Sh:1126

Bollettino dell'Unione Matematica Italiana (2023) 16:521–584
https://doi.org/10.1007/s40574-022-00338-4

Check for updates

Corrected iteration

Saharon Shelah[1,2]

Received: 27 September 2022 / Accepted: 30 September 2022 / Published online: 4 December 2022

Abstract
For λ inaccessible, we may consider $(<\lambda)$-support iteration of some definable in fact specific $(<\lambda)$-complete λ^+-c.c. forcing notions. But do we have "preservation by restricting to a sub-sequence of the iterated forcing"? To regain it we "correct" the iteration. We prove this for a characteristic case for iterations which holds by "nice" for $\lambda = \aleph_0$. This is done generally in a work H. Horowitz and the author Shelah. This work is use in a work of the author in (Trans Am Math Soc 373(8):5351–5369, arXiv:0904.0817, 2020) where we use so called strongly $(<\lambda^+)$-directed $\mathbf{m}$. We could here restrict ourselves to reasonable $\mathbf{m}$ (see 2.13(3)).

Keywords Set theory · Independence · Iterated forcing · Cardinal invariants

Mathematics Subject Classification Primary 03E35; Secondary 03E17 · 03E55

Contents

Partially supported by European Research Council Grant No. 338821, and the Israel Science Foundation Grant 1838/19. Paper 1126 on the Author's list. The author thanks Alice Leonhardt for the beautiful typing of earlier versions (up to 2019) and in later versions the author would like to thank the typist for his work and is also grateful for the generous funding of typing services donated by a person who wishes to remain anonymous. References like [12, 2.7=La32] means the label of Th.2.7 is a32. The reader should note that the version in my website is usually more updated than the one in the mathematical archive. First typed October 18, 2017.

✉ Saharon Shelah
shelah@math.huji.ac.il
http://shelah.logic.at

1 Einstein Institute of Mathematics Edmond J. Safra Campus, Givat Ram, The Hebrew University of Jerusalem, Jerusalem 9190401, Israel

2 Department of Mathematics Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA

0 Introduction

This work is dedicated to proving a theorem on $(< \lambda)$-support iterations of $(< \lambda)$-complete "nicely" definable λ^+-c.c. forcing notions for λ inaccessible. A nicely definable forcing notion can be, for example, random reals forcing (when $\lambda = \aleph_0$). Pedantically, at each stage it is a different forcing notion, but it has the same definition at every step of the iteration. Assume $\mathbb{Q}$ is such a definition, $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \alpha_*, \beta < \alpha_* \rangle$ is such an iteration, $\mathbb{Q}_\beta = \mathbb{Q}^{\mathbf{V}[\mathbb{P}_\beta]}$ has generic $\underset{\sim}{\eta}_\beta$. A question is: assuming $\langle \eta_\beta : \beta < \alpha_* \rangle$ is generic for $\mathbb{P}_{\alpha_*}$, and letting β_* be maximal such that $2\beta_* \leq \alpha_*$, does it follows that also the sequence $\langle \underset{\sim}{\eta}_{2\beta} : \beta$ satisfies $2\beta < \alpha_* \rangle$ is generic for the iteration $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \beta_*, \beta < \beta_* \rangle$?

The point is that in the parallel case for $\lambda = \aleph_0$ so for FS-iterated forcing such a claim is true. In fact, by Judah-Shelah [4], if $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta : \alpha \leq \alpha(*), \beta < \alpha(*) \rangle$ is FS-iteration of Suslin-c.c.c. forcing notions, $\underset{\sim}{\mathbb{Q}}_\beta$ with the generic $\underset{\sim}{\eta}_\beta \in {}^\omega\omega$ and for notational transparency, its definition is with no parameter and the function $\zeta : \beta(*) \to \alpha(*)$ is increasing and $\mathbb{P} = \langle \mathbb{P}'_\alpha, \underset{\sim}{\mathbb{Q}}'_\beta : \alpha \leq \beta(*), \beta < \beta(*) \rangle$ is FS iteration, $\underset{\sim}{\mathbb{Q}}'_\beta$ defined exactly as $\underset{\sim}{\mathbb{Q}}_{\zeta(\beta)}$ but now in $\mathbf{V}^{\mathbb{P}'_\beta}$ rather than $\mathbf{V}^{\mathbb{P}_{\zeta(\beta)}}$ then $\Vdash_{\mathbb{P}_{\alpha(*)}}$ "$\langle \eta_{\zeta(\beta)} : \beta < \beta(*) \rangle$ is generic for $\mathbb{P}'_{\beta(*)}$ over $\mathbf{V}$". For CS iteration of Suslin proper forcing a weaker result holds, see [4, §2] and [10].

