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Different cofinalities of tree ideals

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ABSTRACT

We introduce a general framework of generalized tree forcings, GTF for short, that includes the classical tree forcings like Sacks, Silver, Laver or Miller forcing. Using this concept we study the cofinality of the ideal $\mathcal{I}(\boldsymbol{Q})$ associated with a GTF $\boldsymbol{Q}$. We show that if for two GTF's $\boldsymbol{Q_0}$ and $\boldsymbol{Q_1}$ the consistency of $\operatorname{add}(\mathcal{I}(\boldsymbol{Q_0})) < \operatorname{add}(\mathcal{I}(\boldsymbol{Q_1}))$ holds, then we can obtain the consistency of $\operatorname{cof}(\mathcal{I}(\boldsymbol{Q_1})) < \operatorname{cof}(\mathcal{I}(\boldsymbol{Q_0}))$. We also show that $\operatorname{cof}(\mathcal{I}(\boldsymbol{Q}))$ can consistently be any cardinal of cofinality larger than the continuum.

1. Introduction

The classical tree forcings like Sacks, Silver, Laver or Miller forcing consist of certain subtrees of $2^{<\omega}$ or $\omega^{<\omega}$ (see [2]). They will be denoted by Sa, Si, Mi, La respectively. As usual, for given $Q \in \{Sa, Si, La, Mi\}$ and $p \in Q$, $[p]$ denotes the set of branches of p, so a subset of $\mathbb{R}$, where $\mathbb{R}$ stands for 2^ω or ω^ω appropriately. Then the **tree ideal** $\mathcal{I}(Q)$ consists of all $X \subseteq \mathbb{R}$ such that for every $p \in Q$ there exists $q \in Q$ with $q \subseteq p$ and $[q] \cap X = \emptyset$. By using standard fusion arguments, it is easily seen that $\mathcal{I}(Q)$ is a σ-ideal. Hence we have $\aleph_1 \leq \operatorname{add}(\mathcal{I}(Q)) \leq 2^{\aleph_0}$, where $\operatorname{add}(\mathcal{I}(Q))$ denotes the **additivity** of $\mathcal{I}(Q)$, i.e. the minimal cardinality

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of some $\mathcal{X} \subseteq \mathcal{I}(Q)$ with $\bigcup \mathcal{X} \notin \mathcal{I}(Q)$. By $\mathrm{cof}(\mathcal{I}(Q))$ we denote the minimal cardinality of some $\mathcal{X} \subseteq \mathcal{I}(Q)$ that is cofinal in $(\mathcal{I}(Q), \subseteq)$. The same definitions make sense for many more tree forcings that are studied in set theory. This is one reason for us to introduce in Section 3 the general concept of **generalized tree forcing**. However, some knowledge about the antichain structure of the concrete forcing is needed for this framework to be applicable.

The original motivation for this paper was to gain insight into the cofinalities of classical tree ideals, as very little has been known about them. To our knowledge, the only papers dealing with this topic are [9] and [3]. In [9] it has been shown that $2^{\aleph_0} < \mathrm{cf}(\mathrm{cof}(\mathcal{I}(Sa)))$ holds in ZFC and that consistently $\mathrm{cof}(\mathcal{I}(Sa))$ can be any cardinal with cofinality $> 2^{\aleph_0}$. The same facts are true for Si with essentially the same proofs. Similar results for La, Mi have been obtained in [3]. Here we attack the question whether we can consistently obtain $\mathrm{cof}(\mathcal{I}(Q_0)) \neq \mathrm{cof}(\mathcal{I}(Q_1))$ for different $Q_0, Q_1 \in \{Sa, Si, La, Mi\}$. The main result of this paper implies that $\mathrm{cof}(\mathcal{I}(Q_1)) < \mathrm{cof}(\mathcal{I}(Q_0))$ is consistent for any pair of $Q_0, Q_1 \in \{Sa, Si, La, Mi\}$ for which $\mathrm{add}(\mathcal{I}(Q_0)) < \mathrm{add}(\mathcal{I}(Q_1))$ is consistent.

Unfortunately, distinguishing the additivities of different tree ideals is also a difficult matter. However there are some cases where this has been achieved, as much more work has been done about additivities of tree ideals. Let us mention [19], [9], [4], [10], [5], [6], [13], [14], [18], [15], [17] (chronological order). In [9] for $Q = Sa$ and in [4] for $Q = Si$ it has been shown that MA does not imply $\mathrm{add}(\mathcal{I}(Q)) = 2^{\aleph_0}$, whereas on the other hand, [5] and [6] show that this is true for $Q = La$ or $Q = Mi$. So we can apply our theorem for any choice of $Q_0 \in \{Sa, Si\}$ and $Q_1 \in \{La, Mi\}$. Another such case is when $Q_0 = Si$ and $Q_1 = Sa$. Implicitly in [10], an amoeba for Sa with the Laver property has been constructed. Iterating this with countable supports $\aleph_2$ times one obtains a model for $\mathrm{cov}(\mathcal{M}) = \aleph_1$ and $\mathrm{add}(\mathcal{I}(Sa)) = \aleph_2$. But by [14], $\mathrm{add}(\mathcal{I}(Si)) \leq \mathrm{cov}(\mathcal{M})$ holds in ZFC. (Here $\mathrm{cov}(\mathcal{M})$ is the minimal number of meager sets needed to cover $\mathbb{R}$.)

All the other cases are open. However, by the work of [13] and [15] soft amoebas for $Q \in \{Mi, La\}$ (with the Laver property) and for Si (with the pure decision property) exist. We expect that using these for making $\mathrm{add}(\mathcal{I}(Q)) = \aleph_2$ we can produce more models where our main theorem can be applied.

We expect that the methods and results presented in this paper will prove to be applicable to other tree ideals or similarly defined ideals as, e.g., Mycielski ideals. That is why we try to be as general as possible and, e.g., will introduce two versions of generalized tree forcings, GTF_0 and GTF_1 (see Definition 3.1), and associated amoebas $\mathbb{A}_0$ and $\mathbb{A}_1$ (see Definition 3.2) even though for the four tree forcings mentioned above one version would be enough.

2. $*_{\boldsymbol{d}}$-iterations

In [11], the first author introduced a general framework to iterate forcings that are $(< \lambda)$-closed and have the λ^+-c.c. with supports of size $< \lambda$, where λ is some regular cardinal with $\lambda^{<\lambda} = \lambda$. The main goal is to guarantee that also the iteration is λ^+-c.c. For this the $*_{\boldsymbol{d}}$-property is introduced as follows:

Definition 2.1. Let λ be a regular cardinal with $\lambda^{<\lambda} = \lambda$.

(1) A **c.c.-parameter** is a quintuple $\boldsymbol{d} = (\lambda, D, \varepsilon, \sigma, \boldsymbol{\mathcal{S}})$ such that
  (a) D is a normal filter on λ^+ containing $S^{\lambda^+}_{\lambda}$ and $\varepsilon < \lambda$ is a limit ordinal,
  (b) σ is a cardinal with $2 \leq \sigma \leq \lambda$ and $\boldsymbol{\mathcal{S}} \subseteq [S^{\lambda^+}_{\lambda}]^{<(1+\sigma)}$ has nonempty intersection with $[S]^{<(1+\sigma)}$ for every stationary set $S \subseteq S^{\lambda^+}_{\lambda}$.

(2) Given a forcing notion Q and a c.c.-parameter $\boldsymbol{d}$ we define the game $\mathcal{G}(Q, \boldsymbol{d})$ as follows: It lasts for ε moves. In his ζth move **player I** plays $(\langle q^{\zeta}_i : i < \lambda^+\rangle, f_{\zeta})$ and **player II** plays $\langle p^{\zeta}_i : i < \lambda^+\rangle$, where
  (a) $\forall i < \lambda^+ \forall \zeta < \varepsilon \quad (q^{\zeta}_i, p^{\zeta}_i \in Q \wedge q^{\zeta}_0 = \mathbf{1}_Q)$,
  (b) for every $1 \leq \zeta < \varepsilon \quad f_{\zeta} : \lambda^+ \to \lambda^+$ is regressive, $f_0 : \lambda^+ \to \lambda^+$ is constantly 0, and
  (c) $\forall \xi < \zeta < \varepsilon \forall^D i < \lambda^+ \quad q^{\zeta}_i \leq p^{\xi}_i$ and $\forall \zeta < \varepsilon \forall^D i < \lambda^+ \quad p^{\zeta}_i \leq q^{\zeta}_i$.

(3) **Player I wins** a play $\langle((\langle q_i^\zeta : i < \lambda^+\rangle, f_\zeta), \langle p_i^\zeta : i < \lambda^+\rangle : \zeta < \varepsilon\rangle$ provided that there exists $E \in D$ such that for every $u \in [E]^{<(1+\sigma)} \cap \boldsymbol{\mathcal{S}}$ with the property $\forall i, j \in u \forall \zeta < \varepsilon \quad f_\zeta(i) = f_\zeta(j)$ the set

$$\{p_i^\zeta : \zeta < \varepsilon, i \in u\}$$

has a lower bound in Q.

(4) Given a c.c.-parameter $\boldsymbol{d}$, we say that forcing Q satisfies property $*_{\boldsymbol{d}}$ if player I has a winning strategy in the game $\mathcal{G}(Q, \boldsymbol{d})$.

Remark 2.1. (1) Let Q be a forcing notion satisfying $*_{\boldsymbol{d}}$, where $\boldsymbol{d} = (\lambda, D, \varepsilon, \sigma, \boldsymbol{\mathcal{S}})$ is a c.c.-parameter with $D = CLUB_{\lambda^+}$ and $\boldsymbol{\mathcal{S}} = [S_\lambda^{\lambda^+}]^\kappa$ for some cardinal κ with $2 \le \kappa < 1 + \sigma$.

Note that given $\langle p_i : i < \lambda^+\rangle$, a sequence in Q, there exists a club $E \subseteq \lambda^+$ such that for every stationary $S \subseteq E \cap S_\lambda^{\lambda^+}$ there is $u \in \mathcal{P}(S) \cap \boldsymbol{\mathcal{S}}$ with the property that the set $\{p_i : i \in u\}$ has a lower bound.

Indeed, let $\langle((\langle q_i^\zeta : i < \lambda^+\rangle, f_\zeta), \langle p_i^\zeta : i < \lambda^+\rangle : \zeta < \varepsilon\rangle$ be a play of $\mathcal{G}(Q, \boldsymbol{d})$ where player I uses his winning strategy and player II plays $\langle p_i^0 : i < \lambda^+\rangle = \langle p_i : i < \lambda^+\rangle$ and afterwards just repeats the moves of player I. By Definition 2.1(3) there exists a club E as there. Given any stationary set $S \subseteq E \cap S_\lambda^{\lambda^+}$, for every $i \in S$ we can find $\alpha_i < i$ such that the sequence $\langle f_\zeta(i) : \zeta < \varepsilon\rangle$ is bounded by α_i. By the Pressing-down-Lemma there exist a stationary set $S_* \subseteq S$ and α_* such that $\alpha_i = \alpha_*$ for every $i \in S_*$. By our assumption $\lambda^{<\lambda} = \lambda$, there exists $U \subseteq S_*$ of size λ^+ such that $\langle f_\zeta(i) : \zeta < \varepsilon\rangle = \langle f_\zeta(j) : \zeta < \varepsilon\rangle$ for any $i, j \in U$. By construction and Definition 2.1(3), every $u \in \mathcal{P}(U) \cap \boldsymbol{\mathcal{S}}$ is as desired. By the choice of $\boldsymbol{\mathcal{S}}$, such u exist. In particular, Q is λ^+-c.c.

(2) Suppose that Q is **strongly λ-closed**, i.e., every decreasing sequence of length $< \lambda$ has a largest lower bound (llb for short) and, moreover, **strongly λ-centered** which means that $Q = \bigcup_{\mu<\lambda} Q_\mu$ where every Q_μ is **λ-strongly centered**, i.e., every subset of Q_μ of size $< \lambda$ has a llb. Then Q satisfies $*_{\boldsymbol{d}}$ for every c.c.-parameter $\boldsymbol{d} = (\lambda, D, \varepsilon, \sigma, \boldsymbol{\mathcal{S}})$.

Indeed, if such Q is given, in his ζth move player I plays $(\langle q_i^\zeta : i < \lambda^+\rangle, f_\zeta)$ such that q_i^ζ is a lower bound of player II's moves $\langle p_i^\xi : \xi < \zeta\rangle$ and $f_\zeta(i) = \mu$ such that $q_i^\zeta \in Q_\mu$. We claim that this is a winning strategy for player I. We apply normality of D to the (almost everywhere) regressive function

$$i \mapsto \langle f_\zeta(i) : \zeta < \varepsilon\rangle \in \lambda^{<\lambda} = \lambda$$

to find $E \in D$ and $\bar{f} = \langle f(\zeta) : \zeta < \varepsilon\rangle$ such that

$$\forall i \in E \forall \zeta < \varepsilon \quad f^\zeta(i) = f(\zeta).$$

Given any $u \subseteq E$ of size $< \lambda$ and any $\zeta < \varepsilon$ we have

$$q^{\zeta,u} := \left\{q^\zeta(i) : i \in u\right\} \in \left[Q_{f(\zeta)}\right]^{<\lambda},$$

and hence $q^{\zeta,u}$ has a llb, say r^ζ. Clearly $\langle r^\zeta : \zeta < \varepsilon\rangle$ is decreasing, hence has a llb, say r. But then r is a lower bound of

$$\left\{p_i^\zeta : \zeta < \varepsilon, i \in u\right\}.$$

In [11], the first author has proved the following preservation theorem:

Theorem 2.1. *Suppose that λ is a cardinal with $\lambda^{<\lambda} = \lambda$, $\boldsymbol{d} = (\lambda, D, \varepsilon, \sigma, \boldsymbol{\mathcal{S}})$ is a c.c.-parameter and $\langle P_\alpha, \dot{Q}_\beta : \alpha \le \mu, \beta < \mu\rangle$ is a $(< \lambda)$-support iteration such that for every $\beta < \mu$, $\Vdash_{P_\beta}$ "$\dot{Q}_\beta$ satisfies $*_{\boldsymbol{d}}$". Then P_α satisfies $*_{\boldsymbol{d}}$.*

3. Amoebas for generalized tree forcings

Definition 3.1. Let $\lambda = 2^{\aleph_0}$. (1) A $\mathbf{GTF_0}$ (here GTF stands for **generalized tree forcing**) is a quintupel $\boldsymbol{Q} = (Q, \dot{\zeta}, \mathrm{set}, Q^*, \perp)$ such that

(a) $Q = (Q, <_Q)$ is a forcing notion, $Q \subseteq H(\lambda)$ and $\dot{\zeta}$ is a Q-name such that $\Vdash_Q \dot{\zeta} \in \mathbb{R}$;
(b) Q^* is a dense subset of Q;
(c) set is a function from Q^* to Borel subsets of $\mathbb{R}$ such that
 (α) if $p \leqslant q$ then $\mathrm{set}(p) \subseteq \mathrm{set}(q)$,
 (β) $p \Vdash_Q \dot{\zeta} \in \mathrm{set}(p)$,
 (γ) $\Vdash_Q \{\dot{\zeta}\} = \bigcap\{\mathrm{set}(p) : p \in Q^* \cap \dot{G}_Q\}$, (where $\dot{G}_Q$ is the canonical Q-name of the generic filter);
(d) for every $A \in [\mathbb{R}]^{<\lambda}$ the set $\{p \in Q^* : \mathrm{set}(p) \cap A = \emptyset\}$ is dense in Q;
(e) $\perp$ is a binary, symmetric relation on Q^* such that
 (α) if $p \perp q$, then p and q are incompatible in Q,
 (β) if $p \perp q$, then $\mathrm{set}(p) \cap \mathrm{set}(q) = \emptyset$,
 (γ) if $\beta < \lambda$ and $\langle p_\alpha : \alpha < \beta \rangle$ is a sequence in Q^*, then there is $q \in Q^*$ such that $\forall \alpha < \beta\ p_\alpha \perp q$,
 (δ) if $\beta < \lambda$, $\langle p_\alpha : \alpha < \beta \rangle$ is a sequence in Q^* and $p \in Q$ is incompatible with every p_α, then there is $q \in Q^*$ such that $q \leq p$ and $\forall \alpha < \beta\ p_\alpha \perp q$.

