CO-HOPFIAN AND BOUNDEDLY ENDO-RIGID MIXED ABELIAN GROUPS

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Abstract. For a given cardinal λ and a torsion abelian group K of cardinality less than λ, we present, under some mild conditions (for example $\lambda = \lambda^{\aleph_0}$), boundedly endo-rigid abelian group G of cardinality λ with $\mathrm{tor}(G) = K$. Essentially, we give a complete characterization of such pairs (K, λ). Among other things, we use a twofold version of the black box. We present an application of the construction of boundedly endo-rigid abelian groups. Namely, we turn to the existence problem of co-Hopfian abelian groups of a given size, and present some new classes of them, mainly in the case of mixed abelian groups. In particular, we give useful criteria to detect when a boundedly endo-rigid abelian group is co-Hopfian and completely determine cardinals $\lambda > 2^{\aleph_0}$ for which there is a co-Hopfian abelian group of size λ.

Contents

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§ 1. Introduction

By a torsion (resp. torsion-free) group we mean an abelian group such that all its non-zero elements are of finite (resp. infinite) order. A mixed group G contains both non-zero elements of finite order and elements of infinite order, and these are connected via the celebrated short exact sequence

$$(*) \qquad 0 \longrightarrow \operatorname{tor}(G) \longrightarrow G \longrightarrow \frac{G}{\operatorname{tor}(G)} \longrightarrow 0.$$

Despite the importances of $(*)$, there are series of questions concerning how to glue the issues from torsion and torsion-free parts and put them together to check the desired properties for mixed groups.

Reinhold Baer was interested to find an interplay between abelian groups and rings, see [2] and [3]. In this regard, he raised the following general problem:

Problem 1.1. Which rings can be the endomorphism ring of a given abelian group G?

There are a lot of interesting research papers and books that study this problem, see for example the books [12] and [18]. According to the recent book of Fuchs [16], for mixed groups, only very little can be said. As an achievement, we cite the works of Corner-Göbel [8] and Franzen-Goldsmith [13].

For any group G, by $E_f(G)$ we mean the ideal of $\operatorname{End}(G)$ consisting of all elements of $\operatorname{End}(G)$ whose image is finitely-generated. In [9], Corner has constructed an abelian group $G := (M, +)$, for some ring R and an R-module M, such that any of its endomorphisms is of the form multiplication by some $r \in R$ plus a distinguished function from $E_f(G)$. One can allow such a distinguished function ranges over other classes such as finite-range, countable-range, inessential range or even small homomorphism, and there are a lot of work trying to clarify such situations. As a short list, we may mention the papers Corner-Göbel [8], Dugas-Göbel [11], Corner [9], Thome [35] and Pierce [20].

Here, by a bounded group, we mean a group G such that $nG = 0$ for some fixed $0 < n \in \mathbb{N}$. By a theorem of Baer and Prüfer a bounded group is a direct sum of cyclic groups. The converse is not true. However, there is a partial converse for countable p-groups. For more details see the book of Fuchs [16]. A homomorphism $h \in G_1 \to G_2$ of abelian groups is called bounded if $\mathrm{Rang}(h)$ is bounded.

Definition 1.2. An abelian group G is *boundedly rigid* when every endomorphism of it has the form $\mu_n + h$, where μ_n is multiplication by $n \in \mathbb{Z}$ and h has bounded range. By $\mathrm{E_b}(G)$ we mean the ideal of $\mathrm{End}(G)$ consisting of all elements of $\mathrm{End}(G)$ whose image is bounded.

Let us explain some motivation. The concept of a rigid system of torsion-free groups has a natural analogue for the class of separable p-primary groups: a family $\{G_i : i \in I\}$ of separable p-primary groups is called rigid-like if for all $i \neq j \in I$ every homomorphism $G_i \to G_j$ is small, and also for all $i \in I$, every endomorphism of G_i is the sum of a small endomorphism and multiplication by a p-adic integer. In his paper [27], Shelah confirmed a conjecture of Pierce [20] by showing that if μ is an uncountable strong limit cardinal, then there is a rigid-like system $\{G_i : i \in I\}$ of separable p-primary groups such that $|G_i| = \mu$ and $|I| = 2^\mu$, see also [25] for more results in this direction.

Let us now turn to the paper and state our main results. Section 2 contains the preliminaries, basic definitions and notations that we need. The reader may skip it, and come back to it when needed later. In Section 3, and as a main result, we prove the following.

Theorem 1.3. *Given a cardinal λ such that $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$ and a torsion group K of cardinality less than λ, there is a boundedly rigid abelian group G of cardinality λ with* $\mathrm{tor}(G) = K$.

To prove this, we introduce a series of definitions and present several claims. The first one is the rigidity context, denoted by $\mathbf{k}$, see Definition 3.1. Also, the main technical tool is a variation of "*Shelah's black box*", and we refer to it as *twofold black box*. For its definition (resp. its existence), see Definition 3.13 (resp.

Lemma 3.15). It may be worth to mention that the black boxes were introduced by Shelah in [31], where he showed that they follow from ZFC (here, ZFC means the Zermelo–Fraenkel set theory with the axiom of choice). We can consider black boxes as a general method to generate a class of diamond-like principles provable in ZFC. Then, we continue by introducing the approximation blocks, denoted by AP, more precisely, see Definition 3.18. There is a distinguished object $\mathbf{c}$ in AP that we call it full. The twofold black box, helps us to find such distinguished objects, see Lemma 3.30. Here, one may define the group $G := G_{\mathbf{c}}$. Let $h \in \mathrm{End}(G)$. In order to show h is boundedly rigid, we apply a couple of reductions (see Lemmas 3.35–3.43), to reduce to the case that h factors throughout $G \to \mathrm{tor}(G)$. Finally, in Lemma 3.31 we handle this case, by showing that any map $G \to \mathrm{tor}(G)$ is indeed boundedly rigid.

In the course of the proof of Theorem 1.3, we develop a general method that allows us to prove $0 \to \mathbb{Z} \to \mathrm{End}(G) \to \frac{\mathrm{End}(G)}{\mathrm{E_b}(G)} \to 0$ is exact, and also enables us to present a connection to Problem 1.1. In order to display the connection, let R be a ring coming from the rigidity context. For the propose of the introduction, we may assume that $(R, +)$ is cotorsion-free, see Definition 2.8 (with the convenience that the argument becomes easier if we work with $R := \mathbb{Z}$, or even $(R, +)$ is $\aleph_1$-free). Following our construction, every endomorphism of G has the form $\mu_r + h$, where μ_r is multiplication by $r \in R$ and h has bounded range, i.e., the sequence

$$0 \longrightarrow R \longrightarrow \mathrm{End}(G) \longrightarrow \frac{\mathrm{End}(G)}{\mathrm{E_b}(G)} \longrightarrow 0$$

is exact.

Definition 1.4. A group G is called *Hopfian* (resp. *co-Hopfian*) if its surjective (resp. injective) endomorphisms are automorphisms.

Essentially, we give complete characterization of the pairs (K, λ) by relating our work with the recent works of Paolini and Shelah, see [22], [23] and [24]. To this end, first we recall the following folklore problem:

Problem 1.5. Construct co-Hopfian groups of a given size.

Baer [4] was the first to investigate Problem 1.5 for abelian groups. A torsion-free abelian group is co-Hopfian if and only if it is divisible of finite rank, hence the problem naturally reduces to the torsion and mixed cases. In their important paper [5], Beaumont and Pierce proved that if G is co-Hopfian, then $\mathrm{tor}(G)$ is of size at most continuum, and further that G cannot be a p-groups of size $\aleph_0$. This naturally left open the problem of the existence of co-Hopfian p-groups of uncountable size $\leq 2^{\aleph_0}$, which was later solved by Crawley [7] who proved that there exist co-Hopfian p-groups of size $2^{\aleph_0}$. Braun and Strüngmann [6] showed that the existence of three types of infinite abelian p-groups of size $\aleph_0 < |G| < 2^{\aleph_0}$ are independent of ZFC:

(a) both Hopfian and co-Hopfian,

(b) Hopfian but not co-Hopfian,

(c) co-Hopfian but not Hopfian.

Also, they proved that the above three types of groups of size $2^{\aleph_0}$ exist in ZFC. So, in the light of Theorem 1.3, the remaining part is $2^{\aleph_0} < \lambda < \lambda^{\aleph_0}$. Very recently, and among other things, Paolini and Shelah [23] proved that there is no co-Hopfian group of size λ for such a λ. As an application, in Section 4, we determine cardinals $\lambda > 2^{\aleph_0}$ for which there is a co-Hopfian group of size λ. For the precise statement, see Corollary 4.13.

Let us recall a connection between the concepts boundedly endo-rigid groups and (co-)Hopfian groups. First, recall from the seminal paper [26], for any λ less than the first beautiful cardinal, Shelah proved that there is an endo-rigid torsion-free group of cardinality λ. By definition, for any $f \in \mathrm{End}(G)$ there is $m_f \in \mathbb{Z}$ such that $f(x) = m_f x$. So, f is onto iff $m_f = \pm 1$. In other words, G is Hopfian. This naturally motives us to detect co-Hopfian property by the help of some boundedly endo-rigid groups. This is what we want to do in §4. Namely, our first result on co-Hopfian groups is stated as follows:

Construction 1.6. Let $K := \oplus\{\frac{\mathbb{Z}}{p^n\mathbb{Z}} : p \in \mathbb{P} \text{ and } 1 \leq n < m\}$, where $m < \omega$, and $\mathbb{P}$ is the set of prime numbers. Let G be a boundedly endo-rigid abelian group such that $\mathrm{tor}(G) = K$. Then G is co-Hopfian.

W may recall from Theorem 1.3 that such a group exists for any $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$. In fact, the size of G is λ.

Let h be a natural number. One of the tools that we use is the h-power torsion subgroup of G:

$$\Gamma_h(G) := \{g \in G : \exists n \in \mathbb{N} \textit{ such that } h^n g = 0\}.$$

The assignment $G \mapsto \Gamma_h(G)$ defines a functor from the category of abelian groups to itself. It may be worth to mention that, in the style of Grothendieck, this is called section functor and some authors use $\mathrm{Tor}_h(-)$ to denote it.

In our study of the co-Hopfian property of G, the following subset of prime numbers appears:

$$S_G := \{p \in \mathbb{P} : G/\Gamma_p(G) \textit{ is not } p\textit{-divisible}\}.$$

The set S_G helps us to present a useful criteria to detect when a boundedly endo-rigid abelian group is co-Hopfian:

Proposition 1.7. *Assume $\lambda > 2^{\aleph_0}$ and G is a boundedly endo-rigid abelian group of size λ. Then G is co-Hopfian if and only if:*

(a): *S_G is a non-empty set of primes,*

(b): (b_1) $\Gamma_p(G) \neq G$,

(b_2) *if $p \in S_G$, then $\Gamma_p(G)$ is not bounded,*

(b_3) *if $\Gamma_p(G)$ is bounded, then it is finite.*

Let G be an abelian group. In order to show that G is (not) co-Hopfian, and also to see a connection to bounded morphisms, we introduce a useful set $\mathrm{NQr}_{(m,n)}(\mathrm{G})$ consisting of those bounded $h \in \mathrm{End}(\Gamma_n(G))$ such that:

(1) $h' := m \cdot \mathrm{id}_{\Gamma_n(G)} + h \in \mathrm{End}(\Gamma_n(G))$ is 1-to-1,

(2) h' is not onto or $m > 1$ and $G/\Gamma_n(G)$ is not m-divisible.

In a series of nontrivial cases we check $\mathrm{NQr}_{(m,n)}(\mathrm{G})$ and its negation. This enables us to present some new classes of co-Hopfian and non co-Hopfian groups (see below, items 4.4–4.11).

For all unexplained definitions from set theoretic algebra see the books by Eklof-Mekler [12] and Göbel-Trlifaj [18]. Also, for unexplained definitions from the group theory see the books of Fuchs [16], [15] and [14].

§ 2. Preliminaries

In this paper all groups are abelian, otherwise specialized. In this section we recall some basic definitions and facts that will be used for later sections of the paper.

Definition 2.1. An abelian group G is called $\aleph_1$-free if every countable subgroup of G is free. More generally, an abelian group G is called λ-free if every subgroup of G of cardinality $< \lambda$ is free.

Definition 2.2. Let κ be a regular cardinal. An abelian group G is said to be strongly κ-free if there is a set $\mathcal{S}$ of $< \kappa$-generated free subgroups of G containing 0 such that for any subset S of G of cardinality $< \kappa$ and any $N \in \mathcal{S}$, there is an $L \in \mathcal{S}$ such that $S \cup N \subset L$ and L/N is free.

A group G is pure in an abelian group H if $G \subseteq H$ and $nG = nH \cap G$ for every $n \in \mathbb{Z}$. The common notation for this notion is $G \subseteq_* H$.

Fact 2.3. *Suppose G is a torsion-free group. Then the intersection of pure subgroups of G is again pure. In particular, for every $S \subset G$, there exists a minimal pure subgroup of G containing S. The common notation for this subgroup is $\langle S \rangle_G^*$.*

Fact 2.4 (See [17, Theorem 7])**.** *Let G be an abelian group and H a pure and bounded subgroup of G. Then H is a direct summand of G.*

The notation $\mathrm{tor}(G)$ stands for the full torsion subgroup of G. There is a natural connection with the functor $\mathrm{Tor}_1^{\mathbb{Z}}(-, \sim)$:

$$\mathrm{tor}(G) = \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, G).$$

Fact 2.5 (See [17, Theorem 8])**.** *Let G be an abelian group and $T \subseteq_* \mathrm{tor}(G)$. If T is the direct sum of a divisible group and a group of bounded exponent, then T is a direct summand of G. The same result holds if $T \subseteq_* G$.*

Fact 2.6 (See [5]). *(i) Let G be a countable p-group. Then G is co-Hopfian if and only if G is finite.*

(ii) If a group G is co-Hopfian, then $\operatorname{tor}(G)$ is of size at most continuum, and further that G cannot be a p-groups of size $\aleph_0$.

Fact 2.7 (See [15, Theorem 17.2]). *If G is a p-group of bounded exponent, then G is a direct sum of (finitely many, up to isomorphism) finite cyclic groups.*

Definition 2.8. (i) An abelian group G is called *cotorsion* if $\operatorname{Ext}(J, G) = 0$ for all torsion-free abelian groups J.

(ii) An abelian group G is called *cotorsion-free* if it has no nonzero co-torsion subgroup.

In other words, G is cotorsion provided that it is a direct summand of every abelian group H containing G with the property that H/G is torsion-free. Here, we recall a useful source to produce a cotorsion-free group:

Fact 2.9. *(See* [12, Corollary 2.10(ii)]*). Any $\aleph_1$-free group is cotorsion-free.*

The p-torsion parts of a group G are important sources to produce pure subgroups.

Notation 2.10. Let $\mathbb{P}$ denote the set of all prime numbers.

(i) Let $p \in \mathbb{P}$. The p-power torsion subgroup of G is

$$\Gamma_p(G) := \{g \in G : \exists n \in \mathbb{N} \text{ such that } p^n g = 0\}.$$

(ii) For each $1 \leq m < \omega$, we let $\Gamma_m(G) := \bigoplus\{\Gamma_p(G) : p \,|\, m\}$.

Recall that the assignment $G \mapsto \Gamma_h(G)$ defines a functor from the category of abelian groups to itself, which is also called section functor. It has the following important property. Suppose $f : G \to H$ is a homomorphism of abelian groups. Then the following diagram of natural short exact sequences is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Gamma_h(H) & \xrightarrow{\subseteq} & H & \longrightarrow & H/\Gamma_h(H) & \longrightarrow & 0 \\
 & & {\scriptstyle f\restriction}\uparrow & & {\scriptstyle f}\uparrow & & {\scriptstyle \overline{f}}\uparrow & & \\
0 & \longrightarrow & \Gamma_h(G) & \xrightarrow{\subseteq} & G & \longrightarrow & G/\Gamma_h(G) & \longrightarrow & 0,
\end{array}$$

where $\overline{f}(g+\Gamma_h(G)) := f(g)+\Gamma_h(H)$.

The connection from p-power torsion functors and the classical torsion functor is read as follows:

$$\operatorname{Tor}_1^{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}, G) = \operatorname{tor}(G) = \bigoplus_{p\in\mathbb{P}} \Gamma_p(G).$$

Notation 2.11. In this paper, by $\operatorname{End}(-)$ we mean $\operatorname{End}_{\mathbb{Z}}(-)$ where $(-)$ is at least an abelian group, otherwise we specify it.

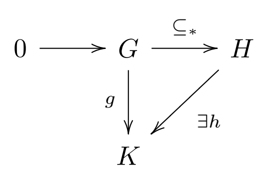
The following notion of boundness plays an important role in establishing the main theorems:

Definition 2.12. Let G be an abelian group of size λ. We say G is *boundedly endo-rigid* when for every $f \in \operatorname{End}(G)$ there is $m \in \mathbb{Z}$ such that the map $x \mapsto f(x) - mx$ has bounded range.

The next fact follows from the definition.

Fact 2.13. *An abelian group G is boundedly endo-rigid if and only if for every $f \in \operatorname{End}(G)$ there is $m \in \mathbb{Z}$ and bounded $h \in \operatorname{End}(G)$ such that $f(x) = mx + h(x)$.*

Fact 2.14. *Let K be a bounded torsion abelian group and let $G \subseteq_* H$. If $g \in$ $\operatorname{Hom}(\mathrm{G},\mathrm{K})$, then there is $h \in \operatorname{Hom}(\mathrm{H},\mathrm{K})$ extending g. This property is conveniently summarized by the subjoined diagram:*



Fact 2.15. *Let G be abelian group and suppose that G is not bounded, then the bounded endomorphisms of G (i.e., those $f \in \operatorname{End}(G)$ with bounded range) form an ideal of the ring $\operatorname{End}(G)$, we denote this ideal by $\mathrm{E_b}(G)$. With respect to this terminology, G is boundedly rigid if and only if the quotient ring $\operatorname{End}(G)/\mathrm{E_b}(G) \cong$ $\mathbb{Z}$.*

Remark 2.16. Recall that torsion subgroups are pure. Let f be a bounded endomorphism of $\mathrm{tor}(G)$. By Fact 2.14, we have

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathrm{tor}(G) & \xrightarrow{\subseteq_*} & G \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \exists h} & \\ & & \mathrm{tor}(G) & & \end{array}$$

Let $\hat{f} : G \xrightarrow{h} \mathrm{tor}(G) \xrightarrow{\subseteq} G$. In sum, f extends to an endomorphisms $\hat{f}$ of G with the same range:

$$\begin{array}{ccc} \mathrm{tor}(G) & \xrightarrow{f} & \mathrm{tor}(G) \\ {\scriptstyle \subseteq}\downarrow & & \downarrow{\scriptstyle \subseteq} \\ G & \xrightarrow[\hat{f}]{} & G \end{array}$$

Hence, the notion of boundedly rigid is really the right notion of endo-rigidity for mixed groups (for G torsion-free abelian group, we say that G is endo-rigid when $\mathrm{End}(G) \cong \mathbb{Z}$). For instance, we look at $K = \bigoplus\{\frac{\mathbb{Z}}{p^{\ell+1}\mathbb{Z}} : \ell < m\}$, for some $m < \omega$, and recall that this has many bounded endomorphisms. The same will happen for any G extending it.

In what follows we will use the concept of reduced group several times. Let us recall its definition.

Definition 2.17. Let G be an abelian group.

(a) G is called reduced if it contains no divisible subgroup other than 0.

