# PARTITION THEOREMS FOR EXPANDED TREES 1176

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Abstract. We look for partition theorems for large subtrees for suitable uncountable trees and colourings.
We concentrate on sub-trees of ${}^{\kappa \geq}2$ expanded by a well ordering of each level. Unlike earlier works, we do not ask the embedding to preserve the height of the tree but the equality of levels is preserved. We get consistency results without large cardinals.
The intention is to apply it to model theoretic problems.

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Annotated Content

# § 0. Introduction

## § 0(A). **Background and Results.**

We continue two lines of research. One is set theoretic: pure partition relations on trees and the other is model theoretic: Hanf numbers and non-deniability of well ordering, in particular related to $\omega_1$. This is related to the existence of GEM (generalized Eherenfuecht-Mostowski) for suitable templates (see [Sheb]), and applications to descriptive set theory.

Halpern-Levy [HL71] had proved a milestone theorem on independence of versions of the axiom of choice: in ZF, AC is strictly stronger than the maximal prime ideal theorem (i.e. every Boolean algebra has a maximal ideal).

This work isolated a partition theorem[1] on the tree ${}^{\omega>}2$, necessary for the proof. This partition theorem was subsequently proved by Halpern-Lauchli [HL66] and was a major and early theorem in Ramsey theory, (so the proof above relies on it).

See more in Laver [Lav71], [Lav73] and [She78b, AP,§2] and Milliken [Mil79], [Mil81].

The [HL66] proof uses induction, later Harrington found a different proof using forcing: adding many Cohen reals and a name of a (non-principal) ultrafilter on $\mathbb{N}$. Earlier, (on using adding many reals and a partition theorem) see Silver's proof of $\pi^1_1$-equivalence relations, in [Sil80].

Now [She92, §4] turn to uncountable trees, i.e. for some $\kappa > \aleph_0$, we consider trees $\mathscr{T}$ which are sub-trees of $({}^{\kappa>}2, \triangleleft)$, such that (as in [HL66]) for every level $\varepsilon < \kappa$, either $(\forall \eta \in \mathscr{T} \cap {}^{\varepsilon}2)(\eta\hat{}\langle 0\rangle, \eta\hat{}\langle 1\rangle \in \mathscr{T})$ or $(\forall \eta \in \mathscr{T} \cap {}^{\varepsilon}2)(\exists! \iota < 2)[\eta\hat{}\langle \iota\rangle \in \mathscr{T}]$; (and of course the first occurs unboundedly often). But a new point is that we have to use a well ordering of $\mathscr{T} \cap {}^{\varepsilon}2$ for $\varepsilon < \kappa$.

Naturally we add "is closed enough (that is under unions of increasing sequences of length $< \kappa$)". Also colouring with infinite number of colours, the proof uses "measurable $\kappa$ which remains so when we add $\lambda$ many $\kappa$-Cohens for appropriate $\lambda$"; it generalizes Harrington's proof. This was continued in several works, see Dobrinen-Hathaway [DH17] and references there.

We are here mainly interested in a weaker version which is enough for the model theoretic applications we have in mind, we start with a large tree and get one of smaller cardinality, in a sense this is solving the "equations" $X$ /(Erdös-Rado theorem) = [She92]/ (the partition relation of a weakly compact cardinal) = [HL66] / (Ramsey theorem). On other consistent partition relation see Boney-Shelah [S$^+$a], in preparation.

Turning to model theory see [She75], [She76] and Dzamonja-Shelah [DS04] where such indiscernibility is considered in a model theoretic context.

A central direction in model theory in the sixties were two cardinal theorems. For infinite cardinals $\mu > \lambda$, let $K_{\mu,\lambda}$ be the class of models $M$ such that $M$ is of cardinality $\mu$ and $P^M$ of cardinality $\lambda$. The main problems were transfer, compactness and completeness. For connection to partition theorems, Morley's proof of [Vau65], the Vaught far apart two cardinal theorem used Erdös-Rado theorem; generally see [She71b], [She71a], [She78a] and the survey [DS79]. Jensen's celebrated gap $n$ two

[1] Using not splitting to 2 but other finite splitting make a minor difference; similarly here.

cardinal theorem solve those problems for e.g $(\aleph_n, \aleph_0)$ <u>when</u> $\mathbf{V} = \mathbf{L}$. But can we get a nice picture in different universes?

Note that by [She89], [Sheb], consistently we have GEM (generalized Eherenfuecht-Mostowski) models for ordered graphs as index models, even omitting types.

On a different direction Douglas Ulrich has asked me on $(*)_n$ below (and told me it has descriptive set theoretic consequences, see [SU19]). We intend to prove (in the sequel [S⁺b]) that for $n < \omega$:

$(*)_n$ consistently
- (a) if $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$ has a model $M$ of cardinality $\beth_{n+1}$ with $(P^M, <^M)$ having order type $\omega_1$ <u>then</u> $\psi$ has a model $N$ of cardinality $\beth_{n+1}$ and $(P^N, <^N)$ is not well ordered,
- (b) moreover, it is enough that $M$ will have cardinality $\aleph_\delta, \delta \geq \beth_n^{++}$,
- (c) of course, preferably not using large cardinals.

This requires consistency of many cases of partition relations on trees and more complicated structures, analysing GEM models. Much earlier we have intended (mentioned in [She00, 1.15] to prove the parallel for first order logic; and $(\beth_n, \aleph_0)$, using (many Cohen's indestructible) measurables $\kappa_1 < \ldots \kappa_n$ as in [She92, §4] and forcing by blowing $2^{\aleph_0}$ to $\kappa_1$, $2^{\kappa_1}$ to $\kappa_2$ etc relying on [She92]; but have not carried out that.

In preparation are also solutions to the two cardinal problems above and

$(*)$
- (a) $\alpha_\bullet < \omega_1, \beth_{\alpha+1} = (\beth_\alpha)^{+\omega_1+1}$ for $\alpha < \alpha_\bullet$ and well ordering of $\omega_1$ is not definable in $\{\mathrm{EC}_\psi(\beth_{\alpha_\bullet}) : \psi \in \mathbb{L}_{\aleph_1,\aleph_0}\}$ or at least,
- (b) as above but for $\beth_{\alpha+1} = \aleph_{\beth_\alpha^{++}}$,
- (c) parallel results replacing $\aleph_0$ by $\mu$.

Contrary to the a priori expectation no large cardinal is used.

In a sequel ([S⁺c] we intend also to deal with other partition relations and with weakly compact cardinals.

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## § 0(B). **Preliminaries.**

**Definition 0.1.** If $\mu = \mu^{<\kappa}$ then "*for a* $(\mu, \kappa)$*-club of* $u \subseteq X$ *we have* $\varphi(u)$" means that: for some $\chi$ such that $\mu, u \in \mathscr{H}(\chi)$ and e.g. $\beth_3(\mu + |u|) < \chi$ and some $x \in \mathscr{H}(\chi)$, if $x \in \mathscr{B} \prec (\mathscr{H}(\chi), \in)$, $\|\mathscr{B}\| = \mu$, $[\mathscr{B}]^{<\kappa} \subseteq \mathscr{B}$ and $\mu + 1 \subseteq \mathscr{B}$, <u>then</u> the set $u = \mathscr{B} \cap X$ satisfies $\varphi(u)$; there are other variants.

**Definition 0.2.** For $\kappa$ regular (usually $\kappa = \kappa^{<\kappa}$) and an ordinal $\gamma$, the forcing $\mathbb{P} = \mathrm{Cohen}(\kappa, \gamma)$ of adding $\gamma$ many $\kappa$-Cohen reals is defined as follows:

- (A) $p \in \mathbb{P}$ <u>iff</u>:
  - (a) $p$ is a function with domain from $[\gamma]^{<\kappa}$,
  - (b) if $\alpha \in \mathrm{dom}(p)$ then $p(\alpha) \in {}^{\kappa>}2$,
- (B) $\mathbb{P} \models p \leq q$ <u>iff</u>:
  - (a) $p, q \in \mathbb{P}$,
  - (b) $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$,
  - (c) if $\alpha \in \mathrm{dom}(p)$ then $p(\alpha) \trianglelefteq q(\alpha)$.
- (C) for $\alpha < \gamma$ let $\underset{\sim}{\eta}_\alpha = \bigcup\{p(\alpha) : p \in \underset{\sim}{\mathbf{G}}_\mathbb{P}$ satisfies $\alpha \in \mathrm{dom}(p)\}$, so $\Vdash_\mathbb{P}$ "$\underset{\sim}{\eta}_\alpha \in {}^\kappa 2$",

(D) for $u \subseteq \gamma$ let $\mathbb{P}_u := \{p \in \mathbb{P} \colon \mathrm{dom}(p) \subseteq u\}$, so $\mathbb{P}_u \lessdot \mathbb{P}$ and $\bar{\eta}_u = \langle \underset{\sim}{\eta}_\alpha : \alpha \in u \rangle$ is generic for $\mathbb{P}_u$.

*Notation* 0.3. 1) We denote infinite cardinals by $\kappa, \lambda, \mu, \chi, \theta, \partial$, and $\sigma$ denotes a possibly finite cardinal.

2) We denote ordinals by $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi$ and sometimes $i, j$.

3) We denote natural numbers by $k, \ell, m, n$ and sometimes $i, j$.