(2) If $\boldsymbol{Q} = (Q, \dot{\zeta}, \mathrm{set}, Q^*, \perp)$ is as in (1) except that in (e), (γ) and (δ) are replaced by the weaker $(\gamma)_1$ and $(\delta)_1$ which ask the same thing as those, but only for **orthogonal sequences** $\langle p_\alpha : \alpha < \beta \rangle$, i.e. $p_\alpha \perp p_{\alpha'}$ for any $\alpha < \alpha' < \beta$, then we call $\boldsymbol{Q}$ a $\mathbf{GTF_1}$.

(3) If $\boldsymbol{Q} = (Q, \dot{\zeta}, \mathrm{set}, Q^*, \perp)$ is a GTF_1 we define

$$\mathcal{I}(\boldsymbol{Q}) = \{X \subseteq \mathbb{R} : \forall p \in Q^*\ \exists q \in Q^*(q \leqslant p \wedge \mathrm{set}(q) \cap X = \emptyset)\}.$$

Clearly $\mathcal{I}(\boldsymbol{Q})$ is an ideal on $\mathbb{R}$ and hence we can define $\mathrm{add}(\mathcal{I})$ and $\mathrm{cof}(\mathcal{I})$ as in the introduction.

Remark 3.1. (1) Clearly we have $\mathrm{GTF}_0 \subseteq \mathrm{GTF}_1$. By Theorem 6.1 below, *Sa* and *Si* can be considered as GTF_0's provided $\mathfrak{d} = 2^{\aleph_0}$, and if $\mathfrak{b} = 2^{\aleph_0}$, *La*, *Mi* can be considered as GTF_1's.

(2) Clearly in the definition of $\mathcal{I}(\boldsymbol{Q})$ we could replace Q^* by Q, and by Definition 3.1(d) we have $[\mathbb{R}]^{<\lambda} \subseteq \mathcal{I}(\boldsymbol{Q})$.

(3) Given $I \subseteq Q^*$ let

$$X(I) = \mathbb{R} \setminus \bigcup\{\mathrm{set}(p) : p \in I\}.$$

Then clearly the following sets are bases of $\mathcal{I}(\boldsymbol{Q})$:

$$\{X(I) : I \subseteq Q^* \text{ is predense}\},$$

$$\{X(I) : I \subseteq Q^* \text{ is a maximal antichain }\}.$$

Note that by applying (d) and (e) of Definition 3.1(1) we can obtain the following:

Claim 1. *Let $2^{\aleph_0} = \lambda = \lambda^{<\lambda}$. Given a GTF_1 $\boldsymbol{Q} = (Q, \dot{\zeta}, \mathrm{set}, Q^*, \perp)$ and a dense open subset $D \subseteq Q$, there exists a maximal antichain (with respect to $(Q, <_Q)$) $\langle q_\varepsilon : \varepsilon < \lambda \rangle$ in $Q^* \cap D$ such that*

(a) $\forall \varepsilon < \xi < \lambda\ q_\varepsilon \perp q_\xi$;
(b) $\forall r \in Q^*(\mathrm{set}(r) \nsubseteq \bigcup\{\mathrm{set}(q_\varepsilon) : \varepsilon < \lambda\} \vee \exists B \in [\lambda]^{<\lambda}\ \mathrm{set}(r) \subseteq \bigcup\{\mathrm{set}(q_\varepsilon) : \varepsilon \in B\})$.

For classical tree forcings this has been proved and applied first in [JMSh] and later was applied frequently.

Definition 3.2. Let $\lambda = 2^{\aleph_0}$.

(1) Given a GTF$_0$ $\boldsymbol{Q} = (Q, \dot{\zeta}, \mathrm{set}, Q^*, \perp)$ we define an amoeba forcing for $\boldsymbol{Q}$, denoted by $\mathbb{A}_0(\boldsymbol{Q})$, as follows:

Elements of $\mathbb{A}_0(\boldsymbol{Q})$ are pairs $\boldsymbol{p} = (\bar{p}, \mathcal{A}) = (\bar{p}_{\boldsymbol{p}}, \mathcal{A}_{\boldsymbol{p}})$ such that $\bar{p}$ is a sequence of length $< \lambda$ of members of Q^* and $\mathcal{A} \subseteq \mathcal{I}(\boldsymbol{Q})$ is a set of size $< \lambda$. Sometimes we write $\bar{p}_{\boldsymbol{p}} = \langle p_{\boldsymbol{p},\varepsilon} : \varepsilon < \lg(\bar{p}_{\boldsymbol{p}})\rangle$.

The order on $\mathbb{A}_0(\boldsymbol{Q})$ is defined by letting $\boldsymbol{p} \leq \boldsymbol{q}$ iff $\bar{p}_{\boldsymbol{q}}$ is an initial segment of $\bar{p}_{\boldsymbol{p}}, \mathcal{A}_{\boldsymbol{q}} \subseteq \mathcal{A}_{\boldsymbol{p}}$ and for every $B \in \mathcal{A}_{\boldsymbol{q}}$ and $\varepsilon \in [\lg(\bar{p}_{\boldsymbol{q}}), \lg(\bar{p}_{\boldsymbol{p}}))$ we have $\mathrm{set}(p_{\boldsymbol{p},\varepsilon}) \cap B = \emptyset$.

(2) Letting $\dot{G}$ denote the canonical $\mathbb{A}_0(\boldsymbol{Q})$-name for the generic filter, we let $\dot{\bar{p}} = \dot{\bar{p}}_{\dot{G}}$ be a name for $\bigcup\{\bar{p}_{\boldsymbol{p}} : \boldsymbol{p} \in \dot{G}\}$, which we also denote by $\langle \dot{p}_\varepsilon : \varepsilon < \dot{\mu}\rangle$, where $\dot{\mu} = \dot{\mu}_{\dot{G}} = \lg(\dot{\bar{p}})$, and for $\varepsilon < \dot{\mu}$ we let $\dot{B}_\varepsilon$ be a name for $\mathbb{R} \setminus \bigcup\{\mathrm{set}(\dot{p}_\zeta) : \zeta \in [\varepsilon, \dot{\mu})\}$. Finally, $\dot{\bar{B}} = \langle \dot{B}_\varepsilon : \varepsilon < \dot{\mu}\rangle$.

(3) Given a GTF$_1$ $\boldsymbol{Q}$, we define $\mathbb{A}_1(\boldsymbol{Q})$ as $\mathbb{A}_0(\boldsymbol{Q})$ except that for its members $\boldsymbol{p} = (\bar{p}, \mathcal{A})$ we require that $\bar{p}$ is an antichain (in Q^*) with respect to $\perp$. If $\dot{G}$ denotes the canonical $\mathbb{A}_1(\boldsymbol{Q})$-name for the generic filter and $\dot{\bar{p}} = \dot{\bar{p}}_{\dot{G}} = \langle \dot{p}_\varepsilon : \varepsilon < \dot{\mu}\rangle$ is defined for it as in (2), we define $\dot{B}_0$ as $\mathbb{R} \setminus \bigcup\{\mathrm{set}(\dot{p}_\varepsilon) : \varepsilon < \dot{\mu})\}$ and $\dot{B}_\varepsilon = \dot{B}_0$ for every $\varepsilon < \dot{\mu}$.

Remark 3.2. In Definition 3.2 the notion "amoeba forcing" is somewhat abused. In the context of some classical tree forcing P like Sa, Si, La or Mi, an amoeba for P is a forcing $\mathbb{A}(P)$ adding some tree in P such that all its branches are P-generic. If $\mathbb{A}(P)$ is reasonably nice, its countable support iteration will increase $\mathrm{add}(\mathcal{I}(P))$ to $\aleph_2$, where $\mathcal{I}(P)$ is the tree ideal associated to P.

The amoebas $\mathbb{A}_0(\boldsymbol{Q})$ or $\mathbb{A}_1(\boldsymbol{Q})$ from Definition 3.2 will be applied in a model where $\mathrm{add}(\mathcal{I}(\boldsymbol{Q})) = 2^{\aleph_0} = \lambda = \lambda^{<\lambda}$. Then they will have the effect that, if iterated with $< \lambda$ supports, they increase $\mathrm{cof}(\mathcal{I}(\boldsymbol{Q}))$ and preserve $\mathrm{add}(\mathcal{I}(\boldsymbol{Q}))$, i.e., won't let it drop to some smaller cardinal.

Lemma 3.1. *Suppose that* $\lambda = 2^{\aleph_0} = \lambda^{<\lambda}$.

(A) Let $\boldsymbol{Q}$ *be a GTF*$_0$ *and* $\mathrm{add}(\mathcal{I}(\boldsymbol{Q})) = \lambda$.

(1) $\mathbb{A}_0(\boldsymbol{Q})$ *is strongly* λ*-closed, i.e., every decreasing sequence of length* $< \lambda$ *has a llb; moreover,* $\mathbb{A}_0(\boldsymbol{Q})$ *is strongly* λ*-centered. Hence it satisfies* $*_{\boldsymbol{d}}$ *for every c.c.-parameter* $\boldsymbol{d} = (\lambda, D, \varepsilon, \sigma, \boldsymbol{\mathcal{S}})$ *(see Remark 2.1(2)).*

(2) $\Vdash_{\mathbb{A}_0(\boldsymbol{Q})}$ *"*$\dot{\mu} = \lambda \wedge \forall \varepsilon < \zeta < \lambda\,(\dot{B}_\varepsilon \in \mathcal{I}(\boldsymbol{Q}) \wedge \dot{B}_\varepsilon \subseteq \dot{B}_\zeta$*", and for every* $B \in \mathcal{I}(\boldsymbol{Q}) \cap \boldsymbol{V}$, $\Vdash_{\mathbb{A}_0(\boldsymbol{Q})} \exists \varepsilon < \lambda \quad B \subseteq \dot{B}_\varepsilon$.

(3) $\forall B \in \mathcal{I}(\boldsymbol{Q}) \cap \boldsymbol{V} \Vdash_{\mathbb{A}_0(\boldsymbol{Q})} \dot{B}_0 \nsubseteq B$.

(B) Let $\boldsymbol{Q}$ *be a GTF*$_1$ *and* $\mathrm{add}(\mathcal{I}(\boldsymbol{Q})) = \lambda$*. Then (A)(1), the first part of (A)(2) and (A)(3) also hold for* $\mathbb{A}_1(\boldsymbol{Q})$*, and, as for the second part of (A)(2), for every* $\mathcal{A} \in \boldsymbol{V}$ *such that* $\mathcal{A} \subseteq \mathcal{I}(\boldsymbol{Q})$ *and* $|\mathcal{A}| < \lambda$ *we have* $(\emptyset, \mathcal{A}) \Vdash_{\mathbb{A}_1(\boldsymbol{Q})} \bigcup \mathcal{A} \subseteq \dot{B}_0$.

Proof. (A)(1) Given $\langle \boldsymbol{p}_\alpha : \alpha < \gamma\rangle$ a descending chain in $\mathbb{A}_0(\boldsymbol{Q})$ with $\gamma < \lambda$, clearly we have that $(\bigcup\limits_{\alpha<\gamma} \bar{p}_{\boldsymbol{p}_\alpha}, \bigcup\limits_{\alpha<\gamma} \mathcal{A}_{\boldsymbol{p}_\alpha})$ is its largest lower bound in $\mathbb{A}_0(\boldsymbol{Q})$. Moreover, given $A \subseteq \mathbb{A}_0(\boldsymbol{Q})$ of size $< \lambda$ with $\bar{p}_{\boldsymbol{p}} = \bar{p}_{\boldsymbol{q}} =: \bar{p}$ for every $\boldsymbol{p}, \boldsymbol{q} \in A$, clearly $(\bar{p}, \bigcup_{\boldsymbol{p} \in A} \mathcal{A}_{\boldsymbol{p}})$ is the llb of A. By $\lambda^{<\lambda} = \lambda$ we conclude that $\mathbb{A}_0(\boldsymbol{Q})$ is strongly λ-centered. By Remark 2.1(2) we conclude that $\mathbb{A}_0(\boldsymbol{Q})$ satisfies $*_{\boldsymbol{d}}$.