(b) G is called injective if for any inclusion $G_1 \subseteq G_2$ of abelian groups, any morphism $f : G_1 \to G$ can be extended into G_2:

$$\begin{array}{ccccc} 0 & \longrightarrow & G_1 & \xrightarrow{\subseteq} & G_2 \\ & & {\scriptstyle f}\downarrow & \swarrow {\scriptstyle \exists h} & \\ & & G & & \end{array}$$

Fact 2.18. *(See* [16]*). An abelian group G is divisible if and only if it is injective.*

Here, we recall a connection between reduced and co-torsion-free abelian groups:

Fact 2.19. *(See* [12, theorem V.2.9]*). An abelian group G is cotorsion-free if and only if it is reduced and torsion-free and does not contain a subgroup isomorphic to $\widehat{\mathbb{Z}}_p$ for any prime p.*

Recall that $\widehat{\mathbb{Z}}_p$ means completion of $\mathbb{Z}$ in the p-adic topology. Here, we collect more basic facts about injective groups that we need:

Discussion 2.20. Let $p \in \mathbb{P}$ be a prime number.

(i) (See [12, page 11]). By the structure theorem for an injective abelian group I, we mean the following decomposition:

$$I = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{\oplus x_p} \oplus \mathbb{Q}^{\oplus x},$$

where x_p and x are index sets.

(ii) (See [19, Theorem 3.7]). Let $p, q \in \mathbb{P}_0 := \mathbb{P} \cup \{0\}$, and set $\mathbb{Z}(0^\infty) := \mathbb{Q}$. Then

$$\operatorname{Hom}(\mathbb{Z}(p^\infty), \mathbb{Z}(q^\infty)) = \begin{cases} \widehat{\mathbb{Z}}_p & \text{if } p = q \\ 0 & \text{otherwise} \end{cases}$$

with the convenience that $\widehat{\mathbb{Z}}_p = \mathbb{Q}$.

(iii) Combining i) and ii) gives us:

$$\operatorname{End}(I) = \prod_{p\in\mathbb{P}_0} \widehat{\mathbb{Z}}_p^{\oplus x_p},$$

where $x_0 := x$.

§ 3. The ZFC construction of boundedly rigid mixed groups

In this section we show that for any cardinal $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$ and any torsion abelian group K of size less than λ, there exists a boundedly rigid abelian group G with $\operatorname{tor}(G) = K$, see Theorem 3.11.

To this end, we define the notion of rigidity context $\mathbf{k}$ which in particular codes a torsion group K, and assign to it a collection of objects $\mathbf{m}$, which among other things have a group G with $\operatorname{tor}(G) = K$. We show that under the above assumptions on λ and K, we can always find such an $\mathbf{m}$ such that the associated group G is boundedly rigid.

Definition 3.1. (1) We say a tuple $\mathbf{k}$ is a *rigidity context* when

$$\mathbf{k} = \big(K_{\mathbf{k}}, R_{\mathbf{k}}, \phi_r^{\mathbf{k}}, \Psi_{r,s}^{\mathbf{k}}, \Psi_{(r,s)}^{\mathbf{k}}, S_{\mathbf{k}}\big)_{r,s\in R_{\mathbf{k}}} = \big(K, R, \phi_r, \Psi_{r,s}, \Psi_{(r,s)}, S\big)_{r,s\in R}$$

where

(a) K is a reduced torsion abelian group,

(b) R is a ring,

(c) S is a set of prime numbers, $S_{\mathbf{k}}^{\perp} = \mathbb{P} \setminus S$ is its complement, and R is $S_{\mathbf{k}}^{\perp}$-divisible. This means that R is divisible for any $p \in S_{\mathbf{k}}^{\perp}$,

(d) for $r \in R$, the map $\phi_r \in \mathrm{End}(K)$ has bounded range,

(e) if $r, s \in R$, then $\Psi_{r,s} = \phi_r + \phi_s - \phi_{r+s} \in \mathrm{End}(K)$,

(f) if $r, s \in R$, then $\Psi_{(r,s)} \in \mathrm{End}(K)$ has bounded range and, letting $t = rs$, for $x \in K$ we have

$$\Psi_{(r,s)}(x) = \phi_r(\phi_s(x)) - \phi_t(x).$$

(2) We say $\mathbf{k}$ is nontrivial when for some prime $p \in S_{\mathbf{k}}$ the p-torsion $\Gamma_p(K)$ is infinite, or the set

$$\{p \in S_{\mathbf{k}} : \Gamma_p(K) \neq 0\}$$

is infinite.

(3) By $\mathbb{Z}_{\mathbf{k}}$ we mean the subring of $\mathbb{Q}$ generated by $\{1\} \cup \{\frac{1}{p} : p \in S_{\mathbf{k}}^{\perp}\}$.

Observation 3.2. *Suppose $(R_{\mathbf{k}}, +)$ is cotorsion-free as an abelian group. Then $S_{\mathbf{k}} \neq \emptyset$.*

Proof. Suppose on the way of contradiction that $S_{\mathbf{k}} = \emptyset$. In other words, $S_{\mathbf{k}}^{\perp}$ is the set of prime numbers. By Definition 3.1(1.c), R is $S_{\mathbf{k}}^{\perp}$-divisible. This means that $\mathbb{Q} \subseteq R_{\mathbf{k}}$. It turns out from Fact 2.19 that $(R_{\mathbf{k}}, +)$ is not cotorsion-free, a contradiction. □

Definition 3.3. Let $\mathbf{k}$ be a rigidity context. By $\mathbf{M}_\mathbf{k}$ we mean the family of all tuples

$$\mathbf{m} = \left(\mathbf{k}_\mathbf{m}, G_\mathbf{m}, F^\mathbf{m}_r, F^\mathbf{m}_{r,s}, F^\mathbf{m}_{(r,s)}\right)_{r,s\in R_{\mathbf{k}_\mathbf{m}}} = \left(\mathbf{k}, G, F_r, F_{r,s}, F_{(r,s)}\right)_{r,s\in R_\mathbf{k}}$$

where

(a) G is an abelain group,

(b) $\mathrm{tor}(G) = K_\mathbf{k}$,

(c) for $r \in R_\mathbf{k}$, F_r is an endomorphism of G extending $\phi^\mathbf{k}_r$:

$$\begin{array}{ccc} K & \xrightarrow{\phi_r} & K \\ \subseteq\downarrow & & \downarrow\subseteq \\ G & \xrightarrow[F_r]{} & G \end{array}$$

(d) for $r, s \in R_\mathbf{k}$, $F_{r,s} \in \mathrm{End}(G)$ extends $\Psi r, s$

$$\begin{array}{ccc} K & \xrightarrow{\Psi r,s} & K \\ \subseteq\downarrow & & \downarrow\subseteq \\ G & \xrightarrow[F_{r,s}]{} & G \end{array}$$

and they have the same range $F_{r,s}[G] = \Psi_{r,s}[K]$.

(e) for $r, s \in R_\mathbf{k}$, $F_{(r,s)} \in \mathrm{End}(G)$ extends $\Psi^\mathbf{k}_{(r,s)}$:

$$\begin{array}{ccc} K & \xrightarrow{\Psi(r,s)} & K \\ \subseteq\downarrow & & \downarrow\subseteq \\ G & \xrightarrow[F_{(r,s)}]{} & G \end{array}$$

and thereby they have the same range $F_{(r,s)}[G] = \Psi_{(r,s)}[K]$.

(f) if $r, s, t \in R$ and $t = r + s$, then for $x \in G$,

$$F_{r,s}(x) = F_r(x) + F_s(x) - F_t(x),$$

(g) if $r, s, t \in R$ and $t = rs$, then for $x \in G$,

$$F_{(r,s)}(x) = F_r(F_s(x)) - F_t(x).$$

Definition 3.4. Adopt the previous notation, and let

$$\mathbf{M} = \bigcup \{\mathbf{M_k} : \mathbf{k} \textit{ is a rigidity context}\}.$$

(1) We define $\leq_{\mathbf{M}}$ as the following partial order on $\mathbf{M}$. Namely, $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ iff
   (a) $\mathbf{m}, \mathbf{n} \in \mathbf{M}$,
   (b) $\mathbf{k_m} = \mathbf{k_n}$,
   (c) $G_{\mathbf{m}} \subseteq G_{\mathbf{n}}$,
   (d) $F_r^{\mathbf{m}} \subseteq F_r^{\mathbf{n}}$.
(2) By $\leq_{\mathbf{M_k}}$ we mean $\leq_{\mathbf{M}} \restriction \mathbf{M_k}$.

Notation 3.5. Let $r \in R$ and $x \in G_{\mathbf{m}}$. By rx we mean $rx := F_r^{\mathbf{m}}(x) \in G_{\mathbf{m}}$.

Definition 3.6. Suppose $\mathbf{k}$ is a rigidity context and $\mathbf{m} \in \mathbf{M_k}$.

(1) We say $\mathbf{m}$ is *boundedly rigid* when for every $f \in \mathrm{End}(G_{\mathbf{m}})$ there are $r \in R$ and $h \in \mathrm{End}_b(G_{\mathbf{m}})$[1] and

$$x \in G_{\mathbf{m}} \implies f(x) = rx + h(x).$$

(2) We say $\mathbf{m}$ is *free* when it has a base B which means that the set $\{x + K_{\mathbf{k}} : x \in B\}$ is a free base of the abelian group $G_{\mathbf{m}}/K$.
(3) We say $\mathbf{m}$ is *λ-free* when $G_{\mathbf{m}}/K$ is.
(4) We say $\mathbf{m}$ is *strongly λ-free* when $G_{\mathbf{m}}/K$ is.
(5) Let $M_{\mathbf{m}}$ be the R-module obtained by expanding $G_{\mathbf{m}}/K$ such that for $x, y \in G_{\mathbf{m}}$ and $r \in R$

$$rx + K = y + K \iff F_r^{\mathbf{m}}(x) = y.$$

The next easy lemma shows that $M_{\mathbf{m}}$ as defined above is well-defined.

Lemma 3.7. *Suppose* $\mathbf{k}$ *is a rigidity context and* $\mathbf{m} \in \mathbf{M_k}$. *Then* $M_{\mathbf{m}}$ *can be turn to an R-module structure.*

[1]so, h has a bounded range.

Proof. Since $M_{\mathbf{m}}$ is an expansion of $G_{\mathbf{m}}/K$, it is an abelian group. Let $r \in R$ and $m := g + K \in M_{\mathbf{m}}$ where $g \in G$. The assignment

$$(r, m) \mapsto rm := F_r^{\mathbf{m}}(g) + K \in G_{\mathbf{m}}/K = M_{\mathbf{m}},$$

defines the desired module structure on $M_{\mathbf{m}}$. □

Lemma 3.8. *Suppose* $\mathbf{k}$ *is a rigidity context and* $\mathbf{m} \in \mathbf{M}_{\mathbf{k}}$. *The following assertions hold.*

1. *Suppose* $R_{\mathbf{k}} = \mathbb{Z}$ *(so,* $S_{\mathbf{k}}^{\perp} = \emptyset$*). Then* $\mathbf{m}$ *is boundedly rigid iff* $G_{\mathbf{m}}$ *is boundedly rigid.*
2. *Let* $R_{\mathbf{k}} = \mathbb{Z}_{\mathbf{k}}$ *(see Definition 3.1(3)). Then* $\mathbf{m}$ *is boundedly rigid iff* $G_{\mathbf{m}}$ *is boundedly rigid.*
3. *if* $\phi_r^{\mathbf{k}}$ *is zero for every* $r \in R$*, then* $G_{\mathbf{m}}$ *is an* R*-module.*

Proof. (1) and (2) are trivial and follow from the definitions.

(3): For each $x \in G_{\mathbf{m}}$ and $r \in R$, we set $rx := F_r^{\mathbf{m}}(x)$. It is straightforward to furnish the following three properties:

- the identity $r(x + y) = rx + ry$ follows from Definition 3.1(2)(c),
- the equality $(r + s)x = rx + sx$ follows from Definition 3.1(2)(d),
- the equality $r(sm) = (rs)m$ follows from items (e) and (f) from Definition 3.1(2).

From these, $G_{\mathbf{m}}$ is equipped with an R-module structure. □

In what follows, the notation $\lg(-)$ stands for the length function.

Definition 3.9. Let $\alpha \in \mathrm{Ord}$.

(1) By $\Lambda_\omega[\alpha]$ we mean

$$\big\{\eta : \lg(\eta) = \omega \textit{ and } \eta(n) = (\eta(n,1), \eta(n,2)) \textit{ where } \eta(n,1) \leq \eta(n,2) < \eta(n{+}1,1) < \alpha\big\}.$$

(2) For each $\eta \in \Lambda_\omega[\alpha]$, we let $\mathbf{j}(\eta) = \bigcup\{\eta(n,1) : n < \omega\}$.

(3) $\Lambda_{<\omega}[\alpha] := \{\langle\rangle\} \cup \bigcup\limits_{k<\omega} \Lambda_k[\alpha]$, where $\Lambda_k[\alpha]$ is the set of all η furnished with the following four properties:

(a) $\lg(\eta) = k + 1$,

(b) $\eta(k) < \alpha$,

(c) for any $\ell < k$ we suppose $\eta(\ell)$ is furnished with a pairing property in the following sense:

(c.1) $\eta(\ell) = (\eta(\ell,1), \eta(\ell,2))$, where $\eta(\ell,1) \leq \eta(\ell,2) < \alpha$, and

(c.2) Suppose in addition $\ell + 1 < k$, we may and do assume that $\eta(\ell,2) < \eta(\ell+1,1)$,

(d) if $\ell < k$, then $\eta(\ell,1) = \eta(\ell,2) \iff \ell = 0$.

(4) $\Lambda[\alpha] := \Lambda_\omega[\alpha] \cup \Lambda_{<\omega}[\alpha]$.

(5) For any $\eta \in \Lambda[\alpha]$ and $k+1 < \lg(\eta)$ we set

(5.1) $\eta \restriction_L k := \big\langle(\eta(\ell,1), \eta(\ell,2)) : \ell < k\big\rangle^\frown\langle\eta(k,1)\rangle$, and

(5.2) $\eta \restriction_R k := \big\langle(\eta(\ell,1), \eta(\ell,2)) : \ell < k\big\rangle^\frown\langle\eta(k,2)\rangle$.

Note that $\eta \restriction_L k$ and $\eta \restriction_R k$ belong to $\Lambda_{k+1}[\alpha]$.

(6) We say $\Lambda \subseteq \Lambda[\alpha]$ is *downward closed* while for each $\eta \in \Lambda$ and $k+1 < \lg(\eta)$ we have $\eta \restriction_L k, \eta \restriction_R k \in \Lambda$.

We next define when a subset of $\Lambda_\omega[\alpha]$ is free.

Definition 3.10. Suppose $\alpha \in \mathrm{Ord}$ and $\Lambda \subseteq \Lambda_\omega[\alpha]$.

(1) We say Λ is free whenever there is a function $h : \lambda \to \omega$ such that the sequence

$$\big\langle\{\eta\restriction_L n, \eta\restriction_R n : h(\eta) \leq n < \omega\} : \eta \in \Lambda\big\rangle$$

is a sequence of pairwise disjoint sets.

(2) We say Λ is μ-free when every $\Lambda' \subseteq \Lambda$ of cardinality $< \mu$ is free.

We can now state the main result of this section as follows.

Theorem 3.11. *Let $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$. Let $\mathbf{k}$ be a nontrivial rigidity context such that $K := K_{\mathbf{k}}$ and $R := R_{\mathbf{k}}$ are of cardinality $\leq \lambda$. Then there exists an abelian group G such that $\mathrm{tor}(G) = K$ and G is boundedly rigid. In particular, the sequence*

$$0 \longrightarrow R \longrightarrow \mathrm{End}(G) \longrightarrow \frac{\mathrm{End}(G)}{\mathrm{E_b}(G)} \longrightarrow 0$$

is exact.

The rest of this section is devoted to the proof of above theorem.

Definition 3.12. For any ordinal γ, a sequence $\eta \in \Lambda[\lambda]$ and a family $\Lambda \subseteq \Lambda[\lambda]$ we define:

(1) S_γ is the closure of $\omega \cup \gamma$ under taking finite subsets, so including finite sequences.
(2) $\gamma(\eta) = \eta(0,1)$.
(3) $\Lambda_\gamma = \{\eta \in \Lambda : \gamma(\eta) < \gamma\}$.
(4) We set
  (4.1) $\Lambda_{<\omega} = \Lambda \cap \Lambda_{<\omega}[\alpha]$, and
  (4.2) $\Lambda_\omega = \Lambda \cap \Lambda_\omega[\alpha]$.

In order to prove Theorem 3.11, we need a twofold version of black box, that we now introduce. On simple black boxes see [29], [32] and [33]. The presentation here is a special case of the n-fold λ-black box from [34], when $n = 2$.

Definition 3.13. We say $\mathbf{b}$ is a *twofold λ-black box* when it consists of:

(1) $\bar{g} = \langle g_\eta : \eta \in \Lambda_\omega[\lambda]\rangle$, where
(2) g_η is a function from ω into S_λ,
(3) Suppose $g : \Lambda_{<\omega}[\lambda] \to S_\lambda$ is a function and $f : \Lambda_{<\omega}[\lambda] \to \gamma$ where $\gamma < \lambda$. Then for some $\eta \in \Lambda_\omega[\lambda]$ the following hold:
  (a) $\gamma(\eta) > \gamma$,
  (b) $g_\eta(0) = g(\langle\rangle)$,
  (c) $g_\eta(n+1) = \big(g(\eta \restriction_L n), g(\eta \restriction_R n)\big)$,
  (d) $\eta(n,1) < \eta(n,2)$ and $f(\eta \restriction_L n) = f(\eta \restriction_R n)$ for all $1 \leq n < \omega$.

Hypothesis 3.14. For the rest of this section we adopt the following hypotheses, otherwise specializes:

- $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$.
- $\mathbf{k}$ is a rigidity context as in Definition 3.1.
- $K = K_{\mathbf{k}}, R = R_{\mathbf{k}}$ are of cardinality $< \lambda$. Without loss of generality, we may assume that the set of elements of K and R are subsets of λ.

- $(R,+)$ is cotorsion-free.
- **b** is a twofold λ-black box.

The following result was proved in [34, Lemma 1.14], with a setting more general than here. As this plays a crucial ingredient, we sketch its proof.

Lemma 3.15. *There exists a twofold λ-black box.*

Proof. For notational simplicity, we set $S := S_\lambda$, and look at the following fixed partition of λ into λ-many sets, each of cardinality λ:

$$\big\langle W_{s_1,s_2} : s_1, s_2 \in S\big\rangle.$$

For each $\eta \in \Lambda_\omega[\lambda]$, we define $g_\eta(n) \in S$, by induction on $n < \omega$.

To start, set

$$(*)_1 \qquad g_\eta(0) = s \iff \eta(0,1) = \eta(0,2) \in W_{s,s}.$$

Now suppose that $n < \omega$ and $g_\eta \restriction (n+1)$ is defined. We are going to define $g_\eta(n+1)$. It is enough to note that

$$(*)_2 \qquad g_\eta(n+1) = (s_1, s_2) \iff \eta(n+1,1) \in W_{s_1,s_2}.$$

We show that $\bar{g} = \langle g_\eta : \eta \in \Lambda_\omega[\lambda]\rangle$ is as required. Suppose that $g : \Lambda_{<\omega}[\lambda] \to S_\lambda$ is a function and $f : \Lambda_{<\omega}[\lambda] \to \gamma$ where $\gamma < \lambda$. We define $\eta \in \Lambda_\omega[\lambda]$, by defining $\eta(n)$, by induction on n.

