SET THEORETIC DREAMS 2023

SAHARON SHELAH

Abstract. Those informal notes are indented to help the author speaking in the Coll. of the E.S.T.S. (European Set Theory Society) Sept 2023.

Date: September 29, 2023.

2020 *Mathematics Subject Classification.* 03E99.

Key words and phrases. set theory, uncountable combinatorics, forcing, partition calculus, axiom of choice.

First typed: September 21, 2023. The author would like to thank the Israel Science Foundation for partial support of this research by grant 1838/19: The author would like to thank the typist for his work and is also grateful for the generous funding of typing services donated by a person who wishes to remain anonymous. References like [Sh:950, Th0.2=Ly5] mean that the internal label of Th0.2 is y5 in Sh:950. The reader should note that the version in my website is usually more up-to-date than the one in arXiv. This is publication number E110 in author's list.

§ 0. Introduction

I will start with directions on which I have very little to say; largely repeating what I had written earlier.

Then, as one who does write all the time do not suffice to read, will give my skewed outlook - move to things on which I have too much to say – topics

I have been working on in recent years. I will give references essentially only to my work, out of laziness (and making last minute work, not the organizers fault).

I will interpret this forum as quite informal, so "I" will be quite dense.

§ 1. Directions I know little about

Problem 1.1.

(A) The parallel of forcing for PA.
(B) The parallel of forcing for $\mathbf{V} = \mathbf{L}$.

It is very important but I do not have any idea what to do; probably like everybody else.

For the first we have lots of candidates for being an independent statement: Riemann hypothesis or e.g.:

Conjecture 1.2. Is it independent of ZFC that there are infinitely many primes of the form $n^2 + 1$? or $p = 2^n + 1$?

For the second, a preliminary question is:

Problem 1.3. Find good questions which are candidates for being independent from $\mathbf{V} = \mathbf{L}$ (not connected to consistency strength, of course).

Question 1.4. *(Andres Villaveces) In Conjecture 1.2, do you really mean "independent of* ZFC*" rather then "independent of* PA*"?*

Answer 1.5. I really intend ZFC (see Thesis 1.8) but even if I prove it just for PA I will be dancing on the roof.

A question were others can say much is:

Problem 1.6.

(A) Continue the inner models of set theory program.
(B) Find good enough way to prove equi-consistency with super-compact cardinals as we know for inaccessible/measurable, etc.

Certainly AD give a fascinating descriptive set theory. BUT I do not think AD give the "true theory", just an extremely nice one and I believe there are more.

Problem 1.7.

(A) Prove the consistency of additional interesting/nice cases of descriptive set theory.
(B) Developed descriptive set theory for $\mathscr{P}(\lambda)$ for uncountable κ.

Concerning Problem 1.7(B), forcing destroying stationary sets are a problem, showing Levy collapse is not homogeneous.

Generally I believe:

Thesis 1.8. What we prove in ZFC is true; there are many interesting semi-axioms; some more natural than others; a major criterion for judging them is if they resolve many statements in interesting ways.

Large cardinals have additional crucial role: they give a scale for determining consistency strength and so play a major role in independent results.

Question 1.9. *(Grigor Sargsyan) Does the proof of "*MM *implies* AD$(*)$*" change your view on* AD*.*

Answer 1.10. In short the answer is not, not at all.

The long answer will take longer. While I think $\mathbf{V} = \mathbf{L}$ is "improbable, a very extreme case, of zero probability" (and you agree), I will give $\neg 0^\sharp$ a positive probability, such universes are certainly very interesting.

Even if you like to have "GCH fail everywere" you need as I recall just high enough hyper-measurable.

Let me repeat: AD is a wonderful axiom with deep beautiful theory involved, one for which the intuition of descriptive set theorist is fully fulfilled. True, $\mathbf{V} = \mathbf{L}$ give answers to the relevant question, however they feel it is a dull one, a "wrong" one. But other families of problems draw you in other orthogonal direction.

E.g. for homological Abelian group theory, again $\mathbf{V} = \mathbf{L}$ give you full answers, erasing distinctions between various notions, see early [Nun77] and quite up to date [EM02]; see more in Problem 8.2.

You may like to stick to the reals, but then "Cichoń's maximum" indicate to me we better have the continuum not too small among the alephs. Just note that if $\mathrm{inv}_0, \ldots, \mathrm{inv}_9$ list the cardinal invariants in Cichoń's diagram and $2^{\aleph_0} < \aleph_{10}$, then we can prove "for some $i < j < 10$ we have $\mathrm{inv}_i = \mathrm{inv}_j$".

