ON COMPLICATED MODELS AND COMPACT QUANTIFIERS SH800

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ABSTRACT. What we do can be looked at as:

(A) finding and classifying compact second order logic quantifiers on automorphisms of definable models of ψ which are already definable,

(B) building a model M such that if we define in M a model $N = N_{M,\bar{\psi}}$ of ψ, then any automorphism of N is inner (that is, first order definable in M) at least in some respect.

(C) This can be looked at as classifying the ψ-s; so for more complicated ψ-s we have fewer such automorphisms.

(D) As a test case, we consider the specific examples of "the model completion of the theory of triangle-free graphs."

More elaborately, we look here again at building models M with second order properties. In particular, M such that every isomorphism between two interpretations of a theory t in M is definable in M or at least is "somewhat" definable (e.g. having a dense linear order, saying this holds for a dense family of intervals). For transparency we can concentrate on t-s of finite vocabulary. If we restrict ourselves to finite t-s, this implies that we get a compact logic when we add to first order logic the second order quantifiers on isomorphisms from one interpretation. We already know this in some instances (e.g. t the theory of Boolean Algebras or the theory of ordered fields) but here we try to analyze a general t. Hence, at least for the time being, we try to sort out what we get can get by forcing rather than really proving it (in ZFC).

We may consider the question: for a given T if there is an κ-iso-rigid-model of T (so κ-full), then our constructions give one. For more details, see the introduction to [Shea].

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Annotated Content

(c) For suitable $\tau_1 \subseteq \tau_M$, $M_1 \equiv (M \restriction \tau_1)^{\mathbb{N}}/D$ has only internal automorphisms; i.e. for every automorphism F of M_1, for some $F_n \in \mathrm{aut}(M \restriction \tau_1)$, $\prod\limits_n (M \restriction \tau_n, F_n)/D = (M_1, F)$.

(d) Parallel variants for a sequence $\langle M_n : n \in \mathbb{N} \rangle$.

There we mainly deal with case of the strong independence property, e.g. a sequence of finite fields. Here we like to generalize this.]

Glossary

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Definition 0.5: iso-rigid

Claim 0.7: connection to compact quantifiers

Definition 0.10: definably-isomorphic transfer

Discussion 0.11: additions?

Definition 0.12: (λ, κ)-compact

Definition 0.13: $\mathbf{S}^\alpha(A, M)$

Definition 0.16: interpretation added

Claim 0.18: Why not for stable T? Because for κ-full model, $\kappa > \kappa(t)$, $t = \mathrm{Th}(N)$, $N = \mathfrak{C}^{\bar{\varphi}}$ gives $N^{\bar{\varphi}}$ is saturated.

Definition 0.20: The general case:

1) M is $\bar{\varphi}$-$(t_1, t_2, \mathscr{L}_1, \mathbf{L}_2)$-rigid.

2) (t_1, t_2) has definability transfer.

Claim 0.21: In Definition 0.5, 0.13 are special cases of Definition 0.20

Claim 0.22: On interpretations: basic properties

Claim 0.23: Sufficient conditions for (t_1, t_2) to have transfer

Observation 0.25: If R is $\mathbb{L}_{\kappa^+,\kappa^+}(\tau)$-definable in $M \restriction \tau, M$ is κ^+-saturated, R is first order definable in M then M is first order definable in $M \restriction \tau$

Discussion 0.27:

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§(1A) **Complicated models** pg.18

Definition 1.1: Local bigness notion

Definition 1.2: Global bigness notion

Definition 1.3: $\Gamma_{t,\bar{\varphi},\bar{\psi}}$

Claim 1.4:

(1) Local bigness notion induces a global one.

(2) $\Gamma_{t,\bar{\varphi},\bar{\psi}}$ is a local $(\mathfrak{C}, \kappa)$-bigness notion

Definition 1.6: Orthogonality of global bigness notion

Definition 1.9: Δ-freedom for Γ_1, Γ_2

Definition 1.10: $\mathfrak{C}$ is (Ω, Γ_1)-complicated κ-embedding for (N_1, N_2)

Definition 1.12: When a first order t is $(\infty, \mathbb{L}_{\infty,\kappa})$-rigid for isomorphic/for embedding

Claim 1.13: Consequences of 1.12, we define $E_{p,\varphi^1_R,\varphi^2_R}$

§(1B) **More on Bigness Notions**

Definition 1.17: $\sum\{\Gamma_\alpha : \alpha < \alpha^*\}$ for bigness notions

Claim 1.18: $\sum(\bar{\Gamma})$ works

Definition 1.19: Lifting Γ to $\Gamma^{[\bar{\varphi}]}$

Claim 1.20: $\Gamma^{[\bar{\varphi}]}$ works

Claim 1.21: For unstable t there is Γ

§2 Triangle free graphs and more general examples pg.28

Definition 2.1: $T^0_{\mathscr{K}}$ and its model completion $T_{\mathscr{K}}$

Claim 2.2: Basic properties of $T_{\mathscr{K}}$

Definition 2.3: $\mathscr{K}$ is interesting

§ 0. Introduction: Semi Rigid models

This continues [Shea], see history there. We try to get a model M of a given (first order complete) T such that any automorphism of a model of (another first order theory) t interpretable in M is inner (i.e. definable by a first order formula with parameters in M); similarly for any isomorphism from one interpretation of t in M to another.

In those works the main cases were $t =$ the theory of Boolean algebras (or the strong independence property or atomless Boolean Algebras) and $t =$ the theory of ordered fields (or just ordered sets). A major theme there was reducing the extra set theoretic assumptions (like diamonds or GCH). This gives results like "$\mathbb{L}(\mathbf{Q})$ is a compact logic" for $\mathbf{Q}$ a second order quantifier of the form "there is an automorphism f of $M^{[\bar\varphi]}$ such that ..." (see [Shea, §0]; f is a second order variable).

We may consider various statements expressing some second order properties like considering complete embedding of one Boolean Algebra into another.

Our main interest is in first order theories, so the reader may first assume we use only that (so $\mathscr{L}_1 = \mathscr{L}_2 = \mathbb{L}$). But proving results like "all automorphisms of $M^{[\bar\varphi]}$ are definable in M," we have to consider first being definable by an $\mathbb{L}_{\infty,\chi}$-formula with parameters (usually in a λ-saturated model). We may with some extra assumptions (e.g. having a specific t) get first order definable.

We consider here, and give some information on:

Question 0.1. 1) For unstable T can we as in [Sheb] get some form of definability?
2) At least assuming some uniformity, see [She00b, §3, 3.9, 3.10].

One of the questions is (really a variant):

Question 0.2. Give[n] a pair (τ, τ_t), $\tau \subseteq \tau_t$, t complete and T vary[ing] on theories (maybe rich enough such that there are interpretations of t):

(a)? When do we have, for every λ and T, a λ-universal model of T which is t-iso-rigid?

§ 0(A). **Reader instructions.** To show some quantifier; i.e. extensions of first order logic by restricted second order quantifiers (see [Shea, §0]) we use 0.7 below, which tells us it suffices to build t-iso-rigid models of a given T for relevant t-s.

In §2 we deal with a wide family of theories t for which we can get results; a new test case is "the random triangle-free graph".

Theorem 0.8 tells us sufficient conditions for a κ-saturated model M to be t-iso-rigid. Those include:

- t has $(\mathbb{L}_{\infty,\kappa}, \mathbb{L}, \kappa)$-def-iso-transfer.
- there are enough models M which are (Ω, Γ, κ)-complicated models (see 1.10). For a so-called bigness notion Γ relevant to t, see Definition 1.1.

In §3 we construct such a model by forcing:

- t has a (Ω, κ)-uniformity[1] or connected (see Definition 1.13, 1.15; by Claim 1.16, start with 1.12(2) and deduce from it).

[1] One canonical case is the order fields via order: an automorphism of an ordered field is determined by any restriction of it to an interval.

From another outlook we may try to classify the complete t-s (equivalently, the quantifiers). By 1.21, for every unstable t there are non-trivial bigness notions relevant to it, hence for every so-called (t, Ω, κ)-complicated model M and interpretations $\bar{\varphi}^1, \bar{\varphi}^2$ of t in M and isomorphism π from $M^{[\bar{\varphi}^1]}, M^{[\bar{\varphi}^2]}$, π is "densely definable" by 1.12.

Why? As for every T, Ω, and κ there is a (t, Ω, κ)-complicated model M, and for every such model and relevant Γ and for $N = M^{[\bar{\varphi}]}$ and automorphism π of N we have definability on a so-called dense set of places. Is the non-stability necessary? Easily yes by 0.18.

We may like to consider models which are only $\aleph_0$-saturated. This is considered in §4. For every unsuperstable (complete) t, there are relevant bigness notions (they are not from the same family of the ones considered earlier).

For this we may also consider successor of singulars: see 5.1-5.6 in §5.

In §1C we connect to older results. We may consider continuing [She08] (see §7). We have not addressed here the trees with no undefinable branches, but see [She78] and also [Sheb].

We may sort out "t unstable in ZFC."

We may sort out the idea of starting with bigness notions in $\mathfrak{C}^+$ and projecting them to $\mathfrak{C} = \mathfrak{C}^+ \restriction \tau_{\mathfrak{C}}$.

§ 0(B). **The Frame.**

Definition 0.3. 1) We say a model M is t-iso-rigid <u>when</u>: if $\bar{\varphi}^1, \bar{\varphi}^2$ are interpretations of t in M, with parameters (see Definition 0.16) and π is an isomorphism from $M^{[\bar{\varphi}^1]}$ onto $M^{[\bar{\varphi}^2]}$, (see 0.16) <u>then</u> π is inner (i.e. definable in M with parameters).
2) We say t is (λ, κ)-rigid <u>when</u> every T has a (λ, κ)-saturated t-iso-rigid model.
3) We say t is rigid <u>when</u> t is (λ, κ)-rigid for every $\lambda \geq \kappa$.

Discussion 0.4. 1) We may interpret 0.3(3) in several ways:

(a) Provably in ZFC, or at least
(b) in some forcing extension
(c) or in the forcing from 3.2(2) (see §3 for λ).
(d) Like (c), but replacing $(\lambda^+, <)$ with a quite homogeneous λ^+-like linear order.

2) It is not clear that the answer to those variants are equivalent. In (c),(d) above we use the largest Ω getting the result for all t-s at once but maybe we can prove for two cases but not for both.

It seems reasonable to start with (c), so start with $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{\pi}$ is an isomorphism from $\underset{\sim}{M}^{[\bar{\varphi}^1]}$ onto $\underset{\sim}{M}^{[\bar{\varphi}^2]}$". But this leads us to (d) as automorphism π of I induces an automorphism of $\hat{\underset{\sim}{\pi}}$ of $\underset{\sim}{M}$ or at least the forcing (see §3; particularly 3.10).

So there are few $\underset{\sim}{\pi}$-s definable in a forcing sense. Moreover, in the forcing approach we can assume $2^\lambda > \lambda^+$, so necessarily there is $p_* \in \mathbb{P}$ such that any automorphism of I over $\mathrm{dom}(p_*)$ the induced automorphism of $\mathbb{P}$ maps $\underset{\sim}{\pi}$ to itself. Can we deduce from it a model theoretic definition of π? even first order ones?
3) We may wonder

(a)$'$ For any $M_* \models T$ there is an $\aleph_0$-saturated t-iso-rigid model of $\mathrm{Th}(M_*, c)_{c \in M_*}$.
(a)$''$ For every T and λ there is a t-iso-rigid model which $(\lambda, \aleph_0)$-saturated; i.e. it is the direct limit of λ-saturated elementary sub-models. (Usually it is first order <u>or</u> is $\mathbb{L}_{\infty,\lambda}$ when M is λ-saturated.)

On $M^{\bar\varphi}$, see Definition 0.16(2) below.

4) But as said above, we need to allow other logics in Definition 0.3.

Definition 0.5. 1) We say that a model M is $(\bar\varphi^1,\bar\varphi^2)$-$(t,\mathscr{L}_1,\mathscr{L}_2)$-isomorphism-rigid (or iso-rigid) when:

- (i) t is a theory (in a vocabulary τ_t, which has nothing to do with τ_M), usually finite.
- (ii) $\mathscr{L}_1$ is a logic, usually first order; $\bar\varphi^1,\bar\varphi^2$ are $\mathscr{L}_1$-interpretations of the theory t in M, possibly with parameters (see Definition 0.16 below).[2]
- (iii) Every isomorphism f from $M^{\bar\varphi^1}$ onto $M^{\bar\varphi^2}$ is definable in M by an $\mathscr{L}_2(\tau_M)$-formula with parameters, where $\mathscr{L}_2$ is a logic.

1A) We qualify "restricted to $\vartheta(x)$" if $\vartheta(x)$ is a formula in the vocabulary τ_t and we replace (iii) by

- $(iii)_{\vartheta(\bar x)}$ if f is an isomorphism from $M^{\bar\varphi^1}$ onto $M^{\bar\varphi^2}$ then $f\restriction\{c: M^{\bar\varphi^1}\models\vartheta[c]\}$ is definable in M by an $\mathscr{L}_2$-formula with parameters.

2) We may omit $(\bar\varphi^1,\bar\varphi^2)$ if this holds for any such $\bar\varphi^1,\bar\varphi^2$. If $\bar\varphi^1=\bar\varphi^2$ we may write $\bar\varphi$.

3) We may omit $\mathscr{L}_1$ if it is first order. We may write t instead of $(t,\mathscr{L}_1,\mathscr{L}_2)$ if $\mathscr{L}_1=\mathscr{L}_2=$ first order.

4) We may replace isomorphism-rigid by embedding rigid in the obvious way.

5) We may replace isomorphism-rigid by weakly-embedding-rigid if in part (1) we have $(i),(ii)$ and

- $(iii)_{\text{wem}}$ For every embedding f from $M^{\bar\varphi^1}$ into $M^{\bar\varphi^2}$ there is a function $F: M^{\bar\varphi^2}\to M^{\bar\varphi^1}$, definable in M by an $\mathscr{L}_2$-formula with parameters, such that $f(a)=b\Rightarrow F(b)=a$.

Similarly for other variants.

6) We can qualify the "embedding" (in part (4)) in various ways; e.g.

- (a) "Complete embeddings" for Boolean algebras.
- (b) "Has dense range" for $t=$ linear orders.

Discussion 0.6. How do we connect this to compact logics (note: if $t\subseteq\mathbb{L}(\tau)$ is computably enumerable, (for long has been called recursively enumerable), τ finite then there are $\tau_*\supseteq\tau$, $\psi\in\mathbb{L}(\tau_*)$ such that $\psi\vdash t$).

Claim 0.7. *1) A sufficient condition for the logic $\mathbb{L}(\mathbf{Q}^{\text{aut}}_{\psi,\tau})$ (see* [Shea, §0]*) to be compact is:*

- (a) $\psi\in\mathbb{L}(\tau_\psi)$
- (b) *τ_ψ finite, $\tau\subseteq\tau_\psi$*
- (c) *The quantifier $\mathbf{Q}^{\text{aut}}_{\psi,\tau}$ says: 'There is a τ-isomorphism f from $M^{[\bar\varphi_1]}$ onto $M^{[\bar\varphi_\ell]}$ where $\bar\varphi_\ell$ is an interpretation of a model of ψ in M' (with parameters: see more in* [Shea]*).*
- (d) *Every (first order) T which codes enough set theory has a model M such that:*
  - $(*)$ *For every $\bar\varphi^1,\bar\varphi^2$ as above, every τ-isomorphism from $M^{[\bar\varphi_1]}$ onto $M^{[\bar\varphi_2]}$ is inner (i.e. definable in M by a (first order) formula with parameter).*

[2] If we would like to avoid this, just stipulate that $\mathscr{L}$ has no formulas which are not sentences.

2) We can weaken (*d*) *to, e.g.,*

- (*d*)′ *For every first order T_1 there are $T_2 \supseteq T_1$ and $|T_2|^+$-universal model M_2 of T_2 such that the statement* (*d*)(∗) *holds when $\bar\varphi^1, \bar\varphi^2$ are such that every M_2-inner isomorphism is M_1-inner.*

3) Similarly, but using $\mathbf{Q}^{\rm rigid}_{\psi,\tau}$ and

- (*c*)′ *says: $M^{[\bar\varphi]}$ is a model of ψ and is τ-rigid.*
- (*d*)′ *Changed naturally.*

How do we get cases of clause (d) of 0.7(1)? The following breaks the work to two.

Theorem 0.8. *1) The model M is t-iso-rigid (see 0.3) <u>when</u>:*

- (*a*) *M is κ-saturated, with $\kappa > |\tau_M| + \aleph_0$.*
- (*b*) *t has $(\mathbb{L}_{\infty,\kappa}, \mathbb{L}, \kappa)$-def-iso-transfer (see Definition 0.10).*
- (*c*) *M is $(t, \mathbb{L}, \mathbb{L}_{\infty,\kappa})$-iso-rigid (see Definition 0.5).*

2) Above, we can replace (*c*) *by*

- (*d*) *M is $(t, \Omega, \mathbb{L}_{\infty,\kappa})$-complicated (see 1.10).*
- (*e*) *$\mathrm{Th}(M)$ has $(t, \Omega, \mathbb{L}_{\infty,\kappa})$-uniformity (see Definition 1.13).*

Discussion 0.9. Our main aim is to investigate when T (normally a complete first order theory) has a t-iso-rigid model, preferably for many t-s.

There are several choices discussed below, but we shall concentrate on the following:

⊞
- (a) All theories are first order.
- (b) The theory T codes enough set theory.
- (c) The theory t has a finite vocabulary.
- (d) Getting only consistency results (rather than ZFC ones).
- (e) Building models $\mathfrak{B}$ of T such that
  - if N_1, N_2 are models of t interpreted in $\mathfrak{B}$ (for transparency, with set of elements $\subseteq \mathfrak{B}$ rather than a set of m-tuples divided by an equivalence relation), <u>then</u> any isomorphism from N_1 onto N_2 is definable in $\mathfrak{B}$, at least to some extent. (E.g. only for a so-called "dense set of big types" p on the set $p(\mathfrak{B}) \subseteq N_1$ divided by a definable equivalence relation with "small" equivalence classes — see below.)

* * *

Part of the proof of instances of "$\mathfrak{B}$ is t-iso-rigid" is the "transfer" (used in 0.8(1)(b)), which we shall now define.

Definition 0.10. 1) We say F is $(\mathscr{L}_1, \kappa_1)$-definable in M <u>when</u>:

- (a) F is a partial function from M to M.
- (b) For some $\tau_F \subseteq \tau_M$ with $|\tau_F| < \kappa_1$, the function F is definable in M by a formula in $\mathscr{L}_1(\tau_F)$ with $< \kappa_1$ parameters.

2) We say that t has $(\mathscr{L}_1, \mathscr{L}_2, \kappa_1, \kappa_2)$-definably-isomorphic transfer (or just transfer) <u>if</u>:

⊠ F is $(\mathscr{L}_2, < \kappa_2)$-definable in M <u>when</u>:

(i) M is a κ_2-saturated$^+$ (or just $(<\kappa_2)$-saturated[3]) model.
(ii) $\bar\varphi^1, \bar\varphi^2$ are interpretations of t in M by first order formulas.
(iii) F is an isomorphism from $M^{\bar\varphi^1}$ onto $M^{\bar\varphi^2}$.
(iv) F is $(\mathscr{L}_1, <\kappa_1)$-definable in M.

3) Above we may omit κ_2 if $\kappa_2 = \aleph_0$, we may omit $\mathbb{L}_2$ if $\mathbb{L}_2 = \mathbb{L}_1$, we may omit $\mathbb{L}$ if $\mathbb{L}_1 = \mathbb{L}_{\infty,\kappa_2}$. So "$t$ has κ-transfer" when it has $(\mathbb{L}_{\infty,\kappa}, \mathbb{L}, \kappa, \aleph_0)$-transfer. Omitting κ means $\kappa = (2^{\tau(M)+\aleph_0})^+$. Similarly for embedding and weak embedding.

4) We may qualify restricting ourselves to M with $\mathrm{Th}(M)$ rich enough (e.g. for compactness of $\mathbb{L}(\mathbf{Q})$).

Discussion 0.11. 1) Now the transfer: Definition 0.10 holds for t = the first order theory of Boolean Algebras. More generally, first order theories with strong independence property and for ordered fields and partial orders such that there are incompatibilities above each element and which are internally isomorphic to any cone, see [She94].

We shall prove it below for a wide family of theories, a characteristic member of which is the theory of existentially closed triangle-free graphs; many a year it seemed to me that the method of $\mathcal{P}(n)$-diagrams will help (see 0.26(f)$_3$) but this is not the case.

2) More challenging is to find a major dividing line for which t we have rigidity. In some sense, to a large extent the "t stable/unstable" dividing line expresses this because for every Skolemized T and κ, there is a saturated enough model such that every isomorphism from one $\mathbb{L}_{\kappa,\kappa}$-interpretation of t onto another is "locally definable" in a natural sense (see 1.12).

We may consider phrasing the question:

(∗) (a) What can $\mathrm{spec}_{T,t}$ be? ($\mathrm{spec}_{T,t}$ is the class of pairs (λ, κ) such that there is a κ-full model M of t of cardinality λ which is t-iso-rigid.)
(b) Omitting λ means "for arbitrarily large λ."

3) Similarly, the "t is [superstable/unsuperstable]" dividing line expresses: for every Skolemized T and λ there is a $(\lambda, \aleph_0)$-saturated model of T such that every isomorphism from one $\mathbb{L}$-interpretation of t in M onto another is locally definable in a natural sense.

4) But to make it relevant for true rigidity and for compactness of the isomorphism quantifier we need further work. A typical case is that of linear order for which we can get only "locally definable", whereas for ordered fields we get the full result.

5) Defining interpretations, in 0.16 we may use $\varphi_=$, an equivalence relation of a set of k-tuples $\{\bar a \in {}^k\mathfrak{C} : \mathfrak{C} \models \varphi_=[\bar a_0, \bar a]\}$. This does not change the result in any meaningful way. So here for notational simplicity we use $k = 1$ and $\varphi_=$ is the equality on $\{x : \varphi_=(x = x)\}$.

Definition 0.12. 1) M is $(\lambda, \kappa_1, \kappa_2)$-saturated <u>when</u> for every $\tau \subseteq \tau_M$ of cardinality $<\kappa_2$ and $A \subseteq M$ of cardinality $<\kappa_1$ and $N, M \restriction \tau \prec N$ and $B \subseteq N$ with $|B| < \lambda$, <u>there is</u> a $(N, M \restriction \tau)$-elementary mapping f from $A \cup B$ into B such that $f \restriction A = \mathrm{id}_A$.

1A) If $\kappa_2 = |\tau_M|^+ + \aleph_0$ we may omit κ_2.

2) M is $(<\kappa)$-full <u>if</u> every type p in M of cardinality $<\kappa$ is realized by $\|M\|$ elements.

3) M is (λ, κ_1)-full <u>when</u> for every $\tau \subseteq \tau_M$ of cardinality $<\kappa_2$, every 1-type p in $M \restriction \tau$ with $<\lambda$ parameters is realized by $\|M\|$ elements.

[3] I.e. $M \restriction \tau'$ is κ_2-saturated when $\tau' \subseteq \tau_M$ and $|\tau'| < \kappa_2$.

Definition 0.13. 1) For a model M and set $A \subseteq M$,

$$\mathbf{S}^\alpha(A, M) := \{\mathrm{tp}(\bar{a}, A, M) : \bar{a} \in {}^\alpha M\}.$$

1A) Let $S^\alpha(A, M) = \bigcup\{\mathbf{S}^\alpha(A, N) : M \prec N\}$.
2) p is an α-type in M if M is a set of (first order) formulas in $\mathbb{L}(\tau_M)$ in the variables $\{x_i : i < \alpha\}$ and parameters from M, finitely satisfiable in M.

Convention 0.14. 1) Dealing with general vocabularies, without loss of generality they are relational; i.e. function symbols and individual constants are translated to predicates.
2) For first order T let $\mathfrak{C}_T$ be the monster for T and $\mathfrak{C}$ is $\mathfrak{C}_T$.

Notation 0.15. For a model M, a relation (or partial function) is *inner* when it is definable by a first order formula with parameters.

Definition 0.16. 1) We say that $\bar{\varphi}$ is a pure interpretation of t in the model M when:
(We consider only the case that τ_t is relational: an n-place function symbol can be treated as $(n+1)$-place predicate.)

$\boxdot$ (a) $\bar{\varphi} = \langle \varphi_R(\bar{x}_R) : R \in \tau_t\rangle$, where we consider equality as one of the predicates: a two-place one, $\varphi_R(\bar{x}_R) \in \mathbb{L}(\tau_M)$.
(b) $\bar{x}_R = \langle x_\ell : \ell < \mathrm{arity}_t(R)\rangle$
(c) $\varphi_=(x_0, x_1)$ is the equality on the non-empty set

$$\{a \in M : M \models \varphi_=(a, a)\}.$$

(d) if $R \in \tau_t$ is k-place then

$$M \models (\forall x_0) \ldots (\forall x_{k-1})\big[\varphi_R(x_0, \ldots, x_{k-1}) \to \bigwedge_{\ell<k} \varphi_=(x_\ell, x_\ell)\big].$$

(e) $M^{\bar{\varphi}}$ is a model of t: see part (2).

