NATURALITY AND DEFINABILITY III

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Abstract. In this paper, we deal with the notions of naturality from category theory and definablity from model theory and their interactions. In this regard, we present three results. First, we show, under some mild conditions, that naturality implies definablity. Second, by using reverse Easton iteration of Cohen forcing notions, we construct a transitive model of ZFC in which every uniformisable construction is weakly natural. Finally, we show that if F is a natural construction on a class $\mathcal{K}$ of structures which is represented by some formula, then it is uniformly definable without any extra parameters. Our results answer some questions by Hodges and Shelah.

§ 1. Introduction

We are looking to find some interplay between the notions of naturality from category theory and definability from set theory and model theory. This continues the investigation initiated by Hodges and Shelah in [8] and [9].

In this paper, naturality is in the sense of Eilenberg and Mac Lane [3], i.e., for a class $\mathcal{K}$ of structures, the constriction $A \mapsto F(A)$, for $A \in \mathcal{K}$, is natural, if for every $A \in \mathcal{K}$ and every automorphism a of A there is an automorphism $f(a)$ of $F(A)$

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such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\subseteq} & F(A) \\ {\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle f(a)} \\ A & \xrightarrow{\subseteq} & F(A) \end{array}$$

and the assignment $A \mapsto F(A)$ has the following two properties:

(1) $f(\mathrm{id}_A) = \mathrm{id}_{F(A)}$, where id_A is the identity function on A,

(2) for automorphisms a, b of A, $f(ab) = f(a)f(b)$.

The challenging initial example for naturality was the dual of vector spaces, see [3]. In [7, page 11] Lambek observed that divisible hulls of abelian groups are not natural. Despite this, he pointed out that divisible hulls of torsion-free abelian groups are natural, as it is left adjoint to the inclusion functor.

Hodges and Shelah [8] observed that the construction of algebraic closure of fields and the construction of divisible hulls are not natural. Also, they showed that any left adjoint to any functor gives naturality. This may be considered as a source to produce several natural constructions. According to [1] in a category with injective hulls and a cogenerator, the injective hulls is not natural, unless all objects are injective.

Naturality says that there is an splitting for $\mathrm{Aut}(F(A)) \to \mathrm{Aut}(A)$. From this one may define the notion of weak naturality, see [9, Definition 4]. Hodges and Shelah [8] asked:

Question 1.1. When does naturality imply definablity?

We use two-sorted models in order to deal with the question. This approach is not new. For example, Harvey Friedman [4] already has considered some constructions in algebra (e.g., direct product construction of pairs of groups) as operations from relational structures of a certain fixed many sorted relational type to structures of an enlarged many sorted relational type. Also, Hodges and Shelah [9] interpreted Question 1.1 in terms of two-sorted models. In sum, Question 1.1 can be discussed naturally in the context of many sorted model theory.

In §2, and for the convenience of the readers, we review the concept of many sorted models. According to its definition, any 2-sorted model $\mathfrak{B}$ gives us two groups $H := \mathrm{Aut}(\mathfrak{B})$ and $G := \mathrm{Aut}(\mathrm{sort}_1(\mathfrak{B}))$. If the restriction map $\varphi : H \to G$ is well-defined and has a weak lifting ψ, we say $\mathfrak{B}$ gives us a uni-construction problem

$$\mathbf{c} = \langle \mathfrak{B}, \mathrm{sort}_1(\mathfrak{B}), H, K, G, \varphi, \psi \rangle,$$

where K is the center of H (see Definition 2.5). The new concept of uni-construction problem will play the role of weak naturality. Then we introduce the concept of solvability (see Definition 2.8), which plays the role of definablity.

Concerning Question 1.1, in §3, we discuss some special kinds of 3-sorted models, and prove the following as our first main result:

Theorem 1.2. *Let χ be a cardinal and let $\mathbf{c}$ be a uni-construction problem. If $\mathbf{c}$ has no lifting, then in a forcing extension, $\mathbf{c}$ has no χ-solution.*

In [9, Theorem 10], Hodges and Shelah constructed a transitive model of ZFC in which every uniformisable construction of a prescribed size is weakly natural. Since their result has a cardinality restriction, they raised the following natural conjecture.

Conjecture 1.3. (See [9, Page 8]) The cardinality restriction can be removed.

In §4 we settle the conjecture, by iterating the main forcing construction of [9, Theorem 10] using reverse Easton iteration of suitable Cohen forcing notions. It may be nice to note that the use of forcing techniques in the naturality problems comes back to Friedman [4] where he applied Easton product Cohen-forcing, though the term naturalness in [4] is weaker than here.

Let M be a model of set theory. Suppose some formula represents the construction F on the class $\mathcal{K}$ in M. If F is natural and there is only a set of isomorphism types of structures in $\mathcal{K}$, then [8, Theorem 3] states that F is definable in M with parameters. In this regard, Hodges and Shelah raised the following problem:

Problem 1.4. (See [8, Problems (A) and (B)])

(i) Can the restriction "$\mathcal{K}$ contains only a set of isomorphism types" be removed?

(ii) If F has representing formula φ, is it always possible to define F by a formula whose parameters are those in φ and those needed to define $\mathcal{K}$?

Recall that left adjoint defines a natural construction. It may be worth to note that Hodges [6] used this and produced some natural constructions that are set-theoretically definable from a parameter, via right Kan extension along a small functor.

We want to restrict parameters from the defining formula of the natural constructions. In §5 we apply techniques from many sorted models, and reformulate the mentioned result of Hodges-Shelah [8, Theorem 3] about naturality implies definablity. This reformulation helps us to control parameters, and enables us to prove the following solution to Problem 1.4:

Theorem 1.5. *(Uniformity) Assume $\mathfrak{B}$ is a two sorted model in the vocabulary τ and $\mathfrak{A} = \mathrm{sort}_1(\mathfrak{B})$. Suppose that $\mathcal{K}$ is a class of τ-models which is first order definable from a parameter $\mathbf{p}$ such that for every $\mathfrak{B} \in \mathcal{K}$, the natural homomorphism $\pi^*_{\mathfrak{B}} : \mathrm{Aut}(\mathfrak{B}) \twoheadrightarrow \mathrm{Aut}(\mathrm{sort}_1(\mathfrak{B}))$ splits. Then there exists a class function*

$$F : \{\mathrm{sort}_1(\mathfrak{B}) : \mathfrak{B} \in \mathcal{K}\} \longrightarrow \mathcal{K},$$

which is uniformly definable from the parameter $\mathbf{p}$ and $\mathrm{sort}_1(F(\mathfrak{A})) = \mathfrak{A}$.

One may regard this uniformity result as an essential generalization of [8, Theorem 3].

§ 2. The uni-construction problem

In this section we define the concepts of uni-construction problem, lifting, weak lifting and solvability. We use many sorted models, to unify and simplify our treatment.

Let us start by defining n-sorted model structures, where $n \geq 2$ is a natural number. We only work with finite structures.

Definition 2.1. An n-sorted model structure $\mathcal{M}$ is of the form

$$\mathcal{M} = \big(\{M_1; \ldots; M_n\}; R_1; \ldots; R_m; f_1; \ldots; f_k; c_1; \ldots; c_l\big)$$

where

(a) the universes $M_1; \ldots; M_n$ are nonempty,

(b) the relations $R_1; \ldots; R_m$ are between elements of the universes. In other words, for each i, there is some sequence $\langle i_1, \ldots, i_s\rangle$ from $\{1, \ldots n\}$ such that

$$R_i \subseteq M_{i_1} \times \ldots \times M_{i_s},$$

(c) the functions $f_1; \ldots; f_k$ are between elements of the universes. In other words, for each i, there is some sequence $\langle i_1, \ldots, i_s, r\rangle$ from $\{1, \ldots n\}$ such that

$$f_i : M_{i_1} \times \ldots \times M_{i_s} \to M_r,$$

(d) the distinguished constants $\{c_1; \ldots; c_l\}$ are in the universes.

Notation 2.2. Given an n-sorted model structure $\mathcal{M}$ as above, for any $1 \leq i \leq n$, there is a natural model structure with the universe M_i, that we denote by $\mathrm{sort}_i(\mathcal{M})$.

In what follow, we only work with the concepts of 2-sorted and 3-sorted model structures.

Notation 2.3. Let H be a group which is not necessarily abelian.

i) By $\mathcal{Z}(H)$ we mean the center of H.

ii) Since $\mathcal{Z}(H)$ is a normal subgroup of H, it induces a group structure on $H/\mathcal{Z}(H)$. By $\pi_H : H \to H/\mathcal{Z}(H)$ we mean the natural group homomorphism from H onto $H/\mathcal{Z}(H)$.

Definition 2.4. Let H and G be two groups, which are not necessarily abelian. Assume $\varphi : H \twoheadrightarrow G$ is an surjective homomorphism of groups.

1) We say $\psi \in \mathrm{Hom}(G, H)$ splits φ, when $\varphi \circ \psi = \mathrm{id}_G$.

2) We say ψ weakly splits φ provided:

   (a) ψ is a function from G into H,

(b) $\varphi \circ \psi = \mathrm{id}_G$,

(c) the composite mapping $\pi_H \circ \psi$ belongs to $\mathrm{Hom}(G, H/\mathcal{Z}(H))$,

(d) $\psi(x^{-1}) = (\psi(x))^{-1}$ and $\psi(1_G) = 1_H$.