4) Instead e.g. $a_i$ we may write $a[i]$, particularly in sub-script; also $\kappa(+)$ means $\kappa^+$.

## § 1. Partition Theorems

### § 1(A). **The definitions.**

Here we consider partition on trees. For uncountable trees we find the need to consider a well ordering of each level, still preserving equality of level. We may consider embeddings where levels are not preserved, see Dzamonja-Shelah [DS04] (in the web version). Also we may waive completeness of the tree but usually still like to have many branches.

We intend to deal with an intermediate ones (and with the weakly compact cardinal) in a sequel [S⁺c].

**Definition 1.1.** 1) Let $\mathbf{T}$ be the class of structures $\mathscr{T}$ such that:

(a) $\mathscr{T} = (u, <_*, E, <, \cap, S, R_0, R_1) = (u_{\mathscr{T}}, <^*_{\mathscr{T}}, E_{\mathscr{T}}, <_{\mathscr{T}}, \cap_{\mathscr{T}}, S_{\mathscr{T}}, R^0_{\mathscr{T}}, R^1_{\mathscr{T}})$ but we may write $s \in \mathscr{T}$ instead of $s \in u$,
(b) $(u, <_*)$ is a well ordering, so linear, $u$ non-empty,
(c) $<_{\mathscr{T}}$ is a partial order included in $<_*$,
(d) $(u, <_{\mathscr{T}})$ is a tree, i.e. if $t \in \mathscr{T}$ <u>then</u> $\{s : s <_{\mathscr{T}} t\}$ is linearly ordered by $<_{\mathscr{T}}$; and is even well ordered, the level of $t$ is the order type of this set; the tree is with $\mathrm{ht}(\mathscr{T})$ levels,
(e) $E$ is an equivalence relation on $u$, convex under $<_*$,
(f) ($\alpha$) each $E$-equivalence class is the set of $t \in \mathscr{T}$ of level $\varepsilon$ for some $\varepsilon$, so the set of $E$-equivalence classes is naturally well ordered,
  ($\beta$) we denote the $\varepsilon$-th equivalence class by $\mathscr{T}_{[\varepsilon]}$,
  ($\gamma$) $E$ has no last $E$-equivalence class if not said otherwise,
  ($\delta$) let $\mathrm{lev}_{\mathscr{T}}(s) = \mathrm{lev}(s, \mathscr{T})$ be $\varepsilon$ when $s \in \mathscr{T}_{[\varepsilon]}$, equivalently $\{t \colon t <_{\mathscr{T}} s\}$ has order type $\varepsilon$ under the order $<_{\mathscr{T}}$,
  ($\varepsilon$) so $\mathrm{ht}(\mathscr{T})$ is $\bigcup\{\mathrm{lev}(s) + 1 \colon s \in \mathscr{T}\}$.
(g) if $s \in u, \mathrm{lev}_{\mathscr{T}}(s) < \zeta < \mathrm{ht}(\mathscr{T})$ then there is $t \in \mathscr{T}_{[\zeta]}$ which is $<_{\mathscr{T}}$-above $s$,
(h) each $s \in \mathscr{T}$ has exactly two immediate successors by $<_{\mathscr{T}}$,
(i) for $s \in \mathscr{T}$ we let:
  $\bullet_1$ $\mathscr{T}_{\geq s} = \{t \in \mathscr{T} : s \leq_{\mathscr{T}} t\}$,
  $\bullet_2$ $\mathrm{suc}_{\mathscr{T}}(s) = \{t : t \in \mathscr{T}_{[\mathrm{lev}(s)+1]}$ satisfies $s <_{\mathscr{T}} t\}$,
(j) let $s = t{\restriction}\varepsilon$ mean that $\mathrm{lev}_{\mathscr{T}}(s) = \varepsilon \leq \mathrm{lev}_{\mathscr{T}}(t) \wedge (s \leq_{\mathscr{T}} t)$,
(k) for $t_1, t_2 \in \mathscr{T}, t_1 \cap_{\mathscr{T}} t_2$ is the maximal common lower bound of $t_1, t_2$ so we demand it always exists, i.e. $(\mathscr{T}, <)$ is normal,
(l) for $\ell = 0, 1$ we have $R_\ell \subseteq \{(s,t) \colon s \in \mathscr{T}$ and $s <_{\mathscr{T}} t\}$ and if $s \in \mathscr{T}$ then for some $t_0 \neq t_1$ we have $\mathrm{suc}_{\mathscr{T}}(s) = \{t_0, t_1\}$ and $\ell < 2 \Rightarrow (\forall t)(sR_\ell t$ <u>iff</u> $t_\ell \leq_{\mathscr{T}} t)$; so $sR_\ell t$ is the parallel to $\eta\hat{\ }\langle \ell \rangle \trianglelefteq \nu$; we may think of $\{t \colon sR_\ell t\}$ as a division to the left side and the right side of the set of the $t'$s above $s$.

1A) For $\mathscr{T} \in \mathbf{T}, <_{\mathrm{lex}} := <^{\mathrm{lex}}_{\mathscr{T}}$ is the lexicographic order, e.g.,

$$\eta <_{\mathrm{lex}} \nu \text{ \underline{iff} } (\exists \rho)(\rho\, R_0\, \nu \wedge \eta\, R_1\, \nu) \text{ or } (\eta < \nu \wedge \eta R_1 \nu) or (\nu < \eta \wedge \nu R_0 \eta).$$

2) Let $\mathbf{T}_{\theta,\kappa} = \{\mathscr{T} \in \mathbf{T} \colon$ the tree $\mathscr{T}$ has $\delta$ levels, for some ordinal $\delta$ of cofinality $\kappa$ and for every $\varepsilon < \delta$ we have $\theta > |\{s \in \mathscr{T} \colon s$ of level $\leq \varepsilon\}|\}$.

3) Let $\mathscr{T}_1 \subseteq \mathscr{T}_2$ mean:

(a) $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$,
(b) $<_{\mathscr{T}_1} := <_{\mathscr{T}_2} {\restriction} u_{\mathscr{T}_1}$,

(c) if $\mathscr{T}_1 \models$ "$\eta \cap \nu = \rho$" then $\mathscr{T}_2 \models$ "$\eta \cap \nu = \rho$",
(d) $R_{\mathscr{T}_1,\ell} = R_{\mathscr{T}_2,\ell} \restriction u_{\mathscr{T}_1}$ for $\ell = 0, 1$,
(e) $<^*_{\mathscr{T}_1} := <^*_{\mathscr{T}_2} \restriction u_{\mathscr{T}_1}$;
(f) $E_{\mathscr{T}_1} := E_{\mathscr{T}_2} \restriction u_{\mathscr{T}_1}$,

4) For $s \in \mathscr{T}$ and $\ell \in \{0, 1\}$, let $\mathrm{suc}_{\mathscr{T},\ell}(s)$ be the unique immediate successor of $s$ in $\mathscr{T}$ such that $(s, t) \in R^{\mathscr{T}}_\ell$.

5) We say $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$ are *neighbors* <u>when</u> they are equal except that for each $t \in \mathscr{T}_1$ we can change the order $<^*_{\mathscr{T}_1} \restriction (t/E_{\mathscr{T}_1})$ to $<^*_{\mathscr{T}_2} (t/E_{\mathscr{T}_2})$.

**Definition 1.2.** 1) We say $f$ is a $\subseteq$-embedding of $\mathscr{T}_1 \in \mathbf{T}$ into $\mathscr{T}_2 \in \mathbf{T}$ <u>when</u>: $f$ is an isomorphism from $\mathscr{T}_1$ onto $\mathscr{T}_1'$ where $\mathscr{T}_1' \subseteq \mathscr{T}_2$.

2) For any ordinal $\alpha$ (limit, if not said otherwise) and sequence $\bar{<} = \langle <_\beta : \beta < \alpha \rangle$, with $<_\beta$ a well ordering of ${}^\beta 2$ we define $\mathscr{T} = \mathscr{T}_{\alpha,\bar{<}}$ as follows (omitting $\bar{<}$ means "for some"):

(a) universe ${}^{\alpha>}2$,
(b) $<_{\mathscr{T}}$ is $\triangleleft \restriction {}^{\alpha>}2$,
(c) $E_{\mathscr{T}} := \{(\eta, \nu) : \eta, \nu \in {}^\beta 2$ for some $\beta < \alpha\}$,
(d) $<^*_{\mathscr{T}} := \{(\eta, \nu) : \eta, \nu \in {}^{\alpha>}2$ and $\ell g(\eta) < \ell g(\nu)$ or $(\exists \beta < \alpha)(\ell g(\eta) = \beta = \ell g(\nu) \wedge \eta <_\beta \nu)\}$,
(e) $R_\ell := \{\eta, \nu) : \eta\char`^\langle \ell \rangle \trianglelefteq \nu \in \mathscr{T}\}$,
(f) $\eta \cap_{\mathscr{T}} \nu := \eta \cap \nu$.

3) For $\mathscr{T} \in \mathbf{T}$ and $\zeta < \mathrm{ht}(\mathscr{T})$, let $<_{\mathscr{T},\zeta}$ be $<^*_{\mathscr{T}} \restriction \mathscr{T}_{[\zeta]}$.