2) For M and $\bar{\varphi}$ as above, we say that $N = M^{\bar{\varphi}} = M^{[\bar{\varphi}]}$ when:

(A) N is a τ_t-model.
(B) $|N|$ (the universe of N) is $\{a \in M : M \models \varphi_=(a, a)\}$.
(C) For $R \in \tau_t$ with k places, we have

$$R^N = \big\{\langle a_0, \ldots, a_{k-1}\rangle \in {}^k M : M \models \varphi_R[a_0, \ldots, a_{k-1}]\big\}.$$

3) We say $\bar{\varphi}$ is a pure interpretation of t in a first order T if part (1) holds for every model M of T.
4) In part (1), instead of "pure" we can say "with parameters" if we allow the formulas φ_R to have parameters from M. Then we write $\bar{\varphi} = (\bar{\varphi}, \bar{\mathbf{c}})$ and then we consider only the parameter sequence $\bar{\mathbf{c}}$ in M such that $M^{\bar{\varphi}}$ is a model of t. If we omit both "pure" and "with parameters" then we allow parameters.

Discussion 0.17. So by our constructions with κ-full M, we may achieve something for unstable T but not for stable ones. However, for full $\aleph_0$-saturated models we can achieve something for un-superstable T as we may omit countable types.

Claim 0.18. *1) Assume*

(a) $\mathfrak{B}$ is a $(< \kappa)$-full model (see Definition 0.12(2)); i.e. every type of cardinality $< \kappa$ is realized by $\|\mathfrak{B}\|$-many elements.
(b) N is interpretable in $\mathfrak{B}$, and $|\tau_N| + \aleph_0 < \kappa$.
(c) $\mathfrak{B}$ has enough set theory coded in it (or just ε code[s / d] finite sets).

(*d*) $\mathrm{Th}(N)$ *is stable.*

Then N is saturated of cardinality $\|\mathfrak{B}\|$ hence has $> 2^{\|\mathfrak{B}\|} > \|\mathfrak{B}\|$ automorphisms, so some automorphisms of N are not definable in $\mathfrak{B}$.
2) Similarly to part (1), replacing classes $(a),(d)$ by

$(a)^-$ *$\mathfrak{B}$ is κ-saturated.*
$(d)^+$ *$\mathrm{Th}(N)$ is stable without the finite cover property.*

(But on $(c)^-$, see the proof.)

3) Similarly to (1), replacing clauses $(c),(d)$ by:

$(c)^-$ *$\mathfrak{B}$ has Skolem functions.*
$(d)^-$ *$\mathrm{Th}(N)$ is superstable.*

4) Assume

(*a*) *$\mathfrak{B}$ is κ-full.*
(*b*) *N is interpretable in $\mathfrak{B}$ by $\bar\varphi$, which has $< \kappa$ parameters.*
(*c*) *$\mathfrak{B}$ codes enough set theory.*
(*d*) $\kappa(\mathrm{Th}(N)) \le \kappa = \mathrm{cf}(\kappa)$.

Then N is saturated.
5) In (4) we can replace $(a),(d)$ by $(a)^-$ and $(d)^*_{\kappa(\mathrm{Th}(N))\le\kappa=\mathrm{cf}(<\kappa)}$ and $\mathrm{Th}(N)$ fails the fcp.

Remark 0.19. 1) See [Shec].
2) In 0.18(2) but T has the fcp then the conclusion fails.
3) We can strengthen the results of 0.18 to "everywhere have an automorphism π which is everywhere not definable", e.g. in 0.18(x) we may add (that π satisfies):

(∗) If $p(x)$ is a type in $\mathfrak{B}$ in cardinality $< \kappa$, $p(\mathfrak{B}) \cap N$ infinite and E an equivalence relation as earlier, then $\pi \restriction p(\mathfrak{B})/E$ is not definable in $\mathfrak{B}$ by an $\mathbb{L}_{\infty,\kappa}$-formula.

Proof. 1) Obviously N is κ-saturated, because $\kappa > |\tau_N| + \aleph_0$. By [She90, Ch.III,3.10(1)], it is enough to show that

⊛ If $\{a_n : n < \omega\}$ is an indiscernible set in N and $n < m \Rightarrow a_n \ne a_n$ then there is an indiscernible set $\mathbf{I}$ in N of cardinality $\|N\|$ which extends $\{a_n : n < \omega\}$.

So it is enough to find $x \in \mathfrak{B}$ which $\mathfrak{B}$ "considers" a Δ-indiscernible set in $\mathfrak{B}^{[\bar\varphi]}$ for every finite $\Delta \subseteq \mathbb{L}_{\tau(N)}$, and $n < \omega \Rightarrow a_n \in^{\mathfrak{B}} x$. This is easy enough; i.e. there is a type of $\mathfrak{B}$ of cardinality $< \kappa$ expressing this instance of (κ,κ)-saturation of $\mathfrak{B}$ (which follows by $(< \kappa)$-full). So $\{a : \mathfrak{B} \models a \in^{\mathfrak{B}} x\} \subseteq \mathfrak{B}^{[\bar\varphi]}$ is infinite hence (by '$\mathfrak{B}$ is $(< \kappa)$-full') of cardinality $\|\mathfrak{B}\|$, so we are done.
2) Now in the proof of part (1), there is no problem with x being "$\in^M$-pseudo-finite". We can demand that "infinite sets" in the $\in^{\mathfrak{B}}$-sense are of cardinality $\|\mathfrak{B}\|$, but there may be pseudo-finite sets. So we have to add to p_*, "x is infinite in the $\in^{\mathfrak{B}}$-sense".

It is okay because t fails the fcp.

3) Using [She90, III,3.10(2)].
4),5) Similarly. $\square_{0.18}$

§ 0(C). **A More General Frame.** See also §(1B).

Instead of dealing with automorphisms of $N = M^{[\bar\varphi]}$, we may look at other cases. The following is quite general, but we have others: consider a pair of $\bar\varphi$-s, <u>or</u> an automorphism of one $M^{[\bar\varphi]}$.

Definition 0.20. 1) We say that a model M is $\bar\varphi$-$(t_1, t_2, \mathscr{L}_1, \mathscr{L}_2)$-rigid <u>if</u>:

- (i) t_1, t_2 are theories such that $t_1 \subseteq t_2$ and $\tau_{t_1} \subseteq \tau_{t_2}$.
- (ii) $\mathscr{L}_1$ is a logic and $\bar\varphi$ is an $\mathscr{L}_1$-interpretation of t_1 in M, possibly with parameters.
- (iii) If N is an expansion of $M^{[\bar\varphi]}$ to a model of t_2 then $R \in \tau_{t_2} = \tau_N \Rightarrow R^N$ is definable in M by an $\mathscr{L}_2(\tau_M)$-formula, possibly with parameters.

2) We say that (t_1, t_2) has $(\mathscr{L}_1, \mathscr{L}_2, \kappa_1, \kappa_2)$-definability transfer <u>when</u>: $|\tau_{t_2}| < \kappa_1$ and

- $\boxtimes$ Every R^N is $\mathscr{L}_2(\tau_M)$-definable (in M, with parameters) <u>when</u>:
  - (i) M is a κ_2-saturated model.
  - (ii) $\bar\varphi$ is an interpretation of t_1 in M by first order formulas.
  - (iii) N is an expansion of $M^{[\bar\varphi]}$ to a model of t_2.
  - (iv) Every R^N is $\mathscr{L}_1(\tau_M)$-definable (in M, with parameters).

2) We adopt the conventions of 0.5 + 0.10.

Claim 0.21. *Definitions 0.5, 0.10 are special cases of Definition 0.20 (as we allow interpretation by k-tuples).*

The choice of "isomorphisms from $M^{[\bar\varphi_1]}$ onto $M^{[\bar\varphi_2]}$" is a very natural one, but a more general definition is (we may consider (e.g.) homomorphisms [one to one], [onto] endomorphisms) i.e. $\tau_{t_2} = \tau_{t_1} \cup \{F\}$, $t_2 = t_1$ + sentences saying the above.

Claim 0.22. *1) Assume that*

- (a) *M_1 is κ-saturated with κ regular.*
- (b) *$\bar\varphi^1$ is an interpretation of t in M_1 with $< \kappa$ parameters from $A_1 \subseteq M_1$, $|A_1| < \kappa$.*
- (c) *M_2 is a κ-saturated model of $\mathrm{Th}(M_1)$ and f is an (M_1, M_2)-elementary mapping (e.g. $M_1 \prec M_2$, $f = \mathrm{id}_A$).*
- (d) *t is first order or even $\subseteq \mathbb{L}_{\kappa,\kappa}$.*
- (e) *$A_2 = f(A_1)$ and $\bar\varphi^2 = f(\bar\varphi^1)$.*

<u>*Then*</u>

- (α) *$M_2^{[\bar\varphi^2]}$ is a model of t.*
- (β) *If $M_1 \prec M_2$ and $f = \mathrm{id}_A$ then $M_2^{[\bar\varphi]}$ is a model of t such that $M_1^{[\bar\varphi]} \prec_{\mathbb{L}_{\kappa,\kappa}} M_2^{[\bar\varphi]}$.*

Proof. Easy. $\square_{0.22}$

The following may help to prove cases of transfer.

Claim 0.23. *A sufficient condition for (t_1, t_2) to have transfer (see Definition 0.20(2)) is:*

- $\circledast$ *For every model M_1 and interpretation $\bar\varphi$ of t in M_1, we can find a first order theory T such that:*
  - (α) *In some model of T we can interpret a model of $\mathrm{Th}(M_1)$.*

(β) *For every κ we can find a κ-saturated model of T for which the transfer works.*

Remark 0.24. The point is that in checking Definition 0.20, we may like to use models M with "enough Skolem functions and enough set theory coded".

Proof. This follows by 0.22 and the observation below. $\square_{0.23}$

Observation 0.25. *Assume M is κ^+-saturated, $A \subseteq M$ with $|A| \leq \kappa$, and R is an n-ary relation definable in $M \restriction \tau$ by an $\mathbb{L}_{\kappa^+,\kappa^+}(\tau)$-formula with parameters from A, where $\tau \subseteq \tau_M$.*

If R is first order definable in M then R is first order definable in $M \restriction \tau$ with parameters from A.

Proof. Let $\psi(\bar{x}, \bar{c})$ define R in M, $n = \ell g(\bar{x})$ is the arity of R and $\psi(\bar{x}, \bar{y}) \in \mathbb{L}(\tau_M)$ so first order. For every $\bar{a} \in {}^nM$, define

$$\Gamma_{\bar{a}} = \big\{\varphi(\bar{x}, \bar{b}) : \bar{b} \subseteq A,\ M \models \varphi[\bar{a}, \bar{b}],\ \text{and } \varphi \in \mathbb{L}(\tau)\big\} \cup \big\{\psi(\bar{x}, \bar{c}) \equiv \neg\psi(\bar{a}, \bar{c})\big\}.$$

Now the set $\Gamma_{\bar{a}}$ is not realized in M.

[Why? If $\bar{a}'$ realizes it we have $\mathrm{tp}(\bar{a}', A, M \restriction \tau) = \mathrm{tp}(\bar{a}, A, M \restriction \tau)$ by the first part of $\Gamma_{\bar{a}}$, so as R is definable in $M \restriction \tau$ by an $\mathbb{L}_{\kappa^+,\kappa^+}(\tau)$-formula with parameters from A, we get $M \models R(\bar{a}') \equiv R(\bar{a})$. Hence by the choice of $\psi(\bar{x}, \bar{c})$ we get

$$M \models \psi(\bar{a}', \bar{c}) \equiv \psi(\bar{a}, \bar{c}),$$

but this contradicts the last part of $\Gamma_{\bar{a}}$.]

Also, the first part of $\Gamma_{\bar{a}}$ is closed under conjunctions. As M is κ^+-saturated, $\Gamma_{\bar{a}}$ is not finitely satisfied in M, hence by the last sentence for some $\varphi_{\bar{a}}(x, \bar{y}) \in \mathbb{L}(\tau)$ and $\bar{b}_{\bar{a}} \in {}^{\ell g(\bar{\varphi})}A$ we have $M \models \varphi_{\bar{a}}[\bar{a}, \bar{b}_{\bar{a}}]$ and $M \models (\forall \bar{x})\big[\varphi_{\bar{a}}(\bar{x}, \bar{b}_{\bar{a}}) \to \psi(\bar{x}, \bar{c}) \equiv \psi(\bar{a}, \bar{c})\big]$.

Let

$$\Phi = \Big\{\varphi(\bar{x}, \bar{b}) : \varphi(\bar{x}, \bar{z}) \in \mathscr{L}(\tau),\ \bar{x} = \langle x_0, \dots, x_{n-1}\rangle,\ \bar{b} \in {}^{\ell g(\bar{y})}A,$$
$$\text{and } M \models (\forall \bar{x})\big[\varphi(\bar{x}, \bar{b}) \wedge \varphi(\bar{z}, \bar{b}) \to \psi(\bar{x}, \bar{c}) \equiv \psi(\bar{z}, \bar{c})\big]\Big\}.$$

By the previous paragraph, as $M \restriction \tau$ is κ^+-saturated every $p \in \mathbf{S}^n(A, M \restriction \tau)$ has a member from Φ; so every $\bar{a} \in {}^nM$ satisfies some formula from Φ. As M is κ^+-saturated this holds for some finite $\Phi' \subseteq \Phi$, the rest should be clear (definition by cases). Let $\Phi' = \{\varphi_\ell(x, \bar{b}_\ell) : \ell < \ell(*)\}$ where $\ell(*) < \omega$. So for each ℓ,

$$\varphi_\ell(M, \bar{b}_\ell) \subseteq R^M \text{ or } \varphi_\ell(M, \bar{b}_\ell) \subseteq {}^nM \setminus R^M;$$

we define $u \subseteq n$ by $u = \{\ell : \varphi_\ell(M, \bar{b}_\ell) \subseteq R^M\}$.

Now let $\varphi(\bar{x}, \bar{b}^*) = \bigvee\limits_{\ell \in u} \varphi_\ell(\bar{x}, \bar{b}_\ell)$. $\square_{0.25}$

Discussion 0.26. Continuing 0.9, possible choices (of our frame) are

- (A) (a) a fixed pair of interpretations <u>or</u>
  - (b) all pairs of interpretation <u>or</u>
  - (c) we can consider only $M^{\bar{\varphi}^1} = M^{\bar{\varphi}^2} = M$, $t = T$ (i.e. automorphisms of M, so a weaker demand than (a)).
- (B) (a) Any (complete first order) T <u>or</u>
  - (b) "rich", T say with Skolem functions and more. (So having enough set theory coded inside; enough for proving compactness of suitable restricted second order quantifiers. E.g. T is the complete first order theory of some expansion of $(\mathcal{H}(\chi), \in, <^*_\chi)$, χ strong limit.)
  - (c) Concentrate on proving the compactness of $\mathbb{L}(\mathbf{Q})$ for some second order quantifier $\mathbf{Q}$.

(C) Concentrate on
  (a) isomorphisms which are onto, <u>or</u> consider
  (b) nice enough embeddings [e.g. for Boolean Algebras, complete embeddings; for dense orders, ones with dense range]. Is there a general definition of "nice embeddings"? Or a large class of such definitions? <u>Or</u> even
  (c) consider other second order properties of $M^{\bar{\varphi}}$.

  (so Definition 0.5(1) is relevant if we choose (a), Definition 0.5(4) is relevant if we choose (b), Definition 0.5 if we choose (c).

Toward this we construct complicated models; this is closely connected to bigness notions (see Definition 1.1 below).

Regarding the construction:

(D) It is easier if
  (α) we force the model (see §3).
  (β) It is hard if we build using, say, instances of GCH.
  (γ) If is harder if we try to do it in ZFC.
(E) we may look for
  (α) κ-saturated models with $\kappa > |T|$; so if we look into models of expansions (of T) then see [Shear] (interesting only for unstable T) <u>or</u> we may restrict ourselves less, only to
  (β) $\aleph_0$-saturated models (so also the parallel of un-superstability plays a role) <u>or</u>
  (γ) look at ultraproducts of models as in [She08].
(F) we need constructions of a kind depending on such choices:
(F)$_1$ for κ-saturated models, $\kappa > |T|$, (i.e. choice (E)(α)) we need the omitting of types of a large size so may choose, as in (E):
  (α) Forcing a κ-saturated model of T (with $|T| < \kappa$) by approximations of size $< \kappa$, we necessarily used bigness notions (so omitting types is automatic).
  (β) Build such a model using $\lambda = \lambda^{<\lambda}$ and/or $2^\lambda = \lambda^+$ and/or $\diamondsuit_{S^{\lambda^+}_\lambda}$, and so omitting types with no support $< \lambda$.
  (γ) Like [Sheb] (in $(2^\lambda)^{++}$ with the model being λ^+-saturated), so omitting types indirectly by orthogonality, working in ZFC.
(F)$_2$ For $\aleph_0$-saturated models, (i.e. choice (E)(β)) we need omitting types of small size (*a priori* hard). So naturally, as in (E), we may choose:
  (α) Use black boxes.
  (β) Build a model M of universe λ by approximations $M \restriction \alpha$ ($\alpha < \lambda$) using non-reflecting stationary $S \subseteq S^\lambda_{\aleph_0}$.
  (γ) force such M.
(F)$_3$ we may need more complicated constructions like $\mathcal{P}^-(n)$-amalgamation, (for use with weak diamond, see [She83a], [She83b]; with forcing in a barebones way, see [MS88]).

Discussion 0.27. 1) If we like to prove that the isomorphism quantifier for (a finite) t is compact, we can use "rich" T (see clause (b)). So this means $(\beta) \Rightarrow (\alpha)$ there.

2) So our program includes:

(A) Build complicated models of (usually first order) T, relative to various bigness notions.

(B) Find more first order theory t such that: if M is a complicated enough model of T then for any two interpretations of N_1, N_2 of t in M, any suitable isomorphism (or morphism) from N_1 into N_2 is definable in M.

$(C)_1$ Try to characterize such t, or at least

$(C)_2$ Get families of such t-s.

3) Now (B) is usually done in two steps. The first step is

(B) (α) Get some definability; say, in the infinitary logic $\mathbb{L}_{\infty,\kappa}$ in κ-saturated models.

The second step of (B) is:

(B) (β) Proving that "some definability implies even a better definability;" i.e. the definable transfer which may be formalized as: [End of Line]

Discussion 0.28. We can continue the discussion in 0.26.

In more detail:

(a) For any (complete first order) t which is unstable there are suitable non-trivial bigness notions.

(b) Suitable bigness notions for t can be lifted to $\mathfrak{C}$ for every interpretation of t inside $\mathfrak{C}$ (see Definition 0.16).

(c) If $\mathfrak{C}^+$ is κ-complicated (see Definition 1.10 or use other variants), $N = (\mathfrak{C}^+)^{[\bar\varphi]}$ is a model of t, and $\bar\varphi$ is an interpretation in $\mathfrak{C}^+$, then N is κ-complicated in the relevant sense.

$(c)^+$ Hence (see full definitions later) for every automorphism π of N, every relevant bigness notion Γ, for a dense set of $p(x,y), \Gamma$-big types in $\mathfrak{C}^+$ for the variable x with $|\mathrm{dom}(p)| < \kappa$, π is "locally" definable in the following sense. [Compare with Definition 1.10]

$\circledast_2$ For any $\varphi = \varphi(x,\bar z)$, we define the relation $E^1_{p,\varphi}$ by

$$\bar c_1 \; E^1_{p,\varphi} \; \bar c_2 \Leftrightarrow [\text{both } p(x,y) \cup \{\varphi(x,\bar c_1), \neg\varphi(x,\bar c_2)\} \text{ and } p(x,y) \cup \{\neg\varphi(x,\bar c_1), \varphi(x,\bar c_2)\} \text{ are not } \Gamma\text{-big}].$$

$\circledast_3$ the following relation R satisfies

(α) R is $\mathbb{L}_{\kappa,\kappa}$-definable in $\mathfrak{C}^+$ by

$$\big\{(\bar c, \bar d) : \text{for every } \Gamma\text{-big } p'(x,\bar y) \supseteq p(x,\bar y) \text{ complete over } \mathrm{dom}(p) \cup \bar c \cup \bar d, \text{ we have } \varphi(x,\bar c) \in p' \Rightarrow \varphi(\bar y, \bar d) \in p'\big\}.$$

(β) $\big\{(\bar c, \pi(\bar c)) : \bar c \in N\big\} \subseteq R$

(γ) If $\bar c_1, \bar c_2 \in N$ and $\neg(\bar c_1 \; E^1_{p,\varphi} \; \bar c_2)$, then for no $\bar d$ do we have $(\bar c_1, \bar d), (\bar c_2, \bar d) \in R$.

So R is the graph of a function from $N/E^1_{p,\varphi}$ into $N/E^1_{p,\varphi}$ [compare with 1.10].

§ 1. Complicated models and bigness notions

§ 1(A). **Complicated Quite Saturated Models.** We here formalize some notions of M being a κ-complicated model; so it is relatively easy if we force such a model, but of course, proving existence assuming instances of GCH is better and even more so doing it in ZFC, but we can expect some price. Our main tool is omitting types but it is easy to omit countable types in countable models, also types of cardinality λ in λ-compact models of cardinality λ. But it is more involved to omit countable types in models of cardinality $\aleph_1$. It is even harder to deal with types of cardinality λ in models of cardinality λ^+ (we need $\diamondsuit_\lambda$, or just $[D\ell]_\lambda$). But there is no real need to understand omitting general small types; we can restrict ourselves to special ones.

For this we define some version of so-called 'bigness notions.' We then stated what some constructions give. Those constructions should have some model theoretic content which here is done by bigness notions and in particular by pairs of orthogonal bigness notions which is useful in omitting types.

In this section we consider κ-saturated models of T a first order complete T; see [She83c], [Sheb].

The local version of bigness speaks on formulas and will be used for the ultraproduct case, continuing [She08], i.e. for local bigness notion Γ, a type p is Γ-big iff every $\varphi(x,\bar a) \in p$ is Γ-big. For global bigness notions this may fail. Bigness is a property of types (still $p \in \mathbf{S}(A,M)$ is Γ-big iff every $p \restriction B$ is for $B \subseteq A$ finite. Usually we start with local ones but the family of global ones has better closure properties.

If the following definitions are too general for your taste, look at the examples starting in 1.22.

Definition 1.1. Recall that $\mathfrak{C} = \mathfrak{C}_T$ denotes a monster model of T (so a $\bar\kappa$-saturated model, and so below $\kappa < \bar\kappa$).

1) We say Γ is a local $(\mathfrak{C},\kappa)$-bigness notion over $A = A_\Gamma \subseteq \mathfrak{C}$ where $|A| < \kappa$, and let $\bar{\mathbf{a}}_p$ be a sequence listing A_Γ of length $\beta(\Gamma) = \beta_p$ when:

- (A) Γ consists of
  - (a) $\bar x = \bar x_\Gamma$, a sequence of $< \kappa$ variables (usually $\langle x_i : i < \alpha(\Gamma)\rangle$, so $\alpha(\Gamma) = \ell g(\bar x)$). If $\ell g(\bar x) = 1$ we may write x_0 or x. Let $\ell g(\bar x)$ be called the -*arity* of Γ (written $\ell g(\Gamma)$).
  - (b) Γ^+, Γ^- such that $\Gamma^+ \cap \Gamma^- = \varnothing$ and $\Gamma^+ \cup \Gamma^-$ is the set $\mathbb{L}(\tau_{\mathfrak{C}}, \mathfrak{C})$ of first order formulas in $\tau_{\mathfrak{C}}$ with parameters from $\mathfrak{C}$ in the sequence of variables $\bar x$.
  - (c) Γ^- is an ideal; that is, if $\varphi_\ell(\bar x_\Gamma, \bar a_\ell) \in \mathbb{L}(\tau_{\mathfrak{C}})$ for $\ell = 1,2,3$ and $\varphi_\ell(\bar x_\Gamma, \bar a_\ell) \in \Gamma^-_\ell$ for $\ell = 1,2$ and $\varphi_1(\bar x_\Gamma, \bar a_1) \vee \varphi_2(\bar x_\Gamma, \bar a_2) \vdash \varphi_3(x, \bar a_3)$ in $\mathfrak{C}$ then $\varphi_3(\bar x_\Gamma, \bar a_3) \in \Gamma^-$ and $\Gamma^- \neq \varnothing$.
  - (d) For some $\tau_\Gamma \subseteq \tau_{\mathfrak{C}}$ of cardinality $< \kappa$ we have:
    - $(*)$ If $\varphi_\ell = \varphi(\bar x, \bar a_\ell) \in \Gamma^+ \cup \Gamma^-$ for $\ell = 1,2$, then $\varphi_1 \in \Gamma^+ \Leftrightarrow \varphi_2 \in \Gamma^+$ when:
      - (A) $\varphi_1 = \varphi_2$ and $\bar a_1, \bar a_2$ realizes the same type in $\mathfrak{C}$ over A_Γ or just (this definition is different if $|\tau_\Gamma| \geq \kappa$)
      - (B) $\varphi_1(\bar x, \bar a_1)$, $\varphi_2(\bar x, \bar a_2)$ are similar over τ_Γ, where
        - $\boxdot$ $\varphi_1 = \varphi_2(\bar x, \bar a_1)$, $\varphi_2 = \varphi_2(\bar x, \bar a_2)$ are *similar* over τ_Γ if there is a mapping $\mathbf{F}$ from $\tau_\Gamma \cup \tau_{\varphi_1}$ onto $\tau_\Gamma \cup \tau_{\varphi_2}$ which preserves -arity (and being a predicate/function symbol), is

the identity on τ_Γ, the mapping $\hat{\mathbf{F}}$ it induces on formulas maps $\varphi_1(\bar{x},\bar{y})$ to $\varphi_2(\bar{x},\bar{y})$ (and so $\ell g(\bar{a}_1) = \ell g(\bar{a}_2)$), and it maps the $\mathbb{L}_{\omega,\omega}(\tau_\Gamma \cup \tau_{\varphi_1})$-type which $\bar{a}_1$ realizes in $\mathfrak{C} \restriction (\tau_\Gamma \cup \tau_{\varphi_1})$ over A_Γ onto $\mathbb{L}(\tau_\Gamma \cup \tau_{\varphi_2})$-type which $\bar{a}_2$ realizes in $\mathfrak{C} \restriction (\tau_\Gamma \cup \tau_{\varphi_2})$ over A_Γ.

