text{NF}_{\lambda,\langle 1,1,\kappa \rangle}(M^\ell_i, M^\ell_{i+1}, N^\ell_i, N^\ell_{i+1})$ respectively for $i < \lambda^+, \ell = 1, 2$. Let M^*_0 be such that $p \in \mathcal{S}^{\text{bs}}(M^*_0), M^*_0 \in K_\lambda, M^*_0 \le_{\mathfrak{k}} M_0$; without loss of generality $M^*_0 \le_{\mathfrak{k}} M^0_0$ and $a \in M^2_0 \le_{\mathfrak{k}} N^2_0$. Now N^ℓ_0 is (λ, κ)-brimmed over M^ℓ_0 hence over M^0_0 (for $\ell = 1, 2$) so there is an isomorphism f_0 from N^2_0 onto N^1_0 extending $\text{id}_{M^0_0}$. There is $a' \in N^1_0$ such that $\text{ortp}(a', M^1_0, N^1_0)$ is a non-forking extension of p and without loss of generality $f_0(a) = a'$ hence $\text{ortp}(f_0(a), M^1_0, N^1_0) \in \mathcal{S}^{\text{bs}}(M^1_0)$ does not fork over M^0_0.

[24] The meaning of this will be that types over $M \in K^{\text{nice}}_{\lambda^+}$ for $(K^{\text{nice}}_{\lambda^+}, \le^*_{\lambda^+})$ can be reduced to basic types over a model in K_λ, i.e., locality.

We continue as in the proof of part (2). In the end $f = \bigcup_{i<\lambda^+} f_i$ is an isomorphism of N_2^* onto N_1^* over M_0 and as $f_0(a)$ is well defined and in $N_0^1 \setminus M_0^1$ clearly $\mathrm{ortp}(f(a), M_i^1, N_i^1)$ does not fork over M_0^1 and extends p hence the pair $(N_1^*, f \restriction M_2)$ is as required.

[**Proof of part (3), (4)**] Like part (2). $\square_{7.7}$

Claim 7.8. *1) If δ is a limit ordinal $< \lambda^{+2}$ and $\langle M_i : i < \delta\rangle$ is a $\leq^*_{\lambda^+}$-increasing continuous (in K_{λ^+}) and $M_\delta = \bigcup_{i<\delta} M_i$ (so $M_\delta \in K_{\lambda^+}$), then $M_i \leq^*_{\lambda^+} M_\delta$ for each $i < \delta$.*
2) If δ is a limit ordinal $< \lambda^{+2}$ and $\langle M_i : i < \delta\rangle$ is a $\leq^*_{\lambda^+}$-increasing sequence, each M_i is in $K^{\rm nice}_{\lambda^+}$, then $\bigcup_{i<\delta} M_i$ is in $K^{\rm nice}_{\lambda^+}$.
3) If δ is a limit ordinal $< \lambda^{+2}$ and $\langle M_i : i < \delta\rangle$ is a $<^+_{\lambda^+}$-increasing continuous (or just $<^*_{\lambda^+}$-increasing continuous, and $M_{2i+1} <^+_{\lambda^+} M_{2i+2}$ for $i < \delta$), then $i < \delta \Rightarrow M_i <^+_{\lambda^+} \bigcup_{j<\delta} M_j$.

Proof. 1) We prove it by induction on δ. Now if C is a club of δ, (as $\leq^*_{\lambda^+}$ is transitive) then we can replace $\langle M_j : j < \delta\rangle$ by $\langle M_j : j \in C\rangle$ so without loss of generality $\delta = \mathrm{cf}(\delta)$, so $\delta \leq \lambda^+$; similarly it is enough to prove $M_0 \leq^*_{\lambda^+} M_\delta := \bigcup_{j<\delta} M_j$. For each $i \leq \delta$ let $\langle M^i_\zeta : \zeta < \lambda^+\rangle$ be a $<^*_{\mathfrak{k}}$-representation of M_i.

Case A: $\delta < \lambda^+$.
Without loss of generality (see 7.3(1)) for every $i < j < \delta$ and $\zeta < \lambda^+$ we have: $M^j_\zeta \cap M_i = M^i_\zeta$ and $\mathrm{NF}_\lambda(M^i_\zeta, M^i_{\zeta+1}, M^j_\zeta, M^j_{\zeta+1})$. Let $M^\delta_\zeta = \bigcup_{i<\delta} M^i_\zeta$, so $\langle M^\delta_\zeta : \zeta < \lambda^+\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous sequence of members of K_λ with limit M_δ, and for $i < \delta, M^\delta_\zeta \cap M_i = M^i_\zeta$. By symmetry (see 6.27) we have $\mathrm{NF}_\lambda(M^i_\zeta, M^{i+1}_\zeta, M^i_{\zeta+1}, M^{i+1}_{\zeta+1})$ so as $\langle M^i_\zeta : i \leq \delta\rangle$,$\langle M^i_{\zeta+1} : i \leq \delta\rangle$ are $\leq_{\mathfrak{k}}$-increasing continuous, by 6.30, the transitivity of $\mathrm{NF}_{\mathfrak{s}}$, we know $\mathrm{NF}_\lambda(M^0_\zeta, M^\delta_\zeta, M^0_{\zeta+1}, M^\delta_{\zeta+1})$ hence by symmetry (6.27) we have $\mathrm{NF}_\lambda(M^0_\zeta, M^0_{\zeta+1}, M^\delta_\zeta, M^\delta_{\zeta+1})$.
So $\langle M^0_\zeta : \zeta < \lambda^+\rangle, \langle M^\delta_\zeta : \zeta < \lambda^+\rangle$ are witnesses to $M_0 \leq^*_{\lambda^+} M_\delta$.

Case B: $\delta = \lambda^+$.
By 7.3(1) (using normality of the club filter, restricting to a club of λ^+ and renaming), without loss of generality for $i < j \leq 1 + \zeta < 1 + \xi < \lambda^+$ we have $M^j_\zeta \cap M_i = M^i_\zeta$, and $\mathrm{NF}_\lambda(M^i_\zeta, M^i_\xi, M^j_\zeta, M^j_\xi)$. Let us define $M^{\lambda^+}_\zeta = \bigcup_{j<1+\zeta} M^j_\zeta$. So $\langle M^{\lambda^+}_\zeta : \zeta < \lambda^+\rangle$ is a $<_{\mathfrak{k}}$-representation of $M_{\lambda^+} = M_\delta$ and continue as before.

2) Again without loss of generality $\delta = \mathrm{cf}(\delta)$ call it κ. Let $\langle M^i_\zeta : \zeta < \lambda^+\rangle$ be a $<_{\mathfrak{k}}$-representation of M_i for $i < \delta$.

Case A: $\delta = \kappa < \lambda^+$.
Easy by now, yet we give details, noting 7.9. So without loss of generality (see 7.3(1)) for every $i < j < \delta$ and $\zeta < \xi < \lambda^+$ we have: $M^j_\zeta \cap M_i = M^i_\zeta$, $\mathrm{NF}_\lambda(M^i_\zeta, M^i_\xi, M^j_\zeta, M^j_\xi)$ and $M^i_{\zeta+1}$ is (λ,λ)-brimmed over M^i_ζ. Let $M^\delta_\zeta = \bigcup_{\beta<\delta} M^\beta_\zeta$.
Let $\xi < \lambda^+$. Now if $p \in \mathcal{S}^{\rm bs}(M^\delta_\xi)$ then by the local character Axiom (E)(c) + the

uniqueness Axiom (E)(e), for some $i < \delta, p$ does not fork over M^i_ξ. As M_i is λ^+-saturated above λ, the type $p \restriction M^i_\xi$ is realized in M_i. So let $b \in M_i$ realize $p \restriction M^i_\xi$ and by Axiom $(E)(h)$, continuity, it suffices to prove that for every $j \in (i, \delta), b$ realizes $p \restriction M^j_\xi$ in M_j which holds by 6.34 (note that $b \in M_i \le_{\mathfrak{k}} M_j$ as $j \in [i, \delta)$). So p is realized in $M_\delta = \bigcup\limits_{i<\delta} M_i$. As this holds for every $\xi < \lambda^+$ and $p \in \mathcal{S}^{\rm bs}(M^\delta_\xi)$, the model M_δ is saturated.

Case B: $\mathrm{cf}(\delta) = \lambda^+$.

Straight: in fact true for $\mathfrak{k}$ AEC with the λ-amalgamation property.

3) Similar. $\square_{7.8}$

Remark 7.9. Note that in Ax(E)(c),Ax(E)(h) the continuity of the sequences is not required.

