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ON SLOW MINIMAL REALS I

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(Communicated by Heike Mildenberger)

ABSTRACT. Answering a question of Harrington, we show that there exists a proper forcing notion, which adds a minimal real $\eta \in \prod_{i < \omega} n_i^*$, which is eventually different from any old real in $\prod_{i < \omega} n_i^*$, where the sequence $\langle n_i^* \mid i < \omega \rangle$ grows slowly.

1. INTRODUCTION

The method of creature forcing was introduced by Shelah for solving problems related to cardinal invariants like the unbounded number or the splitting number, as well as questions of the existence of special kinds of P-points. We refer to [9] for a more complete history about the development of the subject and its wide applications.

Let us call a real $\mathbf{r} : \omega \to \omega$ is slow, if for each $n > 0, \mathbf{r}(n) \leq n^{g(n)}$ where $g : \omega \to \omega$ is non-decreasing with $\lim_{n\to\infty} g(n) = \infty$ and $\lim_{n\to\infty} \frac{g(n)}{\log_2(n)} = 0$.

In this paper, we use the method of tree creature forcing to answer a question of Leo Harrington[1]. There are several examples of forcing notions like Silver-Prikry forcing [3], Laver forcing [4], Jensen's minimal forcing [5], Rational perfect set forcing [8], Sacks forcing [10], splitting forcing [11] and so on, which add a minimal real into the ground model. In these examples, the real is fast growing and cannot be dominated by the ground model reals. On the other hand, given a sequence $\langle n_i^* \mid i < \omega \rangle$ of natural numbers which grows very fast, one can define a forcing notion which adds a real $\eta \in \prod_{i < \omega} n_i^*$, which is minimal and is eventually different from any old real in $\prod_{i < \omega} n_i^*$; see for example [1] and [6]. Motivated by these results, Harrington asked the second author if there is a proper forcing notion which adds a minimal real as above into a sequence $\langle n_i^* \mid i < \omega \rangle$ which grows slowly. In this paper we give abstract conditions, called local minimality condition and global minimality condition, such that if a tree creature forcing notion satisfies them, then it adds a slow minimal real. In a further paper we prove similar results for the measured creature forcing.

The paper is organized as follows. In Section 2, we introduce some preliminary results on creature forcing. Given a sequence $\langle n_i^* \mid i < \omega \rangle$ of natural numbers which

[1]Harrington asked the question from the second author in personal communication.

grows slowly[2], we define, in Section 3, a class of creature forcing notions, where each of them adds a real $\eta \in \prod_{i<\omega} n_i^*$, which is minimal and is eventually different from any old real in $\prod_{i<\omega} n_i^*$. A very special case of the results of this section is proved in [2]. In Sections 4 and 5, we use probabilistic arguments, to show that the class of such forcing notions is non-empty.

We assume no familiarity with creature forcing, and present all required preliminaries, to make the paper as self-contained as possible.

2. CREATURES AND TREE CREATURES

In this section, we briefly review some concepts from Shelah's creature forcing, that will be used in the rest of the paper. Our presentation follows [7] and [9].

Definition 2.1.

(a) A quasi-tree $(T, \lhd)$ over X is a set $T \subseteq X^{<\omega}$ with the initial segment relation, such that T has a $\lhd$-minimal element, called the root of T, $\mathrm{rt}(T)$. It is called a tree, if it is closed under initial segments of length $\geq \mathrm{lh}(\mathrm{rt}(T))$.

(b) For $\eta \in T, \mathrm{Suc}_T(\eta)$ is the set of all immediate successors of η in T:

$$\mathrm{Suc}_T(\eta) = \{\nu \in T : \eta \lhd \nu \text{ and } \neg \exists \rho \in T(\eta \lhd \rho \lhd \nu)\}.$$

(c) $\max(T)$ is the set of all maximal nodes (if there are any) in T:

$$\max(T) = \{\eta \in T : \neg \exists \rho \in T, \eta \lhd \rho\}.$$

(d) $\hat{T} = T \setminus \max(T)$.

(e) For $\eta \in T, T^{[\eta]} = \{\nu \in T : \nu \text{ and } \eta \text{ are comparable}\}$.

(f) $\mathrm{Lim}(T)$ is the set of all cofinal branches through T:

$$\mathrm{Lim}(T) = \{\eta \in X^\omega : \exists^\infty n, \ \eta \restriction n \in T\}.$$

(g) T is well-founded if it has no cofinal branches.

(h) A subset $J \subseteq T$ is a front of T, if J consists of $\lhd$-incomparable elements and for any branch η of T, $\eta \restriction n \in J$, for some n.

Let χ be a sufficiently large enough regular cardinal. Let also $\mathbf{H} : \omega \to V \setminus \{\emptyset\}$ be a function such that for each $i < \omega$, $|\mathbf{H}(i)| \geq 2$. We call $\mathbf{H}(i)$ the reservoir at i. The forcing notions we define aim to add a function $g \in \prod_{i<\omega} \mathbf{H}(i)$.

Definition 2.2.

(a) A (weak) creature for $\mathbf{H}$ is a tuple $t = (\mathbf{nor}[t], \mathbf{val}[t], \mathbf{dis}[t])$, where
 (1) $\mathbf{nor}[t] \in \mathbb{R}^{\geq 0}$.
 (2) $\mathbf{val}[t] \subseteq \bigcup_{m_0 < m_1 < \omega} \{(w, u) \in \prod_{i<m_0} \mathbf{H}(i) \times \prod_{i<m_1} \mathbf{H}(i) : w \lhd u\}$.
 (3) $\mathbf{dis}[t] \in H(\chi)$.
 The family of all creatures for $\mathbf{H}$ is denoted by $\mathrm{CR}[\mathbf{H}]$.