Let $\eta(0) := \langle \eta(0,1), \eta(0,2)\rangle$, where

$$(*)_3 \qquad \gamma < \eta(0,1) = \eta(0,2) \in W_{g(\langle\rangle),g(\langle\rangle)}.$$

Now, suppose that $n < \omega$ and we have defined $\eta \restriction n+1$. We define

$$\eta(n+1) = \langle \eta(n+1,1), \eta(n+1,2)\rangle.$$

Set

a) $s_1 := g(\eta \restriction_L n)$,
b) $s_2 := g(\eta \restriction_R n)$, and

c) $\mathbf{c}_n : W_{s_1,s_2} \to \gamma$ is defined via the following assignment

$$\mathbf{c}_n(\alpha) := f\big((\eta \restriction n+1)^\frown\langle\alpha\rangle\big) \quad (+)$$

As $\gamma < \lambda$ and W_{s_1,s_2} has size λ, we can find an unbounded subset W_n of W_{s_1,s_2} such that $\mathbf{c}_n \restriction W_n$ is constant. Let $\eta(n+1,1) < \eta(n+1,2)$ be such that

$$(*)_4 \qquad \eta(n,2) < \eta(n+1,1), \eta(n+1,2) \in W_n \subseteq W_{g(\eta\restriction_L n), g(\eta\restriction_R n)}.$$

We claim that the η we constructed as above, satisfies the required conditions of Definition 3.13(3). Indeed, thanks to our construction, $\gamma(\eta) = \eta(0,1) > \gamma$. We also have

$$g_\eta(0) = g(\langle\rangle) \iff \eta(0,1) = \eta(0,2) \in W_{g(\langle\rangle),g(\langle\rangle)},$$

which is true by $(*)_3$. We also have

$$g_\eta(n+1) = \big(g(\eta\restriction_L n), g(\eta\restriction_R n)\big) \iff \eta(n+1,1) \in W_{g(\eta\restriction_L n),g(\eta\restriction_R n)},$$

which is again true by $(*)_4$. Finally note that, clearly $f(\eta\restriction_L 1) = f(\eta\restriction_R 1)$, and for all n,

$$\begin{aligned} f(\eta\restriction_L n+2) &= f(\eta \restriction n+1^\frown\langle\eta(n+1,1)\rangle) \\ &\overset{(+)}{=} \mathbf{c}_n\big(\eta(n+1,1)\big) \\ &\overset{(*)_4}{=} \mathbf{c}_n\big(\eta(n+1,2)\big) \\ &\overset{(+)}{=} f(\eta \restriction n+1^\frown\langle\eta(n+1,2)\rangle) \\ &= f(\eta\restriction_R n+2). \end{aligned}$$

The Lemma follows. □

Assuming hypotheses beyond ZFC, we can get stronger versions of twofold λ-black box (see again [34]).

Observation 3.16. *Assume $\lambda = \mathrm{cf}(\lambda) \geq \aleph_1$. Let*

$$S \subseteq \{\alpha < \lambda : \mathrm{cf}(\alpha) = \aleph_0\}$$

be a stationary and non-reflecting subset of λ such that the principle $\diamondsuit_S$ holds. Then there is a λ-free twofold λ-black box $\mathbf{b}$ such that $\Lambda_{\mathbf{b}} = \{\eta_\delta : \delta \in S\}$ and $\mathbf{j}(\eta_\delta) = \delta$ for every $\delta \in S$.

Recall that Jensen's diamond principle $\diamondsuit_S$ is a kind of prediction principle whose truth is independent of ZFC. The point in the above proof is that if $\Lambda_{\mathbf{b}} = \{\eta_\delta : \delta \in S\}$ and $\mathbf{j}(\eta_\delta) = \delta$ for every $\delta \in S$, then as S does not reflect, the set $\Lambda_{\mathbf{b}}$ is λ-free.

Remark 3.17. Recall from [6] that a (co-)Hopfian group of size $\lambda = 2^{\aleph_0}$ exists in ZFC. We can also deal with the case of $\lambda = 2^{\aleph_0}$, but all is known in this case, so we just concentrate on the case $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$.

Definition 3.18. Let AP $:=$ AP$_{\mathbf{k},\lambda}$ be the set of all quintuples

$$\mathbf{c} = \left(\Lambda_{\mathbf{c}}, \mathbf{m}_{\mathbf{c}}, \Gamma_{\mathbf{c}}, X_{\mathbf{c}}, \langle a^{\mathbf{c}}_{\eta,n} : \eta \in \Lambda_{\mathbf{c}}, n < \omega\rangle\right)$$

such that:

(a) $\Lambda_{\mathbf{c}} \subseteq \Lambda[\lambda]$ is downward closed.
(b) $\mathbf{m}_{\mathbf{c}} \in \mathbf{M}_{\mathbf{k}}$. We may write $G_{\mathbf{c}}, M_{\mathbf{c}}$ instead of $G_{\mathbf{m}_{\mathbf{c}}}, M_{\mathbf{m}_{\mathbf{c}}}$ respectively, etc.
(c) $X_{\mathbf{c}}$ is the following set:

$$\{rx_\nu : r \in R, \nu \in \Lambda_{\mathbf{c},<\omega}\} \cup \{ry_{\eta,n} : r \in R, \eta \in \Lambda_{\mathbf{c},\omega}, n < \omega\}.$$

(d) $G_{\mathbf{c}}$ is generated, as an abelian group, by the sets K and $X_{\mathbf{c}}$. The relations presented in item (f), see below.
(e) for any ordinal α, let $G_{\mathbf{c},\alpha}$ be the subgroup of $G_{\mathbf{c}}$ generated by the set K and

$$\{rx_\nu : r \in R, \nu \in \Lambda_{\mathbf{c},<\omega} \cap \Lambda[\alpha]\} \cup \{ry_{\rho,n} : r \in R, \rho \in \Lambda_{\mathbf{c},\omega} \cap \Lambda[\alpha], n < \omega\}.$$

(f) $M_{\mathbf{c}}$, as an R-module, is generated by $X_{\mathbf{c}} \cup K$, freely except the following set $\Gamma_{\mathbf{c}}$ of equations:
   - $y_{\eta,n} = a^{\mathbf{c}}_{\eta,n} + (n!)y_{\eta,n+1} + (x_{\eta\restriction_L n} - x_{\eta\restriction_R n})$,

   where $a^{\mathbf{c}}_{\eta,n} \in G_{\mathbf{c},\eta(0,1)}$.

The following is clear:

Lemma 3.19. *Suppose* $\mathbf{c} \in \mathrm{AP}_{\mathbf{k},\lambda}$. *Then* $G_{\mathbf{c}}$ *is of size* $\lambda^{\aleph_0}$.

Definition 3.20. For any $\mathbf{c} \in \mathrm{AP}_{\mathbf{k},\lambda}$, we define the following:

(1) $\gamma_{\mathbf{c}} := \min\{\gamma \leq \lambda : \Lambda_{\mathbf{c}} \subseteq \Lambda[\gamma]\}$.

(2) Let $\Omega_{\mathbf{c}} := \Lambda_{\mathbf{c},<\omega} \cup \left(\Lambda_{\mathbf{c},\omega} \times \omega\right)$ and define $\langle x_\rho : \rho \in \Omega_{\mathbf{c}}\rangle$ by the following rule

(2.1) If $\rho \in \Lambda_{\mathbf{c},<\omega}$, then x_ρ is defined as in Definition 3.18(c).

(2.2) If $\rho = (\eta, n) \in \Lambda_{\mathbf{c},\omega} \times \omega$, we define $x_\rho := y_{\eta,n}$.

(3) For $b \in G_{\mathbf{c}}$ choose the sequence

$$\left\langle r_{b,\ell}, \eta_{b,\ell}, m_{b,\ell} : \ell < n_b \right\rangle$$

such that

$$b - \sum_{\ell < n_b} r_{b,\ell} y_{\eta_{b,\ell}, m_{b,\ell}} \in \sum_{\rho \in \Lambda_{\mathbf{c},<\omega}} R x_\rho + K,$$

where $r_{b,\ell} \in R \setminus \{0\}$ and $(\eta_{b,\ell}, m_{b,\ell}) \in \Lambda_{\mathbf{c},\omega} \times \omega$.

(4) By $\mathrm{supp}_\circ(b)$ we mean $\{\eta_{b,\ell} : \ell < n_b\}$.

Definition 3.21. Suppose $\mathbf{c} \in \mathrm{AP}_{\mathbf{k},\lambda}$ and let $a \in G_{\mathbf{c}}$.

(a) There is a finite set $\Lambda_a \subseteq \Lambda_{\mathbf{c}}$, a sequence $S := \langle r_\rho : \rho \in \Lambda_a\rangle$ of non-zero elements of R, an $n(a) < \omega$ and $d_a \in K$ such that

$$a = \sum_{\eta \in \Lambda_{a,<\omega}} r_\eta x_\eta + \sum_{\nu \in \Lambda_{a,\omega}} r_\nu y_{\nu,n(a)} + d_a,$$

where $\Lambda_{a,<\omega} = \Lambda_a \cap \Lambda_{\mathbf{c},<\omega}$ and $\Lambda_{a,\omega} = \Lambda_a \cap \Lambda_{\mathbf{c},\omega}$.

(b) Let $\mathrm{supp}_{\mathbf{c}}(a) = \mathrm{supp}(a)$ be the minimal set $\Lambda \subseteq \Lambda_{\mathbf{c}}$ minimal with respect to the following two properties:

(b.1) $\Lambda_a \subseteq \Lambda$.

(b.2) If $\nu \in \Lambda_a \cap \Lambda_{\mathbf{c},\omega}$ and $n < \omega$ then $\Lambda_{a^{\mathbf{c}}_{\nu,n}} \subset \Lambda$ and $\eta{\restriction}_L n, \eta{\restriction}_R n \in \Lambda$.

Remark 3.22. Adopt the previous notation, and $a \in G_{\mathbf{c}}$. Then $\mathrm{supp}_{\mathbf{c}}(a)$ is the minimal set $\Lambda \subseteq \Lambda_{\mathbf{c}}$ such that

$$a \in \left\langle \{x_\eta, y_{\nu,n} : \eta \in \Lambda(L,R), \nu \in \Lambda, n < \omega\} \cup K \right\rangle^*_{G_{\mathbf{c}}}.$$

Remark 3.23. Adopt the previous notation. The following holds.

(1) The set $\mathrm{supp}_{\mathbf{c}}(a)$ is countable.

(2) If $a = x_\nu$ for some $\nu \in \Lambda_{\mathbf{c}}$, then

$$\mathrm{supp}(a) \setminus S_{\eta(\nu,1)} = \{\nu\} \cup \{\nu{\restriction}_L, n, \nu{\restriction}_R, n : n < \omega\}.$$

Definition 3.24. Let $\leq_{\mathrm{AP}}$ be the following partial order on $\mathrm{AP} = \mathrm{AP}_{\mathbf{k},\lambda}$. For any $\mathbf{c}, \mathbf{d} \in \mathrm{AP}$ we say $\mathbf{c} \leq_{\mathrm{AP}} \mathbf{d}$ when the following holds:

(a) $\Lambda_{\mathbf{c}} \subseteq \Lambda_{\mathbf{d}}$,

(b) $\mathbf{m}_{\mathbf{c}} \leq_{\mathbf{M}} \mathbf{m}_{\mathbf{d}}$, hence $G_{\mathbf{c}} \subseteq G_{\mathbf{d}}$, etc.

(c) $a^{\mathbf{c}}_{\eta,\ell} = a^{\mathbf{d}}_{\eta,\ell}$ for $\eta \in \Lambda_{\mathbf{c}}, \ell < \omega$,

(d) $x^{\mathbf{c}}_{\eta} = x^{\mathbf{d}}_{\eta}$ for $\eta \in \Lambda_{\mathbf{c},<\omega}$,

(e) $y^{\mathbf{c}}_{\eta,\ell} = y^{\mathbf{d}}_{\eta,\ell}$ for $\eta \in \Lambda_{\mathbf{c},\omega}$ and $\ell < \omega$.

Lemma 3.25. *The following two assertions are valid:*

(1) $\leq_{\mathrm{AP}}$ *is indeed a partial order,*

(2) If $\bar{\mathbf{c}} = \langle \mathbf{c}_\alpha : \alpha < \delta \rangle$ *is* $\leq_{\mathrm{AP}}$*-increasing, then there exists* $\mathbf{c}_\delta = \bigcup_{\alpha<\delta} \mathbf{c}_\alpha$ *in* AP *which is the* $\leq_{\mathrm{AP}}$*-least upper bound of the sequence* $\bar{\mathbf{c}}$.

Proof. Clause (1) is clear, for clause (2), let

$$\mathbf{c}_\delta := \big(\Lambda, \mathbf{m}, \Gamma, X, \langle a_{\eta,n} : \eta \in \Lambda, n < \omega \rangle\big),$$

where:

- $\Lambda := \bigcup_{\alpha<\delta} \Lambda_{\mathbf{c}_\alpha}$,
- $\mathbf{m} =: (G, F_r, F_{r,s}, F_{(r,s)})$, where
  - $G := \bigcup_{\alpha<\delta} G_{\mathbf{c}_\alpha}$,
  - $F_r := \bigcup_{\alpha<\delta} F^{\mathbf{c}_\alpha}_r$,
  - $F_{r,s} := \bigcup_{\alpha<\delta} F^{\mathbf{c}_\alpha}_{r,s}$,
  - $F_{(r,s)} := \bigcup_{\alpha<\delta} F^{\mathbf{c}_\alpha}_{(r,s)}$.
- $\Gamma := \bigcup_{\alpha<\delta} \Gamma_{\mathbf{c}_\alpha}$,
- $X := \bigcup_{\alpha<\delta} X_{\mathbf{c}_\alpha}$,
- for $\eta \in \Lambda_\omega$ and $n < \omega$, we have $a_{\eta,n} = a^{\mathbf{c}_\alpha}_{\eta,n}$, for some and hence any $\alpha < \delta$ such that $\eta \in \Lambda_{\mathbf{c}_\alpha,\omega}$.

It is easily seen that $\mathbf{c}_\delta$ is as required. □

An R-module M is called $\aleph_1$-free, if every countably generated submodule of M is contained in a free submodule of M. Similarly, μ-free can be defined. For more details, see [12, IV. Definition 1.1].

Lemma 3.26. *Let* $\mathbf{c} \in \mathrm{AP}$. *The following claims hold:*

(1) $\mathrm{tor}(G_{\mathbf{c}}) = K$.

(2) The group

$$G_{\mathbf{c}}/\big\langle K \cup \{rx_\nu : r \in R, \nu \in \Lambda_{\mathbf{c},<\omega}\}\big\rangle$$

is divisible and torsion-free. Also, the parallel result holds for the R*-module:*

$$M_{\mathbf{c}}/\big\langle K \cup \{rx_\nu : r \in R, \nu \in \Lambda_{\mathbf{c},<\omega}\}\big\rangle.$$

(3) The following three properties are satisfied:

(a) $\Lambda_{\mathbf{c}}$ *is* $\aleph_1$*-free.*

(b) If $\Lambda_{\mathbf{c}}$ *is* μ*-free, then* $M_{\mathbf{c}}$ *is* μ*-free.*

(c) If $\Lambda_{\mathbf{c}}$ *is* μ*-free and* $(R, +)$ *is* μ*-free, then* $G_{\mathbf{c}}/K$ *is a* μ*-free abelian group.*

(4) If $\gamma \le \gamma_{\mathbf{c}}$ *and* $\Lambda \subseteq \Lambda_{\mathbf{c}}$, *then there exists a unique* $\mathbf{d} \in \mathrm{AP}$ *such that*

(a) $\Lambda_{\mathbf{d}} = \Lambda \cap \Lambda[\gamma]$,

(b) $G_{\mathbf{d}} \subseteq G_{\mathbf{c}}$.

Such a unique object is denoted by $\mathbf{d} := \mathbf{c} \restriction (\gamma, \Lambda)$.

(5) Assume $\eta \in \Lambda_\omega[\lambda] \setminus \Lambda_{\mathbf{c}}$, $\ell < \omega$ *and* $a_\ell \in G_{\mathbf{c}}$ *are such that* $a_\ell \in G_{\mathbf{c},\eta(0,1)}$ *for each* ℓ. *Then there is* $\mathbf{d} \in \mathrm{AP}$ *equipped with the following three properties:*

(a) $\Lambda_{\mathbf{d}} = \Lambda_{\mathbf{c}} \cup \{\eta\} \cup \{\eta{\restriction_L} n, \eta{\restriction_R} n : n < \omega\}$,

(b) $\mathbf{c} \le_{\mathrm{AP}} \mathbf{d}$ *and so* $G_{\mathbf{c}} \subseteq G_{\mathbf{d}}$,

(c) $a^{\mathbf{d}}_{\eta,\ell} = a_\ell$ *for* $\ell < \omega$.

(6) The group $G_{\mathbf{c}}$ *is of size* λ.

Proof. (1)-(2): These are easy.

(3): (a) : Let $\Lambda \subseteq \Lambda_{\mathbf{c},\omega}$ be countable, and let $\{\eta_n : n < \omega\}$ be an enumeration of it. Define the maps h_1 and h_2 from Λ to ω as follows:

$$h_1(\eta_n) := \min\Big\{k : \forall j < n,\ \forall \ell,\ r \in \{L, R\} \text{ we have } \eta_j \restriction_\ell k \neq \eta_n \restriction_r k\Big\},$$

and

$$h_2(\eta_n) := \min\Big\{k : \eta_n{\restriction_L} k \neq \eta_n \restriction_R k\Big\}.$$

Finally, we set

$$h(\eta_m) := \max\{h_1(\eta_n), h_2(\eta_n)\} + 1.$$

Having Definition 3.10 in mind, we are going to show h is as required. Let $j < i < \omega$ and let

- $h(\eta_j) \leq n_j < \omega$
- $h(\eta_i) \leq n_i < \omega$.

We will show that $\eta_j \restriction_\ell n_i \neq \eta_i \restriction_r n_j$, where $\ell, r \in \{L, R\}$. To see this, we note that there is nothing to prove if $n_i \neq n_j$. So, we may and do assume that $n := n_i = n_j$. Thus, $h(\eta_j), h(\eta_i) \leq n$. W look at $m := h_1(\eta_i)$. According to the definition of h_1, we know that $\eta_j \restriction_\ell m \neq \eta_i \restriction_r m$. As $m \leq n$ one has

$$\eta_i \restriction_\ell n \neq \eta_j \restriction_r n.$$

Also given any $i < \omega$, if $n \geq h(\eta_i)$, then by the definition of h_2 and as $n \geq h_2(\eta_i)$, we have

$$\eta_i{\restriction}_L n \neq \eta_i \restriction_R n.$$

It follows that the sequence

$$\Big\langle \{\eta{\restriction}_L n, \eta{\restriction}_R n : h(\eta) \leq n < \omega\} : \eta \in \Lambda \Big\rangle$$

is a sequence of pairwise disjoint sets. By definition, $\Lambda_{\mathbf{c}}$ is $\aleph_1$-free.