**Claim 1.3.** *If $\theta = \sup\{(2^{|\alpha|})^+ : \alpha < \kappa\}$ and $\bar{<} = \langle <_\beta : \beta < \kappa \rangle$ as above, <u>then</u> $\mathscr{T}_{\kappa,\bar{<}}$ is well defined and belongs to $\mathbf{T}_{\theta,\kappa}$.*

*Proof.* It is clear. $\square_{1.3}$

**Definition 1.4.** 1) For $\mathscr{T} \in \mathbf{T}$ let $\mathrm{eseq}_n(\mathscr{T})$ be the set of sequences $\bar{a}$ such that:

(a) $\bar{a}$ is an $<^*_{\mathscr{T}}$-increasing sequence of length $n$ of members of $\mathscr{T}$,
(b) $k < \ell < n \Rightarrow a_k \cap a_\ell \in \{a_m : m < n\}$,
(c) $k, \ell < n \wedge \mathrm{lev}(a_k) \le \mathrm{lev}(a_\ell) \Rightarrow a_\ell \restriction \mathrm{lev}(a_k) \in \{a_m : m < n\}$,

1A) For $\mathscr{U} \subseteq \mathscr{T}$ we let $\mathrm{eseq}_n(\mathscr{U}, \mathscr{T})$ be $\mathrm{eseq}_n(\mathscr{T}) \cap ({}^n\mathscr{U})$, similarly for part (2).

2) Let $\mathrm{eseq}(\mathscr{T}) = \mathrm{eseq}_{<\omega}(\mathscr{T}) = \cup\{\mathrm{eseq}_n(\mathscr{T}) : n < \omega\}$.

2A) For finite $A \subseteq \mathscr{T}$ we define the sequence $\bar{b} = \mathrm{cl}(A) = \mathrm{cl}_{\mathscr{T}}(A) = \mathrm{cl}(A, \mathscr{T})$ as the unique $\bar{b}$ such that:

(a) $\bar{b} \in \mathrm{eseq}(\mathscr{T})$,
(b) $A \subseteq \mathrm{Rang}(\bar{b})$,
(c) $\mathrm{Range}(\bar{b})$ is minimal under those restrictions.

Also let $\mathrm{pos}(A) = \mathrm{pos}(A, \mathscr{T})$ be the unique function $h$ from $A$ into $\lg(\bar{b})$ such that for every $a \in A$ we have: $i = h(a)$ <u>iff</u> $b_i = a$.

2B) We may replace above $A$ by a finite sequence $\bar{a}$, and let $\mathrm{pos}_{\mathscr{T}}(\bar{a})$ be $\mathrm{pos}_{\mathscr{T}}(\mathrm{rang}(\bar{a}))$.

3) We say $\bar{a}, \bar{b} \in \mathrm{eseq}(\mathscr{T})$ are $\mathscr{T}$-similar or $\bar{a} \sim_{\mathscr{T}} \bar{b}$ <u>when</u> for some $n$ we have:

(a) $\bar{a}, \bar{b} \in \mathrm{eseq}_n(\mathscr{T})$,
(b) for any $k, i, m < n$ we have:

- $\bullet_1$ $a_k \le_{\mathscr{T}} a_i$ iff $b_k \le_{\mathscr{T}} b_i$,
- $\bullet_2$ $(a_k, a_i) \in R_\ell^{\mathscr{T}}$ iff $(b_k, b_i) \in R_\ell^{\mathscr{T}}$ for $\ell = 0, 1$,
- $\bullet_3$ $a_k \cap_{\mathscr{T}} a_\ell = a_m$ iff $b_k \cap_{\mathscr{T}} b_\ell = b_m$, actually follows,
- $\bullet_4$ $a_k = a_\ell \restriction \mathrm{lev}(a_m)$ iff $b_k = b_\ell \restriction \mathrm{lev}(b_m)$, actually follows,
- $\bullet_5$ $(a_k \cap a_m) R_{\mathscr{T},\ell} a_i$ iff $(b_k \cap b_m) R_{\mathscr{T},\ell} b_i$ for $\ell = 0, 1$; actually follows,
- $\bullet_6$ $\mathrm{lev}_{\mathscr{T}}(a_k) \le \mathrm{lev}_{\mathscr{T}}(a_\ell)$ iff $\mathrm{lev}_{\mathscr{T}}(b_k) \le \mathrm{lev}_{\mathscr{T}}(b_\ell)$; actually follows.

3A) We say that $\bar{a}, \bar{b} \in {}^n\mathscr{T}$ are *$\mathscr{T}$-similar* when $\bar{a}' = \mathrm{cl}(\bar{a}), \bar{b}' = \mathrm{cl}(\bar{b})$ are $\mathscr{T}$-similar and $a'_\ell = a_k \Leftrightarrow b'_\ell = b_k$ for any $\ell < \lg(\bar{a}'), k < n$.

4) For $\bar{a} \in {}^n\mathscr{T}$ let $\mathrm{Lev}(\bar{a})$ be the set $\{\mathrm{lev}_{\mathscr{T}}(a_\ell) \colon \ell < \lg(\bar{a})\}$.

5) We say that $\mathscr{T} \in \mathbf{T}$ is weakly $\aleph_0$-saturated when:

(∗) for every $\varepsilon < \mathrm{ht}(\mathscr{T})$ and $s_0, \dots, s_{n-1}$ from $\mathscr{T}_{[\varepsilon]}$, there are $\zeta \in (\varepsilon, \mathrm{ht}(\mathscr{T})$ and $t_0 <^*_{\mathscr{T}} \dots <^*_{\mathscr{T}} t_{n-1}$ from $\mathscr{T}_{[\zeta]}$ satisfying $k < n \Rightarrow s_k <_{\mathscr{T}} t_k$,

6) For $\mathscr{T} \in \mathbf{T}$ let:

(a) $\mathrm{incr}_n(\mathscr{T})$ be the set of $<^*_{\mathscr{T}}$-increasing $\bar{a} \in {}^n\mathscr{T}$ and let $\mathrm{seq}_n(\mathscr{T}) = {}^n\mathscr{T}$ that is the set sequences of length $n$ from $\mathscr{T}$,
(b) $\mathrm{incr}(\mathscr{T}) = \cup\{\mathrm{incr}_n(\mathscr{T}) \colon n < \omega\}$ and $\mathrm{seq}(\mathscr{T}) = \cup\{\mathrm{seq}_n(\mathscr{T}) \colon n < \omega\}$.

7) For $\mathscr{T} \in \mathbf{T}$:

(a) for $\bar{t} \in \mathrm{incr}(\mathscr{T})$ or just $\bar{t} \in {}^n\mathscr{T}$ let $\mathrm{sim}-\mathrm{tp}(\bar{t}, \mathscr{T})$ be the pair (the similarity type of $\mathrm{cl}(\bar{t}, \mathscr{T}), \mathrm{pos}(\bar{t}, \mathscr{T})$), that is all the information from part (3) (of 1.4) (including pos),
(b) if in addition, $\mathscr{U} \subseteq \mathscr{T}$ then we let $\mathrm{sim}-\mathrm{tp}(\bar{t}, \mathscr{U}, \mathscr{T})$ be the function mapping $\bar{s} \in {}^{\omega>}\mathscr{U}$ to $\mathrm{sim}-\mathrm{tp}(\bar{t}\hat{\ }\bar{s}, \mathscr{T})$.

8) Let $\mathbb{S}^n$ be the set of similarity types of sequences of length $n$ in some $\mathscr{T} \in \mathbf{T}$, so the sequences are not necessarily increasing.

9) Naturally $\mathbb{S} = \cup\{\mathbb{S}^n \colon n < \omega\}$.

Now comes the main property.

**Definition 1.5.** 1) For $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$ and $n < \omega$ and a cardinal $\sigma$ let $\mathscr{T}_1 \to (\mathscr{T}_2)^n_\sigma$ mean:

(∗) if $\mathbf{c} \colon \mathrm{eseq}_n(\mathscr{T}_1) \to \sigma$, then there is a $\subseteq$-embedding $g$ of $\mathscr{T}_2$ into $\mathscr{T}_1$ such that the colouring $\mathbf{c} \circ g$ is homogeneous for $\mathscr{T}_2$, which means:
- if $\bar{a}, \bar{b} \in \mathrm{eseq}_n(\mathscr{T}_2)$ are $\mathscr{T}_2$-similar, then $\mathbf{c}(g(\bar{a})) = \mathbf{c}(g(\bar{b}))$.

2) For $\mathscr{T}_1, \mathscr{T}_2 \in \mathbf{T}$, $k < \omega$ and $\sigma$, let $\mathscr{T}_1 \to (\mathscr{T}_2)^{\mathrm{end}(k)}_\sigma$ mean that:

(∗) if $\mathbf{c} \colon \mathrm{eseq}(\mathscr{T}_1) \to \sigma$ then there is an embedding $g$ of $\mathscr{T}_2$ into $\mathscr{T}_1$ such that the colouring $\mathbf{c}' = \mathbf{c} \circ g$ (see below) satisfies $\mathbf{c}'(\bar{\eta})$ does not depend on the last $k$ levels, that is:
- $\bullet_1$ the meaning of $\mathbf{c}' = \mathbf{c} \circ g$ is that for every $s_0, \dots, s_{m-1} \in \mathscr{T}_2$ we have $\mathbf{c}'(\langle s_0, \dots, s_{m-1}\rangle) = \mathbf{c}(\langle g(s_0), \dots, g(s_{m-1})\rangle)$,
- $\bullet_2$ if $n < \omega$ and $\bar{a}, \bar{b} \in \mathrm{eseq}_n(\mathscr{T}_2)$ are $\mathscr{T}_2$-similar and
$$\ell < n \wedge (k \le |\mathrm{Lev}(\bar{a}) \setminus \mathrm{lev}(a_\ell)|) \Rightarrow b_\ell = a_\ell,$$
then $\mathbf{c}'(\bar{a}) = \mathbf{c}'(\bar{b})$.

