USUBA'S PRINCIPLE UB_λ CAN FAIL AT SINGULAR CARDINALS

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Abstract. We answer a question of Usuba by showing that the combinatorial principle UB_λ can fail at a singular cardinal. Furthermore, λ can be taken to be $\aleph_\omega$.

§ 1. introduction

In [5], Usuba introduced a new combinatorial principle, denoted UB_λ.[1] He showed that UB_λ holds for all regular uncountable cardinals and that for singular cardinals, some very weak assumptions like weak square or even ADS_λ imply it. It is known that ADS_λ can fail for singular cardinals, for example if κ is supercompact and $\lambda > \kappa$ is such that $\mathrm{cf}(\lambda) < \kappa$. Motivated by this results, Usuba asked the following question:

Question 1.1. ([5, Question 2.11]) Is it consistent that UB_λ fails for some singular cardinal λ?

In this paper we give a positive answer to the above question by showing that Chang's transfer principle $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ implies the failure of $\mathrm{UB}_{\aleph_\omega}$ if $\aleph_\omega$ is strong limit, see Theoem 3.1, where a stronger result is proved.

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[1]See Section 2 for the statement of the principle.

The paper is organized as follows. In Section 2, we present some preliminaries and results and then in Section 3, we prove our main result.

§ 2. Some preliminaries

In this section we present some definitions and results that are needed for the later section of this paper. Let us start by introducing Usuba's principle.

Definition 2.1. Let λ be an uncountable cardinal. The principle UB_λ is the statement: there exists a function $f : [\lambda^+]^{<\omega} \to \lambda^+$ such that if $x, y \subseteq \lambda^+$ are closed under f, $x \cap \lambda = y \cap \lambda$ and $\sup(x \cap \lambda) = \lambda$, then $x \subseteq y$ or $y \subseteq x$.

It turned out this principle has many equivalent formulations. To state a few of it, let $S = \{x \subseteq \lambda : \sup(x) = \lambda\}$, $\theta > \lambda$ be large enough regular and let $\lhd$ be a well-ordering of $H(\theta)$. Then we have the following.

Lemma 2.2. ([5]) *The following are equivalent:*

(1) UB_λ,
(2) If $M, N \prec (H(\theta), \in, \lhd, \lambda, S, \cdots)$ *are such that* $M \cap \lambda = N \cap \lambda \in S$, *then either* $M \cap \lambda^+ \subseteq N \cap \lambda^+$ *or* $N \cap \lambda^+ \subseteq M \cap \lambda^+$,
(3) If $M, N \prec (H(\theta), \in, \lhd, \lambda, S, \cdots)$ *are such that* $M \cap \lambda = N \cap \lambda \in S$, *and* $\sup(M \cap \lambda^+) \leq \sup(N \cap \lambda^+)$, *then* $M \cap \lambda^+$ *is an initial segment of* $N \cap \lambda^+$.

The principle UB_λ has many nice implications. Here we only consider its relation with the Chang's transfer principles which is also related to our work.

Definition 2.3. Suppose $\lambda > \mu$ are infinite cardinal. The Chang's transfer principle $(\lambda^+, \lambda) \twoheadrightarrow (\mu^+, \mu)$ is the statement: if $\mathcal{L}$ is a countable first order language which contains a unary predicate U, then for any $\mathcal{L}$ -structure $\mathcal{M} = (M, U^{\mathcal{M}}, \cdots)$ with $|M| = \lambda^+$ and $|U^{\mathcal{M}}| = \lambda$, there exists an elementary submodel $\mathcal{N} = (N, U^{\mathcal{N}}, \cdots)$ of $\mathcal{M}$ with $|N| = \mu^+$ and $|U^{\mathcal{N}}| = \mu$.

Given an infinite cardinal ν, The transfer principle $(\lambda^+, \lambda) \twoheadrightarrow_{\leq \nu} (\mu^+, \mu)$ is defined similarly, where we allow the language $\mathcal{L}$ to have size at most ν.