(b) : For simplicity, we present the proof when $\mu := \aleph_1$. Let $X \subseteq M_{\mathbf{c}}$ be countable. We are going to show that it is included into a countably generated free R-submodule of $M_{\mathbf{c}}$. As X countable,

- $\exists \Lambda \subseteq \Lambda_{\mathbf{c},\omega}$ countable,
- $\exists \Lambda_* \subseteq \Lambda_{\mathbf{c},<\omega}$ countable

such that

$$X \subseteq \sum\{Ry_{\eta,n} : \eta \in \Lambda \textit{ and } n < \omega\} + \sum\{Rx_\rho : \rho \in \Lambda_*\}.$$

As $\Lambda_{\mathbf{c}}$ is $\aleph_1$-free and Λ is countable, there is a function $h : \Lambda \to \omega$ such that the sequence

$$\Big\langle \{\eta{\restriction}_L n, \eta{\restriction}_R n : h(\eta) \leq n < \omega\} : \eta \in \Lambda \Big\rangle$$

is a sequence of pairwise disjoint sets. Now, we note the following two properties:

$(b)_1$: The R-module

$$M_\Lambda := \left\langle x_{\eta\restriction_L n}, x_{\eta\restriction_R n}, y_{\eta,n} : \eta \in \Lambda : h(\eta) \leq n < \omega \right\rangle$$

is free;

$(b)_2$: Set $M_{\Lambda\cup\Lambda_*} := \left\langle M_\Lambda \cup \{x_\nu : \nu \in \Lambda_*\}\right\rangle$. Then the R-module $M_{\Lambda\cup\Lambda_*}/M_{\Lambda_*}$ is free.

In view of $(b)_2$ the short exact sequence

$$0 \longrightarrow M_\Lambda \longrightarrow M_{\Lambda\cup\Lambda_*} \longrightarrow M_{\Lambda\cup\Lambda_*}/M_\Lambda \longrightarrow 0,$$

splits. Combining this along with $(b)_1$, we observe that $M_{\Lambda\cup\Lambda_*}$ is free. Since it includes X, we get the desired claim.

(c) : Now, suppose $(R,+)$ is μ-free. Let H be a subset of $(G_{\mathbf{c}}/K,+)$ of size $< \mu$. There is a free R-module F such that $H \subset F$. There is a subset S of R of size $< \mu$ such that any element of H can be written from a linear combination from F with coefficients taken from S. As $(R,+)$ is μ-free, there is a free subgroup $(T,+)$ of it containing S. In other words,

$$H \subseteq T * F := \left\langle \sum\{t_i f_i : t_i \in T, f_i \in F\} \right\rangle.$$

Since $(T * F,+)$ is free as an abelian group, we get the desired claim.

(4): Let $\mathbf{d}$ be such that:

4.1) $\Lambda_{\mathbf{d}} = \Lambda \cap \Lambda[\gamma]$,

4.2) $X_{\mathbf{d}}$ is defined using $\Lambda_{\mathbf{d}}$ naturally,

4.3) for $\nu \in \Lambda_{\mathbf{d},\omega}$ and $n < \omega$, $a^{\mathbf{d}}_{\nu,n} = a^{\mathbf{c}}_{\nu,n}$,

4.4) $\Gamma_{\mathbf{d}}$ is defined naturally as the set of equations in (1), but only for $\eta \in \Lambda_{\mathbf{d},\omega}$.

This is straightforward to check that $\mathbf{d}$ is as required.

(5): Let $\mathbf{d}$ be defined in the natural way, so that:

5.1) $\Lambda_{\mathbf{d}} = \Lambda_{\mathbf{c}} \cup \{\eta\} \cup \{\eta\restriction_L n, \eta\restriction_R n : n < \omega\}$,

5.2) $X_{\mathbf{d}} = X_{\mathbf{c}} \cup \{x_{\eta\restriction_L n}, x_{\eta\restriction_R n} : n < \omega\} \cup \{y_{\eta,n} : n < \omega\}$,

5.3) for $\nu \in \Lambda_{\mathbf{c},\omega}$ and $n < \omega$, $a^{\mathbf{d}}_{\nu,n} = a^{\mathbf{c}}_{\nu,n}$,

5.4) $a^{\mathbf{d}}_{\eta,n} = a_n$ for $n < \omega$, 5.5) in addition to the equations displayed in $\Gamma_{\mathbf{c}}$, $\Gamma_{\mathbf{d}}$ contains equations of the following forms

$$y_{\eta,n} = a_n + (n!)y_{\eta,n+1} + (x_{\eta\restriction_L n} - x_{\eta\restriction_R n}),$$

where $n < \omega$.

The assertion is now obvious by the above definition of $\mathbf{d}$.

(6). In view of Lemma 3.19, the group $G_{\mathbf{c}}$ is of size $\lambda^{\aleph_0}$. Recall from Hypothesis 3.14 that $\lambda^{\aleph_0} = \lambda$. So, the desired claim is clear. □

Lemma 3.27. *Let* $\mathbf{c} \in \mathrm{AP}$. *Then the abelian group* $G_{\mathbf{c}}/K$ *is reduced.*

Proof. Suppose on the way of contradiction that $G_{\mathbf{c}}/K$ is not reduced. Then it has a divisible direct summand, say I. By Fact 2.18, I is injective. We apply the structure theorem for injective abelian groups (see Discussion 2.20(i)) to find the following decomposition:

$$I = \bigoplus_{p\in\mathbb{P}} \mathbb{Z}(p^\infty)^{\oplus x_p} \oplus \mathbb{Q}^{\oplus x},$$

where x_p and x are index sets. Since $G_{\mathbf{c}}/K$ is torsion-free, I is torsion-free. So, I has no p-torsion part. This shows that $x_p = \emptyset$ for all $p \in \mathbb{P}$. In other words, $I = \mathbb{Q}^{\oplus x}$. Since I is nonzero, $x \neq \emptyset$. This yields that $(\mathbb{Q}, +)$ is a directed summand of $G_{\mathbf{c}}/K$. Thanks to Lemma 3.26(3)(a) $\Lambda_{\mathbf{c}}$ is $\aleph_1$-free. We combine this with Lemma 3.26(3)(b) to deduce that $M_{\mathbf{c}}$ is $\aleph_1$-free as an R-module.

We have two possibilities: 1) $\mathbf{k}$ is trivial, and 2) $\mathbf{k}$ is nontrivial.

1) $\mathbf{k}$ is trivial: Then $R := \mathbb{Z}$. Recall that $M_{\mathbf{c}} = G_{\mathbf{c}}/K$ is $\aleph_1$-free. Since $(\mathbb{Q}, +)$ is countable, it should be free, a contradiction.

2) $\mathbf{k}$ is nontrivial: Recall that R is $S_{\mathbf{k}}^{\perp}$-divisible. Since the context is nontrivial, there is $p \in S_{\mathbf{k}}^{\perp}$ such that $\{1/p^n : n \gg 0\} \subseteq R$. For simplicity, we assume that $\{1/p^n : n > 0\} \subseteq R$. Since $M_{\mathbf{c}}$ is $\aleph_1$-free and that $\{1/p^n : n > 0\} \subseteq \mathbb{Q} \subseteq M_{\mathbf{c}}$, there is a free R-module $F \subseteq M_{\mathbf{c}}$ such that $\{1/p^n : n > 0\} \subseteq F$. Let $F = \bigoplus R$. So, the

desired contraction follows by:

$$\begin{aligned}\{r/p^n : n > 0, r \in R\} &= \bigcap_{\ell>0} p^\ell \{r/p^n : n > 0, r \in R\}\\ &\subseteq \bigcap_{\ell>0} p^\ell F\\ &= \bigoplus(\bigcap_{\ell>0} p^\ell R)\\ &\subseteq \bigoplus(\bigcap_{\ell>0} \ell R)\\ &= 0,\end{aligned}$$

where the last equality comes from the fact that $(R, +)$ is cotorsion-free, in fact by Fact 2.19, the abelain group $(R, +)$ is reduced, and so $\bigcap_{\ell>0} \ell R = 0$. The proof is now complete. $\square$

The following easy lemma will be used later at several places.

Lemma 3.28. *Let $\mathbf{c} \in \mathrm{AP}_{\mathbf{k},\lambda}$. Then the following equation*

$$y^{\mathbf{c}}_{\eta,0} = \sum_{i=0}^{n} \big(\prod_{j<i} j!\big) a^{\mathbf{c}}_{\eta,i} + (\prod_{i=1}^{n} i!) y^{\mathbf{c}}_{\eta,n+1} + \sum_{i=0}^{n} \big(\prod_{j<i} j!\big)(x^{\mathbf{c}}_{\eta\restriction_L i} - x^{\mathbf{c}}_{\eta\restriction_R i}),$$

is valid for any $n < \omega$.

Proof. We proceed by induction on n. The desired claim is clearly holds for $n = 0$. Suppose inductively that it holds for n. We are going to show the claim for $n + 1$. To this end, we apply the induction assumption along with the relation

$$y^{\mathbf{c}}_{\eta,n+1} = a^{\mathbf{c}}_{\eta,n+1} + (n + 1)! y^{\mathbf{c}}_{\eta,n+2} + (x^{\mathbf{c}}_{\eta\restriction_L n+1} - x^{\mathbf{c}}_{\eta\restriction_R n+1})$$

to deduce

$$\begin{aligned} y^{\mathbf{c}}_{\eta,0} &= \sum_{i=0}^{n} \big(\prod_{j<i} j!\big) a^{\mathbf{c}}_{\eta,i} + \big(\prod_{i=1}^{n} i!\big) y^{\mathbf{c}}_{\eta,n+1} + \sum_{i=0}^{n+1} (x^{\mathbf{c}}_{\eta\restriction_L i} - x^{\mathbf{c}}_{\eta\restriction_R i})\\ &= \sum_{i=0}^{n} \big(\prod_{j<i} j!\big) a^{\mathbf{c}}_{\eta,i} + \big(\prod_{i=0}^{n} i!\big) a^{\mathbf{c}}_{\eta,n+1} + \big(\prod_{i=1}^{n} i!\big)(n + 1)! y^{\mathbf{c}}_{\eta,n+2}\\ &\quad + \big(\prod_{i=0}^{n} i!\big)(x^{\mathbf{c}}_{\eta\restriction_L n+1} - x^{\mathbf{c}}_{\eta\restriction_R n+1}) + \sum_{i=0}^{n} \big(\prod_{j<i} j!\big)(x^{\mathbf{c}}_{\eta\restriction_L i} - x^{\mathbf{c}}_{\eta\restriction_R i})\\ &= \sum_{i=0}^{n+1} \big(\prod_{j<i} j!\big) a^{\mathbf{c}}_{\eta,i} + \big(\prod_{i=1}^{n+1} i!\big) y^{\mathbf{c}}_{\eta,n+2} + \sum_{i=0}^{n+1} \big(\prod_{j<i} j!\big)(x^{\mathbf{c}}_{\eta\restriction_L i} - x^{\mathbf{c}}_{\eta\restriction_R i}).\end{aligned}$$

Thus the claim holds for $n + 1$ as well. $\square$

There are some distinguished and useful objects in $\mathrm{AP}_{\mathbf{k},\lambda}$:

Definition 3.29. We say $\mathbf{c} \in \mathrm{AP}_{\mathbf{k},\lambda}$ is full when:

(a) $\Lambda_{\mathbf{c}} \supseteq \Lambda_{<\omega}[\lambda]$,

(b) if $a_n \in G_{\mathbf{c}}$ for $n < \omega$ and $f : \Lambda_{<\omega}[\lambda] \to \gamma$, where $\gamma < \lambda$, then for some $\eta \in \Lambda_{\mathbf{c}}$ and all $n < \omega$ we have $a^{\mathbf{c}}_{\eta,n} = a_n$ and $f(\eta \restriction_L n) = f(\eta \restriction_R n)$.

Now, we study the existence problem for fullness in AP:

Lemma 3.30. *Adopt the notation from Hypothesis 3.14. Then there are some full* $\mathbf{c} \in \mathrm{AP}_{\mathbf{k},\lambda}$.

Proof. Let $\mathbf{b}$ be a twofold λ-black box, which exists by Lemma 3.15. We look at

$$\Omega := \Lambda_{<\omega}[\lambda] \cup \big(\Lambda_{\omega}[\lambda] \times \omega\big),$$

and for each ordinal $\alpha < \lambda$ we set

$$\Omega_\alpha := \Lambda_{<\omega}[\alpha] \cup \big(\Lambda_{\omega}[\alpha] \times \omega\big).$$

Fix a bijection map

$$h : S_\lambda \xrightarrow{\cong} (\oplus_{\rho\in\Omega} R x_\rho) \oplus K$$

such that for each ordinal $\alpha < \lambda$ one has

$$h''[S_\alpha] \subseteq (\oplus_{\rho\in\Omega_\alpha} R x_\rho) \oplus K \quad (*),$$

This is possible, as for each α,

$$|S_\alpha| \le \aleph_0 + |\alpha| \le |(\oplus_{\rho\in\Omega_\alpha} R x_\rho) \oplus K| < \lambda.$$

Let $\mathbf{c}$ be defined as

(1) $\Lambda_{\mathbf{c}} = \Lambda_\omega[\lambda] \cup \Lambda_{<\omega}[\lambda]$.

(2) $X_{\mathbf{c}}$ is the following set:

$$\{r x_\nu : r \in R, \nu \in \Lambda_{\mathbf{c},<\omega}\} \cup \{r y_{\eta,n} : r \in R, \eta \in \Lambda_{\mathbf{c},\omega}, n < \omega\}.$$

(3) $a^{\mathbf{c}}_{\eta,n} = h(g^{\mathbf{b}}_\eta(n+1))$, where $g^{\mathbf{b}}_\eta$ is given by the twofold λ-black box.

(4) $G_{\mathbf{c}}$ is generated, as an abelian group, freely by the sets K and $X_{\mathbf{c}}$ except the following set of relations:

$$y_{\eta,n} \quad = a^{\mathbf{c}}_{\eta,n} + (n!) y_{\eta,n+1} + (x_{\eta\restriction_L n} - x_{\eta\restriction_R n}),$$

with the convenience that $a^{\mathbf{c}}_{\eta,n}$ is regarded as an element of $G_{\mathbf{c}}$ via the quotient map

$$(\bigoplus_{\rho\in\Omega} Rx_\rho) \oplus K \twoheadrightarrow G_{\mathbf{c}}.$$

From this identification and (*), $a^{\mathbf{c}}_{\eta,n} \in G_{\mathbf{c},\eta(0,1)}$.

(5) $\Gamma_{\mathbf{c}}$ is defined naturally as in Definition 3.18.

Let us show that $\mathbf{c}$ is as required. It clearly satisfies clause (a) of Definition 3.29. To show that clause (b) of Definition 3.29 is satisfied, let $\langle a_n : n < \omega\rangle \in {}^{\omega}G_{\mathbf{c}}$ and $f : \Lambda_{<\omega}[\lambda] \to \gamma$, where $\gamma < \lambda$. Let $g : \Lambda_{<\omega}[\lambda] \to S_\lambda$ be defined such that for all $\nu \in \Lambda_{<\omega}[\lambda] \setminus \{\langle\rangle\}$,

$$h(g(\nu)) = a_{\lg(\nu)-1} \quad (+).$$

We are going to apply the twofold λ-black box $\mathbf{b}$. According to its properties, there is an $\eta \in \Lambda_\omega[\lambda]$ such that:

(6) $\gamma(\eta) > \gamma$,

(7) $g^{\mathbf{b}}_\eta(0) = g(\langle\rangle)$,

(8) $g^{\mathbf{b}}_\eta(n+1) = g(\eta \restriction_L n)$[2],

(9) $\eta(n,1) < \eta(n,2)$ and $f(\eta \restriction_L n) = f(\eta \restriction_R n)$ for all $1 \le n < \omega$.

Applying h to the both sides of (8), one has

$$a^{\mathbf{c}}_{\eta,n} \overset{(3)}{=} h(g^{\mathbf{b}}_\eta(n+1)) = h(g(\eta\restriction_L n)) \overset{(+)}{=} a_n,$$

thereby completing the proof. □

Lemma 3.31. *Assume* $\mathbf{c} \in \mathrm{AP}$ *is full and let* $h \in \mathrm{Hom}(G_{\mathbf{c}}, K)$ *be unbounded. Then there is a sequence*

$$\langle a_n : n < \omega\rangle \in {}^{\omega}\mathrm{Rang}(h)$$

such that the following set of equations Γ *has no solution, not only in* $G_{\mathbf{c}}$*, but in any* $G_{\mathbf{d}}$ *with* $\mathbf{c} \le \mathbf{d} \in \mathrm{AP}$*, where*

$$\Gamma := \{z_n = a_n + n! z_{n+1} : n < \omega\}.$$

[2]Here we are using a modified version of the twofold λ-black box $\mathbf{b}$, which can be easily obtained from the original one.

Proof. We have two possibilities. First, suppose for some prime number p, the group $\Gamma_p(\mathrm{Rang}(h))$ is infinite, and let p be the first such prime number. Also, let $p_n = p$ for all $n < \omega$. Otherwise, we let

$$p_n \in \{p : \Gamma_p(\mathrm{Rang}(h)) \neq 0\}$$

be a strictly increasing sequence of prime numbers. We refer this as a second possibility.

In the first part of the proof, we argue for both possibilities at the same time. Then, we consider each scenario separately.

Since h is not bounded, we can find by induction on n, the pair (H_n, a_n) such that:

(+) (a) $H_0 = \mathrm{Rang}(h)$,
(b) $H_n = a_n\mathbb{Z} \oplus H_{n+1}$,
(c) a_n has order $p_n^{\mathbf{l}_n}$,
(d) for $n = m + 1$ we have

$$(d_n) : \quad \mathbf{l}_n > \mathbf{l}_m + \left(\prod_{i=0}^{n+1} i!\right).$$

To see this, let $H_0 := \mathrm{Rang}(h)$ and let $a_0 \in \Gamma_{p_0}[\mathrm{Rang}(h)]$ be any nonzero element. Now, suppose inductively that $n > 0$ and we have defined $\langle H_i : i \leq n\rangle$ and $\langle a_i : i < n\rangle$ satisfying the above items. We shall now define a_n and H_{n+1}. By our induction assumption,

$$\mathrm{Rang}(h) = (\bigoplus_{i<n} a_i\mathbb{Z}) \oplus H_n.$$

In particular, H_n is torsion. Using Fact 2.5 (and also Fact 2.7 in the second possibility case), we can find for some ℓ_n and an element a_n such that a_n has order $p_n^{\mathbf{l}_n}$ and $a_n\mathbb{Z}$ is a direct summand of H_n. We may further suppose that

$$\mathbf{l}_n > \mathbf{l}_m + \left(\prod_{i=0}^{n+1} i!\right).$$

Since (a_n) is a direct summand of H_n, there is an abelian group H_{n+1} so that $H_n = a_n\mathbb{Z} \oplus H_{n+1}$.

To prove that the sequence $\langle a_n : n < \omega \rangle$ is as required, assume towards a contradiction that there is $\mathbf{c} \leq \mathbf{d} \in \mathrm{AP}$ such that $\langle c_n : n < \omega \rangle$ is a solution of Γ in $G_{\mathbf{d}}$. So

$$G_{\mathbf{d}} \models \bigwedge_{n<\omega} \big(c_n = a_n + n! c_{n+1}\big) \quad (*)$$

Since for each $n, a_n \in K$, it follows that

$$G_{\mathbf{d}}/K \models \bigwedge_{n<\omega} \big(c_n + K = n! c_{n+1} + K\big).$$

By Lemma 3.27, $G_{\mathbf{c}}/K$ is reduced, hence necessarily,

$$\bigwedge_{n<\omega} \big(c_n + K = 0 + K\big).$$

In other words, $c_n \in K$ for all $n < \omega$.