Claim 7.10. *1) If $M_0 \in K_{\lambda^+}$ <u>then</u> there is M_1 such that $M_0 <^+_{\lambda^+} M_1 \in K^{\rm nice}_{\lambda^+}$, and any such M_1 is universal over M_0 in $(K_{\lambda^+}, \le^*_{\lambda^+})$.*

2) Assume $\boxtimes_{\bar N_1, \bar N_2, M_1, M_2}$ below holds. <u>Then</u> $M_1 <^+_{\lambda^+} M_2$ <u>iff</u> for every $\alpha < \lambda^+$ for stationarily many $\beta < \lambda^+$ there is N such that $N^1_\beta \cup N^2_\alpha \subseteq N \le_{\mathfrak{k}} N^2_\beta$ and N^2_β is $(\lambda, *)$-brimmed over N where

$\boxtimes_{\bar N_1, \bar N_2, M_1, M_2}$ *$M_1 \le^*_{\lambda^+} M_2$ is being witnessed by $\bar N_1, \bar N_2$ that is $\bar N_\ell = \langle N^\ell_\alpha : \alpha < \lambda^+ \rangle$ is a $\le_{\mathfrak{k}}$-representation of M_ℓ for $\ell = 1, 2$ and $\alpha < \lambda^+ \Rightarrow \mathrm{NF}_\lambda(N^1_\alpha, N^1_{\alpha+1}, N^2_\alpha, N^2_{\alpha+1})$ (hence $\alpha \le \beta < \lambda^+ \Rightarrow \mathrm{NF}_\lambda(N^1_\alpha, N^1_\beta, N^2_\alpha, N^2_\beta))$.*

Proof. 1) The existence by 7.4(1). Why "any such $M_1, \ldots$?" if $M_0 \le^*_{\lambda^+} M_2$ then for some $M^+_2 \in K^{\rm nice}_{\lambda^+}$ we have $M_2 <^+_\lambda M^+_2 \in K^{\rm nice}_{\lambda^+}$ so $M_0 \le^*_{\lambda^+} M_1 <^+_{\lambda^+} M^+_2$ hence by 7.4(5) we have $M_0 <^+_\lambda M^+_2$; so by 7.7(2) the models M^+_2, M_1 are isomorphic over M_0, so M_2 can be $\le^*_{\lambda^+}$-embedded into M_1 over M_0, so we are done.

2) Not hard. $\square_{7.10}$

§ 8. Is $K^{\rm nice}_{\lambda^+}$ with $\le^*_{\lambda^+}$ an AEC?

Hypothesis 8.1. The hypothesis 6.9.

An important issue is whether $(K^{\rm nice}_{\lambda^+}, \le^*_{\lambda^+})$ satisfies Ax IV of AEC. So a model $M \in K_{\lambda^{++}}$ may be the union of a $\le^*_{\lambda^+}$-increasing chain of length λ^{++}, but we still do not know if there is a continuous such sequence.

E.g. let $\langle M_\alpha : \alpha < \lambda^{++}\rangle$ be $\le^*_{\lambda^+}$-increasing with union $M \in K_{\lambda^{++}}$ let $M'_n = M_n, M'_{\omega+\alpha+1} = M_{\omega+\alpha}$ and $M'_\delta = \cup\{M_\beta : \beta < \delta\}$ for δ limit. So $\langle M'_\alpha : \alpha < \lambda^{++}\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous, $\langle M'_{\alpha+1} : \alpha < \lambda^{++}\rangle$ is $\le^*_{\lambda^+}$-increasing, but we do not know whether $M'_\delta \le^*_{\lambda^+} M'_{\delta+1}$ for limit $\delta < \lambda^{++}$.

Definition 8.2. Let $M \in \mathfrak{k}_{\lambda^{++}}$ be the union of an $\le_{\mathfrak{k}}$-increasing continuous chain from $(K^{\rm nice}_{\lambda^+}, \le^*_{\lambda^+})$ or just $(K_{\lambda^+}, \le^*_{\lambda^+})$, $\overline{M} = \langle M_i : i < \lambda^{++}\rangle$ such that $\langle M_i : i < \lambda^{++}$ non-limit$\rangle$ is $\le^*_{\lambda^+}$-increasing.

1) Let $S(\overline{M}) = \{\delta : M_\delta eq^*_{\lambda^+} M_{\delta+1}$ (see 8.3(3) below)$\}$, so $S(\overline{M}) \subseteq \lambda^{++}$.

2) For such M let $S(M)$ be $S(\overline{M})/\mathcal{D}_{\lambda^{++}}$ where $\overline{M}$ is a $\le_{\mathfrak{k}}$-representation of M and $\mathcal{D}_{\lambda^{++}}$ is the club filter on λ^{++}; it is well defined by 8.3 below.

3) We say $\langle M_i : i < \delta\rangle$ is non-limit $<^*_{\lambda^+}$-increasing <u>if</u> for non-limit $i < j < \delta$ we have $M_i \le^*_{\lambda^+} M_j$.

Claim 8.3. *1) If $\overline{M}^\ell = \langle M^\ell_i : i < \lambda^{++}\rangle$ for $\ell \in \{1,2\}$ is $\le_{\mathfrak{k}}$-increasing continuous and $i < j < \lambda^{++} \Rightarrow M_0 \le^*_{\lambda^+} M_{i+1} \le^*_{\lambda^+} M_{j+1}$ and $M = \bigcup_{i<\lambda^{++}} M^1_i = \bigcup_{i<\lambda^{++}} M^2_i$ has cardinality λ^{++} <u>then</u> $S(\overline{M}^1) = S(\overline{M}^2) \mod \mathcal{D}_{\lambda^{++}}$.*

2) If $M, \overline{M}$ are as in 8.2 hence $M = \bigcup_{i<\lambda^{++}} M_i$ <u>then</u> $S(\overline{M})/\mathcal{D}_{\lambda^{++}}$ depends just on $M/\cong$.

3) If $\overline{M}$ is as in 8.2 or, equivalently as in part (1), and $i < j < \lambda^{++}$, <u>then</u> $M_i \le^*_{\lambda^+} M_{i+1} \Leftrightarrow M_i \le^*_{\lambda^+} M_j$.

4) If $M \in \mathfrak{k}_{\lambda^{++}}$ is the union of a $\le^*_{\lambda^+}$-increasing chain from $(K^{\rm nice}_{\lambda^+}, \le^*_{\lambda^+})$, not necessarily continuous, <u>then</u> there is $\overline{M}$ as in Definition 8.2, that is $\overline{M} = \langle M_i : i < \lambda^{++}\rangle$, a $\le_{\mathfrak{k}}$-representation of M with $M_i \le^*_{\lambda^+} M_j$ for non-limit $i < j$.

Proof. 1) We can find a club E of λ^{++} consisting of limit ordinals such that $i \in E \Rightarrow M^1_i = M^2_i$. Now if $\delta_1 < \delta_2$ are from E then $\delta_1 \in S(\overline{M}^1) \Leftrightarrow M^1_{\delta_1} \le^*_{\lambda^+} M^1_{\delta_1+1} \Leftrightarrow M^1_{\delta_1} \le^*_{\lambda^+} M^1_{\delta_2} \Leftrightarrow M^2_{\delta_1} \le^*_{\lambda^+} M^2_{\delta_2} \Leftrightarrow M^2_{\delta_1} \le^*_{\lambda^+} M^2_{\delta_1+1} \Leftrightarrow \delta_1 \in S(M^2)$.

[Why? By the definition of $S(\overline{M}^1)$, by part (3), by "$\delta_1, \delta_2 \in E$", by part (3), by the definition of $S(\overline{M}^2)$, respectively.] So we are done.

2) Follows by parts (1) and (3).

3) The implication $\Leftarrow$ is by 7.4(3); for the implication $\Rightarrow$, note that assuming $M_i <^*_{\lambda^+} M_{i+1}$, as $\le^*_{\lambda^+}$ is a partial order, noting that by the assumption on $\overline{M}$ we have $M_{i+1} \le^*_{\lambda^+} M_{j+1}$, and by 7.4(3) we are done.

4) Trivial. $\square_{8.3}$

Claim 8.4. *If $(*)$ below holds <u>then</u> for every stationary $S \subseteq S^{\lambda^{++}}_{\lambda^+} (= \{\delta < \lambda^{++} : \mathrm{cf}(\delta) = \lambda^+\})$ for some λ^+-saturated $M \in K_{\lambda^{++}}$ we have $S(M)$ is well defined and equal to $S/\mathcal{D}_{\lambda^{++}}$, where*

$(*)$ *we can find* $\langle M_i : i \le \lambda^+ + 1\rangle$ *which is* $<_{\mathfrak{k}}$*-increasing continuous sequence of members of* $K^{\rm nice}_{\lambda^+}$ *such that* $i < j \le \lambda^+ + 1$ *and* $(i,j) \ne (\lambda^+, \lambda^+ + 1) \Rightarrow M_i <^+_{\lambda^+} M_j$ *but* $\neg(M_{\lambda^+} \le^*_{\lambda^+} M_{\lambda^+ + 1})$.

