ON THE EXISTENCE OF UNCOUNTABLE HOPFIAN AND CO-HOPFIAN ABELIAN GROUPS

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This paper is dedicated to Moshe Jarden, in honor of his many contributions to field arithmetic.

ABSTRACT. We deal with the problem of existence of uncountable co-Hopfian abelian groups and (absolute) Hopfian abelian groups. Firstly, we prove that there are no co-Hopfian reduced abelian groups G of size $< \mathfrak{p}$ with infinite $\mathrm{Tor}_p(G)$, and that in particular there are no infinite reduced abelian p-groups of size $< \mathfrak{p}$. Secondly, we prove that if $2^{\aleph_0} < \lambda < \lambda^{\aleph_0}$, and G is abelian of size λ, then G is not co-Hopfian. Finally, we prove that for every cardinal λ there is a torsion-free abelian group G of size λ which is absolutely Hopfian, i.e., G is Hopfian and G remains Hopfian in every forcing extensions of the universe.

1. Introduction

A group G is said to be Hopfian (resp. co-Hopfian) if every onto (resp. 1-to-1) endomorphism of G is 1-to-1 (resp. onto), equivalently G is Hopfian if it has no proper quotient isomorphic to itself and co-Hopfian if it has no proper subgroup isomorphic to itself. For example, $\mathbb{Z}$ is Hopfian but not co-Hopfian, while the Prüfer p-group $\mathbb{Z}(p^\infty)$ is co-Hopfian but not Hopfian. The notions of Hopfian and co-Hopfian groups have been studied for a long time, under different names. In the context of abelian group theory they were first considered by Baer in [1], where he refers to them as Q-groups and S-groups. The modern terminology arose from the work of the German mathematician H. Hopf, who showed that the defining property holds of certain two dimensional manifolds. The research on Hopfian and co-Hopfian abelian groups has recently been revived thanks to its recently discovered connections with the study of algebraic entropy and its dual (see [6, 12]), as e.g. groups of zero algebraic entropy are necessarily co-Hopfian (for more on the connections between these two topics see [13]). In this paper we will focus exclusively on abelian groups and for us "group" will mean "abelian group".

We briefly recall the relevant state of the art in this area and then introduce our motivation and state our theorems. We start by considering the co-Hopfian property. An easy observation shows that a torsion-free abelian group is co-Hopfian if and only if it is divisible of finite rank, hence the problem naturally reduces to the torsion and mixed cases. A major progress in this line of research was made by Beaumont and Pierce in [2] where the authors proved several general important results, in particular that if G is co-Hopfian, then $\mathrm{Tor}(G)$ is of size at

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most continuum, and further that G cannot be a p-groups of size $\aleph_0$. This naturally left open the problem of existence of co-Hopfian p-groups of uncountable size $\leqslant 2^{\aleph_0}$, which was later solved by Crawley [4] who proved that there exist p-groups of size $2^{\aleph_0}$. But the question remained: what about p-groups G of size $\aleph_0 < |G| < 2^{\aleph_0}$? Interestingly enough this question remained open until recently, when it was shown by Braun and Strüngmann [3] that this is independent from ZFC. Finally, at the best of our knowledge there are no results on the existence of co-Hopfian groups of size $> 2^{\aleph_0}$ (and possibly for a "good reason", see the discussion after Theorem 1.3).

Moving to Hopfian groups, the situation is quite different, most notably (improving a result of Fuchs [8], who proved this for $\lambda <$ the first beautiful cardinal) the second author showed in [24] that for every infinite cardinal λ there is an endorigid torsion-free group of cardinality λ, i.e., a group G such that for every endomorphism f of G there is $m_f \in \mathbb{Z}$ such that $f(x) = m_f x$ (and such that f is onto iff $m_f \in \{1, -1\}$), evidently such groups are Hopfian and so there are Hopfian groups in every cardinality (recall that finite groups are Hopfian). Hence, the existence of Hopfian groups seems to be settled, but the construction from [24] uses stationary sets, so one may wonder about the "effectiveness" of the construction from [24] or any other known construction of arbitrarily large Hopfian groups. We focus here on a specific notion of "effectiveness" which was suggested for abelian groups by Nadel in [16], i.e., the preservation under any forcing extension of the universe V. We refer to this as the problem of absolute existence (of a group satisfying a certain property). These kind of problems were considered by Fuchs, Göbel, Shelah and others (see e.g. [7, 10, 11]), probably the most important problem in this area is the problem of existence of absolutely indecomposable groups in every cardinality which remains open to this day (despite several partial answers are known).

Relying on the picture sketched above in this paper we consider three major problems on the existence of Hopfian and co-Hopfian groups, namely:

Problem. *(1) Despite the known necessary restrictions, can we improve (in* ZFC*!) the result from* [2] *that there are no co-Hopfian p-groups of size $\aleph_0$ or $> 2^{\aleph_0}$?*
(2) Are there co-Hopfian groups in every (resp. arbitrarily large) cardinality?
(3) Are there absolutely Hopfian groups in every cardinality?

We give solutions to the three problems above with the following three theorems.

Notation. *We denote by* AB *the class of abelian groups and by* TFAB *the class of torsion-free abelian groups. Also, given a cardinal λ, we denote by AB_λ the class of $G \in \mathrm{AB}$ of cardinality λ and by TFAB_λ the class of $G \in \mathrm{TFAB}$ of cardinality λ.*

Theorem 1.1. *Suppose that $G \in \mathrm{AB}$ is reduced and $\aleph_0 \leqslant |G| < 2^{\aleph_0}$. If $\mathfrak{p} > |G|$ and there is a prime p such that $\mathrm{Tor}_p(G)$ is infinite, then G is not co-Hopfian. In particular there are no infinite reduced co-Hopfian p-groups G of size $\aleph_0 \leqslant |G| < \mathfrak{p}$.*

Theorem 1.2. *If $2^{\aleph_0} < \lambda < \lambda^{\aleph_0}$, and $G \in \mathrm{AB}_\lambda$, then G is not co-Hopfian.*

Theorem 1.3. *For all $\lambda \in \mathrm{Card}$ there is $G \in \mathrm{TFAB}_\lambda$ which is absolutely Hopfian.*

We comment on the theorems above. Theorem 1.3 can be considered conclusive in some respect (see also Remark 5.6), while Theorems 1.1 and 1.2 leave room for further investigations. First of all, Theorem 1.1 gives important new information also on the countable case, and in fact in our work in preparation on countable co-Hopfian groups [17] we crucially base our investigations upon this result. Also concerning Theorem 1.1, we might ask: is $\mathfrak{p}$ the right cardinal invariant of the continuum? The answer to this question is: yes, but not quite. In a work in preparation [22] the second author introduces some new $\mathfrak{p}$-like cardinal invariants of the continuum that are tailored exactly to this purpose. Finally, Theorem 1.2 leaves open the question of existence of arbitrarily large co-Hopfian groups, in another work in preparation [21] the second author deals with questions surrounding this problem. Finally, a last word on the existence of arbitrarily large co-Hopfian groups: in [23] the second author proves that there are no arbitrarily large absolutely co-Hopfian groups, in fact he proves that there are no such groups above the first beautiful cardinal, and so a construction of arbitrarily large co-Hopfian groups has to necessarily use some "non-effective methods", such as e.g. Black Boxes [20].

As briefly mentioned above, in a work in preparation [17] we deal with classification and anti-classification results for countable co-Hopfian groups, from the point of view of descriptive set theory, extending on results of our recent paper [18].

The structure of the paper is simple, in Section 2 we introduce the necessary notations and preliminaries, in Section 3 we prove Theorem 1.1, in Section 4 we prove Theorem 1.2, and finally in Section 5 we prove Theorem 1.3.

2. Notations and preliminaries

For readers of various backgrounds, we collect here a number of definitions, notations and (well-known) facts which will be used in the proofs of our theorems.

Notation 2.1. *We denote by $\mathbb{P}$ the set of prime numbers.*

Notation 2.2. *Let G and H be groups.*

(1) $H \leqslant G$ means that H is a subgroup of G.
(2) We let $G^+ = G \setminus \{0_G\}$, where $0_G = 0$ is the neutral element of G.

Definition 2.3. *Let $H \leqslant G \in \mathrm{AB}$, we say that H is pure in G, denoted by $H \leqslant_* G$, when if $k \in H$, $n < \omega$ and $G \models ng = k$, then there is $h \in H$ such that $H \models nh = g$.*

Observation 2.4. *If $H \leqslant_* G \in \mathrm{TFAB}$, $k \in H$ and $0 < n < \omega$, then:*

$$G \models ng = k \Rightarrow g \in H.$$

Observation 2.5. *Let $G \in \mathrm{TFAB}$ and let:*

$$G_{(1,p)} = \{a \in G_1 : a \text{ is divisible by } p^m, \text{ for every } 0 < m < \omega\},$$

then $G_{(1,p)}$ is a pure subgroup of G_1.

Proof. This is well-known, see e.g. the discussion in [14, pg. 386-387]. ■

Notation 2.6. *Given $G \in \mathrm{AB}$ and $p \in \mathbb{P}$, we denote by $\mathrm{Tor}(G)$ the torsion subgroup of G and by $\mathrm{Tor}_p(G)$ the p-torsion subgroup of G.*

Notation 2.7. *Given $G \in \mathrm{AB}$, $g \in G$ and $p \in \mathbb{P}$, we write $p^\infty | \, g$ to mean that g is p^n-divisible (in G) for every $0 < n < \omega$.*

Definition 2.8. *We say that G has bounded exponent or simply that G is bounded if there is $n < \omega$ such that $nG = \{0\}$.*

Notation 2.9. *Given $G, H \in \mathrm{AB}$, we denote by $\mathrm{Hom}(G, H)$, the set of homomorphisms between G and H. We denote by $\mathrm{End}(G)$ the set $\mathrm{Hom}(G, G)$.*

Items 2.10-2.17 below, which will be used in Section 3, are well-known to group theorists, we state them here (with references) for completeness of exposition.

Fact 2.10 ([15, pg. 18, Theorem 7]). *Let $G \in \mathrm{AB}$ and H a pure subgroup of G of bounded exponent. Then H is a direct summand of G.*

Fact 2.11 ([15, pg. 18, Theorem 8]). *Let $G \in \mathrm{AB}$ and $T \leqslant_* \mathrm{Tor}(G)$. If T is the direct sum of a divisible group and a group of bounded exponent, then T is a direct summand of G.*

Fact 2.12 ([2]). *Let $G \in \mathrm{AB}$ be a countable p-group. Then G is co-Hopfian if and only if G is finite.*

Fact 2.13 ([9, Theorem 17.2]). *If $A \in \mathrm{AB}$ is a p-group of bounded exponent, then A is a direct sum of (finitely many, up to isomorphism) finite cyclic groups.*

Fact 2.14. *Let $G \in \mathrm{AB}$ and $p \in \mathbb{P}$.*

(1) If G is an unbounded p-group, then G has a pure cyclic subgroup of arbitrarily large finite size.
(2) $\mathrm{Tor}_p(G) \leqslant_* \mathrm{Tor}(G) \leqslant_* G$ (for $\leqslant_*$ cf. Definition 2.3).