We now show that for each n,

$$\left(\prod_{i<n} i!\right) c_n \in H_n \quad (**)$$

This is true for $n = 0$, because $c_0 \in K = H_0$. Suppose it holds for n. Then multiplying both sides of $(*)$ into $\prod_{i<n} i!$ we get

$$\left(\prod_{i<n} i!\right) c_n = \left(\prod_{i<n} i!\right) a_n + \left(\prod_{i<n+1} i!\right) c_{n+1}.$$

Using the induction hypothesis and $(\star)(b)$ we get

$$\left(\prod_{i<n+1} i!\right) c_{n+1} \in H_{n+1},$$

as requested.

By an easy induction, for each n we have

$$c_0 = a_0 + \sum_{\ell \leq n} \left(\prod_{i=1}^{\ell} i!\right) a_\ell + \left(\prod_{i=1}^{n} i!\right) c_{n+1} \quad (***)_n$$

Indeed this is true for $n = 0$, as $c_0 = a_0 + c_1$. Suppose it holds for n, then using $(*)$ and the induction hypothesis

$$\begin{aligned} c_0 &= a_0 + \sum_{\ell \leq n} \left(\prod_{i=1}^{\ell} i!\right) a_\ell + \left(\prod_{i=1}^{n} i!\right) c_{n+1} \\ &= a_0 + \sum_{\ell \leq n} \left(\prod_{i=1}^{\ell} i!\right) a_\ell + \left(\prod_{i=1}^{n} i!\right) \left(a_{n+1} + (n+1)! c_{n+2}\right) \\ &= a_0 + \sum_{\ell \leq n+1} \left(\prod_{i=1}^{\ell} i!\right) a_\ell + \left(\prod_{i=1}^{n+1} i!\right) c_{n+2}. \end{aligned}$$

We are now ready to complete the proof. Let $m(*)$ be the order of c_0.

Now, we consider each case separately:

Case 1. $p_n = p$ **for all** n**:** Let t be an integer such that

$$m(*) = tp^{\ell(*)} > 1,$$

where $\ell(*) \geq 0$ and $(p,t) = 1$, i.e., p does not divide t. Let k be the least natural number such that $\mathbf{l}_k > \ell(*)$. By multiplying both sides of $(***)_{k+1}$ into $tp^{\mathbf{l}_k}$, we get to

$$tp^{\mathbf{l}_k} c_0 = tp^{\mathbf{l}_k} a_0 + tp^{\mathbf{l}_k} \sum_{\ell \leq k+1} \left(\prod_{i=1}^{\ell} i! \right) a_\ell + tp^{\mathbf{l}_k} \left(\prod_{i=1}^{k+1} i! \right) c_{k+2}.$$

Since the sequence $\langle \mathbf{l}_\ell : \ell \leq k \rangle$ is increasing, we have $p^{\mathbf{l}_k} a_\ell = 0$ for all $\ell \leq k$. Consequently,

$$0 = tp^{\mathbf{l}_k} \left(\prod_{i=1}^{k+1} i! \right) a_{k+1} + tp^{\mathbf{l}_k} \left(\prod_{i=1}^{k+1} i! \right) c_{k+2} \quad (\dagger)$$

According to $(+)_b$, we know $a_{k+1}\mathbb{Z} \cap H_{k+2} = 0$, and by using $(**)$ along with $(\dagger)$ we get that

$$tp^{\mathbf{l}_k} \left(\prod_{i=1}^{k+1} i! \right) a_{k+1} = 0.$$

Recall that the order of a_{k+1} is a power of p. We apply this along with the equality $(p,t) = 1$ to get that

$$p^{\mathbf{l}_k} \left(\prod_{i=1}^{k+1} i! \right) a_{k+1} = 0.$$

Moreover,

$$p^{\mathbf{l}_{k+1}} = \mathrm{ord}(a_{k+1}) \leq p^{\mathbf{l}_k} \left(\prod_{i=1}^{k+1} i! \right) \leq p^{\mathbf{l}_k + (\prod_{i=1}^{k+1} i!)}.$$

Taking $\log_p(-)$ from both sides, we have $\mathbf{l}_{k+1} \leq \mathbf{l}_k + \left(\prod_{i=1}^{k+1} i! \right)$. But, this contradicts $(d_{\mathbf{l}_{k+1}})$. The result follows.

Thereby, without loss of generality we deal with:

Case 2. Otherwise: Then the sequence $\langle p_n : n < \omega \rangle$ is strictly increasing. Let k be the least integer such that

$$p_{k+1} > m(*) \times \left(\prod_{i=1}^{k+1} i! \right) \quad (\dagger\dagger)$$

By multiplying both sides of $(***)_{k+1}$ into $m(*) \times \left(\prod_{i=1}^{k} p_i^{\mathbf{l}_i} \right)$ we get

$$\begin{aligned} 0 &= m(*) \times \left(\prod_{i=1}^{k} p_i^{\mathbf{l}_i}\right) c_0 \\ &= m(*) \times \left(\prod_{i=1}^{k} p_i^{\mathbf{l}_i}\right) a_0 + m(*) \times \left(\prod_{i=1}^{k} p_i^{\mathbf{l}_i}\right) \sum_{\ell \leq k+1} \left(\prod_{i=1}^{\ell} i!\right) a_\ell \\ &+ m(*) \times \left(\prod_{i=1}^{k} p_i^{\mathbf{l}_i}\right) \left(\prod_{i=1}^{k+1} i!\right) c_{k+2}. \end{aligned}$$

We have that $m(*) \times \left(\prod_{i=1}^{k} p_i^{\mathbf{l}_i}\right) a_0 = 0$ and

$$m(*) \times \left(\prod_{i=1}^{k} p_i^{\mathbf{l}_i}\right) \left(\prod_{i=1}^{\ell} i!\right) a_\ell = 0,$$

for all $\ell \leq k$, thus

$$0 = m(*) \times \left(\prod_{i=1}^{k} p_i^{\mathbf{l}_i}\right) \left(\prod_{i=1}^{k+1} i!\right) a_{k+1} + m(*) \times \left(\prod_{i=1}^{k} p_i^{\mathbf{l}_i}\right) \left(\prod_{i=1}^{k+1} i!\right) c_{k+2}.$$

Again, according to $(+)_b$, we know $a_{k+1}\mathbb{Z} \cap H_{k+2} = 0$, and by using $(**)$ along with the previous formula, we lead to the following vanishing formula

$$m(*) \times \left(\prod_{i=1}^{k} p_i^{\mathbf{l}_i}\right) \left(\prod_{i=1}^{k+1} i!\right) a_{k+1} = 0.$$

As the order of a_{k+1} is a power of p_{k+1} and it is different from all p_ℓ's, for $\ell \leq k$, we have

$$m(*) \times \left(\prod_{i=1}^{k+1} i!\right) a_{k+1} = 0.$$

So,

$$p_{k+1} < p_{k+1}^{\mathbf{l}_{k+1}} = \operatorname{ord}(a_{k+1}) \leq m(*) \times \left(\prod_{i=1}^{k+1} i!\right).$$

But this contradicts $(\dagger\dagger)$. The result follows. □

To prove the endo-rigidity property, we first deal with the following special case, and then we reduce things to this situation:

Lemma 3.32. *Let* $\mathbf{c} \in \mathrm{AP}$ *be full. Then every* $h \in \operatorname{Hom}(G_{\mathbf{c}}, K)$ *is bounded.*

Proof. Towards a contradiction assume $h \in \operatorname{Hom}(G_{\mathbf{c}}, K)$ is not bounded. In view of Lemma 3.31, this implies that there is a sequence

$$\langle a_n : n < \omega \rangle \in {}^{\omega}\mathrm{Rang}(h)$$

such that the set of equations

$$\Gamma := \{z_n = a_n + n! z_{n+1} : n < \omega\}$$

has no solutions in $G_{\mathbf{c}}$. Let $\gamma = |K|$, and define $f : \Lambda_{<\omega}[\lambda] \to \gamma$ such that

$$f(\eta) = f(\nu) \Longleftrightarrow h(x_\eta) = h(x_\nu) \quad (*)$$

Since $a_n \in \mathrm{Rang}(h)$ there is b_n such that

$$\forall n < \omega,\ a_n = h(b_n) \qquad (+)$$

As $\mathbf{c}$ is full, we can find some η such that

(1) $f(\eta{\restriction}_L n) = f(\eta{\restriction}_R n)$,

(2) $a^{\mathbf{c}}_{\eta,n} = b_n$ for each n.

Let us combining $(*)$ and (1). This yields that

$$\forall n < \omega,\ h(x_{\eta{\restriction}_L n}) = h(x_{\eta{\restriction}_R n}) \qquad (\dagger).$$

Moreover, by applying h to the both sides of the equation

$$y_{\eta,n} = a^{\mathbf{c}}_{\eta,n} + (n!)y_{\eta,n+1} + (x_{\eta{\restriction}_L n} - x_{\eta{\restriction}_R n}),$$

we lead to the following equation:

$$\begin{aligned} h(y_{\eta,n}) &= h(a^{\mathbf{c}}_{\eta,n}) + n!h(y_{\eta,n+1}) + \big(h(x_{\eta{\restriction}_L n}) - h(x_{\eta{\restriction}_R n})\big) \\ &\overset{(2)}{=} h(b_n) + n!h(y_{\eta,n+1}) + \big(h(x_{\eta{\restriction}_L n}) - h(x_{\eta{\restriction}_R n})\big) \\ &\overset{(\dagger)}{=} h(b_n) + (n!)h(y_{\eta,n+1}) \\ &\overset{(+)}{=} a_n + (n!)h(y_{\eta,n+1}). \end{aligned}$$

In other words, $h(y_{\eta,n})$ is a solution for

$$\Gamma = \{z_n = a_n + n! z_{n+1} : n < \omega\}.$$

This is a contradiction with the choice of the sequence $\langle a_n : n < \omega \rangle$. □

Notation 3.33. Suppose $\mathbf{c} \in \mathrm{AP}$. For each $n < \omega$, we define

$$G_n := \frac{G_{\mathbf{c}}}{K + \big(\prod_{i=1}^n i!\big) G_{\mathbf{c}}}.$$

Also, the notation π_n stands for the natural projection $G_{\mathbf{c}} \twoheadrightarrow G_n$.

Fact 3.34. *Adopt the above notation, let $n < \omega$ and $g \in G_{\mathbf{c}}$.*

(a) *The abelian group G_n is a torsion abelian group with the following minimal generating set*

$$\{x_\rho : \rho \in \Lambda_{\mathbf{c},<\omega}\} \cup \{y_{\eta,k} : \eta \in \Lambda_{\mathbf{c},\omega} \text{ and } k \geq n+2\}.$$

(b) *Similar to Definition 3.20, we can define $\mathrm{supp}_\circ(\pi_n(g))$ with respect to generating set presented in clause (a).*

(c) *According to its definition, it is easy to see that $\mathrm{supp}_\circ(\pi_n(g)) \subseteq \mathrm{supp}_\circ(g)$.*

(d) *Recall from Lemma 3.27 that G_c/K is reduced. This in turns gives us an integer $m_n > n$ such that $\mathrm{supp}_\circ(g) \subseteq \mathrm{supp}_\circ(\pi_{m_n}(g))$.*

Proof. This is straightforward. □

Lemma 3.35. *Suppose $\mathbf{c} \in \mathrm{AP}$ is full and $h \in \mathrm{End}(G_{\mathbf{c}})$. Then for some countable $\Lambda_h \subseteq \Omega_{\mathbf{c}}$ we have:*

$$r \in R,\ \nu \in \Omega_{\mathbf{c}} \setminus \Lambda_h \implies \mathrm{supp}_\circ(h(rx_\nu)) \subseteq \{\nu\} \cup \Lambda_h.$$

Proof. Towards contradiction assume $h \in \mathrm{End}(G_{\mathbf{c}})$ but there is no Λ_h as promised. We define a sequence

$$\big\langle(\eta_i, Y_i, \nu_i, r_i) : i < \omega_1\big\rangle,$$

by induction on $i < \omega_1$, such that

(∗) (a) $\eta_i \in \Omega_{\mathbf{c}}$ and $r_i \in R \setminus \{0\}$,

(b) $Y_i = \bigcup\{\mathrm{supp}_\circ(h(r_j x_{\eta_j})) : j < i\} \cup \{\eta_j : j < i\}$,

(c) $\nu_i \in \mathrm{supp}_\circ(h(r_i x_{\eta_i}))$ but $\nu_i \neq \eta_i$, $\nu_i \notin Y_i$.

To this end, suppose that $i < \omega_1$ and we have defined $\langle(\eta_j, Y_j, \nu_j, r_j) : j < i\rangle$. Set

$$Y_i = \bigcup\{\mathrm{supp}_\circ(h(r_j x_{\eta_j})) : j < i\} \cup \{\eta_j : j < i\}.$$

Following its definition, we know Y_i is at most countable. Thus, due to our assumption, we can find some $\eta_i \in \Omega_{\mathbf{c}} \setminus Y_i$ and $r_i \in R \setminus \{0\}$ such that

$$\mathrm{supp}_\circ(h(r_i x_{\eta_i})) \nsubseteq (\{\eta_i\} \cup Y_i).$$

This allow us to define ν_i, namely, it is enough to take ν_i be any element of $\mathrm{supp}_\circ(h(r_i x_{\eta_i})) \setminus (\{\eta_i\} \cup Y_i)$. This completes the definition of $(\eta_i, Y_i, \nu_i, r_i)$.

Combining the facts $\nu_i \in \mathrm{supp}_\circ(h(r_i x_{\eta_i}))$ and $\nu_i \notin (Y_i \cup \{\eta_i\})$ along with the finiteness of $\mathrm{supp}_\circ(h(x_{\eta_i}))$ we are able to find a subset $W \subseteq \omega_1$ of cardinality ω_1 such that

(∗) If $i \neq j \in W$ then $\nu_j \notin \mathrm{supp}_\circ(h(r_i x_{\eta_i}))$.

Without loss of generality we may and do assume that $W = \omega_1$. Let $a_i = r_i x_{\eta_i}$. We can find

$$f : \Lambda_{\mathbf{c},<\omega} \longrightarrow |R| + \aleph_0 < \lambda$$

such that if $b \in G_{\mathbf{c}}$[3], then from $f(b)$ we can compute

$$\big\langle n_b, \{(\ell, m_{b,\ell}, r_{b,\ell}) : \ell < n_b\}\big\rangle.$$

Recall that $\mathbf{c}$ is full, and that $\mathrm{Rang}(f)$ has size less than λ. From these, there is some $\eta \in \Lambda_{\mathbf{c},\omega}$ furnished with the the following two properties:

(1) $f(\eta \restriction_L n) = f(\eta \restriction_R n)$, for $n < \omega$,
(2) $a^{\mathbf{c}}_{\eta,n} = a_n$ for all $n < \omega$.

Now, we bring the following claim.

Claim: $\nu_i \in \mathrm{supp}_0(h(y_{\eta 0})) \quad \forall i < \omega$.

Note that this will give us the desired contradiction, as $\mathrm{supp}_0(h(y_{\eta 0}))$ is finite. Now we turn to the proof of the claim.

Proof of Claim. By Lemma 3.28 we first observe that:

$$y_{\eta,0} \;= \sum_{i=0}^{n} \big(\textstyle\prod_{j<i} j!\big) r_i x_{\eta_i} + \big(\textstyle\prod_{i=1}^{n} i!\big) y_{\eta,n+1} + \sum_{i=0}^{n} \big(\textstyle\prod_{j<i} j!\big)(x_{\eta\restriction_L i} - x_{\eta\restriction_R i}).$$

Let ℓ be any integer. We are going to use the notation presented in Notation 3.33 for $n = m_\ell$. Applying $\pi_n h(-)$ to it, yields that

[3] Recall we have chosen

$$b - \sum_{\ell<n_b} r_{b,\ell} y_{\eta_{b,\ell}, m_{b,\ell}} \in \sum_{\rho \in \Lambda_{\mathbf{c},<\omega}} R x_\rho + K.$$

$$\begin{aligned}(3)\quad \pi_n(h(y_{\eta,0})) &= \sum_{i=0}^{n}\big(\textstyle\prod_{j<i} j!\big)\pi_n h(r_i x_{\eta_i}) + (\prod_{i=1}^{n} i!)\pi_n h(y_{\eta,n+1}) \\ &\quad+\sum_{i=0}^{n}\big(\textstyle\prod_{j<i} j!\big)\pi_n h(x_{\eta\restriction_L i} - x_{\eta\restriction_R i}) \\ &= \sum_{i=0}^{n}\big(\textstyle\prod_{j<i} j!\big)\pi_n h(r_i x_{\eta_i}) + \sum_{i=0}^{n}\big(\textstyle\prod_{j<i} j!\big)\pi_n h(x_{\eta\restriction_L i} - x_{\eta\restriction_R i}),\end{aligned}$$

where the last equality follows by Definition 3.33. Now, we recall from the construction $(*)$ that:

(3.1) $\nu_i \in \mathrm{supp}_\circ(h(r_i x_{\eta_i}))$,

(3.2) $\nu_i \neq \eta_i$ and $\nu_i \notin Y_i$.

Thanks to Fact 3.34(d) we have

(4) $\nu_i \in \mathrm{supp}_\circ(\pi_n h(r_i x_{\eta_i}))$.

By clause (1) above, $\mathrm{supp}_\circ(h(x_{\eta\restriction_L i} - x_{\eta\restriction_R i})) = \emptyset$. In view of Fact 3.34(c), we deduce that

(5) $\mathrm{supp}_\circ\big(\pi_n(h(x_{\eta\restriction_L i} - x_{\eta\restriction_R i}))\big) = \emptyset$.

First, we plug items (4) and (5) in the clause (3), then we use $(*)$. These enable us to observe that

$$\begin{aligned}\nu_i &\in \mathrm{supp}_\circ\left(\sum_{i=0}^{n}\big(\textstyle\prod_{j<i} j!\big)\pi_n h(r_i x_{\eta_i}) + \sum_{i=0}^{n}\big(\textstyle\prod_{j<i} j!\big)\pi_n h(x_{\eta\restriction_L i} - x_{\eta\restriction_R i})\right) \\ &= \mathrm{supp}_\circ(\pi_n h(y_{\eta,0})).\end{aligned}$$

Another use of Fact 3.34(c), shows that $\nu_i \in \mathrm{supp}_\circ(h(y_{\eta,0}))$. This completes the proof of the claim. □

The lemma follows. □

The following lemma can be proved easily.

Lemma 3.36. *Let* $\mathbf{c} \in \mathrm{AP}$ *be full and* $h \in \mathrm{End}(G_{\mathbf{c}})$. *Let* $Y_0 \subseteq \Omega_{\mathbf{c}}$ *be the downward closure of* Λ_h, *where* Λ_h *is as in Lemma 3.35 and set*

$$K^+ := K + \sum_{\rho \in Y_0 \cap \Lambda_{\mathbf{c},<\omega}} R x_\rho + \sum_{\rho \in Y_0 \cap \Lambda_{\mathbf{c},\omega} n<\omega} R y_{\rho,n}.$$

If $b \in G_{\mathbf{c}}$, *then there are choices*

- $\bar{r}_b := \langle r^2_{b,\rho} : \rho \in \Lambda_b \rangle$, *and*
- $\Lambda_{\mathbf{b}} \subseteq \Lambda_{\mathbf{c},<\omega} \setminus Y_0$ *finite*

such that

$$b - \sum_{\rho \in \Lambda_{\mathbf{b}}} r^2_{b,\rho} x_\rho \in K^+.$$

Proof. This is straightforward. □

Hypothesis 3.37. For the rest of this section, we fix a well-ordering $\prec$ of the large enough part of the universe, and for each:

- $\mathbf{c} \in \mathrm{AP}$ which is full,
- $h \in \mathrm{End}(G_{\mathbf{c}})$, and
- $b \in G_{\mathbf{c}}$,

we let $\bar{r}_b := \langle r^2_{b,\rho} : \rho \in \Lambda_b \rangle$ be the $\prec$-least sequence satisfying the conclusions of Lemma 3.36.