Proof. Fix $S \subseteq S^{\lambda^{++}}_{\lambda^+}$ and $\langle M_i : i \le \lambda^+ + 1\rangle$ as in $(*)$.

Without loss of generality $|M_{\lambda^+ + 1} \setminus M_{\lambda^+}| = \lambda^+$.

We choose by induction on $\alpha < \lambda^{+2}$ a model M^S_α such that:

(a) $M^S_\alpha \in K^{\rm nice}_{\lambda^+}$ has universe an ordinal $< \lambda^{++}$

(b) for $\beta < \alpha$ we have $M^S_\beta \le_{\mathfrak{k}} M^S_\alpha$

(c) if $\alpha = \beta + 1$, $\beta \notin S$ then $M^S_\beta <^+_{\lambda^+} M^S_\alpha$

(d) if $\alpha = \beta + 1, \beta \in S$ then $(M^S_\beta, M^S_\alpha) \cong (M_{\lambda^+}, M_{\lambda^+ + 1})$

(e) if $\beta < \alpha, \beta \notin S$ then $M^S_\beta \le^+_{\lambda^+} M^S_\alpha$

(f) if α is a limit ordinal, then $M_\alpha = \bigcup\{M_\beta : \beta < \alpha\}$.

We use freely the transitivity and continuity of $\le^*_\lambda$ and of $<^+_\lambda$.

For $\alpha = 0$ no problem.

For α limit no problem; choose an increasing continuous sequence $\langle \gamma_i : i < {\rm cf}(\alpha)\rangle$ of ordinals with limit α each of cofinality $< \lambda, \gamma_i \notin S$, and use 7.8(3) for clause (e).

For $\alpha = \beta + 1, \beta \notin S$ no problem.

For $\alpha = \beta + 1, \beta \in S$ so ${\rm cf}(\beta) = \lambda^+$, let $\langle \gamma_i : i < \lambda^+\rangle$ be increasing continuous with limit β and ${\rm cf}(\gamma_i) \le \lambda$, hence $\gamma_i \notin S$ and each γ_{i+1} a successor ordinal. By clause (e) above and 7.4(5) we have $M^S_{\gamma_i} <^+_{\lambda^+} M^S_{\gamma_{i+1}}$, hence $\langle M_{\gamma_i} : i < \lambda^+\rangle$ is $<^+_{\lambda^+}$-increasing continuous. Now there is an isomorphism f_β from M_{λ^+} onto M^S_β mapping M_i onto $M^S_{\gamma_i}$ for $i < \lambda$ (why? choose $f_\beta \restriction M_i$ by induction on i, for $i = 0$ by 7.3(0), for i successor $M^S_{\gamma_i} <^+_\lambda M^S_{\gamma_{i+1}}$ by 7.4(3) as $M^S_{\gamma_i} <^*_{\lambda^+} M^S_{\gamma_{i+1}} <^+_{\lambda^+} M^S_{\gamma_{i+1}}$ so we can use 7.7(2)). So we can choose a one-to-one function f_α from $M_{\lambda^+ + 1}$ onto some ordinal $< \lambda^{++}$ extending f_β and let $M_\alpha = f_\alpha(M_{\lambda^+ + 1})$.

Finally having carried the induction, let $M_S = \bigcup\limits_{\alpha < \lambda^{+2}} M^S_\alpha$, it is easy to check that $M_S \in K_{\lambda^{++}}$ is λ^+-saturated and $\overline{M} = \langle M^S_\alpha : \alpha < \lambda^{++}\rangle$ witnesses that $S(M_S)/\mathcal{D}_{\lambda^{++}}$ is well defined and $S(M_S)/\mathcal{D}_{\lambda^{++}} = S(\langle M^S_\alpha : \alpha < \lambda^{++}\rangle)/\mathcal{D}_{\lambda^{++}} = S/\mathcal{D}_{\lambda^{++}}$ as required. $\square_{8.4}$

Below we prove that some versions of non-smoothness are equivalent.

Claim 8.5. *1) We have* $(**)_{M^*_1, M^*_2} \Rightarrow (***)$ *(see below).*

2) If $(*)$ *then* $(**)_{M^*_1, M^*_2}$ *for some* M^*_1, M^*_2 *and trivially* $(***) \Rightarrow (*)$.

3) In part (1) we get $\langle M_i : i \le \lambda^+ + 1\rangle$ *as in* $(***)$*, see below, such that* $M_{\lambda^+} = M^*_1, M_{\lambda^+ + 1} = M^*_2$ *if we waive* $i < \lambda^+ \Rightarrow M_i <^+_\lambda M_{\lambda+1}$ *or assume* $M^*_1 <_{\mathfrak{k}} M^* <^+_\lambda M^*_2$ *for some* M^*.

4) If $M^*_1 \le^*_{\lambda^+} M^*_2$ *and* $M^*_1 \in K^{\rm nice}_{\lambda^+}$ *and* $N_1 \le_{\mathfrak{k}} N_2 \in K_\lambda, N_\ell \le M^*_\ell$ *for* $\ell = 1, 2$ *and* $p \in \mathcal{S}^{\rm bs}(N_2)$ *does not fork over* N_1 <u>*then*</u> *some* $c \in M^*_1$ *realizes* p

<u>*where*</u>

$(*)$ *there are limit* $\delta < \lambda^{++}, N$ *and* $\overline{M} = \langle M_i : i \le \delta\rangle$ *a* $\le^*_{\lambda^+}$*-increasing continuous sequence with* $M_i, N \in K^{\rm nice}_{\lambda^+}$ *such that:* $M_i \le^*_{\lambda^+} N \Leftrightarrow i < \delta$

$(**)_{M^*_1, M^*_2}$ (i) $M^*_1 \in K^{\rm nice}_{\lambda^+}, M^*_2 \in K^{\rm nice}_{\lambda^+}$

(ii) $M^*_1 \le_{\mathfrak{k}} M^*_2$

(iii) $M^*_1 eq^*_{\lambda^+} M^*_2$

(iv) *if* $N_1 \leq_{\mathfrak{k}} N_2$ *are from* K_λ, $N_\ell \leq_{\mathfrak{k}} M^*_\ell$ *for* $\ell = 1, 2$ *and* $p \in \mathcal{S}^{\rm bs}(N_2)$ *does not fork over* N_1, *then some* $a \in M^*_1$ *realizes* p *in* M^*_2

$(***)$ *there is* $\overline{M} = \langle M_i : i \leq \lambda^+ + 1\rangle$, $\leq_{\mathfrak{k}}$*-increasing continuous, every* $M_i \in K^{\rm nice}_{\lambda^+}$ *and* $M_{\lambda^+} eq^*_{\lambda^+} M_{\lambda^++1}$ *but* $i < j \leq \lambda^+ + 1$ *and* $i \neq \lambda^+ \Rightarrow M_i <^+_{\lambda^+} M_j$; *note that this is* $(*)$ *of 8.4.*

Proof. 1),3) Let $\langle a^\ell_i : i < \lambda^+\rangle$ list the elements of M^*_ℓ for $\ell = 1, 2$. Let $\langle N^*_{2,i} : i < \lambda^+\rangle$ be a $\leq_{\mathfrak{k}}$-representation of M^*_2.

Let $\langle (p_\zeta, N^*_\zeta, \gamma_\zeta) : \zeta < \lambda^+\rangle$ list the triples (p, N, γ) such that $\gamma < \lambda^+, p \in \mathcal{S}^{\rm bs}(N), N \in \{N^*_{2,i} : i < \lambda^+\}$ with each such triple appearing λ^+ times. By induction on $\alpha < \lambda^+$ we choose $\langle N^\alpha_i : i \leq \alpha\rangle, N_\alpha$ such that:

(a) $N^\alpha_i \in K_\lambda$ and $N^\alpha_i \leq_{\mathfrak{k}} M^*_1$

(b) $N_\alpha \leq_{\mathfrak{k}} M^*_2$ and $N_\alpha \in K_\lambda$

(c) $\langle N^\alpha_i : i \leq \alpha\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous

(d) $N^\alpha_\alpha \leq_{\mathfrak{k}} N_\alpha, N_\alpha \cap M^*_1 = N^\alpha_\alpha$

(e) if $i \leq \alpha$ then $\langle N^\beta_i : \beta \in [i, \alpha]\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous

(f) $\langle N_\beta : \beta \leq \alpha\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous

(g) if $\alpha = \beta + 1, i \leq \beta$ then $\mathrm{NF}_\lambda(N^\beta_i, N_\beta, N^\alpha_i, N_\alpha)$

(h) if $\alpha = 2\beta + 1$ then $a^2_\beta \in N_{\alpha+1}$

(i) if $\alpha = 2\beta + 2$ and $i < \alpha$ then N^α_{i+1} is brimmed over $N^\alpha_i \cup N^{2\beta+1}_{i+1}$ and N^α_0 is brimmed over $N^{2\beta}_0$.