All of this does not say that even Berkeley cardinals are not interesting/exciting, but this is not the worthwhile direction.

Let me add, in the fifties and sixties the GCH was prominently use in set theory (e.g. the partition calculus) and in model theory (e.g. isomorphic ultra-powers). But later the picture changes.

§ 2. On pcf theory/cardinal arithmetic

Probably many will ask about:

Conjecture 2.1.

(A) $\mathrm{pp}(\aleph_\omega) < \aleph_{\omega_1}$ instead $\mathrm{pp}(\aleph_\omega) < \aleph_{\omega_4}$, or at least,
(B) If $2^{\aleph_0} < \aleph_\omega$ then $(\aleph_\omega)^{\aleph_0} < \aleph_{\omega_1}$.

Nice, but I think the real question is:

Problem 2.2. Given an *aleph bound* to $\mathrm{pcf}(\mathfrak{a})$ for $\mathfrak{a}$ a set of $< \min(\mathfrak{a})$ regular cardinals, or at least to $\mathrm{pcf}_{\aleph_1\text{-complete}}(\mathfrak{a})$.

By "*aleph bound*" we mean e.g. $|\mathfrak{a}|^{+\omega_4}$ or just $\left(|\mathfrak{a}|^{\aleph_0}\right)^{+\omega_4}$ but even "$|\mathfrak{a}|^{\aleph_0} <$ first weakly inaccessible above $|\mathfrak{a}|$" will be great, at least for me.

Conjecture 2.3. The $\mathrm{RGCH}_{\aleph_\omega}$ is true; that is the revised GCH for $\aleph_\omega$: see below. Recall that it was proved for $\beth_\omega$.

On revised GCH, see [She00b], [She02], [She06].

Recall

Definition 2.4. For a limit cardinal μ, the RGCH_μ (that is, the *revised general continuum hypotheses* for μ) <u>means</u>: for every cardinal $\lambda > \mu$ and every large enough regular cardinal $\kappa < \mu$, we have:

(a) there is a family of $\leq \lambda$ subsets of λ of cardinality $< \mu$ such that any other one is included in the union of $< \kappa$ of them, equivalently:
(b) if $\mathfrak{a}$ is a set pf $< \mu$ regular cardinals from the interval (μ, λ) then

$$\sup\left(\mathrm{pcf}_{\kappa\text{-complete}}(\mathfrak{a})\right) \leq \lambda.$$

Question 2.5.

(A) Is RGCH *the right formulation of Hilbert first problem considering what the well known consistency results telling us we cannot prove.*
(B) Is it the time for independence results or for ZFC?

I believed the answer to part (A) is yes, and still do; probably being a missionary is not my calling, but we all can hope that history will eventually justify us.

Of course, there are many more questions; some of them may be crucial in various applications, in particular see Problem 7.2 below.

Question 2.6. *(Tanmay Inamdar)*

(A) Is there any evidence that suggests that something like this may be true?
(B) Another question is what would be the right analogue of 'revised power' in this case?

<u>**Answer**</u> **2.7.** For the second question, yes, let me stress: the definition (at least in the present version) of the RGCH_μ is such that it make sense for any limit cardinal. For strong limit cardinals we can replace "included in the union of $< \kappa$ many members" by "is equal to such a union".

About the first question, well, a real supporting evidence would be a proof (or disproof for the negations). But:

(A) On the positive side, Gitik independence proofs are far from this and I had conjecture this and not the strong conjecture; see [She00a].

(B) On the negative side, I have tried quite hard to prove it, failing. I did not succeed to prove even "the cardinality of $\mathrm{pcf}(\mathfrak{a})$ is smaller than the first weakly inaccessible above the cardinality of $\mathfrak{a}$".

(C) Looking at analog situation it take not few years to move from

$$\text{``}\aleph_\delta = (\aleph_\omega)^{\aleph_0} \Rightarrow \delta < (2^{\aleph_0})^+\text{''} \quad \text{to} \quad \text{``}(\aleph_\omega)^{\aleph_0} < (2^{\aleph_0})^+ + \aleph_{\omega_4}\text{''}$$

§ 3. With weak choice

Thesis 3.1. We all, (at least I) do know that the axiom of choice is true.

So why bother investigating set theory with restricted choice? Well, first I believe ideology and/or good taste should never stop you from proving a good theorem.