(2) Given $\boldsymbol{p} \in \mathbb{A}_0(\boldsymbol{Q})$, $\gamma < \lambda, p \in Q_{\boldsymbol{Q}} \cap \boldsymbol{V}$ and $B \in \mathcal{I}(\boldsymbol{Q}) \cap \boldsymbol{V}$, by assumption we have that $X := \bigcup \mathcal{A}_{\boldsymbol{p}} \in \mathcal{I}(\boldsymbol{Q})$. By Definition 3.1(1)(b) we can find $\langle p_\varepsilon : \varepsilon < \gamma\rangle$ in Q^* such that $p_0 \leqslant_Q p$ and $\forall \varepsilon < \gamma \quad X \cap \mathrm{set}(p_\varepsilon) = \emptyset$, and hence, letting $\boldsymbol{q} := (\bar{p}_{\boldsymbol{p}}{}^\frown\langle p_\varepsilon : \varepsilon < \gamma\rangle, \mathcal{A}_{\boldsymbol{p}} \cup \{B\})$, we have $\boldsymbol{q} \in \mathbb{A}_0(\boldsymbol{Q})$, $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\boldsymbol{q} \Vdash$ "$\dot{\mu} \geqslant \gamma \wedge \forall \varepsilon < \lg(\bar{p}_{\boldsymbol{p}})\, \exists q \in Q^*(q \leqslant p \wedge \dot{B}_\varepsilon \cap \mathrm{set}(q) = \emptyset) \wedge \forall \varepsilon \geqslant \lg(\bar{p}_{\boldsymbol{q}})\, B \subseteq \dot{B}_\varepsilon$".

Hence by genericity and as $\mathbb{A}_0(\boldsymbol{Q})$ does not add new elements to $H(\lambda)$, we conclude that (2) holds.

(3) Given $\boldsymbol{p} \in \mathbb{A}_0(\boldsymbol{Q})$ and $B \in \mathcal{I}(\boldsymbol{Q}) \cap \boldsymbol{V}$, by Definition 3.1(1)(e) there is $q \in Q^*$ such that $\forall \varepsilon < \lg(\overline{p}_{\boldsymbol{p}}) \quad p_{\boldsymbol{p},\varepsilon} \perp q$. By Definition 3.1 there exists some singleton $X \subseteq \operatorname{set}(q)$ such that $X \cap B = \emptyset$, and hence $\boldsymbol{q} := (\overline{p}_{\boldsymbol{p}}, \mathcal{A}_{\boldsymbol{p}} \cup \{X\}) \in \mathbb{A}_0(\boldsymbol{Q}), \boldsymbol{q} \leqslant \boldsymbol{p}$ and $\boldsymbol{q} \Vdash \dot{B}_0 \nsubseteq B$ (note that by Definition 3.1(1)(e)(β) we have $\forall \varepsilon < \lg(\overline{p}_{\boldsymbol{p}}) \quad \operatorname{set}(p_{\boldsymbol{p},\varepsilon}) \cap X = \emptyset$).

(B) The proof is almost the same as for $\mathbb{A}_0(\boldsymbol{Q})$ in (A). $\square$

Theorem 3.1. *Suppose that* $\boldsymbol{Q}$ *is a* GTF_1, $2^{\aleph_0} = \lambda = \lambda^{<\lambda} < \mu = \operatorname{cf}(\mu) < \chi = \chi^{<\chi}$ *and* $\operatorname{add}(\mathcal{I}(\boldsymbol{Q})) = \lambda$. *There exists a forcing P such that*

(a) $|P| = \chi, P$ *is* λ*-closed and* λ^+*-c.c.*
(b) $\boldsymbol{V}^P \models 2^\lambda = \chi \wedge \operatorname{cof}(\mathcal{I}(\boldsymbol{Q})) = \mu$.

Proof. Let us first assume that $\boldsymbol{Q}$ is even GTF_0 (recall $\mathrm{GTF}_0 \subseteq \mathrm{GTF}_1$). Let P be the limit of a $(<\lambda)$-support iteration $\langle P_\alpha, \dot{Q}_\beta : \alpha < \mu, \beta < \mu \rangle$ where $Q_0 = Fn(\chi, 2, \lambda)$ (which is the standard forcing for adding χ Cohen subsets of λ with conditions of size $< \lambda$) and $\dot{Q}_{1+\beta}$ denotes $\mathbb{A}_0(\boldsymbol{Q})$ in $\boldsymbol{V}^{P_{1+\beta}}$.

It is easy to check that $Fn(\chi, 2, \lambda)$ satisfies $*_{\boldsymbol{d}}$ for every c.c.-parameter $\boldsymbol{d} = (\lambda, D, \varepsilon, \sigma, \boldsymbol{\mathcal{S}})$. Actually, a simplified version of the proof of Lemma 4.2 below can be used. Hence by Lemma 3.1(A)(1) and Theorem 2.1, P satisfies $*_{\boldsymbol{d}}$ and hence, letting $\boldsymbol{\mathcal{S}} = [S^{\lambda^+}_\lambda]^\kappa$ for $\kappa = 2$, by Remark 2.1(1) P is λ^+-c.c. Clearly, by Lemma 3.1(A)(1) and as we have $(<\lambda)$-supports, P is also λ-closed.

Let G be a P-generic filter over $\boldsymbol{V}$. For $1 \leqslant \beta < \mu$ let $\langle B^\beta_\varepsilon : \varepsilon < \lambda \rangle$ be the generic sequence in $\mathcal{I}(\boldsymbol{Q})^{V[G_{\beta+1}]}$ determined by $G(\beta)$. Note that by λ-closedness $\boldsymbol{P}$ does not add new elements to $H(\lambda)$ and hence we have $\mathcal{I}(\boldsymbol{Q})^{\boldsymbol{V}[G_\beta]} = \mathcal{I}(\boldsymbol{Q})^{\boldsymbol{V}[G]} \cap \boldsymbol{V}[G_\beta]$ for every $\beta < \lambda$. By the λ^+-c.c. of P and the regularity of μ, every $X \in \boldsymbol{V}[G]$ of size $< \mu$ with $X \subset \boldsymbol{V}$ belongs to $\boldsymbol{V}[G_\beta]$ for some $\beta < \mu$. Hence by Lemma 3.1(A)(2) we conclude that $\{B^\beta_\varepsilon : 1 \leqslant \beta < \mu, \varepsilon < \lambda\}$ is cofinal in $\mathcal{I}(\boldsymbol{Q})^{\boldsymbol{V}[G]}$, thus $\operatorname{cof}(\mathcal{I}(\boldsymbol{Q})) \leqslant \mu$ in $\boldsymbol{V}[G]$.

For the same reason, given $\mathcal{X} \in \boldsymbol{V}[G]$ such that $\boldsymbol{V}[G] \models \mathcal{X} \subseteq \mathcal{I}(\boldsymbol{Q}) \wedge |\mathcal{X}| < \mu$, there is $\beta < \mu$ such that $\mathcal{X} \subset \boldsymbol{V}[G_\beta]$ (and actually $\mathcal{X} \in \boldsymbol{V}[G_\beta]$). By Lemma 3.1(A)(3) we conclude that no member of $\mathcal{X}$ contains B^β_0. Hence $\boldsymbol{V}[G] \models \operatorname{cof}(\mathcal{I}(\boldsymbol{Q})) = \mu$.

If $\boldsymbol{Q}$ is only GTF_1 we define the iteration P as above except that iterand $\dot{Q}_{1+\beta}$ denotes $\mathbb{A}_1(\boldsymbol{Q})$ in $\mathbb{V}^{P_{1+\beta}}$. The proof is almost the same as in the first case, except that now we argue that $\{B^\beta_0 : 1 \leqslant \beta < \mu\}$ is cofinal in $\mathcal{I}(\boldsymbol{Q})^{\boldsymbol{V}[G]}$, where B^β_0 denotes "B_0 defined by $G(\beta)$" (see Definition 3.2(3)). In fact, given $X \in \mathcal{I}(\boldsymbol{Q})^{\boldsymbol{V}[G]}$, as $X \in \boldsymbol{V}[G_\beta]$ for some $\beta < \mu$, by genericity we have $(\emptyset, \{X\}) \in G(\gamma)$ for some $\beta < \gamma < \mu$, and hence $X \subseteq B^\gamma_0$ by Lemma 3.1(B). $\square$

Lemma 3.2. *Suppose* $2^{\aleph_0} = \lambda = \lambda^{<\lambda}$, $\boldsymbol{Q}$ *is a* GTF_1 *and P is a* λ*-closed forcing. If* $\operatorname{add}(\mathcal{I}(\boldsymbol{Q})) = \mu$, *then* $\boldsymbol{V}^P \models \operatorname{add}(\mathcal{I}(\boldsymbol{Q})) = \mu$.

Proof. We assume $\mu = \lambda$. The case $\mu < \lambda$ is similar. Let $\boldsymbol{Q} = (Q, \dot{\zeta}, \operatorname{set}, Q^*, \perp)$. Suppose $p \in P, \beta < \lambda$ and $\langle \dot{X}_\alpha : \alpha < \beta \rangle$ are P-names such that

$$p \Vdash_P \forall \alpha < \beta \, \dot{X}_\alpha \in \mathcal{I}(\boldsymbol{Q}).$$

By Claim 1, wlog we may assume that there are $\dot{I}_\alpha = \langle \dot{q}^\alpha_\varepsilon : \varepsilon < \lambda \rangle$ for $\alpha < \beta$ such that the following hold:

(1) $p \Vdash_P \forall \alpha < \beta \, (\dot{I}_\alpha \text{ is a maximal antichain in } Q^* \wedge \dot{X}_\alpha = X(\dot{I}_\alpha))$;
(2) for every $\alpha < \beta$, $p \Vdash \forall r \in Q^* (\exists x \in \operatorname{set}(r) \quad x \notin \bigcup_{\varepsilon<\lambda} \operatorname{set}(\dot{q}^\alpha_\varepsilon) \vee \exists B \in [\lambda]^{<\lambda} \quad \operatorname{set}(r) \subseteq \bigcup_{\varepsilon \in B} \operatorname{set}(\dot{q}^\alpha_\varepsilon))$;
(3) $p \Vdash \forall \varepsilon < \xi < \lambda \quad \dot{q}^\alpha_\varepsilon \perp \dot{q}^\alpha_\xi$.

Note that as P does not add new reals nor elements of $H(\lambda)$, by absoluteness we have $\operatorname{set}(r)^{\boldsymbol{V}} = \operatorname{set}(r)^{\boldsymbol{V}^P}$ for every $r \in Q^*$. Moreover, for every $r \in Q^*$, by strengthening p in (2) we can decide which alternative holds and also the witness for this (so some $x \in \operatorname{set}(r)$ or $B \in [\lambda]^{<\lambda}$).

Let $\langle r_\varepsilon : \varepsilon < \lambda \rangle$ list Q^*. By the λ-closedness of P and the remark just made we can easily construct a decreasing sequence $\langle p_\varepsilon : \varepsilon < \lambda \rangle$ in P and a sequence $\langle \zeta_\varepsilon : \varepsilon < \lambda \rangle$ of ordinals in λ such that

(4) $p_0 = p, \zeta_\varepsilon \geqslant \varepsilon$;
(5) for all $\alpha < \beta$ and $\varepsilon < \lambda$, $p_{\varepsilon+1}$ decides $\langle \dot{q}^\alpha_\xi : \xi < \zeta_\varepsilon \rangle$, say as $\langle q^{\alpha,\xi} : \xi < \zeta_\varepsilon \rangle$;
(6) for all $\alpha < \beta$ and $\varepsilon < \lambda$ there is $\xi < \zeta_\varepsilon$ such that r_ε and $q^{\alpha,\xi}$ are compatible (in Q);
(7) for all $\alpha < \beta$ and $\varepsilon < \lambda$, $p_{\varepsilon+1}$ decides which alternative of (2) for $r = r_\varepsilon$ holds and also in either case the witness for this (so either $x^{\alpha,\varepsilon} \in \operatorname{set}(r_\varepsilon)$ or $B^{\alpha,\varepsilon} \in [\lambda]^{<\lambda}$).

For $\alpha < \beta$ we let $A_\alpha = \langle q^{\alpha,\xi} : \xi < \lambda \rangle$. Then by construction every A_α is a maximal antichain (with respect to $(Q, <_Q)$) in Q^* and hence $X(A_\alpha) \in \mathcal{I}(\boldsymbol{Q})$. By hypothesis, $\bigcup_{\alpha<\beta} X(A_\alpha) \in \mathcal{I}(\boldsymbol{Q})$. Choose $r \in Q^*$ such that $\operatorname{set}(r) \cap \bigcup_{\alpha<\beta} X(A_\alpha) = \emptyset$, thus

(8) $\operatorname{set}(r) \subseteq \bigcup\{\operatorname{set}(q^{\alpha,\xi}) : \xi < \lambda\}$ for every $\alpha < \beta$. Let $r = r_\varepsilon$.

Note that by (7), we must have

(9) for every $\alpha < \beta$, $p_{\varepsilon+1} \Vdash \operatorname{set}(r) \subseteq \bigcup_{\xi \in B^{\alpha,\varepsilon}} \operatorname{set}(\dot{q}^\alpha_\xi)$.

Indeed, otherwise we had $\alpha < \beta$ and $x^{\alpha,\varepsilon} \in \operatorname{set}(r_\varepsilon)$ such that

$$p_{\varepsilon+1} \Vdash x^{\alpha,\varepsilon} \notin \bigcup_{\xi<\lambda} \operatorname{set}(\dot{q}^\alpha_\xi).$$

By (8) there is $\xi_0 < \lambda$ such that $x^{\alpha,\varepsilon} \in \operatorname{set}(q^{\alpha,\xi_0})$. Letting $\mu > \max\{\varepsilon, \xi_0\}$ we have $p_\mu \leqslant p_{\varepsilon+1}$ and $p_\mu \Vdash \dot{q}^\alpha_{\xi_0} = q^{\alpha,\xi_0}$, thus $p_\mu \Vdash x^{\alpha,\varepsilon} \in \operatorname{set}(\dot{q}^\alpha_{\xi_0})$, which is a contradiction.

As (9) holds for a dense set of $r \in Q^*$, we conclude that $p \Vdash \bigcup_{\alpha<\beta} \dot{X}_\alpha \in \mathcal{I}(\boldsymbol{Q})$. □

4. Small additivity and large cofinality - the antiamoeba

In this section we shall show that the assumption $\operatorname{add}(\mathcal{I}(\boldsymbol{Q})) \leq \kappa < 2^{\aleph_0} = \kappa^+ < \chi$ for some GTF_1 $\boldsymbol{Q}$ enables us to define some forcing $\mathbb{AA}(\boldsymbol{Q})$, which we call the antiamoeba for $\boldsymbol{Q}$, that introduces some family $\langle X_\alpha : \alpha < \chi \rangle$ in $\mathcal{I}(\boldsymbol{Q})$ that is hard to cover, i.e., for many increasing sequences $\langle \beta_\iota : \iota < \kappa \rangle$ in χ we have

$$\bigcup_{\iota<\kappa} X_{\beta_\iota} \notin \mathcal{I}(\boldsymbol{Q}).$$

This will imply $\operatorname{cof}(\mathcal{I}(\boldsymbol{Q})) \geq \chi$.