Notation 3.38. Suppose $\mathbf{c} \in \mathrm{AP}$ and $\Lambda \subseteq \Lambda_{\mathbf{c}}$. By $G_{\mathbf{c},\Lambda}$ we mean

$$G_{\mathbf{c},\Lambda} := G_\Lambda := \Big\langle \{ r x_\nu, r y_{\eta,n} : r \in R, \nu \in \Lambda_{<\omega}, \eta \in \Lambda_\omega \text{ and } n < \omega \} \Big\rangle.$$

We have the following observation, but as we do not use it, we leave its proof.

Observation 3.39. *Suppose $\Lambda \subseteq \Lambda[\lambda]$ is downward closed. Then $G_{\mathbf{c},\Lambda}$ is a pure subgroup of $G_{\mathbf{c}}$.*

Lemma 3.40. *Let $\mathbf{c} \in \mathrm{AP}$ be full, and $h \in \mathrm{End}(G_{\mathbf{c}})$. Then for some countable $\Lambda_h \subseteq \Lambda[\lambda]$ we have:*

$$r \in R, \nu \in \Omega_{\mathbf{c}} \setminus \Lambda_h \implies h(r x_\nu) \in G_{\mathbf{c}, \Lambda_h \cup \{\nu\}} + K.$$

Proof. Suppose on the way of contradiction that the lemma fails. Let Y_0 be as Lemma 3.36. We define a sequence

$$\langle (Y_i, \nu_i, \rho_i, r_i) : i < \omega_1 \rangle,$$

by induction on $i < \omega_1$, such that

($\natural$) (a) $r_i \in R \setminus \{0\}$,

(b) $Y_i = \bigcup\{\mathrm{supp}(h(r_j x_{\nu_j})) : j < i\} \cup \{\rho_j : j < i\} \cup Y_0$,

(c) $\nu_i \in \Omega_{\mathbf{c}} \setminus Y_i$,

(d) $h(r_i\nu_i) \notin G_{\mathbf{c},Y_i\cup\{\nu_i\}} + K$,

(e) let $b_i := h(r_i\nu_i)$, and let $\bar{r}_{b_i} := \langle r^2_{b_i,\rho} : \rho \in \Lambda_i\rangle$ be as Lemma 3.36 applied to b_i. Then $\rho_i \in \Lambda_i \setminus (Y_i \cup \{\nu_i\})$, and even

$$r^2_{b_i,\rho_i} x_{\rho_i} \notin G_{\mathbf{c},Y_i\cup\{\nu_i\}} + K.$$

To construct this, suppose $i < \omega$ and we have constructed the sequence up to i. Now, $(\natural)_b$ gives the definition of Y_i. Since we assume that the lemma fails, there is an $r_i \in R$ and $\nu_i \in \Omega_{\mathbf{c}} \setminus Y_i$ such that $h(r_i x_{\nu_i}) \notin G_{\mathbf{c},\Lambda_h\cup\{\nu\}} + K$. Now, we define $b_i := h(r_i\nu_i)$. Thanks to Lemma 3.36, there is a finite set $\Lambda_i \subseteq \Lambda_{\mathbf{c},<\omega} \setminus Y_i$ and a sequence $\langle r^2_{b_i,\rho} : \rho \in \Lambda_i\rangle$ such that

$$b_i - \sum_{\rho\in\Lambda_i} r^2_{b_i,\rho} x_\rho \in K^+.$$

As $b_i \notin G_{\mathbf{c},Y_i\cup\{\nu_i\}} + K$ and due to the following containment

$$b_i - \sum_{\rho\in\Lambda_i} r^2_{b_i,\rho} x_\rho \in K^+ \subseteq G_{\mathbf{c},Y_i\cup\{\nu_i\}} + K,$$

there is $\rho_i \in \Lambda_i$ such that $\rho_i \notin (Y_i \cup \{\nu_i\})$, and indeed

$$r^2_{b_i,\rho_i} x_{\rho_i} \notin G_{\mathbf{c},Y_i\cup\{\nu_i\}} + K.$$

This completes the proof of construction. By shrinking the sequence, we may and do assume in addition that

- for all $i \neq j < \omega_1, \rho_j \notin \Lambda_i$.

Let $a_n := r_n x_{\nu_n}$ and define

$$f : \Lambda_{\mathbf{c},<\omega} \to \mid R \mid + \mid K \mid + \aleph_0 < \lambda$$

be such that for any $\rho \in \Lambda_{\mathbf{c},<\omega}$, $f(\rho)$ codes

- $\langle r^2_{b,\rho} : \rho \in \Lambda_b\rangle$, and
- $b - \sum_{\nu\in\Lambda_i} r^2_{b,\nu} x_\nu$,

where $b := h(x_\rho)$. To see such a function f exists, first we define:

- $f_1 : R^{<\omega} \times K^+ \to \mid R \mid + \mid K \mid + \aleph_0$ is a bijection, and

- $f_2 : \Lambda_{\mathbf{c},<\omega} \to R^{<\omega} \times K^+$ is defined as

$$f_2(b) = \big(\langle r^2_{b,\rho} : \rho \in \Lambda_b\rangle, b - \sum_{\nu\in\Lambda_i} r^2_{b,\nu} x_\nu\big).$$

Then, we set $f := f_1 \circ f_2$. Suppose $\rho_1, \rho_2 \in \Lambda_{\mathbf{c},<\omega}$ are such that $f(\rho_1) = f(\rho_2)$. We claim that $h(x_{\rho_1}) = h(x_{\rho_2})$. To see this, it is enough to apply $f(\rho_1) = f(\rho_2)$, and conclude that

(1) $\langle r^2_{b_1,\nu} : \nu \in \Lambda_{b_1}\rangle = \langle r^2_{b_2,\nu} : \nu \in \Lambda_{b_2}\rangle$

(2) $b_1 - \sum_{\nu\in\Lambda_{b_1}} r^2_{b,\nu} x_\nu = b_2 - \sum_{\nu\in\Lambda_{b_2}} r^2_{b,\nu} x_\nu$,

where $b_i = h(x_{\rho_i})$. But, then we have

$$\begin{aligned} b_1 &= b_1 - \textstyle\sum_{\nu\in\Lambda_{b_1}} r^2_{b,\nu} x_\nu + (\sum_{\nu\in\Lambda_{b_1}} r^2_{b,\nu} x_\nu) \\ &\overset{(2)}{=} b_2 - \textstyle\sum_{\nu\in\Lambda_{b_2}} r^2_{b,\nu} x_\nu + (\sum_{\nu\in\Lambda_{b_2}} r^2_{b,\nu} x_\nu) \\ &= b_2, \end{aligned}$$

i.e., $h(x_{\rho_1}) = h(x_{\rho_2})$.

Since $\mathbf{c}$ is full, and in the light of Definition 3.29(b), we are able to find an $\eta \in \Lambda_{\mathbf{c},\omega}$ such that

(3) $a_n = a^{\mathbf{c}}_{\eta,n}$, and

(4) $f(\eta{\restriction_L} n) = f(\eta{\restriction_R} n)$,

for all $n < \omega$. Thanks to the previous paragraph and clause (4) we deduce

$$h(x_{\eta\restriction_L n}) = h(x_{\eta\restriction_R n}) \quad (\sharp)$$

By applying h to the both sides of the following equation

$$y_{\eta,0} \quad = \sum_{i=0}^{n} \big(\textstyle\prod_{j<i} j!\big) r_i x_{\nu_i} + \big(\prod_{i=1}^{n} i!\big) y_{\eta,n+1} + \sum_{i=0}^{n} \big(\prod_{j<i} j!\big)(x_{\eta\restriction_L i} - x_{\eta\restriction_R i}),$$

we get

$$\begin{aligned} h(y_{\eta,0}) &= \sum_{i=0}^{n} \big(\textstyle\prod_{j<i} j!\big) h(r_i x_{\nu_i}) + \big(\prod_{i=1}^{n} i!\big) h(y_{\eta,n+1}) \\ &\quad + \big(\textstyle\prod_{j<i} j!\big)\big(h(x_{\eta\restriction_L n}) - h(x_{\eta\restriction_R n})\big) \\ &\overset{(\sharp)}{=} \sum_{i=0}^{n} \big(\textstyle\prod_{j<i} j!\big) h(r_i x_{\nu_i}) + \big(\prod_{i=1}^{n} i!\big) h(y_{\eta,n+1}) \quad (+) \end{aligned}$$

For each $i < \omega_1$ let $b_i = h(r_i x_{\nu_i})$. Let also $b = h(y_{\eta,0})$ and let Λ_b be as in Lemma 3.36. As Λ_b is finite, for some large enough n, we have

$$\{\rho_i : i < n\} \setminus \Lambda_b \neq \emptyset.$$

Let $i < n$ be such that $\rho_i \notin \Lambda_b$. Here, we apply the argument presented in items (3)-(5) from Lemma 3.35 to the displayed formula (+). So, on the one hand, it turns out that

$$\rho_i \in \Lambda_i \subseteq \Lambda_b.$$

On the other hand by the choice of i, $\rho_i \notin \Lambda_b$. This is a contraction that we searched for it. $\square$

Lemma 3.41. *Let* $\mathbf{c} \in \mathrm{AP}$ *be full, and* $h \in \mathrm{End}(G_{\mathbf{c}})$. *Then for some* $m_* \in R$ *and some countable* $\Lambda_h = \mathrm{cl}(\Lambda_h) \subseteq \Lambda[\lambda]$ *we have:*

$$r \in R, \nu \in \Omega_{\mathbf{c}} \setminus \Lambda_h \implies h(rx_\nu) - m_* x_\nu \in G_{\Lambda_h} + K.$$

Proof. In view of Lemma 3.40, there is some countable downward closed subset Λ of $\Lambda_{\mathbf{c}}$ such that for every $r \in R$ and $\eta \in \Omega_{\mathbf{c}} \setminus \Lambda$, we have $h(rx_\eta) \in G_{\Lambda \cup \{\nu\}} + K$. Thus, for such r and η, there are $m^r_\eta \in R$ and b^r_η satisfying the following two properties:

- $h(rx_\eta) = m^r_\eta x_\eta + b^r_\eta$,
- $b^r_\eta \in G_\Lambda + K$.

Suppose on the way of contradiction that the desired conclusion fails. By induction on $i < \omega_1$ we define a sequence

$$\langle Y_i, r_{i,1}, r_{i,2}, \eta_{i,1}, \eta_{i,2} : i < \omega_1 \rangle$$

such that:

(†) (a) $Y_i = \Lambda \cup \{\eta_{j,\ell} : j < i, \ell \in \{1, 2\}\}$,
(b) $r_{i,1}, r_{i,2} \in R \setminus \{0\}$,
(c) $\eta_{i,\ell} \in \Omega_{\mathbf{c}} \setminus Y_i$, for $\ell \in \{1, 2\}$,
(d) $m^{r_{i,1}}_{\eta_{i,1}} \neq m^{r_{i,2}}_{\eta_{i,2}}$.

The construction is easy, but we elaborate. Let us start with the case $i = 0$. We set $Y_0 = \Lambda$ and then choose $r_{0,1}, r_{0,2} \in R \setminus \{0\}$ and $\eta_{0,1}, \eta_{0,2} \in \Lambda_{<\omega}[\lambda] \setminus \Lambda_h$ such that $m^{r_{0,1}}_{\eta_{0,1}} \neq m^{r_{0,2}}_{\eta_{0,2}}$. Now suppose $i < \omega_1$ and we have define the sequence for all $j < i$. Define Y_i as in clause (†)(a). By our assumption, we can find

i) $r_{i,1}, r_{i,2} \in R \setminus \{0\}$ and

ii) $\eta_{i,1}, \eta_{i,2} \in \Omega_{\mathbf{c}} \setminus Y_i$,

so that $m^{r_{i,1}}_{\eta_{i,1}} \neq m^{r_{i,2}}_{\eta_{i,2}}$. This completes the induction construction.

Let

$$f : \Lambda_{\mathbf{c},<\omega} \to | R | + | K | + \aleph_0 < \lambda$$

be such that if $r \in R$ and $\eta \in \Omega_{\mathbf{c}}$, then $f(rx_\eta)$ is defined in a way that one can compute m^r_η and b^r_η. Again we can define f as

$$f = f_1 \circ f_2 \circ f_3,$$

where:

- $f_1 : R \times (G_\Lambda + K) \to | R | + | K | + \aleph_0$ is a bijection,
- $f_2 : R \times \Lambda_{\mathbf{c},<\omega} \to R \times (G_\Lambda + K)$ is defined as $f_2(r, \eta) = (m^r_\eta, b^r_\eta)$,
- $f_3 : \Lambda_{\mathbf{c},<\omega} \to R \times \Lambda_{\mathbf{c},<\omega}$ is a bijection.

For each $n < \omega$, we set

$$a_n := r_{n,1}x_{\eta_{n,1}} - r_{n,2}x_{\eta_{n,2}}.$$

Applying h to it yields:

$$h(a_n) = m^{r_{n,1}}_{\eta_{n,1}} x_{\eta_{n,1}} - m^{r_{n,2}}_{\eta_{n,2}} x_{\eta_{n,2}} + b_n \quad (+),$$

where $b_n := b^{r_{n,1}}_{\eta_{n,1}} - b^{r_{n,1}}_{\eta_{n,1}}$. Since $\mathbf{c}$ is full, there is an $\eta \in \Lambda_{\mathbf{c},\omega}$ such that

(1) $a_n = a^{\mathbf{c}}_{\eta,n}$, and

(2) $f(\eta{\restriction}_L n) = f(\eta{\restriction}_R n)$

for all $n < \omega$. By clause (2) we deduce:

(3) $\mathrm{supp}_\circ(h(x_{\eta{\restriction}_L n} - x_{\eta{\restriction}_R n})) = \emptyset$ for all $n < \omega$.

Applying h to

$$y_{\eta,0} = \sum_{i=0}^{n} a_i + \big(\prod_{i=1}^{n} i!\big) y_{\eta,n+1} + \sum_{i=0}^{n} \big(\prod_{j<i} j!\big)(x_{\eta{\restriction}_L i} - x_{\eta{\restriction}_R i}),$$

yields that

$$(\natural)\quad h(y_{\eta,0}) \begin{aligned}[t] &= \sum_{i=0}^{n} h(a_i) + \big(\textstyle\prod_{i=1}^{n} i!\big)h(y_{\eta,n+1}) + \big(\textstyle\prod_{j<i} j!\big)\big(h(x_{\eta\restriction_L n}) - h(x_{\eta\restriction_R n})\big)\\ &\overset{(3)}{=} \sum_{i=0}^{n} h(a_i) + \big(\textstyle\prod_{i=1}^{n} i!\big)h(y_{\eta,n+1})\\ &\overset{(+)}{=} \sum_{i=0}^{n} \big(m_{\eta_{n,1}}^{r_{n,1}} x_{\eta_{n,1}} - m_{\eta_{n,2}}^{r_{n,2}} x_{\eta_{n,2}} + b_n\big) + \big(\textstyle\prod_{i=1}^{n} i!\big)h(y_{\eta,n+1}).\end{aligned}$$

Let $n < \omega$ be large enough. Here, we are going to apply the arguments taken from (3)-(5) in Lemma 3.35 to the displayed formula ($\natural$). Then, it turns out that

(4) $\mathrm{supp}_\circ(h(y_{\eta,0})) \supseteq \mathrm{supp}_\circ(h(a_n))$, and

(5) $\mathrm{supp}_\circ(h(a_n)) \cap \{\eta_{n,1}, \eta_{n,2}\} \neq \emptyset$.

Without loss of generality, let us assume that for each $n < \omega$, $\eta_{n,1} \in \mathrm{supp}_\circ((h(a_n))$. So,

$$\{\eta_{n,1} : n < \omega\} \subseteq \mathrm{supp}_\circ(h(y_{\eta,0})),$$

which is infinite. This is a contraction. $\square$

Lemma 3.42. *Assume* $\Lambda = \mathrm{cl}(\Lambda) \subseteq \Lambda_{\mathbf{c}}$ *is countable and* $h \in \mathrm{Hom}(G_{\mathbf{c}}, G_\Lambda + K)$. *Then* h *is bounded.*

Proof. Towards a contradiction we assume that h is unbounded. It follows from Lemma 3.32 that $\mathrm{Rang}(h) \nsubseteq K$. Let $b_* \in \mathrm{Rang}(h) \setminus K$. Then, for some $d_* \in K$, a finite set Λ_* and two sequences $\langle r_\eta \in R \setminus \{0\} : \eta \in \Lambda_*\rangle$ and $\langle m_\eta \in \omega : \eta \in \Lambda_*\rangle$, we can represent b_* as

$$b_* = \sum\{r_\eta x_\eta : \eta \in \Lambda_* \cap \Lambda_{<\omega}\} + \sum\{r_\eta y_{\eta,m(\eta)} : \eta \in \Lambda_* \cap \Lambda_\omega\} + d_*.$$

Let

(1) $J_0 = G_\Lambda + K$,

(2) $J_1 = J_0/K$, which is torsion free.

So, $b_* \in J_0$. Let $\pi : J_0 \to J_1$ be the natural map defined by the assignment $b \mapsto \pi(b) := b + K$. Since $b_* \in \mathrm{Rang}(h) \setminus K$, we have $\pi(b_*) \neq 0$. Suppose on the way of contradiction that for any sequence $\langle e_n : n < \omega\rangle \in {}^\omega\mathbb{Z}$ the following system of equations

$$\Gamma := \{y_n = n! y_{n+1} + e_n b_* : n < \omega\}$$

is solvable in J_1. Say for example, $\{y_n : n < \omega\}$ is such a solution.

Thanks to Lemma 3.26(3)(a) $\Lambda_{\mathbf{c}}$ is $\aleph_1$-free. We combine this with Lemma 3.26(3)(b) to deduce that $M_{\mathbf{c}}$ is $\aleph_1$-free as an R-module. Now, since J_1 is countably generated, we can find a solution to

$$\Gamma = \{y_n = n!y_{n+1} + e_n\bar{b}_* : n < \omega\}$$

in R. Since R is cotorsion-free, a such system of equations has no solution the ring. So, there is a sequence $\langle e_n : n < \omega\rangle \in {}^\omega\mathbb{Z}$ the following equations

$$\Gamma = \{y_n = n!y_{n+1} + e_n b_* : n < \omega\}$$

is not solvable in J_1.

Let $a_* \in G_{\mathbf{c}}$ be such that $b_* = h(a_*)$. Let also $f : \Lambda_{\mathbf{c},<\omega} \to \omega$ be such that for all $\nu, \rho \in \Lambda_{\mathbf{c},<\omega}$,

$$f(\nu) = f(\rho) \Leftrightarrow \pi \circ h(x_\nu) = \pi \circ h(x_\rho).$$

As $\mathbf{c}$ is full, there is some $\eta \in \Lambda_{\mathbf{c},\omega}$ such that:

(3) $a^{\mathbf{c}}_{\eta,n} = e_n a_*$, for all $n < \omega$, and

(4) $f(\eta \restriction_L n) = f(\eta \restriction_R n)$, for $n < \omega$.