3) We say φ splits (resp. weakly splits), if some ψ splits (resp. weakly splits) it.

We now define the concept of uni-construction problem, which plays a key role in this paper.

Definition 2.5. 1) We say $\mathbf{c}$ is a uni-construction problem (ucp in short), when

$$\mathbf{c} = \langle \mathfrak{B}_{\mathbf{c}}, \mathfrak{A}_{\mathbf{c}}, H_{\mathbf{c}}, K_{\mathbf{c}}, G_{\mathbf{c}}, \varphi_{\mathbf{c}}, \psi_{\mathbf{c}} \rangle = \langle \mathfrak{B}, \mathfrak{A}, H, K, G, \varphi, \psi \rangle$$

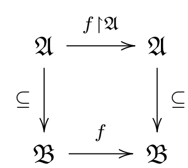
and it satisfies the following conditions:

(a) $\mathfrak{B}$ is a two-sorted model,

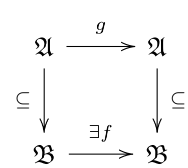
(b) $\mathfrak{A} := \mathrm{sort}_1(\mathfrak{B})$,

(c) $H = \mathrm{Aut}(\mathfrak{B}), K = \mathcal{Z}(H)$ and $G = \mathrm{Aut}(\mathfrak{A})$,

(d) φ is the natural restriction map from H into G, i.e. $\varphi(f) = f{\restriction}\mathfrak{A}$, furthermore, φ is a group homomorphism:



(e) φ is onto G. So, for any $g : \mathfrak{A} \to \mathfrak{A}$ there is an f such that the following diagram commutes



(f) ψ weakly split φ.

2) We say $\mathbf{c}$ is a weak uni-constructive problem, if $\mathbf{c}$ is as above and it satisfies items (a)-(e) above.

Definition 2.6. Let $\mathbf{c}$ be a uni-construction problem. Then the classes $K^1_{\mathbf{c}}$ and $K^2_{\mathbf{c}}$ are defined as follows:

(a) $K^1_{\mathbf{c}} := \{\mathfrak{A} : \mathfrak{A} \text{ isomorphic to } \mathfrak{A}_{\mathbf{c}}\}$,

(b) $K^2_{\mathbf{c}} := \{\mathfrak{B} : \mathfrak{B} \text{ isomorphic to } \mathfrak{B}_{\mathbf{c}}\}$.

Remark 2.7. The assignment $\mathrm{sort}_1(\mathfrak{B}) \overset{F}{\mapsto} \mathfrak{B}$ on the domain of $K^1_{\mathbf{c}}$ is not in general single-valued; but by clause (e) from Definition 2.5 it is single-valued up to isomorphism over $\mathfrak{B}$. In particular, suppose $\mathfrak{A} \cong \mathrm{sort}_1(\mathfrak{B})$. Then

$$F(\mathrm{sort}_1(\mathfrak{B})) = \mathfrak{B} \cong F(\mathfrak{A}).$$

Definition 2.8. Let $\mathbf{c}$ be a uni-construction problem.

(a) We say $\mathbf{c}$ is solvable, when there is a class function $F : K^1_{\mathbf{c}} \to K^2_{\mathbf{c}}$, such that for each $\mathfrak{B} \in K^2_{\mathbf{c}}$, $F(\mathrm{sort}_1(\mathfrak{B})) = \mathfrak{B}$.

(b) Let F be as in clause (a). We say F is definable, if there is a formula θ such that

$$F(\mathfrak{A}) = \mathfrak{B} \Leftrightarrow \theta(\mathfrak{A}, \mathfrak{B}),$$

for all two sorted models $\mathfrak{B}$ with $\mathfrak{A} := \mathrm{sort}_1(\mathfrak{B})$.

(c) We say $\mathbf{c}$ is purely solvable, when there is an F as above which is definable using only $(\mathfrak{B}_{\mathbf{c}}, \mathfrak{A}_{\mathbf{c}})$ as a parameter.

(d) We say $\mathbf{c}$ is χ-solvable, when there is F as above, which is definable by some parameter $a \in \mathscr{H}(\chi)$.

Let us present with the following easy observation.

Observation 2.9. *Let* $\mathbf{c}$ *and* $\mathbf{d}$ *be two uni-construction problems, satisfying the following conditions:*

(a) $\mathfrak{A}_{\mathbf{c}} = \mathfrak{A}_{\mathbf{d}}$,

(b) $\mathfrak{B}_{\mathbf{d}}$ *expands* $\mathfrak{B}_{\mathbf{c}}$.

If $\mathbf{d}$ *is solvable, then* $\mathbf{c}$ *is solvable.*

Proof. Assume $\mathbf{d}$ is solvable, and let $F_{\mathbf{d}} : K^1_{\mathbf{d}} \to K^2_{\mathbf{d}}$ witness it. Let $\mathfrak{A} \in K^1_{\mathbf{c}}$. Since $K^1_{\mathbf{d}} = K^1_{\mathbf{c}}$, we can define $F_{\mathbf{d}}(\mathfrak{A})$. Let $F_{\mathbf{c}}(\mathfrak{A})$ be the reduction of $F_{\mathbf{d}}(\mathfrak{A})$ into the language of $\mathfrak{B}_{\mathbf{c}}$. Then $F_{\mathbf{c}} : K^1_{\mathbf{c}} \to K^2_{\mathbf{c}}$ is well-defined, and it witnesses that $\mathbf{c}$ is solvable. □

§ 3. From uni-construction to naturality

The main result of this section is Theorem 3.9, which shows that for a given cardinal χ, if a uni-construction problem has no lifting, then it has no χ-solution in some forcing extension of the universe.

Hypothesis 3.1. Let $\mathfrak{C}$ be a 3-sorted model such that $\mathbf{c}_{1,2}, \mathbf{c}_{2,3}, \mathbf{c}_{1,3}$ are weak uni-construction problems, where:

(1) $\mathfrak{A}_{\mathbf{c}_{1,2}} = \mathrm{sort}_1(\mathfrak{C}), \mathfrak{B}_{\mathbf{c}_{1,2}} = \mathrm{sort}_{1,2}(\mathfrak{C})$,

(2) $\mathfrak{A}_{\mathbf{c}_{2,3}} = \mathrm{sort}_{1,2}(\mathfrak{C}), \mathfrak{B}_{\mathbf{c}_{2,3}} = \mathfrak{C}$,

(3) $\mathfrak{A}_{\mathbf{c}_{1,3}} = \mathrm{sort}_1(\mathfrak{C}), \mathfrak{B}_{\mathbf{c}_{1,3}} = \mathfrak{C}$.

Lemma 3.2. *(Transitivity) Let $\mathfrak{C}$ be as in Hypothesis 3.1. If $\mathbf{c}_{1,2}$ and $\mathbf{c}_{2,3}$ are solvable, then $\mathbf{c}_{1,3}$ is solvable.*

Proof. According to Definition 2.6 there are definable class functions

$$F_{\mathbf{c}_{1,2}} : K^1_{\mathbf{c}_{1,2}} \to K^2_{\mathbf{c}_{1,2}}, \text{ and}$$

$$F_{\mathbf{c}_{2,3}} : K^1_{\mathbf{c}_{2,3}} \to K^2_{\mathbf{c}_{2,3}}.$$

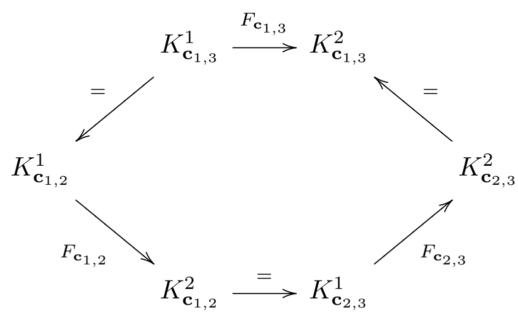
Recall from Hypothesis 3.1 that the following three equalities are satisfied

(i) $K^1_{\mathbf{c}_{1,2}} = \mathrm{sort}_1(\mathfrak{C}) = K^1_{\mathbf{c}_{1,3}}$,

(ii) $K^1_{\mathbf{c}_{2,3}} = \mathrm{sort}_{1,2}(\mathfrak{C}) = K^2_{\mathbf{c}_{1,2}}$,

(iii) $K^2_{\mathbf{c}_{2,3}} = \mathfrak{C} = K^2_{\mathbf{c}_{1,3}}$.

This enables us to get the composition $F_{\mathbf{c}_{1,3}} = F_{\mathbf{c}_{2,3}} \circ F_{\mathbf{c}_{1,2}}$. Let us summarize things in the following commutative diagram:



Then the function $F_{\mathbf{c}_{1,3}}$ witnesses that $\mathbf{c}_{1,3}$ is solvable. □

Conclusion 3.3. *Let $\mathfrak{C}$ be as in Hypothesis 3.1 and let χ be a cardinal. If $\mathbf{c}_{1,3}$ is not χ-solvable and $\mathbf{c}_{2,3}$ is solvable. Then $\mathbf{c}_{1,2}$ is not χ-solvable.*

Proof. Assume by the way of contradiction that $\mathbf{c}_{1,2}$ is not χ-solvable. It follows from Lemma 3.2 that $\mathbf{c}_{1,3}$ is χ-solvable, a contradiction. □

Corollary 3.4. *Let $\mathfrak{C}$ be as in Hypothesis 3.1, and let χ be an infinite cardinal. Assume in addition that*

(a) $\varphi_{\mathbf{c}_{1,3}}$ has no weak lifting, and

(b) $\varphi_{\mathbf{c}_{2,3}}$ has a lifting.