Proof. (1) follows by 2.10 and (2) is well-known. ■

Claim 2.15. *Let $G \in \mathrm{AB}$. Then:*

(1) If $G = G_1 \oplus G_2$ and G_1 is not co-Hopfian (resp. Hopfian), then G is not co-Hopfian (resp. Hopfian);
(2) If $G \in \mathrm{TFAB}$, then G is co-Hopfian iff G is divisible of finite rank;
(3) If $G = G_1 \oplus G_2$ and $G_1 \neq \{0\} \neq G_2$, then G has a non-trivial automorphism.

Proof. Each item is either easy or well-known, see e.g. [2]. ■

Fact 2.16. *Let $K \in \mathrm{AB}$ be a bounded torsion group and let $G \leqslant_* H \in \mathrm{AB}$. If $g \in \mathrm{Hom}(G, K)$, then there is $h \in \mathrm{Hom}(H, K)$ extending g.*

Proof. This is because, by Fact 2.10, K is algebraically compact (cf. [9, Section 38]) and such groups are exactly the pure-injective groups in AB (see [9, Theorem 38.1]). ■

Observation 2.17. *Let $G \in \mathrm{AB}$. Then G is non-co-Hopfian if and only if:*

($\star$) there are f and $z \in G$ such that:
- *(a) $f \in \mathrm{End}(G)$;*
- *(b) $f(x) \neq x$ for every $x \in G \setminus \{0\}$;*
- *(c) for every $x \in G$, $z \neq x - f(x)$.*

Proof. For the direction right-to-left, notice that letting $g = id_G - f \in \mathrm{End}(G)$ we have that by (b) g is 1-to-1 and by (c) g is not onto. The other direction is also easy as if g is a witness for non-co-Hopfianity of G, then $\mathrm{id}_G - g$ satisfies ($\star$). ■

The following notation will be relevant in Section 5.

Notation 2.18. *By τ_{AB} we denote the vocabulary of abelian groups $\{+, -, 0\}$. Given $\lambda, \kappa \in \mathrm{Card}$ we denote by $\mathfrak{L}_{\lambda,\kappa}(\tau_{\mathrm{AB}})$ the corresponding infinitary τ_{AB}-formulas (see e.g. [5]). Sometimes we simply write $\varphi \in \mathfrak{L}_{\kappa,\lambda}$ instead of $\varphi \in \mathfrak{L}_{\kappa,\lambda}(\tau_{\mathrm{AB}})$.*

We now introduce the cardinal invariant $\mathfrak{p}$ (which occurs in Theorem 1.1).

Definition 2.19. *The cardinal invariant of the continuum* $\mathfrak{p}$ *is the minimum size of a family* $\mathcal{F}$ *of infinite subsets of* ω *such that:*

(i) every non-empty finite subfamily of $\mathcal{F}$ *has infinite intersection;*
(ii) there is no infinite $A \subseteq \omega$ *s.t., for every* $B \in \mathcal{F}$, $\{x \in A : x \notin B\}$ *is finite.*

3. Co-Hopfian abelian groups of size $\aleph_0 < \lambda < 2^{\aleph_0}$

As mentioned in the introduction, in this section we aim at proving Theorem 1.1, to this extent we first prove Claim 3.2 which deals with the countable case and then detail on how to modify the proof in order to get Claim 3.3 which gives Theorem 1.1.

Remark 3.1. *By 2.15, the assumption "G is reduced" is without loss of generality. This applies e.g. to Claim 3.2 below.*

Claim 3.2. *Let* $G \in \mathrm{AB}$ *be countable and reduced. Let also* $p \in \mathbb{P}$, *and suppose that* $\mathrm{Tor}_p(G)$ *is infinite. Then:*

(1) G *is not co-Hopfian;*
(2) If in addition $\mathrm{Tor}_p(G)$ *is not bounded, then we can find* $\bar{K}$ *and* K *such that:*
 (a) $\bar{K} = (K_n : n < \omega)$ *and* $K = \bigoplus_{n<\omega} K_n \leqslant_* G$;
 (b) $K_n \leqslant G$ *is a non-trivial finite p-group;*
 (c) there is $f \in \mathrm{End}(G)$ *such that* $\mathrm{ran}(f) \subseteq K$ *and for every* $n < \omega$ *we have that* $\{0\} \neq f(K_n) \subseteq K_n$;
 (d) f *is as in 2.17.*
(3) If in addition to (2) $\mathrm{Tor}_p(G)$ *has height* $\geqslant \omega$, *then in (2)(b) we have that for some increasing* $k(n)$, $p^n(p^{k(n)}K_n) \neq \{0\}$ *and* $x \in K_n \Rightarrow f(x) = p^{k(n)}(x)$.

Proof. If $\mathrm{Tor}_p(G)$ is a bounded infinite group, then by Fact 2.13 we have part (1). So assume that $\mathrm{Tor}_p(G)$ is infinite and not bounded, we prove Items (2) and (3) simultaneously, as Item (2) implies (1) by Observation 2.17, recalling (1)(d), this suffices. As $\mathrm{Tor}_p(G)$ is infinite and not bounded, we can choose (L_n, H_{n+1}, y_n) s.t.:

$(*_1)$ (a) $H_0 = G$;
 (b) $H_n = H_{n+1} \oplus L_n$;
 (c) $L_n = \mathbb{Z}y_n$ and y_n has order $p^{\ell(n)}$, so $L_n, H_{n+1}, H_n \leqslant_* G$;
 (d) without loss of generality $\ell(n) \leqslant \ell(n+1)$;
 (e) moreover $\ell(n) < \ell(n+1)$.

[Why we can do this? By induction on $n < \omega$, using Facts 2.11 and 2.14.]

$(*_2)$ We can find $f_0 \in \mathrm{End}(G)$ such that:
 (a) f_0 maps H_2 into itself;
 (b) f_0 maps $L_0 \oplus L_1$ into itself;
 (c) for some $z \in L_0 \oplus L_1$, $z \notin \{x - f_0(x) : x \in L_0 \oplus L_1\}$;
 (d) $x \in L_0 \oplus L_1$ implies $x \neq f_0(x)$.

[Why? First, $G = H_2 \oplus L_1 \oplus L_0$. Now, let $f_0 \restriction H_2$ be 0 and $f_0 \restriction L_0 \oplus L_1$ be defined by $f_0(m_0 y_0 + m_1 y_1) = p^{\ell(1)-\ell(0)} m_0 y_1$. Then f_0 is as wanted letting $z = y_0$.]

$(*_3)$ if $A \subseteq G$ is finite and $n_0 < \omega$, then we can find $n_2 > n_0$ and h such that:
 (a) h is an hom. from G onto $\mathbb{Z}(p^{\ell(n_2)-\ell(n_0)} y_{n_2})$ (cyclic grp. of order $p^{\ell(n_0)}$);
 (b) $h(a) = 0$, for $a \in A$;
 (c) $h(y_{n_2}) = p^{\ell(n_2)-\ell(n_0)} y_{n_2}$
 (d) $n_2 - n_0 \leqslant (p^{\ell(n_0)})^{|A|}$;
 (e) $h(y_\ell) = 0$, for $\ell < n_0$.

[Why? By $(*_1)$, for each $n < \omega$ we can find a projection h_n of G onto $\mathbb{Z}y_n$, mapping $y_0, ..., y_{n-1}$ to zero and y_n to y_n. So, for every $n_0 \leqslant n < \omega$, $h'_{(n,n_0)} := p^{\ell(n)-\ell(n_0)} h_n$ is an homorphism from G onto $\mathbb{Z}(p^{\ell(n)-\ell(n_0)} y_n)$, which has order $p^{\ell(n_0)}$. Moreover, fixed $n < \omega$, for every $a \in A$ there is $m_n(a) \in \{0, ..., p^{\ell(n_0)} - 1\}$ such that $h'_{(n,n_0)}(a) = m_n(a) p^{\ell(n)-\ell(n_0)} y_n$ (recall that $\mathbb{Z}(p^{\ell(n)-\ell(n_0)} y_n)$ has order $p^{\ell(0)}$). Thus, by the pigeon-hole principle there are $n_1, n_2 < \omega$ such that:

$(\cdot_1)$ $n_0 \leqslant n_1 < n_2 \leqslant n_0 + (p^{\ell(n_0)})^{|A|}$;

$(\cdot_2)$ if $a \in A$, then $m_{n_1}(a) = m_{n_2}(a)$.

Now, let $h \in \mathrm{End}(G)$ be defined as follows:

$$h(x) = h'_{(n_2,n_0)}(x) - f'_{(n_2,n_1)}(h'_{(n_1,n_0)}(x)),$$

where $f'_{(n_2,n_1)}(m y_{n_1}) = m p^{\ell(n_2)-\ell(n_1)} y_{n_2}$, for $m \in \mathbb{Z}$. Then h is as wanted in $(*_3)$.]

Now we can finish the proof. Let $(a_i : i < \omega)$ list the elements of G. We choose $(f_i, K_i, k_i, m_i, n_i)$ by induction on $i < \omega$ as follows:

$(*_4)$ (a) For $i = 0$, let f_0 be as in $(*_2)$ and $K_0 = L_0 \oplus L_1$;

(b) for $i > 0$, $K_i = L_{n(i)}$, $n(i) \geqslant 2$ and n_i is strictly increasing with i;

(c) $f_i \in \mathrm{End}(G)$ has range in K_i and:

$$f_i(y_{n(i)}) = p^{k(i)} y_i, \text{ with } i \leqslant k(i) = \ell(n(i)) - m(i) < \ell(n(i)),$$

so that $f_i(y_{n(i)})$ has order $p^{m(i)}$, where $m(i)$ is as in $(*_{3.1})$;

(d) f_i maps H_2 into itself and $L_0 \oplus L_1$ to zero;

(e) f_i maps $a_0, ..., a_i$ to 0.

[Why can we carry the induction? For $i = 0$ use $(*_2)$, for $i = j + 1$ use $(*)_3$.]