The next lemma shows the relation between $\mathrm{UB}_{\aleph_\omega}$ and Chang's transfer principles.

Lemma 2.4. *([5, Corollary 4.2]) Suppose* $\mathrm{UB}_{\aleph_\omega}$ *holds. Then the Chang transfer principles* $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$ *fail for all* $1 \leq n < \omega$.

Remark 2.5. By [4], $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$ fails for all $n \geq 3$.

Since the consistency of the transfer principle $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_{n+1}, \aleph_n)$ is open for $n = 1, 2$, one can not use the above result to get the consistent failure of $\mathrm{UB}_{\aleph_\omega}$. In the next section we show that if $\aleph_\omega$ is strong limit, then $\mathrm{UB}_{\aleph_\omega}$ implies the failure of $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ as well, and hence by the results of [3] (see also [1] and [2], where the consistency of $\mathrm{GCH} + (\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is proved using weaker large cardinal assumptions) $\mathrm{UB}_{\aleph_\omega}$ can fail. We also need the following notion.

Definition 2.6. An uncountable cardinal κ is said to be Jonsson, if for every function $f : [\kappa]^{<\omega} \to \kappa$ there exists a set $H \subseteq \kappa$ of order type κ such that for each n, $f''[H]^n \neq \kappa$.

Notation 2.7. Given a model M and a subset A of M, by $cl(A, M)$ we mean the least substructure of M which includes A as a subset.

Lemma 2.8. *Assume λ is a singular strong limit cardinal of cofinality κ. Then there is a model M_0 with vocabulary $\mathcal{L}_0$ such that:*

(a) $|\mathcal{L}_0| = \kappa$ *and* $|M_0| = \lambda^+$,

(b) if M is an $\mathcal{L}$-structure which expands M_0, $|\mathcal{L}| = \kappa$ and M has Skolem functions, then for $\alpha_1, \alpha_2 < \lambda^+$, the following statements are equivalent:

$(\dagger)_{\alpha_1,\alpha_2}$ *for some submodels N_1, N_2 of M we have:*

(α) $N_1 \cap \lambda = N_2 \cap \lambda$ *is unbounded in* λ,

(β) $\alpha_1 \in N_1 \setminus N_2$ *and* $\alpha_2 \in N_2 \setminus N_1$.

$(\ddagger)_{\alpha_1,\alpha_2}$ *if* $V_\ell = cl(\{\alpha_\ell\}, M) \cap \lambda$, $\ell = 1, 2$, *and* $V = V_1 \cup V_2$, *then*

$$\alpha_1 \notin cl(\{\alpha_2\} \cup V, M) \ \& \ \alpha_2 \notin cl(\{\alpha_1\} \cup V, M).$$

Proof. Let $\langle \lambda_i : i < \kappa \rangle$ be an increasing sequence cofinal in λ such that for all $i < \kappa, 2^{\lambda_i} < \lambda_{i+1}$. For each $0 < n < \omega$, let

$$\langle F_{n,\alpha} : \alpha \in [\lambda_i, 2^{\lambda_i}) \rangle$$

enumerate all functions from λ_i into λ_i. Let M_0 be defined as follows:

- the universe of M_0 is λ^+,
- $<^{M_0} = \{(\alpha, \beta) : \alpha < \beta < \lambda^+\}$,
- $c_i^{M_0} = \lambda_i$,
- $P^{M_0} = \{\alpha : \alpha < \lambda\}$,
- $F_n^{M_0}$ is an $(n+1)$-ary function such that:
  - if $i < \kappa, \alpha \in [\lambda_i, 2^{\lambda_i})$ and $\beta_0, \cdots, \beta_{n-1} < \lambda_i$, then
  $$F_n^{M_0}(\beta_0, \cdots, \beta_{n-1}, \alpha) = F_{n,\alpha}(\beta_0, \cdots, \beta_{n-1}),$$
  - in all other cases, $F_n^{M_0}(\beta_0, \cdots, \beta_{n-1}, \beta_n) = \beta_n$.