<u>Why is this enough?</u>

We let $M_{\lambda^+} = M^*_1, M_{\lambda^++1} = M^*_2$ and let $M'_{\lambda^++1} \in K^{\rm nice}_{\lambda^+}$ be such that $M_{\lambda^++1} <^+_{\lambda^+} M'_{\lambda^++1}$ and for $i < \lambda^+$ we let $M_i = \cup\{N^\alpha_i : \alpha \in [i, \lambda^+)\}$; now

(α) $M^*_1 = \bigcup\limits_{\alpha<\lambda^+} N^\alpha_\alpha = \bigcup\limits_{i<\lambda^+} M_i$ and $M^*_2 = \bigcup\limits_{\alpha<\lambda^+} N_\alpha$

[why? the second by clause (h) (and (b) of course), the first as $N_\alpha \cap M^*_1 = N^\alpha_\alpha$].

Now:

(β) $\langle M_i : i \leq \lambda^+ + 1\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous

[trivial by clauses (c) + (e) if $i < \lambda^+$ and (d) if $i = \lambda^+$]

(γ) for $i < \lambda^+, M_i$ is saturated, i.e., $\in K^{\rm nice}_{\lambda^+}$.

[Why? Clearly $\langle N^\alpha_i : \alpha \in (i, \lambda^+)\rangle$ is a $\leq_{\mathfrak{k}}$-representation of M_i by clause (e) and the choice of M_i. If $i = 0$ this follows by clauses (i) + (e). If $i = j + 1$ this follows by clauses (e) + (i). If i is a limit ordinal use 7.8(2) and clause (g)]

(δ) for $i < \lambda^+, i < j \leq \lambda^+ + 1$ we have $M_i \leq^*_{\lambda^+} M_j$.

[Why? Let $N^\alpha_{\lambda^+} := N^\alpha_\alpha, N^\alpha_{\lambda^++1} = N_\alpha$ for $\alpha < \lambda^+$ and let γ be i if $j = \lambda^+, \lambda^+ + 1$ and be j if $j < \lambda^+$; so in any case $\gamma < \lambda^+$. Now as $\langle N^\alpha_i : \alpha \in [\gamma, \lambda^+)\rangle$ is a $\leq_{\mathfrak{k}}$-representation of M_i and $\langle N^\alpha_j : \alpha \in [\gamma, \lambda^+)\rangle$ is a $\leq_{\mathfrak{k}}$-representation of M_j and if $\gamma \leq \beta < \lambda^+$ then by clause (g) we have $\mathrm{NF}_\lambda(N^\beta_i, N_\beta, N^{\beta+1}_i, N_{\beta+1})$ hence by symmetry $\mathrm{NF}_\lambda(N^\beta_i, N^{\beta+1}_i, N_\beta, N_{\beta+1})$ hence by monotonicity

$\mathrm{NF}_\lambda(N^\beta_i, N^{\beta+1}_i, N^\beta_j, N^{\beta+1}_j)$; this suffices]

(ε) if $i < j \le \lambda^+$ then $M_i <^+_{\lambda^+} M_j$

[why? by 7.8(3) it suffices to prove this in the cases $j = i+1$. Now claim 7.10(2), clause (i) guaranteed this.]

Clearly $\langle M_i : i \le \lambda^+ + 1\rangle$ is as required for part (1) and for part (3) for first possibility (with waiving) obviously. For the second possibility in part (2), easily $\langle M_i : i \le \lambda^+\rangle\hat{}\langle M'_{\lambda^++1}\rangle$ is as required but $M^*_2, M^1_{\lambda+1}$ are isomorphic over M^*, so also $\langle M_i : i \le \lambda^+ + 1\rangle$ is O.K.

So we are done.

So let us carry the construction.

For $\alpha = 0$ trivially.

For α limit: straightforward.

For $\alpha = 2\beta + 1$ we let $N^\alpha_i = N^{2\beta}_i$ for $i \le 2\beta$ and $N_\alpha \in K_\lambda$ is chosen such that $N_{2\beta} \cup \{a^2_\beta\} \subseteq N_\alpha \le_{\mathfrak{k}} M^*_2$ and $N_\alpha \restriction M^*_1 \le_{\mathfrak{k}} M^*_1$, easy by the properties of abstract elementary class and we let $N^\alpha_{2\beta+1} = N_\alpha \restriction M^*_1$. For $\alpha = 2\beta + 2$ we choose by induction on $\varepsilon < \lambda^2$, a triple $(N^\oplus_{\alpha,\varepsilon}, N^\otimes_{\alpha,\varepsilon}, a_{\alpha,\varepsilon})$ such that:

(A) $N^\otimes_{\alpha,\varepsilon} \le_{\mathfrak{k}} M^*_2$ belongs to K_λ and is $\le_{\mathfrak{k}}$-increasing continuous with ε

(B) $N^\otimes_{\alpha,0} = N_{2\beta+1}$ and $N^\otimes_{\alpha,\varepsilon} \restriction M^*_1 \le^*_{\mathfrak{k}} M^*_1$

(C) $N^\oplus_{\alpha,\varepsilon} \le_{\mathfrak{k}} M^*_1$ belongs to K_λ and is $\le_{\mathfrak{k}}$-increasing continuous with ε

(D) $N^\oplus_{\alpha,0} = N^{2\beta+1}_{2\beta+1}$

(E) $(N^\oplus_{\alpha,\varepsilon}, N^\oplus_{\alpha,\varepsilon+1}, a_{\alpha,\varepsilon}) \in K^{3,\mathrm{uq}}_\lambda$

(F) $\mathrm{ortp}(a_{\alpha,\varepsilon}, N^\otimes_{\alpha,\varepsilon}, M^*_2)$ does not fork over $N^\oplus_{\alpha,\varepsilon}$

(G) $N^\oplus_{\alpha,\varepsilon} \le_{\mathfrak{k}} N^\otimes_{\alpha,\varepsilon}$

(H) for every $p \in \mathcal{S}^{\mathrm{bs}}(N^\oplus_{\alpha,\varepsilon})$ for some odd $\zeta \in [\varepsilon, \varepsilon+\lambda)$ the type $\mathrm{ortp}(a_{\alpha,\zeta}, N^\otimes_{\alpha,\zeta}, N^\otimes_{\alpha,\zeta+1})$ is a non-forking extension of p.

No problem to carry this. [Why? For $\varepsilon = 0$ and ε limit there are no problems. In stage $\varepsilon+1$ by bookkeeping gives you a type $p_\varepsilon \in \mathcal{S}^{\mathrm{bs}}(N^\oplus_{\alpha,\varepsilon})$ and let $q_\varepsilon \in \mathcal{S}^{\mathrm{bs}}(N^\otimes_{\alpha,\varepsilon})$ be a non-forking extension of p_ε. By assumption (iv) of $(**)_{M^*_1,M^*_2}$ there is an element $a_{\alpha,\varepsilon} \in M^*_1$ realizing q_ε. Now M^*_1 is saturated hence there is a model $N^\oplus_{\alpha,\varepsilon+1} \in K_\lambda$ such that $N^\oplus_{\alpha,\varepsilon+1} \le_{\mathfrak{k}} M^*_1$ and $(N^\oplus_{\alpha,\varepsilon}, N^\oplus_{\alpha,\varepsilon+1}, a_{\alpha,\varepsilon}) \in K^{3,\mathrm{uq}}_\lambda$.

Lastly, choose $N^\otimes_{\alpha,\varepsilon+1}$ satisfying clauses (A),(B),(G) so we have carried the induction on ε.]

Note that $\mathrm{NF}_\lambda(N^\oplus_{\alpha,\varepsilon}, N^\otimes_{\alpha,\varepsilon}, N^\oplus_{\alpha,\varepsilon+1}, N^\otimes_{\alpha,\varepsilon+1})$ for each $\varepsilon < \lambda^2$ by clauses (E),(F) and 6.35, hence $\mathrm{NF}(N^{2\beta+1}_{2\beta+1}, N_{2\beta+1}, \bigcup\{N^\oplus_{\alpha,\varepsilon} : \varepsilon < \lambda^2\}, \bigcup\{N^\otimes_{\alpha,\varepsilon} : \varepsilon < \lambda^2\})$ by 6.30 as $(N^\oplus_{\alpha,0}, N^\otimes_{\alpha,0}) = (N^{2\beta+1}_{2\beta+1}, N_{2\beta+1})$ and the sequences $\langle N^\oplus_{\alpha,\varepsilon} : \varepsilon < \lambda^+\rangle$, $\langle N^\otimes_{\alpha,\varepsilon} : \varepsilon < \lambda^+\rangle$ are increasing continuous.

Now let $N_\alpha = \bigcup\{N^\otimes_{\alpha,\varepsilon} : \varepsilon < \lambda^2\}, N^\alpha_\alpha = N_\alpha \cap M^*_1$ recalling clauses (A)+(B).