(b) If $t \in \mathrm{CR}[\mathbf{H}]$, then $\mathrm{basis}(t) = \mathrm{dom}(\mathbf{val}[t])$ and $\mathrm{pos}(t) = \mathrm{range}(\mathbf{val}[t])$.

(c) A creature t is a tree creature, if $\mathrm{basis}(t) = \{\eta\}$ is a singleton and no distinct elements of $\mathrm{pos}(t)$ are $\lhd$-comparable. The family of all tree creatures for $\mathbf{H}$ is denoted by $\mathrm{TCR}[\mathbf{H}]$. We also set

$$\mathrm{TCR}_\eta[\mathbf{H}] = \{t \in \mathrm{TCR}[\mathbf{H}] : \mathrm{basis}(t) = \{\eta\}\}.$$

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[2]See Section 5 for some examples of such sequences.

Definition 2.3.

(a) Let $K \subseteq \mathrm{CR}[\mathbf{H}]$. A function $\Sigma : [K]^{\leq \omega} \to \mathcal{P}(K)$ is called a sub-composition operation on K, if
  (1) $\Sigma(\emptyset) = \emptyset$, and for each $s \in K, s \in \Sigma(s)$.
  (2) (transitivity) If $\mathcal{S} \in [K]^{\leq \omega}$ and for each $s \in \mathcal{S}, \mathcal{S}_s \in [K]^{\leq \omega}$ is such that $s \in \Sigma(\mathcal{S}_s)$, then $\Sigma(\mathcal{S}) \subseteq \Sigma(\bigcup_{s \in \mathcal{S}} \mathcal{S}_s)$.

(b) A pair (K, Σ) is called a creating pair for $\mathbf{H}$ if $K \subseteq \mathrm{CR}[\mathbf{H}]$ and Σ is a sub-composition operation on K.

(c) A sub-composition operation Σ on $K \subseteq \mathrm{TCR}[\mathbf{H}]$ is called a tree-composition operation on K. Such a pair (K, Σ) is called a tree-creating pair for $\mathbf{H}$, if in addition:
  (1) If $\mathcal{S} \in [K]^{\leq \omega}$ and $\Sigma(\mathcal{S}) \neq \emptyset$, then there exists a well-founded quasi-tree $T \subseteq \bigcup_{n < \omega} \prod_{i < n} \mathbf{H}(i)$ and a sequence $\langle s_\nu : \nu \in \hat{T} \rangle \subseteq K$ such that $\mathcal{S} = \{ s_\nu : \nu \in \hat{T} \}$, and for each $\nu \in \hat{T}, s_\nu \in \mathrm{TCR}_\nu[\mathbf{H}]$ and $\mathrm{pos}(s_\nu) = \mathrm{Suc}_T(\nu)$.
  (2) If $t \in \Sigma(s_\nu : \nu \in \hat{T})$, then $t \in \mathrm{TCR}_{\mathrm{rt}(T)}[\mathbf{H}]$ and $\mathrm{pos}(t) \subseteq \max(T)$.

To each tree-creating pair (K, Σ), we assign the forcing notion $\mathbb{Q}^{\mathrm{tree}}(K, \Sigma)$ as follows:

Definition 2.4. Assume (K, Σ) is a tree-creating pair for $\mathbf{H}$. Then $\mathbb{Q}^{\mathrm{tree}}(K, \Sigma)$ consists of all sequences $p = \langle \mathfrak{c}_\eta^p : \eta \in T^p \rangle$, where

(1) $T^p \subseteq \bigcup_{n < \omega} \prod_{i < n} \mathbf{H}(i)$ is a tree with no maximal nodes, i.e., $\max(T^p) = \emptyset$.
(2) $\mathfrak{c}_\eta^p \in \mathrm{TCR}_\eta[\mathbf{H}] \cap K$ and $\mathrm{pos}(\mathfrak{c}_\eta^p) = \mathrm{Suc}_{T^p}(\eta)$.

Given $p, q \in \mathbb{Q}^{\mathrm{tree}}(K, \Sigma)$, we set $p \leq q$ (p is an extension of q), if $T^p \subseteq T^q$ and for each $\eta \in T^p$, there exists a well-founded quasi-tree $T \subseteq (T^p)^{[\eta]}$ such that $\mathfrak{c}_\eta^p \in \Sigma(\mathfrak{c}_\nu^q : \nu \in \hat{T})$.

The forcing notion $\mathbb{Q}^{\mathrm{tree}}(K, \Sigma)$ is maximal among all forcing notions that we assign to (K, Σ), in the sense that all other forcing notions that we define are subsets of $\mathbb{Q}^{\mathrm{tree}}(K, \Sigma)$.

3. Minimality with creatures

In this section, we define a class of creature forcing notions which add a minimal real with slow splitting.

Definition 3.1. Let (K, Σ) be a tree-creating pair for $\mathbf{H}$. The forcing notion $\mathbb{Q}^*(K, \Sigma)$ consists of all conditions $p = \langle \mathfrak{c}_\eta^p : \eta \in T^p \rangle \in \mathbb{Q}^{\mathrm{tree}}(K, \Sigma)$ such that

(1) $T^p \subseteq \omega^{<\omega}$ is a finite branching tree with $\max(T^p) = \emptyset$.
(2) Each $\mathfrak{c}_\eta^p$ is finitary, i.e., $|\mathbf{val}[\mathfrak{c}_\eta^p]|$ is finite.
(3) $\eta \in \mathrm{Lim}(T^p) \implies \liminf(\mathbf{nor}[\mathfrak{c}_{\eta \upharpoonright n}^p] : n \geq \mathrm{lh}(\mathrm{rt}(T^p))) = \infty$.
(4) Each $\mathfrak{c}_\eta^p$ is 2-big, i.e., if $\mathbf{nor}[\mathfrak{c}_\eta^p] \geq 1$ and $\mathrm{pos}(\mathfrak{c}_\eta^p) = u_0 \cup u_1$ is a partition of $\mathrm{pos}(\mathfrak{c}_\eta^p)$, then for some $\mathfrak{d} \in \Sigma(\mathfrak{c}_\eta^p), \mathbf{nor}[\mathfrak{d}] \geq \mathbf{nor}[\mathfrak{c}_\eta^p] - 1$ and for some $l < 2, \mathrm{pos}(\mathfrak{d}) \subseteq u_l$.