We show that the model M_0 is as required. Clause (a) clearly holds. To show that clause (b) is satisfied, let M be an $\mathcal{L}$-structure which expands M_0, $|\mathcal{L}| = \kappa$ and suppose M has Skolem functions. Let also $\alpha_1, \alpha_2 < \lambda^+$.

First suppose that $(\dagger)_{\alpha_1,\alpha_2}$ holds, and suppose that the models N_1, N_2 witness it. Let also $V_\ell = cl(\{\alpha_\ell\}, M) \cap \lambda$, $\ell = 1, 2$. Clearly each V_ℓ is an unbounded subset of λ. Let $V = cl(V_1 \cup V_2, M) \cap \lambda$ and set $N^*_\ell = cl(\{\alpha_\ell\} \cup V, M)$.

Claim 2.9. *$N^*_\ell \subseteq N_\ell$, for $\ell = 1, 2$.*

Proof. Fix ℓ. Sine $\alpha_\ell \in N_\ell$,

$$V_\ell = cl(\{\alpha_\ell\}, M) \cap \lambda \subseteq N_\ell \cap \lambda.$$

On the other hand, $N_1 \cap \lambda = N_2 \cap \lambda$, hence

$$V_{3-\ell} = cl(\{\alpha_{3-\ell}\}, M) \cap \lambda \subseteq N_{3-\ell} \cap \lambda = N_\ell \cap \lambda.$$

It follows that $V_1 \cup V_2 \subseteq N_\ell \cap \lambda$, and hence

$$V = cl(V_1 \cup V_2, M) \cap \lambda \subseteq N_\ell.$$

Thus, as $\{\alpha_\ell\} \cup V \subseteq N_\ell$, we have

$$N^*_\ell = cl(\{\alpha_\ell\} \cup V, M) \subseteq N_\ell.$$

The result follows. □

Claim 2.10. *$\alpha_1 \in N_1^* \setminus N_2^*$ and $\alpha_2 \in N_2^* \setminus N_1^*$.*

Proof. Fix $\ell \in \{1,2\}$. Clearly $\alpha_\ell \in N_\ell^*$. On the other hand, by our assumption, $\alpha_\ell \notin N_{3-\ell}$, and by Claim 2.9, $N_{3-\ell}^* \subseteq N_{3-\ell}$. Thus $\alpha_\ell \notin N_{3-\ell}^*$. □

Thus $(\ddagger)_{\alpha_1,\alpha_2}$ is satisfied.

Conversely suppose that $(\ddagger)_{\alpha_1,\alpha_2}$ holds, and for $\ell = 1,2$, set $N_\ell = cl(\{\alpha_\ell\} \cup V, M)$. By our assumption, clause (β) of $(\dagger)_{\alpha_1,\alpha_2}$ holds.

Claim 2.11. *For $\ell \in \{1,2\}$, $N_\ell \cap \lambda = V$.*

Proof. Fix $\ell \in \{1,2\}$. Clearly $N_\ell \cap \lambda \supseteq V$. Now suppose towards a contradiction that $N_\ell \cap \lambda \neq V$, and let $\gamma \in N_\ell \cap \lambda \setminus V$. As M has Skolem functions, there are $n, \beta_0, \cdots, \beta_{n-1} \in V$ and $(n+1)$-ary function symbol F in $\mathcal{L}$ such that

$$\gamma = F^M(\beta_0, \cdots, \beta_{n-1}, \alpha_\ell).$$

As $\beta_0, \cdots, \beta_{n-1} \in V \subseteq \lambda$ and $\gamma < \lambda$, there is $i < \kappa$ such that $\beta_0, \cdots, \beta_{n-1}, \gamma < \lambda_i$. Define an n-ary function $G : \lambda_i \to \lambda_i$ as follows:

$$G(\xi_0, \cdots, \xi_{n-1}) = \begin{cases} F^M(\xi_0, \cdots, \xi_{n-1}, \alpha_\ell) & \text{if } F^M(\xi_0, \cdots, \xi_{n-1}, \alpha_\ell) < \lambda_i; \\ 0 & \text{otherwise.} \end{cases}$$