Now $\bigcup\{N^\oplus_{\alpha,\varepsilon} : \varepsilon < \lambda^2\} \le_{\mathfrak{k}} M^*_1$ is $(\lambda,*)$-brimmed over $N^{2\beta+1}_{2\beta+1}$ by 4.3 (and clause (H) above). Hence there is no problem to choose $N^\alpha_i \le_{\mathfrak{k}} N^\alpha_\alpha$ for $i \le 2\beta+1$ as required, that is $N^{2\beta+1}_i \le_{\mathfrak{k}} N^\alpha_i, \langle N^\alpha_i : i \le 2\beta+1\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous, $\mathrm{NF}_\lambda(N^{2\beta+1}_i, N^{2\beta+1}_{i+1}, N^\alpha_i, N^\alpha_{i+1})$ and N^α_{i+1} is $(\lambda,*)$-brimmed over $N^{2\beta+1}_{i+1} \cup N^\alpha_i$ and N^α_0 is $(\lambda,*)$-brimmed over $N^{2\beta+1}_0$. So we have finished the induction step on $\alpha = 2\beta+2$.

Having carried the induction we are done.

2) So assume $(*)$ and let $M_{\delta+1} := N$ from $(*)$. It is enough to prove that $(**)_{M_\delta,M_{\delta+1}}$ holds. Clearly clauses (i), (ii), (iii) hold, so we should prove (iv). Without loss of generality $\delta = \mathrm{cf}(\delta)$ so $\delta = \lambda^+$ or $\delta \le \lambda$. For $i \le \delta+1$ let $\langle M_{i,\alpha} : \alpha < \lambda^+\rangle$

be a $\le_{\mathfrak{k}}$-representation of M_i and for $i < \delta, j \in (i, \delta+1]$ let $E_{i,j}$ be a club of λ^+ witnessing $M_i \le^*_{\lambda^+} M_j$ for $\overline{M}^i, \overline{M}^j$. First assume $\delta \le \lambda$. Let $E = \cap\{E_{i,j} : i < \delta, j \in (i, \delta+1]\}$, it is a club of λ^+. So assume $N_2 \le_{\mathfrak{k}} M_{\delta+1}, N_1 \le_{\mathfrak{k}} N_2, N_1 \le_{\mathfrak{k}} M_\delta$ and $N_1, N_2 \in K_\lambda$ and $p \in \mathcal{S}^{\rm bs}(N_2)$ does not fork over N_1. We can choose $\zeta \in E$ such that $N_2 \subseteq M_{\delta+1,\zeta}$, let $p_1 \in \mathcal{S}^{\rm bs}(M_{\delta+1,\zeta})$ be a non-forking extension of p, so p_1 does not fork over N_1 hence (by monotonicity) over $M_{\delta,\zeta}$ so $p_2 := p_1 \restriction M_{\delta,\zeta} \in \mathcal{S}^{\rm bs}(M_{\delta,\zeta})$. By Axiom $(E)(c)$ for some $\alpha < \delta, p_2$ does not fork over $M_{\alpha,\zeta}$ hence $p_2 \restriction M_{\alpha,\zeta} \in \mathcal{S}^{\rm bs}(M_{\alpha,\zeta})$. As $M_\alpha \in K^{\rm nice}_{\lambda^+}$, i.e., M_α is λ^+-saturated (above λ), clearly for some $\xi \in (\zeta, \lambda^+) \cap E$ some $c \in M_{\alpha,\xi}$ realizes $p_2 \restriction M_{\alpha,\zeta}$ but ${\rm NF}_\lambda(M_{\alpha,\zeta}, M_{\delta+1,\zeta}, M_{\alpha,\xi}, M_{\delta+1,\xi})$ hence by 6.34 we know that ${\rm ortp}(c, M_{\delta+1,\zeta}, M_{\delta+1,\xi})$ belongs to $\mathcal{S}^{\rm bs}(M_{\delta+1,\zeta})$ and does not fork over $M_{\alpha,\zeta}$ hence c realizes p_2 and even p_1 hence p and we are done.

Second, assume $\delta = \lambda^+$, then for some $\delta^* < \delta$ we have $N_1 \le_{\mathfrak{k}} M_{\delta^*}$, and use the proof above for $\langle M_i : i \le \delta^*\rangle, M_{\delta+1}$ (or use $M_{\delta^*} \le^*_{\lambda^+} M_{\delta+1}$).

4) Straight, in fact included the proof of 7.8(2). $\square_{8.5}$

The definition below has affinity to "blowing $\mathfrak{k}_\lambda$ to $\mathfrak{k}^{\rm up}_\lambda$" in §1.

Definition 8.6. 0) $K^{3,{\rm cs}}_{\lambda^+} = \{(M,N,a) \in K^{3,{\rm bs}}_{\lambda^+} : M, N$ are from $K^{\rm nice}_{\lambda^+}\}$; we say $N' \in K_\lambda$ (or p') witness $(M,N,a) \in K^{3,{\rm cs}}_{\lambda^+}$ if it witnesses $(M,N,a) \in K^{3,{\rm bs}}_\lambda$.

1) $\mathcal{S}^{\rm cs}_{\lambda^+} := \{{\rm ortp}(a,M,N) : M \le^*_{\lambda^+} N$ are in $K^{\rm nice}_{\lambda^+}, a \in N$ and $(M,N,a) \in K^{3,{\rm cs}}_{\lambda^+}\}$, the type being for $\mathfrak{k}^{\rm nice}_{\lambda^+} = (K^{\rm nice}_{\lambda^+}, \le^*_{\lambda^+})$, see below [25] so the notation is justified by 8.7(1).

2) We define $\mathfrak{k}^\otimes = (K^\otimes, \le^\otimes)$ as follows

(a) $K^\otimes = \mathfrak{k} \restriction \{M \in K : M = \cup\{M_s : s \in I\}$ where $M_s \in K^{\rm nice}_{\lambda^+}, I$ is a directed partial order and $s <_I t \Rightarrow M_s \le^*_{\lambda^+} M_t\}$

(b) Let $M_1 \le^\otimes M_2$ if $M_1, M_2 \in K^\otimes, M_1 \le_{\mathfrak{k}} M_2$ and:

$(*)_{M_1,M_2}$ if $N_\ell \in K_\lambda, N_\ell \le_{\mathfrak{k}} M_\ell$, for $\ell = 1,2, p \in \mathcal{S}^{\rm bs}(N_2)$ does not fork over N_1 and $N_1 \le_{\mathfrak{k}} N_2$ <u>then</u> some $a \in M_1$ realizes p in M_2

(c) let $\le^\otimes_{\lambda^+} = \le^\otimes \restriction K^\otimes_{\lambda^+}$.

3) $\bigcup\limits_{\lambda^+} = \{(M_0, M_1, a, M_3) : M_0 \le^*_{\lambda^+} M_1 \le^*_{\lambda^+} M_3$ are in $K^{\rm nice}_{\lambda^+}$ and $(M_1, M_3, a) \in K^{3,{\rm cs}}_{\lambda^+}$ as witnessed by some $N \le_{\mathfrak{k}} M_0$ from $K_\lambda\}$.

4) $\mathfrak{k}^{\rm nice}_{\lambda^+} = (K^{\rm nice}_{\lambda^+}, \le^*_{\lambda^+})$, that is $(K^{\rm nice}_{\lambda^+}, \le^*_{\lambda^+} \restriction K^{\rm nice}_{\lambda^+})$.

5) We say that M' or p' witness $p = {\rm ortp}_{\mathfrak{k}^{\rm nice}_{\lambda^+}}(a,M,N)$ when $M' \le_{\mathfrak{k}} M, M' \in K_\lambda$ and $[M' \le_{\mathfrak{k}_\lambda} M'' \le_{\mathfrak{k}} M \Rightarrow {\rm ortp}_{\mathfrak{s}}(a, M'', N)$ does not fork over M' and $p' = {\rm ortp}_{\mathfrak{s}}(a, M', N)$.

Conclusion 8.7. *Assume* [26] *(recalling 8.4):*

$\boxtimes$ *not for every* $S \subseteq S^{\lambda^{++}}_{\lambda^+}$ *is there* λ^+*-saturated* $M \in K_{\lambda^{++}}$ *such that* $S(M) = S/\mathcal{D}_{\lambda^{++}}$.

0) On $K^{\rm nice}_{\lambda^+}$*, the relations* $\le^*_{\lambda^+}, \le^\otimes$ *agree.*

1) $\mathfrak{k}^{\rm nice}_{\lambda^+} = (K^{\rm nice}_{\lambda^+}, \le^*_{\lambda^+})$ *is a* λ^+*-abstract elementary class and is categorical in* λ^+ *and has no maximal member and has amalgamation.*

2) $K^\otimes$ *is included in the class of* λ^+*-saturated models in* $\mathfrak{k}$ *and* $K^\otimes_{\lambda^+} = K^{\rm nice}_{\lambda^+}$.