Given $p, q \in \mathbb{Q}^*(K, \Sigma), p \leq q$ iff

(1) $T^p \subseteq T^q$.
(2) For each $\eta \in T^p, \mathfrak{c}_\eta^p \in \Sigma(\mathfrak{c}_\eta^q)$.

A proof of the next lemma can be found in [7] and [9], and is essentially based on the fact that all creatures involved in the forcing are 2-big.

Lemma 3.2.

    (a) $(\mathbb{Q}^*(K,\Sigma),\leq)$ *is proper.*

    (b) *(continuous reading of names) For any condition* $p \in \mathbb{Q}^*(K,\Sigma)$, *any* $k < \omega$ *and any* $\mathbb{Q}^*(K,\Sigma)$-*name* $\underset{\sim}{f} : \omega \to \check{V}$, *there exists* q *such that*

        (α) $q \leq_k p$, *i.e.,* $q \leq p$ *and if* $\eta \in T^q$ *and* $\mathrm{lh}(\eta) \leq k \vee \mathbf{nor}[\mathfrak{c}^q_\eta] \leq k$ *then* $\mathfrak{c}^q_\eta = \mathfrak{c}^p_\eta$, *so* $\mathrm{Suc}_{T^q}(\eta) = \mathrm{Suc}_{T^p}(\eta)$.

        (β) *For every* n, *there exists a front* J_n *of* T^q *such that*

$$\eta \in J_n \implies \exists a \in V, q^{[\eta]} \Vdash \text{``}\underset{\sim}{f}(n) = a\text{''}.$$

We now state and prove the main result of this section, which gives sufficient conditions for the forcing notion $\mathbb{Q}^*(K,\Sigma)$ to add a minimal real.

Theorem 3.3. *Assume* $\mathbb{Q}^*(K,\Sigma)$ *satisfies the following conditions:*

    (a) *(the local minimality condition) If* $\mathbf{nor}[\mathfrak{c}^p_\eta] \geq 1$ *and* E *is an equivalence relation on* $\mathrm{pos}(\mathfrak{c}^p_\eta)$, *then for some* $\mathfrak{d} \in \Sigma(\mathfrak{c}^p_\eta)$ *we have*

        (α) $\mathbf{nor}[\mathfrak{d}] \geq \mathbf{nor}[\mathfrak{c}^p_\eta] - 1$.

        (β) $E \restriction \mathrm{pos}(\mathfrak{d})$ *is trivial: either every equivalence class is a singleton or it has one equivalence class.*

    (b) *(the global minimality condition) For every* $p \in \mathbb{Q}^*(K,\Sigma)$ *and* $k < \omega$ *there are* $q \in \mathbb{Q}^*(K,\Sigma)$ *and* $m > k, \mathrm{lh}(\mathrm{rt}(T^p))$ *such that*

        (α) $q \leq_k p$.

        (β) *If* J *is a front of* T^q *consisting of sequences of length* $> m$, S *is a finite set and* $\langle f_\rho : \rho \in J \rangle$ *is a sequence of one-to-one functions from* $\mathrm{Suc}_{T^q}(\rho)$ *into* S, *then there exists a partition* $S = S_0 \cup S_1$ *of* S *such that for every* $\eta \in T^q \cap \omega^m$, *there are* r_0, r_1 *such that each* $r_l \leq_k q^{[\eta]}$ *and*

$$\eta \lhd \rho \in J \cap T^{r_l} \implies f_\rho \text{``}[\mathrm{Suc}_{T^{r_l}}(\rho)] \subseteq S_l.$$

Then $\Vdash_{\mathbb{Q}^*(K,\Sigma)} \text{``}\underset{\sim}{\eta} = \bigcup\{\mathrm{rt}(T^p) : p \in \dot{G}_{\mathbb{Q}^*(K,\Sigma)}\}$ *is a minimal real''.*

Proof. Assume $p \in \mathbb{Q}^*(K,\Sigma)$ and $p \Vdash_{\mathbb{Q}^*(K,\Sigma)} \text{``}\underset{\sim}{f}$ is a new real''. Without loss of generality, we may assume that $p \Vdash_{\mathbb{Q}^*(K,\Sigma)} \text{``}\underset{\sim}{f} \in {}^\omega 2$. We show that there is $q \leq p$, such that $q \Vdash_{\mathbb{Q}^*(K,\Sigma)} \text{``}\underset{\sim}{\eta} \in V[\underset{\sim}{f}]\text{''}$.

By Lemma 3.2(a) and [9] and extending p if necessary, we can assume that there exists a sequence $\langle (J_n, H_n) : n < \omega \rangle$ such that:

    (1) J_n is a front of T^p.

    (2) J_{n+1} is above J_n, i.e., $\forall \nu \in J_{n+1} \exists l < \mathrm{lh}(\nu), \nu \restriction l \in J_n$.

    (3) $H_n : J_n \to \mathbf{H}(n)$ is such that for each $\nu \in J_n, p^{[\nu]} \Vdash_{\mathbb{Q}^*(K,\Sigma)} \text{``}\underset{\sim}{f}(n) = H_n(\nu)\text{''}$.