Then there is a forcing extension of the universe in which $\mathbf{c}_{1,2}$ is not χ-solvable.

Proof. Fix an infinite cardinal χ. As $\varphi_{\mathbf{c}_{2,3}}$ has a lifting, and in the light of [8, Theorem 3] we observe that it is solvable. Hence, for some cardinal $\chi' \geq \chi$, it is χ'-solvable. As $\varphi_{\mathbf{c}_{1,3}}$ has no weak lifting, thus by [9, Theorem 4], we can find a generic extension $V[G]$ of the universe, in which $\varphi_{\mathbf{c}_{1,3}}$ is not χ'-solvable. In view of Conclusion 3.3 we know that $\mathbf{c}_{1,2}$ is not χ'-solvable. Hence not χ-solvable in $V[G]$, as well. □

Definition 3.5. Let G' be a group, and let ψ be an automorphism of G'. Let G be a $(\mathbb{Z}, G')$-construction. We are going to use the multiplicative structure of $\mathbb{Z}$. In this regard, we identify $\mathbb{Z}$ with $\{y^n : n \in \mathbb{Z}\}$. For simplicity, we denote (y^n, x) with $y^n x$. So, as a set we have

$$G := \{y^n x : n \in \mathbb{Z} \text{ and } x \in G'\}.$$

The multiplication of $x \in G'$ from left hand side is defined via the role

$$x \times y := y\psi(x) \in G.$$

An easy induction shows that $x \times y^n := y^n \psi^n(x) \in G$, and consequently, $(G, \times)$ becomes a group, denoted by $\mathbb{Z} \ltimes_\psi G'$.

Proposition 3.6. *Let G_1 and G_2 be two groups and suppose there is an onto homomorphism $\varphi_{1,2} : G_2 \twoheadrightarrow G_1$ that has no lifting but has a weak lifting. Then, there are G_3 and $\varphi_{2,3}$, such that:*

- *(a) G_3 is a group;*
- *(b) $\varphi_{2,3} : G_3 \twoheadrightarrow G_2$ is a surjective homomorphism which has a lifting;*
- *(c) $\varphi_{1,3} = \varphi_{1,2} \circ \varphi_{2,3} \in \mathrm{Hom}(G_3, G_2)$ is a surjective homomorphism with no weak lifting.*

Proof. (a): First, we introduce the auxiliary group $G'_3 := \bigoplus\{G_{2,n} : n \in \mathbb{Z}\}$, where for each $n \in \mathbb{Z}$, $G_{2,n} \cong G_2$. Let also ψ_n denote the corresponding isomorphism $\psi_n : G_2 \to G_{2,n}$. Note that:

1. $\bigcup\limits_{n \in \mathbb{Z}} G_{2,n}$ generates G'_3,
2. $G_{2,n} \cap \Sigma\{G_{2,k} : k \in \mathbb{Z}\backslash\{n\}\} = \{1_{G'_3}\}$, and
3. the family of groups $\{G_{2,n} : n \in \mathbb{Z}\}$ pairwise commutes.

Let ψ_* be the automorphism of G'_3 such that, for each m,

$$\psi_* {\restriction} G_{2,m} = \psi_{m+1} \circ \psi_m^{-1}.$$

Set $G_3 := \mathbb{Z} \ltimes_{\psi_*} G'_3$. Recall from Definition 3.5 that G_3 is generated by $G'_3 \cup \{y\}$, subject to the following relations

$$y^{-1} x y = \psi_*(x) \quad \forall x \in G'_3.$$

(b): Let $\varphi_{2,3} : G_3 \to G_2$ be the unique homomorphism from G_3 onto G_2 such that:

- $\bullet_1$ $\varphi_{2.3}\restriction G_{2,n} = \psi_n^{-1}$ for $n \in \mathbb{Z}$,
- $\bullet_2$ $\varphi_{2.3}(y) = 1_{G_2}$.

Now ψ_0 is clearly a lifting of $\varphi_{2,3}$, i.e. clause (b) holds.

(c): In order to prove clause (c), we first prove the following claim.

Claim 3.7. *G_3 has trivial center.*

Proof. Suppose $x \in G_3'\backslash\{1_{G_3'}\}$. Clearly $y^{-1}xy \neq x$ (using $\mathrm{supp}(x)$). Suppose $\alpha \in G_3\backslash G_3'$. By Definition 3.5, α has the form $y^n x_1$ with $n \in \mathbb{Z}, x_1 \in G_3'$ and $n \neq 0 \vee x_1 \neq 1_{G_3'}$. Let $x_2 \in G_2'\backslash\{1_{G_2'}\}$ be such that

$$\ell := \min(\mathrm{supp}(x_2)) > \max(\mathrm{supp}(x_1)) + n.$$

Suppose on the way of contradiction that $y^n x_1$ and x_2 commute with each other. This yields that

$$\begin{aligned} y^n(x_1x_2) &= (y^n x_1)x_2 \\ &= x_2(y^n x_1) \\ &= x_2(yy^{n-1}x_1) \\ &= (x_2y)y^{n-1}x_1 \\ &= (y\psi_*(x_2))y^{n-1}x_1 \\ &= y^n(\psi_*^n(x_2)x_1). \end{aligned}$$

By definition, this implies

$$\psi_*^n(x_2)x_1 = x_1x_2 \quad (+)$$

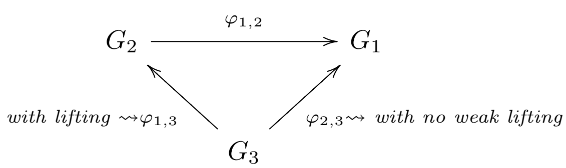
Recall that $\psi_*\restriction G_{2,m} = \psi_{m+1}\circ\psi_m^{-1}$. By applying this to (+), we see some coordinate of its left hand side is nonzero, but the corresponding coordinate in the right hand side is nonzero. This contradiction shows that G_3 has trivial center, as claimed. $\square$

By definition $\varphi_{1,3} \in \mathrm{Hom}(G_3, G_2)$ is onto. We show that it has no weak lifting. Suppose towards contradiction that there is a function $\psi : G_1 \to G_3$ such that $\varphi_{1,3}\circ\psi = \mathrm{id}_{G_1}$ and the composite mapping $\pi_H\circ\psi$ belongs to $\mathrm{Hom}(G_1, G_3/\mathcal{Z}(G_3))$, where $\pi_H : G_3 \to G_3/\mathcal{Z}(G_3)$ is the canonical homomorphism. But, according to Claim 3.7, G_3 has trivial center. So ψ is a homomorphism from G_1 into G_3, hence

$\varphi_{2,3} \circ \psi$ is a homomorphism from G_1 into G_2; in fact, is one to one as $\varphi_{1,3} \circ \psi$ is. This contradicts our assumption that $\varphi_{1,2}$ has no lifting. $\square$

Corollary 3.8. *Let χ be a cardinal. Suppose $\varphi \in \mathrm{Hom}(G_2, G_1)$ is onto, and does not split. Then there exists a weak uni-construction problem $\mathbf{c}$, such that in some forcing extension, $\mathbf{c}$ is not χ-solvable.*

Proof. Let G_1 and G_2 be as above and set $\varphi_{1,2} := \varphi$. By Proposition 3.6, we can find a group G_3 and two homomorphisms $\varphi_{2,3}$ and $\varphi_{1,3}$, which fit in the following diagram



Without loss of generality the groups G_1, G_2 and G_3 are pairwise disjoint. We define the three sorted model $\mathfrak{C}$ as follows:

- $(*)$ (a) the set of elements of $\mathrm{sort}_\ell(\mathfrak{C})$ is the set of elements of G_ℓ for $\ell = 1, 2, 3$;
  - (b) $F_\ell^{\mathfrak{C}} = \varphi_{\ell,\ell+1}$ for $\ell = 1, 2$;
  - (c) for $\ell = 1, 2, 3$ and $a \in G_\ell$, the homomorphism $F_{\ell,a}^{\mathfrak{C}} : G_\ell \to G_\ell$ is defined as $F_{\ell,a}^{\mathfrak{C}}(b) = a \cdot b$ for $b \in G_\ell$.

Now, we show that the assumptions of Corollary 3.4 hold. To this end, note that we consider the sorts only as a set of elements and forget about the multiplication in G_i. The entire structure is given by the functions. We claim that

- (i) $G_1 \cong \mathrm{Aut}(\mathrm{sort}_1(\mathfrak{C})), G_2 \cong \mathrm{Aut}(\mathrm{sort}_{1,2}(\mathfrak{C}))$ and $G_3 \cong \mathrm{Aut}(\mathfrak{C})$ and
- (ii) $\varphi_{i,j} : \mathrm{Aut}(\mathrm{sort}_{i,j}(\mathfrak{C})) \to \mathrm{Aut}(\mathrm{sort}_i(\mathfrak{C}))$ is the natural restriction map.