3) Let $\mathscr{T}_1 \to (\mathscr{T}_2)^{\text{end}(k,m)}_\sigma$ is defined as in part (2), but we restrict in $\bullet_2$ demanding that $\bar a, \bar b$ satisfy $m \ge |\{\ell < n \colon k \le |\text{Lev}(\bar a) \setminus \text{lev}(a_\ell)|\}|$.

4) We define $\mathscr{T}_1 \to' (\mathscr{T}_2)^n_\sigma$ as in part (1), but $\bar a, \bar b \in \text{seq}_n(\mathscr{T}_2)$. We define similarly $\mathscr{T}_1 \to (\mathscr{T}_2)^{\le n}$ and $\mathscr{T}_1 \to' (\mathscr{T}_2)^{\le n}$.

We may mention some implications among the $\to$,

*Remark* 1.6. Of course, the equality $\mathbf{c}(g(\bar a)) = \mathbf{c}(g(\bar b))$ is required only if $\bar a$ and $\bar b$ are $\mathscr{T}_2$-similar since this is the best possible homogeneity, as one can define a coloring according to similarity types.

**Claim 1.7.** *Let* $\mathscr{T} \in \mathbf{T}$.

*1) If $A \subseteq \mathscr{T}$ is finite non-empty with $m$ elements* *<u>then</u>:*

(a) *For some* $n \le (2m-1)m^2$ *and* $\bar a \in \text{eseq}_n(\mathscr{T})$ *we have* $A \subseteq \text{Rang}(\bar a)$; *moreover* $\max\{\text{lev}_{\mathscr{T}}(a) \colon a \in A\} = \max\{\text{lev}_{\mathscr{T}}(a_\ell) \colon \ell < n\}$; *in fact* $\bar a = \text{cl}_{\mathscr{T}}(A)$,

(b) *If* $\mathscr{T} \in \mathbf{T}$ *and finite* $A \subseteq \mathscr{T}$ *then* $\text{cl}(A, \mathscr{T}), \text{pos}(A, \mathscr{T})$ *are well defined.*

*2) The number of quantifier free complete $n$-types realized in some $\mathscr{T} \in \mathbf{T}$ by some $\bar a \in \text{eseq}_n(\mathscr{T})$ is, e.g. $\le 2^{2n^2+n}$ but $\ge n$.*

*3) If $\mathscr{T} \in \mathbf{T}$ is weakly $\aleph_0$-saturated* *<u>then</u>* *$\mathscr{T}$ realizes all possible such types, i.e. each type realized in some $\mathscr{T}' \in \mathbf{T}$; here "$\text{ht}(\mathscr{T})$ is a limit ordinal" follows.*

*4) Assume $\mathscr{T} \in \mathbf{T}$ and $\mathscr{U}$ is a subset of $\mathscr{T}$ closed under $<_{\mathscr{T}}$, (that is $s <_{\mathscr{T}} t \in \mathscr{U} \Rightarrow s \in \mathscr{U}$). Let $\text{lev}(\mathscr{U}, \mathscr{T}) = \sup\{\text{lev}(s, \mathscr{T}) + 1 \colon s \in \mathscr{U}\}$.*

*If $\text{lev}(\mathscr{U}, \mathscr{T}) < \text{lev}(t, \mathscr{T})$ and $A \subseteq \mathscr{U}$ is finite* *<u>then</u>* *$\bar b = \text{cl}(A \cup \{t\})$ has the form $\bar c\hat{\ }\langle t\rangle$ with $\bar c \in \text{eseq}(\mathscr{T}) \cap {}^{\omega>}\mathscr{U}$.*

*Proof.* Clearly (4) holds and we shall use it freely.

1) Let $B_1 = \{\eta \cap_{\mathscr{T}} \nu \colon \eta, \nu \in A\}$ and note that $\eta \in A \Rightarrow \eta = \eta \cap \eta \in B_1$. Now by induction on $|A|$ easily $|B_1| \le 2m - 1$. Let $B_2 := \{\eta \restriction \text{lev}_{\mathscr{T}}(\nu) \colon \eta \in A, \nu \in B_1$ and $\text{lev}_{\mathscr{T}}(\eta) \ge \text{lev}_{\mathscr{T}}(\nu)\}$.

Easily $B_2 = \text{cl}(A, \mathscr{T})$, also $|B_2| \le m|B_1| = m(2m-1)$.

We may improve the bound but this does not matter here; similarly below.

2) Considering the class of such pairs $(\bar a, \mathscr{T})$ (fixing $n$) the number of $E_{\bar a} = \{(k,i) : a_k E_{\mathscr{T}} a_i\}$ is $\le 2^{n^2}$ and the number of $<_{\bar a} = \{(k,i) : a_k <_{\mathscr{T}} a_i\}$ is $\le 2^{n^2}$ and the number of $\{(a_k, a_i) : (a_k, a_i) \in R_1^{\mathscr{T}}$ and for no $j, a_k <_{\mathscr{T}} a_j <_{\mathscr{T}} a_i\}$ is $\le 2^n$.

Lastly, from those we can compute $\{(a_k, a_i) : (a_k, a_i) \in R_0^{\mathscr{T}}\}$ as $\{(a_k, a_i) \colon (a_k \cap_{\mathscr{T}} a_i = a_k) \wedge ((a_k, a_i) \notin R_1) \wedge a_k \ne a_i\}$, so together the number is $\le 2^{2n^2+n}$.

Clearly we can get a better bound, e.g. letting $m^\bullet_n(\mathscr{T}) = |\{\text{tp}_{\text{qf}}(\bar a \restriction n, \emptyset, \mathscr{T}) : \bar a \in \text{eseq}(\mathscr{T})$ has length $\ge n\}|$ then:

$(*)_1$
- $\bullet_1$ $m^\bullet_n(\mathscr{T}) = 1$ for $n = 0, 1$,
- $\bullet_2$ $m^\bullet_{n+1}(\mathscr{T}) \le 4n(m^\bullet_n(\mathscr{T}))$,
- $\bullet_3$ hence $m^\bullet_n(\mathscr{T}) \le 4^{n-1}(n-1)!$.

[Why? e.g. for $\bullet_2$ notice that $\text{tp}_{\text{qf}}(\bar a \restriction (n+1), \emptyset, \mathscr{T})$ is determined by $q = \text{tp}_{\text{qf}}(\bar a \restriction n, \emptyset, \mathscr{T})$ and the unique triple $(m, \iota, \ell) \in n \times 2 \times 2$ such that:

$(*)_{1.1}$ (a) $m < n$ is such that $\text{lev}(a_m \cap a_n)$ is maximal, hence $a_m <_{\mathscr{T}} a_n$,

(b) $a_m R_\iota a_n$,
(c) $\ell$ iff $\text{Lev}_{\mathscr{T}}(a_n) > \text{Lev}_{\mathscr{T}}(a_{n-1})$.

As there are $\leq 4n$ possibilities we are done].

It suffices to consider the case $\mathscr{T}$ is weakly $\aleph_0$-saturated (see 1.4(5), 1.7(3)) and then we can get exact values.

Now for $n \geq k \geq 1$ let,

$m^*_{n,k}(\mathscr{T}) := |\{\text{tp}_{\text{qf}}(\bar{a}, \emptyset, \mathscr{T}) \colon \bar{a} \in \text{eseq}_n(T) \text{ such that } |\{\ell \colon \text{lev}(a_\ell) = \max(\text{Lev}(\bar{a}))\}| = k\}|$.