Claim 3.4. For every $k < \omega$ and $p' \leq p$ there are $q, \bar{J}, \bar{f}$ and $\bar{n}$ such that:

    (a) $q \leq_k p'$,

    (b) $\bar{J} = \langle J_l : l < \omega \rangle$, where each J_l is a front of T^q,

    (c) J_{l+1} is above J_l,

    (d) $\bar{n} = \langle n(l) : l < \omega \rangle$,

    (e) $\bar{f} = \langle f_\nu : \nu \in \bigcup_{l<\omega} J_l \rangle$,

    (f) If $\nu \in J_l$, then $f_\nu : \mathrm{Suc}_{T^q}(\nu) \to^{n(l)} 2$ is such that:

        • f_ν is one-to-one,

- if $\rho \in \mathrm{Suc}_{T^q}(\nu), \nu \in J_l$, then

$$q^{[\rho]} \Vdash_{\mathbb{Q}^*(K,\Sigma)} \text{``} \underset{\sim}{f} \upharpoonright n(l) = f_\nu(\rho)\text{''}.$$

Proof. To start, let $q' \leq_k p'$ and $m > k, \mathrm{lh}(\mathrm{rt}(T^{p'}))$ witness (b)(β). Let n_0 be such that

- $\nu \in T^{q'} \wedge \mathrm{lh}(\nu) \geq n_0 \implies \mathbf{nor}[\mathfrak{c}_\nu^{q'}] > k + |T^{q'} \cap \omega^m|$.
- $n_0 > m$.
- J_{n_0} is above $T^{q'} \cap \omega^m$.

For every $n > n_0$, we define a function H_n^+ with domain a subset of Λ_n and values in ${}^n 2$, a sequence $\langle \mathfrak{d}_\nu^n : \nu \in \Lambda_n \rangle$, by downward induction on $\mathrm{lh}(\nu)$ as follows:

Case 1. $\underline{\nu \in J_n :}$ Let $\rho = H_n^+(\nu) \in {}^n 2$ be such that

$$l < n \wedge \eta \in J_l \wedge \eta \trianglelefteq \nu \implies \rho(l) = H_l(\eta)$$

and set $\mathfrak{d}_\nu^n = \mathfrak{c}_\nu^{q'}$.

Case 2. $\underline{\mathrm{Suc}_{T^{q'}}(\nu) \subseteq \Lambda_n}$ and $\langle H_n^+(\rho), \mathfrak{d}_\rho^n : \rho \in \mathrm{Suc}_{T^{q'}}(\nu) \rangle$ is defined: Define an equivalence relation E_ν on $\mathrm{Suc}_{T^{q'}}(\nu)$ by

$$(\rho_1, \rho_2) \in E_\rho \iff H_n^+(\rho_1) = H_n^+(\rho_2).$$

By the local minimality condition (a), there exists $\mathfrak{d}_\nu^n \in \Sigma(\mathfrak{c}_\nu^q)$ such that $\mathbf{nor}[\mathfrak{d}_\nu] \geq \mathbf{nor}[\mathfrak{c}_\nu^{q'}] - 1$ and $H_n^+ \upharpoonright \mathrm{pos}(\mathfrak{d}_\nu)$ is constant or one-to-one. If $H_n^+ \upharpoonright \mathrm{pos}(\mathfrak{d}_\nu)$ is constant, then choose such a $\mathfrak{d}_\nu^n$ and let $H_n^+(\nu)$ be that constant value. Otherwise let $H_n^+(\nu)$ be undefined. So $\mathrm{dom}(H_n^+)$ is an upward closed subset of Λ_n which includes J_n.

Next we show that there exists $n < \omega$ such that there is no $\eta \in T^{q'} \cap \omega^{n_0}$ with $\eta \in \mathrm{dom}(H_n^+)$. Suppose otherwise. Then as $T^{q'} \cap \omega^{n_0}$ is finite, there is $\eta \in T^{q'} \cap \omega^{n_0}$ such that $\exists^\infty n > n_0, \eta \in \mathrm{dom}(H_n^+)$. Let D be an ultrafilter on ω such that $\{n : \eta \in \mathrm{dom}(H_n^+)\} \in D$. Let

$$T = \{\rho \in T^{q'} : \eta \triangleleft \rho \text{ and } \forall l(\mathrm{lh}(\eta) \leq l < \mathrm{lh}(\rho) \implies \{n : \rho \upharpoonright (l+1) \in \mathrm{pos}(\mathfrak{d}_{\rho \upharpoonright l}^n)\} \in D)\}.$$

Let $q'' \leq q'$ be such that $T^{q''} = T$ and for every $\nu \in T, \mathfrak{c}_\nu^{q''} = \mathfrak{d}_\nu^n$. Then $q'' \Vdash \text{``} \underset{\sim}{f} \in V\text{''}$. To see this, let $\rho \in \mathrm{Lim}(T)$. Then,

$$q'' \Vdash \text{``} \underset{\sim}{f} \upharpoonright (n_0, \omega) = \{(n, H_n^+(\rho \upharpoonright n)) : n > n_0\}\text{''},$$

from which the result follows. We get a contradiction by the fact that f is a new real.

So we can find $n_1 > m$ such that $\mathrm{dom}(H_{n_1}^+) \cap (T^{p'} \cap \omega^{n_0}) = \emptyset$. By extending q', let us assume that for each $n \geq n_1$ and $\nu \in \mathrm{dom}(H_n^+) \setminus J_n, \mathrm{Suc}_{T^{q'}}(\nu) = \mathrm{pos}(\mathfrak{d}_\nu^n)$. It then follows that, for each $n \geq n_1$,

$$\nu \in T^{q'} \cap \mathrm{dom}(H_n^+) \ \& \ \nu \triangleleft \nu' \in T^{q'} \cap \Lambda_n \implies H_n^+(\nu) = H_n^+(\nu').$$

For $l < \omega$ set $n(l) = n_1 + l$ and

$$J_l = \{\rho \in \Lambda_{n(l)} : \rho \notin \mathrm{dom}(H_{n(l)}^+) \text{ but } \mathrm{Suc}_{T^{q'}}(\rho) \subseteq \mathrm{dom}(H_{n(l)}^+)\}.$$

Then J_l is a front of $T^{q'}$ above $T^{q'} \cap \omega^{n_0}$ and by our construction, for every $\rho \in J_l$, $H_{n(l)}^+ \upharpoonright \mathrm{pos}(\mathfrak{d}_\rho^{n(l)})$ is one-to-one.