But more to the point, while the existential quantifier is great and true interpretation, a constructive existential proof is better; and weakening the axiom of choice is a major way to express this problem. I do not know how to advance on some old question; e.g:

Problem 3.2.

(A) Is Quine set theory consistent?
(B) Does ZF + "if there is a function from a non empty A onto B then there is a one to one function from B into A" implies the axiom of choice?

Also I do not know enough on the quite new "assuming choice fail only for large enough cardinals, e.g. if Reinhardt elementary embedding of $\mathbf{V}$ into itself".

I had been interested in trying to develop pcf theory in ZF or ZF + DC. The results have not been so satisfying.

But this lead me to Ax_4 assuming ZF + DC.

This axiom can be described as dual to ZF + DC + AD: that approach was closely related to $\mathbf{L}[\mathbb{R}]$ for which we have full choice <u>except</u> that we cannot well order the set of reals.

While

Definition 3.3. Ax_4 tell you that we can well ordered each $[\lambda]^{\aleph_0}$.

So consider the set theory ZF + DC + Ax_4.

A prototype for such universe is the Easton model where starting with $\mathbf{V}$ satisfying GCH we for each regular uncountable cardinal λ, blew up it's power set $\mathscr{P}(\lambda)$ without putting a well ordering of it.

Thesis 3.4. This set theory is not as dull as ZF; in fact I think is not so far from ZFC; we know it implies:

(A) There is a class of successor regular cardinals, essentially successor of singular are regular;
(B) While $\mathscr{P}(\lambda)$ is chaotic, if $\lambda \gg \kappa = \mathrm{cf}(\kappa) > \aleph_0$ then $[\lambda]^\kappa$ can be decomposed to "few" well ordered sets, in fact uniformly, (let me stress their number depend just on κ);
(C) We can generalized the "pcf existence of a generating sequence theorem"; now on the one hand we have to use bigger lower bound to $\min(\mathfrak{a})$ and a cardinal may appear twice in the spectrum <u>but</u> on the other hand the scales are defined canonically.

See [She97], [She12b], [She11a], [LS09], and [She16].

Not a great success at proselytizing.

Related to this is:

Problem 3.5. Classify definable forcing notions by the consistency strength of each of the following (pedantically the pair $(\mathbb{Q}, \underset{\sim}{\eta})$: where $\underset{\sim}{\eta}$ being a $\mathbb{Q}$-name of a real.

(A) ZF + DC + every set is equivalent to a Borel set modulo the ideal related to the forcing $\mathbb{Q}$ and $\underset{\sim}{\eta}$.
(B) Similarly replacing $\tilde{\mathrm{DC}}$ by $\mathrm{AC}_{\aleph_0}$; have to be careful defining the ideal even of the random real forcing.
(C) Similarly omitting DC altogether.

See e.g. my works with Haim Horowitz and references there.

§ 4. Set theory of the reals: Polish set theory

There are many dichotomies of the form:

- $(*)_1$ if the relevant cardinal is uncountable then it is the continuum (e.g. the cardinality of a closed set of reals, $|\mathbb{R}/E|$ where E is a Π^1_1-equivalence relation. Also, as above for Σ^1_1; if E remain an equivalence relation after adding a Cohen real, even Π^1_2), or:
- $(*)_2$ if the relevant cardinal is $\leq \aleph_1$ then it is the continuum.

We shall assume that the continuum is large enough among the alephs (not like e.g. "real valued measurable") and maybe MA, to clean the air.

Long ago I have anticipate that: if I prove

- $(*)_1$ $(\aleph_\omega, \aleph_0) \to (2^{\aleph_0}, \aleph_0)$,

then clearly I will shortly be able to prove:

- $(*)_2$ if $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$ has a model of cardinality $\aleph_{\omega_1}$ then it has a model of cardinality continuum, $2^{\aleph_0}$.

Note that $\aleph_{\omega_1}$ is a lower bound.

Alas, while $(*)_1$ had been proved , concerning $(*)_2$, more than fifty years later I have not seen the light. What has been done is pointing out a cardinal which control some interesting phenomena.

Definition 4.1. Let $\lambda(\aleph_0)$ be the first cardinal λ such that, if $\psi \in \mathbb{L}_{\aleph_1,\aleph_0}$ has a model of cardinality λ, then it has a model of cardinality continuum, $2^{\aleph_0}$.

If we ask about existence of large square in Borel subsets of $\mathbb{R} \times \mathbb{R}$ this is analysed in [522] and under our assumptions this cardinal is the right bound.