For clause (i), we only show that $G_3 \cong \mathrm{Aut}(\mathfrak{C})$, as the other cases can be proved in a similar way.

In order to see $G_3 \cong \mathrm{Aut}(\mathfrak{C})$, define the map

$$\theta : G_3 \to \mathrm{Aut}(\mathfrak{C})$$

which sends some element $c \in G_3$ to the map $\sigma_c : \mathfrak{C} \to \mathfrak{C}$, which is defined via

$$\sigma_c(g) = \begin{cases} c \cdot g & \text{if } g \in G_3 \\ \varphi_{2,3}(c) & \text{if } g \in G_2 \\ \varphi_{1,3}(c) & \text{if } g \in G_1 \end{cases}$$

We have to show that θ is well-defined. Let $c \in G_3$. First we show that $\theta(c)$ is a homomorphism of $\mathfrak{C}$. Consider $a \in G_3$, and show it behaves well with respect to $F^{\mathfrak{C}}_{a,3}$:

$$\sigma_c(F^{\mathfrak{C}}_{a,3}(b)) = c \cdot b \cdot a = \sigma_c(b) \cdot a = F^{\mathfrak{C}}_{a,3}(\sigma_c(b)),$$

where, $b \in G_3$. Furthermore, it behaves well with respect to $F^{\mathfrak{C}}_{2,3}$:

$$\sigma_c(F^{\mathfrak{C}}_{2,3}(a)) = \sigma_c(\varphi_{2,3}(a)) = \varphi_{2,3}(c) \cdot \varphi_{2,3}(a) = \varphi_{2,3}(c \cdot a) = \varphi_{2,3}(\sigma_c(a)) = F^{\mathfrak{C}}_{2,3}(\sigma_c(a)).$$

Similar arguments apply for $a, b \in G_i$ with $i = 1, 2$, whence σ_c is indeed a homomorphism of $\mathfrak{C}$. It is easy to see that each $\theta(c)$ is a bijection, as it corresponds to left translation in the groups. Thus $\theta(c) \in \mathrm{Aut}(\mathfrak{C})$, and θ is well-defined.

Now it remains to show that θ is an isomorphism of groups. In order to be an homomorphism, it has to respect the function symbols. Let $c_1, c_2 \in G_3$ be arbitrary. Then

$$\theta(c_1 \cdot c_2)(a) = c_1 \cdot c_2 \cdot a = \theta(c_1)(c_2 \cdot a) = (\theta(c_1)\theta(c_2))(a),$$

for any $a \in G_3$ and similarly for $a \in G_i$ with $i = 1, 2$.

Clearly, $\theta(c^{-1}) = \theta(c)^{-1}$. Finally, the map θ is injective, as for $c \neq c'$ and $a \in G_3$, we have that

$$\sigma_c(a) = c \cdot a \neq c' \cdot a = \sigma_{c'}(a).$$

For surjectivity, consider an arbitrary automorphism $\sigma \in \mathrm{Aut}(\mathfrak{C})$. It is easy to check that for $c = \sigma(1_{G_3})$, we have $\sigma = \sigma_c$, by using the identity

$$\sigma(F^{\mathfrak{C}}_a(1_{G_3})) = F^{\mathfrak{C}}_a(\sigma(1_{G_3})).$$

This concludes that $\theta : G_3 \to \mathrm{Aut}(\mathfrak{C})$ is an isomorphism of groups.

To prove clause (ii), we have to show that the maps $\varphi_{i,j}$ correspond to the natural restriction maps between the automorphism groups. This is immediate, as we saw that any automorphism of $\mathfrak{C}$ corresponds to the right translation on each of the sorts and if σ restricted to the j^{th} sort is the translation by $b \in G_j$, then it is translation by $\varphi_{i,j}(b)$ on the sort i.

In the light of Proposition 3.6, and in view of the way we defined the 3-sorted model $\mathfrak{C}$, we observe that $\varphi_{\mathbf{c}_{1,3}}$ has no weak lifting, and $\varphi_{\mathbf{c}_{2,3}}$ has a lifting. This allows us to apply Corollary 3.4 and deduce that $\mathbf{c}_{1,2}$ is not χ-solvable in some forcing extension. But $\mathbf{c}_{1,2} = \mathbf{c}$, and the corollary follows. □

Now, we are ready to prove the main result of this section.

Theorem 3.9. *Let χ be a cardinal and $\mathbf{c}$ be a uni-construction problem. If $\mathbf{c}$ has no lifting, then in some forcing extension, $\mathbf{c}$ has no χ-solution.*

Proof. Recall that $G_{\mathbf{c}} := \mathrm{Aut}(\mathfrak{A}_{\mathbf{c}})$ and $H_{\mathbf{c}} := \mathrm{Aut}(\mathfrak{B}_{\mathbf{c}})$. We set $G_1 := G_{\mathbf{c}}, G_2 := H_{\mathbf{c}}$ and $\varphi_{1,2} := \varphi_{\mathbf{c}}$. In the light of Proposition 3.6 we can find a group G_3, an a surjective homomorphism $\varphi_{2,3} : G_3 \twoheadrightarrow G_2$ which has a lifting such that the homomorphism $\varphi_{1,3} = \varphi_{1,2} \circ \varphi_{2,3} \in \mathrm{Hom}(G_3, G_2)$ has no weak lifting. Without loss of generality we may assume that

$$a \in \mathfrak{B}_{\mathbf{c}} \wedge b \in G_3 \Rightarrow a \neq b.$$

Let $\langle a_\alpha : \alpha < \alpha_* \rangle$ list the elements of $\mathfrak{B}_{\mathbf{c}}$. We define a 3-sorted model $\mathfrak{C}$ as follows:

$(*)$ (a) $\mathrm{sort}_{1,2}(\mathfrak{C}) = \mathfrak{B}_{\mathbf{c}}$ so $\mathrm{sort}_1(\mathfrak{C}) = \mathfrak{A}_{\mathbf{c}}$;

(b) the set of elements of $\mathrm{sort}_3(\mathfrak{C})$ is the set of elements of G_3;

(c) $F^{\mathfrak{C}}_{1,\alpha} : G_3 \to \mathrm{sort}_1(\mathfrak{C})$ is defined by the help of $\varphi_{1,3}$. More precisely, suppose $b \in G_3$, then we set $F^{\mathfrak{C}}_{1,\alpha}(b) = (\varphi_{1,3}(b))(a_\alpha)$;

(d) $F^{\mathfrak{C}}_{2,\alpha} : G_3 \to \mathfrak{B}_{\mathbf{c}}$ is defined by:

$$b \in G_3 \Rightarrow F^{\mathfrak{C}}_{2,\alpha}(b) = (\varphi_{2,3}(b))(a_\alpha);$$

(e) $F^{\mathfrak{C}}_{3,c} : G_3 \to G_3$, where $c \in G_3$, is defined by:

$$b \in G_3 \Rightarrow F^{\mathfrak{C}}_{3,c}(b) = cb.$$

By the same vein as in the proof of Corollary 3.8, we observe that

$(**)$ (a) $\mathfrak{A}_{\mathbf{c}_{1,2}} = \mathrm{sort}_1(\mathfrak{C}), \mathfrak{B}_{\mathbf{c}_{1,2}} = \mathrm{sort}_{1,2}(\mathfrak{C})$,

(b) $\mathfrak{A}_{\mathbf{c}_{2,3}} = \mathrm{sort}_{1,2}(\mathfrak{C}), \mathfrak{B}_{\mathbf{c}_{2,3}} = \mathfrak{C}$,

(c) $\mathfrak{A}_{\mathbf{c}_{1,3}} = \mathrm{sort}_1(\mathfrak{C}), \mathfrak{B}_{\mathbf{c}_{1,3}} = \mathfrak{C}$.

By our construction, it is easily seen tha $\varphi_{\mathbf{c}_{1,3}}$ has no weak lifting, and $\varphi_{\mathbf{c}_{2,3}}$ has a lifting.

It follows from Corollary 3.4 that there exists a forcing extension $V[G]$ of the universe in which $\mathbf{c}_{1,2}$ has no χ-solution. But $\mathbf{c}_{1,2} = \mathbf{c}$, and hence $\mathbf{c}$ has no χ-solution in $V[G]$. The theorem follows. □

§ 4. A global consistency result

For an infinite cardinal λ, let $\mathbb{S}_\lambda$ be the forcing notion

$$\mathbb{S}_\lambda = \{p : \lambda^{++} \times \lambda^{++} \times \lambda^{++} \to 2 : |p| \leq \lambda\},$$

ordered by reverse inclusion. Thus $\mathbb{S}_\lambda$ is forcing equivalent to $\mathrm{Add}(\lambda^+, \lambda^{++})$, the Cohen forcing for adding λ^{++}-many Cohen subsets of λ^+. Furthermore, it is λ^+-closed and satisfies the λ^{++}-c.c. In [9], the following is proved.