So,

- $m^*_{1,1}(\mathscr{T}) = 1$, $m^*_{1,0}(\mathscr{T}) = 0$ and stipulate $m^*_{0,k}(\mathscr{T}) = 0$,
- if $n = k \geq 1$, then $m^*_{n,k}(\mathscr{T}) = 1$,
- if $2k - 1 > n \geq k \geq 1$, then $m^*_{n,k}(\mathscr{T}) = 0$,
- if $n \geq 1$, then $m^*_{n+1,1}(\mathscr{T}) = \Sigma\{2k \cdot m^*_{n,k}(\mathscr{T}) \colon k \in [1, n]\}$,

and more generally,

- if $n > k \geq 1$, then

$$m^*_{n+k,k} = \sum\left\{\ell! \cdot \binom{\ell}{\ell_1} \cdot \binom{\ell - \ell_1}{\ell_2} \cdot 2^\ell \cdot m^*_{n,\ell}(\mathscr{T}) \colon \ell, \ell_0, \ell_1, \ell_2 \in [0, n),\ \ell = \ell_0 + \ell_1 + \ell_2\right\}.$$

[Why? Considering $p = \text{tp}_{\text{qf}}(\bar{a}{\restriction}(n+k), \emptyset, \mathscr{T})$ we fix $q = \text{tp}(\bar{a}{\restriction}n, \emptyset, \mathscr{T})$, let $\ell$ be maximal such that $n - \ell \leq i < n \Rightarrow \text{lev}(a_i) = \text{lev}(a_{n-1})$ (equivalently $\text{lev}(a_{n-\ell})) = \text{lev}(a_{n-1})$). For $\iota = 0, 1, 2$, let $S_\iota = \{m \colon n - \ell \leq m < n \text{ and } \iota = |\{j < k \colon a_m <_{\mathscr{T}} a_j\}|\}$, so $(S_0, S_1, S_2)$ is a partition of $[n - \ell, n)$. Let $S^\bullet_1 = \{m \in S_1 \colon \text{ if } j < k \text{ then } a_m R_1 a_s\}$. Fixing $\ell$ the number of possibles $q$'s in $m^*_{n,\ell}(\mathscr{T})$ and fixing $q$ (and so $\ell$) the freedom left is choosing $\ell_0, \ell_1, \ell_2 \geq 0$ such that $\ell_1 + 2\ell_2 = k$ and then choosing the partition $(S_0, S_1, S_2)$ which have $\binom{\ell}{\ell_1}\binom{\ell - \ell_2}{\ell_2}$ possibilities we have $2^{\ell_1}$ possible choices of $S'_1$ and lastly $k$ possible linear orders of $\{a_i \colon i \in [n, n+k)\}$ clearly we are done.]

3), 4), Clear. $\square_{1.7}$

**Claim 1.8.** *Let*[2] $\sigma \geq \aleph_0$

*1) If $\mathscr{T} \to (\mathscr{T})^{\text{end}(1)}_\sigma$, then $\mathscr{T} \to (\mathscr{T})^{\text{end}(k)}_\sigma$ for every $k < \omega$.*

*2) If $\mathscr{T} \in \mathbf{T}$ and $\mathscr{T} \to (\mathscr{T})^{\text{end}(1)}_\sigma$, then $\mathscr{T} \to (\mathscr{T})^n_\sigma$ for every $n < \omega$.*

*3) If $k \geq 1$ and $\mathscr{T}_\ell \in \mathbf{T}$ for $\ell = 0, \ldots, k$ and $\mathscr{T}_{\ell+1} \to (\mathscr{T}_\ell)^{\text{end}(1,m)}_\sigma$ for $\ell < k$, then $\mathscr{T}_k \to (\mathscr{T}_0)^{\text{end}(k,m)}_\sigma$, hence $\mathscr{T}_k \to (\mathscr{T}_0)^{\leq m}_\sigma$.*

*Proof.* 1.8 Clear. $\square$

## § 1(B). **Forcing in** ZFC.

*Remark* 1.9. Concerning the choice of $m(*)$ in 1.10 below (given $\mathbf{m}$), its value is immaterial for the model theoretic results.

In [She92, §4] we use Erdös-Radó theorem for $m = 2n$, but what we really need in essence is the canonical version, see [EHMR84], [DH17] and references there.

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[2] If $\sigma < \aleph_0$ we have parallel results depending decreasing $\sigma$ in the conclusion on the bounds from a4, that is, for part (2): if $\mathscr{T} \to (\mathscr{T})^{\text{end}(1,m(1))}_{\sigma(1)}$, then $\mathscr{T} \to (\mathscr{T})^{\text{end}(n,m(2))}_{\sigma(2)}$.

**Claim 1.10.** *In $\mathbf{V}^{\mathbb{P}}$ we have $\mathscr{T}_2 \to (\mathscr{T}_1)^{\mathrm{end}(1,\mathrm{m})}_\sigma$ <u>when</u> for a suitable $m(*)$:*

(a) $\kappa = \kappa^{<\kappa}, \lambda \to (\kappa_1)^{m(*)}_\sigma, \sigma = 2^\kappa$ *and* $\kappa_1 \to (\kappa^+)^m_\kappa$,
(b) $\mathbb{P} = \mathrm{Cohen}(\kappa, \lambda)$, *noting* $\delta*$ *has cofinality* $\kappa$ *because* $u_* = \mathscr{B} \cap \lambda$, $[\mathscr{B}]^{<\kappa} \subseteq \mathscr{B}$ *and* $\|\mathscr{B}\| = \kappa$,
(c) $\mathscr{T}_2 \in \mathbf{T}$ *expands* $({}^{\kappa(+)>}2, \lhd)$ *in* $\mathbf{V}^{\mathbb{P}}$,
(d) *In* $\mathbf{V}^{\mathbb{P}}$, $\mathscr{T}_1 \in \mathbf{T}_{\kappa,\kappa}$ *and* $\mathscr{T}_1 \subseteq \mathscr{T}_1^+$, *where* $\mathscr{T}_1^+$ *expands* $({}^{\kappa>}2, \lhd)$ *and so* $\mathrm{otp}(\mathscr{T}_{1,[\alpha]}, <_{\mathscr{T}_{1,\alpha}}) < \kappa$ *for* $\alpha < \kappa$, *see 1.2(3),*
(e) *let* $\mathfrak{A}_1 = \mathfrak{A}_0[\mathbf{G}_{u_*}]$ *and* $\mathfrak{A}_2 = \mathfrak{A}_0[\mathbf{G}]$.

*Proof.* First,

$\boxdot_1$ Without loss of generality, $\mathscr{T}_1 \in \mathbf{V}$ is an object (not just a $\mathbb{P}$-name).

[Why? Let $\underset{\sim}{\mathscr{T}}_1$ be a $\mathbb{P}$-name, then for some $u \in [\lambda]^{\le\kappa}$ we have $\underset{\sim}{\mathscr{T}}_1$ is a $\mathbb{P}_u$-name. We can force by $\mathbb{P}_u$, so as $\mathbb{P}/\mathbb{P}_u = \mathrm{Cohen}(\kappa, \lambda)$, we are done.]

So $\kappa, \mathscr{T}_1, \underset{\sim}{\mathscr{T}}_2$ are well defined ($\underset{\sim}{\mathscr{T}}_2$ a $\mathbb{P}$-name). Let $\underset{\sim}{\mathbf{c}}$ be a $\mathbb{P}$-name, $\underset{\sim}{\mathbf{c}}\colon \mathrm{eseq}(\underset{\sim}{\mathscr{T}}_2) \to \sigma$, without loss of generality be such that:

$\boxdot_2$ if $\bar{t} \in \mathrm{eseq}(\mathscr{T}_2)$, then from $\underset{\sim}{\mathbf{c}}(\bar{t})$ we can compute the similarity type of $\bar{t}$ in $\mathscr{T}_2$ and of $\underset{\sim}{\mathbf{c}}(\bar{t}{\restriction}u)$ when $u \subseteq \mathrm{dom}(\bar{t}), \bar{t}{\restriction}u \in \mathrm{eseq}(\underset{\sim}{\mathscr{T}}_2)$.

Let $\underset{\sim}{\bar{\eta}} = \langle \underset{\sim}{\eta}_\alpha \colon \alpha < \lambda\rangle$ be the generic of $\mathbb{P}$, so $\Vdash$"$\underset{\sim}{\eta}_\alpha \in {}^\kappa 2$" and let $\underset{\sim}{\bar{\eta}}_u = \langle \underset{\sim}{\eta}_\alpha \colon \alpha \in u\rangle$ for $u \subseteq \lambda$.

Next, in $\mathbf{V}$, we choose:

$(*)_1$ (a) let $\chi > \lambda$ and $<^*_\chi$ a well ordering of $\mathscr{H}(\chi)$,
(b) let $\mathfrak{B} \prec \mathfrak{A}_0 = (\mathscr{H}(\chi), \in, <^*_\chi)$ be of cardinality $\kappa$ such that $[\mathfrak{B}]^{<\kappa} \subseteq \mathfrak{B}$ and $\lambda, \kappa, \mu, \sigma, \mathscr{T}_1, \underset{\sim}{\mathscr{T}}_2, \underset{\sim}{\mathbf{c}} \in \mathfrak{B}$,
(c) let $u_* = \mathfrak{B} \cap \lambda \in [\lambda]^\kappa$,
(d) let $\mathbf{G}_{u_*} \subseteq \mathbb{P}_{u_*}$ be generic over $\mathbf{V}_0 = \mathbf{V}$, $\mathbf{G} \subseteq \mathbb{P}$ be generic over $\mathbf{V}_0$ such that $\mathbf{G}_{u_*} \subseteq \mathbf{G}$,
(e) let $\bar{\eta}_u = \langle \underset{\sim}{\eta}_\alpha[\mathbf{G}_u] \colon \alpha \in u\rangle$, for $u \subseteq \lambda$,
(f) let $\mathbf{V}_1 = \mathbf{V}_0[\mathbf{G}_{u_*}] = \mathbf{V}_0[\bar{\eta}_{u_*}]$,
(g) let $\mathbf{V}_2 = \mathbf{V}_0[\bar{\eta}_\lambda] = \mathbf{V}_1[\bar{\eta}_{\lambda\setminus u_*}]$.