Now, let $K = \bigoplus_{i<\omega} K_i$ define f as follow: for $x \in G$, we let $f(x) = \sum\{f_i(x) : i < \omega\}$. This infinite sum is well-defined because by clause $(*_4)(e)$, $f_i(x) = 0$, for every large enough i. It is easy to see that (f, K) are as wanted for clauses (2)(a)-(c) and (3), finally we show that clause (2)(d) holds, i.e., that f satisfies the hypotheses of 2.17. First, 2.17(a) is obvious. Concerning 2.17(b), as $\mathrm{ran}(f) \leqslant K$, clearly if $x \notin K$ we have that $f(x) \neq x$. On the other hand if $x \in K$ use the equation in $(*_4)(c)$, when $x \notin K_0$ and $(*_2)(d)$ when $x \in K_0$. Finally, concerning 2.17(c), let $x \in G$, we want to show that $z \neq x - f(x)$, where z is as in $(*_2)(c)$. Recall that $G = H_2 \oplus L_1 \oplus L_0 = H_2 \oplus K_0 = K_0 \oplus H_2$, so $x = x_1 + x_2$, with $x_1 \in K_0$ and $x_2 \in H_2$. Thus, $f(x) = (x_1 - f(x_1)) + (x_2 - f(x_2))$. If $x_2 - f(x_2) \neq 0$, then clearly $z \neq x - f(x)$, as $z \in K_0$. On the other hand, if $x_2 - f(x_2) = 0$, then $f(x) = x_1 - f(x_1)$ and by $(*_2)(c)$ we are done. This concludes the proof, as (1) is clear and (3) is also satisfied as we can let $m(i) = i$ in $(*_4)$. ∎

Claim 3.3. *In the context of Claim 3.2.*

(A) We can omit "reduced" if we strengthen "$\mathrm{Tor}_p(G)$ is infinite" to "$\mathrm{Tor}_p(G)$ is of infinite rank and $\mathrm{div}(G) \cap \mathrm{Tor}_p(G)$ is of finite rank".

(B) We can omit "countable" if $|G| < 2^{\aleph_0}$ and MA holds or at least $\mathfrak{p} > |G|$.

(C) We can apply both (A) and (B) simultaneously.

Proof. Concerning (A), let $G = G_1 \oplus G_2$, with G_1 divisible and G_2 reduced. As $\mathrm{Tor}_p(G) = \mathrm{Tor}_p(G_1) \oplus \mathrm{Tor}_p(G_2)$ and $\mathrm{Tor}_p(G_1)$ is of finite rank, necessarily $\mathrm{Tor}_p(G_2)$ is of infinite rank, hence it is in particular infinite and so we can apply Claim 3.2 to G_2 and thus conclude by 2.15(1) that G is not co-Hopfian recalling that $G = G_1 \oplus G_2$.

Concerning clause (B), it suffices to run the proof of 3.2 up to $(*_4)$. First of all, recalling that MA $\wedge\, 2^{\aleph_0} \Rightarrow \mathfrak{p} > |G|$, we can assume that $\mathfrak{p} > |G|$. Now, let $(y_n : n < \omega)$, $(\ell(n) : n < \omega)$, $(h'_{(m,n)} : n < m < \omega)$ and $(f'_{(m,n)} : n < m < \omega)$ be as in $(*_1)$ and the proof of $(*_3)$ from the proof of 3.2. Now, for every finite $A \subseteq G$ and $n < \omega$, let $X_{(A,n)}$ be the following set:

$$\{(n_2, n_1, n_0) : n_2 > n_1 \geqslant n_0 \geqslant n \text{ and } (h'_{(n_2,n_0)} - f'_{(n_2,n_1)} h'_{(n_1,n_0)})(A) = \{0\}\}.$$

Now, we have:

$(+_1)$ (a) for (A, n) as above $X_{(A,n)}$ is infinite; [Why? By the proof of $(*_3)$ in 3.2.]
(b) if $n \leqslant m < \omega$ and $A \subseteq B \subseteq_\omega G$, then $X_{(B,m)} \subseteq X_{(A,n)}$.

As the set $\{(A, n) : A \subseteq_\omega G, n < \omega\}$ has cardinality $|G|$ and by $\mathfrak{p} > |G|$, recalling that $|G|$ is finite, by the definition of $\mathfrak{p}$ and $(+_1)$ we have:

$(+_2)$ there is an infinite $X_* \subseteq \{(n_2, n_1, n_0) : n_0 \leqslant n_1 < n_2 < \omega\}$ such that for every (A, n) as above we have $X_* \subseteq X_{(A,n)}$ modulo finitely many elements.

Now, by induction on $i < \omega$, choose $(n_{(i,2)}, n_{(i,1)}, n_{(i,1)}) \in X_*$ such that $j < i$ implies $n_{(j,2)} < n_{(i,0)}$. Finally, let $f_0 \in \mathrm{End}(G)$ be as in $(*_4)(a)$ of the proof of Claim 3.2 and, for $0 < i < \omega$, let $f_i \in \mathrm{End}(G)$ be as follows:

$$h'_{(n_{(i,2)},n_{(i,0)})} - f'_{(n_{(i,2)},n_{(i,1)})} h'_{(n_{(i,1)},n_{(i,0)})}$$

Now, let $K = \bigoplus_{i<\omega} K_i$, i.e., as in the proof of 3.2, and let f be such that for $x \in G$ we have $f(x) = \sum\{f_i(x) : i < \omega\}$. Notice that f is well-defined (and so clearly $f \in$ End(G))), as for every $x \in G$ we have that $\{i < \omega : f_i(x) \neq 0\}$ is finite, given that $X_* \subseteq X_{(\{a\},0)}$ modulo finite, by construction. It is now easy to see that (f, K) are as wanted, arguing as in the proof of this is as in the proof 3.2. This concludes the proof of (B), finally clause (C) is by combining the proofs of (A) and (B). ■

We are now in the position to prove our first main theorem.

Theorem 1.1. *Suppose that $G \in$ AB is reduced and $\aleph_0 \leqslant |G| < 2^{\aleph_0}$. If $\mathfrak{p} > |G|$ and there is a prime p such that $\mathrm{Tor}_p(G)$ is infinite, then G is not co-Hopfian. In particular there are no infinite reduced co-Hopfian p-groups G of size $\aleph_0 \leqslant |G| < \mathfrak{p}$.*

Proof. Immediate by Claims 3.2 and 3.3. ■

Remark 3.4. *In the context of Claim 3.3(B) we ask ourselves: is $\mathfrak{p}$ the right cardinal invariant? The answer is: yes, but not quite. On this see* [22].

The following claim is essentially known, in particular (1), see e.g. [2, pg. 213] on this, but we mention it as it follows from the proofs of the claims above.

Claim 3.5. *Let $G \in$ AB be reduced.*

(1) If $\mathrm{Tor}_p(G)$ is of cardinality $> 2^{\aleph_0}$, then G is not co-Hopfian;
(2) If $|\mathrm{Tor}_p(G)| \geqslant \lambda$, $\mathrm{cof}(\lambda) = \aleph_0$ and $\alpha < \lambda \Rightarrow |\alpha|^{\aleph_0} < \lambda$, then $|\mathrm{End}(G)| \geqslant 2^\lambda$.

This claim will be relevant in what follows and it is of independent interest.

Claim 3.6. *Let $G \in$ AB and $p \in \mathbb{P}$. If $\mathrm{Tor}_p(G)$ is bounded and $G/\mathrm{Tor}_p(G)$ is not p-divisible, then G is not co-Hopfian.*

Proof. Let $K = \mathrm{Tor}_p(G)$, then, recalling that by assumption K is bounded, by Fact 2.10, K is a direct summand of G, say $G = H \oplus K$. Now, $H \in$ AB and $\mathrm{Tor}_p(H) = \{0\}$, hence $x \mapsto px$ is a 1-to-1 endomorphism of H which is not onto (as otherwise $G/\mathrm{Tor}_p(G)$ would be p-divisible). By Claim 2.15(1) we are done. ■

4. Non-existence of co-Hopfian abelian groups

As mentioned in the introduction, in this section we aim at proving Theorem 1.2, to this extent we prove two theorems: 4.1 and 4.2, from which Theorem 1.1 follows. Theorem 4.1 has stronger assumptions and a simpler proof, while Theorem 4.2 has weaker assumptions but a more complicated proof, but it is needed for Theorem 1.2.

Theorem 4.1. *Suppose that $\lambda = \sum_{n<\omega} \lambda_n > 2^{\aleph_0}$, and, for every $n < \omega$, $\lambda_n = \lambda_n^{\aleph_0} < \lambda_{n+1}$. If $G \in \mathrm{AB}_\lambda$, then G is not co-Hopfian.*

Proof. The proof splits into cases.
Case 1. $\mathrm{Tor}_p(G) = \{0\}$ and $pG \neq G$.
In this case $x \mapsto px$ is a 1-to-1 endomorphism of G which is not onto.
Case 2. $|\mathrm{Tor}(G)| > 2^{\aleph_0}$.
In this case G is not co-Hopfian, see e.g. [2].
Case 3. G has an infinite rank divisible subgroup which is torsion-free or a p-group. This case is easy.
Case 4. For some $p \in \mathbb{P}$, $\mathrm{Tor}_p(G)$ is finite and $G/\mathrm{Tor}_p(G)$ is not p-divisible.
Also in this case G is not co-Hopfian, cf. Claim 3.6.
Case 5. For some $p \in \mathbb{P}$, $\mathrm{Tor}_p(G)$ is infinite and bounded.
Also in this case G is not co-Hopfian, cf. Claim 3.2.

Hence, recalling 2.15, w.l.o.g. for the rest of the proof we can assume:

(+) G is reduced and G does not fall under Cases 1, 2, 3, 4, 5.

So we have:

$(*_0)$ For each $p \in \mathbb{P}$ we have (a) or (b), where:
- $(a)_p$ $\mathrm{Tor}_p(G)$ is infinite of cardinality $\leqslant 2^{\aleph_0}$;
- $(b)_p$ $\mathrm{Tor}_p(G)$ is finite and $G/\mathrm{Tor}_p(G)$ is p-divisible.

$(*_1)$ (a) Let $\mathbb{A} = \{p \in \mathbb{P} : \mathrm{Tor}_p(G) \text{ is infinite}\}$;
- (b) For every $p \in \mathbb{A}$ there is $K_p = \bigoplus\{K_{(p,n)} : n < \omega\} \leqslant_* \mathrm{Tor}_p(G)$ such that for every $n < \omega$, $K_{(p,n)} \cong \mathbb{Z}_{p^{k(p,n)}} z_{(p,n)}$, with $1 \leqslant k(p,n) < \omega$ and $k(p,n)$ increasing with n, as $p \in \mathbb{A}$ and not Case 5 $\Rightarrow$ $\mathrm{Tor}_p(G)$ is not bounded.
  [Why we can get $K_p = \bigoplus\{K_{(p,n)} : n < \omega\} \leqslant_* \mathrm{Tor}_p(G)$? By Fact 2.14(2).]