Recent works deal with giving a Borel set of reals, what is the maximum number of translates with pairwise non trivial intersection? see [RS18], [RSa], [RSb], [RSc] and in preparation [S$^+$a].

Problem 4.2. Is $\lambda(\aleph_0) = \aleph_{\omega_1}$ (recall we are assuming the continuum is larger)?

§ 5. Set theory of the reals: forcing

In this context I am particularly interested in preservation theorems which naturally lead to forcing axioms; though of course many problems require specific forcing.

Thesis 5.1. I think that the parallel to general topology/algebraic topology/the topology of $\mathbb{R}^n$/examples, in this context is:

(a) general iteration theorems like properness,
(b) reasonably definable forcing Suslin, nep,
(c) creature forcing,
(d) specific forcing.

While many properties are preserved for iterating proper forcing notions with CS (that is, countable support), two glaring omissions are no new real and adding no Cohen. In both cases we do not have preservation in the limit stage.

While still much had been said in the case of no new real, for no Cohen the picture is opaque.

Thesis 5.2. While the case of the continuum being $\aleph_2$ was easiest for forcing, and we know much about forcing preserving the continuum being $\aleph_1$, the situation for larger values of the continuum seem opaque.

If you truly believe that the continuum is $\aleph_1$ and/or $\aleph_2$ this is not a problem,

However while I know we cannot determine the values of the continuum, it seem to me that:

Thesis 5.3.

(A) The value of the continuum naturally/probably much larger than $\aleph_2$.
(B) A strong argument for me is that if the continuum is at most $\aleph_2$, then in e.g. Cichoń's diagram there are many relations between those cardinal invariants, which are artificial as proved recently by "Cichoń's Maximum".

See [GKS19], [KLS19], [GKMS21a], [RS87], and [GKMS21b] and references there. There are other such results.

We have various additional ways to classify pairs $(\mathbb{Q}, \underset{\sim}{\eta})$ where $\mathbb{Q}$ is a definition of a forcing notion and $\underset{\sim}{\eta}$ is a $\mathbb{Q}$-name of a reals.

Problem 5.4. Classify them by being sweet/sour/saccharinity.

See on this [DS03], [RS06], [KS11], and [HS].

Problem 5.5. Classify such pairs by asking is there an $\aleph_1$-complete ideal I on a set λ such that the Boolean algebra $\mathscr{P}(A)/I$ is isomorphic to the ideal on $\mathbb{R}$ which the pair $(\mathbb{Q}, \underset{\sim}{\eta})$ define on $\mathbb{R}$.

See [GS99], [GS93], [GS01] and [KS17] and references there.

§ 6. Combinatoric for cardinal above the continuum

The old Hungarian school list of problems (see [EH71] and [EH74]) have strong influence in particular on my work; originally I have started to be interested for the applications in model theory. Komjath has lately update it.

In this tradition I have thought:

Thesis 6.1.

(A) It is worthwhile to prove consistency of strong partition theorems in particular between μ and 2^μ.
(B) This include relatives of partition theorems true for large cardinals.

See [She81], [SS01], [She12a], [She], and references there.

Such a problem is:

Problem 6.2. Prove consistency of a partition theorem improving the classical Hajnal - Juhasz general topology result $|X| \leq 2^{(2^{s(X)})}$.

Problem 6.3. Can we generalize the dichotomies in $(*)_1, (*)_2$ from §4 replacing $\aleph_0$ by $\kappa > \aleph_0$.

Tries in this direction are: [MSS83], [MSS84], [GS89], [GS98b], [HS01], [She01], [SV02b], [SV02a], [She04], [She11b] and references there.

§ 7. Black boxes

Having started in model theory, I have tried to build many non-isomorphic models of suitable theories; this lead me to constructing principles. Having learn later on the earlier Jensen diamond, I considered it wonderful but ‘‘expensive”, that is not provable in ZFC. I have aimed at getting weaker guessing principles which are cheap provable in ZFC; when they do not speak of subsets of δ for $\delta \in S$ (as is the case of weak diamond [DS78]) I have been calling them black boxes.

Thesis 7.1. It is good to find black boxes = guessing principle provable to exist in ZFC.

See [She74], [She84a], [She84b], [She22], [She05], [She13], and the books [EM02] and [GT06].