3) $\mathfrak{k}^\otimes$ *is an AEC with* ${\rm LS}(K^\otimes) = \lambda^+$ *and is the lifting of* $\mathfrak{k}^{\rm nice}_{\lambda^+}$.

[25] actually to define ${\rm ortp}_{\mathfrak{k}_\lambda}(a,M,N)$ where $M \le_{\mathfrak{k}_\lambda} N, \bar a \in N$ we need less that "$\mathfrak{k}_\lambda$ is a λ-AEC", and we know on $(K^{\rm nice}_{\lambda^+}, \le^*_{\lambda^+})$ more than enough

[26] this is like $(**)_{M_1,M_2}$ from 8.5, particularly see clause (iv) there

4) On $K^{\rm nice}_{\lambda^+}, (\mathcal{S}^{\rm cs}_{\lambda^+}, \bigcup\limits_{\lambda^+})$ *are equal to* $(\mathcal{S}^{\rm bs} \restriction K^{\rm nice}_{\lambda^+}, \bigcup\limits_{<\infty} \restriction K^{\rm nice}_{\lambda^+})$ *where they are defined in 2.4, 2.5.*

5) $(\mathfrak{k}^{\rm nice}_{\lambda^+}, \mathcal{S}^{\rm cs}_{\lambda^+}, \bigcup\limits_{\lambda^+})$ *is a good* λ^+*-frame.*

6) For $M_1 \le^*_{\lambda^+} M_2$ *from* $K^\otimes_{\lambda^+}$ *and* $a \in M_2 \setminus M_1$*, the type* $\mathrm{ortp}_{K^\otimes}(a, M_1, M_2)$ *is determined by* $\mathrm{ortp}_{\mathfrak{k}_\lambda}(a, N_1, M_2)$ *for all* $N_1 \le_{\mathfrak{k}} M_1, N_1 \in K_\lambda$*.*

Proof. 0) By 8.4 and our assumption ⊠, we have $M_1, M_2 \in K^{\rm nice}_{\lambda^+}$ and $M_1 \le^\otimes M_2 \Rightarrow M_1 \le^*_{\lambda^+} M_2$ (otherwise $(**)_{M_1,M_2}$ of 8.5 holds hence $(***)$ of 8.5 holds and by 8.4 we get ¬⊠, contradiction). The other direction is easier just see 8.5(4).

1) We check the axioms for being a λ^+-AEC: Ax 0: (Preservation under isomorphisms) Obviously. Ax I: Trivially. Ax II: By 7.4(2). Ax III: By 7.8(2) the union belongs to $K^{\rm nice}_{\lambda^+}$ and it $\le^*_{\lambda^+}$-extends each member of the union by 7.8(1). Ax IV: Otherwise $(*)$ of 8.5 holds, hence by 8.5 also $(***)$ of 8.5 holds. So by 8.4 our assumption ⊠ fail, contradiction; this is the only place we use ⊠ in the proof of (1). Ax V: By 7.4(3) and Ax V for $\mathfrak{k}$.

Also $\mathfrak{k}^{\rm nice}_{\lambda^+}$ is categorical by the uniqueness of the saturated model in λ^+ for $\mathfrak{k}$ has no maximal model by 7.4(1). $\mathfrak{k}^{\rm nice}_{\lambda^+}$ has amalgamation by 7.7(1).

2) Every member of $K^\otimes$ is λ^+-saturated in $\mathfrak{k}$ by 7.8(2) (prove by induction on the cardinality of the directed family in Definition 8.6(2), i.e. by the LS-argument it is enough to deal with the index family of $\le \lambda^+$ models each of cardinality λ^+, which holds by part (0) + (1)). If $M \in K_{\lambda^+}$ is λ^+-saturated, clearly $\in K^{\rm nice}_{\lambda^+}$.

3),4) Easy by now (or see §1).

5) We have to check all the clauses in Definition 2.1. We shall use parts (0)-(3) freely. Axiom (A):

By part (3) (of 8.7). Axiom (B):

There is a superlimit model in $K^\otimes_{\lambda^+} = K^{\rm nice}_{\lambda^+}$ by part (1) and uniqueness of the saturated model. Axiom (C):

By part (1), i.e., 7.7(1) we have amalgamation; JEP holds as $K^{\rm nice}_{\lambda^+}$ is categorical in λ^+. "No maximal member in $\mathfrak{k}^\otimes_{\lambda^+}$" holds by 7.4(1). Axiom (D)(a),(b):

By the definition 8.6(1). Axiom (D)(c):

By 2.9 (and Definition 8.6(1)). Clearly $K^{3,\rm cs}_{\lambda^+} = K^{3,\rm bs} \restriction K^{\rm nice}_{\lambda^+}$. Axiom (D)(d):

For $M \in \mathfrak{k}^\otimes_{\lambda^+}$ let $\overline{M} = \langle M_i : i < \lambda^+\rangle$ $\le_{\mathfrak{k}}$-represent M, so if $M \le^\otimes N \in K^\otimes_{\lambda^+}$, (hence $M \le^*_{\lambda^+} N \in K^\otimes_{\lambda^+} = K^{\rm nice}_{\lambda^+}$) and $a \in N$, $\mathrm{ortp}_{\mathfrak{k}^{\rm nice}_{\lambda^+}}(a, M, N) \in \mathcal{S}^{\rm cs}_{\lambda^+}(M)$, we let $\alpha(a, N, \overline{M}) = \min\{\alpha : \mathrm{ortp}(a, M_\alpha, N) \in \mathcal{S}^{\rm bs}(M_\alpha)$ and for every $\beta \in (\alpha, \lambda^+)$, $\mathrm{ortp}(a, M_\beta, N) \in \mathcal{S}^{\rm bs}(M_\beta)$ is a non-forking extension of $\mathrm{ortp}(a, M_\alpha, N)\}$.

Now

- (a) $\alpha(a, N, \overline{M})$ is well defined for a, N as above
  [Why? By Defintion 2.7 + 8.6(1)]
- (b) if a_ℓ, N_ℓ are above for $\ell = 1, 2$ and $\alpha(a_1, N_1, \overline{M}) = \alpha(a_2, N_2, \overline{M})$ call it α and $\mathrm{ortp}_{\mathfrak{s}}(a_1, M_\alpha, N) = \mathrm{ortp}_{\mathfrak{s}}(a_2, M_\alpha, N_2)$ then
  - $(*)$ for $\beta < \lambda^+$ we have $\mathrm{ortp}_{\mathfrak{s}}(a_1, M_\beta, N_1) = \mathrm{ortp}_{\mathfrak{s}}(a_1, M_\beta, N_2) \in \mathcal{S}^{\rm bs}(M_\beta)$
    [Why? By the non-forking uniqueness (Ax(E)(e)) when $\beta \ge \alpha$ by monotonicity if $\beta \le \alpha$]
- (c) if a_ℓ, N_ℓ are as above for $\ell = 1, 2$ and $(*)$ above holds then

$(**)$ $\mathrm{ortp}_{\mathfrak{k}^{\otimes}_{\lambda^+}}(a_1, M, N_1) = \mathrm{ortp}_{\mathfrak{k}^{\otimes}_{\lambda^+}}(a_2, M, N_2)$
[Why? Use 7.7(3) or by part (6) below].

As $\alpha < \lambda \Rightarrow |\mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(M_\alpha)| \leq \lambda$ (by the stability Axiom (D)(d) for $\mathfrak{s}$), clearly $|\mathcal{S}^{\mathrm{cs}}_{\lambda^+}(M)| \leq \sum\limits_{\alpha<\lambda^+} |\mathcal{S}^{\mathrm{bs}}(M_\alpha)| \leq \lambda^+ = \|M\|$ as required.

The reader may ask why do we not just quote the parallel result from §2: The answer is that the equality of types there is "a formal, not the true one". The crux of the matter is that we prove locality (in clause (c) above). Axiom (E)(a):

By 2.4 - 2.7. Axiom (E)(b); monotonicity:

Follows by Axiom (E)(b) for $\mathfrak{s}$ and the definition. Axiom (E)(c); local character:

By 2.11(5) or directly by translating it to the $\mathfrak{s}$-case. Axiom (E)(d); (transitivity):

By 2.11(4). Axiom (E)(e); uniqueness:

By 7.7(3) or by part (6) below. Axiom (E)(f); symmetry:

So assume $M_0 \leq^*_{\lambda^+} M_1 \leq^*_{\lambda^+} M_2$ are from $K^{\otimes}_{\lambda^+}$ and for $\ell = 1,2$ we have $a_\ell \in M_\ell$, $\mathrm{ortp}_{\mathfrak{k}^{\mathrm{nice}}_{\lambda^+}}(a_\ell, M_0, M_\ell) \in \mathcal{S}^{\mathrm{cs}}_{\lambda^+}(M_0)$ as witnessed by $p_\ell \in \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}}(N^*_\ell), N^*_\ell \in K_\lambda, N^*_\ell \leq_{\mathfrak{k}} M_0$ and $\mathrm{ortp}_{\mathfrak{k}^{\otimes}_{\lambda^+}}(a_2, M_1, M_2)$ does not fork (in the sense of $\bigcup\limits_{\lambda^+}$) over M_0 (note that M_0, M_1, M_2 here stand for M_0, M_1, M'_3 in clause (i) of Ax(E)(f) from Definition 2.1). As we know by monotonicity without loss of generality $M_1 <^+_{\lambda^+} M_2$. We can finish by 7.7(4) (and Axiom (E)(e) for $\mathfrak{s}$).