Definition 4.1. Let $2^{\aleph_0} = \lambda = \lambda^{<\lambda} = \kappa^+$ and $\boldsymbol{Q} = (Q, \dot{\zeta}, \operatorname{set}, Q^*, \perp)$ be a GTF_1. We say that $\boldsymbol{Q}$ **has a strong witness $\mathcal{W}$ for** $\operatorname{add}(\mathcal{I}(\boldsymbol{Q})) \leqslant \kappa$ if $\mathcal{W} = (\overline{q}^*, \langle \overline{q}^*_{\iota,\varepsilon} : \iota < \kappa, \varepsilon < \lambda \rangle)$ such that the following hold: $\overline{q}^* = \langle q^*_\varepsilon : \varepsilon < \lambda \rangle$ is an orthogonal maximal antichain (w.r.t. $(Q, \leq)$) in Q^* and for every $\iota < \kappa$ and $\varepsilon < \lambda$ $\overline{q}^*_{\iota,\varepsilon} = \langle q^*_{\iota,\varepsilon,\zeta} : \zeta < \lambda \rangle$ is some family in Q^* below q^*_ε such that $\overline{q}^*_{\iota,\varepsilon}$ is predense (w.r.t. $(Q, \leq)$) below q^*_ε, hence

$$X_{\iota,\varepsilon} := \mathrm{set}(q^*_\varepsilon) \setminus \bigcup\{\mathrm{set}(q^*_{\iota,\varepsilon,\zeta}) : \zeta < \lambda\}$$

belongs to $\mathcal{I}(\boldsymbol{Q})$, but $Y_\varepsilon := \bigcup\limits_{\iota<\kappa} X_{\iota,\varepsilon} \notin \mathcal{I}(\boldsymbol{Q})$.

Definition 4.2. (1) Let $\chi > 2^{\aleph_0} = \lambda = \lambda^{<\lambda} = \kappa^+$ and $\boldsymbol{Q} = (Q, \dot{\zeta}, \mathrm{set}, Q^*, \perp)$ a GTF$_1$ with a strong witness $\mathcal{W}$ for $\mathrm{add}(\mathcal{I}(\boldsymbol{Q})) \leqslant \kappa$, and let $\mathcal{W} = (\overline{q}^*, \langle \overline{q}^*_{\iota,\varepsilon} : \iota < \kappa, \varepsilon < \lambda\rangle)$ be as in Definition 4.1. We define a forcing notion $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ as follows ("$\mathbb{AA}$" stands for "anti-amoeba"):

(A) (a) Conditions $p \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ have the form $p = (u, \zeta, \overline{r}, S, f) = (u_p, \zeta_p, \overline{r}_p, S_p, f_p)$ where
  (b) $u \in [\chi]^{\leqslant\kappa}$ and $\zeta < \lambda$,
  (c) $\overline{r} = \langle r_{\alpha,\varepsilon} : \alpha \in u, \varepsilon < \zeta\rangle$ and $\overline{r}^{[\alpha]} := \langle r_{\alpha,\varepsilon} : \varepsilon < \zeta\rangle$ (for $\alpha \in u$) are such that every $r_{\alpha,\varepsilon}$ is a member of Q^* below some q^*_ξ (from the strong witness)
  (d) $S \subseteq \{\overline{\alpha} : \overline{\alpha} \in {}^\kappa u \text{ is increasing }\}$ and $|S| \leqslant \kappa$,
  (e) $f : S \to \lambda$ is such that for every $\overline{\alpha}_1, \overline{\alpha}_2 \in S$
    (α) if $f(\overline{\alpha}_1) = f(\overline{\alpha}_2)$ then $(\overline{\alpha}_1, \overline{\alpha}_2)$ is a **Δ-system pair**, i.e. $\forall i, j < \kappa(\overline{\alpha}_1(i) = \overline{\alpha}_2(j) \Rightarrow i = j)$ and
    (β) if $f(\overline{\alpha}_1) \neq f(\overline{\alpha}_2)$, then $|\mathrm{ran}(\overline{\alpha}_1) \cap \mathrm{ran}(\overline{\alpha}_2)| \leqslant 1$.

(B) The order on $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ is defined as follows: For $p_1, p_2 \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ we declare $p_2 \leqslant p_1$ iff
  (a) $u_{p_1} \subseteq u_{p_2}$, $\zeta_{p_1} \leqslant \zeta_{p_2}$, $\overline{r}_{p_1} = \overline{r}_{p_2} \restriction u_{p_1} \times \zeta_{p_1}$, $S_{p_1} \subseteq S_{p_2}$, $f_{p_1} \subseteq f_{p_2}$ and
  (b) if $(\alpha, \varepsilon) \in (u_{p_2} \times \zeta_{p_2}) \setminus (u_{p_1} \times \zeta_{p_1})$, $\xi(p_2, \alpha, \varepsilon)$ is the unique ξ such that $r^{p_2}_{\alpha,\varepsilon} \leqslant q^*_\xi$ and $\overline{\beta} \in S_{p_1}$, $\iota < \kappa$ are such that $f_{p_1}(\overline{\beta}) = \xi(p_2, \alpha, \varepsilon)$ and $\beta_\iota := \overline{\beta}(\iota) = \alpha$ (note that this implies $\alpha \in u_{p_1}$ by (A)(d), and by (A)(e)(α) ι does not depend on $\overline{\beta}$), then $r^{p_2}_{\alpha,\varepsilon} \leqslant q^*_{\iota, f_{p_1}(\overline{\beta}), \zeta}$ for some $\zeta < \lambda$.

(2) Letting $\dot{G}_{\mathbb{AA}(\boldsymbol{Q},\mathcal{W},\chi)}$ the canonical name for the $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$-generic filter, for $\alpha < \chi$ we let $\dot{\overline{p}}_\alpha = \langle \dot{r}_{\alpha,\varepsilon} : \varepsilon < \lambda\rangle$ be the $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$-name $\bigcup\{\overline{r}^{[\alpha]}_p : p \in \dot{G}_{\mathbb{AA}(\boldsymbol{Q},\mathcal{W},\chi)} \wedge \alpha \in u_p\}$ and $\dot{X}_\alpha = \mathbb{R} \setminus \bigcup\{\mathrm{set}(\dot{r}_{\alpha,\varepsilon}) : \varepsilon < \lambda\}$.

Lemma 4.1. *With the notation of Definition 4.2 the following statements are true:*

(1) *Every descending sequence in $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ of length $< \lambda$ has a largest lower bound.*

(2) *$\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ is not empty and for every $r_* \in Q, \alpha_* < \chi$ and $p_1 \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ there exists $p_2 \in \mathbb{AA}(\boldsymbol{Q}, \chi)$ such that*
  (a) *$p_2 \leqslant p_1$,*
  (b) *$\zeta_{p_1} < \zeta_{p_2}$ and $\alpha_* \in u_{p_2}$,*
  (c) *for some $\varepsilon < \zeta_{p_2}$ we have that $r^{p_2}_{\alpha_*,\varepsilon}$ and r_* are compatible.*
  (d) *$\forall \alpha < \chi \quad \Vdash_{\mathbb{AA}(\boldsymbol{Q},\mathcal{W},\chi)} \dot{\overline{p}}_\alpha$ lists a predense subset of Q.*

(3) *Suppose that $p \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$, $\xi < \lambda$, $\overline{\beta} \in {}^\kappa\chi \setminus S_p$ are such that $\xi \notin \{\nu < \lambda : \exists(\alpha, \varepsilon) \in u_p \times \zeta_p \quad r^p_{\alpha,\varepsilon} \leqslant q^*_\nu\} \cup \mathrm{ran}(f_p)$ and, letting*

$$q := (u_p \cup \mathrm{ran}(\bar{\beta}), \zeta_p, \overline{r}_p, S_p \cup \{\overline{\beta}\}, f_p \cup \{(\overline{\beta}, \xi)\}),$$

we have $q \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ and hence $q \leqslant p$. Then

$$q \Vdash_{\mathbb{AA}(\boldsymbol{Q},\mathcal{W},\chi)} \quad \bigcup_{\iota<\kappa} (\mathrm{set}(q^*_\xi) \setminus \bigcup\{\mathrm{set}(\dot{r}_{\beta_\iota,\varepsilon}) : \varepsilon < \lambda\}) \notin \mathcal{I}(\boldsymbol{Q})$$

and hence $q \Vdash_{\mathbb{AA}(\boldsymbol{Q},\mathcal{W},\chi))} \bigcup\limits_{\iota<\kappa} \dot{X}_{\beta_\iota} \notin \mathcal{I}(\boldsymbol{Q})$.

Remark 4.1. Note that in (3), for $q \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi))$ to hold we only need that $\overline{\beta}$ is increasing and for every $\overline{\alpha} \in S_p$ we have $|\mathrm{ran}(\overline{\alpha}) \cap \mathrm{ran}(\overline{\beta})| \leqslant 1$.

Proof. (1) Given a descending chain $\langle p_\alpha : \alpha < \mu < \lambda \rangle$ in $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi))$ we define $q \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi))$ by letting $u_q = \bigcup\{u_{p_\alpha} : \alpha < \mu\}$, $\zeta_q = \sup\{\zeta_{p_\alpha} : \alpha < \mu\}$, $\overline{r}_q$ is such that for every $\beta \in u_q$ $\overline{r}_q^{[\beta]} = \bigcup\{\overline{r}_{p_\alpha}^{[\beta]} : \alpha < \mu \wedge \beta \in u_\alpha\}$, $S_q = \bigcup\{S_{p_\alpha} : \alpha < \mu\}$ and $f_q = \bigcup\{f_{p_\alpha} : \alpha < \mu\}$. By our assumptions it is easily checked that $q \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi))$, $\forall \alpha < \mu \quad q \leqslant p_\alpha$ and q is the largest lower bound.

(2) $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi))$ is not empty as $(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$ is an element. Let us check density. We do it in two steps. First we find $p_2 \leqslant p_1$ with $\zeta_{p_1} < \zeta_{p_2}$. We can choose $\xi \in \lambda \setminus \mathrm{ran}(f_{p_1})$. We let $u_{p_2} = u_{p_1}$, $\zeta_{p_2} = \zeta_{p_1} + 1$, $\overline{r}^{p_2} \restriction u_{p_2} \times \zeta_{p_1} = \overline{r}^{p_1}$ and $r^{p_2}_{\alpha, \zeta_{p_1}} = q^*_{0,\xi,0}$ for every $\alpha \in u_{p_2}$, $S_{p_2} = S_{p_1}$ and $f_{p_1} = f_{p_2}$. Then clearly $p_2 \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi))$ and $p_2 \leqslant p_1$.

Next we construct $p_2 \leqslant p_1$ with $\alpha_* \in u_{p_2}$. We may assume $\alpha_* \notin u_{p_1}$. Let $u_{p_2} = u_{p_1} \cup \{\alpha_*\}$, $\zeta_{p_2} = \zeta_{p_1}$, $\overline{r}^{p_2} \restriction u_{p_1} \times \zeta_{p_2} = \overline{r}^{p_1}$ and $r^{p_2}_{\alpha_*, \varepsilon} = q^*_{0,0,0}$ for every $\varepsilon < \zeta_{p_2}$. As $\alpha_* \notin \mathrm{ran}(\overline{\beta})$ for every $\overline{\beta} \in S_{p_1}, (B)(b)$ of Definition 4.2 vacuously holds, thus $p_2 \leqslant p_1$.

Finally we construct $p_2 \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)), p_2 \leqslant p_1$ such that (c) holds. By what we have just shown, we may assume $\alpha_* \in u_{p_1}$. We also assume that $r^{p_1}_{\alpha_*, \varepsilon}$ and r_* are incompatible for every $\varepsilon < \zeta_{p_1}$, as otherwise we let $p_2 = p_1$. We fix $\xi < \lambda$ such that q^*_ξ and r_* are compatible (recall that $\overline{q}^*$ is a maximal antichain), and fix $r \leqslant q^*_\xi$, r_*.

Case 1. There exist $\overline{\beta} \in S_{p_1}$ and $\iota < \kappa$ such that $f_{p_1}(\overline{\beta}) = \xi$ and $\beta_\iota = \alpha_*$.

Note that by Definition 4.2 (A)(e) ι is uniquely determined. As $\overline{q}^*_{\iota,\xi}$ is a maximal antichain below q^*_ξ, there exists $\zeta < \lambda$ such that $q^*_{\iota,\xi,\zeta}$ and r are compatible. We define p_2 such that $u_{p_2} = u_{p_1}$, $\zeta_{p_2} = \zeta_{p_1} + 1$, $\overline{r}_{p_2} \restriction u_{p_2} \times \zeta_{p_1} = \overline{r}_{p_1}, r^{p_2}_{\alpha_*, \zeta_{p_1}} \leqslant q^*_{\iota,\xi,\zeta}$, r, and $r^{p_2}_{\alpha_*, \zeta_{p_1}}$ is a member of Q^*. We can easily define $r^{p_2}_{\alpha, \zeta_{p_1}}$ for $\alpha \in u_{p_2} \setminus \{\alpha_*\}$ such that, letting $S_{p_2} = S_{p_1}$ and $f_{p_2} = f_{p_1}$, p_2 is as desired.

Case 2. There is no pair $(\overline{\beta}, \iota) \in S_{p_1} \times \kappa$ such that $f_{p_1}(\overline{\beta}) = \xi$ and $\beta_\iota = \alpha_*$.

We construct p_2 as in Case 1 except that $\iota < \kappa$ can be chosen randomly.