I have dream about:

Problem 7.2. Prove that there are unboundedly many regular cardinals λ such that for some $\kappa < \lambda$, (of particular interest is $\kappa = \aleph_0$) and a sequence $\langle M_\delta : \delta < \lambda, \mathrm{cf}(\delta) = \kappa \rangle$ such that:

- M_δ is a τ_δ-model with universe an unbounded subset of λ,
- for every model M with universe λ and vocabulary $\tau \subseteq \mathscr{H}(\kappa)$ for stationarily many δ we have M_δ is an elementary sub-model of M.

If τ consist of predicates only this is provable (for many κ-s, but alas, not for $\aleph_0$ and not for $\aleph_1$.)

Thesis 7.3. For many problems on Abelian groups (and modules, and groups) we need n-dimensional black boxes

On existence theorems and some applications see [She07] for $\aleph_n$-free ones and [She20] for $ha_{\omega_1 \times n}$-free ones and some applications, they are applied in [GHS14], [GSS13], [DHS20] and hopefully (in preparation) for suitable ring R on having an R-module M such that every endomorphism of M as an Abelian groups is multiplication by some $r \in R$. The reason is that the analysis of quite free such algebraic structures lead to failure of n-amalgamation.

§ 8. FORCING ABOVE THE CONTINUUM

There is much to be said for and about the family of Prikry like forcing notion, but better not my me.

Even excluding this there has been much work and I will concentrate on some directions close to my heart.

Problem 8.1. Find consistent forcing axioms with $\aleph_0$ replaced by a strong limit singular μ.

There are some works in this direction, using on the one hand well established forcing to change the cofinality of a large cardinal (and possibly collapsing μ to become e.g. $\beth_\delta$ where δ may be even ω) and, on the other hand, have a preliminary forcing which establish a forcing notion on μ, "when it was a regular large cardinal". The point is that something of the forcing axiom is preserved after in the final model.

See

- $(*)_1$ [MS89] on getting uniformization properties;
- $(*)_2$ [GS98a] on density of box product and P-filters,
- $(*)_3$ [DS03] on getting the existence a graph of cardinality $\mu^{++} < 2^\mu$ universal for graphs of cardinality μ^+ for μ of cofinality $\aleph_0$;
- $(*)_4$ [CDM$^+$17] getting more in particular the cofinality is larger;
- $(*)_5$ [PSb], [PSa] on getting a universal graphs in $\mu^+ < 2^\mu$ or some quite arbitrary $\lambda \in (\mu, 2^\mu)$.

Still non of this look like a satisfying forcing axiom. Maybe we need to mix the preparatory forcing with a sophisticated Radin like forcing.

Another direction is:

Problem 8.2. Force a forcing axiom consistent with GCH, applying to all regular cardinals.

See for an attempt in [GSar] and hopefully (in preparation) [S$^+$b]. A major idea there is that for regular λ, forcing notions which are S-complete for $S = S_\lambda = \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$ are problematic, just demanding this for some S, a stationary subset of λ, (usually of S_λ), is "soft". So we just demand the axiom saying that for every suitable task related to λ there is a stationary sub-set for which this holds. This is sufficient for existence for many problems.

I expect this will give a universe where the answer to many problem in set theoretical homology separating many properties which coincide if $\mathbf{V} = \mathbf{L}$; the one partially written deal with some uniformization and proving having $\mathrm{Ext}(G, \mathbb{Z}) = 0$ may fail compactness in singular strong limit.

Problem 8.3. Develop forcing and iterated forcing methods preserving $\lambda = \lambda^{<\lambda}$ at least for inaccessible cardinals which are bounding, that is, every new $f \in {}^{\lambda}\lambda$ is bound by an "old" such function.

For long I have "known" that for regular uncountable $\lambda = \lambda^{<\lambda} > \aleph_0$ there is a bounding $(< \lambda)$-complete λ^+-c.c. forcing notion not adding a λ-Cohen $\eta \in {}^{\lambda}2$.

This unlike the case $\lambda = \aleph_0$ were random real forcing is such a forcing; but this does not generalize because there is no generalization of Lebesgue measure with the ideal of null being $(< \lambda)$-complete.

After explaining this, quite convincingly by my judgment I have go on and prove the existence.

This is done in [She17] for a weakly compact cardinal λ; the idea is that we define by induction on inaccessible $\kappa < \lambda$ a forcing notion $\mathbb{Q}_\kappa$ which try to be very Cohen, so at first glance this is contrary to our intention. This mean that the restriction of the generic $\underset{\sim}{\eta}$ to κ belong to many dense subsets of ${}^{\kappa}2$. This is continued in [CS19] for every inaccessible assuming a mild condition easily forced. This will hopefully be continued in a work in preparation [S⁺c] where we move to having a forcing be more like iteration. There is more to be said on generalizing Cichoń's diagram in this context.