Note that $G \in \{F_{n,\zeta} : \zeta \in [\lambda_i, 2^{\lambda_i})\}$. Let

$$\zeta_* = \min\{\zeta : (\forall \xi_0, \cdots, \xi_{n-1} < c_i) G(\xi_0, \cdots, \xi_{n-1}) = F_n^{M_0}(\xi_0, \cdots, \xi_{n-1}, \zeta)\}.$$

ζ_* is well-defined and is definable in M (even in M_0) from α_ℓ, so clearly $\zeta_* \in cl(\{\alpha_\ell\}, M)$.

As $\zeta_* \in cl(\{\alpha_\ell\}, M) \cap \lambda = V_\ell \subseteq V$ and $\beta_0, \cdots, \beta_{n-1} \in V$, so

$$\gamma = F^M(\beta_0, \cdots, \beta_{n-1}, \alpha_\ell) = F^M_{n,\zeta_*}(\beta_0, \cdots, \beta_{n-1}) \in V.$$

This contradicts our initial assumption that $\gamma \in N_\ell \cap \lambda \setminus V$. The claim follows. □

Claim 2.12. *$N_1 \cap \lambda = N_2 \cap \lambda$.*

Proof. By Claim 2.11, we have $N_1 \cap \lambda = V = N_2 \cap \lambda$, which concludes the result. □

By Claim 2.12, $N_1 \cap \lambda = N_2 \cap \lambda$, which implies clause (α) of $(\dagger)_{\alpha_1,\alpha_2}$. Thus N_1 and N_2 are as required in clause $(\dagger)_{\alpha_1,\alpha_2}$.

This completes the proof of the lemma. $\square$

§ 3. UB_λ can fail at singular cardinals

In this section we prove the following theorem which answers Usuba's question 1.1.

Theorem 3.1. *Assume λ is a singular strong limit cardinal.* UB_λ *fails if at least one of the following hold:*

(a) $\lambda = \aleph_\omega$ *and the Chang's transfer principle* $(\lambda^+, \lambda) \twoheadrightarrow (\aleph_1, \aleph_0)$ *holds,*

(b) $\lambda > \mu \geq \mathrm{cf}(\lambda)$ *are such that* $(\lambda^+, \lambda) \twoheadrightarrow_{\leq \mathrm{cf}(\lambda)} (\mu^+, \mu)$ *holds,*

(c) $\lambda > \mu \geq \mathrm{cf}(\lambda)$ *and for every model M with universe λ^+ and vocabulary of cardinality* $\mathrm{cf}(\lambda)$*, we can find an increasing sequence $\vec{\alpha} = \langle \alpha_i : i < \mu^+ \rangle$ of ordinals less than λ^+ such that*

$$S^M_{\vec{\alpha}} = \{ i < \mu^+ : cl(\{\alpha_i\}, M) \cap \lambda \subseteq cl(\{\alpha_j : j < i\}, M)\}$$

is stationary in μ^+,

(d) there exists χ with $\lambda > \chi = \mathrm{cf}(\chi) > \mathrm{cf}(\lambda)$ *such that for every model M with universe λ^+ and vocabulary of cardinality* $\mathrm{cf}(\lambda)$*, we can find an increasing sequence $\vec{\alpha} = \langle \alpha_i : i < \chi \rangle$ of ordinals less than λ^+ such that*

$$S^M_{\vec{\alpha}} = \{ i < \chi : cl(\{\alpha_i\}, M) \cap \lambda \subseteq cl(\{\alpha_j : j < i\}, M)\}$$

is stationary in χ,

(e) there is no sequence $\vec{X} = \langle U_i : i < \lambda^+ \rangle$ such that each $U_i \cap \lambda$ is a cofinal subset of λ, $U_i \cap \lambda$ has size $\mathrm{cf}(\lambda)$*, and for every $i < \lambda^+$ there is a sequence $\vec{X}_i = \langle (\alpha_{i,j}, \beta_{i,j}) : j < i \rangle$ such that:*