(3) Given $q' \leqslant q$ and $(\alpha, \varepsilon) \in u_{q'} \times \zeta_{q'}$ such that $\alpha = \beta_\iota$ for some $\iota < \kappa$ and $r^{q'}_{\alpha,\varepsilon} \leqslant q^*_\xi$, then, by Definition 4.2(1)(B)(b), for some $\zeta < \lambda$ we have $r^{q'}_{\alpha,\varepsilon} \leqslant q^*_{\iota,\xi,\zeta}$. By (2)(c) we conclude

$$q \Vdash_{\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)} \forall \varepsilon < \lambda \; ((\dot{r}_{\beta_\iota, \varepsilon} \leqslant q^*_\xi \rightarrow \exists \zeta < \lambda \quad \dot{r}_{\beta_\iota, \varepsilon} \leqslant q^*_{\iota,\xi,\zeta}) \wedge \forall \zeta < \lambda \exists \varepsilon < \lambda \quad \dot{r}_{\beta_\iota, \varepsilon} \leqslant q^*_{\iota,\xi,\zeta}).$$

As $\iota < \kappa$ was arbitrary, by Definition 4.1 we conclude that (3) is true. □

Lemma 4.2. *Suppose* $\chi > 2^{\aleph_0} = \lambda = \lambda^{<\lambda} = \kappa^+$, $\boldsymbol{Q}$, *strong witness* $\mathcal{W}$ *for* $\mathrm{add}(\mathcal{I}(\boldsymbol{Q})) \leqslant \kappa$ *and* $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ *are as in Definition 4.2. If* $\langle p_\alpha : \alpha < \lambda^+ \rangle$ *is a family of conditions in* $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ *there exist a club* $E \subseteq \lambda^+$ *and a regressive function* $h : E \cap S^{\lambda^+}_\lambda \rightarrow \lambda^+$ *such that for every* $w \subseteq E \cap S^{\lambda^+}_\lambda$ *of cardinality at most* κ, *if* $h \restriction w$ *is constant then* $\langle p_\alpha : \alpha \in w \rangle$ *has a largest lower bound in* $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$.

Proof. Let $\langle p_\alpha : \alpha < \lambda^+ \rangle$ be given. We write $p_\alpha = (u_\alpha, \zeta_\alpha, \overline{r}_\alpha, S_\alpha, f_\alpha)$, $\overline{r}_\alpha = \langle r^\alpha_{\gamma,\varepsilon} : \gamma \in u_\alpha, \varepsilon < \zeta_\alpha \rangle$, $\overline{r}^{[\gamma]}_\alpha = \langle r^\alpha_{\gamma,\varepsilon} : \varepsilon < \zeta_\alpha \rangle$. For every $\alpha < \lambda^+$ let $g_\alpha : otp(u_\alpha) \rightarrow u_\alpha$ be the unique increasing surjection. We define a binary relation R_* on λ^+ by letting $\alpha R_* \beta$ iff

(a) $otp(u_\alpha) = otp(u_\beta)$, $otp(\alpha \cap u_\alpha) = otp(\beta \cap u_\beta)$, $\zeta_\alpha = \zeta_\beta$, and

(b) $g_\beta \circ g_\alpha^{-1}$ is an isomorphism from p_α onto p_α, i.e.,

- (α) if $g_\beta \circ g_\alpha^{-1}(\gamma_1) = \gamma_2$, then $\overline{r}^{[\gamma_1]}_\alpha = \overline{r}^{[\gamma_2]}_\beta$ and
- (β) if $\overline{\gamma} = \langle \gamma_\iota : \iota < \kappa \rangle \in {}^\kappa(\lambda^+)$, then $\overline{\gamma} \in S_\alpha$ iff $g_\beta \circ g_\alpha^{-1}(\overline{\gamma}) := \langle g_\beta \circ g_\alpha^{-1}(\gamma_\iota) : \iota < \kappa \rangle \in S_\beta$ and $f_\alpha(\overline{\gamma}) = f_\beta(g_\beta \circ g_\alpha^{-1}(\overline{\gamma}))$.

It is easy to check that R_* is an equivalence relation and that (by our assumption $\lambda = \lambda^{<\lambda}$) E_* has λ many equivalence classes.

For every $\alpha < \lambda^+$ we let $U_{<\alpha} = \bigcup\{u_\beta : \beta < \alpha\}, v_\alpha = \{\iota < otp(u_\alpha) : g_\alpha(\iota) \in U_{<\alpha}\}$.

We define the function $h_1 : \lambda^+ \to \lambda^+$ by letting

$h_1(\alpha) = \min\{\beta \in \boldsymbol{Ord} : \beta \geqslant \alpha \wedge \forall\gamma_1 < \lambda^+ (\operatorname{ran}(g_{\gamma_1} \restriction v_{\gamma_1}) \subseteq U_{<\alpha} \Rightarrow \exists\gamma_2 < \beta\ (\gamma_1 R_* \gamma_2 \wedge g_{\gamma_2} \restriction v_{\gamma_2} = g_{\gamma_1} \restriction v_{\gamma_1}))\}$.

Note that as R_* has λ equivalence classes and $|U_{<\alpha}|^{<\lambda} \leqslant \lambda^{<\lambda}$, the function h_1 maps indeed into λ^+.

Let $E = \{\gamma < \lambda^+ : \gamma \text{ is a limit ordinal and } \forall\alpha < \gamma \quad h_1(\alpha) < \gamma\}$. Thus E is a club on λ^+.

Finally we define our desired function $h : E \cap S^{\lambda^+}_\lambda \to \lambda^+$ by letting

$$h(\gamma) = \min\{\delta < \lambda^+ : g_\delta \restriction v_\delta = g_\gamma \restriction v_\gamma \wedge \gamma R_* \delta\}.$$

Then $h(\gamma) \leqslant \gamma$ holds trivially. By construction even $h(\gamma) < \gamma$, hence h is regressive. Indeed, by definition $\operatorname{ran}(g_\gamma \restriction v_\gamma) \subseteq U_{<\gamma}$. As $|\operatorname{ran}(g_\gamma \restriction v_\gamma)| < \lambda$ and $\operatorname{cf}(\gamma) = \lambda$ we can find $\delta_1 < \gamma$ such that $\operatorname{ran}(g_\gamma \restriction v_\gamma) \subseteq U_{<\delta_1}$. Since $h_1(\delta_1) < \gamma$ there exists $\delta_2 < \gamma$ such that $g_{\delta_2} \restriction v_{\delta_2} = g_\gamma \restriction v_\gamma$ and $\delta_2 R_* \gamma$, and hence $h(\gamma) \leqslant \delta_2 < \gamma$.

Suppose now that $w \subseteq E \cap S^{\lambda^+}_\lambda$, $|w| \leqslant \kappa$ and $h \restriction w$ is constant. By definition of h, $g_\alpha \restriction v_\alpha = g_\beta \restriction v_\beta =: g^*$ for any $\alpha, \beta \in w$. By definition of v_α we conclude that $\langle u_\alpha : \alpha \in w\rangle$ is a Δ-system with root $\operatorname{ran}(g^*)$ and $g_\beta \circ g_\alpha^{-1}$ is the identity on $\operatorname{ran}(g^*)$ for any $\alpha, \beta \in w$.

Moreover, by definition of R_* we have $\zeta_\alpha = \zeta_\beta =: \zeta_w$, and if $\gamma \in u_\alpha \cap u_\beta$, hence $\gamma \in \operatorname{ran}(g^*)$ and $g_\beta \circ g_\alpha^{-1}(\gamma) = \gamma$ then $\overline{r}^{[\gamma]}_\alpha = \overline{r}^{[\gamma]}_\beta$.

Now we define $q = q_w \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ as follows:

(a) $u_q = \bigcup\{u_\alpha : \alpha \in w\}$.

Note that $|u_q| \leqslant \kappa$ as required, as $|w| \leqslant \kappa$.

(b) $\zeta_q = \zeta_w$.
(c) $\overline{r}_q = \langle r^\alpha_{\gamma,\varepsilon} : \alpha \in w, \gamma \in u_\alpha, \varepsilon < \zeta_q\rangle$.

Note that by the remark above this is well defined (i.e. $r^\alpha_{\gamma,\varepsilon} = r^\beta_{\gamma,\varepsilon}$ if $\gamma \in u_\alpha \cap u_\beta$).

(d) $S_q = \bigcup\{S_\alpha : \alpha \in w\}$.

Again $|S_q| \leqslant \kappa$ as required, by $|w| \leqslant \kappa$.

(e) $f_q = \bigcup\{f_\alpha : \alpha \in w\}$.

Note that f_q is a function. Indeed, if $\overline{\gamma} \in S_\alpha \cap S_\beta$ for $\alpha, \beta \in w$ then $\operatorname{ran}(\overline{\gamma}) \subseteq u_\alpha \cap u_\beta = \operatorname{ran}(g^*)$. Since $g_\beta \circ g_\alpha^{-1} \restriction \operatorname{ran}(g^*)$ is the identity, by (b)(β) in the definition of R_* we have $f_\alpha(\overline{\gamma}) = f_\beta(\overline{\gamma})$.

Let us check (A)(e) from Definition 4.2(1). Let $\alpha, \beta \in w, \alpha \neq \beta$, and $\overline{\gamma}^1 \in S_\alpha, \overline{\gamma}^2 \in S_\beta$. Let $\overline{\gamma}^3 := g_\beta \circ g_\alpha^{-1}(\overline{\gamma}^1)$, thus $\overline{\gamma}^3 \in S_\beta, f_\alpha(\overline{\gamma}^1) = f_\beta(\overline{\gamma}^3)$, and $(\overline{\gamma}^1, \overline{\gamma}^3)$ is a Δ-system pair (see Definition 4.2(1)(e)(α)). If $\overline{\gamma}^2 = \overline{\gamma}^3$, hence $f_q(\overline{\gamma}^1) = f_q(\overline{\gamma}^2)$, we are done. Now suppose $\overline{\gamma}^2 \neq \overline{\gamma}^3$. Note that

$$\{(\iota, \nu) \in \kappa^2 : \gamma^1_\iota = \gamma^2_\nu\} \subseteq \{(\iota, \nu) \in \kappa^2 : \gamma^3_\iota = \gamma^2_\nu\}.$$

If $f_\alpha(\overline{\gamma}^1) = f_\beta(\overline{\gamma}^2)$, hence $f_\beta(\overline{\gamma}^2) = f_\beta(\overline{\gamma}^3)$ and thus $(\overline{\gamma}^2, \overline{\gamma}^3)$ is a Δ-system pair, we are done. Otherwise $f_\alpha(\overline{\gamma}^1) \neq f_\beta(\overline{\gamma}^2)$, hence $f_\beta(\overline{\gamma}^2) \neq f_\beta(\overline{\gamma}^3)$ and thus $|\operatorname{ran}(\overline{\gamma}^2) \cap \operatorname{ran}(\overline{\gamma}^3)| \leqslant 1$. But this implies $|\operatorname{ran}(\overline{\gamma}^1) \cap (\overline{\gamma}^2)| \leqslant 1$.

Finally, it is straightforward to verify $q \leqslant p_\alpha$ for every $\alpha \in w$. That q actually is the largest lower bound is also clear. □

5. Different cofinalities if amoeba and antiamoeba interact

Lemma 5.1. *Suppose* $\chi > 2^{\aleph_0} = \lambda = \lambda^{<\lambda} = \kappa^+, \boldsymbol{Q}$, χ *a cardinal, strong witness* $\mathcal{W}$ *for* $\mathrm{add}(\mathcal{I}(\boldsymbol{Q})) \leqslant \kappa$ *and* $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ *are as in Definition 4.2. Moreover let* $\boldsymbol{d} = (\lambda, \mathrm{CLUB}_{\lambda^+}, \varepsilon, \kappa, [S^{\lambda^+}_\lambda]^\kappa)$ *where* $\varepsilon < \lambda$ *(so* $\boldsymbol{d}$ *is a c.c.-parameter). If* $\dot{P}$ *is an* $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$*-name for a forcing such that* $\Vdash_{\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)}$ *"*$\dot{P}$ satisfies $*_{\boldsymbol{d}}$*", then*

$$\Vdash_{\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi) * \dot{P}} \mathrm{cof}(\mathcal{I}(\boldsymbol{Q})) \geq \chi.$$

Proof. Towards a contradiction we assume that there are $p_* \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi) * \dot{P}$, cardinal $\alpha_* < \chi$ and a family $\langle \dot{B}_\alpha : \alpha < \alpha_* \rangle$ of $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi) * \dot{P}$-names such that

$$p_* \Vdash_{\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi) * \dot{P}} \text{"}\langle \dot{B}_\alpha : \alpha < \alpha_* \rangle \text{ is a cofinal sequence in } \mathcal{I}(\boldsymbol{Q})\text{."}$$

We must have $\alpha_* > \lambda$. For $\alpha < \chi$ we can find $p_\alpha \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi) * \dot{P}$ below p_* and $\gamma(\alpha) < \alpha_*$ such that

(a) $p_\alpha \Vdash_{\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi) * \dot{P}} \dot{X}_\alpha \subseteq \dot{B}_{\gamma(\alpha)}$,

where $\dot{X}_\alpha$ is the $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$-name as in Definition 4.2(2). We can find some unbounded $U \subseteq \alpha_*^+$ and $\gamma_* < \alpha_*$ such that $\gamma(\alpha) = \gamma_*$ for every $\alpha \in U$. By renumbering we may assume $U = \alpha_*^+$. In the sequel we only make use of $\langle p_\alpha : \alpha < \lambda^+ \rangle$ to get a contradiction. Let $p_\alpha = (p^1_\alpha, \dot{p}^2_\alpha)$ where $p^1_\alpha \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ and $\Vdash_{\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)} \dot{p}^2_\alpha \in \dot{P}$.

In $\boldsymbol{V}^{\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)}$ we consider the game $\mathcal{G}(\dot{P}, \boldsymbol{d})$ (see Definition 2.1), for which, by assumption, player I has a winning strategy. Let

$$\langle (\langle \dot{t}^\zeta_i : i < \lambda^+ \rangle, \dot{f}_\zeta), \langle \dot{s}^\zeta_i : i < \lambda^+ \rangle : \zeta < \varepsilon \rangle$$

be the play described in Remark 2.1(1) with $\langle \dot{s}^0_i : i < \lambda^+ \rangle = \langle \dot{p}^2_i : i < \lambda^+ \rangle$. As player I wins this play, there exists a $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$-name $\dot{E}_2$ for a club of λ^+ as in the winning rule for $\mathcal{G}(\dot{P}, \boldsymbol{d})$. As by Lemma 4.2 $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ has the λ^+-c.c., wlog we may assume $\dot{E}_2 = E_2 \in \boldsymbol{V}$.

By λ-closedness of $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$, for every $\alpha < \lambda^+$ we can find $p^3_\alpha \in \mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ below p^1_α and $g_\alpha : \varepsilon \to \lambda^+$ in $\boldsymbol{V}$ such that

(b) $p^3_\alpha \Vdash_{\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)} \langle \dot{f}_\zeta(\alpha) : \zeta < \varepsilon \rangle = g_\alpha$.