In $(*_7)$ below we will prove that $\mathbb{A} \neq \emptyset$. Now we move to:

$(*_2)$ Choose $(G_n : n < \omega)$ such that:
- (a) $\bigcup_{n<\omega} G_n = G$;
- (b) for every $n < \omega$, $G_n \leqslant G_{n+1} \leqslant G$ and $|G_n| \leqslant \lambda_n$;
- (c) $G_n \preccurlyeq_{\mathfrak{L}_{\aleph_1,\aleph_1}} G$;
- (d) for every $n < \omega$, if $(a_\ell : \ell < \omega) \in G_n^\omega$, $(x_\ell : \ell < \omega) \in G^\omega$, $(k_\ell : \ell < \omega) \in \mathbb{Z}$ and, for every $\ell < \omega$, $x_\ell = k_\ell x_{\ell+1} + a_\ell$, then for some $(y_\ell : \ell < \omega) \in G_n^\omega$ we have that, for every $\ell < \omega$, $y_\ell = k_\ell y_{\ell+1} + a_\ell$;
- (e) $G_n \leqslant_* G$;
- (f) $\mathrm{Tor}(G) \leqslant G_0$;

[Why $(*_2)$ holds? We can fulfill (a)-(b) because we assume that $\lambda = \sum_{n<\omega} \lambda_n$, and we can fulfill (c) because we assume $\lambda_n = \lambda_n^{\aleph_0}$ (see e.g. [5, Corollary 3.1.2]). Items (d) and (e) follow from (c). Finally, we can fulfill (f) easily recalling that $|\mathrm{Tor}_p(G)| \leqslant 2^{\aleph_0}$ and that by assumption $\lambda_0 = \lambda_0^{\aleph_0}$, which implies that $\lambda_0 \geqslant 2^{\aleph_0}$.]

$(*_3)$ Choose $(H_n : n < \omega)$ such that:
- (a) $G_n \leqslant H_n \leqslant G_{n+1}$;

(b) H_n is a pure subgroup of G;
(c) H_n/G_n is torsion-free of rank 1.

[Why possible? Let $a_n \in G_{n+1} \setminus G_n$ and let H_n be the pure closure of $G_n + \mathbb{Z}a_n$, then recalling $(*_2)(a)$ and $(*_2)(f)$ we are done.]

From here till $(*_8)$ excluded, fix $n < \omega$.

$(*_4)$ Let $h_n \in \operatorname{Hom}(H_n, \mathbb{Q})$ be such that $h_n \neq 0$ and $\ker(h_n) = G_n$.
[Why possible? By $(*_3)$.]

$(*_5)$ There is an homomorphism $g_n : \operatorname{ran}(h_n) \to H_n$ be such that $h_n \circ g_n = id_{\operatorname{ran}(h_n)}$.

We prove $(*_5)$. Let $q_{(n,\ell)} \in \operatorname{ran}(h_n)$ be such that:

$(\cdot_1)$ $\mathbb{Z}q_{(n,\ell)} \subseteq \mathbb{Z}q_{(n,\ell+1)} \subseteq \operatorname{ran}(h_n)$;
$(\cdot_2)$ $\bigcup_{\ell<\omega} \mathbb{Z}q_{(n,\ell)} = \operatorname{ran}(h_n)$.

Let $q_{(n,\ell)} = k_{(n,\ell)} q_{(n,\ell+1)}$, with $1 \leqslant k_{(n,\ell)} < \omega$. Let $x_{(n,\ell)}$ be such that $h_n(x_{(n,\ell)}) = q_{(n,\ell)}$. Thus, for each $\ell < \omega$ we have:

$(*_{5.1})$ $h_n(k_{(n,\ell)} x_{(n,\ell+1)} - x_{(n,\ell)}) = k_{(n,\ell)} q_{(n,\ell+1)} - q_{(n,\ell)} = 0$,

which means that $a_{(n,\ell)} := k_{(n,\ell)} x_{(n,\ell+1)} - x_{(n,\ell)} \in G_n$. By $(*_2)(c)$, there are $y_{(n,\ell)} \in G_n$ such that for $\ell < \omega$ we have $a_{(n,\ell)} = k_{(n,\ell)} y_{(n,\ell+1)} - y_{(n,\ell)}$. Now, define $g_n : \operatorname{ran}(h_n) \to H_n$ as follows, for $\ell < \omega$ and $m \in \mathbb{Z}$, we let:

$$g_n(mq_{(n,\ell)}) = m(x_{(n,\ell)} - y_{(n,\ell)}),$$

clearly g_n is well-defined and it is 1-to-1 homomorphism from $\operatorname{ran}(h_n)$ into H_n.

$(*_6)$ (a) $H_n = G_n \oplus L_n$, where $L_n = \operatorname{ran}(g_n)$;
(b) L_n is torsion-free of rank 1.

[Why? As g_n is 1-to-1 and $G_n \cap L_n = \{0\}$.]

$(*_7)$ (a) L_n is not divisible;
(b) there is a prime p_n such that $p_n L_n \neq L_n$;
(c) we can choose $y_n \in L_n$ not divisible by p_n (in L_n and even in G);
(d) $p_n \in \mathbb{A}$.

[Why? Item (a) is because of (+) (see the beginning of the proof), which implies in particular that G is not reduced, recalling that $\{0\} \neq L_n \leqslant G$. Item (b) is by (a). Item (c) is because by (b) we can choose $y_n \in L_n$ as required (as $L_n \leqslant_* H_n \leqslant_* G$, by $(*_6)(a)$ and $(*_3)(b)$, respectively). Lastly, (d) is because by (b) we have $G/\operatorname{Tor}_{p_n}(G)$ is not p_n-divisible, recalling the definition of $\mathbb{A}$. This proves $(*_7)$.]

$(*_8)$ For $n < \omega$, recalling $(*_1)$, let:
(a) if $n = 0$, then $K_n = K_{(p_n,0)} \oplus K_{(p_n,1)}$;
(b) if $n > 0$, then $K_n = K_{(p_n,n+1)}$;
(c) $K = \bigoplus\{K_n : n < \omega\}$.

$(*_{8.1})$ $K_n \leqslant_* \operatorname{Tor}_{p_n}(G) \leqslant_* G$.

[Why? The first is by $(*_1)(b)$ and the second is by Fact 2.14.]

$(*_9)$ Let $n < \omega$, then:
(a) (i) for $n = 0$, we let:
$(\cdot_1)$ $h_0^0 \in \operatorname{End}(K_0)$ be such that:

$$h_0^0(z_{(p_0,0)}) = z_{(p_0,0)} + p^{k(p_0,1)-k(p_0,0)} z_{(p_0,1)}$$
$$h_0^0(z_{(p_0,1)}) = z_{(p_0,1)};$$

$(\cdot_2)$ $z = z_{(p_0,0)}$, so $x \in K_0 \Rightarrow x - h_0^0(x) \neq z$, as in the proof of 3.2;
$(\cdot_3)$ let $f_0^0 \in \operatorname{Hom}(G_0, K_0)$ extend h_0^0;

(ii) for $n > 0$, we let $f^0_n \in \mathrm{Hom}(G_n)$ be zero;

(b) there is onto $f^1_n \in \mathrm{Hom}(L_n, K_n)$ mapping y_n to $z_{(p_n,n)}$;

(c) there is $f^2_n \in \mathrm{Hom}(H_n, K_n)$ extending f^1_n and f^0_n;

(d) there is $f^3_n = f_n \in \mathrm{Hom}(G, K_n)$ extending f^2_n.

[Why? (a)($\cdot_1$)-($\cdot_2$) is clear and (a)($\cdot_3$) is by 2.16 recalling ($*_{8.1}$). Concerning (b), for every $\ell < \omega$, there are $y_{(n,\ell)} \in L$ such that $(\mathbb{Z}y_{(n,\ell)} : \ell < \omega)$ is increasing with union L and $y_{(n,0)} = y_n$, so let $y_{(n,\ell)} = m_{(n,\ell)}y_{(n,\ell+1)}$ with $1 \leqslant m_{(n,\ell)} < \omega$ and $(m_{(n,\ell)}, p_n) = 1$, by the choice of y_n. Now, let $b_{(n,\ell)} \in K_n$ such that:

$$b_{(n,0)} = z_{(p_n,n)} \text{ and } \ell = k + 1 \Rightarrow b_{(n,k)} = m_{(n,k)}b_{(n,\ell)},$$

where for $\ell = k + 1$ we use that K_n is divisible by $m_{(n,\ell)}$ as $(m_{(n,\ell)}, p_n) = 1$. This proves (b). Clause (c) is because $H_n = G_n \oplus L_n$. Finally, clause (d) is by Fact 2.16.]

Now we can continue as in the proof of Claim 3.2, specifically, we define f as follows:

$$f(x) = \sum\{f_n(x) : n < \omega\}.$$

This infinite sum is well-defined because by $(*_9)(c)$, $n > 0$, $x \in G_n$ implies $f_n(x) = f^2(x) = 0$ and by $(*_2)$, $x \in G$ implies for almost all $n < \omega$, $x \in G_n$. Now we claim that f as in as 2.17. Preliminarily, notice that for every $n < \omega$ we have that:

$(*_{10})$ f maps G into K and $K \leqslant G_0$.

Now, returning to showing that f as in as 2.17, Item 2.17(a) is obvious, concerning 2.17(b), as $\mathrm{ran}(f) \leqslant K$, clearly if $x \notin K$, we have that $f(x) \neq x$. On the other hand, if $x \in K \setminus K_0$, then, recalling $f \restriction \bigoplus_{n>0} K_n$ is zero, as $f(x) \in K_0$, $f(x) \neq x$. Finally if $x \in K_0$, then we use the choice in $(*_9)$. Finally, concerning 2.17(c), set $z = y_0 + z_{(p_0,0)}$, we want to show that for every $x \in G$, $z \neq x - f(x)$. We distinguish cases:

Case 1. $x \in K_0$.

In this case we use the choice of h^0_0 from $(*_9)$.

Case 2. $x \in K \setminus K_0$.

In this case $f(x) \in K_0 \leqslant K$ so $x - f(x) \in K \setminus K_0$ but $z \in K_0$, so $x - f(x) \neq z$.

Case 3. $x \in G \setminus K$.

By $(*_{10})$, $f(x) \in K$, hence $x - f(x) \in G \setminus K$, but $z \in K$, so $x - f(x) \neq z$. ■

To follow there is a strengthening or 4.1 with a more complicated proof.