Theorem 4.1. *(GCH) Let M be an inner model of ZFC + GCH and let λ be an infinite cardinal of M. Let $\mathbb{Q}$ be the forcing notion $\mathbb{S}_\lambda$ as computed in M and let $\mathbf{G}$ be $\mathbb{Q}$-generic over M. Then the following holds in $M[\mathbf{G}]$:*

$(*)_\lambda$: *suppose $\mathbf{c}$ is a uniformisable uni-construction problem, such that $\mathbf{c}$ is defined using parameters from V, $\mathcal{B}_\mathbf{c} \in V$ and $\mathcal{B}_\mathbf{c}$ and $\mathrm{Aut}(\mathcal{B}_\mathbf{c})$ have size $\leq \lambda$. Then $\mathbf{c}$ is weakly natural.*

In this section we are going to prove a global version of this theorem, which removes both the cardinality assumption and the parameter assumption from the above result. The proof uses the reverse Easton iteration of forcing notions, where we refer to [2] and [10, Chapter 21] for more details on this subject.

Theorem 4.2. *(GCH) Then there exists a GCH and cofinality preserving class generic extension $V[\mathbf{G}]$ of the universe in which the following hold:*

$(*)$ *If $\mathbf{c}$ is a uniformisable uni-construction problem, then $\mathbf{c}$ is weakly natural.*

Proof. Given an infinite cardinal λ set $\mathbb{S}_\lambda$ be defined as above. Let

$$\mathbb{P} = \big\langle \langle \mathbb{P}_\lambda : \lambda \in \mathrm{Ord} \rangle, \langle \underset{\sim}{\mathbb{Q}}_\lambda : \lambda \in \mathrm{Ord} \rangle \big\rangle$$

be the reverse Easton iteration of forcing notions, such that for each ordinal $\lambda, \underset{\sim}{\mathbb{Q}}_\lambda$ is forced to be the trivial forcing notion except λ is an infinite cardinal, in which case we let

$$\Vdash_{\mathbb{P}_\lambda} \text{“}\underset{\sim}{\mathbb{Q}}_\lambda = \underset{\sim}{\mathbb{S}}_\lambda\text{”}.$$

Thus a condition in $\mathbb{P}$ is a partial function p such that:

(1) $\mathrm{dom}(p)$ is a set of ordinal,
(2) if $\lambda \in \mathrm{dom}(p)$, then $p \restriction \lambda \Vdash_{\mathbb{P}_\lambda}$ “$p(\lambda) \in \underset{\sim}{\mathbb{Q}}_\lambda$”,
(3) for any regular cardinal κ, $|\mathrm{supp}(p) \cap \kappa| < \kappa$, where

$$\mathrm{supp}(p) = \{\lambda \in \mathrm{dom}(p) : p \restriction \lambda \Vdash_{\mathbb{P}_\lambda} \text{“}p(\lambda) \neq 1_{\underset{\sim}{\mathbb{Q}}_\lambda}\text{”}\}.$$

Let also

$$\mathbf{G} = \big\langle \langle \mathbf{G}_\lambda : \lambda \in \mathrm{Ord}\rangle, \langle \mathbf{H}_\lambda : \lambda \in \mathrm{Ord}\rangle \big\rangle$$

be $\mathbb{P}$-generic over V. Thus for each infinite cardinal λ, $\mathbf{G}_\lambda = \mathbb{P}_\lambda \cap \mathbf{G}$ is $\mathbb{P}_\lambda$-generic over V and H_λ is $\underset{\sim}{\mathbb{S}}_\lambda[\mathbf{G}_\lambda]$-generic over $V[\mathbf{G}_\lambda]$.

We show that $V[\mathbf{G}]$ is as required. To make things easire, we recall the following well-known result from [2] (also see [10, Chapter 21]):

Lemma 4.3. *(1) $V[\mathbf{G}]$ is a GCH and cofinality preserving class generic extension of V,*
(2) $V[\mathbf{G}]$ and $V[\mathbf{G}_\lambda]$ contain the same λ-sequences of ordinals.

Now suppose $\mathbf{c}$ is a uniformisable uni-construction problem. Let λ be a large enough cardinal such that $\mathcal{B}_\mathbf{c}$, $\mathrm{Aut}(\mathcal{B}_\mathbf{c})$ and the parameters occurring in the definition of $\mathbf{c}$ and the formula uniformizing it are all in $V[\mathbf{G}_\lambda]$, and $|\mathcal{B}_\mathbf{c}|, |\mathrm{Aut}(\mathcal{B}_\mathbf{c})| \leq \lambda$.

By Theorem 4.1, applied to the model $V[\mathbf{G}_\lambda]$ and the uni-construction problem $\mathbf{c}$, we can conclude that

$$V[\mathbf{G}_\lambda][H_\lambda] \models \text{“}\mathbf{c} \text{ is weakly natural”}.$$

On the other hand, $V[\mathbf{G}]$ is a generic extension of $V[\mathbf{G}_\lambda][H_\lambda]$ by a class forcing notion which adds no new subsets to λ^+. Thus

$$V[\mathbf{G}] \models \text{“}\mathbf{c} \text{ is weakly natural”}.$$

The theorem follows. □

§ 5. Uniformity

In this section we deal with Problem 1.4. Our main result is Theorem 5.1, which can be considered as a generalization of [8, Theorem 3]. We state our result in terms of two sorted models.

Theorem 5.1. *Assume* $\mathfrak{B}$ *is a two sorted model in the vocabulary* τ *and* $\mathfrak{A} = \mathrm{sort}_1(\mathfrak{B})$. *Suppose that* K *is a class of* τ*-models which is first order definable from a parameter* $\mathbf{p}$ *such that for every* $\mathfrak{B} \in K$*, the natural homomorphism* $\pi^*_{\mathfrak{B}}$ *from* $\mathrm{Aut}(\mathfrak{B})$ *onto* $\mathrm{Aut}(\mathrm{sort}_1(\mathfrak{B}))$ *splits. Then there exists a class function* F*, which is uniformly definable from the parameter* $\mathbf{p}$*, such that:*

- *(a) the domain of* F *is* $K_1 = \{\mathrm{sort}_1(\mathfrak{B}) : \mathfrak{B} \in K\}$,
- *(b) if* $\mathfrak{A} \in K_1$*, then* $F(\mathfrak{A}) \in K$ *and* $\mathrm{sort}_1(F(\mathfrak{A})) = \mathfrak{A}$.

Proof. We prove the theorem in a sequence of claims. We first show that we can be reduced to the case where K has only one equivalence class. To this end, define the class function $\mathbf{H}$, with $\mathrm{dom}(\mathbf{H}) = K$, by

$$\mathbf{H}(\mathfrak{B}) := \Big\{\mathfrak{B}' : \mathfrak{B}' \text{ is a } \tau\text{-model isomorphic to } \mathfrak{B} \text{ with universe the cardinal } \|\mathfrak{B}\|\Big\}.$$

Clearly, $\mathbf{H}$ is definable from the parameter $\mathbf{p}$. For every $\mathbf{x} \in \mathrm{Range}(\mathbf{H})$, set

(1) $K_{\mathbf{x}} := \{\mathfrak{B} \in K : \mathbf{H}(\mathfrak{B}) = \mathbf{x}\}$,

(2) $K_{\mathbf{x},1} := \{\mathrm{sort}_1(\mathfrak{B}) : \mathfrak{B} \in K_{\mathbf{x}}\}$.

We remark that $\langle K_{\mathbf{x}} : \mathbf{x} \in \mathrm{Rang}(\mathbf{H})\rangle$ is a partition of K, which is uniformly definable using the parameter $\mathbf{p}$, so it suffices to uniformly deal with $K_{\mathbf{x}}$ for each $\mathbf{x} \in \mathrm{Range}(\mathbf{H})$. Thus, let us fix some $\mathbf{x} \in \mathrm{Range}(\mathbf{H})$. Here, we consider to Υ. It is the family of all pairs $(\mathfrak{B}, \psi)$ equipped with the following properties:

Υ_1) $\mathfrak{B} \in K$ is such that for some $\mathfrak{B}' \in \mathbf{x}$ the function $b \mapsto (\mathfrak{B}', b)$, for $b \in \mathfrak{B}'$, is an isomorphism from $\mathfrak{B}'$ onto $\mathfrak{B}$.

Υ_2) ψ is a weak lifting for $\mathfrak{B}$.

Let us collect all of them with a new name:

(3) Let $\mathbf{y}_{\mathbf{x}} := \big\{(\mathfrak{B}, \psi) : (\mathfrak{B}, \psi) \in \Upsilon\big\}$.

To make things be easier, we bring a series of claims (see Claim 5.2-5.9):

Claim 5.2. $\mathbf{y}_{\mathbf{x}}$ *is non-empty.*

Proof. By our hypothesis, for each $\mathfrak{B} \in K$, there is a weak lifting for $\mathfrak{B}$, and hence $\mathbf{y}_{\mathbf{x}}$ is non-empty, as requested. □

Let $S = \mathbf{y}_{\mathbf{x}}$, and for each $s \in S$ set $(\mathfrak{B}_s, \psi_s) = s$. It then follows that

$$\mathbf{y}_{\mathbf{x}} = \{(\mathfrak{B}_s, \psi_s) : s \in S\}.$$

For $s \in S$ set

(4) $\mathfrak{A}_s := \mathrm{sort}_1(\mathfrak{B}_s)$.