(1A) Above we may replace "local" by "purely local" when $A_\Gamma = \emptyset$; to stress the general case we may say "with parameters"; so the general case is reduced to the pure case <u>if</u> we work in $(\mathfrak{C}, a)_{a \in A}$ for some $A = A_\Gamma \subseteq \mathfrak{C}$ of cardinality $< \kappa$.

(B) (see 3.2(1A)) We say $\mathbf{\Gamma}$ is a local big notion scheme <u>if</u> we do not specify A_Γ but demand $\bar{\mathbf{c}}_p$ (which lists A_Γ) realizes $r^*_{\mathbf{\Gamma}}$, then we define an instance naturally. I.e. if $\bar{\mathbf{c}} \in {}^{\beta(\Gamma)}\mathfrak{C}_T$ realizes $r_\Gamma(\bar{z}_\Gamma)$ then $\mathbf{\Gamma}_{\bar{\mathbf{c}}}$ is a bigness notion, and if π is an automorphism of $\mathfrak{C}_T$ then $\pi''(\mathbf{\Gamma}_{\bar{\mathbf{c}}})$ is $\mathbf{\Gamma}_{\pi(\bar{\mathbf{c}})}$. Similar to Definition 1.2 [this definition is repeated in 3.2(1A),(1B)].

(C) We may omit κ if $|\tau_{\mathfrak{C}}| + \aleph_0 < \kappa$ (and $|A_\Gamma| < \kappa$ if we have parameters).

(D) We say $p \in \mathbf{S}^\alpha(M, \mathfrak{C})$ is Γ-big <u>if</u> $\alpha = \ell g(\bar{x}_\Gamma)$ and p is a set of formulas in $\bar{x}_\Gamma$ over M and every finite conjunction of members is Γ-big, where

(E) Members of Γ^+ are called Γ-big formulas, members of Γ^- are called Γ-small formulas.

(F) We say Γ is a local (T, M^*, A, κ)-bigness notion <u>if</u> it satisfies the demands in part (1) with M^* in the role of $\mathfrak{C}$, A the set of parameters; we let $A_\Gamma = A$, $\bar{\mathbf{c}}_p \in {}^{\gamma(p)}\mathfrak{C}$, $\bar{\mathbf{c}}_p$ listing A.

Definition 1.2. Γ is a global $(\mathfrak{C}, \kappa)$-bigness notion <u>when</u>: for some $\alpha = \alpha_\Gamma = \ell g(\bar{x}_\Gamma) < \kappa$, $\tau_\Gamma \subseteq \tau_{\mathfrak{C}}$ and $A_\Gamma \subseteq \mathfrak{C}$ are of cardinality $< \kappa$ we have:

(a) $\Gamma \subseteq \big\{p :$ for some $\tau_p \subseteq \tau_{\mathfrak{C}},\ |\tau_p| < \kappa,\ \tau_p \supseteq \tau_\Gamma$ and $A \subseteq \mathfrak{C},\ A \supseteq A_\Gamma,\ |A| < \kappa$ we have $p \in \mathbf{S}^\alpha_{\mathbb{L}(\tau_p)}(A, \mathfrak{C})\big\}$

(b) Γ is downward monotonic; i.e. if $\tau_\Gamma \subseteq \tau_1 \subseteq \tau_2$, $A_1 \subseteq A_2 \subseteq \mathfrak{C}$ and $p_2 = \mathrm{tp}(\bar{a}, A_2, \mathfrak{C} \restriction \tau_2) \in \Gamma$ then $p_1 = \mathrm{tp}(\bar{a}, A_1, \mathfrak{C} \restriction \tau_1) \in \Gamma$.

(c) Membership depends just on restrictions to finite sets: that is, if $\tau_\Gamma \subseteq \tau \subseteq \tau_{\mathfrak{C}}$, $p = \mathrm{tp}(\bar{a}, A, \mathfrak{C} \restriction \tau)$, $|\tau| + |A| < \kappa$, <u>then</u>

$$p \in \Gamma \Leftrightarrow (\forall B \subseteq A)\big[|B| < \aleph_0 \Rightarrow p \restriction (B \cup A_\Gamma) \in \Gamma\big].$$

(d) Γ is invariant in the natural sense; i.e. if f is a $(\mathfrak{C}, \mathfrak{C})$-elementary mapping, $A_\Gamma \subseteq \mathrm{dom}(f)$, $f \restriction A_\Gamma = \mathrm{id}_{A_\Gamma}$, <u>then</u> f maps a member of Γ to a member of Γ.

(e) **[Extension existence:]** If $p \in \mathbf{S}^\alpha_{\tau_p}(A, \mathfrak{C})$, $A \subseteq B$, and $\tau_p \subseteq \tau \subseteq \tau_{\mathfrak{C}}$ <u>then</u> for some $q \in \Gamma$ we have $p \subseteq q \in \mathbf{S}^\alpha_\tau(B, \mathfrak{C})$.

As in [She08], [Sheb], naturally our interest is in pre-t-bigness notions, which are local bigness notions.

Definition 1.3. We define $\Gamma_{t,\psi,\bar{\varphi}}$.

1) $\Gamma = \Gamma_{t,\psi}$ is a <u>pre-t-bigness notion scheme</u> when it consists of:

(a) a first order t and
(b) a sentence ψ_Γ (in possibly infinitary logic) in the vocabulary $\tau(t) \cup \{P_*\}$, where
(c) P_* is a unary predicate; recall that for simplicity we treat n-place function symbols $F \in \tau(t)$ as $(n+1)$-place predicates.

1A) We say a pre-t-bigness scheme $\Gamma = \Gamma_{t,\psi}$ is [locally true / globally true] <u>when</u> in parts 2), 3) below [$\mathbf{\Gamma}^{\rm loc}_{t,\psi,\bar\varphi,\bar{\mathbf{c}}}$ / $\mathbf{\Gamma}^{\rm glb}_{t,\psi,\bar\varphi,\bar{\mathbf{c}}}$] is a [local / global] $(\mathfrak{C},\kappa)$-bigness notion.

2) For an interpretation $\bar\varphi$ with parameters $\bar{\mathbf{c}}$ of t in $\mathfrak{C}$ with $\kappa = \mathrm{cf}(\kappa) > |\tau_\psi|$, we define $\Gamma = \mathbf{\Gamma}^{\rm loc}_{t,\psi,\bar\varphi} = \mathbf{\Gamma}^{\rm loc}_{t,\psi,\bar\varphi,\bar{\mathbf{c}}}$, a local $(\mathfrak{C},\kappa)$-bigness notion, as follows:

$(*)_1$ $r_\Gamma(\bar z_\Gamma)$ is such that $\bar{\mathbf{c}} \in {}^{\ell g(\bar z_1)}\mathfrak{C}$ realizes $r_\Gamma(\bar z_p)$ <u>iff</u> $N^{[\bar\varphi,\bar{\mathbf{c}}]}$ is a model of t.

$(*)_2$ If $\bar{\mathbf{c}}$ realizes $r_{\mathbf{\Gamma}}(\bar z_{\mathbf{\Gamma}})$, $N = \mathfrak{C}^{[\bar\varphi,\bar{\mathbf{c}}]}$, and $\varphi(\bar x_\Gamma,\bar y) \in \mathbb{L}(\tau_T)$, $\bar b \in {}^{\ell g(\bar y)}\mathfrak{C}$, then:
- $\varphi(\bar x_\Gamma,\bar b)$ is Γ-big iff the $\tau_t \cup \{P_*\}$-model $(N, \varphi(\mathfrak{C},\bar b) \cap {}^{\alpha(\Gamma)}N)$ satisfies ψ.

3) We define the global version $\Gamma = \mathbf{\Gamma}^{\rm glb}_{t,\psi,\bar\varphi} = \mathbf{\Gamma}^{\rm glb}_{t,\psi,\bar\varphi,\bar{\mathbf{a}}}$ for $\bar\varphi$ an interpretation of t in $\mathfrak{C}$:

$(*)_3$ If $N = \mathfrak{C}^{[\bar\varphi,\bar{\mathbf{c}}]}$ is a model of t, $B \subseteq \mathfrak{C}$ is finite, and $p \in \mathbf{S}^{\alpha(\Gamma)}(B \cup A_\Gamma,\mathfrak{C})$ then: p is Γ-big iff the $\tau_t \cup \{P_*\}$-model $(N, p(\mathfrak{C}))$ satisfies ψ.

4) We use loc and glb as shorthand for local and global, respectively; omitting [loc/glb] means it works for both.

Claim 1.4. *1) A local $(\mathfrak{C},\kappa)$-bigness notion (induces a) global bigness notion naturally.*

2) If $\mathfrak{C}$ is κ-saturated, t first order, $|\tau_\kappa| < \kappa, \bar\varphi$ an interpretation of t in M, and ψ as in Definition 1.3 <u>then</u> $\mathbf{\Gamma}^{\rm loc}_{t,\psi,\bar\varphi,\bar{\mathbf{a}}}$ is a local $(\mathfrak{C},\kappa)$-bigness notion and $\mathbf{\Gamma}^{\rm glb}_{t,\psi,\bar\varphi,\bar{\mathbf{c}}}$ is a global $(\mathfrak{C},\kappa)$-bigness notion.

3) We can decrease κ as long as $\kappa > |A_\Gamma| + |\tau_{\mathfrak{C}}|$.

Proof. See [Sheb]. $\square_{1.4}$

We can find natural global bigness notions which are not local bigness notions.

Example 1.5. 1) If Γ is a local bigness notion and $\mathbf{S} \subseteq \{p \in \mathbf{S}(A_\Gamma,\mathfrak{C}) : p \text{ is } \Gamma\text{-big}\}$ is dense with dense complement <u>then</u>

$$\Gamma_{[\mathbf{S}]} := \{p : p \in \mathbf{S}(B,\mathfrak{C}) \text{ for some } B,\ p \text{ is } \Gamma\text{-big},\ A_\Gamma \subseteq B \subseteq \mathfrak{C} \text{ and } p \restriction A_\Gamma \in \mathbf{S}\}$$

is a global bigness notion which is not local.

2) Let T be such that $\mathbf{S} = \{p \in \mathbf{S}(\varnothing,\mathfrak{C}) : p \text{ is weakly minimal}^4\}$ be dense in $\mathbf{S}(\varnothing,\mathfrak{C})$ with dense complement, and let

$$\Gamma = \{p \in \mathbf{S}(B,\mathfrak{C}) : p \text{ is non-algebraic and } p \restriction \varnothing \notin \mathbf{S}\}.$$

<u>Then</u> Γ is a global bigness notion which is not a local one.

Definition 1.6. 1) Let Γ_1,Γ_2 be two global bigness notions (for $\mathfrak{C}$), for the sequences of variables $\bar x^1,\bar x^2$ respectively (maybe infinite). We say that Γ_1,Γ_2 are orthogonal (or say Γ_1 is orthogonal to Γ_2, or say $\Gamma_1 \perp \Gamma_2$) <u>if</u> for any model $M \prec \mathfrak{C}, A \subseteq M$, and sequences $\bar a^1,\bar a^2 \in M$ of length $\ell g(\bar x^1),\ell g(\bar x^2)$ respectively such that $\mathrm{tp}(\bar a^\ell,A,M)$ is Γ_ℓ-big for $\ell = 1,2$, <u>there are</u> sequences $\bar b^1,\bar b^2$ (from $\mathfrak{C}$) of length $\ell g(\bar x^1),\ell g(\bar x^2)$ respectively such that for $1 = 1,2$ the sequence $\bar b^\ell$ realizes $\mathrm{tp}(\bar a^\ell,A,N)$ and $\mathrm{tp}(\bar b^\ell,A \cup \bar b^{3-\ell},N)$ is Γ_ℓ-big for $\ell = 1,2$. Similarly "for T".

2) In part (1) we say Γ_1,Γ_2 are *nicely orthogonal* (or we say Γ_1 is nicely orthogonal to Γ_2, or we say $\Gamma_1 \underset{\rm nice}{\perp} \Gamma_2$) <u>if</u>: adding to the assumption $A_{\Gamma_1} \cup A_{\Gamma_2} \subseteq A = \mathrm{acl}_M A$ we can add to the conclusion $\mathrm{acl}_M(A \cup \bar b^1) \cap \mathrm{acl}_M(A \cup \bar b^2) = A$.

[4] A type p is weakly minimal when it does not have $> 2^{|T|}$ pairwise contradictory non-algebraic extensions.

acl stands for *algebraic closure*; i.e.

$$\text{acl}_M(A) = \{b \in M : \text{for some } \bar a \subseteq A \subseteq M \text{ and } \varphi(y,\bar x) \in L \text{ we have } M \models \varphi[b,\bar a] \text{ and } M \models (\exists^{<n} y)\varphi(y,\bar a) \text{ for some finite } n\}.$$

Definition 1.7. 1) We say a bigness notion Γ is *isolated* when for every $B \supseteq A_p$, there is exactly one Γ-big $p \in \mathbf{S}^{\alpha(p)}(G,\mathfrak{C})$.

2) Γ is isolated *above* p_* when p_* is Γ-big and for every $B \supseteq \text{dom}(p) \cup A_\Gamma$, there is exactly one Γ-big $p \in \mathbf{S}^{\alpha(p)}(B,\mathfrak{C})$ extending p_*.

3) We say Γ is nowhere isolated when there is no p_* as in part (2).

Discussion 1.8. For some purposes isolated Γ are helpful; e.g. when we construct M_α increasing with α such that for many α-s some pseudo finite set in $M_{\alpha+1}$ includes M_α. Many times for rigidity models, some interesting bigness notions are such that every Γ-type have many contradictory extensions. Sometimes we need more: Γ_1,Γ_2 are not just orthogonal but in the relevant cases we have freedom; see 4.4.

Definition 1.9. 1) Assume Γ is a global $(\mathfrak{C},\kappa)$-bigness notion and Δ is a set of formulas from $\mathbb{L}(\tau_{\mathfrak{C}})$. We say that Γ has Δ-*freedom* in $(\mathfrak{C},\kappa)$ when:

$B_1 \subseteq \mathfrak{C}$, $|B_1| < \kappa$, $A_\Gamma \subseteq B_1$, $p \in \mathbf{S}^{\alpha(\Gamma)}(B_1,\mathfrak{C})$ is Γ-big then for some $\varphi(\bar x,\bar y) \in \Delta$ and $\bar c \subseteq \mathfrak{C}$, there are Γ-big types $p_0,p_1 \in \mathbf{S}^{\alpha(\Gamma)}(B \cup \bar c)$ extending p such that $\varphi(\bar x,\bar c) \in p_1, \neg\varphi(\bar x,\bar c) \in p_0$.

2) Assume Γ_1,Γ_2 are orthogonal global $(\mathfrak{C},\kappa)$-bigness notions. For a pair (Δ,A) with $A \subseteq \mathfrak{C}$ and Δ a set of formulas of the form $\varphi(\bar x_{\Gamma_1},\bar x_{\Gamma_2},\bar y)$, possibly with parameters we say (Γ_1,Γ_2) has (Δ,A)-freedom[5] over (p_1,p_2) when:

(a) p_ℓ is a Γ_ℓ-big type with $|p_\ell| < \kappa$, for $\ell = 1,2$.

(b) If $A_{\Gamma_1} \cup \text{dom}(p_1) \cup A_{\Gamma_2} \cup \text{dom}(p_2) \cup A \subseteq B \subseteq \mathfrak{C}$ and $|B| < \kappa$ and $p^+_\ell(\bar x_{\Gamma_\ell}) \in \mathbf{S}^{\ell g(x_{\Gamma_\ell})}(B,\mathfrak{C})$ is Γ_ℓ-big extending $p_\ell(\bar x_{\Gamma_\ell})$, then for some $\varphi(\bar x_{\Gamma_1},\bar x_{\Gamma_2},\bar y) \in \Delta$ and $\bar c \in {}^{\ell g(\bar y)}\mathfrak{C}$ and $\tau \subseteq \tau_{\mathfrak{C}}, |\tau| < \kappa, \tau \supseteq \tau_{\Gamma_1} \cup \tau_{p_1} \cup \tau_{\Gamma_2} \cup \tau_{p_2}$ and for $\mathbf{t} \in \{\text{true,false}\}$ there are $\bar a_\ell \in {}^{\ell g(\Gamma_\ell)}\mathfrak{C}$ realizing $p^+_\ell(\bar x_{\Gamma_\ell})$ [such that] the type $\text{tp}(\bar a_\ell, B + \bar a_{3-\ell} + \bar c, \mathfrak{C} \restriction \tau)$ is Γ_ℓ-big for $\ell = 1,2$ satisfying $\mathfrak{C} \models \varphi[\bar a_1,\bar a_2,\bar c]^{\text{if}(\mathbf{t})}$.

3) We may write Δ instead of (Δ,A) if $A = \varnothing$. We say *pure* if $\bar c \in {}^{\ell g(\bar y)}B$ and *very purely* if $\varphi(\bar x_{\Gamma_1},\bar x_{\Gamma_2},\bar y) \in \Delta \Rightarrow \bar y$ empty.

We can define complicated models for a bigness-notion or a family of bigness notions; i.e. as the ones constructed in [She83c] or forced. But we can just state the complications most relevant to trying to have few isomorphisms from N_1 to N_2 as above. This we do now.

Definition 1.10. 1) We say that M is a κ-isomorphism complicated or κ-embedding complicated (κ-iso-complicated or κ-emb-complicated for short) when for every Ω,Γ_1,N_1,N_2 as below, it is (Ω,Γ_1,κ)-isomorphism/embedding complicated for (N_1,N_2), which means:

(a) M is $(<\kappa)$-saturated,

(b) Γ_1 is a global $(\mathfrak{C},\kappa)$-bigness notion, Ω a family of such bigness notion schemes,

(c) N_ℓ is a model interpretable in M as $M^{[\bar\varphi^\ell]}$ for $\ell = 1,2$ for some (first order) $\bar\varphi^\ell$ possibly with parameters from A_{N_ℓ} such that Γ_1 concentrates on N_1 (meaning: if $p \in \mathbf{S}(B)$ is Γ_1-big then $p(\mathfrak{C}) \subseteq N_1$),

[5] We may omit A when $A = \varnothing$.

(d) $\tau_{N_1} = \tau_{N_2}$ has cardinality $< \kappa$

(e) If π is an embedding from N_1 onto/into N_2 (the onto is for the isomorphism version, the into for the embedding versions) and p_1 is a Γ_1-big type over M, $|p_1| < \kappa$, <u>then</u> we can find $\tau \subseteq \tau_{\mathfrak{C}}$ of cardinality $< \kappa$ and $B \subseteq M$ of cardinality $< \kappa$ and Γ_2, an instance of a bigness notion from Ω in $\mathfrak{C}$ over B satisfying $\mathrm{dom}(p_1) \cup A_{\Gamma_1} \cup A_{\Gamma_2} \cup A_{N_1} \cup A_{N_2} \subseteq B \subseteq M$ and $a \in N_1$ such that:

$(*)$ (α) $p'_1 = \mathrm{tp}(a, B, \mathfrak{C} \restriction \tau)$ is Γ_1-big extending p_1.

(β) $p_2(x_0, x_1) = \mathrm{tp}(\langle a, b\rangle, B, \mathfrak{C} \restriction \tau)$ is Γ_2-big, where $b = F(a)$.

(γ) If $\mathrm{tp}(\langle a', b'\rangle, B', \mathfrak{C} \restriction \tau)$ is Γ_2-big and $B \subseteq B'$ then $\mathrm{tp}(a', B', \mathfrak{C} \restriction \tau)$ is Γ_1-big.

(δ) If $R \in \tau_{N_\ell}$ has k-places, $a_2, \ldots, a_k \in N_1$, a' realizes p'_1, $b' \in N_2$ is such that the pair (a', b') realizes $\mathrm{tp}(\langle a, \pi(a)\rangle, B, \mathfrak{C} \restriction \tau)$, $B' = B \cup \{a_\ell, \pi(a_\ell) : \ell = 2, \ldots, k\}$, and $\mathrm{tp}(\langle a', b'\rangle, B', \mathfrak{C} \restriction \tau)$ is Γ_2-big, <u>then</u> $\mathfrak{C} \models \varphi^1_R(a', a_2, \ldots, a_k) \equiv \varphi^2_R(b', \pi(a_2), \ldots, \pi(a_k))$.

2) We omit Ω if it is the family of global $(\mathfrak{C}, \kappa)$-bigness notions schemes.

3) We say M is $(t, \Omega, \Gamma, t, \kappa)$-complicated <u>when</u> it is $(\Omega, \Gamma_1, \kappa)$-complicated for N_1, N_1 wherever $N_\ell = M^{[\bar\varphi^1]}$ for some interpretation $\bar\varphi^\ell$ of t in M, for $\ell = 1, 2$ and Γ_1 is an instance of Ω (i.e. of some member) concentrating on N_1.

Definition 1.11. We say that a first order theory t (in a vocabulary τ_t) is $(\infty, \mathbb{L}_{\infty,\kappa})$-rigid [for isomorphism/for embedding] <u>when</u>: if for some Γ_1, Ω as in (a),(b) of Definition 1.1 for every model M (in any vocabulary) which is $(\Omega, \Gamma_1, \kappa)$-complicated for isomorphisms/for embeddings, if N_1, N_2 are as in clause (c),(d),(e) of Definition 1.10 <u>then</u> F is definable in $\mathfrak{C}$ by some $\mathbb{L}_{\infty,\kappa}$-formula with parameters.

From Definition 1.10, we can deduce more (and if F is onto, *even more*)

Claim 1.12. *1) In Definition 1.10, clause (e) we can add (i.e. it follows that)*

(γ) *there are no $R \in \tau_t$ with $k > 1$ places and $a_1, \ldots, a_{k-1} \in N_1$ and $a'_1, \ldots, a'_{k-1} \in N_1$ such that*

(i) $\mathrm{tp}(\langle a_1, \ldots, a_{k+1}, \pi(a_1), \ldots, \pi(a_{k-1})\rangle, B, \mathfrak{C} \restriction \tau)$
$= \mathrm{tp}(\langle a'_1, \ldots, a'_{k-1}, \pi(a'_1), \ldots, \pi(a'_{k-1})\rangle, B, \mathfrak{C} \restriction \tau$

(ii) $p'_1 \cup \{\varphi^1_R(x, a_1, \ldots, a_{k-1}) \equiv \neg\varphi^1_R(x, a'_1, \ldots, a'_{k-1})\}$ *is Γ_1-big.*

2) Above if in addition $k = 2$ for notational simplicity <u>then</u> the two place relation $E = E_R = E_{\mathfrak{C}}$ and $\pi, \bar\varphi^1, \bar\varphi^2, \Gamma_1, \Gamma_2$ satisfy the following (it is defined in (β) below):

(α) *E is an equivalence relation on N_1.*

(β) *$a'Ea''$ <u>iff</u> $p'_1 \cup \{\varphi^1_R(x, a') \equiv \neg\varphi^1_R(x, a'')$ is not Γ-big.*

(γ) *The truth-value of $a'\ E\ a''$ is determined by $\mathrm{tp}(\langle a', a''\rangle, B, \mathfrak{C} \restriction \tau)$.*

(δ) *If $F(a') = b'$ then a'/E is determined by b' (and $\mathfrak{C}$, p'_1) because*

$\circledast$ *$a'' \in a'/E$ <u>iff</u> there is no (a^*, b^*) realizing $\mathrm{tp}((a, \pi(a)), B, \mathfrak{C} \restriction \kappa)$ such that $\mathrm{tp}(a^*, B \cup \{a''\}, \mathfrak{C} \restriction \tau)$ is Γ-big and $\mathfrak{C} \models$ "$\varphi^1_R(a^*, a'') \equiv \neg\varphi^2_R(b^*, b')$".*

Proof. Easy. $\square_{1.12}$

Definition 1.13. 1) We say t has $\Gamma_{t,\psi}$-uniformity <u>when</u>:

(a) $\Gamma_{t,\psi}$ is a pre-t-bigness notion scheme (see Definition 1.3) so if $\bar\varphi$ is an interpretation of t in $\mathfrak{C}$ then $\Gamma_{t,\psi,\bar\varphi}$ is a bigness notion as in 1.10(b).

(b) If $\mathfrak{C}, \pi, \bar\varphi^1, \bar\varphi^2, R$ are as in clause (e) of 1.10, then:

(α) $E = E_R$ is the equality on its domain.

(β) For some finite $\Delta \subseteq \mathbb{L}(\tau_t)$, for no $a \neq b \in \mathfrak{C}^{[\bar{\varphi}^1]}$, do we have

$$\mathrm{tp}(a, \mathrm{dom}(E), M^{[\bar{\varphi}^1]}) = \mathrm{tp}(b, \mathrm{dom}(E'), M^{[\bar{\varphi}^1]}).$$

2) We say t has half $\Gamma_{t,\psi}$-uniformity if only (a),(b)(α) hold.

Claim 1.14. *If M is $(t, \Omega, \Gamma_1, \kappa)$-complicated, see 1.10(3) and t has Γ-uniformity, then M is t-rigid.*

Proof. Should be clear. $\square_{1.14}$

The following is an alternative to connectivity.

Definition 1.15. We say t has κ-connectivity when $(A) \Rightarrow (B)$, where:

(A) (a) $\kappa > |T| + |\tau_t| + \aleph_0$
  (b) M is a κ-aut-complicated model of T.
  (c) $\bar{\varphi}^\ell$ is an interpretation of t in M for $\ell = \kappa$.
  (d) $\mathscr{X} \subseteq X_* = \big\{(\Gamma_1, p_1'(x), E_R) : \Gamma_1$ as in 1.10, p_1' is Γ_1-big type in M of cardinality $< \kappa$, and E_R as in 1.12 is dense in $X_*\big\}$.