For $l < \omega$ and $\nu \in J_l$ set $f_\nu = H_{n(l)}^+ \upharpoonright \mathrm{pos}(\mathfrak{d}_\nu^{n(l)})$. Then $f_\rho : \mathrm{Suc}_{T^{q'}}(\nu) \to^{n(l)} 2$ is a one-to-one function.

Finally let $q \leq_k p'$ be such that for each $l < \omega$ and $\rho \in J_{n(l)}$, $\mathrm{Suc}_{T^{q'}}(\rho) = \mathrm{pos}(\mathfrak{d}_\rho^{n(l)})$.   □

Claim 3.5. Suppose $p, \underset{\sim}{f}$ are as above, $k, i < \omega$ and $p' \leq p$. There are r, Ψ, n_* and $n_\bullet$ such that:

(a) $r \leq_k p'$,
(b) $\Psi : \prod_{l < n_\bullet} \mathbf{H}(l) \cap T^q \to \{0, 1\}$
(c) If $\nu \in T^r$, $\mathrm{lh}(\nu) = n_*$, then

$$r^{[\nu]} \Vdash_{\mathbb{Q}^*(K, \Sigma)} \text{“}\underset{\sim}{\eta}(i) = \Psi(\underset{\sim}{f} \restriction n_\bullet)\text{”}.$$

Proof. Let p', k, i be given as above. Without loss of generality we may assume that $k > i$. By the global minimality condition 3.3(b), applied to p' and $k + 1$, we can find $q \leq_{k+1} p'$ and $m > k + 1, \mathrm{lh}(\mathrm{rt}(T^{p'}))$ satisfying clauses (α) and (β) there. Let $\bar{J}$ and $\bar{f}$ be as in the conclusion of Lemma 3.4. So for some $l, J = T^q \cap J_l$ is a front of T^q above level m. Let $S = {}^{n(l)}2$. Hence we can apply 3.3(b)(β) and get $(S_0, S_1, \langle r_{\nu,j} : \nu \in T^q \text{ with } \mathrm{lh}(\nu) = m, \ j < 2 \rangle)$ as there. Let r be a k-extension of p' such that

$$T^r = \bigcup \{ T^{r_{\nu,j}} : \nu \in T^q \cap {}^m\omega \text{ and } j = \nu(l) \}.$$

Set also $n_* = \max\{\mathrm{lh}(\nu) : \nu \in J\}$ and $n_\bullet = m$. Finally define $\Psi : \prod_{l < n_\bullet} \mathbf{H}(l) \cap T^q \to \{0, 1\}$ by

$$\Psi(t) = j \iff j = \nu(l) \text{ where } \nu \text{ is such that } t \in T^{r_{\nu,j}}.$$

It is easily seen that r, n, Ψ, n_* and $n_\bullet$ are as required.   □

Claim 3.6. For every k and $p' \leq_k p$, there are $q, \bar{n}$ and $\bar{\Psi}$ such that

(a) $q \leq_k p$,
(b) $\bar{n} = \langle n(l) : l < \omega \rangle$,
(c) $\bar{\Psi} = \langle \Psi_l : l < \omega \rangle$,
(d) Ψ_l is a function from $T^q \cap {}^{n(l)}\omega \to 2$,
(e) If $\nu \in T^q \cap {}^{n(l)}\omega$, then

$$q^{[\nu]} \Vdash_{\mathbb{Q}^*(K, \Sigma)} \text{“}\underset{\sim}{\eta}(l) = \Psi_l(\underset{\sim}{f} \restriction n(l))\text{”}.$$

Proof. By Lemma 3.5 we can find $\bar{r}, \bar{\Psi}, \bar{n_*}$ and $\bar{n_\bullet}$ such that:

• $\bar{r} = \langle r_l : l < \omega \rangle$ is such that $r_0 \leq_k p'$ and for each $l < \omega$, $r_{l+1} \leq_{k+l} r_l$,
• $\bar{n_*} = \langle n_{*,l} : l < \omega \rangle$ and $\bar{n_\bullet} = \langle n_{\bullet,l} : l < \omega \rangle$ are increasing sequences of natural numbers,
• $\bar{\Psi} = \langle \Psi_l : l < \omega \rangle$, where $\Psi_l : \prod_{j < n_{\bullet,l}} \mathbf{H}(l) \cap T^q \to \{0, 1\}$,
• If $\nu \in T^{r_l}$, $\mathrm{lh}(\nu) = n_{*,l}$, then

$$r_l^{[\nu]} \Vdash_{\mathbb{Q}^*(K, \Sigma)} \text{“}\underset{\sim}{\eta}(l) = \Psi(\underset{\sim}{f} \restriction n_{\bullet,l})\text{”}.$$

For $l < \omega$ set $n(l) = n_{*,l}$ and let $q = \lim_{l \to \infty} r_l$ be the natural condition obtained by the fusion argument applied to the sequence $\bar{r}$.   □

Finally we are ready to complete the proof of Theorem 3.3. Suppose that $p \Vdash_{\mathbb{Q}^*(K, \Sigma)} \text{“}\underset{\sim}{f} \in {}^\omega 2\text{”}$. If $p \nVdash_{\mathbb{Q}^*(K, \Sigma)} \text{“}\underset{\sim}{f} \notin V\text{”}$, then for some $q \leq p, q \Vdash \text{“}\underset{\sim}{f} \in \check{V}\text{”}$ and we are done. So suppose otherwise. By Lemma 3.6, there are $q \leq p, \bar{n}$ and $\bar{\Psi}$ as there. So for every $l < \omega$,

$$q \Vdash_{\mathbb{Q}^*(K, \Sigma)} \text{“}\underset{\sim}{\eta}(l) = \Psi_l(\underset{\sim}{f} \restriction n(l))\text{”}.$$

It follows that $q \Vdash_{\mathbb{Q}^*(K, \Sigma)} \text{“}\underset{\sim}{\eta} \in V[\underset{\sim}{f}]\text{”}$. We are done.   □

4. Sufficient conditions for global minimality condition

In this section we give some sufficient conditions to guarantee the global minimality condition of Theorem 3.3.