In more details, we can find N_0, N_1, N_2 such that: $N_\ell \leq_{\mathfrak{k}} M_\ell$ and $N_\ell \in K_\lambda$ for $\ell = 0,1,2$ and $N^*_1 \cup N^*_2 \subseteq N_0 \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} N_2$ and $a_1 \in N_1, a_2 \in N_2$ and N_2 is $(\lambda, *)$-brimmed over N_1 hence over N_0, and $(\forall N \in K_\lambda)[N_0 \leq_{\mathfrak{k}} N \leq_{\mathfrak{k}} M_0 \rightarrow (\exists M \in K_\lambda)(M \leq_{\mathfrak{k}} M_2$ and $\mathrm{NF}_\lambda(N_0, N, N_2, M))]$.

By Axiom (E)(f) for $\mathfrak{s} = (\mathfrak{k}, \mathcal{S}^{\mathrm{bs}}, \bigcup\limits_{\lambda})$ we can find N' such that $N_0 \leq_{\mathfrak{k}} N' \leq_{\mathfrak{k}} N_2$ such that $a_2 \in N'$ and $\mathrm{ortp}_{\mathfrak{s}}(a_1, N', N_2)$ does not fork over N_0. Now we can find f'_0, M'_1 such that $M_0 \leq^+_{\lambda^+} M'_1, f'_0$ is a $\leq_{\mathfrak{k}}$-embedding of N' into M'_1 and $(\forall N \in K_\lambda)[N_0 \leq_{\mathfrak{k}} N \leq_{\mathfrak{k}} M_0 \rightarrow (\exists M \in K_\lambda)(M \leq_{\mathfrak{k}} M'_1$ and $\mathrm{NF}_\lambda(N_0, N, f'_0(N'), M))]$. Next we can find f''_0, M'_2 such that $M'_1 <^+_{\lambda^+} M'_2, f''_0 \supseteq f'_0$ and f''_0 is a $\leq_{\mathfrak{k}}$-embedding of N_2 into M'_2 and $(\forall N \in K_\lambda)[N_0 \leq_{\mathfrak{k}} N \leq_{\mathfrak{k}} M_0 \rightarrow (\exists M \in K_\lambda)(M \leq_{\mathfrak{k}} M'_2$ and $\mathrm{NF}_\lambda(N_0, N, f''_0(N_2), M)]$.

Lastly, by 7.7(4) there is an isomorphism f from M_2 onto M'_2 over M_0 extending f''_0. Now $f^{-1}(M'_1)$ is a model as required. Axiom (E)(g); extension existence:

Assume $M_0 \leq^*_{\lambda^+} M_1$ are from $K^{\mathrm{nice}}_{\lambda^+}, p \in \mathcal{S}^{\mathrm{cs}}_{\lambda^+}(M_0)$, hence there is $N_0 \leq_{\mathfrak{k}} M_0, N_0 \in K_\lambda$ such that $(\forall N \in K_\lambda)(N_0 \leq_{\mathfrak{k}} N <_{\mathfrak{k}} M_0 \rightarrow p \restriction N$ does not fork over $N_0)$. By 7.4(1A) there are $M_2 \in K^{\otimes}_{\lambda^+}$ and $a \in M_2$ such that $M_1 \leq^*_{\lambda^+} M_2$ and $\mathrm{ortp}_{\mathfrak{k}^{\mathrm{nice}}_{\lambda^+}}(a, M_1, M_2) \in \mathcal{S}^{\mathrm{cs}}_{\lambda^+}(M_1)$ is witnessed by $p \restriction N_0$ and by part (6) we have $\mathrm{ortp}_{\mathfrak{k}^{\mathrm{nice}}_{\lambda^+}}(a, M_0, M_2) = p$. Checking the definition of does not fork, i.e., $\bigcup\limits_{\lambda^+}$ we are done. Axiom (E)(h), (continuity):

By 2.11(6). Axiom (E)(i):

It follows from the rest by 2.18.

6) So assume $M \leq^*_{\lambda^+} M_\ell, a_\ell \in M_\ell \setminus M$ for $\ell = 1,2$ and $N \leq_{\mathfrak{k}} M \wedge N \in K_\lambda \Rightarrow \mathrm{ortp}_{\mathfrak{k}}(a_1, N, M_1) = \mathrm{ortp}_{\mathfrak{k}}(a_2, N, M_2)$. By 7.4(1) there are $M^+_1, M^+_2 \in K^{\mathrm{nice}}_{\lambda^+}$ such that $M_\ell <^+_{\lambda^+} M^+_\ell$ for $\ell = 1,2$. By 7.7(2),(3) there is an isomorphism f from M^+_1 onto M^+_2 over M which maps a_1 to a_2. This clearly suffices. $\square_{8.7}$

§ 9. §9 Final conclusions

We now show that we have actually solved our specific test questions about categoricity and few models. First we deal with good λ-frames.

Lemma 9.1. ***Main Lemma***
1) Assume

(a) (α) $2^\lambda < 2^{\lambda^+} < 2^{\lambda^{++}} < \ldots < 2^{\lambda^{+n}}$, *and* $n \geq 2$
(β) *and* $\mathrm{WDmId}(\lambda^{+\ell})$ *is not* $\lambda^{+\ell+1}$*-saturated (normal ideal on* $\lambda^{+\ell}$*) for* $\ell = 1, \ldots, n-1$

(b) $\mathfrak{s} = (\mathfrak{k}, \mathcal{S}^{\mathrm{bs}}, \bigcup)$ *is a good* λ*-frame*

(c) $\dot{I}(\lambda^{+\ell}, \mathfrak{k}(\lambda^+\text{-saturated})) < \mu_{\mathrm{unif}}(\lambda^{+\ell}, 2^{\lambda^{\ell-1}})$ *for* $\ell = 2, \ldots, n$.

Then

(α) *K has a member of cardinality* λ^{+n+1}

(β) *for* $\ell < n$ *there is a good* $\lambda^{+\ell}$*-frame* $\mathfrak{s}_\ell = (\mathfrak{k}^\ell, \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}_\ell}, \bigcup_{\mathfrak{s}_\ell})$ *such that* $K^\ell_{\lambda^{+\ell}} \subseteq K_{\lambda^{+\ell}}$ *and* $\leq_{\mathfrak{k}^\ell} \subseteq \leq_{\mathfrak{k}}$

(γ) $\mathfrak{s}_0 = \mathfrak{s}$ *and if* $\ell < m < n$ *then* $K^\ell_{\lambda^{+m}} \supseteq K^m_{\lambda^{+m}}$ *and* $\leq_{\mathfrak{k}^\ell} \restriction K^m \supseteq \leq_{\mathfrak{k}^m}$.

2) Like part (1) omitting (β) of clause (a).

Proof. 1) We prove this by induction on n.

For $n = m + 1 \geq 2$, by the induction hypothesis for $\ell = 0, \ldots, m-1$, there is a frame $\mathfrak{s}_\ell = (\mathfrak{k}^\ell, \bigcup_{\mathfrak{s}_\ell}, \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}_\ell})$ which is $\lambda^{+\ell}$-good and $K_{\mathfrak{s}_\ell} \subseteq K^{\mathfrak{s}}_{\lambda^{+\ell}}$ and $\leq_{\mathfrak{k}^\ell} \subseteq \leq_{\mathfrak{k}} \restriction \mathfrak{k}^\ell$. By 5.9 and clause (c) of the assumption we know that $\mathfrak{s}$ has density for $K^{3,\mathrm{uq}}_{\mathfrak{s}}$. Now without loss of generality K^{m-1} is categorical in $\lambda^{+(m-1)}$ (by 2.23 really necessary only for $\ell = 0$) and by Observation 5.8 we get the assumption 6.9 of §6 hence the results of §6, §7, §8 apply. Now apply 8.7 to $(\mathfrak{k}^{m-1}, \mathcal{S}^{\mathrm{bs}}_{\mathfrak{s}_{m-1}}, \bigcup_{\mathfrak{s}_{m-1}})$ and get a λ^{+m}-frame $\mathfrak{s}_m$ as required in clause (β). By 4.14 we have $K^m_{\lambda^{+m+1}} \neq \varnothing$ which is clause (α) in the conclusion. Clause (β) has already been proved and clause (γ) should be clear.