(B) We can find $(\Gamma_{1,i}, p'_{1,i}(x), E_{R_i,i}) \in \mathscr{X}$ for $i < i_* < \kappa$ such that:
  (a) Each $E_{R_i,i}$ is equality on its domain.[6]
  (b) For no $a \neq b \in M^{[\bar{\varphi}^1]}$ do we have

$$\mathrm{tp}\big(a, \bigcup_i \mathrm{dom}(E_{R_i,i}), M^{[\bar{\varphi}^1]}\big) = \mathrm{tp}\big(b, \bigcup_i \mathrm{dom}(E_{R_i,i}), M^{[\bar{\varphi}^2]}\big).$$

Claim 1.16. *If M is κ-complicated then t has κ-connectivity.*

§ 1(B). More on Bigness Notions, and Old Examples.

Recall

Definition 1.17. 1) If $\bar{\Gamma} = \langle \Gamma_\varepsilon : \varepsilon < \varepsilon^* \rangle$ is a sequence of $(\mathfrak{C}, \kappa)$-bigness notions, then $\Gamma := \sum \langle \Gamma_\varepsilon : \varepsilon < \varepsilon^* \rangle$ is the following bigness notion (see 1.1 below)

(a) $\bar{x}_\Gamma$ is the concatenation of $\langle \bar{x}_{\Gamma_\varepsilon} : \varepsilon < \varepsilon^* \rangle$ and $\bar{x}'_{\Gamma_\varepsilon}$ is a copy of $\bar{x}_{\Gamma_\varepsilon}$. To make them pairwise disjoint, we will say

$$\alpha_\Gamma = \sum \langle \alpha_{\Gamma_\varepsilon} : \varepsilon < \varepsilon^* \rangle, \quad \bar{x}'_{\Gamma_\varepsilon} = \langle x_{\sum\{\alpha(\Gamma_\zeta):\zeta<\varepsilon\}+\gamma} : \gamma < \alpha_{\Gamma_\varepsilon} \rangle.$$

(b) $A_\Gamma = \bigcup \{A_{\Gamma_\varepsilon} : \varepsilon < \varepsilon^*\}$
(c) The type which $\langle \bar{a}_\varepsilon : \varepsilon < \varepsilon^* \rangle$ realizes over B is Γ-big iff $\mathrm{otp}(\bar{a}_\varepsilon, B \cup \bigcup \{\bar{a}_\zeta : \zeta < \varepsilon\}, \mathfrak{C})$ is Γ_ε-big for every $\varepsilon < \varepsilon^*$.

2) Assume:

(a) $\bar{\Gamma} = \langle \Gamma_\varepsilon : \varepsilon < \varepsilon^* \rangle$ is a sequence of $(\mathfrak{C}, \kappa)$-bigness notion schemes.
(b) Let $\bar{x}_\varepsilon = \bar{x}_{\Gamma_\varepsilon}$, $r_\varepsilon(\bar{z}_\varepsilon) = r^{\Gamma_\varepsilon}$ for $\varepsilon < \varepsilon^*$.
(c) $\bar{x}'_\varepsilon$ (a copy of $\bar{x}_\varepsilon$) and $\bar{x} = \langle \bar{x}'_\varepsilon : \varepsilon < \varepsilon^* \rangle \hat{\ } \langle \bar{x}_\varepsilon : \varepsilon < \varepsilon^* \rangle$ are pairwise disjoint, and $\bar{x}_{\mathbf{\Gamma}} = \bar{x}'_0 \hat{\ } \bar{x}'_1 \hat{\ } \ldots$.
(d) $\bar{z}'_\varepsilon$ (a copy of $\bar{z}_\varepsilon$) and $\bar{z} = \bar{z}'_0 \hat{\ } \bar{z}_0 \hat{\ } \ldots$ are pairwise disjoint.
(e) $\bar{x}$ disjoint to $\bar{z}$.
(f) $r_p = r_p(\bar{z}_p) = \bigcup \{r_{\Gamma_\varepsilon}(\bar{z}'_\varepsilon) : \varepsilon < \varepsilon^*\}$ so $\bar{\mathbf{c}} = \langle \bar{c}_\varepsilon : \varepsilon < \varepsilon^* \rangle$ realizes $r_p(\bar{z}_p)$ iff $\bar{c}_\alpha$ realizes $r_{\Gamma_\varepsilon}(\bar{z}_\varepsilon)$ for every $\varepsilon < \varepsilon^*_\omega$.

[6]This can be weakened.

(g) [As] $\bar{\mathbf{c}} = \langle \bar{c}_\varepsilon : \varepsilon < \varepsilon^* \rangle$ realizes r_p and [we] set $B \subseteq \mathfrak{C}$, $(B \supseteq \bar{\mathbf{c}}_p)$, $\mathrm{tp}(\langle \bar{c}_\varepsilon : \varepsilon < \varepsilon_* \rangle, B, \mathfrak{C})$ is $\Gamma_{\bar{\mathbf{c}}}$-big iff for every α we have that
$$\mathrm{tp}\big(\bar{a}_\alpha, B \cup \bigcup_{\beta<\alpha} \bar{a}_\beta, \mathfrak{C}\big) \text{ is } \Gamma_{\alpha,\bar{\mathbf{c}}_\varepsilon}\text{-big.}$$

Then $\Gamma = \sum \langle \Gamma_\varepsilon : \varepsilon < \varepsilon^* \rangle$, modulo the choices in (b), is a bigness notion scheme.
3) If Γ is a $(\mathfrak{C}, \kappa)$-bigness notion $\bar{x}' \subseteq \bar{x}_\Gamma$, then[7] $\Gamma' := \Gamma \restriction \bar{x}'$ (the projection of Γ to $\bar{x}'$) is the bigness notion defined as follows:

(a) $\bar{x}_{\Gamma'} = \bar{x}'$
(b) $\mathrm{tp}(\bar{a}', B, \mathfrak{C})$ is Γ'-big iff for some $\bar{a} \in {}^{\alpha(\Gamma)}\mathfrak{C}$, $\mathrm{tp}(\bar{a}, B, \mathfrak{C})$ is Γ-big and $\bar{a} \restriction \ell g(\bar{x}') = \bar{a}'$.

4) Similarly for Γ a $(\mathfrak{C}, \kappa)$-bigness notion scheme.

Claim 1.18. *The $(\mathfrak{C}, \kappa)$-bigness notions defined in Definition 1.17(1)-(4) are $(\mathfrak{C}, \kappa)$-bigness notions or notion schemes.*

Definition 1.19. 1) If Γ is a t-bigness notion (i.e. for $\mathfrak{C}_t$) then for any $\mathfrak{C}$ and interpretation $\bar{\varphi} = (\bar{\varphi}, \bar{c})$ of N in $\mathfrak{C}$ we define $\Gamma^{[\bar{\varphi}]}$, the lifting of Γ to $\mathfrak{C}$ through $\bar{\varphi}$ as follows: if $B \subseteq \mathfrak{C}$ is finite $p(\rho_\Gamma, \bar{x}_\Gamma) \in \mathbf{S}^{\alpha(\Gamma)}(B, \mathfrak{C})$ and $p(\mathfrak{C}) \subseteq {}^{\alpha(\Gamma)}N$ then $p(\bar{x}_\Gamma)$ is $\Gamma^{[\bar{\varphi}]}[\mathfrak{C}]$-big iff or some $B \subseteq N$ (so of cardinality $< \bar{\kappa}$) satisfies
$$B \subseteq \{\bar{a} \in {}^{\ell g(\bar{x}_\Gamma)}N : \mathrm{tp}(\bar{a}, B', N) \text{ is } \Gamma\text{-small}\}.$$

2) Similarly, schemes are lifted to schemes — only the new scheme has more parameters: those of Γ from n and those of the interpretation $\bar{\varphi}$.

Claim 1.20. *1) In 1.19(1), $\Gamma^{[\bar{\varphi}]}$ is a $(\mathfrak{C}, \kappa)$-bigness notion.*
2) In 1.19(2), $\Gamma^{[\bar{\varphi}]}$ is a $(\mathfrak{C}, \kappa)$-bigness notion scheme.
3) In 1.19, if Γ is local and has $\{\psi(x, \bar{y})\}$-freedom, then $\Gamma^{\bar{\varphi}}$ has $\{\psi'(x, \bar{y}')\}$-freedom, where $\psi'(x, \bar{y})$ in the result of substituting $\bar{\varphi}$ inside ψ.
4) We can phrase Definition 1.19 as cases of $\Gamma_{t,\psi}$, see Definition 1.3.

As promised, just the assumption that t is unstable suffices for the existence of non-atomic bigness notion. (This is complementary to Claim 0.18.)

Claim 1.21. *If t is (complete first order and) unstable, then there is a bigness notion Γ for t; Γ is everywhere not isolated (see 1.7).*

Proof. If $\varphi = \varphi(\bar{x}, \bar{z})$ is an unstable formula, $\mathfrak{C}$ a model of t, defines $\Gamma = \Gamma_{\varphi,\mathfrak{C}}$ by: a type $p(\bar{x})$ in $\mathfrak{C}$ is Γ-big iff $\bigcup \{p(\bar{x}_\eta) : \eta \in {}^\omega 2\} \cup \{\varphi(\bar{x}_\eta, z_{\eta \restriction n})^{\eta(n)} : \eta \in {}^\omega 2,\ n < \omega\}$ is finitely satisfiable in $\mathfrak{C}$. $\square_{1.21}$

We can phrase some older results in this frame.

Definition 1.22. 1) Let $t = t_{\mathrm{dlo}}$ the theory of dense linear order.
2) $\psi = \psi_{\mathrm{dlo}} \in \mathbb{L}(\tau_t \cup \{P_*\})$ be: $M \models \psi$ iff $(|M|, <^M)$ is a dense linear order and P_*^M is dense in some interval.

Claim 1.23. *1) For $(t, \psi) = (t_{\mathrm{dlo}}, \psi_{\mathrm{dlo}})$ from Definition 1.22:*

(A) $\Gamma_{t,\psi}$ is as in 1.3, a pre-t-bigness notion scheme.
(B) It has half uniformity (see Definition 1.13).
(C) If M is κ-iso-complicated (see Definition 0.5) and π is an isomorphism from $M^{[\bar{\varphi}^1]}$ onto $M^{[\bar{\varphi}^2]}$, where $\bar{\varphi}^\ell$ interprets t, then for a dense set of intervals I of $N^{[\bar{\varphi}^1]}$, $\pi \restriction I$ is M-inner (see Definition 0.15).

[7] On restricting τ, see later.

2) For $t = t_{\rm of}$ = the theory of ordered fields, [if] the above holds we have:

(A) $\Gamma_{t,\psi}$ is as in 1.3, a pre-t-bigness notion scheme.
(B) t has $\Gamma_{t,\psi}$-uniformity (see 1.13).
(C) In (1)(C) we get 'π is M-inner' (see 0.15).
(D) t has transfer (see Definition 0.10).
(E) t is rigid (see Definition 0.3).

Proof. By [She83c]. $\square_{1.23}$

Definition 1.24. 1) Let $t_{\rm ABA}$ be the first order theory of atomic Boolean Algebras.
2) Let $\psi = \psi^{\rm loc}_{\rm ABA} \in \mathbb{L}(\tau(t_{\rm BA}) \cup \{P_*\})$ say:

- For some finite set X of atoms (of the Boolean Algebras), for every n and pairwise distinct atoms $y_0, \ldots, y_{2n-1}$ not from X, there is x such that
$$P_*(x) \wedge \bigwedge_{\ell<n} y_{2\ell} \leq x \wedge \bigwedge_{\ell<n} y_{2\ell+1} \cap x = 0.$$

3) Let $t^{\rm glb}_{{\rm ABA},\theta}$ be defined similarly, but $|X| \leq \theta$.

Claim 1.25. *1) For $t = t^{\rm loc}_{\rm ABA}$, clauses (A)-(E) of 1.22(2) hold.*

Remark 1.26. 1) This is enough for proving that quantification on isomorphisms from one Boolean Algebra onto another, is compact (we need a little more then).
2) We can deal also with complete embeddings.
3) Can use $t_{\rm BA}$ and $\psi^{\rm loc}_{\rm ABA}$ complete.
4) We can say more on the case of atomless Boolean algebras.

Proof. Left to the reader. $\square_{1.25}$

More general is

Definition 1.27. Let $\tau_{\rm ind} = \{P, Q, R\}$ with P, Q unary, R binary.

1) Let $t = t_{\rm ind} \subseteq \mathbb{L}(\tau_{\rm ind})$ be such that $M \models t$ iff P^M, Q^M is a partition of M and for every n and pairwise distinct $a_0, \ldots, a_{2n-1} \in P^M$ there is $b \in Q^M$ such that $M \models$ "$b\ R\ a_\ell^{\text{if}(\ell \text{is even})}$" for $\ell < 2n$.

2) Let $\psi = \psi_{\rm ind} \in \mathbb{L}(\tau_{\rm ind} \cup \{P_*\})$ say

$$(P_* \subseteq Q) \wedge \Big(\bigwedge_m \bigvee_n (\exists x_0, \ldots, x_m \in P)(\forall y_0, \ldots, y_n \in P)$$

$$\Big[\bigwedge_{\substack{i<m \\ j<n}} x_i \neq y_j \wedge \bigwedge_{i<j<n} y_i \neq y_j \Rightarrow (\exists z \in P_*)\big[\bigwedge_{i<n} z\ R\ y_i^{\text{if}(2|n)} \big] \Big] \Big)$$

Claim 1.28. *For $(t, \psi) = (t_{\rm ind}, \psi_{\rm ind})$ from Definition 1.27, we have*

(A) *$\Gamma_{t,\psi}$ is, as in 1.3, a pre-t-bigness notion scheme.*

(B) *t has half uniformity (see 1.13).*

(C) *If M is κ-iso-complicated (see Definition 0.5) and π is an isomorphism from $N_1 = M^{[\bar\varphi^1]}$ to $N_2 = M^{[\bar\varphi^2]}$ (where the $\bar\varphi^\ell$ interpret t), then for some disjoint $A, B \subseteq P^{N_1}$ of cardinality $< \aleph_1$, letting*

$$I = \{c \in Q : a \in A \Rightarrow c\ R^{N_1} a,\ b \in B \Rightarrow \neg c\ R^{N_1} b\},$$

we have that $\pi \restriction I$ is M-inner (see 0.15).

We need the more general frame of §(0C) for the following example. The following will formalize [the statement / our hypothesis / etc.] "in M, every branch of the tree is definable" in two ways.

Claim 1.29. *1) Let $t_1 =$ " $<$ is a tree"; i.e. a partial order, which is a linear order below any element and let $t_2 = t_1 +$ "P_* is a branch (= maximal linearly ordered subset)". Then (t_1, t_2) has def transfer (see 0.10 but really 0.20(2)).*

2) Let t_1 say

(a) *$(P, <_1)$ is a partial order.*

(b) *$(Q, <_2)$ is a linear order.*

(c) *$F : P \to Q$ acts as level assignment:*

   (α) *$x <_1 y \Rightarrow F(x) <_2 F(y)$*

   (β) *$(\forall x \in P)(\forall y \in Q)\big[F(x) <_2 y \to (\exists z)[x <_1 z \wedge F(z) = y]\big]$ (alternatively, consider $t <_2 F(z)$).*

Let t_2 be $t_1 +$ "$P_* \subseteq P$ is directed" $+$ "$\{F(x) : x \in P_*\}$ is cofinal in $(Q, <_2)$".

Then (t_1, t_2) has def. transfer (see 0.20(2)).

Definition 1.30. 1) Let $\tau_{\rm po} = \{<\}$ be a binary predicate (where po stands for 'partial order'). $t_{\rm po} \subseteq \mathbb{L}(\tau_{\rm po})$ is such that $M \models t_{\rm po}$ if:

(a) $<^M$ is a partial order.

(b) $M \models$ "$(\forall x)(\exists y_1, y_2)\big[x < y_1 \wedge x < y_2 \wedge \neg(\exists w)[y_1 < w \wedge y_2 < w_2]\big]$"

(c) Any two members of M have a common $\leq^M$-lower bound.

1A) Let $\psi_{\rm po} = \mathbb{L}(\tau_{\rm po} \cup \{P_*\})$ say that P_* is somewhere dense: that is,

$$(\exists x)(\forall y)\big[x < y \Rightarrow (\exists z \in P_*)[y < z]\big]$$

2) Let $\tau_{\rm hpo} = \{<, P, F\}$, F a three-place function symbol, P unary, where hpo stands for 'homogeneous partial order'. Let $t_{\rm hpo} \subseteq \mathbb{L}(\tau_{\rm hpo})$ be such that $M \models \tau_{\rm hpo}$

iff $M \models t_{\rm po}$ and P^M is dense and for any $a, b \in P^M$, $F(-, a, b)$ is an isomorphism from $(M_{\geq a}, <^M \restriction M_{\geq a})$ onto $(M_{\geq b}, <^M \restriction M_{\geq b})$.
2A) $\psi_{\rm hpo} \in \mathbb{L}(\tau_{\rm hpo} \cup \{P_*\})$ says: P_* is somewhere dense and

$$x \not\leq y \Rightarrow (\exists z)\big[x < z \wedge (\forall w)[z \leq w \Rightarrow y \not\leq w]\big]$$

Claim 1.31. *1) For $(t, \psi) = (t_{\rm po}, \psi_{\rm po})$ from Definition 1.30(1),*

(a) $\Gamma_{t,\psi}$ is as in 1.3, a pre-t-bigness notion scheme.

(b) t has half uniformity (see 1.13).

(c) if M is κ-iso-complicated (see Definition 0.5) and π is an isomorphism from $N_1 = M^{[\bar\varphi^1]}$ to $N_2 = M^{[\bar\varphi^2]}$ (where the $\bar\varphi^\ell$ interpret t), then for a somewhere dense $I \subseteq N_1$, $\pi \restriction I$ is M-inner (see Definition 0.15).

2) For $(t, \psi) = (t_{\rm hpo}, \psi_{\rm hpo})$:

(a) $\Gamma_{t,\psi}$ is as in 1.3, a pre-t-bigness notion scheme.

(b) t has $\Gamma_{t,\psi}$-uniformity (see 1.13).

(c) in (1)(c) we get π is M-inner (see 0.15).

(d) t has transfer (see Definition 0.10).

(e) t is rigid (see Definition 0.3).

Proof. As in [She83c]. $\square_{1.31}$

Discussion 1.32. **[2022-04-10 – Sort out?]** 1) Assume that $\psi(x, y) \in \mathbb{L}(\tau_t)$ has the strict order property in every model of t:

(a) $\Gamma = \Gamma^{\rm sor}_{t,\psi}$ is a local bigness notion scheme, where for $N = M^{[\bar\varphi]}$ a model of t, a formula $\varphi(x, \bar a)$ in M is Γ-big <u>when</u> there are $a_1 <^N_\psi a_2$ (meaning $N \models \psi[a_1, a_2] \wedge \bigwedge_n (\exists x_0 \ldots x_n)\big[\bigwedge_{\ell<n} \psi(x_\ell, x_{\ell+1}) \wedge x_0 = a_1 \wedge x_n = a_2\big]$) such that:

- If $\psi(a_1, a_1')$, $a_1' <^N_\psi a_2'$ and $\psi(a_2', a_1)$ then
$$(\exists x)\big[\varphi(x, \bar a) \wedge \psi(\bar a_1', x) \wedge \psi(x, a_2')\big].$$

(b) If $T_1 \supseteq t$ and $\lambda = \lambda^{<\lambda} > |T_1|$ for transparency, then for some $T_2 \supseteq T_1$, $|T_2| = |T_1|$ and a unary predicate $P \in \tau(T_2)$ such that:

- $\bullet_1$ $M_2 \models T_2$
- $\bullet_2$ $M_2 \restriction \tau(T_1)$ is quite saturated.
- $\bullet_3$ $\psi(-,-)$ linearly orders P^{M_2}.
- $\bullet_4$ P^{M_2} is infinite.
- $\bullet_5$ If $b_1, b_2 \in M$ realize the same $\mathbb{L}(\tau_t)$-type over P^{M_2}, then for some $\mathbf{I} \subseteq P^{M_2}$ of cardinality $< \|M_2\|$ we have: if $\mathbf{I} \subseteq \mathbf{I}' \subseteq M_2$ and $\psi(-,-)$ linearly orders $\bar{\mathbf{T}}'$ then ${\rm tp}(b_1, \mathbf{I}', M_2 \restriction \tau_t) = {\rm tp}(b_2, \mathbf{I}', M_2, \tau_t)$.

3) Assume $\psi(x, y) \in \mathbb{L}(\tau_t)$ has the independence property in t.

(a) $\Gamma = \Gamma^{\rm ind}_{t,\psi}$ is a local bigness scheme (as in [Sheb]).

(b) Like clause (b) of part (1), but

- $\bullet_3'$ The following sequence of formulas $\langle \psi(x, a) : a \in P^M \rangle$ is independent.
- $\bullet_5'$ Analogously.

Remark 1.33. We may sort out what it means that: for κ-saturated a Ω-complicated model and let for dense pair $(p(x), E(x, y))$, $\underset{\sim}{f} \restriction \big(p(M)/E^M\big)$ is $\mathbb{L}_{\infty,\kappa}$-definable.

[I'm guessing Ms. Leonhardt put a mark there because she couldn't read or parse what was on the page. This sentence needs to be rewritten.]

§ 2. Triangle free graphs and more general examples

In this section we try to find some additional cases. It seems that having dealt with Boolean algebras and ordered fields, a natural candidate is the model completion of triangle-free graphs. The idea was that this will require more sophisticated constructions. As it happens, 1.10, 1.12 from §1 suffice. We do this in a more general way, model complete $\mathscr{K}$-free models. The reader may start with the main example in 2.7.

Definition 2.1. Assume

$(*)_{\mathscr{K}}$ $\tau = \tau_{\mathscr{K}}$ a finite vocabulary with predicates only, $\mathscr{K}$ is a finite set of finite τ-structures M which are full; i.e. for every $a \neq b \in M$ for some $R \in \tau$ and $\langle c_1, \ldots, c_{n(R)}\rangle \in R^M$ we have $\{a, b\} \subseteq \{c_1, \ldots, c_{n(R)}\}$.

1) Let $T^0_{\mathscr{K}}$ be the universal theory saying the $\tau_{\mathscr{K}}$-structures are $\mathscr{K}$-free; i.e. $\text{Mod}(T^0_{\mathscr{K}})$ is the family of $\tau_{\mathscr{K}}$-structures with no finite substructure isomorphic to a member of $\mathscr{K}$.

2) Let $T_{\mathscr{K}}$ be the model completion of $T^0_{\mathscr{K}}$.

3) We say $\mathscr{K}$ is non-trivial <u>if</u> for some $R \in \tau_{\mathscr{K}}, |\text{Rang}(\bar{a})| \geq 2$ where $\bar{a} \in R^M \neq \varnothing$ for some $M \models T^0_{\mathscr{K}}$; without loss of generality $|M| = \{a_1, \ldots, a_{n(R)}\}$, $\langle a_1, \ldots, a_{n(R)}\rangle \in R^M$. Replacing R by an atomic formula φ we have $M \models \varphi[\bar{a}]$, $\bar{a} = \langle a_1, \ldots, a_{n(\varphi)}\rangle$ (see part (5) below) is with no repetitions and we call M or $(M, a_\ell)_{\ell=1,\ldots,n(\varphi)}$ a φ-witness.

3A) We say $\mathscr{K}$ is strongly indecomposable when for every model M of $T_{\mathscr{K}}$ and finite $A \subseteq M$ and non-algebraic $p \in \mathbf{S}_{\rm qf}(A, M)$ there is no non-trivial automorphism of M over $M \cup p(M)$.

4) For a model M of $T^0_{\mathscr{K}}$ and $A, B \subseteq M$ let $A \oplus B \subseteq M$ mean: $A \cap B = \varnothing$ and if $R \in \tau_{\mathscr{K}}$ has n-place then $R \cap {}^n(A \cup B) = (R \cap {}^n A) \cup (R \cap {}^n B)$.

Similarly

$\oplus$ $A_1 \oplus_{A_0} A_2 \subseteq M$ means $A_\ell \subseteq M, A_1 \cap A_2 \subseteq A_0$ and for every $R \in \tau_{\mathscr{K}}$ we have $R \cap {}^n(A_0 \cup A_1 \cup A_2) \subseteq (R \cap {}^n(A_0 \cup A_1)) \cup (R \cap {}^n(A_0 \cup A_2))$. Similarly $\bigoplus\limits_A \{A_i : i < i^*\} \subseteq M$.

4A) We say "c is disconnected to A in M" if $\{c\} \oplus A \subseteq M$.

5) We say that $\bar{a} \in {}^{\omega>}M$ strictly satisfies an atomic formula <u>if</u> there is $R \in \tau_M$ and $i_0, \ldots, i_{n(R)-1} \in \{0, \ldots, \ell g(\bar{a}) - 1\}$ such that $\langle a_{i_0}, \ldots, a_{i_{n(R)-1}}\rangle \in R^M$ and for every $\ell < \ell g(\bar{a})$, a_ℓ appears in $\{a_{i_0}, \ldots, a_{i_{n(R)-1}}\}$. (The point is that if $\bar{a} \in R^M$ then some $\bar{a}'$ with no repetition strictly satisfies an atomic formula in M and $\text{Rang}(\bar{a}) = \text{Rang}(\bar{a}')$.)

We may use sequences instead of sets.

Those theories are well known (see [CSS99] and references there).