Definition 4.1. Given a finite set S, we define a probability space $(\Omega, \mathcal{F}, \text{Prob})$ as follows:

- $\Omega = \{(S_0, S_1) : S_0 \cup S_1 \text{ is a partition of } S\}$.
- $\mathcal{F} = \mathcal{P}(\Omega)$.
- For $A \subseteq \Omega$, $\text{Prob}(A) = \dfrac{|A|}{|\Omega|}$.

Lemma 4.2. *Assume that for every $p = \langle \mathfrak{c}_\eta^p : \eta \in T^p \rangle \in \mathbb{Q}^*(K, \Sigma)$ and every $k < \omega$ there are $m > k, \text{lh}(\text{rt}(T^p))$ and q such that:*

(a) $q \leq_k p$.
(b) *If $\eta \in T^p$ and $\text{lh}(\eta) \geq m$, then $\mathbf{nor}[\mathfrak{c}_\eta^q] > k + |T^q \cap \omega^m|$.*
(c) *If J is a front of T^q is such that for $\eta \in J$, $\text{lh}(\eta) \geq m$, S is a finite set and $\langle f_\rho : \rho \in J \rangle$ is a sequence of one-to-one functions from $\text{Suc}_{T^q}(\rho)$ into S, then we can find a sequence $\langle a_\eta : \eta \in \Lambda \rangle$, where*

$$\Lambda = \{\eta \in T^q : \text{lh}(\eta) \geq m \text{ and } \eta \text{ is below } J \text{ or is in } J\}$$

such that:

(α) $a_\eta \in (0, 1)_\mathbb{R}$.
(β) *If $\eta \in J$, then $a_\eta \geq Prob(\text{for some } l \in \{0, 1\}, \text{ there is no } \mathfrak{d} \in \Sigma(\mathfrak{c}_\eta^p)$ with $\mathbf{nor}[\mathfrak{d}] \geq \mathbf{nor}[\mathfrak{c}_\eta^p] - 1$ and $f_\eta \text{``}[pos(\mathfrak{d})] \subseteq S_l)$.*
(γ) *If $\eta \in \Lambda \setminus J$, then $a_\eta \geq Prob(\text{there is no } \mathfrak{d} \in \Sigma(\mathfrak{c}_\eta^p) \text{ with } \mathbf{nor}[\mathfrak{d}] \geq \mathbf{nor}[\mathfrak{c}_\eta^p] - 1$ such that $\nu \in pos(\mathfrak{d}) \implies E_\nu$ does not occur$)$, whenever E_ν are events for $\nu \in pos(\mathfrak{c}_\eta^q)$ each of probability $\leq a_\nu$.*
(δ) $1 > \Sigma\{a_\eta : \eta \in T^q \cap \omega^m\}$.

Then the global minimality condition of Theorem 3.3 holds.

Proof. For $\eta \in \Lambda$, let Good_η be the event: "For $l < 2$, there is $r_l \in \mathbb{Q}^*(K, \Sigma)$ such that $\text{rt}(T^{r_l}) = \eta, r_l \leq_k q^{[\eta]}$ and $\forall \nu \in T^{r_l} \cap J, \text{Suc}_{T^{r_l}}(\nu) \subseteq S_l$".

Claim 4.3. For $\eta \in \Lambda, \text{Prob}(\text{Good}_\eta) \geq 1 - a_\eta$.

Proof. We prove the claim by downward induction on $\text{lh}(\eta)$. Suppose $\eta \in J$. Let Θ_η denote the statement "for some $l \in \{0, 1\}$, there is no $\mathfrak{d} \in \Sigma(\mathfrak{c}_\eta^p)$ with $\mathbf{nor}[\mathfrak{d}] \geq \mathbf{nor}[\mathfrak{c}_\eta^p] - 1$ and $f_\eta \text{``}[\text{pos}(\mathfrak{d})] \subseteq S_l$". So by clause $(c)(\beta)$, $a_\eta \geq \text{Prob}(\Theta_\eta)$. This implies

$$1 - a_\eta \leq 1 - \text{Prob}(\Theta_\eta) \leq \text{Prob}(\text{Good}_\eta).$$

Otherwise, $\eta \in \Lambda \setminus J$ and η is not $\lhd$-maximal in T^q. Let Φ_η denote the statement "there is no $\mathfrak{d} \in \Sigma(\mathfrak{c}_\eta^p)$ with $\mathbf{nor}[\mathfrak{d}] \geq \mathbf{nor}[\mathfrak{c}_\eta^p] - 1$ such that $\nu \in \text{pos}(\mathfrak{d}) \implies E_\nu$ does not occur". By clause $(c)(\gamma)$, $a_\eta \geq \text{Prob}(\Phi_\eta)$, so

$$1 - a_\eta \leq 1 - \text{Prob}(\Phi_\eta) \leq \text{Prob}(\text{Good}_\eta).$$

□

Since $1 > \Sigma\{a_\eta : \eta \in T^q \cap \omega^m\}$, we can find a pair $(S_0, S_1) \in \Omega$ such that for every $\eta \in T^q \cap \omega^m, \text{Good}_\eta$ occurs and hence we can choose r_0, r_1 as guaranteed by Good_η and we are done.   □