Thanks to (4), one has

$$\forall n < \omega,\ \pi \circ h(x_{\eta\restriction_L n}) = \pi \circ h(x_{\eta\restriction_R n}) \qquad (+)$$

By applying $\pi \circ h$ into the equation

$$y_{\eta,n} = a^{\mathbf{c}}_{\eta,n} + n!y_{\eta,n+1} + (x_{\eta\restriction_L n} - x_{\eta\restriction_R n}),$$

and using clause (3) and (+) we get

$$\pi \circ h(y_{\eta,n}) = e_n\pi(b_*) + n!\pi \circ h(y_{\eta,n+1}).$$

This clearly gives a contradiction, as then

$$J_1 \models \text{“}y_n = n!y_{n+1} + e_n b''_*,$$

where $y_n = \pi \circ h(y_{\eta,n})$. □

Lemma 3.43. *Let* $\mathbf{c}$ *be full and* $h \in \mathrm{End}(G_{\mathbf{c}})$. *Then* $\mathrm{Rang}(h)$ *is bounded.*

Proof. Suppose not, it follows that for some countable $\Lambda = \mathrm{cl}(\Lambda) \subseteq \Lambda_{\mathbf{c}}$,

$$h \restriction G \in \mathrm{Hom}(G, G_\Lambda + K)$$

is unbounded, where G is the subgroup of $G_{\mathbf{c}}$ generated by $h^{-1}[G_\Lambda + K]$. This contradicts Lemma 3.42. □

Now, we are ready to prove:

Theorem 3.44. *Adopt the notation from Hypothesis 3.14. Then there is some* $\mathbf{c}$ *such that the abelian group* $G_{\mathbf{c}}$ *is boundedly rigid. In particular, there is an abelian group* G *equipped with the following properties*

(1) $\mathrm{tor}(G) = K$,
(2) *G is of size λ,*
(3) *the sequence*

$$0 \longrightarrow R_{\mathbf{c}} \longrightarrow \mathrm{End}(G) \longrightarrow \frac{\mathrm{End}(G)}{\mathrm{E_b}(G)} \longrightarrow 0$$

is exact.

Proof. According to Lemma 3.30, there is a full $\mathbf{c} \in \mathrm{AP}$. This allows us to apply Lemma 3.43, and deduce that $G := G_{\mathbf{c}}$ is boundedly rigid. By definition, this completes the proof. □

§ 4. Co-Hopfian and boundedly endo-rigid abelian groups

As stated in [16], it is difficult to construct an infinite Hopfian–co-Hopfian p-group. What about mixed groups? In this section, we answer the question. We start by recalling that a group G is called

(i) *Hopfian* if its surjective endomorphisms are automorphisms;
(ii) *co-Hopfian* if its injective endomorphisms are automorphisms.

In what follows we will use the following two items:

Fact 4.1. (i) *(See* [24, Claim 2.15(1)]*). Any direct summand of a co-Hopfian abelian group is again co-Hopfian.*

(ii) *(See* [23, Theorem 1.2]*). Suppose* $2^{\aleph_0} < \lambda < \lambda^{\aleph_0}$*. Then there is no co-Hopfian abelian group of size* λ*.*

Here, we introduce a useful criteria:

Definition 4.2. Let G be an abelian group of size λ and $m, n \geq 1$ be such that $m \mid n$. Then:

(1) $\mathrm{NQr}_{(\mathrm{m,n})}(\mathrm{G})$ means that there is an (m, n)-anti-witness h, which means:
   (a) $h \in \mathrm{End}(\Gamma_n(G))$,
   (b) $\mathrm{Rang}(h)$ is a bounded group,
   (c) $h' := m \cdot \mathrm{id}_{\Gamma_n(G)} + h \in \mathrm{End}(\Gamma_n(G))$ is 1-to-1,
   (d) h' is not onto or $m > 1$ and $G/\Gamma_n(G)$ is not m-divisible.

(2) $\mathrm{NQr}_{\mathrm{m}}(\mathrm{G})$ means $\mathrm{NQr}_{(\mathrm{m,n})}(\mathrm{G})$ for some $n \geq 1$.

(3) $\mathrm{NQr}(\mathrm{G})$ means $\mathrm{NQr}_{\mathrm{m}}(\mathrm{G})$ for some $m \geq 1$.

Definition 4.3. Adopt the previous notation.

(1) $\mathrm{Qr}(\mathrm{G})$ means the negation of $\mathrm{NQr}(\mathrm{G})$.

(2) $\mathrm{Qr}_*(\mathrm{G})$ means $\mathrm{Qr}(\mathrm{G})$ and in addition that $\Gamma_p(G)$ is unbounded, for at least one $p \in \mathbb{P}$.

In items 4.4–4.11 we check $\mathrm{NQr}_{(\mathrm{m,n})}(\mathrm{G})$ and its negation. This enables us to present some new classes of co-Hopfian and non co-Hopfian groups.

Lemma 4.4. *Let G be an abelian group such that the property* $\mathrm{NQr}(\mathrm{G})$ *holds. Then G is not co-Hopfian. Furthermore, let $h \in \mathrm{Hom}(G, \Gamma_n(G))$ be such that $h \restriction \Gamma_n(G)$ is an (m, n)-anti-witness. Then $m \cdot \mathrm{id}_G + h$ witnesses that G is not co-Hopfian.*

Proof. Suppose that G admits an (m, n)-anti-witness $h_0 \in \mathrm{End}(\Gamma_n(G))$ as in Definition 4.2. As h_0 is bounded, by Fact 2.14 we can extend h_0 to $h_1 \in \mathrm{Hom}(\mathrm{G}, \Gamma_{\mathrm{n}}(\mathrm{G}))$. So, the following diagram commutes:

$$\begin{array}{ccccc} 0 & \longrightarrow & \Gamma_n(G) & \xrightarrow{\subseteq_*} & G \\ & & \downarrow h_0 & \swarrow \exists h_1 & \\ & & \Gamma_n(G) & & \end{array}$$

We claim that $f = m \cdot \mathrm{id_G} + \mathrm{h_1} \in \mathrm{End(G)}$ is 1-to-1 but not onto.

$(*)_1$ f is one-to-one.

To see this, suppose $x \in G$ in non-zero and we want to show that $f(x) \neq 0$. Suppose first we deal with the case $x \in \Gamma_n(G) \setminus \{0\}$. According to clause (c) of Definition 4.2(1) we have

$$f(x) = mx + h_1(x) = m \cdot \mathrm{id}_{\Gamma_n(G)}(x) + h_0(x) \Rightarrow f(x) \neq 0.$$

Now, suppose that $x \in G \setminus \Gamma_n(G)$. Recall from Definition 4.2 that m divides n. As $m \mid n$, we have $mx \in G \setminus \Gamma_n(G)$. If $f(x) = 0$, we have $mx + h_1(x) = 0$, thus

$$h_1(x) = -mx \in G \setminus \Gamma_n(G).$$

But, $\mathrm{Rang}(h_1) \subseteq \Gamma_n(G)$, which is impossible. Thus f is 1-to-1, as wanted.

$(*)_2$ f is not onto.

For this, we consider two cases:

Case 1) h_0 is not onto:

By the case assumption, there is

$$y \in \Gamma_n(G) \setminus \mathrm{Rang}\left(\mathrm{id}_{\Gamma_n(G)} + (\mathrm{h_0} \restriction \Gamma_n(G))\right)$$

and it is easy to see that such a y is also a witness for f to be not onto.

Case 2) h_0 is onto:

By Definition 4.2(1)(d), we must have $m > 1$ and $G/\Gamma_n(G)$ is not m-divisible. Let $z \in G$ be such that $z + \Gamma_n(G)$ is not divisible by m in $G/\Gamma_m(G)$. Clearly, z does not belong to $\mathrm{Rang}(f)$.

The lemma follows. □

Lemma 4.5. *Let K be an abelian p-group. The following claims are valid: If* $\mathrm{NQr}(K)$ *holds, then K is infinite.*

Proof. By definition, there are m and n such that $m \mid n$ and that $\mathrm{NQr}_{(m,n)}(K)$ holds. Thanks to Definition 4.2(1), there is $h \in \mathrm{End}(\Gamma_n(G))$ satisfying the following properties:

(a) $\mathrm{Rang}(h)$ is a bounded group,

(b) $h' := m \cdot (\mathrm{id}_{\Gamma_n(K)}) + \mathrm{h} \in \mathrm{End}(\Gamma_n(K))$ is 1-to-1,

(c) h' is not onto or $m > 1$ and $K/\Gamma_n(K)$ is not m-divisible.

We have two possibilities: 1) $p \nmid n$ and 2) $p \mid n$.

(1) Suppose first that $p \nmid n$. As K is a p-group, $\Gamma_n(K) = \{0\}$. This means that h is constantly zero and is onto, as well as h'. Thanks to clause (c) it follows that $m > 1$ and K is not m-divisible. Since $m \mid n$ we deduce that $p \nmid m$. Now, we consider the map $m \cdot \mathrm{id}_K : K \to K$. Since K is not m-divisible, this map is not surjective. Let us show that it is 1-to-1. To this end, let $x \in K$ be such that $mx = 0$. Let ℓ be the order of x so that $p^\ell x = 0$. As $(p^\ell, m) = 1$, we can find r, s such that $rp^\ell + sm = 1$. By multiplying both sides with x, we obtain

$$x = rp^\ell x + smx = 0 + 0 = 0.$$

It follows that $m \cdot \mathrm{id}_K : K \to K$ is 1-to-1 and not onto, hence K is infinite.

(2) Suppose $p \mid n$. As K is a p-group, this implies that $\Gamma_n(K) = K$. Therefore, in the above item (c), the case "$K/\Gamma_n(K)$ is not m-divisible" does not occur. This is in turn implies that h' is not onto K. We proved that the map $h' \in \mathrm{End}(K)$ is 1-to-1 and not onto. Hence K is infinite.

The proof is now complete. □

Discussion 4.6. Keep the notation of Fact 2.5. One can not replace "divisible" with "reduced" and drives a similar result, as some easy examples suggest this. Here, we consider this as an application of the construct of co-Hopfian groups.

(i) Suppose on the way of contradiction that the replacement is valid.

(ii) Let G be a co-Hopfian group such that its reduced part is unbounded (recall from the introduction that a such group exists, see [7]).

(iii) Here, we drive a contradiction by showing from that G is not co-Hopfian. Indeed, let K_2 be the maximal divisible subgroup of K. Recall from Fact 2.18 that K_2 is injective. Since it is injective, we know K_2 is a directed summand. Let us write K as $K = K_1 \oplus K_2$. Due to the maximality of K_2 one may know that K_1 is reduced. We show that K_1 is not co-Hopfian, and hence by Fact 4.1(i), K is not co-Hopfian. Thus by replacing K by K_1 if necessary, we may assume without loss of generality that K is reduced and unbounded. For $\ell < \omega$, we choose by induction H_ℓ, y_ℓ and z_ℓ such that:

- $(iii)_a$ $H_0 = K$,
- $(iii)_b$ if $\ell = k+1$, then $H_k = H_\ell \oplus \mathbb{Z}z_\ell$,
- $(iii)_c$ $z_\ell \in (\mathbb{Z}y_\ell)_*$ recall that $(\mathbb{Z}y_\ell)_*$ denotes the pure closure of $\mathbb{Z}y_\ell$,
- $(iii)_d$ $y_{\ell+1} \in H_\ell$,
- $(iii)_e$ The order of z_i is $\geq p^\ell$.

[Why? For $\ell = 0$, we set $H_0 = K$ and let $y_0 \in K$ be arbitrary. Then $(\mathbb{Z}y_0)_*$ is a pure subgroup of K of bounded exponent. Thanks to Fact 2.5 we know $(\mathbb{Z}y_0)_*$ is a direct summand of K. In view of Fact 2.7 we can find z_0 such that $\mathbb{Z}z_0$ is a direct summand of $(\mathbb{Z}y_0)_*$. In other words, $\mathbb{Z}z_0$ is a direct summand of $H_0 = K$ as well. Consequently, we have $H_0 = H_1 \oplus \mathbb{Z}z_0$ for some H_1. Having defined inductively $\{H_\ell, y_\ell, z_\ell\}$, let $y_{\ell+1} \in H_\ell$. Let χ be a regular cardinal, large enough, so that $H_\ell \in \mathscr{H}(\chi)$. The notation $\mathscr{B}$ stands for $(\mathscr{H}(\chi), \in)$. Let $\mathscr{B}_\ell$ be countable such that $H_\ell \in \mathscr{B}_\ell$. Now, we look at

$$\mathcal{L}_\ell := \mathscr{B}_\ell \cap H_\ell.$$

We find easily that $\mathcal{L}_\ell$ is an unbounded countable abelian p-group. Hence it is of the form $\oplus_i \mathbb{Z}z_{\ell,i}$ where $z_{\ell,i}$ is of order $p^{m(\ell,i)}$. As $\mathcal{L}_\ell$ is unbounded, we may and do assume that $m(\ell, i) > \ell$. This implies that $\mathbb{Z}z_{\ell,i}$ is a pure subgroup of $\mathcal{L}_\ell$, and hence H_ℓ. Consequently, $\mathbb{Z}z_{\ell,i}$ is a direct summand of H_ℓ as well. By definition, we have $H_\ell = H_{\ell+1} \oplus \mathbb{Z}z_{\ell+1}$ for some abelian subgroup $H_{\ell+1}$ of H_ℓ.]

For each $i < \omega$, we let $\ell(i) > 1$ be such that z_i is of order $p^{\ell(i)}$. Following clause (e), clearly we can find some infinite $u \subseteq \omega$ such that the sequence $\langle \ell(i) : i \in u \rangle$ is increasing. For any $j < \omega$, we clearly have $\bigoplus_{i \in u \cap j} \mathbb{Z}z_i \subseteq_* K$, and hence $\oplus_{i \in u} \mathbb{Z}z_i \subseteq_* K$. In the light of part (i), $\bigoplus_{i \in u} \mathbb{Z}z_i$ is a direct summand of K, thus there is some K_3 such that $K = \oplus_{i \in u} \mathbb{Z}z_i \oplus K_3$. Let $\langle j(k) : k < \omega \rangle$ be lists u in an increasing order, and define $h \in \mathrm{End}(K)$ be such that

- $h{\restriction}K_3 = \mathrm{id}_{K_3}$,
- $h(z_{j(k)}) = p^{\ell(k+1)-1} z_{j(\ell+1)}$.

It is easy to check that h is a well-defined endomorphism of K and it satisfies the following properties:

- h is injective,
- h is not surjective.

In sum, h witnesses that K is not co-Hopfian. This is a contradiction that we searched for it.

The following is clear:

Corollary 4.7. *Let G be a p-group such that its reduced part is unbounded and its countable pure subgroups are directed summand. Then G is not co-Hopfian.*

Lemma 4.8. *Let G be an abelian group of size λ and $m \geq 1$. Suppose there is a bounded $h \in \mathrm{End}(G)$ such that $f := m \cdot \mathrm{id}_G + h \in \mathrm{End}(G)$ is 1-to-1 not onto* [4]. *Then for some $n \geq 1$ we have:*

(i) $\mathrm{NQr}_{(m,n)}(G)$,

(ii) Letting $h_0 = h \restriction \Gamma_n(G)$, h_0 is an (m,n)-anti-witness for $\Gamma_n(G)$.

Proof. Let f and h be as above. As $\mathrm{Rang}(h)$ is bounded, for some $n \geq 1$ we have $\mathrm{Rang}(h) \leq \Gamma_n(G)$ and without loss of generality $m \mid n$. Possibly, replacing n with nm, which is possible as $n_1 | n_2$ implies that $\Gamma_{n_1}(G) \leq \Gamma_{n_2}(G)$. Notice now that:

$(*)_1$ (a) f maps $\Gamma_n(G)$ into itself.

(b) if $x \in G \setminus \Gamma_n(G)$, then $f(x) \notin \Gamma_n(G)$.

Clause (a) clearly holds as by the choice of n we have $\mathrm{Rang}(h) \leq \Gamma_n(G)$. To see clause (b), we suppose by contradiction that $f(x) = mx + h(x) \in \Gamma_n(G)$. It follows that $mx = f(x) - h(x) \in \Gamma_n(G)$, and hence as $m \mid n$, $x \in \Gamma_n(G)$, a contradiction.

Let now $h_0 = h \restriction \Gamma_n(G)$. Then we have:

$(*)_2$ (a) $h_0 \in \mathrm{End}(\Gamma_n(G))$,

(b) h_0 is bounded,

(c) Since f is 1-to-1, so is $f_0 = m \cdot \mathrm{id}_{\Gamma_n(G)} + h_0 \in \mathrm{End}(\Gamma_n(G))$.

[4] Thus f witnesses non co-Hopfianity of G.

We are left to show that h_0 is an (m,n)-anti-witness. By $(*)_2$ it suffices show that f_0 is not onto or $G/\Gamma_n(G)$ is not m-divisible. Suppose on the contrary that f_0 is onto and $G/\Gamma_n(G)$ is m-divisible. We are going to show that f is onto, which contradicts our assumption. To this end, let $x \in G$. Since $G/\Gamma_n(G)$ is m-divisible, we can find some $y \in G$ such that

$$x - my \in \Gamma_n(G).$$

We look at

$$w := x - my - h_0(y) \in \Gamma_n(G).$$

As f_0 is onto, we can find some $z \in \Gamma_n(G)$ such that $f_0(z) = w$. So,

$$x - my - h_0(y) = w = f_0(z) = mz + h_0(z).$$

Using this equation, and the additivity of h_0, we observe that

$$x = m(y + z) + h_0(y + z) = f(y + z).$$

In other words, f is onto. This is a contradiction. □

Notation 4.9. Suppose κ and μ are infinite cardinals. The infinitary language $\mathcal{L}_{\mu,\kappa}(\tau)$ is defined so as its vocabulary is the same as τ, it has the same terms and atomic formulas as in τ, but we also allow conjunction and disjunction of length less than μ, i.e., if ϕ_j, for $j < \beta < \mu$ are formulas, then so are $\bigvee_{j<\beta} \phi_j$ and $\bigwedge_{j<\beta} \phi_j$. Also, quantification over less than κ many variables.

Lemma 4.10. *Let G be a reduced abelian group of size λ such that*

(1) $\lambda > 2^{\aleph_0}$,

(2) G is co-Hopfian.

Then the property $\mathrm{Qr}_*(G)$ is valid.

Proof. Thanks to Lemma 4.4 we know $\mathrm{Qr}(G)$ is satisfied, so it is enough to show that for some prime p, $\Gamma_p(G)$ is not bounded. Towards a contradiction, we suppose that $\Gamma_p(G)$ is bounded for every prime $p \in \mathbb{P}$.

Here, we are going to show the pure subgroup $\Gamma_p(G)$ is finite. Suppose on the way of contradiction that $\Gamma_p(G)$ is infinite. Recall that p-torsion subgroups are pure. According to Fact 2.4 $\Gamma_p(G)$ is a direct summand of G, as we assumed that it is bounded. Also, following Fact 2.7 we know that $\Gamma_p(G)$ is a direct summand of cyclic groups. In sum, we observed that $\Gamma_p(G)$ has a direct summand K which is a countably infinite p-group. In view of Fact 2.6(i), we may and do assume that K is not co-Hopfian. Recall that any direct summand of co-Hopfian, is co-Hopfian. This means that G is not co-Hopfian as well, which contradicts our assumption. Thus, it follows that for every $p \in \mathbb{P}$, the group $\Gamma_p(G)$ is finite and therefore a direct summand of G, hence there is a projection h_p from G onto $\Gamma_p(G)$. Recall that $p \in \mathbb{P}$ and also $h_p \restriction \Gamma_p(G) \in \mathrm{End}(\Gamma_p(G))$ is essentially equal to the identity map, so is one-to-one, and hence onto, as $\Gamma_p(G)$ is finite. Since $\mathrm{Qr}(G)$ is satisfied, it follows from Definition 4.2(1)(d) that $G/\Gamma_p(G)$ is p-divisible.