Claim 2.2. *1) $T_{\mathscr{K}}$ is well defined and has elimination of quantifiers.*

2) $T^0_{\mathscr{K}}$ has natural amalgamation; i.e. if $M_0 \subseteq M_\ell$ for $\ell = 1, 2$ are models of $T^0_{\mathscr{K}}$ and $|M_1| \cap |M_2| = |M_0|$ and $M = M_1 \cup M_2$ (i.e. $|M| = |M_1| \cup |M_2|$, $R^M = R^{M_1} \cup R^{M_2}$) <u>then</u> M is a model of $T^0_{\mathscr{K}}$. We may write this as $M = M_1 \bigoplus\limits_{M_0} M_2$; if $M_0 = \varnothing$ we write $M_1 \bigoplus\limits_{M_0} M_2$ and similarly $\oplus\{M_i : i < i^*\}$ or $\bigoplus\limits_M \{M_i : i < i^*\}$.

Definition 2.3. For $\mathscr{K}$ satisfying $(*)$ of Definition 2.1.

0) We say $\mathscr{K}$ (or $T_{\mathscr{K}}$) is interesting when for every complete quantifier free type $p^*(x)$ over the empty set (realized in some model of $T_{\mathscr{K}}$) we have $\mathscr{K}$ is $p^*(x)$-interesting (see below).

1) We say that $\mathscr{K}$ is $p^*(\bar{x}) - m$-interesting if:

(a) $p^* = p^*(\bar{x})$ is a $\tau_{\mathscr{K}}$-quantifier free type realized in some model of $T_{\mathscr{K}}$ hence is realized by infinitely many elements in some models of $T_{\mathscr{K}}$

(b) if N is a model of $T_{\mathscr{K}}$, $A \subseteq N$ is finite, $k < \omega$, for $\ell < 2k$ we have $\bar{d}_\ell \in {}^{\ell g(\bar{x})}(N)$ disjoint to A realizes $p^*(\bar{x})$ for $\ell < k$ we have $\bar{d}_{2\ell} \neq \bar{d}_{2\ell+1}$ and $\bigoplus\{\bar{d}_{2\ell}{}^\frown\bar{d}_{2\ell+1} : \ell < k\} \oplus A \subseteq N$ (see Definition 2.1(4) so in particular $\langle \mathrm{Rang}(\bar{d}_{2\ell}{}^\frown\bar{d}_{2\ell+1}) \setminus A : \ell < k\rangle$ is a sequence of pairwise disjoint sets) then there $\bar{d} \in {}^m N$ such that:

$\circledast$ $i, j < k \Rightarrow$ the quantifier free types which $\bar{d}{}^\frown\bar{d}_{2i}, \bar{d}{}^\frown\bar{d}_{2j+1}$ realize in N are different.

Omitting m means: for some m.

(By compactness, if N is κ-compact [even just for quantifier free formulas] and $|A| < \kappa$ we can allow any $k < \kappa$.)

2) We say $\mathscr{K}$ is $p^*(\bar{x})$-interesting when we add in clause (b) the demands: for $i < k$ for every n, there is an indiscernible sequence over A of length n to which $\bar{d}_{2i}, \bar{d}_{2i+1}$ belongs.

Definition 2.4. Assume $\mathscr{K}$ is $p^*(\bar{x}) - m$-interesting, $t = T_{\mathscr{K}}$ (a complete first order theory), P_* a new predicate with m-places. We define $\psi = \psi_{t,p^*(\bar{x})} = \psi_{\mathscr{K},p^*(\bar{x})}$ as follows (see Definition 1.3).

Now $(N, P_*) \models \psi$ if

(a) N is a model of t.

(b) For every $k < \omega$, for some finite set $A \subseteq N$, we have [End of Line?]

(c) if $\bar{d}_i \in {}^m N$ realizes the type $p^*(\bar{x})$ for $i < 2k$ and $\bar{d}_{2i} \neq \bar{d}_{2i+1}$ for $i < \kappa$ and

$$\bigoplus_{i<k}(\bar{d}_{2i} \cup \bar{d}_{2i+1}) \oplus A \subseteq N$$

(see Definition 2.1(4)), then there is $\bar{d} \in P_*^N$ such that:

$\circledast$ If $i, j < k$ then $\bar{d}{}^\frown\bar{d}_{2i}, \bar{d}{}^\frown\bar{d}_{2j+1}$ realize different quantifier-free types in N.

Claim 2.5. *Assume $(*)_{\mathscr{K}}$ of Definition 2.1 and $t = T_{\mathscr{K}}$ and t is $p^*(\bar{x})$-interesting. $\psi = \psi_{t,p^*(\bar{x})}$ and then $\Gamma_{t,\psi}$ is a pre-bigness notion scheme.*

Proof. Obviously $\bar{x} = \bar{x}$ is Γ_ψ-big by the choice of $p^*(\bar{x})$ (see 2.3(1)); clearly monotonicity holds, so the main point is (N interpreted in $\mathfrak{C}$)

$\boxtimes$ if $\varphi_\ell(\bar{x})$ is Γ_ψ-small for $\ell = 1, 2$ and $\varphi(\bar{x}) = \varphi_1(\bar{x}) \vee \psi_2(\bar{x})$ then $\varphi(\bar{x})$ is Γ-small (all in $\mathfrak{C}$ with parameters).

Why? For $i \in \{1, 2\}$ as the formula $\varphi_\ell(\bar{x})$ is not Γ_ψ-big there is $k_\ell < \omega$ such that: for every finite set $A \subseteq \mathfrak{C}$ there are $\langle \bar{d}_\ell[A, i] : \ell < 2k_i\rangle$ witnessing the failure (see Definition 2.4). Let $k = k_1 + k_2 < \omega$ and we shall show that it exemplifies "$\varphi(\bar{x}) = \varphi_1(\bar{x}) \vee \varphi_2(\bar{x})$ is Γ_ψ-small". So let a finite set $A \subseteq \mathfrak{C}$ be given, and we should find $\langle \bar{d}_\ell : \ell < 2k\rangle$ as required. Let $\bar{d}_\ell$ be $\bar{d}_\ell[A, 1]$ if $\ell < 2k_1$ and let $\bar{d}_\ell$ be $d_{\ell-2k_1}[A', 2]$ if $\ell \in [2k_1, 2k)$ when we let $A' = A \cup \bigcup\{\bar{d}_\ell : \ell < 2k_1\}$.]

Now for any $\bar{d}$ realizing $\varphi(\bar{x})$ we should find $i, j < 2k$ such that $\bar{d} = \bar{d} \wedge \bar{d}_{\langle i\rangle}, \bar{d}{}^\frown\bar{d}_{2i+1}$ realizes the same quantifier type, clearly $\varphi_1(\bar{d}) \vee \varphi_2(\bar{d})$. Now we split the proof to two cases.

Case 1: $\varphi_1(\bar d)$.

Then (by the choice of $\langle \bar d_\ell : \ell < 2k_1\rangle$) for some $i, j < k_1$ the sequences $\bar d\char`^\bar a_{2i}, \bar d\char`^\bar a_{2j+1}$ realize the same quantifier-free type in N, so i, j are as required.[8]

Case 2: $\varphi_2(\bar d)$.

Similarly using $\langle \bar d_{2k_1+\ell} : \ell < 2k_2\rangle$. $\square_{2.5}$

As said above, bigness notion with freedom are helpful.

Claim 2.6. *Assume $(*)_{\mathscr K}$ of 2.1, $t = T_{\mathscr K}$, t is $p^*(\bar x)$-interesting, $\psi = \psi_{t,p^*(x)}$, $N = \mathfrak C^{\bar\varphi}$ a model of t and Γ is the bigness notion $\Gamma_{t,\psi,\bar\varphi}$ in $\mathfrak C$ (recall Definitions 2.4, 1.3).*

If $\varphi(\bar y, \bar a)$ is Γ-big formula (in $\mathfrak C$), <u>then</u> for some countable $A \subseteq \mathfrak C$ we have:

- ⊛ *If $\bar a_0 \ne \bar a_1 \in N$ realizes $p^*(\bar x)$ in N and $A \oplus (\bar a_0\char`^\bar a_1) \subseteq N$ <u>then</u> $\varphi^+(\bar x, \bar a^+) := \varphi(\bar x, \bar a) \wedge \neg \mathrm{eq}_{\bar\varphi}(\bar x\char`^\bar a_0, \bar x\char`^\bar a_1)$ is Γ-big where $\mathrm{eq}_{\bar\varphi}(\bar x', \bar x'')$ says that $\bar x', \bar x''$ realizes the same quantifier free type in $N = \mathfrak C^{\bar\varphi}$.*

Proof. As $\varphi(\bar y, \bar a)$ is Γ-big for each $k < \omega$ there is a finite set $A^\varphi_k \subseteq \mathfrak C$ (in fact $A^\varphi_k \subseteq N$) as required in the definition of Γ (see 2.4). Let $A = \bigcup\limits_{k<\omega} A^\varphi_k$ it is a countable subset of $N = \mathfrak C^{\bar\varphi}$ and assume that $\bar a_0 \ne \bar a_1 \in N$ realizes $p^*(\bar x)$ satisfies $A \oplus (\bar a_0\char`^\bar a_1) \subseteq N$. Let $k < \omega$ and we should find $A^{\varphi^+}_k$ as required. We choose $A^{\varphi^+}_k = A^\varphi_k \cup (\bar a_0\char`^\bar a_1)$, easy to check. $\square_{2.6}$

Claim 2.7. *1) The model completion of the theory of triangle-free graphs is $T_{\mathscr K}$ for some interesting $\mathscr K$.*

2) If $\mathscr K$ satisfies $(*)_{\mathscr K}$ of 2.1 and is non trivial <u>then</u> for some $p^*(x)$ and m we have $T_{\mathscr K}$ is $p^*(x) - m$-interesting.

3) $\mathscr K$ is interesting <u>iff</u> for every quantifier free complete 1-type $p(x)$ we can find $n \in [2, \omega)$ and a model M of $T_{\mathscr K}$ and a sequence $\bar a \in {}^n M$ with no repetitions satisfying some atomic formula and a_0 realizing $p(x)$ <u>iff</u> for some $M \models T_{\mathscr K}$ we have $(p(M) \oplus (M \setminus p(M)) \nsubseteq M$

Proof. 1) Let $\tau_{\mathscr K} = \{R\}$, R a two-place relation.

Let $\mathscr K$ consist of:

(a) $M_0 = (\{0\}, \{\langle 0, 0\rangle\})$; this guarantees irreflexivity.
(b) $M_1 = (\{0, 1\}, \{(0, 1)\})$; this guarantees symmetry.
(c) $M_2 = (\{0, 1, 2\}, \{(i, j) : i \ne j < 3\})$; this guarantees 'triangle-free,' so the universe of M_ℓ is $\{0, \dots, \ell\}$.

So $t = T_{\mathscr K}$ is the theory of triangle-free graphs. Now $t = T_{\mathscr K}$ has a unique complete quantifier free 1-type $p^*(x) = \{x = x\}$. Let $m = 1$ and we shall show that t is $p^*(x)$-interesting.

Let M be a model of t, $A \subseteq M$ finite and assume $k < \omega$, $a_{2\ell} \ne a_{2\ell+1}$ are in $M \setminus A$ and realize $p^*(x)$ for $\ell < k$, and $\bigoplus \big\{\langle a_{2\ell}, a_{2\ell+1}\rangle : \ell < k\big\} \oplus A \subseteq M$.

Now we can find N, b such that (as $T_{\mathscr K}$ has amalgamation)

$(*)$ $N \supseteq M$, $|N| = |M| \cup \{b\}$ for $a \in M$, $a\ R^N b$ <u>iff</u> $a \in \{a_{2\ell} : \ell < k\}$.

Clearly N is triangle-free hence (as T is model complete) there is $b' \in M$ such that $a_\ell\ R\ b' \Leftrightarrow \ell$ even. So clearly we are done.

2) Similarly. Assume $n \ge 2$, $\bar a \in {}^n(M_*)$ without repetitions, $\bar a$ satisfies an atomic formula say $\varphi(x_0, \dots, x_{n-1}) = R(x_{i_0}, \dots, x_{i_{n(R)-1}})$, where $\{i_\ell : \ell < n(R)\} =$

[8][Earlier version: $\bar d$ contradicts the choice of $\langle \bar d_\ell : \ell < 2k_1\rangle$].

$\{0,\dots,n-1\}$ and let a_0 realize the complete quantifier free type $p^*(x_0)$. Let $m = n-1$ and we shall prove that $T_{\mathscr{K}}$ is $p^*(x) - m$-interesting. So assume that M is a model of $T_{\mathscr{K}}, A \subseteq M$ is finite $\{(\bar d_{2\ell}, \bar d_{2\ell+1}) : \ell < k\} \oplus A \subseteq M, d_{2\ell} \neq d_{2\ell+1}$ each d_ℓ realizing $p^*(x)$.

Now we can define $N, \bar b$ such that:

(∗) $\bar b = \langle b_1,\dots,b_{n-1}\rangle$ is with no repetition, $b_2 \notin M$, $|N| = |M| \cup \{b_1,\dots,b_{n-1}\}$, $N \cap |M| = M$, the quantifier-free types of $\langle d_{2\ell}, b_1, b_3,\dots,b_{n-1}\rangle$ in N and $\bar a$ in M are equal, $N = M \bigoplus_{M_0} N_0$ (where $M_0 = M \restriction \{d_{2\ell} : \ell < k\}$ and $N_0 = N \restriction (\{d_{2\ell} : \ell < k\} \cup \{b_2,\dots\}))$, and $\bar b \oplus \{d_{2\ell} : \ell < \omega\} \subseteq N_0$.

[Why is this possible? As we use free amalgamation twice: first to get N_0, second to get N.]

As $T_{\mathscr{K}}$ is model complete, there is such $\bar b' \in {}^{n-1}M$ realizing the quantifier free type of $\langle d_{2\ell}, b_1,\dots,b_{n-1}\rangle$ over $\{d_\ell : \ell < 2k\}$ in M, so we are done.

3) Should be clear. $\square_{2.7}$

Major Claim 2.8. *Assume that $t = T_{\mathscr{K}}, \mathscr{K}$ is interesting; see 2.3 and/or 2.7(3).*

1) The theory t has $(\infty, \mathbb{L}_{\infty,\kappa})$-iso-rigidity.

2) Above, t has $(\mathbb{L}_{\infty,\kappa}, \mathbb{L}, \kappa)$-def-iso-transfer (see Definition 0.10).

3) The theory t has $\aleph_0$-connectivity (see 1.15) provided that it is strongly indecomposable[9].

Proof. 1) Let $\langle p^*_\ell(x) : \ell < \ell^*\rangle$ list the interesting $\tau_{\mathscr{K}}$-complete quantifier free types realized in models of $T_{\mathscr{K}}$. For each $\ell < \ell^*$ let $\vartheta_\ell(x) := \wedge p^*_\ell(x)$. Let $\psi_\ell = \psi_{t,p^*_\ell}(x)$ be the t-bigness notion schemes of the form from Definition 2.4 + 1.3 for $p^*_\ell(x)$.

So we are assuming

(∗) (i) M is κ-compact, with $\kappa > \aleph_0$.
(ii) $N_i = M^{\bar\varphi}$ is a first order interpretation of t in M for $i = 1,2$.
(iii) M is κ-embedding Γ-complicated for the bigness notion $\Gamma_{t,p^*_\ell(x)} = \Gamma_{\psi_\ell}[\bar\varphi^1]$ for each $\ell < \ell^*$ (see Definition 1.10).
(iv) F is an isomorphism from N_1 onto N_2.

We should prove that F is $\mathbb{L}_{\infty,\kappa}(\tau')$-definable in $(M \restriction \tau', c)_{c \in A}$ for some $\tau' \subseteq \tau_M$, $A \subseteq M$ both of cardinality $< \kappa$.

For $A \subseteq \mathfrak{C}$ let $\mathbf{B}^\perp_i[A] := \{c \in N_i : c$ disconnected to $A \cap N_i$ (in N_i, see Definition 2.1(4A))$\}$.

So for each $\ell < \ell^*$ for some $B_\ell \subseteq \mathfrak{C}$, $|B_\ell| < \kappa$, $\tau_\ell \subseteq \tau_M$, $|\tau_\ell| < \kappa$, $p_\ell \in \mathbf{S}(B_\ell, \mathfrak{C} \restriction \tau_\ell)$ as in the definition of κ-embedding-complicated for $\Gamma_{t,p^*_\ell(x)}$ (see 1.10), so by monotonicity, without loss of generality $B_\ell = B_*$, $\tau_\ell = \tau_*$ for $\ell < \ell^*$, $(\mathfrak{C} \restriction \tau_*) \restriction B_* \prec \mathfrak{C} \restriction \tau_*$, B_* is closed under F, F^{-1}, and $A_{N_1} \cup A_{N_2} \subseteq B_*$. Let $B_i = \mathbf{B}^\perp_i[B_* \cap N_i]$.

So for each ℓ we have

$(*)_1$ F maps B_1 onto B_2.

[Why? As F is an isomorphism form N_1 onto N_2 it maps $B_* \cap N_1$ onto $B_* \cap N_2$ and $B^\perp_i$ is defined from the set $B_* \cap N_i$ in N_i in the same way-inspect the definitions, permuting $B_* \cap N_i$ does not matter.]

$(*)_2$ If $(B_* \cap N_1) \oplus \{a', a''\} \subseteq N_1$ and $a' \neq a'' \in B^\perp_i$ then $\neg(a'\ E_\ell\ a'')$ where E_ℓ is the equivalence relation on N_1 from Claim 1.12(2).

[9] We may weaken the assumption; e.g. even if $M \models T_{\mathscr{K}} \Rightarrow M = M_1 \oplus M_2$, still if each M_ℓ is as above the desired conclusion holds. A true criterion should use "interesting" but goes further.

[Why? By 1.12(2) and the definitions of $\Gamma_{t,\bar\varphi^1,\psi_\ell}$-big.]

$(*)_3$ E_ℓ is the equality on $B_1^\perp$

[Why? By $(*)_2$, using 2.6.]

$(*)_4$ there is a formula $\varphi^*(x,y) \in \mathbb{L}_{\kappa,\kappa}(\tau_{\mathfrak{C}})$ with a set of parameters B_* such that $\mathfrak{C} \models (\forall y)(\exists^{\le 1} x)\varphi^*(x,y)$ and $a \in N \setminus B_*^\perp \Rightarrow \mathfrak{C} \models \varphi^*(a, F(a))$.

[Why? Just put together in particular for all $\ell < \ell^*$.]

$(*)_5$ there is a formula $\varphi^*(x,y) \in \mathbb{L}_{\kappa,\kappa}(\tau_{\mathfrak{C}})$ with a set of parameters B_* such that:

- (i) $\mathfrak{C} \models (\forall y)(\exists^{\le 1} x)\varphi^*(x,y)$
- (ii) $\mathfrak{C} \models (\forall x)(\exists^{\le 1} y)(\varphi^*(x,y))$
- (iii) recalling $B_i^\perp = \{a : (B_* \cap N_1) \oplus \{a\} \subseteq N_1\}$ we have F maps B_1^* onto B_2^* and $a \in B_1^* \Rightarrow \mathfrak{C} \models \varphi^*(a, F(a))$.

[Why? As F is onto N_2 clearly F maps $B_1^\perp$ onto $B_2^\perp$ hence by $(*)_5$.]

$(*)_6$ there is a formula $\varphi^{**}(x,y) \in \mathbb{L}_{\kappa,\kappa}(\tau_{\mathfrak{C}})$ with parameters B_* defining F.

[Why? Let $\varphi^{**}(x,y)$ say that ($x \in N_1, y \in N_2$, of course and) for any $n < \omega$ and $a_1,\dots,a_n \in (B_* \cap N_1) \cup B_1^\perp$ the quantifier free type which $\langle x, a_1,\dots,a_n\rangle$ realizes in N_1 is equal to the quantifier free type which $\langle y, F(a_1),\dots,F(a_n)\rangle$ is realized in N_2. Note that $F(a_\ell)$ is definable by φ^* if $a_\ell \in B_1^\perp$ and by a list if $a_\ell \in B_*$. Clearly if $F(a) = b$ then $\mathfrak{C} \models \varphi^{**}(a,b)$. But if $\models \varphi^{**}[a',b']$ and $a' \ne a, b' = F(a)$ then we can find n and $a_2,\dots,a_n \in N_1$ such that $((B_* \cap N_1) \cup \{a',a\}) \bigoplus\limits_{\{a\}} \{a, a_2,\dots,a_n\}$ and $\langle a, a_2,\dots,a_n\rangle$ strictly satisfies an atomic formula and is with no repetition (such situation exists by "interesting").]

So $(B_* \cap N_1) \oplus (N_1 \restriction \{a_2,\dots,a_{n-1}\}) \subseteq N_1$, so $\{a_2,\dots,a_{n-1}\} \subseteq B_1^\perp$. By the definition of φ^{**} we have

$\circledast_1$ $\langle a', a_2,\dots,a_{n-1}\rangle, \langle b', F(a_2),\dots,F(a_{n-1})\rangle$ realizes the same quantifier free types in N_1, N_2, respectively as $F(a) = b'$ and the assumption on t

$\circledast_2$ $\langle a, a_2,\dots,a_{n-1}\rangle, \langle b', F(a_2),\dots,F(a_{n-1})\rangle$ realizes the same quantifier free types in N_1, N_2, respectively. By transitivity of equality of types

$\circledast_3$ $\langle a', a_2,\dots,a_{n-1}\rangle, \langle a, a_2,\dots,a_{n-1}\rangle$ realizes the same quantifier free types in N_1. This contradicts the choice of $a_2,\dots,a_{n-1}$.

2) So we assume

$\boxtimes$ $\mathfrak{C}$ is a κ-saturated model, $\vartheta = \vartheta(x,y) \in \mathbb{L}_{\kappa,\kappa}(\tau_{\mathfrak{C}})$ and $N_1 = \mathfrak{C}^{\bar\varphi^1}$, $N_2 = \mathfrak{C}^{\bar\varphi^2}$ are models of t, ϑ define an isomorphism F from N_1 onto N_2 and we should prove that F is first order definable in $\mathfrak{C}$; without loss of generality

$\circledast_1$ (a) $\bar\varphi^1, \bar\varphi^2, \vartheta$ use no parameters
(b) $\kappa > |\tau_{\mathfrak{C}}| + \aleph_0$ (even $\kappa > 2^{|\tau|}$ or whatever you like).

[Why? For (a) make those $< \kappa$ elements to individual constants. For (b) note that if $\mathfrak{C}_1, \mathfrak{C}_2$ are elementarily equivalent κ-saturated then they are $\mathbb{L}_{\infty,\kappa}$-equivalent hence

$(*)$ if for $\mathfrak{C}_1$ there is an isomorphic π from $M^{\bar\varphi^1}$ onto $M^{\bar\varphi^2}$ which is $\mathbb{L}_{\infty,\kappa}(\tau_{\mathfrak{C}_1})$-definable but not $\mathbb{L}_{\omega,\omega}(\tau_{\mathfrak{C}_1})$-definable <u>then</u> this holds for $\mathfrak{C}_2$ too. So let $\mathfrak{C}'$ be a $(\kappa + |\tau_{\mathfrak{C}}|)^+$-saturated elementary extensions of $\mathfrak{C}$.

[So it suffices to prove the claim for $\mathfrak{C}'$.]

$\circledast_2$ Without loss of generality in $\mathfrak{C}$ we can code finite sets and $\mathfrak{C}$ is a model of $\mathrm{Th}(\mathfrak{B}^+)$ where $\mathfrak{B}^+ = (\mathcal{H}(\chi), \in, \mathfrak{C}, N_1, N_2)), \chi$ strong limit, $\mathfrak{C}^*, N_1, N_2$ as above.

[Why? Let $\mathfrak{B}^+$ be as above, let $\mathfrak{B}$ be κ-saturated model of $\mathrm{Th}(\mathfrak{B}^+)$, let $\mathfrak{C}_1^*$ be $\mathfrak{C}$ interpreted in $\mathfrak{B}$. See $\mathfrak{C}_1$ is κ-saturated, $\mathfrak{C}_1 \equiv \mathfrak{C}$.

Now

$\odot$ define also in $\mathfrak{C}_1^*$ an isomorphism F_1 from $\mathfrak{C}_1^{\bar{\varphi}^2}$ onto $\mathfrak{C}_1^{\bar{\varphi}^2}$ not $\mathbb{L}_{\omega,\omega}(\tau_{\mathfrak{C}})$-definable with parameters in $\mathfrak{C}_1$. So F_1 is $\mathbb{L}_{\infty,\kappa}(\tau_{\mathfrak{B}})$-definable in $\mathfrak{B}$. If F is a first order definable in $\mathfrak{B}$ (even with parameters) then it is first order definable $\mathfrak{B}^+$ (recall that $\tau_{T_{\mathscr{K}}}$ is finite!) hence it is first order definable in $\mathfrak{C}'$.

[Now this works in $\mathfrak{B}^+$.]

$\circledast_3$ $\mathfrak{C} \models \vartheta(a, b)$ <u>then</u> in $\mathfrak{C}, a$ is definable over b and b is definable over a.

[Why? E.g., if b is not definable over a, there is b' such that $\langle a, b\rangle, \langle a', b'\rangle$ realizes the same type in $\mathfrak{C}$. But as $\mathfrak{B}$ is κ-saturated we know that $\langle a, b\rangle, \langle a, b'\rangle$ realizes the same $\mathbb{L}_{\kappa,\kappa}(\tau_{\mathfrak{B}})$-type in $\mathfrak{B}$, so $\mathfrak{C} \models \vartheta(a, b) \equiv \vartheta^{\mathfrak{C}}(a, b')$ easy contradiction.]