Lemma 4.4. *Assume for a dense set of conditions* $q \in \mathbb{Q}^*(K, \Sigma)$ *there exists a sequence* $\bar{a} = \langle a_\eta : \eta \in T^q \rangle$ *such that:*

(a) *Each* $a_\eta \in (0, 1]_\mathbb{R}$.
(b) *If* $\eta \in \text{Lim}(T^q)$, *then* $\lim_n \langle a_{\eta \upharpoonright n} : n \geq \text{lh}(\text{rt}(T^q)) \rangle = 0$.
(c) *For every large enough* m, *we have*
    (α) $1 > \Sigma\{a_\eta : \eta \in T^q \cap \omega^m\}$.
    (β) *For any* $\eta \in T^q \cap \omega^m$, *if* $\langle E_\nu : \nu \in \text{Suc}_{T^q}(\eta) \rangle$ *is a sequence of events each of probability* $\leq a_\nu$, *then we have* $a_\eta \geq \text{Prob}(\text{there is no } \mathfrak{d} \in \Sigma(\mathfrak{c}_\eta^p)$ *with* $\text{nor}[\mathfrak{d}] \geq \text{nor}[\mathfrak{c}_\eta^p] - 1$ *such that* $\nu \in \text{pos}(\mathfrak{d}) \implies E_\nu$ *does not occur*).

Then the global minimality condition of Theorem 3.3 *holds.*

Proof. By Lemma 4.2. $\square$

5. A FORCING NOTION SATISFYING THE CONDITIONS OF THEOREM 3.3

In this section we introduce a forcing notion which satisfies the conditions of local and global minimality, hence it adds a minimal real.

Let $h : \omega \to \omega$ be a non-decreasing function with $\lim h(n) = \infty$ (e.g. $n \mapsto \log_2(\log_2(n))$ or $n \mapsto \log_*(n)$, where $\log_*(n)$ is defined by recursion as $\log_*(0) = \log_*(1) = 1$, $\log_*(\beth_{m+1}(2)) = \log_*(\beth_m(2)) + 1$, for $m < \omega$,[3] and for each n with $\beth_m(2) \leq n < \beth_{m+1}(2)$, $\log_*(n) = \log_*(\beth_m(2))$)).

Let $\mathbf{H} : \omega \to V$ be defined by

$$\mathbf{H}(n) = (\max\{1, n\})^{h(n)}.$$

Let

$$K = \{t \in TCR[\mathbf{H}] : \text{nor}[t] = \log_2(\log_2(|\text{pos}(t)|))\}.$$

Also let $\Sigma : [K]^{\leq \omega} \to \mathcal{P}(K)$ be such that it is non-empty only for singletons, and for every $t \in K$,

$$\Sigma(t) = \{s \in K : \text{val}[s] \subseteq \text{val}[t]\}.$$

Clearly (K, Σ) forms a tree-creating pair. Let $\mathbb{Q}_h$ be defined as follows:

(A) $p \in \mathbb{Q}_h$ if
    (a) $p = \langle \mathfrak{c}_\eta^p : \eta \in T^p \rangle \in \mathbb{Q}^*(K, \Sigma)$.
    (b) $T^p \subseteq \bigcup_{n < \omega} \prod_{i < n} \mathbf{H}(i)$ is a tree, such that for each non-maximal node $\eta \in T^p$, $|\text{Suc}_{T^p}(\eta)| \geq 4$.
    (c) For $\eta \in T$, $\text{nor}[\mathfrak{c}_\eta^p] = \text{nor}(\text{Suc}_{T^p}(\eta)) = \log_2(\log_2(|\text{Suc}_{T^p}(\eta)|))$.
    (d) If $\eta \in \text{Lim}(T^p)$ and $i < \omega$, then $\lim_n \dfrac{\text{nor}(\text{Suc}_{T^p}(\eta))}{n^i} = \infty$.
(B) $p \leq q$ iff $p \leq_{\mathbb{Q}^*(K, \Sigma)} q$.

Lemma 5.1. *The forcing notion* $\mathbb{Q}_h$ *satisfies the local and global minimality conditions of Theorem* 3.3.

Proof. $\underline{\mathbb{Q}_h \text{ satisfies the local minimality condition}}$: Suppose $\text{nor}[\mathfrak{c}_\eta^p] \geq 1$ and E is an equivalence relation on $\text{pos}(\mathfrak{c}_\eta^p)$.

Consider the set $X = \{[\rho]_E : \rho \in \text{pos}(\mathfrak{c}_\eta^p)\}$. If there exists $\rho \in \text{pos}(\mathfrak{c}_\eta^p)$ such that $[\rho]_E$ has size $\geq \sqrt{|\text{pos}(\mathfrak{c}_\eta^p)|}$, then set

$$\mathfrak{d} = \{(\eta, \nu) : \nu \in [\rho_*]_E\}.$$

---

[3]Where $\beth_m(2)$ is defined inductively as $\beth_0(2) = 1$ and $\beth_{m+1}(2) = 2^{\beth_m(2)}$.