By replacing each $\mathfrak{B}_s$ by $\mathfrak{B}_s \times \{s\}$ if necessary, we may assume that the $\mathfrak{B}_s$'s are pairwise disjoint. For $b \in \bigcup\limits_{s \in S} \mathfrak{B}_s$ let

$$r(b) = \text{the unique } s \in S \text{ such that } b \in \mathfrak{B}_s.$$

Claim 5.3. $\mathbf{x}$ *is definable from* $\mathbf{y}_{\mathbf{x}}$.

Proof. Recall that $\mathbf{x}$ is equal to

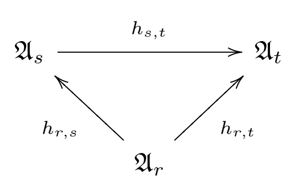
$$\Big\{\mathfrak{B}' : \exists(\mathfrak{B}, \psi) \in \mathbf{y}_{\mathbf{x}}\big(\textit{the function } b \mapsto (\mathfrak{B}', b) \textit{ induces } \mathfrak{B}' \xrightarrow{\cong} \mathfrak{B}\big)\Big\},$$

which concludes the result. □

We say the $\mathfrak{A}_\bullet$-family $\big\langle h_{s,t} : \mathfrak{A}_s \xrightarrow{\cong} \mathfrak{A}_t \big\rangle_{s,t \in S}$ of isomorphisms is commutative provided:

- $h_{r,s} \circ h_{s,t} = h_{r,t}$ and
- $h_{s,s} = \mathrm{id}_{\mathfrak{A}_s}$,

where $r, s, t \in S$. So, the diagram



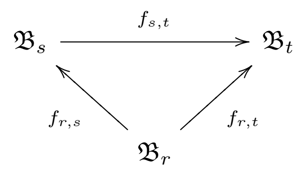
is commutative.

(5) By $\mathrm{Aut}_1(\mathbf{y}_{\mathbf{x}})$ we mean

$$\big\{\bar{h} = \langle h_{s,t} : s, t \in S\rangle : \langle h_{s,t} : s, t \in S\rangle \textit{ is a commutaive } \mathfrak{A}_\bullet - \textit{family}\big\}.$$

Similarly, we say the $\mathfrak{B}_\bullet$-family $\langle f_{s,t} : \mathfrak{B}_s \xrightarrow{\cong} \mathfrak{B}_t\rangle_{s,t\in S}$ of isomorphisms is commutative if

$f_{r,s} \circ f_{s,t} = f_{r,t}$ and $f_{s,s} = \mathrm{id}_{\mathfrak{B}_s}$ for all $r, s, t \in S$. So, the diagram



is commutative.

(6) By $\mathrm{Aut}_2(\mathbf{y}_{\mathbf{x}})$ we mean

$$\big\{\bar{f} = \langle f_{s,t} : s, t \in S\rangle : \langle f_{s,t} : s, t \in S\rangle \textit{ is a commutaive } \mathfrak{B}_\bullet - \textit{family}\big\}.$$

For $\mathfrak{A} \in K_{\mathbf{x},1}$ let

(7) $\mathrm{iso}_{\mathbf{y}_{\mathbf{x}}}(\mathfrak{A})$ be the following

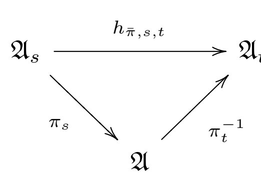
$$\{\bar{\pi} : \bar{\pi} = \langle \pi_s : s \in S\rangle \textit{ such that } \pi_s : \mathfrak{A}_s \xrightarrow{\cong} \mathfrak{A}\}.$$

Suppose $\bar{\pi} \in \mathrm{iso}_{\mathbf{y}_{\mathbf{x}}}(\mathfrak{A})$ and $s, t \in S$. We set

(8) $h_{\bar{\pi},s,t} = \pi_t^{-1} \circ \pi_s \in \mathrm{iso}(\mathfrak{A}_s, \mathfrak{A}_t)$,

(9) $\bar{h}_{\bar{\pi}} = \langle h_{\bar{\pi},s,t} : s, t \in S\rangle$.

This property can be summarized by the subjoined diagram:

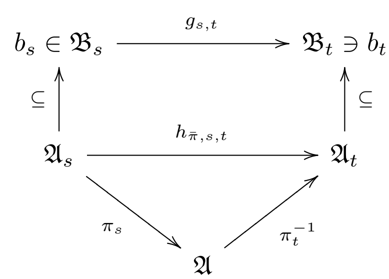


The proof of next claim is evident.

Claim 5.4. *Suppose* $\mathfrak{A} \in K_{\mathbf{x},1}$ *and* $\bar{\pi} \in \mathrm{iso}_{\mathbf{y}_{\mathbf{x}}}(\mathfrak{A})$. *Then* $\bar{h}_{\bar{\pi}} \in \mathrm{Aut}_1(\mathbf{y}_{\mathbf{x}})$.

For $\mathfrak{A} \in K_{\mathbf{x},1}$, suppose $\bar{\pi} \in \mathrm{iso}_{\mathbf{y_x}}(\mathfrak{A})$ and $\bar{g} \in \mathrm{Aut}_2(\mathbf{y_x})$ are such that $g_{s,s} = \psi_s(h_{\bar{\pi},s,s})$ [1] and $h_{\bar{\pi},s,t} \subseteq g_{s,t}$ for $s,t \in S$. Let also $\bar{b} = \langle b_s : s \in S\rangle$. We say the triple $(\bar{\pi}, \bar{g}, \bar{b})$ is *match*, if $\bar{\pi}$, $\bar{g}$ and $\bar{b}$ are as above, $b_s \in \mathfrak{B}_s$ and $g_{s,t}(b_t) = b_s$ for $s,t \in S$.

Let $\Sigma(\mathbf{x}, \mathfrak{A})$ denote the family of all triples $(\bar{\pi}, \bar{g}, \bar{b})$ as above which are match. The following diagram summarizes the above situation:



(10) $X_{\mathbf{y_x}}(\mathfrak{A}) := \big\{(\bar{\pi}, \bar{g}, \bar{b}) : (\bar{\pi}, \bar{g}, \bar{b}) \in \Sigma(\mathbf{x}, \mathfrak{A})\big\}$.

For $\mathfrak{A} \in K_{\mathbf{x},1}$, let $E_{\mathfrak{A}}$ be the following two place relation on $X_{\mathbf{y_x}}(\mathfrak{A})$:

$$(\bar{\pi}, \bar{g}_1, \bar{b}_1) E_{\mathfrak{A}} (\pi_2, \bar{g}_2, \bar{b}_2)$$

if and only if

- $(\bar{\pi}_1, \bar{g}_1, \bar{b}_1)$ and $(\bar{\pi}_2, \bar{g}_2, \bar{b}_2)$ belong to $X_{\mathbf{y_x}}(\mathfrak{A})$,
- letting $\widetilde{\pi}_s = \pi_{1,s}^{-1}\pi_{2,s}$ and $h_s = \psi_s(\widetilde{\pi}_s^{-1})$, we have $h_s(b_{1,s}) = b_{2,s}$.

Note that $\widetilde{\pi}_s$ is an automorphism of $\mathfrak{A}_s$, hence so is $\widetilde{\pi}_s^{-1}$, and hence $h_s = \psi_s(\widetilde{\pi}_s^{-1})$ is a member of $H_s = \mathrm{Aut}(\mathfrak{B}_s)$ which extends $\widetilde{\pi}_s$. Since $b_{1,2}, b_{2,s} \in \mathfrak{B}_s$ so $h_s(b_{1,s}) = b_{2,s} \in \mathfrak{B}_s$ is meaningful.

Claim 5.5. *$E_{\mathfrak{A}}$ is an equivalence relation.*

Proof. It is clearly reflexive and symmetric. To show that $E_{\mathfrak{A}}$ is transitive, assume that

(a) $(\bar{\pi}_1, \bar{g}_1, \bar{b}_1) E_{\mathfrak{A}} (\bar{\pi}_2, \bar{g}_2, \bar{b}_2)$,

(b) $(\bar{\pi}_2, \bar{g}_2, \bar{b}_2) E_{\mathfrak{A}} (\bar{\pi}_3, \bar{g}_3, \bar{b}_3)$.

By the definition of $E_{\mathfrak{A}}$

[1]Recall that ψ_s is a weak lifting for $\mathfrak{B}_s$.

(c) $(\psi_s(\pi_{1,s}^{-1}\pi_{2,s})^{-1})(b_{1,s}) = b_{2,s}$ for $s \in S$

(d) $(\psi_s(\pi_{2,s}^{-1}\pi_{3,s})^{-1})(b_{2,s}) = b_{3,s}$ for $s \in S$.