- *$\vec{X}_i$ has no repetition,*
- *$\alpha_{i,j} \in U_i$,*
- *$\beta_{i,j} \in U_j \cap \lambda$.*

Furthermore, the statement (e) is equivalent to $\neg\mathrm{UB}_\lambda$*, provided that* $\mathrm{cf}(\lambda)$ *is not a Jonsson cardinal.*

Remark 3.2. The assumption "λ is a strong limit cardinal" is only used in the proof of (e) implies $\neg\mathrm{UB}_\lambda$.

Proof. We prove the theorem by a sequence of claims. First note that:

Claim 3.3. *Clause (a) is a special case of clause (b), and clause (c) implies clause (d).*

Claim 3.4. *(b) implies (c).*

Proof. Let M be a model with universe λ^+ and vocabulary of cardinality at most $\mathrm{cf}(\lambda)$. By (b), there exists an elementary submodel $N \prec M$ such that $||N|| = \mu^+$ and $|N \cap \lambda| = \mu$. Let $\vec{\alpha} = \langle \alpha_i : i < \mu^+ \rangle$ list in increasing order the first μ^+ elements of N. So for $i < \mu^+$ we have

$$cl(\{\alpha_i\}, M) \cap \lambda \subseteq N \cap \lambda,$$

and since $N \cap \lambda$ has size μ, we can find some $i(*) < \mu^+$ such that

$$\forall i < \mu^+,\ cl(\{\alpha_i\}, M) \cap \lambda \subseteq \bigcup_{j<i(*)} cl(\{\alpha_j\}, M).$$

Hence the set $S^M_{\vec{\alpha}}$ includes $[i(*), \mu^+)$ and so is stationary in μ^+, as requested. $\square$

Claim 3.5. *(d) implies (e).*

Proof. Suppose towards a contradiction that (d) holds but (e) fails. As (e) fails, we can find sequences $\vec{X} = \langle U_i : i < \lambda^+ \rangle$ and $\vec{X}_i = \langle (\alpha_{i,j}, \beta_{i,j}) : j < i \rangle$ as in clause (e). Let M be a model in a vocabulary $\mathcal{L}$ such that:

(1) $|\mathcal{L}| = \mathrm{cf}(\lambda)$,
(2) M has universe λ^+,
(3) $M = (\lambda^+, \langle \tau_i^M : i < \mathrm{cf}(\lambda) \rangle, H^M)$, where
  (a) $\tau_i^M = i$,
  (b) H^M is a 2-place function such that for all i, $U_i \cap \lambda = \{H^M(i, \alpha) : \alpha < \mathrm{cf}(\lambda)\}$.

Now by (d) applied to the model M, we can find a sequence $\vec{\zeta} = \langle \zeta_i : i < \chi \rangle$ of ordinals less than λ^+ such that the set $S^M_{\vec{\zeta}}$ is stationary in χ. Let $\zeta = \sup\limits_{i<\chi} \zeta_i$. Consider the sequence $\vec{X}_\zeta = \langle (\alpha_{\zeta,\xi}, \beta_{\zeta,\xi}) : \xi < \zeta \rangle$.