Theorem 4.2. *Let $\lambda^{\aleph_0} > \lambda > 2^{\aleph_0}$, then:*

(1) no $G \in \mathrm{AB}_\lambda$ is co-Hopfian;

(2) if $G \in \mathrm{AB}_\lambda$ is reduced, $|G/\mathrm{Tor}(G)|^{\aleph_0} = \lambda^{\aleph_0}$ and there is $\mathbb{A} \subseteq \mathbb{P}$ such that:

(a) if $p \in \mathbb{A}$, then $\exists K_p = \bigoplus_{n<\omega} K_{(p,n)} \leqslant_* G$, $K_{(p,n)} \neq \{0\}$ a finite p-group;

(b) if $p \in \mathbb{P} \setminus \mathbb{A}$, then $G/\mathrm{Tor}_p(G)$ is p-divisible;

then we have that $\lambda^{\aleph_0} \leqslant |\{h \in \mathrm{End}(G, K\}|$, where $K = \bigoplus_{p\in\mathbb{A}} K_p$.

Proof. We first prove (2).

$(*_0)$ W.l.o.g. $K_{(p,0)}$ is as in the proof of 4.2, i.e., $K_{(p,0)} = \mathbb{Z}y_{(p,0)} \oplus \mathbb{Z}y_{(p,1)}$, with $y_{(p,\ell)}$ of order $k_{(p,\ell)}$ with $1 \leqslant k_{(p,0)} \leqslant k_{(p,1)}$.

$(*_1)$ Let $\mu = \min\{\mu \leqslant \lambda : \mu^{\aleph_0} \geqslant \lambda\}$, then:

(a) $\mu > 2^{\aleph_0}$;

(b) $\theta < \mu \Rightarrow \theta^{\aleph_0} < \mu$;

(c) $\mu^{\aleph_0} = \lambda^{\aleph_0}$;

(d) $\mathrm{cf}(\mu) = \aleph_0$;

(e) W.l.o.g. $\mu = \sum_{n<\omega} \lambda_n$, $2^{\aleph_0} < \lambda_n = \lambda_n^{\aleph_0} < \lambda_{n+1}$;
(f) $|G/\mathrm{Tor}(G)| \geqslant \mu$.

Why (a)? As $\lambda > 2^{\aleph_0}$. Why (b)? If $\theta < \mu \leqslant \theta^{\aleph_0}$, then $\lambda \leqslant \mu^{\aleph_0} \leqslant (\theta^{\aleph_0})^{\aleph_0} = \theta^{\aleph_0 \aleph_0} = \theta^{\aleph_0}$, a contradiction. Why (c)? As $\lambda^{\aleph_0} \leqslant (\mu^{\aleph_0})^{\aleph_0} = \mu^{\aleph_0\aleph_0} = \mu^{\aleph_0} \leqslant \lambda^{\aleph_0}$. Why (d)? If not then $\mu^{\aleph_0} = |\{\eta : \eta \in \mu^\omega\}| = |\bigcup_{\alpha<\mu}\{\eta : \eta \in \alpha^\omega\}| \leqslant \sum_{\alpha<\mu} |\alpha|^{\aleph_0} \leqslant \mu \times \mu = \mu \leqslant \lambda < \lambda_0^{\aleph}$. Why (e)? Because of (a)-(d). Why (f)? As $|G/\mathrm{Tor}(G)|^{\aleph_0} = \lambda^{\aleph_0}$.

$(*_{1.5})$ Let $(x^*_\alpha + \mathrm{Tor}(G) : \alpha < \lambda_* = |G/\mathrm{Tor}(G)|)$ be a basis of $G/\mathrm{Tor}(G)$;

$(*_2)$ Let $S_n = \prod_{\ell \leqslant n} \lambda_\ell$.

$(*_3)$ We can find $(G_n, H_n, \bar{x}_n, p_n : n < \omega)$ such that:
(a) $\mathrm{Tor}(G) \leqslant G_0$ and $G_n \prec_{\mathfrak{L}_{\aleph_1,\aleph_0}} G$ (so $G_n \leqslant_* G$);
(b) $G_n \leqslant_* G_{n+1}$;
(c) $|G_n| = \lambda_n$;
(d) $G_n \leqslant_* H_n = G_n \oplus \bigoplus_{\eta \in S_n} L_{(n,\eta)} \leqslant_* G_{n+1}$, where $L_{(n,\eta)} = \langle x_{(n,\eta)}\rangle^*$;
(e) $p_n \in \mathbb{A}$ and $x_{(n,\eta)}$ is not divisible by p_n.

We prove $(*_3)$. Let $G_0 \prec_{\mathfrak{L}_{\aleph_1,\aleph_0}} G$ be of cardinality λ_0 (cf. [5, Corollary 3.1.2]). Suppose that G_n was chosen, we shall choose $(G_{(n,\alpha)}, x_{(n,\alpha)} : \alpha < \lambda_n^+)$ as follows:

$(\cdot_1)$ $G_{(n,\alpha)} \prec_{\mathfrak{L}_{\aleph_1,\aleph_0}} G$;
$(\cdot_2)$ $x_{(n,\alpha)} \in G \setminus G_{(n,\alpha)}$ and such that $x_{(n,\alpha)} + G_{(n,\alpha)} \notin \mathrm{Tor}(G/G_{(n,\alpha)})$;
$(\cdot_3)$ $G_n \cup \bigcup\{G_{(n,\beta)}, x_{(n,\beta)} : \beta < \alpha\} \subseteq G_{(n,\alpha)}$.

As in 4.1, since $\mathrm{Tor}(G) \leqslant G_0$, w.l.o.g. $G_{(n,\alpha)} \oplus \langle x_{(n,\alpha)}\rangle^*_G \leqslant_* G$ and let $L_{(n,\alpha)} = \langle x_{(n,\alpha)}\rangle^*_G$. Let $p_{(n,\alpha)} \in \mathbb{A}$ be such that $L_{(n,\alpha)}$ is not $p_{(n,\alpha)}$-divisible (recalling G is reduced). W.l.o.g. $p_{(n,\alpha)} = p_n$, as λ_n^+ has uncountable cofinality. Lastly, let $H_n = \bigoplus_{\alpha<\lambda^+} L_{(n,\alpha)} \oplus G_{(n,\alpha)}$. We can prove by induction on $\alpha \leqslant \lambda_n^+$ that $G_n \oplus \bigoplus_{\eta \in S_n} L_{(n,\eta)} \leqslant_* G$, so indeed $H_n \leqslant_* G_{n+1}$. As $\lambda_n = |S_n|$ renaming we are done. Choose now $G_{n+1} \prec_{\mathfrak{L}_{\aleph_1,\aleph_0}} G$ such that $|G_{n+1}| = \lambda_{n+1}$ and $\bigcup_{\alpha<\lambda^+} G_{(n,\alpha)} \leqslant G_{n+1}$.

$(*_4)$ For $n < \omega$, let AP_n be the set of $(H, \bar{f})$ such that:
(a) $H_n \leqslant H \leqslant_* G$;
(b) $\bar{f} = \{f_{(n,\eta)} : \eta \in S_n\}$;
(c) $f_{(n,\eta)} \in \mathrm{Hom}(H, K_{(p_n,n)})$;
(d) $f_{(n,\eta)}(x_{(n,\nu)}) = 0$ iff $\eta \neq \nu$;
(e) $f_{(n,\eta)} \restriction (G_n)$ is 0;
(f) if $z \in H$, then $|\{\eta \in S_n : f_{(n,\eta)}(z) \neq 0\}| \leqslant 2^{\aleph_0}$ and even $\leqslant \aleph_0$;
(g) if $n = 0$, so necessarily $\eta = ()$, then $f_{(0,\eta)}$ is as $(*_9)(a)$ of the proof of 4.1.

$(*_{4.5})$ Let $(H, \bar{f}) \leqslant \mathrm{AP}_n(H', \bar{f}')$ be the natural order between objects as in $(*_4)$, that is $H \leqslant H'$ and $\eta \in S_n$ implies $f_{(n,\eta)} \subseteq f'_{(n,\eta)}$.

$(*_5)$ For $n < \omega$, $\mathrm{AP}_n \neq \emptyset$.

We prove $(*_5)$. Let $H = H_n$ and let $f_{(n,\eta)} \in \mathrm{Hom}(H)$ be such that:

(i) $f_{(n,\eta)}$ is zero on G_n;
(ii) $f_{(n,\eta)}$ is zero on $L_{(n,\nu)}$, for $\nu \in S_n \setminus \{\eta\}$;
(iii) $\mathrm{ran}(f_{(n,\eta)}) \leqslant K_{(p_n,n)}$;
(iv) $f_{(n,\eta)}(x_{(n,\eta)}) \neq 0$.

Why we can do this? Cf. $(*_9)$ of the proof of 4.1.

$(*_6)$ If $(H_1, \bar{f}_1) \in \mathrm{AP}_n$, then we can find $H_1 \lneq_* H_2 \lneq_* G$ such that $H_2/H_1 \in$ TFAB is of rank 1 and there is $(H_2, \bar{f}_2) \in \mathrm{AP}_n$ such that:

$$(H_1, \bar{f}_1) <_{\mathrm{AP}_n} (H_2, \bar{f}_2) \text{ and } H_1 \nleq H_2.$$

We prove $(*_6)$. Now, for every $\ell < \omega$, we can find $k_\ell < \omega$ and $y_\ell \in H_2$ such that:

$$(\star) \qquad a_\ell := k_\ell y_{\ell+1} - y_\ell \in H_1 \text{ and } H_2 = \bigcup_{\ell<\omega} (\mathbb{Z}y_\ell \oplus H_1).$$

Now, for every $\ell < \omega$ and $\eta \in S_n$ we can find $f_{(n,\eta,\ell)} \in \mathrm{Hom}(\mathbb{Z}x_\ell \oplus H_1, K_{(p_n,n)})$ extending $f^1_{(n,\eta)}$ such that $f_{(n,\eta,\ell)}(y_\ell) = 0$. As $K_{(p_n,n)}$ is finite, for some infinite $\mathcal{U}_{(n,\eta)} \subseteq \omega$ we have that, for $\ell_1 < \ell_2 \in \mathcal{U}_{(n,\eta)}$, $f_{(n,\eta,\ell_2)}(y_{\ell_1})$ is constant (why? by the Ramsey Theorem applied on the coloring $f_{(n,\eta,\ell_2)}(y_{\ell_1}) \in K_{(p_n,n)}$). Now, $(f_{(n,\eta,\ell)} : \ell \in \mathcal{U}_{(n,\eta)})$ converges, i.e., if $(k_i : i < \omega)$ lists $\mathcal{U}_{(n,\eta)}$, then $f'_{(n,\eta,k_i)} = f_{(n,\eta,k_{i+1})} \restriction (\mathbb{Z}y_{k_i} \oplus H_i)$ is increasing. Let $\bar{f}_2 = (f^2_{(n,\eta)} : \eta \in S_n)$, where we let $f^2_{(n,\eta)} = \bigcup \{f'_{(n,\eta,\ell)} : \ell \in \mathcal{U}_{(n,\eta)}\}$. So $(H_2, \bar{f}_2)$ is well-defined and easily it is as required, where the main point is checking $(*_4)(f)$ which is easy as we have:

$(*_{6.5})$ if $\eta \in S_n$ and $\bigwedge_{\ell<\omega} f^1_{(n,\eta)}(a_\ell) = 0$, then:
- (a) if $\ell < m$, then $f_{(n,\eta,m)}(y_\ell) = 0$;
- (b) $f^2_{(n,\eta)}(y_m) = 0$, for $m < \omega$;
- (c) as in (b) for $ky_\ell + b$ ($k \in \mathbb{Z}$ and $b \in H_2$).