$\circledast_4$ if $\mathfrak{B} \models \vartheta^{\mathfrak{C}}(a, b)$ <u>then</u> for some first definable element f of $\mathfrak{B}$ (over $\varnothing$!) we have $\mathfrak{B} \models$ "f is a partial one to one function from N_1 into N_2 and $f(a) = b$".

[Why? By $\circledast_3$.]

$\circledast_5$ there is a pseudo-finite set $\mathscr{F} \in \mathfrak{C}$ of partial one to one functions from N_1 into N_2 such that

$$(F(a) = b) \Rightarrow \mathfrak{C} \models (\exists f \in \mathscr{F})[f(a) = b].$$

[Why? Just by saturated there is such s to which all such functions first order definable in $\mathfrak{C}$ belongs. (Note: there is a pseudo set of all one to to one functions from N_1 to N_2).]

$\circledast_6$ There is $e \in \mathfrak{C}$ such that for any finite $\Delta \subseteq \mathbb{L}_{\omega,\omega}(\tau_{\mathfrak{C}})$ we have $\mathfrak{C} \models$ "e is an equivalence relation on N_1 with finitely many equivalence classes such that if x is an e-equivalence class and $f_1, f_2 \in \mathscr{F}$ then $(\alpha) + (\beta) + (\gamma)$", where

(α) Any two members of x realize the same Δ-type over $\varnothing$ (in $\mathfrak{C}$).

(β) $(\forall y \in x)[y \in \mathrm{dom}(f_1)] \vee (\forall y \in x)(y \in \mathrm{dom}(f_1))$???

(γ) If $x \subseteq \mathrm{dom}(f_1) \cap \mathrm{dom}(f_2)$ then $(\forall y \in x)[f_1(x) = f_2(x)]$ or $(\forall y \in x)[f_1(y) \neq f_2(y)]$.

[Why? Just define e by the demands and recall obvious facts on finite.]

$\circledast_7$ if x is an e-equivalence class then for some $f \in \mathfrak{C} \models f \in \mathscr{F}$ and $F \restriction \{y : y \in^{\mathfrak{C}} x\} \subseteq f^{\mathfrak{C}}$.

[Why? By $\circledast_6$.]

$\circledast_8$ there is $\bar{f}$ such that $\mathfrak{C} \models$ "$\bar{f} = \langle f_{x,i} : x \in N_1/e, i < i_x\rangle$, each i_x finite, $f_{x,i}$ a partial one-to-one function from N_1 to N_2 and

$$x \in N_1/e,\ i < j < i_x \Rightarrow (\forall y \in x)(f_{x,i}(y) \neq f_{x,j}(y))$$

and

$$x \in N_1/e, f \in \mathscr{F} \wedge x \subseteq \mathrm{dom}(f) \Rightarrow (\exists i < i_x)(f \restriction x = f_{x,i} \restriction x)".$$

[Why? By $\circledast_6$.]

$\circledast_9$ there are $s_0, s_1, s_2 \in \mathfrak{C}$ such that $\mathfrak{C} \models$ "$s_0 = \langle a_j, b_j, c_j : j < j^* \rangle$, j^* finite, $a_j \in N_1$, $b_j \neq c_j \in N_2$ and $s_1 := \bigoplus \big\{\{b_j, c_j\} : j < j^*\big\} \subseteq N_2$ and letting $s_2 = \{y \in N_1$: for some $j < i_{a/e}, f_{e/j}(y) \in \{b_j, c_j : j < j^*\}$ again finite we have: for every $x \in N_1/e$ and $i_0 < i_1 < i_x$ at least one of the following holds:

(α) for some j we have $a_j \in x$ and $f_{x,i_0}(a_j) = b_j$ and $f_{x,i_1}(a_j) = c_j$

(β) $(\forall y \in x)(y \notin s_1 \Rightarrow s_1 \oplus \{f_{i_0}(y)\} \nsubseteq N_2)$

(γ) $(\forall y \in x)(y \notin s_2 \Rightarrow s_1 \oplus \{f_{i_1}(y) \nsubseteq N_2)$".

[Why? Just we list the tasks (x, i_0, i_1) and inductively try to choose (a_j, b_j, c_j).]

$\circledast_{10}$ there is $d_2 \in N_2^{\mathfrak{C}}$ such that $\mathfrak{C} \models$ "if $j < j^*$ then $\langle d_2, b_j \rangle, \langle d_2, c_j \rangle$ does not realize the same quantifier free N_2-type.

[Why? By the properties of t.]

$\circledast_{11}$ there is $d_1 \in N_1$ such that: if $a \in N_1^{\mathfrak{C}}$, $b = F(a)$ then $\langle d_1, a \rangle, \langle d_2, b \rangle$ realize the same quantifier free type in $N_1^{\mathfrak{C}}, N_2^{\mathfrak{C}}$ respectively.

[Why? Let $d_1 = F^{-1}(d_2)$, recalling F is an isomorphism from $N_1^{\mathfrak{C}}$ onto $N_2^{\mathfrak{C}}$.]

$\circledast_{12}$ $\mathfrak{C} \models$ "for each $x \in N_1/e$ for at most one $i < i_x$ we have

(a) $(\forall j < j_*)[a_j \in x \Rightarrow$ (the quantifier free types of $\langle d_1, a_j \rangle$ and $\langle d_2, f_{x,i}(a_j) \rangle$ in N_1, N_2, respectively, are equal)]

(b) $\{y \in x : y \notin s_2$ and $s_1 \oplus f_i(y) \subseteq N_2\}$ is empty.

[Why? By $\circledast_9 + \circledast_{11}$; the choice of d_1 is immaterial as long as $d_1 \in N_1^{\mathfrak{C}}$.]

$\circledast_{13}$ if $F(a) = b$ then $\mathfrak{C} \models$ "if $a \in x \in N_1/e$, (b) of $\circledast_9$ fails (the set is nonempty), $i < i_x$, and $f_{x,i}(a) = b$ (there is such i, see depending only on x, see $\circledast_7$ above) <u>then</u> (a) of $\circledast_9$ holds for i (and x)".

[Why? Check.]

Now let $f^* \in \mathfrak{C}$ be such that

$$\mathfrak{C} \models \text{"}f_* = \bigcup \{f_{x,i} : x \in N_1/e,\ i < i_x^* \text{ and (a) + (b) of } \circledast_{12} \text{ holds}\}\text{"}.$$

So clearly

⊡ (a) $f_*^{\mathfrak{C}} \subseteq F$

(b) $\mathrm{dom}(f_*^{\mathfrak{C}}) = \{y : \mathfrak{C} \models y \in N_1$ and y/e satisfies clause (b) of $\circledast_{12}\}$.

(c) f_* is first order definable in $\mathfrak{C}$.

(d) $\mathfrak{C} \models$ "$N_1 \setminus \mathrm{dom}(f_*)$ is pseudo finite and disjoint to

$$\{y \in N_1 : \{y\} \oplus s_2 \subseteq N_1\}$$

and s_2 is finite".

We can finish as in the end of part (1), the definition of f is $(n^* > \mathrm{arity}(\tau_{\mathscr{K}}))$: $F(a) = b$ <u>iff</u> for any $a_1, \ldots, a_{n^*} \in^{\mathfrak{C}} \mathrm{dom}(f_*)$ the complete quantifier free types which $\langle a, a_1, \ldots, a_r \rangle, \langle b, f_*(a_1), \ldots, f_*(a_{n^*}) \rangle$ realize (in N_1, N_2, respectively) are equal.

3) Left to the reader. $\square_{2.8}$

Question 2.9. 1) Complete embedding?

Existence of embedding ≡ there is a "bad" equivalence relation F on N_i (such that most $a_\ell \in F(a_\ell)/E$ are okay.

2) Return to [Shear, Ch.XI] on characterization.

Observation 2.10. *In verifying "t has $(\mathbb{L}_{\kappa,\kappa}, \mathbb{L}, \kappa)$-definably-isomorphic transfer" we can assume $\circledast_1 + \circledast_2$ from the proof of 2.8(2).*

Proof. As there. $\square_{??}$

§ 3. Construction by forcing or strong assumptions

Here we try to see when we can get complicated models by forcing. So 3.2 is in the line of [She83c] and it is most suitable for the case $\lambda = \lambda^{<\lambda} > |T|$, although with a little more work $\lambda = \lambda^{<\lambda} \geq \aleph_0$ is okay, too. We could alternatively use models with universe $\subseteq u \times \lambda$. We can do this using also $(\mathfrak{C}, \lambda)$-bigness notion.

Question 3.1. Phrase for bigness + orthogonality, but can we omit types $\mathbb{L}(Q^{M,M})$?

Definition 3.2. 1) We say that $\mathfrak{s}$ is a λ-b.n.f. (bigness notion family) if it consists of:

(a) T is a first order complete theory of cardinality $\leq \lambda$ and let $\mathfrak{C}$ be a λ^+-saturated model of T, (for simplicity, every formula is equivalent to a predicate)

(b) a set of $\leq \lambda$ T-bigness notion scheme $\mathfrak{b}(\bar{z})$, see below, including $\Gamma^{\rm tr}, \Gamma^{\rm na}$, each satisfying $\ell g(\bar{z}) < \lambda$.

1A) A T-bigness notion scheme (T-b.n.s.) $\mathfrak{b} = \mathfrak{b}(\bar{z})$ consists of

(a) $r(\bar{z}) = r^{\mathfrak{b}}(\bar{z})$, a type in the (sequence of) variable $\bar{z} = \bar{z}_{\mathfrak{b}}$, in the language $\mathbb{L}(\tau_T)$

(b) a set $\Lambda_{\mathfrak{b}}$ of pairs of the form $(q_1(\bar{y}, \bar{z}), q_2(\bar{x}, \bar{y}, \bar{z}))$ of complete $\mathbb{L}(\tau_T)$-types (in the respective variables) such that: for every $\bar{c} \in \mathfrak{C}$ realizing $r_{\mathfrak{b}}, \Gamma_{\mathfrak{b},\bar{c}}$ is a global bigness notion on the variables $\bar{x}_{\mathfrak{b}}$, such that $F \in \text{Aut}(\mathfrak{b}) \Rightarrow F(\Gamma_{\bar{\mathfrak{b}},\bar{c}}) = \Gamma_{\mathfrak{b},F(\bar{c})}$, where

⊛ we define $\Gamma_{\mathfrak{b},\bar{c}} = \{p(\bar{x}) : p(\bar{x}) = \text{tp}(\bar{a}, B, \mathfrak{C})$ (with $\bar{c} \subseteq B$ for simplicity) such that for every $\bar{b} \subseteq B$ the pair $(\text{tp}((\bar{c}; \bar{c}), \varnothing, \mathfrak{C}), \text{tp}((\bar{a}, \bar{b}, \bar{c}), \varnothing, \mathfrak{C}))$ belongs to $\Lambda_{\mathfrak{b}}\}$; this is a Γ.

1B) A T-local bigness notion scheme (T-l.b.n.s.) $\mathfrak{b}$ is defined similarly (only the member of $\Lambda_{\mathfrak{b}}$ are of the form $(\varphi(\bar{x}, \bar{y}), q(\bar{x}, \bar{y}, \bar{z}))$.

2) Assume $\mathfrak{s}$ is a λ-b.n.f. so $|T| < \lambda$, $\big|\{\mathfrak{b}(\bar{z}) : \mathfrak{b}(\bar{z}) \in \mathfrak{s}\}\big| < \lambda$. We define $\mathbb{P} = \mathbb{P}_{\mathfrak{s},\lambda}$ as the set of triples $\mathbf{p} = (u, p, \bar{\Gamma}) = (u^{\mathbf{p}}, p^{\mathbf{p}}, \bar{\Gamma}^{\mathbf{p}})$ such that

(α) $u \in [\lambda^+]^{<\lambda}$

(β) p is a complete type in the variables $\{x_\varepsilon : \varepsilon \in u\}$

(γ) $\bar{\Gamma} = \langle \Gamma_\varepsilon(\bar{z}_\varepsilon) = \Gamma_{\mathfrak{b}_\varepsilon}(\bar{z}_\varepsilon) : \varepsilon \in u \rangle$

(δ) $\bar{z}_\varepsilon = \langle x_{j(\varepsilon,\alpha)} : \alpha < \alpha_{i(\varepsilon)} \rangle, j(\varepsilon, \alpha) \in u \cap \varepsilon$ and $T, p \vdash r_{\mathfrak{b}_\varepsilon}(\bar{z}_\varepsilon)$

(ε) if $\langle b_\varepsilon : \varepsilon \in u \rangle$ realizes p in $\mathfrak{C}$, a model of T <u>then</u>

  (i) $\text{tp}(b_\varepsilon, \{b_\zeta : \zeta \in u \cap \varepsilon\}, \mathfrak{C})$ is $\Gamma_\varepsilon(\langle b_{j(\varepsilon,\alpha)} : \alpha < \alpha_{i(\varepsilon)} \rangle)$-big

  (ii) if λ / α, $\alpha \leq \varepsilon \in u$ then $b_\varepsilon \notin \text{acl}\{b_\zeta : \zeta \in w \cap \alpha\}$.

3) We define $\mathbb{P}$-name $\underset{\sim}{p}$ by $\underset{\sim}{p} = \bigcup \{p^{\mathbf{p}} : \mathbf{p} \in \underset{\sim}{G}_{\mathbb{P}}\}$ and let $M_{\underset{\sim}{p}}$ be the $\mathbb{P}$-name of the model (using $=^M$ just as an equivalence relation; i.e. allowing repetition) with universe $\{b_\varepsilon : \varepsilon < \lambda^+\}$ where $\langle b_\varepsilon : \varepsilon < \lambda^+ \rangle$ realizes $\underset{\sim}{p} \in N$.

4) If we do not assume $\lambda = \lambda^{<\lambda}$, it is interesting to consider the following. Let $\mathfrak{B}_\varepsilon \prec (\mathcal{H}(\chi), \in, <^*_\chi)$ for $\varepsilon < \lambda^+$ be increasing continuous, $\|\mathfrak{B}_\varepsilon\| = \lambda, \langle \mathfrak{B}_\varepsilon : \varepsilon \leq \zeta \rangle \in \mathfrak{B}_\zeta$, [$\varepsilon$ not limit and $\lambda = \lambda^\theta \Rightarrow {}^\theta(\mathfrak{B}_{\varepsilon+1}) \subseteq \mathfrak{B}_\varepsilon$] and $T, \langle q_i(\bar{y}_i), \Gamma_i(\bar{y}_i) : i < i^* \rangle, \lambda$ belongs to $\mathfrak{B}_0$ and let $\underset{\sim}{\bar{\mathfrak{B}}} = \langle \mathfrak{B}_\varepsilon : \varepsilon < \lambda^+ \rangle$.

We define $\mathbb{P} = \mathbb{P}^-_{\bar{\mathfrak{B}}} = \mathbb{P}^{\mathfrak{s}}_{\bar{\mathfrak{B}}}$ as $\{\mathbf{p} \in \mathbb{P}$: for every ε we have $\mathbf{p} \restriction (\varepsilon + 1) \in \mathfrak{B}_{\varepsilon+1}\}$. Note that λ can be reconstructed from $\mathfrak{B}$.

Claim 3.3. *1)* $\Vdash_{\mathbb{P}}$ "$\underset{\sim}{M}$ *is a model of* T *in which* $\langle b_i : i < \lambda^+\rangle$ *realizes* $\underset{\sim}{p}$".
2) In 3.2 $\underset{\sim}{M}$ *is* λ*-compact; (so if* $|T| < \lambda$, *then* $\underset{\sim}{M}$ *is* λ*-saturated).*
3) $\mathbb{P}$ *in 3.2(1) when* $\lambda = \lambda^{<\lambda}$ *(and* $\mathbb{P}_{\mathfrak{B}}$ *in 3.2(2) in general) satisfies the* λ^+*-c.c.*
4) If $\lambda = \lambda^{<\theta}$ *then* $\mathbb{P}$ *is* θ*-complete.*
5) If $\lambda = \lambda^{<\lambda}$ *then* $\mathbb{P}^{\mathfrak{s}}_{\mathfrak{B}} = \mathbb{P}_{\mathfrak{s},\lambda}$.

Claim 3.4. *In the following game* ∂, *the Ghibellines wins. On the games see* [HLS93], [She94, §3]; *they say we co*

Claim 3.5. *Assume* $\lambda = \lambda^{<\lambda}$, *then in* $\mathbf{V}^{\mathbb{P}}$

(∗) $\underset{\sim}{M}$ *is* λ*-isomorphism complicated; i.e. for all possibilities with* Γ *being the closure under well ordered iteration.*

Remark 3.6. We may add: $\underset{\sim}{M}$ is complicated in the following sense [Fill]

Claim 3.7. *If* $\lambda = \lambda^{<\lambda}, \diamondsuit_{S^{\lambda^+}_{\lambda}}$, *then there is* $G \subseteq \mathbb{P}$ *generic enough.*

Discussion 3.8. Assume $\lambda = \lambda^{<\lambda}$ and $\lambda > \aleph_0$, $(\mathrm{Dl})_\lambda$ for transparency.

(A) The forcing notion from [Wim82](3) fits the frame of [HLS93] and of [She94, §3].
(B) Hence we can define a suitable game between the Guelf and Ghibellines, as in [She94, §3].
(C) So in the following game ∂, the Gbl wins.

* * *

Discussion 3.9. 1) Let $\lambda = \lambda^{<\lambda}$, I a λ^+-like linear order.

We may define a forcing $\mathbb{Q}_J$ for $J \subseteq I$ as in 3.10 below. Is $\mathbb{Q}_J \lessdot \mathbb{Q}_I$? This should help to prove if we cannot force a model M with only inner automorphisms to any interpretation $M^{[\bar\varphi]}$ of t in M, then there are T and interpretation of t with built in many automorphisms as in [She00b, §3].

Why not like a tree? But then there are models for which every branch is definable. But consider a linear order f_t; i.e.

$$\begin{aligned}M \models \text{“}&(F(-,t) \text{ is an automorphism of } M^{[\bar\varphi]} \text{ for every } t \in Q,\\ &< \text{ a linear order } Q \text{ and for } x \in M^{[\bar\varphi]}\\ &e_x = \{(t,s) : F(x,t) = F(x,s), t \in Q, s \in Q\}\\ &\text{has two equivalence classes, each convex”}\end{aligned}$$

Let us return to $\mathbb{Q}_J \lessdot \mathbb{Q}_I$: there are some superficial problems, but inherent is the kind of bigness property that we [desire / can obtain]. If we have for $\Gamma_r (r \in I)$ a possible choice of $x/e = c \in Q$, as in ordered field, this fails.

We should consider bigness notion Γ such that p is Γ-big implies p does not fork over $\varnothing$; recall: for every T, there are definable types: $\mathrm{Av}(\mathfrak{C}, D)$, so are there interesting cases?

2) We may consider in 3.2, what failure implies.

Definition 3.10. Assume

(*a*) $T_{\mathfrak{s}}$ is as in 3.2(1), each $\Gamma_{\mathfrak{b},i}(\bar z_i)$ is local and co-complete, that is:

⊛ if $\mathfrak{b} \in \mathfrak{s}$ and $\bar c \in {}^{\ell g(\bar z_i)}C$ realizes $r^{\mathfrak{b}_i}(\bar z_i)$, and $\varphi(\bar x_{\mathfrak{b}}, \bar y)$ a formula $b \in {}^{\ell g(\bar y)}\mathfrak{C}$ then $\varphi(x_{\mathfrak{b}}, \bar b)$ is $\Gamma_{\mathfrak{b}}(\bar c)$-big iff $\bar b$ realizes $q_{\mathfrak{b},\varphi(\bar x,\bar y)}(\bar y)$

(b) I is a quasi order, with set of elements λ^+, such that

 (i) $\alpha + \lambda^2 \le \beta \Rightarrow \alpha <_I \beta$

 (ii) letting $E = E_I = \{(\alpha,\beta) : \alpha \le \beta \le \alpha\}$, an equivalence relation, we have each equivalence class is cardinality λ and has the form $[\alpha, \alpha+\lambda)$, with $\lambda | \alpha$

 (iii) if $\lambda^2 | \delta$, $\mathrm{cf}(\delta) = \lambda$ then $(\forall \alpha)(\alpha <_I \delta \equiv \alpha < \delta)$

 (iv) we call I λ-dense if: $A, B \subseteq \lambda^+$ non empty, $|A| + |B| = \lambda^+$, $B \nsubseteq [0, \lambda)$ and $a \in A,\ b \in B \Rightarrow a <_I b$ implies that for some $c \in \lambda^+$,

$$a \in A,\ b \in B \Rightarrow a <_I c <_I b.$$

1) For $\ell \in \{1, 2\}$, let $\mathbb{P}^\ell_{\mathfrak{s}}$ be the set of $\mathbf{p} = (u^{\mathbf{p}}, p^{\mathbf{p}}, \bar{\Gamma}^{\mathbf{p}})$ such that:

 (α) $u \in [\lambda^+]^{<\lambda}$

 (β) p is a complete type in the variables $\{x_\varepsilon : \varepsilon \in u^p\}$

 (γ) $\bar{\Gamma} = \langle \Gamma_\varepsilon(\bar{z}_\varepsilon) = \Gamma_{\mathfrak{b}_\varepsilon}(\bar{z}_\varepsilon) : \varepsilon \in u^{\mathbf{p}}/E_I \rangle$

 (δ) $\bar{z}_{\varepsilon/E} = \langle x_{j(\varepsilon,\alpha)} : \alpha < \alpha_{i(\varepsilon)} \rangle$, $j(\varepsilon, \alpha) \in I_{<\varepsilon} = \{j : j <_I \varepsilon\} \cap u$, $T_p \vdash r^{\mathfrak{b}_\varepsilon}(\bar{z}_\varepsilon)$, and $\bar{x}_\varepsilon \subseteq u \cap \{x_\zeta : \zeta \in \varepsilon/E\}$

 (ε) if $\langle b_\varepsilon : \varepsilon \in u \rangle$ realizes p in $\mathfrak{C}$ <u>then</u>

 (i) $\mathrm{tp}(\langle b_\zeta : \zeta \in u \cap (\varepsilon/E) \rangle, \{b_\zeta : \zeta \in u \cap I_{<\varepsilon}\}, \mathfrak{C})$ is $\Gamma_{\mathfrak{b}_\varepsilon}(\langle b_{j(\varepsilon,\alpha)} : \alpha < \alpha_{i(\varepsilon)} \rangle)$-big

 (ii) if $\varepsilon \in u$ then $b_\varepsilon \notin \mathrm{acl}(\{b_\zeta : \zeta \in u \cap I_{<\varepsilon}\}, \mathfrak{C})$

 (iii) if $\ell = 2$ and $\varepsilon_1, \varepsilon_2 \in u$ are E_I-equivalent then
 $b_{\varepsilon_1} \in \mathrm{acl}(\{b_\zeta : \zeta \in u \cap I_{<\varepsilon_1}\} \cup \{b_{\varepsilon_1}\}, \mathfrak{C})$ iff
 $b_{\varepsilon_2} \in \mathrm{acl}(\{b_\zeta : \zeta \in u \cap I_{<\varepsilon_2}\} \cup \{b_{\varepsilon_2}\}, \mathfrak{C})$

 (ζ) $p \Vdash x_\varepsilon \ne x_\zeta$ if $\varepsilon \ne \zeta$ are from u^p.

§ 4. Unsuperstable Case

§ 4(A). **Omitting Countable Types.**

Discussion 4.1. To deal with cases of un-superstability we use a relative of bigness. To motivate it, consider the following example. See more in §4.

Example 4.2. Let t be the theory of abelian groups and $\bar\varphi$ be an interpretation of t in $\mathfrak{C}$, a quite saturated model and $\langle p_n : n < \omega\rangle$ a sequence of distinct primes and $q_n = \prod\limits_{m<n} p_m$ (so $N^{\bar\varphi}$ is the abelian group with universe $\{a : \mathfrak{C} \models \varphi_0(a)\}$ and $N^{\bar\varphi} \models a + b = c$ iff $\mathfrak{C} \models \varphi_{x+y=z}(a, b, c)$).

We define $\Gamma^{\bar\varphi}_n : p \in \mathbf{S}(A)$ is $\Gamma^{\bar\varphi}_n$-big if (parameters of $\bar\varphi$ are $\subseteq A$ and):

(a) $\varphi_0(x) \in p$

(b) $\bigcup\{p(x_\eta) : \eta \in {}^\omega\omega\} \cup \{(q_k \text{ divides } x_\eta - x_\nu) \wedge (p_k \text{ does not divide } x_\eta - x_\nu) : \eta \in {}^\omega\omega, \nu \in {}^\omega\omega, \ell g(\eta \cap \nu) = k < \omega\}$

(Pedantically, the statements $q_k \mid (x_\eta - x_\nu)$ and $p_k \nmid (x_\eta - x_\nu)$ should be translated by $\bar\varphi$ to formulas in $\mathbb{L}(\tau_{\mathfrak{C}})$).

Now

(a) each $\Gamma^{\bar\varphi}_n$ is a global bigness notion, has the extension property (in fact is local); that is, if $p(x)$ is a 1-type over A and every finite conjunction of members of p belongs to some complete $\Gamma^{\bar\varphi}_n$-big type, $\mathrm{dom}(p) \cup \mathrm{dom}(\bar\varphi) \subseteq A \subseteq \mathfrak{C}$, then some $\Gamma^{\bar\varphi}_n$-big $q \in \mathbf{S}^1(A)$ extends p.

But in the previous examples we got interesting conclusions on automorphisms when each Γ-big type has two contradictory extensions. Now here each $\Gamma^{\bar\varphi}_n$ actually fails this property (as t is stable), but there is a weak substitute.