$(*)_2$ (a) let $\underset{\sim}{\mathscr{T}}_0$ be the sub-structure of $\underset{\sim}{\mathscr{T}}_2$ with set of elements $\{\underset{\sim}{\eta}\colon \underset{\sim}{\eta}$ is a canonical $\mathbb{P}$-name of a member of $\underset{\sim}{\mathscr{T}}_2$ and this name belongs to $\mathfrak{B}\}$,
(b) let $\delta_* = \delta(*)$ be $\min(\lambda \setminus u_*) = \min(\kappa^+ \setminus u_*) = \kappa^+ \cap u_*$,
(c) let $\langle \delta_\varepsilon \colon \varepsilon < \kappa\rangle$ be increasing continuous with limit $\delta_*$ in $\mathbf{V}_0$,
(d) $\mathfrak{B}_2 = \mathfrak{B}[\mathbf{G}_\lambda]$, $\mathfrak{B}_1 = \mathfrak{B}_2{\restriction}\{\underset{\sim}{\tau}[\mathbf{G}_\lambda]\colon \underset{\sim}{\tau}$ is a $\mathbb{P}_u$-name from $\mathfrak{B}$ for some $u \in [\lambda]^{\le\kappa} \cap \mathfrak{B}\}$, $\mathfrak{B}_0 = \mathfrak{B}$, so $\mathfrak{B}_2 \prec \mathfrak{A}_2 = \mathscr{H}(\chi)[\mathbf{G}_\lambda]$ and $\mathfrak{B}_1 \cap {}^{\kappa\ge}\lambda = \mathfrak{B}_2 \cap {}^{\kappa\ge}\lambda$.

Clearly,

$(*)_3$ (a) $\Vdash$"$\underset{\sim}{\mathscr{T}}_0 \subseteq \underset{\sim}{\mathscr{T}}_2$ is closed under initial segments, is of cardinality $\kappa$ and has $\delta_*$ levels and is closed under unions of increasing chains of length $< \kappa$ and $\nu \in \underset{\sim}{\mathscr{T}}_0 \Rightarrow \nu^\frown\langle 0\rangle, \nu^\frown\langle 1\rangle \in \underset{\sim}{\mathscr{T}}_0$, and $\alpha < \delta_* \Rightarrow (\forall \nu \in \mathscr{T}_0)(\exists \rho)[\nu \lhd \rho \in \mathscr{T}_0 \wedge \lg(\rho) \ge \alpha]$, so $\underset{\sim}{\mathscr{T}}_0 \in \mathbf{T}$",
(b) $\underset{\sim}{\mathscr{T}}_0$ is actually a $\mathbb{P}_{u_*}$-name and we can use $\delta_* = \mathfrak{B} \cap \kappa^+$ as its set of levels.

$(*)_4$ (a) let $\mathscr{T}_0 = \underset{\sim}{\mathscr{T}}_0[\mathbf{G}_{u_*}]$, $\mathbf{c}_0 = \mathbf{c}{\restriction}\mathrm{eseq}(\mathscr{T}_0)$ so they are from $\mathbf{V}_1$,
(b) let $\mathbb{P}_* = \mathbb{P}/\mathbf{G}_{u_*} = \mathbb{P}_{\lambda\setminus u_*}$.

$(*)_5$ (a) for each $\alpha \in \lambda \setminus u_*$, in $\mathbf{V}_1[\eta_\alpha]$ there is $\eta^\bullet_\alpha \in \lim_{\delta_*}(\mathscr{T}_0)^{\mathbf{V}_1[\eta_\alpha]}$ with $\lg(\eta^\bullet_\alpha) = \delta_*$ such that $\varepsilon < \delta_* \Rightarrow \eta^\bullet_\alpha{\restriction}\varepsilon \in \mathscr{T}_0$ and $\eta^\bullet_\alpha$ is a generic $\delta_*$-branch of $\mathscr{T}_0$ over $\mathbf{V}_1$,
(b) Clearly $\lim_{\delta_*}(\mathscr{T}_0)^{\mathbf{V}_1[\eta_\alpha]} \subseteq \mathscr{T}_2[G_\lambda]$.

We shall work in $\mathbf{V}_1$.

$(*)_6$ (in $\mathbf{V}_1$) for $\alpha \in \lambda \setminus u_*$, let $\underset{\sim}{\eta}^\bullet_\alpha$ be a $\mathbb{P}_{\{\alpha\}}$-name of $\eta^\bullet_\alpha$, so without loss of generality for some $\kappa$-Borel function $\mathbf{B}\colon {}^\kappa 2 \to {}^{\delta(*)}2$, from $\mathbf{V}_1$, we have $\Vdash$"$\underset{\sim}{\eta}^\bullet_\alpha = \mathbf{B}(\underset{\sim}{\eta}_\alpha)$ is as above in $(*)_5$(a)". [Saharon 2022-11-06, recheck]

$(*)_7$ (a) Recall we had in $(*)_2$(c) fix an increasing sequence $\langle \delta(\varepsilon) \coloneqq \delta_\varepsilon \colon \varepsilon < \kappa\rangle$ so that $\delta(*) = \bigcup_{\varepsilon<\kappa} \delta(\varepsilon)$
(b) Let $\langle \mathbf{B}_\varepsilon \colon \varepsilon < \kappa\rangle$ be such that $\mathbf{B}_\varepsilon$ is a $\kappa$-Borel function from ${}^\kappa 2$ to ${}^{\delta(\varepsilon)}2$ from $\mathbf{V}_1$ and $\Vdash$"$\underset{\sim}{\eta}^\bullet_\alpha{\restriction}\delta_\varepsilon = \mathbf{B}_\varepsilon(\underset{\sim}{\eta}_\alpha)$",
(c) Let $\underset{\sim}{\mathbf{B}}, \underset{\sim}{\mathbf{B}}_\varepsilon$ be $\mathbb{P}_{\underset{\sim}{u}_*}$-names forced to be as above can be considered as $\mathbb{P}_{u_*}$-name.

Without loss of generality (using the freedom in choosing $\mathfrak{B}$).

$(*)_8$ If $\varepsilon < \kappa$, $\nu \in {}^\kappa 2, \mathbf{B}_\varepsilon(\nu) = \rho$, then for every large enough $\zeta < \kappa$, we have $\nu{\restriction}\zeta \lhd \varrho \in {}^\kappa 2 \Rightarrow \rho \lhd \mathbf{B}(\varrho)$ and we write $\mathbf{B}_\varepsilon(\nu{\restriction}\zeta) = \rho$.

Recalling we work in $\mathbf{V}_1$ there are $\mathscr{U}, \bar{N}$ such that:

$(*)_9$ (a) $\mathscr{U} \subseteq \lambda \setminus u_*$,
(b) $\mathrm{otp}(\mathscr{U})$ satisfies $\mathrm{otp}(\mathscr{U}) \to (\kappa^+)^m_\kappa$
(c) $\bar{N} = \langle N_u \colon u \in [\mathscr{U}]^{\le m}\rangle$,
(d) $N_u \cap N_v \subseteq N_{u\cap v}$,
(e) $\kappa, \mathscr{T}_1, \underset{\sim}{\mathscr{T}}_2, \underset{\sim}{\mathbf{c}}, \mathscr{B}, \mathscr{B}_1 \in N_u \prec \mathfrak{A}_1, \|N_u\| = \kappa, [N_u]^{<\kappa} \subseteq N_u$,
(f) if $u, v \in [\mathscr{U}]^{\le m}$ and $|u| = |v|$, then there is a unique isomorphism $g_{u,v}$ from $N_v$ onto $N_u$ which is the identity on $(\kappa+1) \cup \{\mathbf{c}, \underset{\sim}{\eta}, \mathbf{B}\}$ hence on $\mathscr{T}_2, \langle \underset{\sim}{\eta}_\alpha, \underset{\sim}{\eta}^\bullet_\alpha \colon \alpha < \lambda \setminus u_*\rangle$ and maps $v$ onto $u$,
(g) $\langle \underset{\sim}{\eta}^\bullet_\alpha \colon \alpha \in \mathscr{U}\rangle$ is $<^*_{\underset{\sim}{\mathscr{T}}_2}$-increasing.

[Why? As in [She89], if $m(*) = 2n$; for smaller $m(*)$ this is close to canonical partitions, see [EHMR84]; [DH17] and references there; for clause (g) recall that $<^*_{\underset{\sim}{\mathscr{T}}_2}$ is a linear well ordering.] [2222-11-06, Saharon elaborate +, see $\boxplus^0_{\mathbf{a}}$.]

$(*)_{10}$ notation:
(a) in $\mathbf{V}_2$ for $\bar{s} \in \mathrm{eseq}(\mathscr{T}_2)$, let $\varepsilon(\bar{s}) \coloneqq \max\{\mathrm{lev}_{\mathscr{T}_2}(s_i) \colon i < \lg(\bar{s})\}$, $w(\bar{s}) = \{i < \lg(\bar{s}) \colon \mathrm{lev}_{\mathscr{T}_2}(s_i) = \varepsilon(\bar{s})\}$ and $u(\bar{s}) = \{s_i \colon i \in w(\bar{s})\}$ and $v(\bar{s}) = \{s_i \colon i < \lg(\bar{s})$ and $i \notin u(\bar{s})\}$,
(b) for finite $u \subseteq \mathscr{T}_{0,[\varepsilon]}$ for some $\varepsilon < \delta_*$, let $\mathbf{H}_u \coloneqq \{h \colon h$ is a one-to-one function from $u$ into $\mathscr{U}\}$,
(c) If in $\mathbf{V}_2$, $\bar{s} \in \mathrm{eseq}(\mathscr{T}_0), u = u[\bar{s}] \in [\mathscr{T}_{0,[\varepsilon]}]^{\le n}$ and $h \in \mathbf{H}_u$, then we let $\bar{s}^{[h]}$ be the $\bar{t} \in \mathrm{eseq}(\mathscr{T}_2)$ such that $\lg(\bar{t}) = \lg(\bar{s})$, $i \in \lg(\bar{s}) \setminus u \Rightarrow t_i = s_i$ and $i \in w(\bar{s}) \Rightarrow t_i = s_i{}^\frown \eta^\bullet_{h(s_i)}{\restriction}[\varepsilon(\bar{s}), \delta_*)$.