By Lemma 4.1(2)(b), we may assume that $\alpha \in u_{p^3_\alpha}$ (see Definition 4.2(1)(A)(a)). Applying Lemma 4.2 to $\langle p^3_\alpha : \alpha \in \lambda^+ \rangle$ we can find a club $E_1 \subseteq \lambda^+$ and a regressive function $f_1 : E_1 \cap S^{\lambda^+}_\lambda \to \lambda^+$ as there. Note that by the construction of f_1 (denoted h in the proof of Lemma 4.2), for given $\delta < \lambda^+$ there is u^* such that whenever $f_1(\alpha) = f_1(\beta) = \delta$ for some $\alpha, \beta \in E_1 \cap S^{\lambda^+}_\lambda$, then $\alpha R_* \beta$ and $u_\alpha \cap u_\beta = u^*$.

As in Remark 2.1(1) we have a regressive function $f_2 : S^{\lambda^+}_\lambda \to \lambda^+$ such that $\mathrm{ran}(g_\alpha)$ is bounded by $f_2(\alpha)$ for every $\alpha \in S^{\lambda^+}_\lambda$.

We shall use notation and proof of Lemma 4.2 below. As $E_1 \cap E_2 \cap S^{\lambda^+}_{\lambda^+}$ is a stationary subset of λ^+, there are ordinals γ^1_*, γ^2_* such that the set

$$S := \{\alpha < \lambda^+ : \alpha \in E_1 \cap E_2 \cap S^{\lambda^+}_\lambda \wedge f_1(\alpha) = \gamma^1_* \wedge f_2(\alpha) = \gamma^2_*\}$$

is stationary. By $\lambda = \lambda^{<\lambda}$ we can find some unbounded set $V \subseteq S$ and g_* such that $g_\alpha = g_*$ for every $\alpha \in V$.

By the above remark about the construction of f_1 in the proof of Lemma 4.2 we have that

(c) all $\alpha \in S$ are R_*-equivalent,
(d) $\langle u_{p^3_\alpha} : \alpha \in S\rangle$ is a Δ-system (hence $\alpha \in u_{p^3_\alpha} \setminus \bigcup\{u_{p^3_\beta} : \beta \in S \wedge \beta \neq \alpha\}$ for all $\alpha \in S$).

We choose $w \subseteq V$ such that $otp(w) = \kappa$ and let $\bar{\alpha}^*$ list w in increasing order. By the proof of Lemma 4.2 we know that $\{p^3_\alpha : \alpha \in w\}$ has a largest lower bound, say p^w. We define $p^1 \leq p^w$ in $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$ as follows:

Let $u_{p^1} = u_{p^w}$, $\zeta_{p^1} = \zeta_{p^w}$, $\bar{r}_{p^1} = \bar{r}_{p^w}$, $S_{p^1} = S_{p^w} \cup \{\bar{\alpha}^*\}$, and $f_{p^1} = f_{p^w} \cup \{(\bar{\alpha}^*, \xi)\}$, where $\xi < \lambda$ is chosen such that no member of $\bar{r}_{p^w}$ is below q^*_ξ, and hence $\forall \beta \in u_{p^w} \forall \varepsilon < \zeta_{p^w} \quad \mathrm{set}(r^{p^w}_{\beta,\varepsilon}) \cap \mathrm{set}(q^*_\xi) = \emptyset$, and moreover $\xi \notin \mathrm{ran}(f_{p^w})$.

By construction (see (c)) we have $|\mathrm{ran}(\bar{\alpha}^*) \cap \mathrm{ran}(\bar{\beta})| \leqslant 1$ for every $\bar{\beta} \in S_{p^w}$ and hence p^1 is as desired (see Remark 4.1).

Let $\bar{\alpha}^* = \langle \alpha_\iota : \iota < \kappa\rangle$. By Lemma 4.1(3) we conclude

$$p^1 \Vdash_{\mathbb{AA}(\boldsymbol{Q},\mathcal{W},\chi)} \quad \bigcup_{\iota<\kappa} \dot{X}_{\alpha_\iota} \notin \mathcal{I}(\boldsymbol{Q}).$$

By construction (especially the definition of p^3_α and g_α in (b)), there exists some $\mathbb{AA}(\boldsymbol{Q}, \mathcal{W}, \chi)$-name $\dot{p}^2$ such that $\Vdash_{\mathbb{AA}(\boldsymbol{Q},\mathcal{W},\chi)} \dot{p}^2 \in \dot{P}$ and

$$p^1 \Vdash_{\mathbb{AA}(\boldsymbol{Q},\mathcal{W},\chi)} \quad \dot{p}^2 \text{ is a lower bound of } \{\dot{p}^2_\alpha : \alpha \in w\}.$$

But now we have a contradiction, as $(p^1, \dot{p}^2) \leq p_*$ and by (a)

$$(p^1, \dot{p}^2) \Vdash_{\mathbb{AA}(\boldsymbol{Q},\mathcal{W},\chi)*\dot{P}} \quad \bigcup_{\iota<\kappa} \dot{X}_{\alpha_\iota} \subseteq \dot{B}_{\gamma_*}. \quad \square$$

As a conclusion of what we proved so far we obtain the following main theorem of this paper:

Theorem 5.1. *Suppose that* $2^{\aleph_0} = \lambda = \lambda^{<\lambda} = \kappa^+ < \mu = \mathrm{cf}(\mu) < \chi = \chi^{<\chi}$. *Moreover we assume the following:*

(1) $\boldsymbol{Q_0} = (Q_0, \dot{\zeta}_0, \mathrm{set}_0, Q^*_0, \perp_0)$ *is a* GTF_1 *such that* $\boldsymbol{Q_0}$ *has a strong witness* $\mathcal{W}$ *for* $\mathrm{add}(\mathcal{I}(\boldsymbol{Q_0})) \leqslant \kappa$,
(2) $\boldsymbol{Q_1} = (Q_1, \dot{\zeta}_1, \mathrm{set}_1, Q^*_1, \perp_1)$ *is a* GTF_1 *such that* $\mathrm{add}(\mathcal{I}(\boldsymbol{Q_1})) = \lambda$,
(3) *Let* $\dot{P}$ *be the* $\mathbb{AA}(\boldsymbol{Q_0}, \mathcal{W}, \chi)$*-name of the limit of a* $(< \lambda)$*-support iteration* $\langle \dot{P}_\alpha, \dot{Q}_\beta : \alpha < \mu, \beta < \mu\rangle$ *in* $\boldsymbol{V}^{\mathbb{AA}(\boldsymbol{Q_0},\mathcal{W},\chi)}$, *where* $\dot{Q}_\beta$ *denotes* $\mathbb{A}_1(\boldsymbol{Q_1})$ *in* $\boldsymbol{V}^{\mathbb{AA}(\boldsymbol{Q_0},\mathcal{W},\chi)*\dot{P}_\beta}$.

Then the following hold:

(4) $\mathbb{AA}(\boldsymbol{Q_0}, \mathcal{W}, \chi) * \dot{P}$ *is* λ*-closed and* λ^+*-c.c.*
(5) $\boldsymbol{V}^{\mathbb{AA}(\boldsymbol{Q_0},\mathcal{W},\chi)*\dot{P}} \models \quad 2^{\aleph_0} = \lambda \wedge \mathrm{cof}(\mathcal{I}(\boldsymbol{Q_0})) = 2^\lambda = \chi \wedge \mathrm{cof}(\mathcal{I}(\boldsymbol{Q_1})) = \mu$.

6. Application to classical tree forcings

Here we study the well-known classical tree forcings Sacks, Silver, Laver and Miller. We abbreviate them by *Sa*, *Si*, *La* and *Mi*, respectively. We shall show that under certain assumptions they are GTF$_1$ in the sense of Definition 3.1. Then we shall explain for which pairs (Q_0, Q_1) of these the assumptions of Theorem 5.1 are known to be consistent, hence we can get the consistency of $\mathrm{cof}(\mathcal{I}(Q_0)) > \mathrm{cof}(\mathcal{I}(Q_1))$.

Theorem 6.1. *(1) Suppose $\mathfrak{d} = 2^{\aleph_0}$. Then both, Sacks and Silver forcing, can be considered as GTF_1's.*

(2) Suppose $\mathfrak{b} = 2^{\aleph_0}$. Then both, Laver and Miller forcing, can be considered as GTF_0's.

Proof. It is well-known that for every $Q \in \{Sa, Si, La, Mi\}$, every $p \in Q$ has continuum many extensions such that any two of them have no common infinite branch.

(1) Let $Q \in \{Sa, Si\}$. Let $\dot{G}_Q$ be the canonical Q-name for the generic filter, let $\dot{\zeta}_Q = \bigcap \dot{G}_Q$, i.e. $\dot{\zeta}_Q$ denotes the Sacks, Silver real, respectively. Let $\mathrm{set}_Q(p) = [p]$, let Q^* be the set of all $p \in Q$ such that $[p]$ is nowhere dense, and let $p \perp_Q q$ mean $[p] \cap [q] = \emptyset$. We claim that $(Q, \dot{\zeta}_Q, \mathrm{set}_Q, Q^*, \perp_Q)$ is GTF$_1$. In fact, (1)(a), (b), (c) and (e)(α), (β) are obvious, for (c)(γ) we use the well-known fact that a Sacks or Silver real determines its generic filter. (1)(d) follows from the remark at the beginning of this proof. Nontrivial are (e)$(\gamma)_1$ and $(\delta)_1$. For these we apply the results in [7] (for Sa) and [18] (for Si) that every maximal antichain in Sa or Si that consists of nowhere dense trees must have size at least $\mathfrak{d}$. Then (e)$(\gamma)_1$ and $(\delta)_1$ follow easily from our assumption, the remark at the beginning of this proof and the fact that if p, q are incompatible Sacks or Silver trees, then $[p] \cap [q]$ is countable.

(2) For $Q \in \{La, Mi\}$ we apply the base matrix tree from [1]. This is a family $\langle \mathcal{A}_\alpha : \alpha < \mathfrak{h} \rangle$ such that every $\mathcal{A}_\alpha$ is a mad family in $[\omega]^\omega$ of size continuum, $\mathcal{A}_\beta$ refines $\mathcal{A}_\alpha$ (i.e. $\forall b \in \mathcal{A}_\beta \exists a \in \mathcal{A}_\alpha\, b \subseteq^* a$) for every $\alpha < \beta < \mathfrak{h}$, and $\bigcup_{\alpha<\mathfrak{h}} \mathcal{A}_\alpha$ is dense in $([\omega]^\omega, \subseteq)$. Actually, by an easy modification of its construction we can achieve the following:

($*$) for every sequence $\langle a_n : n < \omega \rangle$ in $[\omega]^\omega$ there is $\alpha < \mathfrak{h}$ and a sequence $\langle b_n : n < \omega \rangle$ in $\mathcal{A}_\alpha$ such that $\forall n\, b_n \subseteq a_n$.

Note that here we ask for proper inclusion not just almost inclusion. Otherwise ($*$) would follow from $\aleph_0 < \mathfrak{h}$.

Now we let La^* consist of all $p \in La$ with the property that there exists $\alpha < \mathfrak{h}$ such that for every $\sigma \in p$ extending $\mathrm{stem}(p)$ we have $\mathrm{succ}_p(\sigma) \in \mathcal{A}_\alpha$, where $\mathrm{succ}_p(\sigma) = \{n < \omega : \sigma^\frown n \in p\}$. If $\sigma \notin p$ we define $\mathrm{succ}_p(\sigma) = \emptyset$. As for (1) we let $\dot{\zeta}_{La}$ denote the Laver real, $\mathrm{set}_{La}(p) = [p]$, and $p \perp_{La} q$ mean $[p] \cap [q] = \emptyset$. We claim that $(La, \dot{\zeta}_{La}, \mathrm{set}_{La}, La^*, \perp_{La})$ is GTF$_0$. Let us check Definition 3.1(1): (b) follows easily from property ($*$) of the base tree matrix. (c) is well-known. (d) holds by the remark at the beginning of this proof. Nontrivial are (e)(γ) and (δ). Let $\beta < 2^{\aleph_0}$ and $\langle p_\alpha : \alpha < \beta \rangle$ a sequence in La^*. The set

$$S = \{\mathrm{succ}_{p_\alpha}(\sigma) : \mathrm{stem}_{p_\alpha} \subseteq \sigma \in p_\alpha \wedge \alpha < \beta\}$$

has cardinality $< 2^{\aleph_0}$ and is contained in the base matrix tree. As $\mathcal{A}_0$ has size $2^{\aleph_0}$ and the base matrix is a tree with respect to $\supseteq^*$, there exists $a \in \mathcal{A}_0$ such that $a \cap b$ is finite for every $b \in S$. Let $p \in La^*$ be the tree with empty stem and $\mathrm{succ}_p(\sigma) = a$ for every $\sigma \in p$. Then clearly p is incompatible with every p_α. We need the following claim which is folklore wisdom:

Claim 2. *Let $\langle p_\alpha : \alpha < \beta < \mathfrak{b} \rangle$ be a sequence in La. If $p \in La$ is such that p is incompatible (w.r.t. $(La, \leq)$) with p_α for every $\alpha < \beta$, then there exists $q \leq p$, $q \in La$, such that* $\mathrm{stem}(p) = \mathrm{stem}(q)$ *and* $[p_\alpha] \cap [q] = \emptyset$ *for every $\alpha < \beta$.*

Proof. Fix $\alpha < \beta$. We define a rank function rk_α on $p^- := \{\sigma \in p : \mathrm{stem}(p) \subseteq \sigma\}$ as follows:

$\mathrm{rk}_\alpha(\sigma) = 0$ iff $\mathrm{succ}_p(\sigma) \cap \mathrm{succ}_{p_\alpha}(\sigma)$ is finite, and

$\mathrm{rk}_\alpha(\sigma) = \nu$ iff $\nu \in$ ***Ord*** is minimal such that for all except finitely many $n \in \mathrm{succ}_p(\sigma) \cap \mathrm{succ}_{p_\alpha}(\sigma)\, \mathrm{rk}_\alpha(\sigma^\frown n) < \nu$.

If σ gets no ordinal rank we define $\mathrm{rk}_\alpha(\sigma) = \infty$.