(b) if $p \in \mathbf{S}^1(A)$ is $\Gamma^{\bar\varphi}_n$-big, we can find many pairwise contradictory $\Gamma^{\bar\varphi}_{n+1}$-big extensions.

Hence though for every $\aleph_1$-saturated $N \prec \mathfrak{C}$, $N^{[\bar\varphi]}$ is saturated, still:

(c) if $\mathfrak{C}$ is complicated for $\bar\Gamma$ in the sense of omitting many countable types, <u>then</u> for the abelian group $N^{\bar\varphi}$, every automorphism π of it is definable "somewhere" provided that

  ⊛ there is a $\Gamma^{\bar\varphi}_n$-big type for some n.
  [Contrast this with the well known result in [Fuc73]]

  $\circledast_1$ a divisible abelian group H is the direct sum of copies of $\mathbb{Q}$ and the group $\mathbb{Z}^\infty_p$ [p prime] so if the group H is uncountable it has $2^{|H|}$ automorphisms.
  Note

  $\circledast_2$ if for some $N \prec \mathfrak{C}$, $N^{[\bar\varphi]}$ is e.g. the direct sum of infinitely many copies of $\mathbb{Z}$ <u>then</u> there is a $\Lambda^{\bar\varphi}_0$-big type (for any $\bar p$).

In non-trivial cases (by $\aleph_0$-saturation) $N^{\bar\varphi}$ will have large divisible groups (which necessarily are direct summands, if $\mathfrak{C}$ rich enough).

So we cannot get really rigid cases.

Example 4.3. Un-superstable complete first order theories.

The following definition is intended to help deal with examples like the ones in 4.3 and 4.2.

Definition 4.4. 1) We say $\bar\Gamma$ is a global $(\mathfrak{C}, \mathscr{W}, \kappa, \omega)$-bigness notion <u>if</u>:

(a) $\mathfrak{C}$ is κ-saturated, $\mathscr{W}$ the family of subsets of $\mathfrak{C}$ of cardinality $< \kappa$ (then we can omit $\mathscr{W}$) or

$(a)^-$ $\mathfrak{C}$ is $\aleph_0$-saturated, $\mathscr{W}$ a family of "small" sets as in [Shea]

(b) $\bar{\Gamma} = \langle \Gamma_n : n < \omega \rangle$

(c) each Γ_n is a family of global $(\mathfrak{C}, \kappa)$-bigness notion scheme; (each member is called a case of Γ_n, Γ_n-big means for some Γ_n) but only over $\mathfrak{C}$ members of $\mathscr{W}$

(d) if $p \in \mathbf{S}^{\alpha(\Gamma_n)}(B, \mathfrak{C})$ is Γ_n-big, $B \in \mathscr{W}$ then for some $m > n, p$ has a Γ_m-big extension.

2) We say $\bar{\Gamma}$ has $\bar{\Delta}$-freedom, if

(a) $\bar{\Gamma}$ a $(\mathfrak{C}, \mathscr{W}, \kappa, \omega)$-bigness notion

(b) $\bar{\Delta} = \langle \Delta_{n,m} : n < m < \omega \rangle, \Delta_{n,m}$ a set of formulas

(c) if $p \in \mathbf{S}^{\alpha(\Gamma_n)}(B_1, \mathfrak{C})$ is Γ_n-big, B_1 is small, then for some $m \in (n, \omega)$ and small $B_2 \supseteq B_1$ there are Γ_m-big $p_1, p_2 \in \mathbf{S}^{\alpha(\Gamma_n)}(B_2, \mathfrak{C})$ extending p such that $p_1 \restriction \Delta_{n,m} \neq p_2 \restriction \Delta_{n,m}$.

Definition 4.5. 1) Assume that $\bar{\Gamma}$ is a global $(\mathfrak{C}, \mathscr{W}, \kappa, \omega)$-notion with set parameter $A_{\bar{\Gamma}}$.
We say that $\mathfrak{C}$ is $(\bar{\Gamma}, \mathscr{W}, \kappa)$-complicated for embedding for (N_1, N_2) when:

(a) $\mathfrak{C}$ is $\aleph_0$-saturated (follows by (a) of Definition 4.4(1))

(b), (c), (d) As in 1.10

(e) if F is an embedding of N_1 into N_2 and p_1 is a Γ_{n_1}-big type over some member of $\mathscr{W}$ then we can find $B \in \mathscr{W}$ including $\mathrm{dom}(p_1) \cup A_{\bar{\Gamma}} \cup A_{N_1} \cup A_{N_2}$ and $a \in N_1$ and $n_2 < \omega$ such that $(*)$ of 1.10(e) holds for Γ_{n_2}

$(*)$ (α) $p_1' = \mathrm{tp}(a, B, \mathfrak{C} \restriction \tau)$ is Γ_{n_1}-big extending p_1.

(β) $\mathrm{tp}(\langle a, b \rangle, B, \mathfrak{C} \restriction \tau)$ is Γ_{n_2}-big, where $b = F(a)$.

(γ) If $\mathrm{tp}(\langle a', b' \rangle, B', \mathfrak{C} \restriction \tau)$ is Γ_{n_2}-big and $B \subseteq B'$, then $\mathrm{tp}(a', B', \mathfrak{C} \restriction \tau)$ is Γ_1-big.

(δ) If $R \in \tau_{N_\ell}$ has k-places, $a_2, \ldots, a_k \in N_1$, a' realizes p_1', $b' \in N_2$ is such that the pair (a', b') realizes $\mathrm{tp}(\langle a, F(a) \rangle, B, \mathfrak{C} \restriction \kappa)$,

$$B' = B \cup \{a_\ell, F(a_\ell) : \ell = 2, \ldots, k\},$$

and $\mathrm{tp}(\langle a', b' \rangle, B', \mathfrak{C} \restriction \tau')$ is Γ_2-big, then

$$\mathfrak{C} \models \varphi_R^1(a', a_2, \ldots, a_k) \equiv \varphi_R^2(b', F(a_2), \ldots, F(a_1)).$$

Remark 4.6. Can think of the case: for every case Γ_n^* of the scheme Γ_n and $p \in \mathbf{S}_{\Gamma_n^*}(A)$ we can find a case Γ_{n+1}^* of Γ_{n-1} such that p_n has many Γ_{n+1}^*-big extensions. See §4.

Claim 4.7. *We can deduce from Definition 4.5 a result parallel to 1.12.*

Proof. Straightforward. $\square_{4.7}$

§ 4(B). **Using a Stationary Non-reflective set, Getting Little Saturation.** We deal with the $\aleph_0$-saturation alternative in clause (e) of **??**. See 4.4.

Compared to [Shea] = [Shear, Ch.XI] λ is not necessarily a successor of singular.

We think of omitting countable types of cardinality κ. Can replace ε_i by a linear order.

Hypothesis 4.8.

(a) T is first order complete.

(b) $\lambda = \mathrm{cf}(\lambda) > |T|$ (the $> |T|$ for simplicity), and $\kappa = \mathrm{cf}(\kappa) < \lambda$, $S \subseteq S^\lambda_\kappa$ is stationary not reflecting, $\overline{C} = \langle C_\delta : \delta < \lambda \text{ limit}\rangle$ is a square avoiding S: i.e.
- $\alpha \in C_\delta \Rightarrow C_\alpha = C_\delta \cap \alpha$
- C_α a closed subset of α
- $\alpha > \sup(C_\delta) \Rightarrow \mathrm{cf}(\alpha) \leq \aleph_0$
- $C_\alpha \cap S = \varnothing$

Furthermore, $S' \subseteq \lambda \setminus S$ and $S \cup S'$ does not reflect.

(c) $\bar{\Gamma} = \big\langle \big(q_i(\bar{y}_i), \Gamma_i(\bar{y}_i)\big) : i < i^* \leq \lambda \big\rangle$ as in 3.2.

Definition 4.9. We define $\mathbb{P} = \mathbb{P}^T_{\bar{\Gamma}} = \mathbb{P}^T_{\lambda,\bar{\Gamma}}$ as follows:

(A) $\mathbf{p} \in \mathbb{P}$ iff $\mathbf{p} = (\alpha, \overline{M}, \overline{N}, \bar{b}, \bar{\Gamma})$ such that
  (a) $\alpha < \lambda$ limit, $|\alpha|$ divides α
  (b) $\overline{M} = \langle M_i : i \leq \kappa\rangle$ is a $\prec$-increasing sequence of models of T
  (c) $|M_\kappa| = \alpha$
  (d) $\overline{N} = \langle N_{i,\varepsilon} : \varepsilon \leq \varepsilon_i, i < \kappa\rangle$ is increasing; i.e. $i < \kappa$ and $\zeta < \varepsilon \leq \varepsilon_i \Rightarrow M_i \prec N_{i,\zeta} \prec N_{i,\varepsilon} \prec M_{i+1}$ and $\varepsilon_i < \lambda$
  (e) $\bar{b} = \langle b_{i,\varepsilon} : \varepsilon < \varepsilon_i, i < \kappa\rangle$
  (f) $\bar{\Gamma} = \langle \Gamma_{j(i,\varepsilon)}(\bar{a}_{i,\varepsilon}) : \varepsilon < \varepsilon_i, i < \kappa\rangle$
  (g) $\bar{a}_{i,\varepsilon} \subseteq N_{i,\varepsilon}$ realizes $q_i(\bar{y}_{j(i,\varepsilon)})$
  (h) $b_{i,\varepsilon} \in N_{i,\varepsilon+1}$ and $\mathrm{tp}(b_{i,\varepsilon}, N_{i,\varepsilon}, N_{i,\varepsilon+1})$ is $\Gamma_{j(i,\varepsilon)}$-big

(B) $\mathbf{p} \leq_{\mathbb{P}} \mathbf{q}$ <u>if</u>:
  (a) $\alpha^{\mathbf{p}} \leq \alpha^{\mathbf{q}}$
  (b) for every large enough $i < \kappa$ we have
    (α) $M^{\mathbf{p}}_i \prec M^{\mathbf{q}}_i$ and $|M^{\mathbf{p}}_i| \lhd |M^{\mathbf{q}}_i|$ **[Check]**
    (β) $\varepsilon^{\mathbf{p}}_i \leq \varepsilon^{\mathbf{q}}_i$
    (γ) $\varepsilon \leq \varepsilon^{\mathbf{p}}_i \Rightarrow N^{\mathbf{p}}_{i,\varepsilon} \prec N^{\mathbf{q}}_{i,\varepsilon}$
    (δ) $\varepsilon < \varepsilon^{\mathbf{p}}_i \Rightarrow \Gamma^{\mathbf{p}}_{i,\varepsilon} = \Gamma^{\mathbf{q}}_{i,\varepsilon}$
    (ϵ) $\varepsilon < \varepsilon^{\mathbf{p}}_i \Rightarrow b^{\mathbf{p}}_{i,\varepsilon} = b^{\mathbf{p}}_{i,\varepsilon}$.

(C) $\mathbf{p} \leq_j \mathbf{q}$ if the demand in (B)(b) holds for every $i \in [j, \kappa)$ and $\mathbf{p} \leq_{\mathrm{pr}} \mathbf{q}$ means $\mathbf{p} \leq_0 \mathbf{q}$

(D) we say $\langle \mathbf{p}_\beta : \beta < \beta^*\rangle$ is $\overline{C}$-increasing if
  (a) $\mathbf{p}_\beta \in \mathbb{P}$
  (b) $\beta < \gamma \Rightarrow \mathbf{p}_\beta \leq \mathbf{p}_\gamma$
  (c) $\beta \in \mathrm{acc}(C_\gamma) \Rightarrow \mathbf{p}_\beta \leq_{\mathrm{pr}} \mathbf{p}_\gamma$.

(E) Let $\mathbf{p} <_{\mathbb{P}} \mathbf{q}$ mean $\mathbf{p} \leq \mathbf{q}$ and $\alpha^{\mathbf{p}} < \alpha^{\mathbf{q}}$.

Claim 4.10. *Let $\mathbb{P} = \mathbb{P}^{\mathbf{T}}_{\lambda,\bar{\Gamma}}$.*

1) $\mathbb{P}$ is a partial order, $(< \kappa^+)$-complete (really quasi order).

2) If $\mathbf{p} \in \mathbb{P}$ and $\alpha^{\mathbf{p}} \le \beta < \lambda$, then there is $\mathbf{q} \in \mathbb{P}$ such that $\mathbf{p} \le \mathbf{q}$ and $\alpha^{\mathbf{q}} = \beta$.

3) If $\langle \mathbf{p}_\beta : \beta < \beta^* \rangle$ is $\bar{C}$-increasing in $\mathbb{P}$, then there is $\mathbf{p}_{\beta^*} \in \mathbb{P}$ such that $\langle \mathbf{p}_\beta : \beta < \beta^* + 1\rangle$ is $\bar{C}$-increasing.

4) If $\beta^* \le \lambda$ is a limit ordinal and $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha < \beta^* \rangle$ then $\bar{\mathbf{p}}$ is $\bar{C}$-increasing iff $\bar{p} \restriction \beta$ is $\bar{C}$-increasing for every $\beta < \beta^*$.

5) If $\mathbb{P} \models \mathbf{p} \le \mathbf{q}$ then for some $\mathbf{r}$ we have $\mathbf{p} \le_{\rm pr} \mathbf{r}$ and $\mathbf{q} \le \mathbf{r} \le \mathbf{q}$.

Proof. Easy. $\square_{??}$

Claim 4.11. *1) Assume $\diamondsuit_S$. We can find $\langle \mathbf{p}_\beta : \beta < \lambda \rangle$ which is $\bar{C}$-increasing and is generic enough. **[FILL!]***

2) We can express it by games.

Question 4.12. 1) The model is really somewhat rigid?

2) For $\lambda = \mu^+, \mu = \mu^\kappa$?

3) Using middle diamond? Can we combine black box and middle diamond?

Discussion 4.13. Assume λ is strongly inaccessible (or $\lambda = \mu^+, \mu = \beth_\mu$ we can partly imitate this) for $\delta \in S, \diamondsuit_S$ gives a guess: $F_\delta \in \mathrm{Iso}(M^{\bar{\varphi}^1}, M^{\bar{\varphi}^2})$. We first find $\mathbf{p}_\delta$ above $\mathbf{p}_\alpha$ for $\alpha < \delta$ such that: $M^{\mathbf{p}_\delta}_{i+1}$ is $\|N^{\mathbf{p}_\delta}_{i,\varepsilon(i,\mathbf{p}_\delta)}\|^+$-saturated (and close as in the proof after 4.14). Now we can add an element x_δ and omit some types.

We can have a pseudo finite sets $a_\alpha \in M^{\mathbf{p}_{\alpha+1}}_0$ which includes $M^{\mathbf{p}_\alpha}_\kappa$, this helps as for $\delta \in S$ if we guess right $F \restriction \bigcup\limits_{\alpha<\delta} M^{\mathbf{p}_\alpha}_\kappa$. Building $\mathbf{p}_\delta$ by diagonalizing: without loss of generality $\mathrm{otp}(C_\delta) = \kappa, C_\delta = \{\alpha_\zeta : \zeta < j < i\}$ in stage $\zeta < \kappa$ we have Γ_δ-big type $q \in S(M^{\mathbf{p}_{\alpha(\zeta)}}_\zeta)$ and let $\xi \in C_\delta \setminus \kappa + 1$, so $M^{\mathbf{p}_{\alpha(\xi)}}_\zeta$ is quite a saturated extension of $M^{\mathbf{p}_{\alpha(\zeta)}}_\zeta$ and $(a_0, a_\ell), F \cap (M^{\mathbf{p}_{\alpha(\xi)}}_\xi \times M^{\mathbf{p}_{\alpha(\xi)}}_\xi) \Rightarrow a_\ell \in \mathrm{ak}(a_{+\ell}, M^{\mathbf{p}_{\alpha(\zeta)}}_\zeta, M^{\mathbf{p}_{\alpha(\varepsilon)}}_\xi)$.

* * *

§ 4(C). Successor of Strong Limit.

The case λ is a successor of a strong limit singular cardinal of cofinality κ is different; we can get more, at least in some directions. See more in §5.

Definition 4.14. Assume that in addition to 4.8, we add the following three clauses:

(d)) $\lambda = \mu^+$, $\mathrm{otp}(C_\alpha) \le \mu$, $\bar{\lambda} = \langle \mu_i : i < \kappa \rangle$ is increasing continuous, [?] $\mu = \sum\limits_{i<\kappa} \lambda_i$, $[i < j \Rightarrow \mu_i < \lambda_i \le \mu_j]$, λ_i is regular, $\mathrm{cf}(\mu) = \kappa$, and $i < j \Rightarrow \lambda_i < \lambda_j$.

(e) $\lambda = \mathrm{tcf}(\prod \lambda_i / J^{\rm bd}_\kappa)$ and $\bar{f} = \langle f_\beta : \beta < \lambda \rangle$ is a scale of $\prod\limits_{i<\kappa} \lambda_i$ obeying $\bar{C}$.

(f)) $\sum\limits_{j<i} \lambda_j < \mu_i = \mathrm{cf}(\mu_i) < \lambda_i$, $(\forall \alpha < \lambda_{i+1})\big[|\alpha|^{\mu_i} < \lambda_i\big]$, and $(\mathrm{tcf}(\prod\limits_{i<\kappa} \mu_i / J^{\rm bd}_\kappa) = \lambda$[?]).

1) We now define $\mathbb{P} = \mathbb{P}^T_{\bar{\Gamma}} = \mathbb{P}^T_{\lambda,\bar{\lambda},\bar{f}}$ as in Definition 4.9(A): $\mathbf{P} \in \mathbb{P}$ iff $\mathbf{p} = (\alpha, \overline{M})$ satisfies:

(a) $\alpha < \lambda$ limit, $|\alpha|$ divides α.

(b) $\overline{M} = \langle M_i : i \le \kappa \rangle$ is a $\prec$-increasing sequence of models of T.

(c) $|M_\kappa| = \alpha$

(d) $\varepsilon_i < \mu_i$, $\|M_i^{\mathbf{p}}\| < \lambda_i$, $M_i^{\mathbf{p}}$ is μ_i^+ saturated, and $\|N^{\mathbf{p}}_{i,\varepsilon(p,\mathbf{p})}\| < \mu_i$.

(e) $f_\beta(i) \le \mathrm{otp}(|M_i^{\mathbf{p}_\beta}|)$.

Remark 4.15. Guess in λ_i (or $\lambda_i = \mu_i = \chi_i^+$, omit types of cardinality χ_i <u>or</u> better as in [She00a] essentially (so $\mu_i \ll \lambda_i$) <u>or</u> use guessing in S'?.

Definition 4.16. We define $\mathbf{p} \le_j \mathbf{q}$ as in 4.9(C), but in 4.9(D)(c) we have $\beta \in C_\gamma$ and $|C_\gamma| < \mu_j \Rightarrow \mathbf{p}_\beta \le_j \mathbf{p}_\gamma$.

Observation 4.17. *Assume $\langle \mathbf{p}_\beta : \beta < \lambda\rangle$ is $<$-increasing, assume $X \in [\lambda]^\lambda$ and for $\beta < \lambda$ let $g^X_\beta \in \prod\limits_{i \in w} \lambda_i$ be defined by*

$$g^X_\beta(i) = \mathrm{otp}\{\varepsilon \in M_i^{\mathbf{p}_\beta} : (\exists \zeta)[\varepsilon \le \zeta \in X \cap M_i^{\mathbf{p}_\beta}]\}.$$

<u>Then</u> *$\langle g^X_\beta : \beta < \lambda\rangle$ is a $\le_{J^{\mathrm{bd}}_\kappa}$-increasing, unbounded and even cofinal in $(\prod\limits_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa})$.*

Proof. Why? Otherwise there are $Y \in [\kappa]^\kappa$ and $g^* \in \prod\limits_{i \in Y} \lambda_i$ such that $g^X_\beta \restriction Y <_{J^{\mathrm{bd}}_Y} g^*$ for $\beta < \lambda$. So for some $\beta(*)$, $g^* <_{J^{\mathrm{bd}}_\kappa} f_{\beta(*)}$ by clause (e) of Definition 4.14. Now let $\beta \in [\beta(*), \lambda)$, so for every large enough $i \in Y$ we have (with the last inequality by 4.14(e)):

$$g^X_\beta(i) < g^*(i) < f_{\beta(*)}(i) \le \mathrm{otp}(M_i^{\mathbf{p}_{\beta(*)}})$$

and $|M_i^{\mathbf{p}_{\beta(*)}}| \lhd |M_i^{\mathbf{p}_\beta}|$ (recalling $\mathbf{p}_{\beta(*)} \le \mathbf{p}_\beta$) hence $X \cap M_i^{\mathbf{p}_\beta} \subseteq M_i^{\mathbf{p}_{\beta(*)}}$.

As this holds for every $i \in Y$ large enough, $\kappa = \sup(Y)$ and $\langle M_j^{\mathbf{p}_{\beta(*)}} : j < \kappa\rangle$, $\langle M_j^{\mathbf{p}_\beta} : j < \kappa\rangle$ are increasing we get $X \cap M_\kappa^{\mathbf{p}_\beta} \subseteq M_\kappa^{\mathbf{p}_{\beta(*)}}$. But $X \subseteq \lambda$ and $\lambda = \bigcup\{(M_\kappa^{\mathbf{p}_\beta} : \beta < \lambda\}$ and $M_\kappa^{p_\beta}$ is increasing with β so $X \subseteq M_\kappa^{\mathbf{p}_{\beta(*)}}$ hence $|X| \le \|M_\kappa^{\mathbf{p}_{\beta(*)}}\| \le \mu < \lambda$, contradicting the assumption on X. $\square_{4.17}$

Claim 4.18. *Assume that $\mathbb{P} = \mathbb{P}^T_{\lambda,\bar f,\bar\Gamma}$. <u>Then</u> the parallel of 4.10 holds.*

Let $\mathbb{P} = \mathbb{P}^{\mathbf{T}}_{\lambda,\bar\Gamma}$.

1) $\mathbb{P}$ is a partial order, $(< \kappa^+)$-complete (really quasi order).

2) If $\mathbf{p} \in \mathbb{P}$ and $\alpha^{\mathbf{p}} \le \beta < \lambda$, <u>then</u> there is $\mathbf{q} \in \mathbb{P}$ such that $\mathbf{p} \le \mathbf{q}$ and $\alpha^{\mathbf{q}} = \beta$.

3) If $\langle \mathbf{p}_\beta : \beta < \beta^*\rangle$ is $\bar C$-increasing in $\mathbb{P}$, <u>then</u> there is $\mathbf{p}_{\beta^*} \in \mathbb{P}$ such that $\langle \mathbf{p}_\beta : \beta < \beta^* + 1\rangle$ is $\overline{C}$-increasing.

4) If $\beta^* \le \lambda$ is a limit ordinal and $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha < \beta^*\rangle$ <u>then</u> $\bar{\mathbf{p}}$ is $\overline{C}$-increasing <u>iff</u> $\bar p \restriction \beta$ is $\overline{C}$-increasing for every $\beta < \beta^*$.

5) If $\mathbb{P} \models \mathbf{p} \le \mathbf{q}$ <u>then</u> for some $\mathbf{r}$ we have $\mathbf{p} \le_{\mathrm{pr}} \mathbf{r}$ and $\mathbf{q} \le \mathbf{r} \le \mathbf{q}$.

Discussion 4.19. We need more than 4.17. In some sense there are an imaginary $p^*, M_i^{p^*}$ has order type $\lambda_i, X_i \in [\lambda_i]^{\lambda_i}$ and we choose $\beta < \lambda$ such that for every large enough $i < \kappa, (M_i^{\mathbf{p}_\beta}, X_i' \cap M_i^{\mathbf{p}_\beta})$ is a good approximation to $(M_i^{\mathbf{p}^*}, X_i)$. We shall use $2^\mu = \lambda(= \mu^+)$ to show this.

Of course, if $\lambda_i(\lambda_i^-)^+, \lambda_i$ strong limit singular (or λ_i strongly inaccessible) things should be clearer.

There are two kinds of tasks.

<u>Task 1</u>: Building a Boolean algebra $\underset{\sim}{B} = \underset{\sim}{M}$ with $\mathrm{irr}(\underset{\sim}{B}) = \mu$.

Well, do not fit present definitions. So assume $\lambda_i = \theta_i^+, \theta_i = \mathrm{cf}(\theta_i)$ and we like to commit ourselves to some X_i.

Task 2: [DEBT]

Definition 4.20. $\partial_{\lambda,\mu}(T)$ is a game, it lasts μ moves in stage i and a pair $(M_i, \mathscr{P}_i)$ created such that:

(a) M_i is a model of T of cardinality $< \lambda$ with universe on an ordinal $\gamma_i < \lambda$.
(b) $\mathscr{P}_i$ is a set of $< \mu_i$ types omitted by M_i with no support of cardinality $< \mu_i$, increasing with i.
(c) In limit [ordinals] we take unions.
(d) In the i-th move the challenger gives β^i_i and $M^-_{i+1} \leq M_i$, $M^-_{i+1} <_* N_i$, $(N_i) \subseteq N_i$, $N_i \cap M_i = M^-_i$, $\|N_i\| < \mu$ and the defender chooses M_{i+1}, $M_i \leq_* M_{i+1}$, (and N_i sits nicely) and the challenger adds a type to $\mathscr{P}_i$ with no support.

The defender wins if he always has a legal move.

Now suppose $\delta \in S$ and we guess here $X_\delta \subseteq \delta = \bigcup\limits_{\alpha<\delta} M^{\mathbf{p}_\alpha}_i$ we can choose a sequence $j_i < \kappa, \alpha_i < \delta$ such that $\delta = \bigcup \alpha_i$, $j_i \in [i, \kappa)$ and use diagonal limit of $\langle \mathbf{p}_{\alpha_i} : i < \kappa\rangle$ or $\mathbf{p}_\delta$ says $\mathbf{p}_{\alpha_i} \leq_{j_i} \mathbf{p}_\delta$ for $i < \kappa$.