I believe this is the beginning of a theory of iterating such bounding forcing notions.

References

[CDM+17] James Cummings, Mirna Džamonja, Menachem Magidor, Charles Morgan, and Saharon Shelah, *A framework for forcing constructions at successors of singular cardinals*, Trans. Amer. Math. Soc. **369** (2017), no. 10, 7405–7441, arXiv: 1403.6795. MR 3683113

[CS19] Shani Cohen and Saharon Shelah, *Generalizing random real forcing for inaccessible cardinals*, Israel J. Math. **234** (2019), no. 2, 547–580, arXiv: 1603.08362. MR 4040837

[DHS20] Manfred H. Dugas, Daniel Herden, and Saharon Shelah, *$\aleph_k$-free cogenerators*, Rend. Semin. Mat. Univ. Padova **144** (2020), 87–104, arXiv: 1909.00595. MR 4186448

[DS78] Keith J. Devlin and Saharon Shelah, *A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, Israel J. Math. **29** (1978), no. 2-3, 239–247. MR 0469756

[DS03] Mirna Džamonja and Saharon Shelah, *Universal graphs at the successor of a singular cardinal*, J. Symbolic Logic **68** (2003), no. 2, 366–388, arXiv: math/0102043. MR 1976583

[EH71] Paul Erdős and Andras Hajnal, *Unsolved problems in set theory*, Axiomatic Set Theory (Providence, R.I.), Proc. of Symp. in Pure Math., vol. XIII Part I, AMS, 1971, pp. 17–48.

[EH74] ______, *Solved and unsolved problems in set theory*, Proc. of the Symp. in honor of Tarksi's seventieth birthday in Berkeley 1971 (Leon Henkin, ed.), Proc. Symp in Pure Math., vol. XXV, 1974, pp. 269–287.

[EM02] Paul C. Eklof and Alan Mekler, *Almost free modules: Set theoretic methods*, North–Holland Mathematical Library, vol. 65, North–Holland Publishing Co., Amsterdam, 2002, Revised Edition.

[GHS14] Rüdiger Göbel, Daniel Herden, and Saharon Shelah, *Prescribing endomorphism algebras of $\aleph_n$-free modules*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 9, 1775–1816. MR 3273308

[GKMS21a] Martin Goldstern, Jakob Kellner, Diego A. Mejía, and Saharon Shelah, *Controlling cardinal characteristics without adding reals*, J. Math. Log. **21** (2021), no. 3, Paper No. 2150018, 29, arXiv: 2006.09826. MR 4330526

[GKMS21b] ______, *Preservation of splitting families and cardinal characteristics of the continuum*, Israel J. Math. **246** (2021), no. 1, 73–129, arXiv: 2007.13500. MR 4358274

[GKS19] Martin Goldstern, Jakob Kellner, and Saharon Shelah, *Cichoń's maximum*, Ann. of Math. (2) **190** (2019), no. 1, 113–143, arXiv: 1708.03691. MR 3990602

[GS89] Rami P. Grossberg and Saharon Shelah, *On the structure of* $\mathrm{Ext}_p(G, \mathbf{Z})$, J. Algebra **121** (1989), no. 1, 117–128. MR 992319

[GS93] Moti Gitik and Saharon Shelah, *More on simple forcing notions and forcings with ideals*, Ann. Pure Appl. Logic **59** (1993), no. 3, 219–238. MR 1213273

[GS98a] ______, *On densities of box products*, Topology Appl. **88** (1998), no. 3, 219–237, arXiv: math/9603206. MR 1632081

[GS98b] Rami P. Grossberg and Saharon Shelah, *On cardinalities in quotients of inverse limits of groups*, Math. Japon. **47** (1998), no. 2, 189–197, arXiv: math/9911225. MR 1615081

[GS99] Moti Gitik and Saharon Shelah, *Cardinal preserving ideals*, J. Symbolic Logic **64** (1999), no. 4, 1527–1551, arXiv: math/9605234. MR 1780068

[GS01] ______, *More on real-valued measurable cardinals and forcing with ideals*, Israel J. Math. **124** (2001), 221–242, arXiv: math/9507208. MR 1856516

[GSar] Noam Greenberg and Saharon Shelah, *Many forcing axioms for all regular uncountable cardinals*, Israel J. Math. (to appear), arXiv: 2107.05755.