It is clear that as p and p_α are incompatible, every $\sigma \in p^-$ has an ordinal rank. We define $f_\alpha : p^- \to \omega$ as follows: If $\mathrm{rk}_\alpha(\sigma) = 0$ let $n = \sup(\mathrm{succ}_p(\sigma) \cap \mathrm{succ}_{p_\alpha}(\sigma))$ and $n = \sup\{m \in \mathrm{succ}_p(\sigma) \cap \mathrm{succ}_{p_\alpha}(\sigma) : \mathrm{rk}_\alpha(\sigma^\frown m) \geq \mathrm{rk}_\alpha(\sigma)\}$ otherwise. Now let $f_\alpha(\sigma) = n + 1$. It can easily be checked that if $g(\sigma) \geq f_\alpha(\sigma)$ for almost all $\sigma \in p^-$, then, if we prune p using g, i.e. for every $\sigma \in p^-$ deleting everything above $\sigma^\frown m$ for $m < g(\sigma)$, we obtain a Laver tree $q \leq p$ with $[p_\alpha] \cap [q] = \emptyset$. But by $\beta < \mathfrak{b}$ we can get g like this for every $\alpha < \beta$. □

Continuing with the proof of (e)(γ), by the claim and as La^* is dense we can find $q \in La^*$ with $q \leq p$ and $[p_\alpha] \cap [q] = \emptyset$ for every $\alpha < \beta$, as desired. These arguments also prove (e)(δ).

For Mi, analogous arguments work. □

Theorem 6.2. *(1) Suppose $Q \in \{Sa, Si\}$. Then $\operatorname{add}(\mathcal{I}(Q)) \leq \mathfrak{b}$ holds.*

(2) Suppose $2^{\aleph_0} = \mathfrak{b}$ and $Q \in \{La, Mi\}$. Then $\operatorname{add}(\mathcal{I}(Q)) \leq \mathfrak{h}$.

Proof. Let $\kappa(Q)$ the least cardinal κ such that forcing with Q changes the cofinality of $(2^{\aleph_0})^{\boldsymbol{V}}$ to κ.

(1) Simon [12] has proved $\kappa(Sa) \leq \mathfrak{b}$. In [9], $\operatorname{add}(\mathcal{I}(Sa)) \leq \kappa(Sa)$ is proved under the assumption that $2^{\aleph_0}$ is regular. In [7] it is proved that this assumption is not needed.

In [18], $\operatorname{add}(\mathcal{I}(Si)) \leq \mathfrak{b}$ is proved directly. A stronger result has been proved in [16] where it is shown that the nowhere Ramsey ideal is Tukey reducible to the Silver ideal, and hence even $\operatorname{add}(\mathcal{I}(Si)) \leq \mathfrak{h}$ is true.

(2) In [6], $\kappa(Q) \leq \mathfrak{h}$ has been shown for $Q \in \{La, Mi\}$. Similarly as in [9] for Sa, one can prove $\operatorname{add}(\mathcal{I}(Q)) \leq \kappa(Q)$ for $Q \in \{La, Mi\}$, provided that $2^{\aleph_0} = \mathfrak{b}$ holds. Actually, for $Q = Mi$, $\mathfrak{d} = 2^{\aleph_0}$ suffices (see [8], Corollary 13). □

Corollary 6.1. *Suppose $Q_0 \in \{Sa, Si, La, Mi\}$ is such that $\operatorname{add}(\mathcal{I}(Q_0)) = 2^{\aleph_0}$. Then the following are true:*

(1) Every $Q \in \{Sa, Si, La, Mi\}$ is GTF_1 (La and Mi are even GTF_0).

(2) If $Q_1 \in \{Sa, Si, La, Mi\}$ is such that $\operatorname{add}(\mathcal{I}(Q_1)) \leq \kappa < 2^{\aleph_0}$, then there exists a strong witness for this (see Definition 4.1).

Proof. (1) follows from Theorems 6.1 and 6.2. (2) follows from (1) and the homogeneity of the classical tree forcings. □

The following theorem collects all the cases for which the consistency of $\operatorname{add}(\mathcal{I}(Q_0)) < \operatorname{add}(\mathcal{I}(Q_1))$ is known, where $Q_0, Q_1 \in \{Sa, Si, La, Mi\}$.

Theorem 6.3. *If* ZF *is consistent, then the following statements are consistent with* $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_2 = \aleph_2^{\aleph_1}$*:*

(1) $\operatorname{add}(\mathcal{I}(Si)) < \operatorname{add}(\mathcal{I}(Sa))$,

(2) $\forall Q \in \{La, Mi\} \quad \operatorname{add}(\mathcal{I}(Sa)) < \operatorname{add}(\mathcal{I}(Q))$,

(3) $\forall Q \in \{La, Mi\} \quad \operatorname{add}(\mathcal{I}(Si)) < \operatorname{add}(\mathcal{I}(Q))$.

Proof. (1) Implicitly in [10], an amoeba forcing for Sa with the Laver property has been constructed. See also [17] for detailed analysis and proofs. If this forcing is iterated $\aleph_2$ times with countable supports, a model for $\operatorname{cov}(\mathcal{M}) < \operatorname{add}(\mathcal{I}(Sa))$ is obtained (where $\mathcal{M}$ is the meager ideal). In [14], $\operatorname{add}(\mathcal{I}(Si)) \leq \operatorname{cov}(\mathcal{M})$ has been proved in ZFC.

(2) In [6], it has been shown that MA implies $\operatorname{add}(\mathcal{I}(Q)) = 2^{\aleph_0}$ for both $Q \in \{La, Mi\}$. In [9], and independently in [19], it has been shown that MA does not imply $\operatorname{add}(\mathcal{I}(Sa)) = 2^{\aleph_0}$, i.e. a model for $\mathrm{MA} + \operatorname{add}(\mathcal{I}(Sa)) = \aleph_1 < 2^{\aleph_0} = \aleph_2$ is constructed.

(3) In [4] it has been shown that MA does not imply $\operatorname{add}(\mathcal{I}(Si)) = 2^{\aleph_0}$, i.e. a model for $\mathrm{MA}+\operatorname{add}(\mathcal{I}(Si)) = \aleph_1 < 2^{\aleph_0} = \aleph_2$ is constructed.

Alternatively one can use the models in [13], where amoebas for La and Mi with the Laver property have been constructed. In these, $\operatorname{add}(\mathcal{I}(Si)) = \aleph_1$ holds by [14] as in (1). □

As an immediate consequence of Theorems 5.1, 6.1, 6.2 and 6.3 we obtain the following:

Theorem 6.4. *If* ZF *is consistent, then the following statements are consistent with* ZFC*:*
(1) $\mathrm{cof}(\mathcal{I}(Sa)) < \mathrm{cof}(\mathcal{I}(Si))$,
(2) $\mathrm{cof}(\mathcal{I}(Q_1)) < \mathrm{cof}(\mathcal{I}(Q_0))$, *where* $Q_0 \in \{Sa, Si\}$ *and* $Q_1 \in \{La, Mi\}$.

7. Singular cofinality

In this section we shall show that consistently we can have $\mathrm{cof}(\mathcal{I}(\boldsymbol{Q}))$ singular, where $\boldsymbol{Q}$ is a GTF_1. For this we apply the amoeba from Section 3, but we have to use a more elaborate iteration. For Sacks forcing, this result has been obtained in [9].

Theorem 7.1. *Suppose that* $\boldsymbol{Q} = (Q, \dot{\zeta}, \mathrm{set}, Q^*, \perp)$ *is a* GTF_1, $2^{\aleph_0} = \lambda = \lambda^{<\lambda} < \theta = \mathrm{cf}(\mu) < \mu < \chi = \chi^{<\chi}$ *and* $\mathrm{add}\,(\mathcal{I}(\boldsymbol{Q})) = \lambda$. *Moreover we assume* $\forall \alpha < \mu\ |\alpha|^\lambda < \mu$. *There exists a forcing* P *such that*

(a) P *is* λ*-closed and* λ^+*-c.c.*
(b) $\boldsymbol{V}^P \models \quad 2^\lambda = \chi \wedge \mathrm{cof}(\mathcal{I}(\boldsymbol{Q})) = \mu$.

Proof. We fix an increasing sequence $\langle \lambda_\iota : \iota < \theta \rangle$ of regular cardinals $\lambda_\iota < \mu$ with $\lambda < \lambda_0$ and $\sup\{\lambda_\iota : \iota < \theta\} = \lambda$. Let

$$\mathcal{F} = \{f \in \prod_{\iota<\theta} \lambda_\iota : |\{\iota < \theta : f(\iota) \neq 0\}| < \lambda\}.$$

For $f \in \mathcal{F}$ let $\mathrm{supp}(f) = \{\iota < \theta : f(\iota) \neq 0\}$. Let $\leq_{\mathcal{F}}$ denote the natural partial order on $\mathcal{F}$ defined by $f \leq_{\mathcal{F}} g$ iff $\mathrm{supp}(f) \subseteq \mathrm{supp}(g)$ and $\forall \iota \in \mathrm{supp}(f) \quad f(\iota) \leq g(\iota)$. By our assumptions, clearly $|\mathcal{F}| = \mu$ and $\mathcal{F}$ is $(< \lambda^+)$-directed. Let $\langle f^*_\beta : \beta < \mu \rangle$ list $\mathcal{F}$ such that f^*_0 is the constantly 0 function.

Definition 7.1. Let the assumptions of Theorem 7.1 hold.

(1) We call a family $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{Q}) = \langle P_\alpha, \dot{Q}_\beta, u_\beta, \dot{\eta}_\beta, f_\beta : \alpha \leq \alpha_{\boldsymbol{q}}, \beta < \alpha_{\boldsymbol{q}} \rangle$ a **($<\boldsymbol{\lambda}$)-support iteration of $\boldsymbol{Q}$ with memory** if

(a) $\chi < \alpha_{\boldsymbol{q}}$ is a limit ordinal, and $\langle P_\alpha, \dot{Q}_\beta : \alpha \leq \alpha_{\boldsymbol{q}}, \beta < \alpha_{\boldsymbol{q}} \rangle$ is a $(< \lambda)$-support iteration such that for every $\beta < \alpha_{\boldsymbol{q}}$,
$\Vdash_{P_\beta}$ "$\dot{Q}_\beta$ has a subset of $\mathcal{P}(H(\lambda))$ as its set of elements and $\dot{\eta}_\beta \subseteq \dot{Q}_\beta$ is the generic filter".
(b) $u_\beta \subseteq \beta$ such that $\forall \gamma \in u_\beta \quad u_\gamma \subseteq u_\beta$ (**transitivity of the memory** $\langle u_\beta : \beta < \alpha_{\boldsymbol{q}} \rangle$).
(c) $\forall \beta < \chi\ (u_\beta = \emptyset \wedge \Vdash_{P_\beta}$ "$\dot{Q}_\beta = ({}^{<\lambda}\lambda, \supseteq)$").
(d) $\forall \beta \in [\chi, \alpha_{\boldsymbol{q}}) \Vdash_{P_\beta}$ "$\dot{Q}_\beta = \mathbb{A}_1(\boldsymbol{Q})^{\boldsymbol{V}[\dot{\eta}[u_\beta]]}$", where $\dot{\eta}[u]$ denotes $\langle \dot{\eta}_\nu : \nu \in u \rangle$ for $u \subseteq \beta$.
(e) (α) $f_\beta \in \mathcal{F}$ and if $\beta < \mu$ then $f_\beta = f^*_0$.
  (β) If $\beta \in u_\gamma$ then $f_\beta \leq_{\mathcal{F}} f_\gamma$.
  (γ) If $\beta \in u_\gamma$ and $\beta < \mu$ then $\sup\{\lambda_\nu : \nu < \iota\} \leq \beta < \lambda_\iota$ implies $\beta < f_\gamma(\iota)$.

(2) Let $\boldsymbol{q}$ be as in (1) and $\bar{u} = \langle u_\beta : \beta < \alpha_{\boldsymbol{q}} \rangle$. A subset $U \subseteq \alpha_{\boldsymbol{q}}$ is called **$\bar{u}$-closed** if $\forall \beta \in U \quad u_\beta \subseteq U$.

Claim 3. *Let* $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{Q})$ *be as in Definition 7.1 and* $U \subseteq [\chi, \alpha_{\boldsymbol{q}})$ *such that*

(1) $\forall u \in [\alpha_{\boldsymbol{q}}]^{\leq \lambda}\, \exists \beta \in U \quad u \subseteq u_\beta$.

Let $\dot{\bar{p}}^\beta = \langle \dot{p}^\beta_\varepsilon : \varepsilon < \lambda\rangle$ *denote the generic maximal antichain in* Q *added by* $\dot{Q}_\beta$ *and* $\dot{X}_\beta = X(\dot{\bar{p}}^\beta)$ *the associated set in* $\mathcal{I}(\boldsymbol{Q})^{\boldsymbol{V}[\dot\eta[0,\beta]]}$.

Then $\boldsymbol{V}^{P_{\alpha_{\boldsymbol{q}}}} \models$ *"*$\langle \dot{X}_\beta : \beta \in U\rangle$ *is cofinal in* $\mathcal{I}(\boldsymbol{Q})$*, hence* $\operatorname{cof}(\mathcal{I}(\boldsymbol{Q})) \le |U|$*".*

Proof. Note that (1) implies $\operatorname{cf}(\alpha_{\boldsymbol{q}}) > \lambda$ and hence

(1)' $\forall u \in [\alpha_{\boldsymbol{q}}]^{\le\lambda}\, \exists^\lambda \beta \in U \quad u \subseteq u_\beta$.