Why? Clauses (b) and (c) are easy and clause (a) can be proved by downward induction on ℓ, where for $\ell = 1$, the conclusion is true by choice and for $\ell - 1$ we use $(\star)$. Hence, we are done proving $(*_6)$.

$(*_7)$ For each $n < \omega$ we can choose $\bar{f}_n$ such that $(G, \bar{f}_n) \in \mathrm{AP}_n$.

Why? By $(*_5)$ and $(*_6)$ (and their proof). We elaborate. By induction on $\alpha \leqslant \lambda_*$ we choose pairs $(H^n_\alpha, \bar{f}^n_\alpha)$ such that:

$(*_{7.5})$
- (a) $(H^n_\alpha, \bar{f}^n_\alpha) \in \mathrm{AP}_n$;
- (b) if $\beta < \alpha$, then $(H^n_\beta, \bar{f}^n_\beta) \leqslant_{\mathrm{AP}} (H^n_\alpha, \bar{f}^n_\alpha)$;
- (c) if $\alpha = \beta + 1$, then $x^*_\beta \in H^n_\alpha$.

Why we can carry the induction? For $\alpha = 0$, use $(*_5)$. For $\alpha = \beta + 1$, if $x^*_\beta \in H^n_\beta$, let $(H^n_\beta, \bar{f}^n_\beta) = (H^n_\alpha, \bar{f}^n_\alpha)$, while if $x^*_\beta \notin H^n_\beta$, use $(*_6)$. For α limit, let:

$$H^n_\alpha = \bigcup_{\beta<\alpha} H^n_\beta \text{ and } f^n_{(\alpha,\eta)} = \bigcup_{\beta<\alpha} f^n_{(\beta,\eta)}, \text{ for } \eta \in S_n.$$

Having carried the induction, by the definition of AP_n and the choice of $(x^*_\alpha : \alpha < \lambda_*)$ necessarily $H^n_{\lambda_*} = G$ and so we are done proving $(*_7)$.

$(*_8)$ If $z \in G$, then $\Lambda_z = \{\nu \in \prod_{n<\omega} \lambda_n : \exists^\infty n (f_{(n,\nu\restriction n)}(z) \neq 0)\}$ has size $\leqslant 2^{\aleph_0}$.

Why? For each $n < \omega$, $|\{\eta \in \prod_{\ell<n} \lambda_\ell : f_{(n,\eta)}(z) \neq 0\}| \leqslant \aleph_0$.
Let $\Lambda = \bigcup \{\Lambda_z : z \in G\}$, so clearly $\Lambda \subseteq \prod_{\ell<\omega} \lambda_\ell$ and $|\Lambda| \leqslant 2^{\aleph_0} + |G| < \lambda^{\aleph_0} = \prod_{\ell<\omega} \lambda_\ell$ (cf. $(*_1)(c) - (e)$). So for each $\nu \in \prod_{\ell<\omega} \lambda_\ell \setminus \Lambda$ we have:

$(*_9)$ $f_\nu : G \to K$ defined by $f_\nu(z) = \sum \{f_{\nu\restriction n}(z) : n < \omega\}$ is well-defined.

Why? As for each $z \in G$ all but finitely many terms in the sum are zero. Hence:

$(*_{10})$ if $\eta \neq \nu \in \prod_{\ell<\omega} \lambda_\ell \setminus \Lambda$, then:
- (a) $f_\nu \in \mathrm{Hom}(G, K)$;
- (b) then $f_\nu \neq f_\eta$.

As necessarily $\prod_{\ell<\omega} \lambda_\ell \setminus \Lambda$ has cardinality $\prod_{\ell<\omega} \lambda_\ell \setminus \Lambda = \lambda^{\aleph_0}$ we are done proving Part (2) of the theorem. Concerning Part (1), note that for each $\nu \in \prod_{\ell<\omega} \lambda_\ell \setminus \Lambda$ we have that $(f_{\nu\restriction\ell} : \ell < \omega)$ is as in the proof of 4.1, where the only missing part is to justify that f_ν as in $(*_9)$ is well-defined, which we do there. Hence, for each $\nu \in \prod_{\ell<\omega} \lambda_\ell \setminus \Lambda$ we have that f_ν is as in 2.17 and so G is not co-Hopfian. ∎

Remark 4.3. *Similarly to 4.2 we can prove (A) implies (B), where:*

(A) $G \in \mathrm{TFAB}_\lambda$ *is reduced,* $\lambda^{\aleph_0} > \lambda > 2^{\aleph_0}$, $\emptyset \neq \mathbb{A} \subseteq \mathbb{P}$, $p \in \mathbb{A} \Rightarrow G$ *is p-divisible, and* $K = \bigoplus_{p\in\mathbb{A}} K_p$, *with* K_p *as in 4.2(2);*

(B) $|\mathrm{Hom}(G, K)| \geqslant \lambda^{\aleph_0}$.

Theorem 1.2. *If* $2^{\aleph_0} < \lambda < \lambda^{\aleph_0}$, *and* $G \in \mathrm{AB}_\lambda$, *then G is not co-Hopfian.*

Proof. Immediate by Theorem 4.2. ∎

5. Absolutely Hopfian abelian groups

In reading Convention 5.1 and subsequent items recall Notation 2.18.

Convention 5.1. *By a positive conjunctive existential* $\varphi(\bar{x}_n) \in \mathfrak{L}_{\infty,\aleph_0}(\tau_{\mathrm{AB}})$ *we mean a formula of* $\mathfrak{L}_{\infty,\aleph_0}(\tau_{\mathrm{AB}})$ *which does not uses* $\neg$, $\vee$ *and* $\forall$.

Fact 5.2. *Let* $\varphi(\bar{x}_n) \in \mathfrak{L}_{\infty,\aleph_0}(\tau_{\mathrm{AB}})$ *be positive conjunctive existential and* $G \in \mathrm{AB}$.

(A) $\varphi(G) = \{\bar{a} \in G^n : G \models \varphi[\bar{a}]\}$ *is a subgroup of G;*

(B) if $f \in \mathrm{End}(G)$ *and* $G \models \varphi[\bar{a}]$, *then* $G \models \varphi[f(\bar{a})]$.

Proof. Clause (A) is by e.g. [25, Claim 2.3]. Clause (B) is easy. ∎

Fact 5.3. *(1) If λ is beautiful* $> |R|$ *and M is an R-module of cardinality* $\geqslant \lambda$, *then M is not absolutely co-Hopfian.*

(2) If λ is $<$ the first beautiful cardinal, then there is $G \in \mathrm{TFAB}$ *of cardinality λ which is absolutely endo-rigid (and thus Hopfian).*

Proof. (1) is by the proof of [7, Theorem 4]. (2) is by [11]. ∎

The use of the forcing $\mathrm{Levy}(\aleph_0, |G|)$ in the proof of Theorem 1.3 is justified by:

Fact 5.4. *For given* $G \in \mathrm{TFAB}_\lambda$, *the following are equivalent:*

(a) $\mathrm{Levy}(\aleph_0, \lambda)$ *forces "G is not Hopfian";*

(b) some forcing $\mathbb{P}$ forces "G is not Hopfian";

(c) every forcing $\mathbb{P}$ collapsing λ to $\aleph_0$ forces "G is not Hopfian".

Convention 5.5. *In the proof below by "absolutely if* $f \in \mathrm{End}(G)$, *then..." we mean that the forcing* $\mathrm{Levy}(\aleph_0, |G|)$ *forces the statement "if* $f \in \mathrm{End}(G)$, *then ...".*

Theorem 1.3. *For all* $\lambda \in \mathrm{Card}$ *there is* $G \in \mathrm{TFAB}_\lambda$ *which is absolutely Hopfian.*

Proof. Let λ be an infinite cardinal. We want to construct $G \in \mathrm{TFAB}_\lambda$ which is absolutely Hopfian. To this extent, let:

(a) for $n < \omega$, $\mathrm{decr}_n(\lambda) = \{\eta : \eta$ is a decreasing n-sequence of ordinals $< \lambda\}$;

(b) $\mathrm{decr}_{\geqslant 2}(\lambda) = \bigcup_{2\leqslant n<\omega} \mathrm{decr}_n(\lambda)$;

(c) $\mathrm{decr}(\lambda) = \bigcup_{n<\omega} \mathrm{decr}_n(\lambda)$.

$(*_1)$ let p_1, p_2, $p_{(1,n)}$ $(n \geqslant 1)$, $p_{(2,n)}$ $(n \geqslant 1)$, $q_{(1,n)}$ $(n \geqslant 1)$, $q_{(2,n)}$ $(n \geqslant 1)$ be pairwise distinct primes (notice that we can replace $\mathbb{Z}$ by a ring R with such primes, certainly if R is an integral domain);

$(*_2)$ Let $(\eta_\alpha : \alpha < \lambda)$ list $\mathrm{decr}_{\geqslant 2}(\lambda)$ with no repetitions.

$(*_3)$ Now, we define:
- $(\cdot_1)$ $H_2 = \{\mathbb{Q}x_{(\ell,\alpha)} : \alpha < \lambda, \ell \in \{1,2\}\}$;
- $(\cdot_2)$ $H_0 = \{\mathbb{Z}x_{(\ell,\alpha)} : \alpha < \lambda, \ell \in \{1,2\}\}$.

$(*_4)$ For $\eta \in \mathrm{decr}(\lambda) \setminus \{()\}$ and $\ell \in \{1,2\}$ let $x_{(\ell,\eta)}$ be:
- $(\cdot_1)$ $x_{(\ell,(\alpha))} = x_{(\ell,\alpha)}$, for $\alpha < \lambda$;
- $(\cdot_2)$ $x_{(\ell,\eta)} = x_{(3-\ell,\beta)}$, when $\beta < \lambda$ and $\eta = \eta_\beta$ (recall $(*_2)$).