[GSS13] Rüdiger Göbel, Saharon Shelah, and Lutz H. Strüngmann, *$\aleph_n$-free modules over complete discrete valuation domains with almost trivial dual*, Glasg. Math. J. **55** (2013), no. 2, 369–380. MR 3040868

[GT06] Rüdiger Göbel and Jan Trlifaj, *Approximations and endomorphism algebras of modules*, de Gruyter Expositions in Mathematics, vol. 41, Walter de Gruyter, Berlin, 2006.

[HS] Haim Horowitz and Saharon Shelah, *Saccharinity with ccc*, arXiv: 1610.02706.

[HS01] Aapo Halko and Saharon Shelah, *On strong measure zero subsets of* ${}^{\kappa}2$, Fund. Math. **170** (2001), no. 3, 219–229, arXiv: math/9710218. MR 1880900

[KLS19] Jakob Kellner, Anda Latif, and Saharon Shelah, *Another ordering of the ten cardinal characteristics in Cichoń's diagram*, Comment. Math. Univ. Carolin. **60** (2019), no. 1, 61–95, arXiv: 1712.00778. MR 3946665

[KS11] Jakob Kellner and Saharon Shelah, *Saccharinity*, J. Symbolic Logic **76** (2011), no. 4, 1153–1183, arXiv: math/0511330. MR 2895391

[KS17] Ashutosh Kumar and Saharon Shelah, *A transversal of full outer measure*, Adv. Math. **321** (2017), 475–485. MR 3715717

[LS09] Paul B. Larson and Saharon Shelah, *Splitting stationary sets from weak forms of choice*, MLQ Math. Log. Q. **55** (2009), no. 3, 299–306, arXiv: 1003.2477. MR 2519245

[MS89] Alan H. Mekler and Saharon Shelah, *Uniformization principles*, J. Symbolic Logic **54** (1989), no. 2, 441–459. MR 997878

[MSS83] Menachem Magidor, Saharon Shelah, and Jonathan Stavi, *On the standard part of nonstandard models of set theory*, J. Symbolic Logic **48** (1983), no. 1, 33–38. MR 693245

[MSS84] ______, *Countably decomposable admissible sets*, Ann. Pure Appl. Logic **26** (1984), no. 3, 287–361. MR 747687

[Nun77] R.J. Nunke, *Whitehead's problem*, Abelian group theory (Proc. Second New Mexico State Univ. Conf., Lecture Notes in Math., vol. 616, Springer, Berlin, 1977, pp. 240–250.

[PSa] Márk Poór and Saharon Shelah, *Universal graphs at the successors of small singulars*.

[PSb] ______, *Universal graphs between a strong limit singular and its power*, arXiv: 2201.00741.

[RSa] Andrzej Rosłanowski and Saharon Shelah, *Borel sets without perfectly many overlapping translations II*, arXiv: 1909.00937.

[RSb] ______, *Borel sets without perfectly many overlapping translations, III*, arXiv: 2009.03471.

[RSc] ______, *Borel sets without perfectly many overlapping translations IV*, arXiv: 2302.12964.

[RS87] Matatyahu Rubin and Saharon Shelah, *Combinatorial problems on trees: partitions, Δ-systems and large free subtrees*, Ann. Pure Appl. Logic **33** (1987), no. 1, 43–81. MR 870686

[RS06] Andrzej Rosłanowski and Saharon Shelah, *How much sweetness is there in the universe?*, MLQ Math. Log. Q. **52** (2006), no. 1, 71–86, arXiv: math/0406612. MR 2195002

[RS18] ______, *Small-large subgroups of the reals*, Math. Slovaca **68** (2018), no. 3, 473–484, arXiv: 1605.02261. MR 3805955

[S^+a] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F1927.

[S^+b] ______, *Tba*, In preparation. Preliminary number: Sh:F2070.

[S^+c] ______, *Tba*, In preparation. Preliminary number: Sh:F2261.

[She] Saharon Shelah, *Partition theorems for expanded trees*, arXiv: 2108.13955.