Now suppose $p \Vdash_{P_{\alpha_{\boldsymbol{q}}}} \dot\tau \in \mathcal{I}(\boldsymbol{Q})$. Wlog we may assume that there exists a family of $P_{\alpha_{\boldsymbol{q}}}$-names $\langle \dot{q}_\varepsilon : \varepsilon < \lambda\rangle$ such that

$$p \Vdash_{P_{\alpha_{\boldsymbol{q}}}} \langle \dot{q}_\varepsilon : \varepsilon < \lambda\rangle \textit{ is a maximal antichain of } Q \textit{ and } \dot\tau = X(\langle \dot{q}_\varepsilon : \varepsilon < \lambda\rangle).$$

Each $\dot{q}_\varepsilon$ can be viewed as a pair $(A_\varepsilon, h_\varepsilon)$ where A_ε is a maximal antichain in $P_{\alpha_{\boldsymbol{q}}}$ and $h_\varepsilon : A_\varepsilon \to Q$. As $P_{\alpha_{\boldsymbol{q}}}$ has the λ^+-c.c., $|A_\varepsilon| \le \lambda$. Note that by the definition of the $\dot{Q}_\beta$, if $\Vdash_{P_\beta}$ "$\dot\sigma \in \dot{Q}_\beta$" then $\dot\sigma$ can be coded in essentially the same way as $\dot\tau$, i.e. by λ may maximal antichains of P_β. As $\boldsymbol{q}$ is a $(<\lambda)$-support iteration, doing this for every $p(\beta)$ where $p \in A_\varepsilon$ and $\beta \in \operatorname{dom}(p)$ and then proceeding similarly, we obtain a wellfounded tree T on $(\alpha_{\boldsymbol{q}}, >)$ such that every node has at most λ many immediate successors, T has no infinite branch, and $\dot\tau$ can be evaluated from $\langle \dot\eta_\nu : \nu \in T\rangle$. As $|T| \le \lambda$, by (1)' there are λ many $\alpha_\iota \in U$ such that $T \subseteq u_{\alpha_\iota}$. By Lemma 3.1(B) we conclude $p \Vdash_{P_{\alpha_{\boldsymbol{q}}}} \exists \iota < \lambda\ \dot\tau \subseteq \dot{X}_{\alpha_\iota}$. Note that for this argument no memory is needed. □

Definition 7.2. Let $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{Q})$ and $\bar{u}$ be as in Definition 7.1. By induction on $\alpha \le \alpha_{\boldsymbol{q}}$, for all $\bar{u}$-closed $U \subseteq \alpha$, we define $P'_U \subseteq P_\alpha$ and prove

(a) P'_U consists of all $p \in P_\alpha$ such that $\operatorname{dom}(p) \subseteq U$ and for every $\beta \in \operatorname{dom}(p)$, $p(\beta)$ is a P'_{u_β}-name for a subset of $H(\lambda)$ (so either for an element of ${}^{<\lambda}\lambda$ or of $\mathbb{A}_1(\boldsymbol{Q})^{\boldsymbol{V}[\dot\eta[u_\beta]]}$).
(b) If $\alpha_1 < \alpha$ then $P'_{U\cap\alpha_1} \subseteq P'_U$ (clearly $U \cap \alpha_1$ is $\bar{u}$-closed).
(c) P'_α is dense in P_α.
(d) P'_U is a dense subset of the limit of the $(<\lambda)$-support iteration of the form $\langle P^*_\beta, \dot{Q}^*_\beta : \beta \in U\rangle$ such that for every $\beta \in U \cap \chi$, $\Vdash_{P^*_\beta}$ "$\dot{Q}^*_\beta = ({}^{<\lambda}\lambda, \supseteq)$", and for every $\beta \in U \cap [\chi, \alpha_{\boldsymbol{q}})$, $\Vdash_{P^*_\beta}$ "$\dot{Q}^*_\beta = \mathbb{A}_1(\boldsymbol{Q})^{\boldsymbol{V}[\dot\eta[u_\beta]]}$". (Here $\dot\eta_\beta$ and $\dot\eta[u_\beta]$ are defined as in Definition 7.1. Note again that $u_\beta \subseteq U$ as U is $\bar{u}$-closed.) Hence, letting $U = \alpha$, we have (c).
(e) P'_U is a complete suborder of P_α.
(f) For every $q \in P'_\alpha$, $q \restriction U \in P'_U$ and $q \le_{P'_\alpha} q \restriction U$.
(g) For every $q \in P'_\alpha$ and $p \in P'_U$, if $p \le_{P'_U} q \restriction U$, then p and q are compatible in P'_α; in fact, $p \cup q \restriction (\operatorname{dom}(q) \setminus U)$ is a lower bound of p and q.
(h) $p \in P'_U$ iff $p \in P'_\alpha$ and $\operatorname{dom}(p) \subseteq U$.

Proof. We won't use (d), hence we omit its proof. The main point is (c), as (f), (g), and (h) are clear, and hence (e) follows from (c). So let us prove (c) by induction on α. The case $\alpha = 0$ is trivial.

Let $\alpha = \beta + 1$ and $p \in P_\alpha$. Wlog we may assume that $\beta \in \operatorname{dom}(p)$, as otherwise we can apply the induction hypothesis. For the same reason we know that P'_{u_β} is a complete subforcing of P_β and P'_β is dense in P_β. Clearly we have $P'_{u_\beta} \subseteq P'_\beta$. Hence by definition we have

$$\Vdash_{P'_\beta} \text{"}p(\beta) \in \boldsymbol{V}[\langle \dot\eta_\gamma : \gamma \in u_\beta\rangle]\text{"}.$$

As $\langle \dot\eta_\gamma : \gamma \in u_\beta\rangle$ is (forced to be) P'_{u_β}-generic, there exist a P'_{u_β}-name $\dot\tau$ and $p_1 \le_{P_\beta} p \restriction \beta$ in P'_β such that $p_1 \Vdash_{P'_\beta} p(\beta) = \dot\tau$. Let $q = (p_1, \dot\tau)$. Then $q \in P'_\alpha$ and $q \le p$.

Now suppose that α is a limit ordinal and $p \in P_\alpha$. As $|\mathrm{dom}(p)| < \lambda$ we may assume that $\mathrm{cf}(\alpha) < \lambda$. Let $\langle \alpha^*_\iota : \iota < \mathrm{cf}(\alpha)\rangle$ be increasing and cofinal in α. We choose $\langle q_\iota : \iota \leq \mathrm{cf}(\alpha)\rangle$ such that $q_\iota \in P'_{\alpha^*_\iota}$, $q_\iota \leq_{P_{\alpha^*_\iota}} p \restriction \alpha^*_\iota$ and if $\iota < \nu \leq \mathrm{cf}(\alpha)$ then $q_\nu \leq_{P'_{\alpha^*_\nu}} q_\iota$. For the successor step we apply the inductive hypothesis. Suppose that $\nu \leq \mathrm{cf}(\alpha)$ is a limit ordinal and $\langle q_\iota : \iota < \nu\rangle$ have been chosen as desired. Let $\gamma \in \bigcup_{\iota<\nu} \mathrm{dom}(q_\iota)$. Choose $\iota(\gamma)$ such that $\gamma \in q_{\iota(\gamma)}$. Then in $\boldsymbol{V}$, $\langle q_\iota(\gamma) : \iota \in [\iota(\gamma), \nu)\rangle$ is a sequence of P'_{u_γ}-names for members of $\dot{Q}_\gamma$ such that this sequence is forced to be decreasing. But this forcing is forced to be $< \lambda$-complete and can be evaluated in $\boldsymbol{V}^{P_{u_\gamma}}$. Hence we can choose $q_\nu(\gamma)$ as a P'_{u_γ}-name that is forced to be a lower bound of it. Hence we have $q_{\mathrm{cf}(\alpha)} \in P'_\alpha$ and $q_{\mathrm{cf}(\alpha)} \leq p$. $\square$

In order to get $\boldsymbol{V}^{P_{\alpha_{\boldsymbol{q}}}} \models \mathrm{cof}\,(\mathcal{I}(\boldsymbol{Q})) \geq \mu$ we must make $\boldsymbol{q}$ more concrete as follows: We let

(2) $\alpha_{\boldsymbol{q}} = \chi + \mu \cdot \lambda^+$,
(3) if $\beta < \chi$, then $u_\beta = \emptyset$ and $f_\beta = f^*_0$,
(4) if $\beta = \chi + \mu \cdot \iota + \nu$ for $\iota < \lambda^+$ and $\nu < \mu$, then $f_\beta = f^*_\nu$ and $u_\beta = \{\alpha < \mu : \sup\{\lambda_\nu : \nu < \iota\} \leq \alpha < \lambda_\iota \Rightarrow \alpha < f_\beta(\iota)\} \cup \{\alpha \in [\mu, \beta) :\, f_\alpha \leq_{\mathcal{F}} f_\beta\}$.

Note that $\langle u_\beta : \beta < \alpha_{\boldsymbol{q}}\rangle$ is transitive: Let $\beta \in u_\gamma$ and $\alpha \in u_\beta$. We must have $\chi \leq \beta < \gamma$ and hence $f_\beta \leq f_\gamma$. If $\alpha < \mu$, hence $\sup\{\lambda_\nu : \nu < \iota\} \leq \alpha < \lambda_\iota$ for some $\iota < \theta$, we have $\alpha < f_\beta(\iota) \leq f_\gamma(\iota)$. If $\mu \leq \alpha$ we have $f_\alpha \leq f_\beta \leq f_\gamma$ and we are done.

Also note that (1)’ holds for $U = [\chi, \alpha_{\boldsymbol{q}})$: Let $u \subseteq \alpha_{\boldsymbol{q}}$ have size λ. As $\mathcal{F}$ is $(< \lambda^+)$-directed, we can easily find $f \in \mathcal{F}$ such that

(5) $u \cap [\sup\{\lambda_\nu : \nu < \iota\}, \lambda_\iota)$ is bounded by $f(\iota)$ for every $\iota < \theta$, and
(6) $f_\beta \leq_{\mathcal{F}} f$ holds for every $\beta \in u \cap [\mu, \alpha_{\boldsymbol{q}})$.

It follows that for every $\gamma \in [\sup(u) + 1, \alpha_{\boldsymbol{q}})$ such that $f_\gamma = f$, we have $u \subseteq u_\gamma$. As by construction there are at least λ^+ such γ, we are done.

Now let us prove $\boldsymbol{V}^{P_{\alpha_{\boldsymbol{q}}}} \models \mathrm{cof}\,(\mathcal{I}(\boldsymbol{Q})) \geq \mu$, where $\boldsymbol{q}$ is the iteration just defined. By Definition 7.2(e) we have $\boldsymbol{V}^{P_{\alpha_{\boldsymbol{q}}}} = \boldsymbol{V}^{P'_{\alpha_{\boldsymbol{q}}}}$. By contradiction suppose we had $\iota(*) < \theta$, $p \in P'_{\alpha_{\boldsymbol{q}}}$ and a family $\langle \dot{Y}_\alpha : \alpha < \lambda_{\iota(*)}\rangle$ of $P'_{\alpha_{\boldsymbol{q}}}$-names such that

$$p \Vdash_{P'_{\alpha_{\boldsymbol{q}}}} \langle \dot{Y}_\alpha : \alpha < \lambda_{\iota(*)}\rangle \,\text{is cofinal in}\, \mathcal{I}(\boldsymbol{Q}).$$

Wlog we may assume that every $\dot{Y}_\alpha$ is forced to be of the form $X(\langle \dot{q}_{\alpha,\varepsilon} : \varepsilon < \lambda\rangle)$ (see Remark 3.1(3)), where $\langle \dot{q}_{\alpha,\varepsilon} : \varepsilon < \lambda\rangle$ is forced to be a maximal antichain of Q. Since $Q \subseteq \mathbb{R}$ and $P_{\alpha_{\boldsymbol{q}}}$ does not add reals, wlog we may assume that every $\dot{q}_{\alpha,\varepsilon}$ is a nice $P'_{\alpha_{\boldsymbol{q}}}$-name, i.e. has the form $(A_{\alpha,\varepsilon}, f_{\alpha,\varepsilon})$ where $A_{\alpha,\varepsilon}$ is a maximal antichain of $P'_{\alpha_{\boldsymbol{q}}}$ and $f_{\alpha,\varepsilon} : A_{\alpha,\varepsilon} \to Q$. Let $v_\alpha = \bigcup\{\mathrm{dom} : (p) : p \in A_{\alpha,\varepsilon}\}$, thus $v_\alpha \in [\alpha_{\boldsymbol{q}}]^{\leq\lambda}$ and hence, by (1)’ for our memory $\bar{u}$, we find $\gamma_\alpha < \alpha_{\boldsymbol{q}}$ such that $v_\alpha \subseteq u_{\gamma_\alpha}$.

Let $\beta^* = \sup\{f_{\gamma_\alpha}(\iota(*) + 1) + 1 : \alpha < \iota(*)\}$ and $u^* = \bigcup\{u_{\gamma_\alpha} : \alpha < \iota(*)\}$. Then clearly $\beta^* < \lambda_{\iota(*)+1}$, u^* is $\bar{u}$-closed and $u^* \cap [\lambda_{\iota(*)}, \lambda_{\iota(*)+1}) = [\lambda_{\iota(*)}, \beta^*)$. By Definition 7.2(e) we have that P'_{u^*} is a complete subforcing of $P_{\alpha_{\boldsymbol{q}}}$, and hence every η_β for $\beta \in [\beta^*, \lambda_{\iota(*)+1})$ is λ-Cohen, i.e. generic for $({}^{<\lambda}\lambda, \supseteq)$, over $\boldsymbol{V}^{P_{u^*}}$. As $\langle \dot{Y}_\alpha : \alpha < \lambda_{\iota(*)}\rangle$ is forced to belong to $\boldsymbol{V}^{P_{u^*}}$, the following claim will complete the proof of Theorem 7.1:

Claim 4. *If $\boldsymbol{Q}$ is GTF_1, $2^{\aleph_0} = \lambda$ and $\eta : \lambda \to \lambda$ is λ-Cohen, i.e. generic for $({}^{<\lambda}\lambda, \supseteq)$, over $\boldsymbol{V}$, then in $\boldsymbol{V}[\eta]$ there exists $X \in \mathcal{I}(\boldsymbol{Q})$ that is not contained in any member of $\mathcal{I}(\boldsymbol{Q})^{\boldsymbol{V}}$.*

Proof. Let $\langle r_\varepsilon : \varepsilon < \lambda\rangle$, $\langle p_\varepsilon : \varepsilon < \lambda\rangle$ list $\mathbb{R}$, Q respectively. In $\boldsymbol{V}[\eta]$ we define families $\langle s_\varepsilon : \varepsilon < \lambda\rangle$ in $\mathbb{R}$ and $\langle q_\varepsilon : \varepsilon < \lambda\rangle$ in Q as follows: Let $s_0 = r_{\eta(0)}$ and let q_0 be the $\eta(1)$th p_ε that satisfies $p_\varepsilon \leq p_0$ and $s_0 \notin [p_\varepsilon]$.

If $\langle s_\varepsilon : \varepsilon < \nu \rangle$ and $\langle q_\varepsilon : \varepsilon < \nu \rangle$ have been determined for some $\nu < \lambda$, let s_ν be the $\eta(\nu \cdot 2)$th r_ε such that $r_\varepsilon \notin \bigcup_{\varepsilon < \nu} [q_\varepsilon]$. To define q_ν we distinguish two cases. If p_ν is compatible with some q_ε for $\varepsilon < \nu$ we let $q_\nu = q_0$. Otherwise, let q_ν the $\eta(\nu \cdot 2 + 1)$th p_ε such that $p_\varepsilon \leq p_\nu$ and $[p_\varepsilon] \cap \{s_\xi : \xi < \nu\} = \emptyset$. As $\boldsymbol{Q}$ is GTF$_1$, this construction is possible. Now $X = \{s_\xi : \xi < \lambda\}$ is as desired. □

□Theorem 7.1

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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