Clearly, $\mathbf{val}[\mathfrak{d}] \subseteq \mathbf{val}[\mathfrak{c}_\eta^p]$ and so $\mathfrak{d} \in \Sigma(\mathfrak{c}_\eta^p)$. Also, $E \restriction \mathfrak{d}$ is trivial. We also have

$$
\begin{aligned}
\mathbf{nor}[\mathfrak{d}] &= \log_2(\log_2(|\mathrm{pos}(\mathfrak{d})|)) \\
&\geq \log_2(\log_2(\sqrt{|\mathrm{pos}(\mathfrak{c}_\eta^p)|})) \\
&\geq \log_2(\log_2(|\mathrm{pos}(\mathfrak{c}_\eta^p)|)) - \log_2 2 \\
&\geq \log_2(\log_2(|\mathrm{pos}(\mathfrak{c}_\eta^p)|)) - 1 \\
&= \mathbf{nor}[\mathfrak{c}_\eta^\mathfrak{p}] - 1.
\end{aligned}
$$

So we are done. Otherwise, each equivalence class $[\rho]_E$ has size less than $\sqrt{|\mathrm{pos}(\mathfrak{c}_\eta^p)|}$. But then $|X| \geq \sqrt{|\mathrm{pos}(\mathfrak{c}_\eta^p)|}$, so pick $t_\rho \in [\rho]_E$ for each $\rho \in \mathrm{pos}(\mathfrak{c}_\eta^p)$ and set

$$
\mathfrak{d} = \{(\eta, t_\rho) : \rho \in \mathrm{pos}(\mathfrak{c}_\eta^p)\}.
$$

Again, $\mathbf{val}[\mathfrak{d}] \subseteq \mathbf{val}[\mathfrak{c}_\eta^p]$ and so $\mathfrak{d} \in \Sigma(\mathfrak{c}_\eta^p)$. Also, $E \restriction \mathfrak{d}$ is trivial. As before,

$$
\mathbf{nor}[\mathfrak{d}] = \log_2(\log_2(|\mathrm{pos}(\mathfrak{d})|)) \geq \mathbf{nor}[\mathfrak{c}_\eta^\mathfrak{p}] - 1,
$$

and so we are done. The result follows.

$\underline{\mathbb{Q}_h \text{ satisfies the global minimality condition}}$: We check Lemma 4.4. Let $p \in \mathbb{Q}_h$ and $k < \omega$. Then we can find $q \in \mathbb{Q}_h$, $m_* > k, \mathrm{lh}(\mathrm{rt}(T^p))$ and m_0 such that:

- $q \leq_k p$.
- $k < m_0 < m_*$.
- $\eta \in T^q \wedge \mathbf{nor}[\mathfrak{c}_\eta^q] \leq k \implies \mathrm{lh}(\eta) < m_0$.
- If $n \geq m_0$, then $\langle |\mathrm{Suc}_{T^q}(\eta)| : \eta \in T^q \cap \omega^n\rangle$ is constant. Moreover, for some non-decreasing function $h_1 \leq h$, $h_1 : [m_0, \omega) \to \omega \setminus \{0\}$, we have $|\mathrm{Suc}_{T^q}(\eta)| = \mathrm{lh}(\eta)^{2^{h_1(\mathrm{lh}(\eta))}}$.
- $h_1(m_*) > k + |T^q \cap \omega^{m_0}|$.

For $\eta \in T^q$, we define b_η by

$$
b_\eta = \mathrm{Prob}(\text{for a partition } S_0 \cup S_1 \text{ of } m_\eta = |\mathrm{pos}(\mathfrak{c}_\eta^q)|, \text{ we have } \bigvee_{l < 2} |S_l| < \sqrt{m_\eta}).
$$

Then it is clear that

$$
b_\eta \leq \frac{2 \cdot \Sigma\{m_\eta : i < \sqrt{m_\eta}\}}{2^{m_\eta}}
$$

(the 2 is because of $\bigvee_{l<2}$ and the sum is because of $\bigvee_{i<\sqrt{m_\eta}} |S_l = i|$). So

$$
b_\eta \leq \frac{(m_\eta)^{\sqrt{m_\eta}}}{2^{m_\eta}} = \frac{2^{\sqrt{m_\eta} \cdot \log_2(m_\eta)}}{2^{m_\eta}}.
$$

Thus for some n_*,

$$
\eta \in T^q, \mathrm{lh}(\eta) = n \geq n_* \implies b_\eta \leq 2^{\sqrt{n^{2^{h_1(n)}}} \cdot 2^{h_1(n)} \cdot \log_2(n)} \cdot 2^{-(n^{2^{h_1(n)}})} \leq 2^{-(n^{2^{h_1(n)}/2^4})}.
$$

For $\eta \in T^q, \mathrm{lh}(\eta) \geq n_*$ let

$$
a_\eta = \Sigma\{2^{-(n^{2^{h_1(n)}}/3)} : n \geq \mathrm{lh}(\eta)\}.
$$

Then $a_\eta \leq 2^{-(n^{2^{h_1(n)}})}$, where $n = \mathrm{lh}(\eta)$, because

$$
n^{2^{h_1(n)}} + 1 \leq (n+1)^{2^{h_1(n)}} \leq (n+1)^{2^{h_1(n+1)}}.
$$

We show that a_η's, $\eta \in T^q$ are as required. It is clear that each $a_\eta \in (0,1)_\mathbb{R}$. Also, if $\eta \in \mathrm{Lim}(T^q)$, then clearly $\lim_n \langle a_{\eta \restriction n} : n \geq \mathrm{lh}(\mathrm{rt}(T^q)) \rangle = 0$ (as $a_\eta \leq 2^{-(n^{2^{h_1(n)}})}$). Also, for $n \geq n_*$, we have

$$\Sigma\{a_\eta : \eta \in T^q \cap \omega^n\} \leq \Sigma\{2^{-(n^{2^{h_1(n)}})} : \eta \in T^q \cap \omega^n\} = |T^q \cap \omega^n| \cdot 2^{-(n^{2^{h_1(n)}})} < 1.$$

Finally clause (c)(β) of Lemma 4.4 holds by the choice of b_η's and a_η's. $\qquad\square$

Remark 5.2. We could also use the function $h(n) = \log_*(\log_*(n))$ and the norm $\mathbf{nor}[t] = \log_*(\log_*(|\mathrm{pos}(t)|))$. In this case, there is no need to require the tree T^p satisfies the extra property "for each non-maximal node $\eta \in T^p, |\mathrm{Suc}_{T^p}(\eta)| \geq 4$".

Acknowledgments

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