For $i < \chi$, let

$$W_i = cl(\{\zeta_j : j < i\}, M) \cap \lambda.$$

So $\langle W_i : i < \chi \rangle$ is a $\subseteq$-increasing continuous sequence of sets each of cardinality $< \chi$. Note that for each $i \in S^M_{\vec{\zeta}}$,

$$\beta_{\zeta,\zeta_i} \in U_{\zeta_i} \cap \lambda \subseteq cl(\{\zeta_i\}, M) \cap \lambda \subseteq W_i.$$

(The former inclusion $\subseteq$ holds because $\mathrm{cf}(\lambda) \cup \{\zeta_i\} \subseteq cl(\{\zeta_i\}, M)$ and $cl(\{\zeta_i\}, M)$ is closed under H^M. The latter inclusion $\subseteq$ holds because $i \in S^M_{\vec{\zeta}}$). Then since $S^M_{\vec{\zeta}}$ is stationary in χ, there is β_* such that

$$U = \{i \in S^M_{\vec{\zeta}} : \beta_{\zeta,\zeta_i} = \beta_*\}$$

is stationary. Moreover, since $|U_\zeta| = \mathrm{cf}(\lambda) < \chi$, we get some $i_i < i_2$ in U such that $\alpha_{\zeta,\zeta_{i_1}} = \alpha_{\zeta,\zeta_{i_2}}$. This contradicts that $\vec{X}_\zeta$ has no repetition. □

Claim 3.6. *(e) implies* $\neg\mathrm{UB}_\lambda$.

Proof. Suppose not. Thus we can assume that both (e) and UB_λ hold. Let $f : [\lambda^+]^{<\omega} \to \lambda^+$ witness UB_λ. Choose a vocabulary $\mathcal{L}$ of size $\mathrm{cf}(\lambda)$ and an $\mathcal{L}$-model M such that:

(1) M has universe λ^+,

(2) M expands the model M_0 of Lemma 2.8, by expanding $\mathcal{L}_0$ (the vocabulary of M_0) using the constant symbols $\langle d^M_i : i < \mathrm{cf}(\lambda) \rangle$ and the function symbols $(\langle F^M_n : n < \omega \rangle, p^M, G^M_1, G^M_2)$, where:

  (a) $d^M_i = i$ for $i < \mathrm{cf}(\lambda)$,

  (b) F^M_n is an n-ary function such that

$$F^M_n(\alpha_0, \cdots, \alpha_{n-1}) = f(\{\alpha_0, \cdots, \alpha_{n-1}\}),$$

  (c) p^M is a pairing function on λ^+, mapping $\lambda \times \lambda$ onto λ

(d) G_1^M and G_2^M are 2-place functions such that for every $\alpha \in [\lambda, \lambda^+)$, $\langle G_1(\beta, \alpha) : \beta < \alpha \rangle$ enumerates λ and

$$\big(\beta < \alpha \ \& \ \gamma = G_1(\beta, \alpha)\big) \Rightarrow \beta = G_2(\gamma, \alpha).$$

By expanding M further, let us suppose that

(3) M contains Skolem functions.

For $\alpha < \lambda^+$, set $N_\alpha = cl(\{\alpha\}, M)$.

$(*)_1$ N_α belongs to $[\lambda^+]^{\mathrm{cf}(\lambda)}$ and it contains an unbounded subset of λ.

Proof. As $\mathcal{L}$ has size $\mathrm{cf}(\lambda)$, so $|N_\alpha| \le \mathrm{cf}(\lambda)$. On the other hand, by clause (2)(a), $\mathrm{cf}(\lambda) \subseteq N_\alpha$ and hence N_α belongs to $[\lambda^+]^{\mathrm{cf}(\lambda)}$. Also as $\{c_i^{M_0} : i < \mathrm{cf}(\lambda)\} \subseteq N_\alpha$ (see the proof of Lemma 2.8) and $\langle c_i^{M_0} : i < \mathrm{cf}(\lambda)\rangle$ is an unbounded sequence in λ, we have N_α contains an unbounded subset of λ. $\square$

Let

$$E = \{\delta \in (\lambda, \lambda^+) : \delta = cl(\delta, M)\}.$$

E is clearly a club of λ^+ and $E \cap \lambda = \emptyset$. By Lemma 2.8, we have