We define $\mathrm{AP} \coloneqq \bigcup_{\varepsilon<\kappa} \mathrm{AP}_\varepsilon$, where $\mathrm{AP}_\varepsilon$ is the set of objects $\mathbf{a}$ which consists of (so $\varepsilon = \varepsilon_{\mathbf{a}}, \bar{\nu} = \bar{\nu}_{\mathbf{a}}$, etc).

$\boxplus^0_{\mathbf{a}}$ (a) $\varepsilon < \kappa$,
(b) $\bar{\eta} = \langle \eta_\rho : \rho \in \mathscr{T}_{1,[\varepsilon]} \rangle$ is with no repetitions,
(c) $\bar{\nu} = \langle \nu_\rho : \rho \in \mathscr{T}_{1,[\varepsilon]} \rangle$,
(d) $\eta_\rho \in \mathscr{T}_{0,[\delta(\zeta)]}$ and $\nu_\rho \in {}^{\kappa>}2$ for some $\zeta = \zeta_{\mathbf{a}} < \kappa$ but $\zeta < \varepsilon$ (for all $\rho \in \mathscr{T}_{1,[\varepsilon]}$),
(e) $\mathbf{B}^\bullet_\zeta(\nu_\rho) = \eta_\rho$,
(f) $\bar{p} = \langle p_{u,h} : u \in [{}^\varepsilon 2]^{\le n}, h \in \mathbf{H}_u \rangle$, where $p_{u,h} \in \mathbb{P}_{\lambda \setminus u_*}$,
(g) $p_{u,h} \in N_{h[u]}$ and $[\rho \in u \wedge h(\rho) = \alpha \Rightarrow p_{u,h}(\alpha) = \nu_\rho]$,
(h) if $h_1, h_2 \in \mathbf{H}_u$ and $h_1[u] = h_2[u]$ <u>then</u> $\alpha \in \mathrm{dom}(p_{u,h_1}) \setminus h_1[u] \Rightarrow p_{u,h_1}(\alpha) = p_{u,h_2}(\alpha)$,
(i) if $h_1, h_2 \in \mathbf{H}_u$ and $\rho_1, \rho_2 \in u \Rightarrow h_1[\rho_1] < h_1[\rho_2] \equiv h_2[\rho_1] < h_2[\rho_2]$, then $g_{h_2(u),h_1(u)}$ maps $p_{u,h_1}$ to $p_{u,h_2}$,
(j) $p_{u,h}$ forces a value to $\underset{\sim}{\mathbf{c}}(\bar{s}^{[h]})$ when $\bar{s} \in \mathrm{eseq}(\mathscr{T}_0)$ and $u = u[\bar{s}]$,
(k) if $u_1 \subseteq u_2 \in [{}^\varepsilon 2]^{\le n}, h_\ell \in \mathbf{H}_{u_\ell}, h_1 \subseteq h_2$, then $p_{u_1,h_1} \le_{\mathbb{P}_*} p_{u_2,h_2}$.

Let further.

$\boxplus^1_{\mathbf{a}}$ $\mathrm{AP}^+ = \bigcup \{ \mathrm{AP}^+_\varepsilon : \varepsilon < \kappa \}$, where $\mathrm{AP}^+_\varepsilon$ is the set of $\mathbf{a} \in \mathrm{AP}_\varepsilon$ such that:
(l) for some $\zeta \in [\varepsilon, \zeta_{\mathbf{a}})$ if $(\alpha)$ <u>then</u> $(\beta)$, where:
 $(\alpha)$ $\bar{s} \in \mathrm{eseq}(\mathscr{T}_1), u = u(\bar{s})$ and $h \in \mathbf{H}_u$,
 $(\beta)$ $\bullet_1$ for every $\rho, \varrho \in u[\bar{s}]$, we have that $\eta_\rho \le_{\mathscr{T}_{1,\zeta}} \eta_\varrho \Leftrightarrow p_{u,h} \Vdash \underset{\sim}{\eta}^\bullet_{h(\rho)} {\restriction} \zeta <_{\mathscr{T}_{2,\delta(*)}} \underset{\sim}{\eta}^\bullet_{h(\varrho)} {\restriction} \zeta$,
 $\bullet_1$ $p_{u,h} \Vdash \underset{\sim}{\mathbf{c}}(\bar{s}^{[h]}) = j$ <u>iff</u> $\mathbf{c}(\bar{t}) = j$, where $\bar{t} \in \mathrm{eseq}(\mathscr{T}_0), \lg(\bar{t}) = \lg(\bar{s}), i \in \lg(\bar{s}) \setminus u \Rightarrow t_i = s_i$ and $i \in u \Rightarrow t_i = \eta_{s_i} {\restriction} \zeta$ noting $i \in u \Rightarrow s_i \in \mathscr{T}_{1,[\varepsilon]}$.

$\boxplus_2$ we define the two-place relation $\le_{\mathrm{AP}}$ as follows: $\mathbf{a}_1 \le_{\mathrm{AP}} \mathbf{a}_2$ <u>iff</u>:
(a) $\mathbf{a}_1, \mathbf{a}_2 \in \mathrm{AP}$,
(b) $\varepsilon_1 = \varepsilon_{\mathbf{a}_1} \le \varepsilon_{\mathbf{a}_2} = \varepsilon_2$,
(c) if $\iota \in \{0,1\}$, $\rho_1 \in {}^{\varepsilon(1)}2, \rho_1^\frown \langle \iota \rangle \trianglelefteq \rho_2 \in {}^{\varepsilon(2)}2$, then $\eta_{\mathbf{a}_1,\rho_1}{}^\frown \langle \iota \rangle \trianglelefteq \eta_{\mathbf{a}_2,\rho_2}$ and $\nu_{\mathbf{a}_1,\rho_1}{}^\frown \langle \iota \rangle \trianglelefteq \nu_{\mathbf{a}_2,\rho_2}$,
(d) if $m < \omega$, $u_1 \in [{}^{\varepsilon(1)}2]^m, u_2 \in [{}^{\varepsilon(2)}2]^m, u_1 = \{\rho {\restriction} \varepsilon(1) : \rho \in u_2\rangle, h_\ell \in \mathbf{H}_{u_\ell}$ and $\rho \in u_2 \Rightarrow h_2(\rho) = h_1(\rho {\restriction} \varepsilon(1))$, <u>then</u> $p_{u_1,h_1} \le_{\mathbb{P}_*} p_{u_2,h_2}$.

$\boxplus_3$ $(\mathrm{AP}, <_{\mathrm{AP}})$ is a partial order.

[Why? Read the definitions.]

$\boxplus_4$ for $\varepsilon = 0$ there is $\mathbf{a} \in \mathrm{AP}_\varepsilon$ .

[Why? Trivial.]

$\boxplus_5$ if $\varepsilon < \kappa$ is a limit ordinal and $\mathbf{a}_\zeta \in \mathrm{AP}_\zeta$ for $\zeta < \varepsilon$ is $\le_{\mathrm{AP}}$ increasing, <u>then</u> there is $\mathbf{a}_\varepsilon \in \mathrm{AP}_\varepsilon$ such that $\zeta < \varepsilon \Rightarrow \mathbf{a}_\zeta <_{\mathrm{AP}} \mathbf{a}_\varepsilon$.

[Why? Straightforward.]

$\boxplus_6$ if $\varepsilon < \kappa$ and $\mathbf{a} \in \mathrm{AP}_\varepsilon$ there is $\mathbf{b}$ such that:
(a) $\mathbf{b} \in \mathrm{AP}_\varepsilon$,
(b) $\mathbf{a} \le_{\mathrm{AP}} \mathbf{b}$,
(c) $\varepsilon_{\mathbf{a}} = \varepsilon_{\mathbf{b}} + 1$,

[Why? Straightforward.]

$\boxplus_7$ if $\mathbf{a} \in \mathrm{AP}_\varepsilon$, <u>then</u> there is $\mathbf{b} \in \mathrm{AP}^+_\varepsilon$ such that $\mathbf{a} \le_{\mathrm{AP}} \mathbf{b}$.