[She74] ______, *Categoricity of uncountable theories*, Proceedings of the Tarski Symposium, Proc. Sympos. Pure Math., vol. XXV, Amer. Math. Soc., Providence, R.I., 1974, pp. 187–203. MR 0373874

[She81] ______, *$\aleph_\omega$ may have a strong partition relation*, Israel J. Math. **38** (1981), no. 4, 283–288. MR 617675

[She84a] ______, *A combinatorial principle and endomorphism rings. I. On p-groups*, Israel J. Math. **49** (1984), no. 1-3, 239–257. MR 788269

[She84b] ______, *A combinatorial theorem and endomorphism rings of abelian groups. II*, Abelian groups and modules (Udine, 1984), CISM Courses and Lect., vol. 287, Springer, Vienna, 1984, pp. 37–86. MR 789808

[She97] ______, *Set theory without choice: not everything on cofinality is possible*, Arch. Math. Logic **36** (1997), no. 2, 81–125, arXiv: math/9512227. MR 1462202

[She00a] ______, *On what I do not understand (and have something to say). I*, Fund. Math. **166** (2000), no. 1-2, 1–82, arXiv: math/9906113. MR 1804704

[She00b] ______, *The generalized continuum hypothesis revisited*, Israel J. Math. **116** (2000), 285–321, arXiv: math/9809200. MR 1759410

[She01] ______, *Strong dichotomy of cardinality*, Results Math. **39** (2001), no. 1-2, 131–154, arXiv: math/9807183. MR 1817405

[She02] ______, *PCF and infinite free subsets in an algebra*, Arch. Math. Logic **41** (2002), no. 4, 321–359, arXiv: math/9807177. MR 1906504

[She04] ______, *On nice equivalence relations on ${}^\lambda 2$*, Arch. Math. Logic **43** (2004), no. 1, 31–64, arXiv: math/0009064. MR 2036248

[She05] ______, *Super black box (ex. Middle diamond)*, Arch. Math. Logic **44** (2005), no. 5, 527–560, arXiv: math/0212249. MR 2210145

[She06] ______, *More on the revised GCH and the black box*, Ann. Pure Appl. Logic **140** (2006), no. 1-3, 133–160, arXiv: math/0406482. MR 2224056

[She07] ———, *$\aleph_n$-free abelian group with no non-zero homomorphism to $\mathbb{Z}$*, Cubo **9** (2007), no. 2, 59–79, arXiv: math/0609634. MR 2354353

[She11a] ———, *MAD saturated families and SANE player*, Canad. J. Math. **63** (2011), no. 6, 1416–1435, arXiv: 0904.0816. MR 2894445

[She11b] ———, *Polish algebras, shy from freedom*, Israel J. Math. **181** (2011), 477–507, arXiv: math/0212250. MR 2773054

[She12a] ———, *Many partition relations below density*, Israel J. Math. **191** (2012), no. 2, 507–543, arXiv: 0902.0440. MR 3011486

[She12b] ———, *PCF arithmetic without and with choice*, Israel J. Math. **191** (2012), no. 1, 1–40, arXiv: 0905.3021. MR 2970861

[She13] ———, *Pcf and abelian groups*, Forum Math. **25** (2013), no. 5, 967–1038, arXiv: 0710.0157. MR 3100959

[She16] ———, *ZF + DC + AX_4*, Arch. Math. Logic **55** (2016), no. 1-2, 239–294, arXiv: 1411.7164. MR 3453586

[She17] ———, *A parallel to the null ideal for inaccessible λ: Part I*, Arch. Math. Logic **56** (2017), no. 3-4, 319–383, arXiv: 1202.5799. MR 3633799

[She20] ———, *Quite free complicated Abelian groups, pcf and black boxes*, Israel J. Math. **240** (2020), no. 1, 1–64, arXiv: 1404.2775. MR 4193126

[She22] ———, *Black Boxes*, Annales Univ. Sci. Budapest., Sec. Math. **LXV** (2022), 69–130, arXiv: 0812.0656 Ch. IV of The Non-Structure Theory" book [Sh:e].

[SS01] Saharon Shelah and Lee J. Stanley, *Forcing many positive polarized partition relations between a cardinal and its powerset*, J. Symbolic Logic **66** (2001), no. 3, 1359–1370, arXiv: math/9710216. MR 1856747

[SV02a] Saharon Shelah and Pauli Väisänen, *On equivalence relations second order definable over $H(\kappa)$*, Fund. Math. **174** (2002), no. 1, 1–21, arXiv: math/9911231. MR 1925484

[SV02b] ———, *The number of $L_{\infty\kappa}$-equivalent nonisomorphic models for κ weakly compact*, Fund. Math. **174** (2002), no. 2, 97–126, arXiv: math/9911232. MR 1927234

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, 9190401, Jerusalem, Israel; and, Department of Mathematics, Rutgers University, Piscataway, NJ 08854-8019, USA

URL: https://shelah.logic.at/