$(*)_2$ Suppose $\xi < \zeta$ are in E. Then

$$\xi \in cl\Big(\{\zeta\} \cup (N_\xi \cap \lambda) \cup (N_\zeta \cap \lambda), M\Big).$$

Proof. Suppose by the way of contradiction that $\xi \notin cl\big(\{\zeta\} \cup (N_\xi \cap \lambda) \cup (N_\zeta \cap \lambda), M\big)$. Let $V_1 = N_\xi \cap \lambda$, $V_2 = N_\zeta \cap \lambda$ and $V = V_1 \cup V_2$. By our assumption,

$$\xi \notin cl(\{\zeta\} \cup V, M),$$

also, it is clear that

$$\zeta \notin cl(\{\xi\} \cup V, M).$$

Thus by Lemma 2.8, we can find submodels N_1^*, N_2^* of M such that

(1) $N_1^* \cap \lambda = N_2^* \cap \lambda$ is unbounded in λ,
(2) $\xi \in N_1^* \setminus N_2^*$ and $\zeta \in N_2^* \setminus N_1^*$.

The models N_1^* and N_2^* are clearly f-closed, and by clause (1) above and UB_λ, we have $N_1^* \subseteq N_2^*$ or $N_2^* \subseteq N_1^*$, which contradicts clause (2) above. $\square$

Let $\langle \sigma_i(x_0, \cdots, x_{n(i)-1}) : i < \mathrm{cf}(\lambda) \rangle$ list all terms of $\mathcal{L}$. By $(*)_2$, for each $\xi < \zeta$ from E, we can choose some $i(\xi, \zeta) < \mathrm{cf}(\lambda)$ together with sequences $\vec{a}_{\xi,\zeta} \in (N_\zeta \cap \lambda)^{<\omega}$ and $\vec{b}_{\xi,\zeta} \in (N_\xi \cap \lambda)^{<\omega}$ such that

$$(\oplus)_1 \qquad \xi = \sigma_{i(\xi,\zeta)}(\zeta, \vec{a}_{\xi,\zeta}, \vec{b}_{\xi,\zeta}).$$

For $\xi \in E$ set $U_\xi = N_\xi = cl(\{\xi\}, M)$. It follows that $U_\xi = cl(U_\xi, M)$. For $\xi < \zeta$ use the pairing function p^M to find $\alpha_{\zeta,\xi}$ and $\beta_{\zeta,\xi}$ such that $\alpha_{\zeta,\xi}$ codes $\langle i(\xi,\zeta)\rangle^\frown \vec{a}_{\xi,\zeta}$ and $\beta_{\zeta,\xi}$ codes $\vec{b}_{\xi,\zeta}$.

Now the sequences

$$\vec{X} = \langle U_\xi : \xi \in E \rangle$$

and

$$\langle \langle (\alpha_{\zeta,\xi}, \beta_{\zeta,\xi}) : \xi \in \zeta \cap E \rangle : \zeta \in E \rangle$$

witness the failure of (e). We get a contradiction and the claim follows. $\square$

Thus so far we have shown that

$$(a) \implies (b) \implies (c) \implies (d) \implies (e) \implies \neg \mathrm{UB}_\lambda.$$

Claim 3.7. *Suppose that* $\mathrm{cf}(\lambda)$ *is not a Jonsson cardinal. Then* $\neg \mathrm{UB}_\lambda$ *implies (e).*