Having chosen α_i, j_i we ask: are there $j \in (j_i, \kappa), \beta \in (\alpha_j, \delta)$ such that:

$\circledast$ (i) $\mathbf{p}_{\alpha_i} \leq_j \mathbf{p}_\beta$
(ii) $M^{\mathbf{p}_\beta}_j \cap X_\delta$ is large.

What does large mean? For the irrendundancy of the Boolean algebra it is

$(*)$ We can add the promise: if $\mathbf{p}_\beta \leq_j \mathbf{p}_\gamma$ in $M^{\mathbf{p}_\gamma}_j$, $X_\delta \cap M^{\mathbf{p}_\beta}_j$ is still a maximal irredundant set.

It seems

(a) better to force
(b) **Case 1**: $\lambda_i = \theta^+_1$ and instead orthogonality we use $\leq \theta_i$ types of cardinality θ_i with no support of cardinality $< \theta_1$ (as usual in "Model with second order properties III".)
(c) **Case 2**: $\beth_{\mu^+_i} < \lambda_i$ and $\|N^{\mathbf{p}}_{i,\varepsilon}\| \leq \beth_{2\varepsilon+1}(\mu_i)$.

In Case 1, in the above scheme arriving to β, j we find $\leq_j$-increasing $\langle \mathbf{p}_{\beta_\varepsilon} : \varepsilon < \varepsilon\rangle$, dealing with all possible supports (so better have $\theta_j = \theta^{<\theta_j}_j$).

Question 4.21. Phrase omitting type theorem for (λ_i, μ_i) meaning: an approx is a model of cardinality $< \lambda_i$.

Remark 4.22. $\theta_i = \theta^{<\theta_i}_i$ is not such a bad assumption if $\neg \mathbf{0}^\#$, etc., for any $\mu = \beth_\delta$, ω^2/δ we can find such λ_i, μ_i. But above we can choose $\beta > \alpha_j$ and use $(M^{\mathbf{p}_\beta}_j, P^{\mathbf{p}_\alpha}_j)$!

§ 5. Toward Ghibellines and Guelfs for successor of singulars

Context 5.1. The parameter $\mathfrak{x}$ consists of:

(a) $\lambda = \mu^+$, $\mu > \mathrm{cf}(\mu) = \kappa$
(b) $\langle \mu_i : i < \kappa \rangle$ is increasing continuous with limit μ.
(c) $\mu_i < \lambda_i = \mathrm{cf}(\lambda_i) < \mu_{i+1}$
(d) $\lambda = \mathrm{tcf}(\prod\limits_{i<\kappa} \lambda_i, <_{J^{\mathrm{bd}}_\kappa})$
(e) $\bar{f} = \langle \bar{f}^*_\alpha : \alpha < \lambda \rangle$ is $<_{J^{\mathrm{bd}}_\kappa}$-increasing and cofinal in $\prod\limits_{i<\kappa} \lambda_i$.

Definition 5.2. We say $\mathbb{P}$ is $\mathfrak{x}$-uniform (forcing notion or approximation system) when it consists of the following objects satisfying the following conditions:

(a) A set P (but we may write $\mathbf{p} \in \mathbb{P}$ instead of $\mathbf{p} \in P$),
(b) for $p \in \mathbb{P}$ we have $\mathrm{dom}(p) \in [\lambda]^{\leq \mu_i}$ for some $i < \kappa$,
(c) quasi order $\leq^{\mathbb{P}}$ on P,
(d) $p \leq q$ implies $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$,
(e) $\leq^{\mathbb{P}}_{\mathrm{pr}} \subseteq \leq^{\mathbb{P}}$ a quasi order, $p \leq^{\mathbb{P}}_{\mathrm{pr}} q \Rightarrow \mathrm{dom}(p) \trianglelefteq \mathrm{dom}(q)$.
(f) any $\leq^{\mathbb{P}}$-increasing sequence in $\{p \in \mathbb{P} : |\mathrm{dom}(p)| \leq \mu\}$ has an upper bound in it.

Definition 5.3. 1) Let $\mathrm{app}(\mathbb{P})$ be the set of $\mathbf{p}$ such that for some ε:

(a) $\mathbf{p} = \langle p_i : i < \kappa \rangle$; we may write $p^{\mathbf{p}}_i$ or $p_i[\mathbf{p}]$
(b) $\mathrm{dom}(p_i) \in [\lambda]^{\leq \mu_i}$ for $p_i \in \mathbb{P}$.
(c) p_i is increasing (in $\mathbb{P}$) with i
(d) For some ordinal $\alpha = \alpha^{\mathbf{p}} = \alpha[\mathbf{p}]$ we have $\alpha = \bigcup\{\mathrm{dom}(p_i) : i < \kappa\}$

2) We define two place relations $\leq, \leq_j, \leq_{\mathrm{pr}}$ on $\mathrm{app}(\mathbb{P})$ as follows:

(a) $\mathbf{p} \leq_j \mathbf{p}$ <u>iff</u> for every $i \in [j, \kappa)$ we have $p_i[\mathbf{p}] = p_i[\mathbf{p}] \restriction \alpha^{\mathbf{p}}$
(b) $\mathbf{p} \leq_{\mathrm{pr}} \mathbf{p}$ iff $\mathbf{p} \leq_0 \mathbf{p}$
(c) $\mathbf{p} \leq \mathbf{p}$ iff $(\exists j < \kappa)(\mathbf{p} \leq_j \mathbf{p})$.

3) For a square sequence $\overline{C}$ we say $\bar{\mathbf{p}}$ is $\overline{C}$-increasing if

(a) $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha < \ell g(\bar{\mathbf{p}}) \rangle$
(b) $\alpha < \beta < \ell g(\bar{\mathbf{p}}) \Rightarrow \mathbf{p}_\alpha \leq \mathbf{p}_\beta$
(c) if $\alpha \in C_\beta$, $\beta < \ell g(\bar{\mathbf{p}})$ and $\mu_j > |C_\beta|$ then $\mathbf{p}_\alpha \leq_j \mathbf{p}_\beta$.

Claim 5.4. *1) $\leq, \leq_j, \leq_{\mathrm{pr}}$ are quasi orders on* $\mathrm{app}(\mathbb{P})$.
2) Any $\leq$-increasing sequence in $\mathrm{app}(\mathbb{P})$ *of length $\leq x$ has an upper bound.*
2A) If $j < x$ then any $\leq_j$-increasing sequence in $\mathrm{app}(\mathbb{P})$ *of length $< \mu^+_j$ has an upper bound.*
3) If $\mathbf{p}, \mathbf{q} \in$ $\mathrm{app}(\mathbb{P})$ *and $\mathbf{p} \leq \mathbf{q}$ and $j < \kappa$ then for some $\mathbf{r}$ we have $\mathbf{p} \leq_j \mathbf{r} \leq \mathbf{q}$, $\mathbf{q} \leq \mathbf{r}$ [and $\mathbf{r} \restriction j = \mathbf{q} \restriction j$?]*
4) If $\bar{\mathbf{p}}$ is $\overline{C}$-increasing, $\overline{C}$ a square, $\delta^* = \ell g(\bar{\mathbf{p}})$ a limit ordinal and $\mathrm{cf}(\delta^*) > \kappa \Rightarrow \delta^* = \sup(C_{\delta^*})$ *then there is a $\overline{C}$-increasing $\bar{\mathbf{p}}'$ of length $\delta^* + 1$ such that $\bar{\mathbf{p}} \triangleleft \bar{\mathbf{p}}'$.*

Proof. Straightforward. $\square_{5.4}$

For examples of F as below (and why the defender can win in the game below).

For $\lambda = \theta^+$, $\theta = \theta^{<\theta}$ the omitting type theorem for λ^+, $\lambda = \lambda^{<\lambda} \wedge \diamondsuit_\lambda$ works, but undesirable.

Definition 5.5. 1) We say that F is an i-a (= abstract type automorphism), or (λ, μ_i)-auto if

(a) F is a function.
(b) $\mathrm{dom}(F) = \{(p, A) : p \in \mathbb{P},\ \mathrm{dom}(p) \in [\lambda]^{<\lambda_i}$, and $A \subseteq [\mathrm{dom}(p)]^{<\kappa}\}$
(c) $F(p, a) \subseteq \{q \in \mathbb{P} : \mathrm{dom}(p) \trianglelefteq \mathrm{dom}(q)$ and $p = q \restriction \alpha$ for some $\alpha\}$.

2) F is good i-auto if in addition it satisfies:

(α) $F(p, A)$ is downward closed.
(β) $F(p, A)$ is closed under unions of $<^{\mathbb{P}}_{\mathrm{dr}}$-increasing chains of length $< \lambda_i$.
(γ) If $\langle P_i : i \le \gamma < \theta\rangle$ is $<_{\mathrm{dr}}$-increasing, $A_i \subseteq [\mathrm{dom}(p)]^{<\kappa}$, and $p_\gamma \in \bigcap\limits_{i<j} F(p_i, A_i)$ then there is q such that $p_\gamma <_{\mathrm{dr}} q \in \bigcap\limits_{i<\gamma} F(p_i, A_i)$.

3) An i-auto F is weakly good if in the following game the defender has a winning strategy.

A play lasts λ_i moves. Before the α-th move, $\langle (p_\beta, q_\beta, A_\beta, u_\beta, w_\beta) : \beta < \alpha\rangle$ and p_α is defined such that

$(*)_\alpha$ (a) $p_\beta \in \mathbb{P}$ is $<_{\mathrm{dir}}$-increasing, $\mathrm{dom}(p_\beta) \in [\lambda]^{<\lambda_i}$.
(b) p_β is $<_{\mathrm{dir}}$-increasing continuous.
(c) $A_\beta \subseteq [\mathrm{dom}(p_\beta)]^{<\kappa}$
(d) $w_\beta \subseteq \beta$, $|w_\beta| < \theta$, and $\gamma < \beta \Rightarrow w_\beta \cap \gamma \subseteq w_\gamma$.
(e) $q_\beta \in \mathbb{P}$, $[\mathrm{dom}(q_\beta)]^{<\theta}$, $q_\beta \restriction \alpha(p) \le_{\mathbb{P}} p_\beta$, $q_\beta \le p_{\beta+1}$
(f) If $\gamma \in w_\beta$ then $q_\gamma <_{\mathrm{dir}} q_\beta$. [Or use another system: $\langle u_\beta : \beta < \alpha\rangle$ a partial square.]
(g) If $\beta \in w_\gamma$ then $p_\gamma \in F(p_\beta, A_B)$.

In the α-th move:

The challenger chooses $q'_\alpha \le_{\mathrm{dir}} q_\alpha \in \mathbb{P}, \mathrm{dom}(q_\alpha) \in [\lambda]^{<\theta}, q'_\alpha$ is an earlier q_β or direct limit of such and $A_\alpha \subseteq [\mathrm{dom}(p_\alpha)]^{<\kappa}$ and $\gamma_\alpha < p$ and $w'_\alpha \subseteq \lim\langle w_\beta : \beta < \alpha\rangle$ of cardinality $< \theta$ (initial segment, closed).

The defender chooses $p_{\alpha+1} \in P$ such that $p_\alpha <_{\mathrm{dir}} p_{\alpha+1}, q_\alpha \subseteq P_{\alpha+1}$ such that $p_{\alpha+1} \in \bigcap\{F(p_\beta, A_\beta) : \beta \in w' \cup \{\alpha\}\}$.

Claim 5.6. *Assume*

(a) $\mathfrak{x}$ is as in 5.1
(b) $\mathbb{P}$ is as in 5.2,
(c) F_i is weakly good i-auto (for $(\mathfrak{x}, \mathbb{P})$) for $i < \kappa$, St_i a winning strategy of the defender witnessing it in the game from 5.4
(d) $\bar{C}$ is a partial square on λ, (so $\bar{C} = \langle C_\alpha : \alpha < \lambda\rangle$, C_α a closed subset of α, $\beta \in C_\alpha \Rightarrow C_\beta = C_\alpha \cap \beta$) such that $\delta < \lambda$ and $\mathrm{cf}(\delta) > \kappa \Rightarrow \delta = \sup(C_\delta)$ and $\alpha < \lambda \Rightarrow |C_\alpha| < \mu$
(e) $S \subseteq S^\lambda_\kappa$ is stationary, $\alpha \in S \Rightarrow \sup(C_\alpha) < \alpha$ and $\diamondsuit_S$.

Then there are $\bar{\mathbf{p}}$ such that $\langle A_{\alpha,i} : \alpha < \lambda, i < \kappa\rangle$

(a) $\bar{\mathbf{p}} = \langle \mathbf{p}_\alpha : \alpha < \lambda\rangle$ is $\bar{C}$-increasing, let $\mathbf{p}_\alpha = \langle p_{\alpha,i} : i < \kappa\rangle$

(b) *$A_{\mathbb{P}}$ is as in 5.2 $\subseteq [\mathrm{dom}(p_{\mathbb{P}})]^{<\kappa}$*
(c) *each i, the $(p_{\alpha,i}, A_{\alpha,i})$ obeys St_i in the natural way*
(d) *if $A \subseteq [\lambda]^{<\kappa}$, <u>then</u> for stationarily many $\delta \in S$ we have*
   (i) $\alpha(\mathbf{p}_\delta) = \delta = \bigcup\{\alpha(p_\beta) : \beta < \delta\}$
   (ii) $A_{\delta,i} = A \cap [\mathrm{dom}(p_{\delta,i})]^{<\lambda}$
   (iii) *for i large enough, each $A_{\delta,i}$ is quite closed.*

§ 6. Games and Boolean algebras irr(B)

Here we shall apply §5.

The following game is defined such that a winning strategy for the defendant helps in building a Boolean algebra B of cardinality $\lambda = \mu^+$ with $\mathrm{irr}(B) = \mu$.

Definition 6.1. Assume $\theta \le \lambda$ are regular cardinals. We define a game $\partial = \partial^{\mathrm{irr,ba}}_{\lambda,\theta}$ as follows: $\sigma(\lambda) = \min\{\sigma : (\exists \alpha < \lambda)[|\alpha|^\sigma \ge \lambda]\}$.

A play of the game lasts λ^+ moves; in the α-th move we already have $(\beta_\beta, B_\beta, B^-_\beta, \bar{\mathscr{A}}_\beta, w_\beta)$ for $\beta < \alpha$ and p_α such that B^-_α has the role of q in Definition 5.5 (compare).

$(*)_\alpha$
(a) β_α is an ordinal $< \lambda$, increasing continuous in α, $\beta_0 = 0$
(b) B_α is a Boolean algebra generated by $\{x_\beta : \beta < \beta_\alpha\}$ such that $x_\beta \notin \langle \{x_\gamma : \gamma < \beta\}\rangle_{B_\alpha}$ for $\beta < \beta_2$ (can decide that the set of elements of B_α is an ordinal)
(c) if $\alpha_1 < \alpha$ then $B_{\alpha_1} \subseteq B_2$
(d) $\bar{\mathscr{A}}_\alpha = \langle \mathscr{A}_\beta : \beta \in w_\alpha\rangle, w_\alpha \subseteq \alpha, |w_\alpha| < \theta$
(e) if $\alpha_1 < \alpha$ then $w_\alpha \cap \alpha_1 \subseteq w_{\alpha_1}$ and if α is limit then $w_\alpha \subseteq \lim\langle w_\beta : \beta < \alpha\rangle$
(f) $\mathscr{A}_\beta \subseteq B_\beta$ is an irredundant subset of B_β
(g) if $\beta < \alpha$, $b \in B_\alpha$, <u>then</u> either $\mathscr{A}_\alpha \cup \{x\}$ is redundant <u>or</u> there is $A \subseteq B_2$, $|A| < \sigma < (\lambda)$ such that: there is no $a' \in \mathscr{A}_\beta$ such that a, a' realizes the same quantifier-free type over A in B_α
[<u>or</u> more strict no relevant small support].

In Stage α:

<u>The challenger</u> gives $\gamma_\alpha < \lambda^+$ and possibly $\mathscr{A}_\alpha \subseteq B_\alpha$ and $w'_\alpha \subseteq w_{\alpha-1}$ if α is a successor, $w'_\alpha \subseteq \lim\langle w_\beta : \beta < \alpha\rangle, |w'_\alpha| < \theta$ if α is a limit ordinal.

<u>The defender</u> chooses $\gamma_{\alpha+1}, B_{\alpha+1}$ as above and let

$$w_\alpha = \{\beta \in w'_\alpha \cup \{\alpha\} : \mathscr{A}_\alpha \text{ satisfies clause (f) above.}\}$$

§ 7. CONTINUING [She08]

Consider finding a model $\mathfrak{C}$ of T which is t-rigid; i.e. characterize the t for which they work for every T.

We think of how to find enough bigness notion derived from t such that if $\mathfrak{C}$ is complicated for them.

If t has the independence property (say, $N = \mathfrak{C}^{\bar\varphi}$ is a model of t, $\langle \bar a_i : i < \omega\rangle$ is an indiscernible sequence in N, D a uniform ultrafilter on ω, $\vartheta(\bar x, \bar y)$ a formula in $\mathbb{L}(\tau_t)$ such that $\langle \vartheta(\bar x, \bar b_i) : i < \omega\rangle$ is independent) then we can define (in $\mathfrak{C}, \Gamma$): $\varphi(\bar x, \bar a)$ is Γ-big when for some $A \supseteq \bar a \cup \bigcup_i \bar b_i$, we have $\circledast_1 \Rightarrow \circledast_2$, where

$\circledast_1$ $\bar b'_i$ realizes $\mathrm{Av}\big(A_i, \bigcup_{j<i} \bar b'_j, D, N[\mathfrak{C}]\big)$ for $i < \omega$.

$\circledast_2$ $\varphi(\bar x, \bar a) \cup \{\theta(\bar x, b'_i)^{\mathrm{if}(i \text{ is even})} : i < \omega\}$ is finitely satisfiable in $\mathfrak{C}$.

This is a "high" way to use the independence property.

A "low" way is just to require that we can find some indiscernible sequence $\langle \bar b'_i : i < \omega\rangle$ over $\bar a$ as in $\circledast_2$.

[So Γ above gives us: if $\mathfrak{C}$ is complicated then for any interpretation of t in $\mathfrak{C}$, in many $p, q(N[\mathfrak{C}])$, F is definable.] Can we move to a formula?

We may consider definition like §3:

$\varphi(\bar x, \bar a)$ is the beginning <u>iff</u> $\langle \mathfrak{C}_i : i < \delta\rangle$ is increasing fast enough, $\bar c_i \in N[\mathfrak{C}_{i+2}]$ realizes an appropriate $L\tau_t$)-type over $N[\mathfrak{C}_i]$ in $N[\mathfrak{C}]$ then

$$\{\varphi(x, \bar a)\} \cup \bigcup \{q_i(x, \bar c) : i < 0\}$$

is finitely satisfiable.

Well, we may be more elaborate. We may consider the following game of length λ: for given $q_i = \{\varphi(x, \bar a)\}$ or given 1-type q_0: during a play of the game q, in the i-th move p_i is chosen, p_i a 1-type in $\mathfrak{C}$, increasing with i, $|p_i| < \lambda$. In stage [?] our player gives $M_i \supseteq \mathrm{dom}(q_i), |M_i| < \kappa$, the opponent $p(x) \in \Phi(M_i) \le \mathbf{S}^{<\omega}(M_i)$ and $\varphi'_i \in \{\pm\varphi(x, \bar c_i)\}$ and our player has to choose $q_i \supseteq \bigcup_{j<i} q_j \cup \{\varphi'_i\}$. A good point will be if the $\Phi(M_i)$ depends just on the situation in $N = \mathfrak{C}^{[\bar\varphi]}$.

* * *

After defining such bigness notions, if $F \in \mathrm{ISO}(N_1, N_0)$ and $N_\ell = \mathfrak{C}^{[\bar\varphi^\ell]}$, for some $\bar\Gamma$-big type $\bar q(x, y)$ with $\Gamma_0 = \Gamma$ we have

$$F(A) = b \Rightarrow q \cup \{\theta^{N_i}(x, a)\} \vdash_{\bar\Gamma} \Theta(y, F(a)).$$

This induces $R_1 \subseteq N_1 \times N_2$ which really gives an equivalence relation on N_ℓ such that F maps N_1/E_1 into N_2/E_2 and is $L_{\infty,\kappa}$-definable, where E_1, E_2 are equivalence relations defined naturally from R_1.

We may present it by

Definition 7.1. Assume

$\boxtimes$ (a) $\mathfrak{C}$ is a $\kappa, \bar x = \langle x_i : \alpha < \alpha(*)\rangle$

(b) I is κ-closed partial order

(c) $\bar p = \langle p_t(\bar x) : t \in I\rangle$ is a sequence of α-type in $\mathfrak{C}$ such that

$$t \models \text{“}x < t \Rightarrow p_t(\bar x) \subseteq p_s(\bar x).\text{”}$$

1) Let $\Gamma = \Gamma_{\kappa, \bar p}$ is the family of types $q(\bar x)$ in $\mathfrak{C}$ of cardinality $< \kappa$ such that some $t \in I$ witnesses it, i.e. $q(\bar x) \cup p_t(\bar x)$ is finitely satisfiable in $\mathfrak{C}$.

Discussion 7.2. 1) Does [She08, §4] exhaust all the genericity conclusions by usually being enough for Δ-embedding?

Clearly not by phrase the fully-κ-complicated (in addition to rigid/endomorphisms).

2) Also in [She08, 3.6=L3.1D] (and similar cases) implicate is a pair of equivalence relations $\bar{E} = (E_1, E_2)$, E_ℓ on N_ℓ and $F'_{\bar{E}'} : N'_1/E_1 \to N_2/E_2$. We can guess for $\delta \in S^{\lambda^+}_\lambda$ a close enough family of $\bar{E}$-s (and all their less fine ones). Can we find a bigness notion Γ which guarantees more?

3) We should think on games:

given $M_i \prec \mathfrak{C}$, $|M_i| < \kappa$, we give $\bar{a}_i$ such that we have freedom to add to $q, \langle \varphi_i(\bar{a}_i) : \ell < \ell^* \rangle$ so ℓ^*-possibilities.

We can use $\langle (\mathfrak{C}_i, \bar{a}_i) : i < \delta \rangle$, $\mathrm{cf}(\delta) > \kappa$, $\mathfrak{C} = \bigcup\limits_{i<\delta} \mathfrak{C}_i$ and $q(x)$ is big if

$$\bigcup_{i<\delta} q(x) \cup \big\{ \varphi(x, \bar{a}_i) : j \in [i, \delta) \big\}$$

is consistent. Rich enough, but presently no dichotomy.

References

[CSS99] Gregory L. Cherlin, Saharon Shelah, and Niandong Shi, *Universal graphs with forbidden subgraphs and algebraic closure*, Adv. in Appl. Math. **22** (1999), no. 4, 454–491, arXiv: math/9809202. MR 1683298

[Fuc73] Laszlo Fuchs, *Infinite abelian groups*, vol. I, II, Academic Press, New York, 1970, 1973.

[HLS93] Bradd T. Hart, Claude Laflamme, and Saharon Shelah, *Models with second order properties. V. A general principle*, Ann. Pure Appl. Logic **64** (1993), no. 2, 169–194, arXiv: math/9311211. MR 1241253

[MS88] Alan H. Mekler and Saharon Shelah, *Diamond and λ-systems*, Fund. Math. **131** (1988), no. 1, 45–51. MR 970913

[Shea] Saharon Shelah, *Compact logics in ZFC: Constructing complete embeddings of atomless Boolean rings*, Ch. X of "The Non-Structure Theory" book [Sh:e].

[Sheb] ———, *Compactness of the Quantifier on "Complete embedding of BA's"*, arXiv: 1601.03596 Ch. XI of "The Non-Structure Theory" book [Sh:e].

[Shec] ———, *On spectrum of κ-resplendent models*, arXiv: 1105.3774 Ch. V of [Sh:e].

[She78] ———, *Models with second-order properties. II. Trees with no undefined branches*, Ann. Math. Logic **14** (1978), no. 1, 73–87. MR 501098

[She83a] ———, *Classification theory for nonelementary classes. I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part A*, Israel J. Math. **46** (1983), no. 3, 212–240. MR 733351

[She83b] ———, *Classification theory for nonelementary classes. I. The number of uncountable models of $\psi \in L_{\omega_1,\omega}$. Part B*, Israel J. Math. **46** (1983), no. 4, 241–273. MR 730343

[She83c] ———, *Models with second order properties. IV. A general method and eliminating diamonds*, Ann. Pure Appl. Logic **25** (1983), no. 2, 183–212. MR 725733

[She90] ———, *Classification theory and the number of nonisomorphic models*, 2nd ed., Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990, Revised edition of [Sh:a]. MR 1083551

[She94] ———, *Vive la différence. II. The Ax-Kochen isomorphism theorem*, Israel J. Math. **85** (1994), no. 1-3, 351–390, arXiv: math/9304207. MR 1264351

[She00a] ———, *Cellularity of free products of Boolean algebras (or topologies)*, Fund. Math. **166** (2000), no. 1-2, 153–208, arXiv: math/9508221. MR 1804709

[She00b] ———, *On what I do not understand (and have something to say), model theory*, Math. Japon. **51** (2000), no. 2, 329–377, arXiv: math/9910158. MR 1747306

[She08] ———, *Vive la différence. III*, Israel J. Math. **166** (2008), 61–96, arXiv: math/0112237. MR 2430425

[Shear] ———, *Non-structure theory*, Oxford University Press, to appear.

[Wim82] Edward L. Wimmers, *The Shelah P-point independence theorem*, Israel J. Math. **43** (1982), no. 1, 28–48. MR 728877

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