$(*_5)$ Let $G = H_1 \leqslant H_2$ be generated by $X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5 \cup X_6$, where:

$$X_1 = \{p_1^{-m}x_{(1,\alpha)} : \alpha < \lambda, m < \omega\};$$

$$X_2 = \{p_2^{-m}x_{(2,\alpha)} : \alpha < \lambda, m < \omega\};$$

$$X_3 = \{p_{(1,n)}^{-m}(x_{(1,\eta)} - x_{(1,\nu)}) : n \geqslant 1, \eta \in \mathrm{decr}_n(\lambda), \nu \in \mathrm{decr}_{n+1}(\lambda), \eta \triangleleft \nu, m < \omega\};$$

$$X_4 = \{p_{(2,n)}^{-m}(x_{(2,\eta)} - x_{(2,\nu)}) : n \geqslant 1, \eta \in \mathrm{decr}_n(\lambda), \nu \in \mathrm{decr}_{n+1}(\lambda), \eta \triangleleft \nu, m < \omega\};$$

$$X_5 = \{q_{(1,n)}^{-m}x_{(1,\eta)} : \eta \in \mathrm{decr}_{\geqslant n}(\lambda), m < \omega, 2 \leqslant n < \omega\};$$

$$X_6 = \{q_{(2,n)}^{-m}x_{(2,\eta)} : \eta \in \mathrm{decr}_{\geqslant n}(\lambda), m < \omega, 2 \leqslant n < \omega\}.$$

$(*_6)$ For $\ell \in \{1,2\}$ and $\alpha < \lambda$, let:
- (a) $G^0_\ell = \langle X_\ell \rangle_G$, $G_\ell = \langle X_\ell \rangle^*_G$, where $X_\ell = \{x_{(\ell,\beta)} : \beta < \lambda\}$;
- (b) $G^0_{(\ell,\alpha)} = G^0_{(\ell,\alpha,1)} = \langle \{x_{(\ell,\beta)} : \alpha \leqslant \beta < \lambda\} \rangle_G$ and $G_{(\ell,\alpha)} = G_{(\ell,\alpha,1)} = \langle \{x_{(\ell,\beta)} : \alpha \leqslant \beta < \lambda\} \rangle^*_G = \langle \{x_{(\ell,\beta)} : \lg(\eta_\beta) = 1 \text{ and } \alpha \leqslant \min(\mathrm{ran}(\eta_\beta))\} \rangle^*_G$;
- (c) for $n \geqslant 2$, $G_{(\ell,\alpha,n)} = \langle \{x_{(3-\ell,\beta)} : \lg(\eta_\beta) = n \text{ and } \alpha \leqslant \min(\mathrm{ran}(\eta_\beta))\} \rangle^*_G$.

Notice that:

$(*_7)$ (a) $G = \langle G_1 \oplus G_2 \rangle^*_G$;
- (b) $G_{(\ell,\alpha,n)} \leqslant_* G$;
- (c) for $\ell \in \{1,2\}$ and $n \geqslant 1$, the sequence $(G_{(\ell,\alpha,n)} : \alpha < \lambda)$ is $\subseteq$-decreasing, continuous and with intersection $\{0\}$.

$(*_8)$ For $\ell \in \{1,2\}$ let $\psi_\ell(x) = \bigwedge_{n<\omega} p^n_\ell \,|\, x$.

$(*_9)$ For $\ell \in \{1,2\}$ we have:
- (a) $\psi_\ell(x)$ is a formula in $\mathfrak{L}_{\aleph_1,\aleph_0}(\tau_{\mathrm{AB}})$;
- (b) $\psi_\ell(x)$ is positive conjunctive existential;
- (c) $\psi_\ell(G) = G_\ell$;
- (d) if $f \in \mathrm{End}(G_\ell)$, then f maps G_ℓ into G_ℓ.

[Why? Clauses (a), (b) are clear, clause (d) follows by clause (c) and Fact 5.2(B), and clause (c) is clear by Fact 5.2(A) and the definitions, recalling that if $L \in \mathrm{TFAB}$, and p is a prime, then the p^∞-divisible elements of L form a pure subgroup of L.]

$(*_{10})$ For $1 \leqslant n < \omega$ and $\ell \in \{1,2\}$, by induction on $\alpha < \lambda$ we define $\varphi_{(\ell,\alpha,n)}(x)$ as:
- (a) if $\alpha = 0$ and $n = 1$, then $\varphi_{(\ell,\alpha,n)}(x) = \psi_\ell(x)$;
- (b) if $\alpha = 0$ and $n > 1$, then $\varphi_{(\ell,\alpha,n)}(x) = \psi_{3-\ell}(x) \wedge \bigwedge_{m \geqslant 1} \exists y (q^m_{(\ell,n)} y = x)$;
- (c) if $\alpha > 0$, then $\varphi_{(\ell,\alpha,n)}(x)$ is the formula:

$$\bigwedge_{\beta<\alpha} \exists y (\varphi_{(\ell,\beta,n+1)}(y) \wedge p^\infty_{(\ell,n)} | (x - y) \wedge \varphi_{(\ell,\beta,n)}(x) \wedge q^\infty_{(\ell,n+1)} \,|\, y);$$

- (d) $\varphi^*_{(\ell,0,n)}(x) = \varphi_{(\ell,0,n)}(x)$ and $\alpha > 0 \Rightarrow \varphi^*_{(\ell,\alpha,n)}(x) = \bigvee_{m \geqslant 1} \varphi_{(\ell,\alpha,n)}(mx)$.

$(*_{10.5})$ $\varphi^*_{(\ell,\alpha,n)}(G)$ is the pure closure of $\varphi_{(\ell,\alpha,n)}(G)$ (we shall use this freely).

$(*_{11})$ (a) $\varphi_{(\ell,\alpha,n)}(x) \in \mathfrak{L}_{\lambda,\aleph_0}(\tau_{AB})$ is positive conjunctive existential;

(b) $\varphi^*_{(\ell,0,1)}(G) = G_{(\ell,0,1)} = G_{(\ell,0)} = G_\ell$;
(c) $\varphi^*_{(\ell,\alpha,1)}(G) = G_{(\ell,\alpha,1)} = G_{(\ell,\alpha)}$;
(d) $\varphi^*_{(\ell,\alpha,n)}(G) = G_{(\ell,\alpha,n)}$.

We prove $(*_{11})$ by induction on $\alpha < \lambda$.
Case 1. $\alpha = 0$. Easy.
Case 2. α limit. Easy.
Case 3. $\alpha = \beta + 1$.
Case (a) of $(*_{11})$. Just read the definition of $\varphi_{(\ell,\alpha,1)}$.
Case (b) of $(*_{11})$. Just read the definition of $\varphi_{(\ell,\alpha,1)}$ and $\varphi^*_{(\ell,\alpha,1)}$.
Case (c), (d) of $(*_{11})$. The proofs of (c) and (d) are similar, so we write only the proof of (c). Note that proving (c) we use clauses (c) and (d) for all $\beta < \alpha$.
Thus, we want to prove:

(i) if $\gamma \in [\alpha, \lambda)$, then $x_{(\ell,\gamma)} \in \varphi_{(\ell,\alpha,1)}(G)$;
(ii) if $x \in H_0$ (cf. $(*_3)$) and $x \in \varphi_{(\ell,\alpha,1)}(G)$, then $x \in G_{(\ell,\alpha,1)}$.

We prove (i). We have to show that letting $x = x_{(\ell,\gamma)}$ for every $\beta_1 \leqslant \beta$ we have:

$$(\star_1) \qquad G \models \exists y(\varphi_{(\ell,\beta_1,2)}(y) \wedge p^\infty_{(\ell,1)} \,|\, (x-y) \wedge \varphi_{(\ell,\beta_1,1)}(x) \wedge q^\infty_{(\ell,2)} \,|\, y).$$

Hence, we have to find a witness for $(\star_1)$, to this extent we let $y = x_{(\ell,(\gamma,\beta_1))}$ (cf. $(*_4)(\cdot_2)$) and show that this choice of y is as wanted. Now, the first conjunct $\varphi_{(\ell,\beta_1,2)}(y)$ holds by the inductive hypothesis noticing that $y = x_{(\ell,(\alpha,\beta_1))} \in G_{(\ell,\beta_1,2)}$ (and recalling that we are doing an induction on α for all $1 \leqslant n < \omega$ for clauses (c) and (d) simultaneously). The second conjunct $p^\infty_{(\ell,1)} \,|\, (x-y)$ holds by the choice of G (cf. X_3 and X_4 of $(*_5)$). The third conjunct $\varphi_{(\ell,\beta_1,1)}(x)$ holds by the inductive hypothesis (as $x = x_{(\ell,\gamma)} \in G_\ell$). Finally, the fourth conjunct $q^\infty_{(\ell,2)} \,|\, y$ holds by the choice of G (cf. X_5 and X_6) of $(*_5)$. This concludes the proof of (i).

We now prove (ii). So let $x \in H_0$ and $x \in \varphi_{(\ell,\alpha,1)}(G)$, we want to show that $x \in G_{(\ell,\alpha,1)}$. Clearly $x \in \varphi_{(\ell,\alpha,1)}(G)$ implies that $x \in G_\ell$, in fact as $x \in \varphi_{(\ell,\alpha,1)}(G)$ in particular $G \models \varphi_{(\ell,\beta,1)}(x)$ (as this is the third conjunct of $\varphi_{(\ell,\alpha,1)}$, see $(\star_1)$ above with $\beta_1 = \beta$), so by the inductive hypothesis we have that $x \in G_\ell$ and in fact as $x \in H_0$ we have that $x \in G^0_\ell$ (cf. $(*_6)(b)$). Now, toward contradiction assume $x \notin G_{(\ell,\alpha,1)}$, so $x \neq 0$. As $x \in G^0_\ell = \langle\{x_{(\ell,\gamma)} : \gamma < \lambda\rangle_G$ and $x \neq 0$ there are $k < \omega$ and $\alpha_0 < \cdots < \alpha_k < \lambda$ such that we have the following equation:

$$(\star_2) \qquad x = \sum_{i \leqslant k} n_i x_{(\ell,\alpha_i)},$$

with $n_i \in \mathbb{Z} \setminus \{0\}$. Now, if $\alpha_0 \geqslant \alpha$ we get the desired conclusion, so we assume that $\alpha_0 < \alpha$. Now, if $\alpha_0 < \beta$, clearly $x \notin G_{(\ell,\beta,1)}$, as $\{x_{(\ell,\gamma)} : \gamma \in [\beta, \lambda)\}$ is a basis of $G_{(\ell,\beta,1)}$, but this contradicts the inductive hypothesis. Hence, w.l.o.g. we can assume that $\alpha_0 = \beta$. Now, as $x \in \varphi_{(\ell,\alpha,1)}(G)$ and $\beta < \alpha$ there is $y_0 \in G$ such that:

$$(\star_3) \qquad G \models \varphi_{(\ell,\beta,2)}(y_0) \wedge p^\infty_{(\ell,1)} \,|\, (x-y_0) \wedge \varphi_{(\ell,\beta,1)}(x) \wedge q^\infty_{(\ell,2)} \,|\, y_0.$$