Proof. Suppose towards a contradiction that (e) fails and let $\vec{X} = \langle U_i : i < \lambda^+ \rangle$ and $\langle \vec{X}_i : i < \lambda^+ \rangle$, where $\vec{X}_i = \langle (\alpha_{i,j}, \beta_{i,j}) : j < i \rangle$ as in clause (e) witness this failure. Let $\langle \lambda_i : i < \mathrm{cf}(\lambda) \rangle$ be an increasing sequence cofinal in λ and define the function $c : \lambda \to \mathrm{cf}(\lambda)$ as

$$c(\alpha) = \min\{i < \mathrm{cf}(\lambda) : \alpha < \lambda_i\}.$$

For $\xi < \lambda^+$ let $\langle \gamma_{\xi,i} : i < \mathrm{cf}(\lambda) \rangle$ enumerate U_ξ such that each element of U_ξ appears cofinally many often. Let $f : [\lambda^+]^{<\omega} \to \lambda^+$ be such that:

(1) if $\xi < \zeta < \lambda^+$, then

$$f(\alpha_{\zeta,\xi}, \beta_{\zeta,\xi}, \zeta) = \xi,$$

(2) if $\zeta < \lambda^+$ and $\alpha < \lambda$, then for arbitrary large $j < \mathrm{cf}(\lambda)$, we have

$$\sup_{i<j} \lambda_i < \alpha < \lambda_j \implies f(\alpha, \zeta) = \gamma_{\zeta,j}.$$

(3) if $A \in [\mathrm{cf}(\lambda)]^{\mathrm{cf}(\lambda)}$, $c(\alpha_i) = i$ for $i \in A$ and $j < \mathrm{cf}(\lambda)$, then for some n and some sequence $\vec{\xi} = \langle \xi_0, \cdots, \xi_{n-1} \rangle \in A^n$, we have

$$j = c(f(\alpha_{\xi_0}, \cdots, \alpha_{\xi_{n-1}})).$$

Since $\mathrm{cf}(\lambda)$ is not a Jonsson cardinal, we can define such a function f[2]. Let us show that the pair (f, c) witnesses UB_λ holds,[3] which contradicts our assumption. To see this, suppose $x, y \subseteq \lambda^+$ are closed under f, $x \cap \lambda = y \cap \lambda$ and $\sup(x \cap \lambda) = \lambda$. Assume towards a contradiction that $x \nsubseteq y$ and $y \nsubseteq x$. Let $\xi = \min(x \setminus y)$ and $\zeta = \min(y \setminus x)$, and let us suppose that $\xi < \zeta$.

By clause (3), $\mathrm{cf}(\lambda) \subseteq y$, and then by clause (2), and since $y \cap \lambda$ is cofinal in λ, we have $U_\zeta \subseteq y$. Similarly $U_\xi \subseteq x$. As $x \cap \lambda = y \cap \lambda$ and $U_\xi \subseteq \lambda$, we conclude that $U_\xi \subseteq y$ as well. Thus by item (1), and since $\alpha_{\zeta,\xi}, \beta_{\zeta,\xi}, \zeta \in y$ we have $\xi \in y$, which contradicts the choice of $\xi \in x \setminus y$. This completes the proof of the claim. $\square$

The theorem follows. $\square$

Remark 3.8. The above proof shows that the following are equivalent:

(1) clause (e) of Theorem 3.1,
(2) for each model M with universe λ^+ and vocabulary of cardinality $\mathrm{cf}(\lambda)$, there are substructures N_0, N_1 of M such that $N_0 \cap \lambda = N_1 \cap \lambda$, $N_0 \nsubseteq N_1$ and $N_1 \nsubseteq N_0$.

As we noticed earlier, it is consistent relative to the existence of large cardinals that Chang's transfer principle $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ holds with $\aleph_\omega$ being strong limit. Hence by our main theorem, we have the following corollary.

Corollary 3.9. *It is consistent, relative to the existence of large cardinals, that* $\mathrm{UB}_{\aleph_\omega}$ *fails.*

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[2] this assumption is used to guarantee clause (3) in definition of f holds.

[3] We can define a function $\tilde{f} : [\lambda^+]^{<\omega} \to \lambda^+$ which codes (f, c) so that a set is closed under $\tilde{f}$ if and only if it is closed under both of f and c.

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