Now, we take χ be a regular cardinal, large enough, such that $G \in \mathscr{H}(\chi)$ and let

$$M \prec_{\mathcal{L}_{\aleph_1,\aleph_1}} (\mathscr{H}(\chi), \in)$$

be such that:

- M has cardinality $2^{\aleph_0}$,
- $G, \mathrm{tor}(G) \in M$,
- $2^{\aleph_0} + 1 \subseteq M$.

In the light of Fact 2.6(ii), we may and do assume that $|\mathrm{tor}(G)| = \mu \leq 2^{\aleph_0}$. Recall that $2^{\aleph_0} + 1 \subseteq M$ and $\mathrm{tor}(G) \in M$. These imply that $\mathrm{tor}(G) \subseteq M$. Now, as $G/\Gamma_p(G)$ is p-divisible, then so is

$$\frac{G/\Gamma_p(G)}{(G \cap M)/\Gamma_p(G)},$$

which by the Third Isomorphism Theorem, is canonically isomorphic to $G/G \cap M$. As $\mathrm{tor}(G) \subseteq M$, $G/(G \cap M)$ is torsion-free, it is divisible. Let $x \in G \setminus M$ and define the sequence $(x_n : n < \omega)$ such that:

- $x_0 = x$,
- If $n = m + 1$ then

$$G/(G \cap M) \models \text{“}n! x_n + (G \cap M) = x_m + (G \cap M)\text{''}.$$

So, letting $a_0 = 0$ and for $n = m + 1 < \omega$,

$$a_n = n!x_n - x_m \in G \cap M,$$

we have that $(a_n : n < \omega) \in M^\omega \subseteq M$ and so, as

$$M \prec_{\mathcal{L}_{\aleph_1,\aleph_1}} (\mathscr{H}(\chi), \in),$$

we can find

$$\bar{y} = (y_n : n < \omega) \in (G \cap M)^\omega$$

such that $a_n = n!y_n - y_m$, but then for every $m < \omega$:

$$G \models \text{“}m!(x_{m+1} - y_{m+1}) = x_m - y_m\text{''}.$$

Hence,

$$\bigcup \{\mathbb{Z}(x_m - y_m) : m < \omega\}$$

is a non-trivial divisible subgroup of G, contradicting the assumption that G is reduced. So we have proved the desired claim. □

Proposition 4.11. *Let $G \in$ be a boundedly endo-rigid abelian group. The following assertions are valid:*

(1) G is co-Hopfian iff $\mathrm{Qr}(G)$,

(2) If $|G| > 2^{\aleph_0}$, then G is co-Hopfian iff $\mathrm{Qr}_*(G)$.

Proof. (1). If G is co-Hopfian, then by Lemma 4.4, $\mathrm{Qr}(G)$ holds. For the other direction, suppose that G is boundedly rigid and $\mathrm{Qr}(G)$ holds. Let $f \in \mathrm{End}(G)$ be 1-to-1, we want to show that f is onto. As G is boundedly rigid we have m, h and L such that the following items hold:

(a) $m \in \mathbb{Z}$, $h \in \mathrm{End(G)}$,

(b) $f(x) = mx + h(x)$,

(c) $L = \mathrm{Rang}(h)$ is a bounded subgroup of G (and so of $\mathrm{tor}(G)$).

If f is not onto, then by Lemma 4.8, there is $n \geq 1$ such that $\mathrm{NQr}_{(m,n)}(G)$ holds, which is not possible (as we are assuming $\mathrm{Qr}(G)$). Thus f is onto as required.

(2). It follows from clause (1) and Lemma 4.10. □

Construction 4.12. Let $K := \oplus\{\frac{\mathbb{Z}}{p^n\mathbb{Z}} : p \in \mathbb{P} \text{ and } 1 \leq n < m\}$, where $m < \omega$, and $\mathbb{P}$ is the set of prime numbers. Let G be a boundedly endo-rigid abelian group such that $\mathrm{tor}(G) = K$[5]. Then G is co-Hopfian.

Proof. For any $p_1 \in \mathbb{P}$ and $n_1 < m$, let us define

$$(x_{(p_1,n_1)})_{(p,n)} = \begin{cases} 1 + p^n\mathbb{Z} & \text{if } (p,n) = (p_1,n_1) \\ 0 & \text{otherwise} \end{cases}$$

For simplicity, we abbreviate it by $x_{(p_1,n_1)}$. Assume towards a contradiction that there exists $f \in \mathrm{End}(G)$ such that f is 1-to-1 and not onto. As G is boundedly endo-rigid, there are $m \in \mathbb{Z}$ and $h \in \mathrm{E_b}(G)$ such that $f = m \cdot \mathrm{id}_G + h$. As f is 1-to-1 and K has no infinite bounded subgroup, we can conclude that $m \neq 0$.

$(*)_1$ $m \in \{1, -1\}$.

To see $(*)_1$, suppose on the contrary that there is $p \in \mathbb{P}$ such that $p|m$ and let m_1 be such that $m = m_1 p$. Now, as $\mathrm{Rang}(h)$ is bounded, there is $k \geq 1$ such that

$$p^k(\mathrm{Rang}(h)) \cap \Gamma_p(G) = \{0\}.$$

Let $n \geq k+1$, then:

$$\begin{aligned} f(p^{n-1}x_{(p,n)}) &= mp^{n-1}x_{(p,n)} + h(p^{n-1}x_{(p,n)}) \\ &= m_1pp^{n-1}x_{(p,n)} + p^k h(p^{n-1-k}x_{(p,n)}) \\ &= 0, \end{aligned}$$

which contradicts the fact that f is 1-to-1. This completes the argument of $m \in \{1, -1\}$ and without loss of generality we may assume that $m = 1$. Thus $f = \mathrm{id}_G + h$.

$(*)_2$ f maps $G \setminus \mathrm{tor}(G)$ into itself.

This is because f is 1-to-1. Indeed let $x \in G \setminus \mathrm{tor}(G)$. If $f(x) \in \mathrm{tor}(G)$, then for some $k, f(kx) = kf(x) = 0$, thus $kx = 0$, i.e., $x \in \mathrm{tor}(G)$ which contradicts $x \in G \setminus \mathrm{tor}(G)$.

$(*)_3$ $f \restriction \mathrm{tor}(G) \in \mathrm{End}(\mathrm{tor}(G))$ is 1-to-1 not onto.

[5]In the light of our main result such a group exists for any $\lambda = \lambda^{\aleph_0} > 2^{\aleph_0}$ and the size of G should be λ.

Clearly $f \restriction \mathrm{tor}(G) \in \mathrm{End}(\mathrm{tor}(G))$, and since f is 1-to-1, $f \restriction \mathrm{tor}(G)$ is 1-to-1 as well. Now, suppose by contradiction that $f \restriction \mathrm{tor}(G)$ is onto. Then

(1) $\mathrm{tor}(G) \subseteq \mathrm{Rang}(f)$,

(2) $x \in G \Rightarrow f(x) = x + h(x) \in \mathrm{tor}(G)$.

Recall that $h(x) \in \mathrm{tor}(G)$. Apply this along with (1), we deduce that $h(x) \in \mathrm{Rang}(f)$. Also, recall that $\mathrm{Rang}(f)$ is a group. Now, let $x \in G$. Thanks to (2), we observe that

$$x = f(x) - h(x) \in \mathrm{Rang}(f).$$

In other words, f is onto, a contradiction. So, $f \restriction \mathrm{tor}(G)$ is not onto.

$(*)_4$ (a) for every $p \in \mathbb{P}$, f maps $\Gamma_p(G)$ into itself and so $f \restriction \Gamma_p(G)$ is 1-to-1,

(b) for some $p \in \mathbb{P}$, $f \restriction \Gamma_p(G)$ is not onto.

Item (a) above is simply because f is 1-to-1. To see (b) holds, note that if $f \restriction \Gamma_p(G)$ is onto for all prime number p, then so is $f \restriction \mathrm{tor}(G)$, which contradicts $(*)_3$.

Thus, let us fix some prime $p \in \mathbb{P}$ such that $f \restriction \Gamma_p(G)$ is not onto and let $h_p = h \restriction \Gamma_p(G)$. Then by the above observations, it equipped with the following properties:

$(*)_5$ (a) $h_p \in \mathrm{End}(\Gamma_\mathrm{p}(\mathrm{G}))$,

(b) $\mathrm{Rang}(h_p)$ is bounded,

(c) $h'_p = m \cdot \mathrm{id}_{\Gamma_p(G)} + h_p = \mathrm{id}_{\Gamma_p(G)} + h_p$ is 1-to-1,

(d) h'_p is not onto.

In the light of Definition 4.2 and $(*)_5$ we observe that

$(*)_6$ h_p is a $(1, p)$-anti-witness for $\Gamma_p(G)$ and so $\mathrm{NQr}(\Gamma_p(G))$.

Thanks to Lemma 4.5, $\Gamma_p(G)$ is infinite. But,

$$\Gamma_p(G) = \Gamma_p(K) = \bigoplus\{\tfrac{\mathbb{Z}}{p^n\mathbb{Z}} : 1 \leq n < m\},$$

which is finite. Thus we get a contradiction, and hence f is onto. It follows that G is co-Hopfian and the lemma follows. $\square$

Corollary 4.13. *For any cardinals $\lambda > 2^{\aleph_0}$, there is a co-Hopfian abelian group G of size λ iff $\lambda = \lambda^{\aleph_0}$.*

Proof. Let $\lambda > 2^{\aleph_0}$ be given. Suppose first that $\lambda < \lambda^{\aleph_0}$. In other words, $2^{\aleph_0} < \lambda < \lambda^{\aleph_0}$. According to Fact 4.1(ii), there is no co-Hopfian abelian group of size λ. Now, assume that $\lambda = \lambda^{\aleph_0}$. Let

$$K := \oplus\{\frac{\mathbb{Z}}{p^n\mathbb{Z}} : p \in \mathbb{P} \text{ and } 1 \leq n < m\},$$

where $m < \omega$. In the light of Theorem 3.11, there exists a boundedly endo-rigid abelian group G with $\operatorname{tor}(G) = K$. By Construction 4.12, G is co-Hopfian. □

Notation 4.14. (Harrison) For each group G, we set

$$S := S_G := \{p \in \mathbb{P} : G/\Gamma_p(G) \textit{ is not } p\textit{-divisible}\}.$$

Now, we are ready to present the following promised criteria:

Proposition 4.15. *Let $\lambda > 2^{\aleph_0}$, and suppose G is a boundedly endo-rigid abelian group of size λ. Then G is co-Hopfian if and only if:*

- *(a): S_G is a non-empty set of primes,*
- *(b):* (b_1) $\operatorname{tor}(G) \neq G$,
  - (b_2) *if $p \in S$, then $\Gamma_p(G)$ is not bounded,*
  - (b_3) *if $\Gamma_p(G)$ is bounded, then it is finite (and $p \notin S_G$).*

Proof. Let $K := \operatorname{tor}(G)$, and for each prime number p, we set $K_p := \Gamma_p(G)$.

First, we assume that G is co-Hopfian, and we are going to show items (a) and (b) are valid. As G is co-Hopfian, and recall from the introduction that Beaumont and Pierce (see [5]) proved that for the co-Hopfian group G, we know $\operatorname{tor}(G)$ is of size at most continuum. In other words, $|\operatorname{tor}(G)| \leq 2^{\aleph_0}$. We combine this along with our assumption $|G| = \lambda > 2^{\aleph_0}$, and conclude that $K = \operatorname{tor}(G) \neq G$, as claimed by (b_1).

To prove (b_2), let $p \in S$ and suppose by contradiction that K_p is bounded. As K_p is pure in G, and following Fact 2.4, the boundedness property guarantees that K_P is a direct summand of G. By definition, there is G_p such that $G = K_p \oplus G_p$. Now, we look at $\operatorname{id}_{K_p} + p \cdot \operatorname{id}_{G_p} \in \operatorname{End}(G)$. Let

$$(k, g) \in \operatorname{Ker}(\operatorname{id}_{K_p} + p \cdot \operatorname{id}_{G_p}).$$

Following definition,

$$(0,0) = (\mathrm{id}_{K_p} + p \cdot \mathrm{id}_{G_p})(k,g) = (k,pg).$$

In other words, $k = 0$ and as G_p is p-torsion-free, $g = 0$. This means that

$$\mathrm{Ker}(\mathrm{id}_{K_p} + p \cdot \mathrm{id}_{G_p}) = 0,$$

and hence $\mathrm{id}_{K_p} + p \cdot \mathrm{id}_{G_p}$ is 1-to-1. Since $p \in S$, $G_p := G/\Gamma_p(G)$ is not p-divisible, thus there is g in G_p such that $g \notin \mathrm{Rang}(p \cdot \mathrm{id}_{G_p})$. Consequently, $\mathrm{id}_{K_p} + p \cdot \mathrm{id}_{G_p}$ is 1-to-1 not onto. This is in contradiction with the co-Hopfian assumption, so K_p is not bounded and (b_2) follows.

In order to check (b_3), suppose $K_p = \Gamma_p(G)$ is bounded. Then it is a direct summand of G, say $G = K_p \oplus G_p$. Since G is co-Hopfian, and in view of Fact 4.1, we observe that K_p is co-Hopfian. Thanks to Fact 2.6 K_p is finite.

Lastly, we check clause (a). Suppose on the way of contradiction that S is empty. Let $G_1 \prec_{\mathcal{L}_{\aleph_1,\aleph_1}} G$ be of cardinality $2^{\aleph_0}$ containing $\mathrm{tor}(G)$, recalling $|\mathrm{tor}(G)| \leq 2^{\aleph_0}$, so G/G_1 is divisible of cardinality λ.

As $G_1 \neq G$, there is $x_0 \in G \setminus G_1$, and note that $x \notin \mathrm{tor}(G)$. Now as $G/\mathrm{tor}(G)$ is divisible, we can choose the sequence $\langle x_n : n \geq 1 \rangle$ of elements of G, by induction on n, such that $x_0 = x$ and for each n,

$$G/\mathrm{tor}(G) \models \text{“}n!x_{n+1} + \mathrm{tor}(G) = x_n + \mathrm{tor}(G)\text{''}.$$

Set

$$a_n := n!x_{n+1} - x_n \in \mathrm{tor}(G).$$

Note that $\langle a_n : n < \omega \rangle \in G_1$, thus as $G_1 \prec_{\mathcal{L}_{\aleph_1,\aleph_1}} G$, we can find elements $y_n \in G_1$ for $n < \omega$ such that

$$n!y_{n+1} = y_n + a_n.$$

Subtracting the last two displayed formulas, shows that the group

$$L = \bigcup \{\mathbb{Z}(x_n - y_n) : n < \omega\}$$

is a non-zero divisible subgroup of G. Since L is an injective group, the sequence

$$0 \longrightarrow L \xrightarrow{g} G \longrightarrow \mathrm{Coker}(g) \longrightarrow 0,$$

splits. Recall from Discussion 2.20 that

$$\mathrm{End}(I) = \prod_{p\in\mathbb{P}_0} \widehat{\mathbb{Z}}_p^{\oplus x_p},$$

where $\mathbb{P}_0 := \mathbb{P} \cup \{0\}$ and x_p are some index sets. This turns out that I is not boundedly endo-rigid, provided it is nonzero. As the property of boundedly endo-rigid behaves well with respect to direct sum, it obviously implies G is not boundedly endo-rigid. This contradiction implies that S is not empty. So clause (a) holds. All together, we are done proving the left-right implication.

For the right-left implication, assume items (a) and (b) hold, and we show that G is co-Hopfian. Suppose on the way of contradiction that there exists $f \in \mathrm{End}(G)$ such that f is 1-to-1 and not onto. As G is boundedly endo-rigid, there are $m \in \mathbb{Z}$ and $h \in \mathrm{E_b}(G)$ such that $f = m \cdot \mathrm{id}_G + h$.

$(*)_1$ $m \neq 0$.

To see $(*)_1$, suppose $m = 0$. Then $f = h$, and since $\mathrm{Rang}(h)$ is bounded and f is 1-to-1, we can conclude that G is bounded and therefor $G = \mathrm{tor}(G)$. This contradicts clause (b_1).

$(*)_2$ If $\Gamma_p(G)$ is infinite, then $p \nmid m$.

In order to see $(*)_2$, first note that $\mathrm{tor}(G)$ is unbounded, as otherwise $\Gamma_p(G)$ is also bounded, hence by (b_3) it is finite, contradicting our assumption. Suppose on the way of contradiction that $p \mid m$. Then there is m_1 such that $m = m_1 p$. Now, as $\mathrm{Rang}(h)$ is bounded, there exists $k \geq 1$ such that

$$p^k\big(\mathrm{Rang}(h){\restriction}\Gamma_p(G)\big) = \{0\}.$$

Recall that K_p is unbounded. This gives us an element $x \in \Gamma_p(G)$ of order p^n for some $n \geq k+1$. But then

$$\begin{aligned} f(p^{n-1}x) &= mp^{n-1}x + h(p^{n-1}x) \\ &= m_1 pp^{n-1}x + p^k h(p^{n-1-k}x) \\ &= 0, \end{aligned}$$

which contradicts the fact that f is 1-to-1.

As before, we have the following properties:

$(*)_3$ f maps $G \setminus \mathrm{tor}(G)$ into itself.

$(*)_4$ $f{\restriction}\mathrm{tor}(G) \in \mathrm{End}(\mathrm{tor}(G))$ is 1-to-1 not onto.

$(*)_5$ (a) for every $p \in \mathbb{P}$, f maps $\Gamma_p(G)$ into itself and so $f \restriction \Gamma_p(G)$ is 1-to-1,

(b) for some $p \in \mathbb{P}$, $f \restriction \Gamma_p(G)$ is not onto.

Fix $p \in \mathbb{P}$ such that $f \restriction \Gamma_p(G)$ is not onto. Then $h_p := h \restriction \Gamma_p(G)$ is equipped with the following properties:

$(*)_6$ (a) $h_p \in \mathrm{End}(\Gamma_{\rm p}(\rm G))$,

(b) $\mathrm{Rang}(h_p)$ is bounded,

(c) $h'_p = m \cdot \mathrm{id}_{\Gamma_p(G)} + h_p = \mathrm{id}_{\Gamma_p(G)} + h_p$ is 1-to-1,

(d) h'_p is not onto.

In the light of its definition, h_p is a $(1, p)$-anti-witness and so $\mathrm{NQr}(\Gamma_p(G))$ holds. Thanks to Lemma 4.5:

$(*)_7$ $\Gamma_p(G)$ is infinite.

This is in contradiction with $(*)_2$. $\square$

In [1] we studied absolutely co-Hopfian abelian groups. Recall an abelian group is absolutely co-Hopfian if it is co-Hopfian in any further generic extension of the universe. Also, see [22] for the existence of absolutely Hopfian abelian groups of any given size. Similarly, one may define absolutely endo-rigid groups. Despite its simple statement, one of the most frustrating problems in the theory infinite abelian groups is as follows:

Problem 4.16. Are there absolutely endo-rigid abelian groups of arbitrary large cardinality?

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