Also, for some $m < \omega$ we have that $y = my_0 \in H_0$ and easily we have:

$$(\star_4) \qquad G \models \varphi_{(\ell,\beta,2)}(y) \wedge p^\infty_{(\ell,1)} \,|\, (mx-y) \wedge \varphi_{(\ell,\beta,1)}(mx) \wedge q^\infty_{(\ell,2)} \,|\, y.$$

Now, by the fact that $G \models q^{\infty}_{(\ell,2)} \,|\, y$ and $y \in H_0$ there are pairwise distinct $\eta_0, ..., \eta_{i-1} \in \mathrm{decr}_2(\lambda)$ and $m_j \in \mathbb{Q}$ such that we have the following:

$$(\star_5) \qquad y = \sum_{j<i} m_j x_{(\ell,\eta_j)}.$$

By $(\star_4)$, $G \models p^{\infty}_{(\ell,1)}|(mx - y)$. Now, $\{z \in G : p^{\infty}_{(\ell,1)}|z\}$ is a pure subgroup of G and its intersection with H_0 is generated by (recalling that $x_{(\ell,(\xi))} = x_{(\ell,\xi)}$, cf. $(*_4)(\cdot_1)$):

$$(\star_6) \qquad \{x_{(\ell,(\zeta))} - x_{(\ell,(\zeta,\epsilon))} : \epsilon < \zeta < \lambda\}.$$

Why $(\star_6)$? By X_3 and X_4 in $(*_5)$. So for some $\epsilon_j < \zeta_j < \lambda$, with $j < j_*$, we have:

$$(\star_7) \qquad mx - y = \sum_{j<j_*} n'_j(x_{(\ell,\zeta_j)} - x_{(\ell,(\zeta_j,\epsilon_j))}),$$

for $n'_j \in \mathbb{Z} \setminus \{0\}$. Also, by $(\star_2)$ and $(\star_5)$ we have that:

$$(\star_8) \qquad mx - y = m \sum_{i\leqslant k} n_i x_{(\ell,\alpha_i)} - \sum_{j<i} m_j x_{(\ell,\eta_j)}.$$

Recall also (crucially) that we are under the following assumption:

$$(\star_9) \qquad \alpha_0 = \beta.$$

W.l.o.g. $(\zeta_j : j < j_*)$ is non decreasing and $j_1 < j_2 \wedge \zeta_1 = \zeta_2$ implies $\epsilon_{j_1} < \epsilon_{j_2}$. We now compare the supports in $(\star_7)$ and $(\star_8)$. There are three cases:
Case A. $\zeta_0 < \beta$.
In this case $x_{(\ell,\zeta_0)}$ appears in $(\star_7)$ but not in $(\star_8)$ (recall $\alpha_0 = \beta$), a contradiction.
Case B. $\zeta_0 > \beta$.
In this case $x_{(\ell,\beta)}$ appears in $(\star_8)$ but not in $(\star_7)$ (recall $\alpha_0 = \beta$), a contradiction.
Case C. $\zeta_0 = \beta = \alpha_0$.
In this case we compare for $\nu \in \mathrm{decr}_2(\lambda)$ when $x_{(\ell,\nu)}$ is in the support of $(\star_7)$ and when it is in the support of $(\star_8)$. We restrict ourselves to the case $\nu = (\zeta_0, \epsilon_0) = (\beta, \epsilon_0)$. As $x_{(\zeta_0,\epsilon_0)}$ appears in $(\star_7)$ it has to appear also in $(\star_8)$, so for some $j < i$ we have that $\eta_j = (\beta, \epsilon_0)$, so $x_{(\ell,\eta_j)}$ appears in the support of y, but, by $(\star_4)$, $G \models \varphi_{(\ell,\beta,2)}(y)$ and so we get a contradiction to clause (d) for β, as $\epsilon_0 < \beta = \zeta_0$ (recalling that $\nu \in \mathrm{decr}_2(\lambda)$). This concludes the proof of $(*_{11})$.

From here on we may work in $\mathbf{V}^{\mathrm{Levy}(\aleph_0,\lambda)}$, toward proving that G is absolutely Hopfian, alternatively all the claims below about $f \in \mathrm{End}(G)$ can be considered as absolute statements in the sense of Convention 5.5.

$(*_{12})$ if $f \in \mathrm{End}(G)$, $\ell \in \{1, 2\}$ and $\alpha < \lambda$, then:
 - (α) f maps $G_{(\ell,\alpha)}$ into $G_{(\ell,\alpha)}$;
 - (β) f maps $G_{(\ell,\alpha,n)}$ into $G_{(\ell,\alpha,n)}$;

[Why? By Fact 5.2(B), $(*_{11})$ and: $\varphi^*_{(\ell,\alpha,n)}(G)$ is the pure closure of $\varphi_{(\ell,\alpha,n)}(G)$.]

$(*_{12.5})$ if $f \in \mathrm{End}(G)$, then there is a unique $\hat{f} \in \mathrm{End}(H_2)$ extending f.

Why? This is because H_2 is the divisible hull of G (so H_2/G is torsion) and by the following fact: if $L_1 \leqslant L_2 \in \mathrm{TFAB}$, L_2/L_1 is torsion, L_2 is divisible and $f \in \mathrm{End}(L_1)$, then f has exactly one extension to a map in $\mathrm{End}(L_2)$.

$(*_{13})$ if $f \in \mathrm{End}(G)$ is onto, then f maps G_ℓ onto G_ℓ.

[Why? Let $\ell \in \{1,2\}$ and $x \in G_\ell$, so for some $y \in G$ we have $f(y) = x$. As $y \in G$, for some $q_1, q_2 \in \mathbb{Q}$ and $y_1 \in G_1$, $y_2 \in G_2$ we have that $y = q_1 y_1 + q_2 y_2$ (recall that $G = \langle G_1 + G_2 \rangle^*_G \leqslant H_2$ so that $q_\ell y_\ell$ make sense). Thus, $f(y) = \hat{f}(y) = \hat{f}(q_1 y_1 + q_2 y_2) = q_1 \hat{f}(y_1) + q_2 \hat{f}(y_2) = q_1 \hat{f}(y_1) + q_2 \hat{f}(y_2) = q_1 f(y_1) + q_2 f(y_2)$, but, as $x = f(y) \in G_\ell$ and $G_\ell \cap G_{3-\ell} = \{0\}$, necessarily $q_{3-\ell} = 0$ so that $y = q_\ell y_\ell \in G_\ell$.]

$(*_{14})$ if $f \in \mathrm{End}(G)$ is onto and $\alpha < \lambda$, then f maps $G_{(\ell,\alpha)} \setminus G_{(\ell,\alpha+1)}$ into itself.

[Why? We prove this by induction on $\alpha < \lambda$. By the inductive hypothesis and $(*_7)(c)$ f maps $G_\ell \setminus G_{(\ell,\alpha)}$ into itself, so by $(*_{13})$ we have that:

$(*_{14.5})$ f maps $G_{(\ell,\alpha)}$ onto $G_{(\ell,\alpha)}$.

By $(*_{14.5})$, $x_{(\ell,\alpha)} \in \mathrm{ran}(f \restriction G_{(\ell,\alpha)})$, let then $z \in G_{(\ell,\alpha)}$ be such that $f(z) = x_{(\ell,\alpha)}$. Now, $x_{(\ell,\alpha)} \notin G_{(\ell,\alpha+1)}$ by $(*_6)(b)$ and $(*_3)$, so $f(z) = x_{(\ell,\alpha)} \notin G_{(\ell,\alpha+1)}$. As $z \in G_{(\ell,\alpha)}$ and $G_{(\ell,\alpha)} = \langle G_{(\ell,\alpha+1)} \cup \{x_{(\ell,\alpha)}\} \rangle^*_G$, necessarily for some rational $q \neq 0$ and $b \in G_{(\ell,\alpha+1)}$ we have that $z = q x_{(\ell,\alpha)} + b$. This implies the following:

$$\begin{aligned} y \in G_{(\ell,\alpha)} \setminus G_{(\ell,\alpha+1)} \quad &\Rightarrow \quad \exists q_y \in \mathbb{Q}^+, y \in q_y x_{(\ell,\alpha)} + G_{(\ell,\alpha+1)} \\ &\Rightarrow \quad f(y) \in q q_y x_{(\ell,\alpha)} + G_{(\ell,\alpha+1)} \\ &\Rightarrow \quad f(y) \in G_{(\ell,\alpha)} \setminus G_{(\ell,\alpha+1)}. \end{aligned}$$

So f maps $G_{(\ell,\alpha)} \setminus G_{(\ell,\alpha+1)}$ into $G_{(\ell,\alpha)} \setminus G_{(\ell,\alpha+1)}$, as wanted in $(*_{14})$.

$(*_{15})$ if $f \in \mathrm{End}(G)$ is onto and $\alpha < \lambda$, then f maps $G_{(\ell,\alpha)} \setminus G_{(\ell,\alpha+1)}$ onto itself.

Why? As f maps $G_{(\ell,\alpha)}$ onto $G_{(\ell,\alpha)}$ (by $(*_{14.5})$) and $G_{(\ell,\alpha+1)}$ into $G_{(\ell,\alpha)}$ (by $(*_{15})$).

$(*_{16})$ If $f \in \mathrm{End}(G)$ is onto, then f is 1-to-1.

[Why? By $(*_{13})$ and $G = \langle G_1 \oplus G_2 \rangle^*_G$, it suffices to show that, fixed $\ell \in \{1,2\}$, $0 \neq x \in G_\ell$ implies that $f(x) \neq 0$. Let $\alpha < \lambda$ be minimal such that $x \in G_{(\ell,\alpha)} \setminus G_{(\ell,\alpha+1)}$, which is justified as $\bigcap_{\alpha<\lambda} G_{\ell,\alpha} = \{0\}$ and $(G_{(\ell,\alpha)} : \alpha \leqslant \lambda)$ is $\subseteq$-decreasing continuous, then by $(*_{14})$ we are done, as $f(x) \in G_{(1,\alpha)} \setminus G_{(1,\alpha+1)}$, and so $f(x) \neq 0$.]

As, from $(*_{12})$ on we have been assuming to work in $\mathbf{V}^{\mathrm{Levy}(\aleph_0,\lambda)}$, we conclude that G is indeed not only Hopfian but absolutely Hopfian, and so we are done. ■

Remark 5.6. *Using ideas on the line of* [10] *(cf. the use of four primes), the proof of Theorem 1.3 can be simplified using only a small (finite) number of primes, and thus in particular it works for R-modules with R having at least that amount of primes, but we choose not to follow this route in order to simplify the proof.*

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