Hence

$$\begin{aligned} b_{3,s} &= (\psi_s(\pi_{2,s}^{-1}\pi_{3,s})^{-1})(b_{2,s}) \\ &= (\psi_s(\pi_{2,s}^{-1}\pi_{3,s})^{-1})((\psi_s(\pi_{1,s}^{-1}\pi_{2,s})^{-1})(b_{1,s})) \\ &= (\psi_s\big((\pi_{2,s}^{-1}\pi_{3,s})^{-1} \circ (\pi_{1,s}^{-1}\pi_{2,s})^{-1}\big)(b_{1,s})) \\ &= \psi_s((\pi_{1,s}^{-1}\pi_{3,s})^{-1})(b_{1,s}), \end{aligned}$$

as requested. □

We define a model $\mathfrak{B}'_{\mathfrak{A}}$ as follows:[2]

(11.1) the universe of $\mathfrak{B}'_{\mathfrak{A}}$ is $X_{\mathbf{y}_\mathbf{x}}(\mathfrak{A})$,

(11.2) if $R \in \tau$ is an n-place relation, then

$$R^{\mathfrak{B}'_{\mathfrak{A}}} := \Big\{ \big\langle (\bar{\pi}_\ell, \bar{g}_\ell, \bar{b}_\ell) : \ell < n \big\rangle \in X_{\mathbf{y}_\mathbf{x}}(\mathfrak{A})^n : \exists s \in S\big(\langle b_{\ell,s} : \ell < n\rangle \in R^{\mathfrak{B}_s}\big) \Big\}.$$

Claim 5.6. *In the definition of $R^{\mathfrak{B}'_{\mathfrak{A}}}$, we can replace "$\exists s \in S$" by "$\forall s \in S$".*

Proof. Suppose $s, t \in S$ and $\langle (\bar{\pi}_\ell, \bar{g}_\ell, \bar{b}_\ell) : \ell < n\rangle \in X_{\mathbf{y}_\mathbf{x}}(\mathfrak{A})^n$. We have to show that

$$\langle b_{\ell,s} : \ell < n\rangle \in R^{\mathfrak{B}_s} \Leftrightarrow \langle b_{\ell,t} : \ell < n\rangle \in R^{\mathfrak{B}_t}.$$

This follows from the fact that $g_{s,t} : \mathfrak{B}_s \to \mathfrak{B}_t$ is an isomorphism and $g_{s,t}(b_{\ell,s}) = b_{\ell,t}$. □

Claim 5.7. *Assume $(\bar{\pi}^\iota_\ell, \bar{g}^\iota_\ell, \bar{b}^\iota_\ell) \in X_{\mathbf{y}_\mathbf{x}}(\mathfrak{A})$, for $\ell < n, \iota = 1, 2$, and suppose that for each $\ell < n$, $(\bar{\pi}^1_\ell, \bar{g}^1_\ell, \bar{b}^1_\ell) E_{\mathfrak{A}} (\bar{\pi}^2_\ell, \bar{g}^2_\ell, \bar{b}^2_\ell)$. Then*

$$\langle (\bar{\pi}^1_\ell, \bar{g}^1_\ell, \bar{b}^1_\ell) : \ell < n\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}} \Leftrightarrow \langle (\bar{\pi}^2_\ell, \bar{g}^2_\ell, \bar{b}^2_\ell) : \ell < n\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}}.$$

Proof. Assume $\langle (\bar{\pi}^1_\ell, \bar{g}^1_\ell, \bar{b}^1_\ell) : \ell < n\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}}$. For each $\ell < n$, we have

$$(\bar{\pi}^1_\ell, \bar{g}^1_\ell, \bar{b}^1_\ell) E_{\mathfrak{A}} (\bar{\pi}^2_\ell, \bar{g}^2_\ell, \bar{b}^2_\ell).$$

This in turns imply that

$$\psi_s((\pi^2_{\ell,s})^{-1}\pi^1_{\ell,s})(b^1_{\ell,s}) = b^2_{\ell,s},$$

[2]For simplicity, we assume that the structures $\mathfrak{B}$ are relational, so they only contain relations.

where $s \in S$. As $\psi_s((\pi^2_{\ell,s})^{-1}\pi^1_{\ell,s})$ is an automorphism of $\mathfrak{B}_s$, we have

$$\langle b^1_{\ell,s} : \ell < n\rangle \in R^{\mathfrak{B}_s} \Leftrightarrow \langle b^2_{\ell,s} : \ell < n\rangle \in R^{\mathfrak{B}_s}.$$

Hence:

$$\begin{aligned} \big\langle(\bar{\pi}^1_\ell, \bar{g}^1_\ell, \bar{b}^1_\ell) : \ell < n\big\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}} &\Rightarrow \exists s \in S \textit{ such that } \langle b^1_{\ell,s} : \ell < n\rangle \in R^{\mathfrak{B}_s} \\ &\Rightarrow \langle b^2_{\ell,s} : \ell < n\rangle \in R^{\mathfrak{B}_s} \\ &\Rightarrow \big\langle(\bar{\pi}^2_\ell, \bar{g}^2_\ell, \bar{b}^2_\ell) : \ell < n\big\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}}. \end{aligned}$$

By symmetry

$$\big\langle(\bar{\pi}^2_\ell, \bar{g}^2_\ell, \bar{b}^2_\ell) : \ell < n\big\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}} \Rightarrow \big\langle(\bar{\pi}^1_\ell, \bar{g}^1_\ell, \bar{b}^1_\ell) : \ell < n\big\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}}.$$

The claim follows. □

It follows from the above claim that $E_{\mathfrak{A}}$ is a congruence relation on $\mathfrak{B}'_{\mathfrak{A}}$. Define a function $\mathbf{k}$ with domain $\mathfrak{A}$ as follows:

$$\mathbf{k}(a) = \{(\bar{\pi}, \bar{g}, \bar{b}) \in X_{\mathbf{y_x}}(\mathfrak{A}) : b_s = \pi_s^{-1}(a) \text{ for all } s \in S\}.$$

Claim 5.8. *If $a \in \mathfrak{A}$, then $\mathbf{k}(a)$ is an $E_{\mathfrak{A}}$-equivalence class.*

Proof. Let us first show that it is closed under the relation $E_{\mathfrak{A}}$, in the sense that if $(\bar{\pi}_1, \bar{g}_1, \bar{b}_1) \in \mathbf{k}(a)$ and $(\bar{\pi}_1, \bar{g}_1, \bar{b}_1) E_{\mathfrak{A}} (\bar{\pi}_2, \bar{g}_2, \bar{b}_2)$, then $(\bar{\pi}_2, \bar{g}_2, \bar{b}_2) \in \mathbf{k}(a)$. Thus suppose $s \in S$. Then

- $b_{1,s} = \pi^{-1}_{1,s}(a)$,
- $\psi_s(\pi^{-1}_{2,s}\pi_{1,s})(b_{1,s}) = b_{2,s}$.

It then follows that

$$\begin{aligned} b_{2,s} &= \psi_s(\pi^{-1}_{2,s}\pi_{1,s})(b_{1,s}) \\ &= \psi_s(\pi^{-1}_{2,s}\pi_{1,s})(\pi^{-1}_{1,s}(a)) \\ &= \pi^{-1}_{2,s}\pi_{1,s}\pi^{-1}_{1,s}(a) \\ &= \pi^{-1}_{2,s}(a). \end{aligned}$$

Thus $(\bar{\pi}_2, \bar{g}_2, \bar{b}_2) \in \mathbf{k}(a)$, as requested.

Next we show that if $(\bar{\pi}_1, \bar{g}_1, \bar{b}_1), (\bar{\pi}_2, \bar{g}_2, \bar{b}_2) \in \mathbf{k}(a)$, then $(\bar{\pi}_1, \bar{g}_1, \bar{b}_1) E_{\mathfrak{A}} (\bar{\pi}_2, \bar{g}_2, \bar{b}_2)$. Let $s \in S$ and set $h_s = \psi_s(\pi^{-1}_{2,s}\pi_{1,s})$. We have to show that $h_s(b_{1,s}) = b_{2,s}$. But we have

$$\begin{aligned} h_s(b_{1,s}) &= \psi_s(\pi_{2,s}^{-1}\pi_{1,s})(b_{1,s}) \\ &= \psi_s(\pi_{2,s}^{-1}\pi_{1,s})(\pi_{1,s}^{-1}(a)) \\ &= \pi_{2,s}^{-1}\pi_{1,s}\pi_{1,s}^{-1}(a) \\ &= \pi_{2,s}^{-1}(a) \\ &= b_{2,s}. \end{aligned}$$

We are done. The claim follows immediately. $\square$

Claim 5.9. *$\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}}$ is isomorphic to $\mathfrak{B}_s$, for $s \in S$.*

Proof. Fix $s \in S$, and let $\phi : \mathfrak{A}_s \simeq \mathfrak{A}$ be an isomorphism. Set

$$Y_{s,\phi} := \{(\bar{\pi}, \bar{g}, \bar{b}) \in X_{\mathbf{y_x}}(\mathfrak{A}) : \pi_s = \phi\},$$

and define a function

$$\rho_{s,\phi} : Y_{s,\phi} \to \mathfrak{B}_s$$

as $\rho_{s,\phi}((\bar{\pi}, \bar{g}, \bar{b})) = b_s$. Clearly, $\rho_{s,\phi}$ is well-defined. Next, we bring the following two auxiliary observations:

$(*)_{5.9.1}$ Suppose $x_1 = (\bar{\pi}_1, \bar{g}_1, \bar{b}_1)$ and $x_2 = (\bar{\pi}_2, \bar{g}_2, \bar{b}_2)$ are in $Y_{s,\phi}$. Then

$$x_1 E_{\mathfrak{A}} x_2 \Rightarrow \rho_{s,\phi}(x_1) = \rho_{s,\phi}(x_2).$$

Proof. Suppose that $x_1 E_{\mathfrak{A}} x_2$. It then follows that $\psi_s(\pi_{2,s}^{-1}\pi_{1,s})(b_{1,s}) = b_{2,s}$. But as $\pi_{1,s} = \phi = \pi_{2s}$, we have

$$\psi_s(\pi_{2,s}^{-1}\pi_{1,s})(b_{1,s}) = \psi_s(\mathrm{id}_{\mathfrak{A}_s})(b_{1,s}) = \mathrm{id}_{\mathfrak{B}_s}(b_{1,s}) = b_{1,s},$$

and thus $\rho_{s,\phi}(x_1) = b_{1,s} = b_{2,s} = \rho_{s,\phi}(x_2)$. $\square_5$

Suppose $\bar{\pi}_2 = \langle \pi_s^2 : s \in S \rangle \in \mathrm{iso}_{\mathbf{y_x}}(\mathfrak{A})$ and let $x_1 = (\bar{\pi}_1, \bar{g}_1, \bar{b}_1) \in X_{\mathbf{y_x}}(\mathfrak{A})$. Recall that $\widetilde{\pi}_s = \pi_{1,s}^{-1}\pi_{2,s}$. For simplicity, we set:

- $\widetilde{\Pi}_s := \psi_s(\widetilde{\pi}_s) \in \mathrm{Aut}(\mathfrak{B}_s)$.

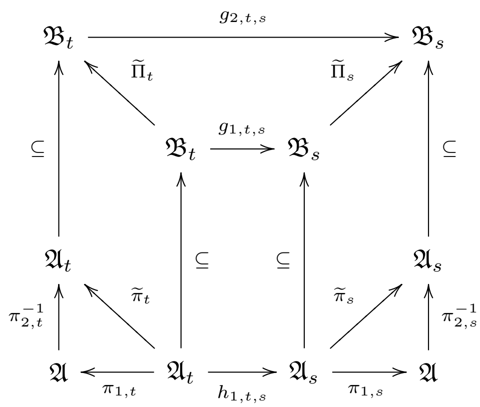
$(*)_{5.9.2}$ Suppose $\bar{\pi}_* = \langle \pi_s^* : s \in S \rangle \in \mathrm{iso}_{\mathbf{y_x}}(\mathfrak{A})$ and $s(*) \in S$. Let $x_1 = (\bar{\pi}_1, \bar{g}_1, \bar{b}_1) \in X_{\mathbf{y_x}}(\mathfrak{A})$. Then there is $x_2 = (\bar{\pi}_2, \bar{g}_2, \bar{b}_2) \in [x_1]_{E_{\mathfrak{A}}}$ such that
  - $\bar{\pi}_2 = \bar{\pi}_*$ and
  - $b_{2,s(*)} = \widetilde{\Pi}_{s(*)}^{-1} b_{1,s(*)}$.

Proof. In order to define x_2, we set

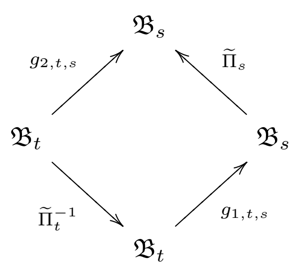
(i) $\bar{\pi}_2 = \bar{\pi}_*$,

(ii) for $s, t \in S$, $g_{2,t,s} = \psi_t((\pi_{1,t}^{-1}\pi_{2,t})^{-1}) \circ g_{1,t,s} \circ \psi_s(\pi_{1,s}^{-1}\pi_{2,s})$.

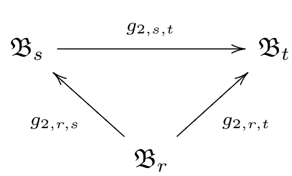
The following commutative diagram summarizes the above situation:



Let us depict the resulting commutative diagram:



It is easily seen that x_2 as defined above is as required. Indeed, by (i), $\bar{\pi}_2 = \bar{\pi}_*$. Also, it is clear that for each $s \in S$, $g_{2,s,s} = \mathrm{id}_{\mathfrak{B}_s}$. Next, we show the following diagram



is commutative where $r, s, t \in S$. To check this, we first recall that $\widetilde{\pi}_s = \pi_{1,s}^{-1}\pi_{2,s}$, then we have

$$\begin{aligned} g_{2,r,s} \circ g_{2,s,t} &= \big(\psi_r(\widetilde{\pi}_r)^{-1} \circ g_{1,r,s} \circ \psi_s(\widetilde{\pi}_s)\big) \circ \big(\psi_s(\widetilde{\pi}_s)^{-1} \circ g_{1,s,t} \circ \psi_t(\widetilde{\pi}_t)\big) \\ &= \psi_r(\pi_{1,r}^{-1}\pi_{2,r})^{-1} \circ g_{1,r,s} \circ g_{1,s,t} \circ \psi_t(\pi_{1,t}^{-1}\pi_{2,t}) \\ &= \psi_r(\pi_{1,r}^{-1}\pi_{2,r})^{-1} \circ g_{1,r,t} \circ \psi_t(\pi_{1,t}^{-1}\pi_{2,t}) \\ &= g_{2,r,t}. \end{aligned}$$

Also,

$$\begin{aligned} g_{2,s(*),s}(b_{2,s(*)}) &= g_{2,s(*),s}(\widetilde{\Pi}_{s(*)}^{-1} b_{1,s(*)}) \\ &= \widetilde{\Pi}_s^{-1} g_{1,s(*),s}(b_{1,s(*)}) \\ &= \widetilde{\Pi}_s^{-1}(b_1 s) \\ &= b_2 s. \end{aligned}$$

In sum, we proved that

$$x_2 = (\bar{\pi}_2, \bar{g}_2, \bar{b}_2) \in X_{\mathbf{y_x}}(\mathfrak{A}).$$

Next, we are going to show that $x_2 E_{\mathfrak{A}} x_1$. To this end, let $s \in S$, and set

- $\pi_s = \pi_{1,s}^{-1}\pi_{2,s}$,
- $h_s = \psi_s(\pi_s^{-1})$.

Then:

$$\begin{aligned} h_s(b_{1,s}) &= \psi_s(\pi_{2,s}^{-1}\pi_{1,s})(b_{1,s}) \\ &= \widetilde{\Pi}_s(g_{1,s(*),s}(b_{1,s(*)})) \\ &= g_{2,s,s(*)}(\widetilde{\Pi}_{s(*)} b_{1,s(*)}) \\ &= g_{2,s,s(*)}(b_{2,s(*)}) \\ &= b_{2,s}, \end{aligned}$$

i.e., $h_s(b_{1,s}) = b_{2,s}$, from which the equivalence $x_2 E_{\mathfrak{A}} x_1$ follows. $\square_5$

It follows from $(*)_{5.9.1}$ and $(*)_{5.9.2}$ that

$$\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}} = Y_{s,\phi}/E_{\mathfrak{A}} \simeq \mathfrak{B}_s.$$

Claim 5.9 follows. $\square$

Given any $\mathfrak{A} \in K_{\mathbf{x},1}$, we are going to define an isomorphism

$$F_{\mathfrak{A}} : \mathfrak{A} \to \mathrm{sort}_1(\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}}),$$

uniformly definable from $\mathbf{p}$. To this end, we set

$$F_{\mathfrak{A}}(a) := \Big\{ \big[(\bar{\pi}, \bar{g}, \bar{b})\big]_{E_{\mathfrak{A}}} : (\bar{\pi}, \bar{g}, \bar{b}) \in X_{\mathbf{y}_{\mathbf{x}}}(\mathfrak{A}) \text{ and } \bigwedge_{s \in S} \pi_s(b_s) = a \Big\}.$$

In other words,

$$F_{\mathfrak{A}}(a) = \Big\{ \big[(\bar{\pi}, \bar{g}, \bar{b})\big]_{E_{\mathfrak{A}}} : (\bar{\pi}, \bar{g}, \bar{b}) \in \mathbf{k}(a) \Big\}.$$

By Claim 5.8, $\mathbf{k}(a)$ is an $E_{\mathfrak{A}}$-equivalence class, in particular, $F_{\mathfrak{A}}(a)$ is singleton. Now for $\mathfrak{A} \in K_{\mathbf{x},1}$, let $\mathfrak{B}_{\mathfrak{A}}$ be defined as follows:

- $\mathfrak{B}_{\mathfrak{A}}$ has as universe $|\mathfrak{A}| \cup \mathrm{sort}_2(\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}})$,
- the mapping $F_{\mathfrak{A}} \cup \mathrm{id}_{\mathrm{sort}_2(\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}})}$ is an isomorphism from $\mathfrak{B}_{\mathfrak{A}}$ onto $\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}}$.

Then $F(\mathfrak{A}) = \mathfrak{B}_{\mathfrak{A}}$ is the required function we were looking for, and the theorem follows. □

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