2) Similarly but we use 5.11 instead of 5.9, i.e. we use the full version. $\square_{9.1}$

Second (this fulfills the aim of [She01] — equivalently, [She09c]).

theorem 9.2. *1) Assume* $2^{\lambda^{+\ell}} < 2^{\lambda^{+(\ell+1)}}$ *for* $\ell = 0, \ldots, n-1$ *and the normal ideal* $\mathrm{WDmId}(\lambda^{+\ell})$ *is not* $\lambda^{+\ell+1}$*-saturated for* $\ell = 1, \ldots, n-1$.

If $\mathfrak{k}$ is an abstract elementary class with $\mathrm{LS}(\mathfrak{k}) \leq \lambda$ *which is categorical in* λ, λ^+ *and* $1 \leq \dot{I}(\lambda^{+2}, K)$ *and* $\dot{I}(\lambda^{+m}, \mathfrak{k}) < \mu_{\mathrm{unif}}(\lambda^{+m}, 2^{\lambda^{+(m-1)}})$*, see* [She09a, 88r-0.wD]*(3). For* $m \in [2, n)$ *(or just* $\dot{I}(\lambda^{+m}, \mathfrak{k}(\lambda^+\text{-saturated})) < \mu_{\mathrm{unif}}(\lambda^{+m}, 2^{\lambda^{+(m-1)}})$*), then* $\mathfrak{k}_{\lambda^{+n}} \neq \varnothing$ *(and there are* $\mathfrak{s}_\ell (\ell < n)$ *as in (γ) of 9.1).*

2) We can omit the assumption "not $\lambda^{+\ell+1}$*-saturated".*

Proof. 1) By 3.10 and 9.1(1).

2) See by 3.10 and 9.1(2), i.e. using the full version of [She09d]. $\square_{9.2}$

Next we fulfill an aim of [She09a].

theorem 9.3. *1) Assume $2^{\aleph_\ell} < 2^{\aleph_{(\ell+1)}}$ for $\ell = 0, \ldots, n-1$ and $n \geq 2$ and $\mathrm{WDmId}(\lambda^{+\ell})$ is not $\lambda^{+\ell+1}$-saturated for $\ell = 1, \ldots, n-1$.*

If $\mathfrak{k}$ is an abstract elementary class which is $\mathrm{PC}_{\aleph_0}$ and $1 \leq \dot{I}(\aleph_1, \mathfrak{k}) < 2^{\aleph_1}$ and $\dot{I}(\aleph_\ell, \mathfrak{k}) < \mu_{\mathrm{unif}}(\aleph_\ell, 2^{\aleph_{\ell-1}})$, for $\ell = 2, \ldots, n$, then $\mathfrak{k}$ has a model of cardinality $\aleph_{n+1}$ (and there are $\mathfrak{s}_\ell(\ell < n)$ as in 9.2.

2) We can omit the assumption "not $\lambda^{+\ell+1}$-saturated".

Remark 9.4. Compared with Theorem 9.2 our gains are no assumption on $\dot{I}(\lambda, K)$ and weaker assumption on $\dot{I}(\lambda^+, K)$, i.e., $< 2^{\aleph_1}$ (and ≥ 1) rather than $= 1$. The price is $\lambda = \aleph_0^+$ and being $\mathrm{PC}_{\aleph_0}$.

Proof. 1) By 3.5 and 9.1(1).
2) See by 3.5 and 9.1(2), i.e. using the full version of [She09d]. $\square_{9.3}$

Lastly, we fulfill an aim of [She75].

theorem 9.5. *1) Assume $2^{\aleph_\ell} < 2^{\aleph_{\ell+1}}$ for $\ell \leq n-1$ and $\mathrm{WDmId}(\lambda^{+\ell})$ is not $\lambda^{+\ell+1}$-saturated for $\ell = 1, \ldots, n-1, \psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q}), \dot{I}(\aleph_1, \psi) \geq 1$ and $\dot{I}(\aleph_\ell, \psi) < \mu_{\mathrm{unif}}(\aleph_\ell, 2^{\aleph_{\ell-1}})$, for $\ell = 1, \ldots, n$. Then ψ has a model in $\aleph_{n+1}$ and there are $\mathfrak{s}_1, \ldots, \mathfrak{s}_{n-1}$ as in 9.3 with $K_{\mathfrak{s}_\ell} \subseteq \mathrm{Mod}_\psi$ and appropriate $\leq_{\mathfrak{k}}$.*

2) We can omit the assumption "not $\lambda^{+\ell+1}$-saturated".

Proof. 1) By 3.8 mainly clauses (c)-(d) and 9.1(1). Note that this time in 9.1 we use the $\dot{I}(\lambda^{+\ell}, \mathfrak{k}(\lambda^+\text{-saturated})) < \mu_{\mathrm{unif}}(\aleph_\ell, 2^{\aleph_{\ell-1}})$.
2) As in part (1) using 9.1(2). $\square_{9.5}$

References

[GSar] Noam Greenberg and Saharon Shelah, *Many forcing axioms for all regular uncountable cardinals*, Israel J. Math. (to appear), arXiv: 2107.05755.

[HS89] Ehud Hrushovski and Saharon Shelah, *A dichotomy theorem for regular types*, Ann. Pure Appl. Logic **45** (1989), no. 2, 157–169. MR 1044122

[JS13] Adi Jarden and Saharon Shelah, *Non-forking frames in abstract elementary classes*, Ann. Pure Appl. Logic **164** (2013), no. 3, 135–191, arXiv: 0901.0852. MR 3001542

[Kei71] H. Jerome Keisler, *Model theory for infinitary logic. logic with countable conjunctions and finite quantifiers*, Studies in Logic and the Foundations of Mathematics, vol. 62, North–Holland Publishing Co., Amsterdam–London, 1971.

[Mak85] Johann A. Makowsky, *Abstract embedding relations*, Model-Theoretic Logics (J. Barwise and S. Feferman, eds.), Springer-Verlag, 1985, pp. 747–791.

[S+] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F888.

[She] Saharon Shelah, *Compactness of the Quantifier on "Complete embedding of BA's"*, arXiv: 1601.03596 Ch. XI of "The Non-Structure Theory" book [Sh:e].

[She75] ______, *Categoricity in $\aleph_1$ of sentences in $L_{\omega_1,\omega}(Q)$*, Israel J. Math. **20** (1975), no. 2, 127–148. MR 0379177

[She83a] ______, *Classification theory for nonelementary classes. I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A*, Israel J. Math. **46** (1983), no. 3, 212–240. MR 733351

[She83b] ______, *Classification theory for nonelementary classes. I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B*, Israel J. Math. **46** (1983), no. 4, 241–273. MR 730343

[She87a] ______, *Classification of nonelementary classes. II. Abstract elementary classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Math., vol. 1292, Springer, Berlin, 1987, pp. 419–497. MR 1033034

[She87b] ______, *Universal classes*, Classification theory (Chicago, IL, 1985), Lecture Notes in Math., vol. 1292, Springer, Berlin, 1987, pp. 264–418. MR 1033033

[She90] ______, *Classification theory and the number of nonisomorphic models*, 2nd ed., Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990, Revised edition of [Sh:a]. MR 1083551

[She01] ______, *Categoricity of an abstract elementary class in two successive cardinals*, Israel J. Math. **126** (2001), 29–128, arXiv: math/9805146. MR 1882033

[She09a] ______, *Abstract elementary classes near $\aleph_1$*, Classification theory for abstract elementary classes, Studies in Logic (London), vol. 18, College Publications, London, 2009, arXiv: 0705.4137 Ch. I of [Sh:h], pp. vi+813.

[She09b] ______, *Categoricity and solvability of A.E.C., quite highly*, 2009, arXiv: 0808.3023 Ch. IV of [Sh:h].

[She09c] ______, *Categoricity of an abstract elementary class in two successive cardinals, revisited*, 2009, Ch. 6 of [Sh:i].

[She09d] ______, *Non-structure in λ^{++} using instances of WGCH*, 2009, arXiv: 0808.3020 Ch. VII of [Sh:i].

[She09e] ______, *Toward classification theory of good λ frames and abstract elementary classes*, 2009, arXiv: math/0404272 Ch. III of [Sh:h].

[She09f] ______, *Universal Classes: Axiomatic Framework [Sh:h]*, 2009, Ch. V (B) of [Sh:i].

[SV] Saharon Shelah and Sebastien Vasey, *Categoricity and multidimensional diagrams*, arXiv: 1805.06291.

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 9190401, Jerusalem, Israel; and, Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA

URL: https://shelah.logic.at/