Why? Noting,

$\boxplus_{7.1}$ if $\bar\eta' = \langle \eta'_\rho : \rho \in \mathscr{T}_{1,[\varepsilon]}\rangle$ and $\eta_{\mathbf{a},\rho} \trianglelefteq \eta'_\rho \in \mathscr{T}_1$ for $\rho \in \mathscr{T}_{1,[\varepsilon]}$, <u>then</u>:
- (a) there is $\mathbf{b} \in \mathrm{AP}_\varepsilon$ such that $\mathbf{a} \le_{\mathrm{AP}} \mathbf{b}$ and $\eta'_\rho \triangleleft \eta_{\mathbf{b},\rho}$,
- (b) moreover there is $\mathbf{b}$ as above such that for some $\zeta \in (\zeta_\mathbf{a}, \zeta_\mathbf{b})$ such that:
  - if $\rho, \varrho \in \mathscr{T}_{1,[\varepsilon]}$, then $\rho <^*_{\mathscr{T}_1} \varrho \Leftrightarrow \eta_{\mathbf{b},\rho}{\restriction}\zeta <^*_{\mathscr{T}_0} \eta_{\mathbf{b},\varrho}{\restriction}\zeta$,
  - $\langle \eta_{\mathbf{b},\rho}{\restriction}\zeta : \rho \in \mathscr{T}_{1,[\varepsilon]}\rangle$ is as in clause (l) (see $\boxplus^1_\mathbf{a}$).

[Why? Let $\alpha_\rho \in \mathscr{U}$ for $\rho \in T_{1,[\varepsilon]}$ be such that $\rho <_{\mathscr{T}_{1,\varepsilon}} \varrho \Rightarrow \alpha_\rho < \alpha_\varrho$. Let $\bar h := \langle h_u : u \subseteq T_{1,[\varepsilon]}$ is finite$\rangle$ be such that $\rho \in u \in [\mathscr{T}_{1,[\varepsilon]}]^{<\aleph_0} \Rightarrow h_u(\ell) = \alpha_\rho$. Let $q_* = \bigcup\{p_{u,h_u} : u \subseteq \mathscr{T}_{1,[\varepsilon]}$ is finite$\}$, so $q_* \in \mathbb{P}_*$.

Clearly there is $p_* \in \mathbf{G}_0$ such that $p_* \cup q_* \in \mathbb{P}_\lambda$ forces all the relevant information related to $\mathbf{a}$. Now, as $\mathfrak{B}_0 \prec \mathfrak{A}_0$, there is a 1-to-1 function $\mathbf{g}$ from $\mathrm{dom}(p_* \cup q_*)$ such that $(p_* \cup q_\bullet) = \mathbf{g}(p_* \cup q_*)$ satisfies all the relevant information; necessarily $q_\bullet \in \mathbb{P}_{u_*}$; without loss of generality $q_\bullet \in \mathbf{G}_0$ and the rest follows.]

Now $\boxplus_7$ follows because $\langle \underset{\sim}{\eta}_\alpha : \alpha \in \mathscr{U}\rangle$ is generic over $\mathbf{V}_1$ for $\mathbb{P}_\mathscr{U}$.

$\boxplus_8$ we can choose $\mathbf{a}_\varepsilon \in \mathrm{AP}_\varepsilon$ by induction on $\varepsilon < \kappa$ such that $\zeta < \varepsilon \Rightarrow \mathbf{a}_\zeta \le_{\mathrm{AP}} \mathbf{a}_\varepsilon$ and $\varepsilon = \zeta + 1 \Rightarrow \mathbf{a}_\varepsilon \in \mathrm{AP}^+_\varepsilon$.

[Why? Use $\boxplus_4$ for $\varepsilon = 0$, use $\boxplus_5$ for $\varepsilon$ a limit ordinal and $\boxplus_6 + \boxplus_7$ for $\varepsilon = \zeta + 1$.]

To finish let $\zeta_\varepsilon \in [\varepsilon, \zeta_{\mathbf{a}_{\varepsilon+1}})$ be as in clause (j) of $\boxplus^1_\mathbf{a}$. Let $g : \kappa \to \kappa$ be increasing such that $\zeta_{(\mathbf{a}_{g(\varepsilon)})} < g(\varepsilon + 1)$.

Now define $h : \mathscr{T}_1 \to \mathscr{T}_2$ by $\rho \in \mathscr{T}_{1,[\varepsilon]} \Rightarrow h(\rho) = \eta_{\mathbf{a}_{g(\varepsilon)+1}}{\restriction}\zeta_{g(\varepsilon)}$ and check. The indiscernibility holds by $\boxplus^1_\mathbf{a}(\ell)(\beta)$ not just $\boxplus^0_\mathbf{a}(\mathbf{g})$. $\square_{1.10}$

*Remark* 1.11. 1) Using the end of §1A we get the desired conclusions.

2) In 1.10 we may state and prove the variant with the square bracket. In more details,

- (A) We say $\bar a, \bar b \in \mathrm{eseq}_m(\mathscr{T})$ are weakly $\mathscr{T}$-similar as before but omitting "$a_\ell <^*_\mathscr{T} a_k \Leftrightarrow b_\ell <^*_\mathscr{T} b_k$"; that is, when $\lg(\bar a) = \lg(\bar b)$ and for some permutation $\pi$ of $\lg(\bar a)$ for $k, \ell, m < \lg(\bar a)$, we have:
  - (a) $\mathrm{lev}_\mathscr{T}(a_\ell) = \mathrm{lev}_\mathscr{T}(a_k) \Rightarrow \mathrm{lev}_\mathscr{T}(b_{\pi(\ell)}) = \mathrm{lev}_\mathscr{T}(b_{\pi(k)})$,
  - (b) $a_\ell <_\mathscr{T} a_k \Leftrightarrow b_{\pi(\ell)} <_\mathscr{T} b_{\pi(k)}$.
- (B) We replace "$\mathscr{T}_2 \to (\mathscr{T}_1)^{\mathrm{end(k,m)}}_\sigma$" by "$\mathscr{T}_2 \to [\mathscr{T}_1]^{\mathrm{end(k,m)}}_{\sigma,j}$" for suitable finite $j$ which means that 1.5(1)$(*)\bullet_2$ is replaced by:
  - $\bullet_2$ if $n < \omega$ and $\bar a \in \mathrm{eseq}_n(\mathscr{T}_2)$, <u>then</u> the following set has at most $j$ elements: $\{\mathbf{c}'(\bar b) : \bar b \in \mathrm{eseq}_n(\mathscr{T}_2)$ is weakly $\mathscr{T}_1$-similar to $\bar a$ and $\ell < n \wedge (k \le |\mathrm{Lev}(\bar a)) \setminus \mathrm{lev}(a_\ell)|) \Rightarrow b_\ell = a_\ell\}$.

3) This will be enough for the model theory and if we use minimal j (well, depends on $\bar s{\restriction}v(\bar s)$) we get back to 1.10.

**Conclusion 1.12.** *Assume $\chi \le \infty$ is limit and $\theta < \chi \Rightarrow 2^\theta = \theta^+$.*

*We can find a forcing notion $\mathbb{P}$ such that:*

- (a) *$\mathbb{P}$ collapses no cardinal, changes no cofinality,*
- (b) $2^\theta < \theta^{+\omega}$, $2^{2^\theta} = (2^\theta)^+$ *for $\theta < \chi$,*

(c) *in* $\mathbf{V}^{\mathbb{P}}$, *if* $\theta^{+\omega} \le \chi$, *then for some* $n < \omega$, *for every* $\mathscr{T}_1$ *expanding* $({}^{\theta>}2, \triangleleft)$ *there is* $\mathscr{T}_2$ *expanding* $({}^{\theta(+n)>}2, \triangleleft)$ *such that* $\mathscr{T}_2 \to (\mathscr{T}_1)^{\mathrm{end(n)}}_\theta$.

*Remark* 1.13. Concerning the order on each level.

**Discussion 1.14.** We may like to replace ${}^{\kappa>}2$ by ${}^{\kappa>}I$ and even use creature tree forcing, see Roslanowski-Shelah [RS99], [RS07], Goldstern-Shelah [GS05]) but (in second thought, for $\kappa = \aleph_0$ maybe see the paper with Zapletal [SZ11]). That is, for $\kappa > \aleph_0$ in each node we have a forcing notion which is quite complete, but of cardinality $< \kappa =$ set of levels.

So we do not have a tree but a sequence of creatures, $\langle \mathfrak{c}_\varepsilon : \varepsilon < \mathrm{ht}(\mathscr{T}) \rangle$, such that for a colouring we like to find $\mathfrak{d}_\varepsilon \in \Sigma(\mathfrak{c}_\varepsilon)$ for $\varepsilon < \kappa$, which induces a sub-tree in which the colouring is 1-end-homogeneous. Alternatively we have $\langle \mathfrak{c}_\eta : \eta \in \mathscr{T} \rangle$ where $\mathfrak{c}_\eta$ is a creature a with set of possible values being in $\mathrm{suc}_{\mathscr{T}}(\eta)$, see [RS07].

Clearly the answer is that we can, but it is not clear how interesting it is. We can just,

$\boxdot$ replace 2 by $\Upsilon \in [2, \kappa)$ and ${}^{\kappa>}2$ by ${}^{\kappa>}\Upsilon$; in the Definition 1.1 replace $R_{\mathscr{T},\ell}$ $(\ell < 2)$ by $R_{\mathscr{T},\ell}$ $(\ell < \Upsilon)$ and add: if $s \in \mathscr{T}$, then $\mathrm{suc}_{\mathscr{T}}(s)$ is either a singleton or is $\{s_\ell : \ell < \Upsilon\}$, where $s_\ell R_{\mathscr{T},\ell} s$ for $\ell < \Upsilon$.

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