Now this is not clear to us for $(< \lambda)$-support iteration of $(< \lambda)$-strategically complete forcing notions. The solution is essentially to change the iteration to what we call "corrected iteration". We use a "quite generic" $(< \lambda)$-support iteration which "includes" the one we like and use the complete sub-forcing it generates. Here we deal with a characteristic case (used in [12]). The proof applies also to partial memory iteration. On wide generalization (including the case $\lambda = \aleph_0$) and application (for $\lambda = \aleph_0$) this is continued in a work of H. Horowitz and the author [3]; more fully [3] generalizes §1, §2, §3A, §3B, §3D of the present work whereas §3C, §3E, §3F were added later, and §3C is inverse engineering of [3, 4.2,4.4]. Our main result is 2.12, proving that there is "corrected iteration", i.e. one satisfying the promised property or see 2.11 and more in 2.16, 2.17.

The problem arises as follows. In [12] it is proved that for λ inaccessible, consistently cov_λ(meagre), the covering number of the meagre ideal on λ is strictly smaller than $\mathfrak{d}_\lambda$, the dominating number. The result here is used there but the editor prefers to separate it. In §3F we have an alternative proof of the main theorem, for this we noted in some earlier places what rely on what.

We have two extreme versions of our frameworks, one we call fat, that is, in Definition 1.10, $\mathscr{P}_{\mathbf{m},t} = [u_{\mathbf{m},t}]^{\leq\lambda}$ (used in [12]). The other is the lean one when the $\mathscr{P}_{\mathbf{m},t}$ are restricted to the leaves (i.e. $t/E'_{\mathbf{m}}$). This was the original version and is the one continued in Horowitz-Shelah [3].

The interest in having "**m** is strongly λ^+-directed" is that it implies $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$\{\underset{\sim}{\eta}_s : s \in M\}$ cofinal in $\left(\Pi_{\varepsilon<\lambda}\theta_\varepsilon, <_{J^{\mathrm{bd}}_\lambda}\right)$", by 1.29. Now using $\mathbf{m} \in \mathbf{M}_{\mathrm{ec}}$ (being full and wide) as

Sh:1126

constructed in §1C, does not give this, e.g. because there may be $t \in L_{\mathbf{m}}$ above all members of $M_{\mathbf{m}}$. This is circumvented in 2.6 by having, on the one hand for cofinaly many $c \in M_{\mathbf{m}}$, $\mathbf{m}(< c) \in \mathbf{M}_{\text{ec}}$ and on the other hand having "$\mathbf{m}$ is strongly $(< \lambda^+)$-directed" (see 2.13(2)). An alternative approach is to restrict ourselves to the fat context.

This work is continued in [3] and lately in [7], which in particular sort out when corrected iteration is necessary; we have lecture on this in the Set Theoretic Conference, in Jerusalem, July 2022.

We thank Shimoni Garti and Haim Horowitz for helpful comments. We thank Johannes Schürz and Martin Goldstern for pointing out several times problem with the application to [12], in particular in 2019 that an earlier version of the proof of [12, 2.7=La32] the statement $\circledast_4'$ was insufficient; and later pointing out a problem in earlier version of §3E. We thanks Mark Poór for pointing out many points which need correction.

For a reader of [12] we try to give exact references to the places here we rely on there (pages refer to the 2022-08 version; there we assume that $\mathbf{m}$ is ordinary, that is, $L_{\mathbf{m}}$ has set of elements an ordinal $\alpha(\mathbf{m})$ and $\beta < \gamma < \alpha(\mathbf{m})$ implies $\beta <_{L_{\mathbf{m}}} \gamma$).

(a) on [12, 1.8=Lz32, page 6], the definition of $\mathbf{Q} = \mathbf{Q}_{\lambda,\bar{\theta},\alpha(*)}$, see here Def 1.10, Claim 1.11, page 9, so $\mathbf{q}$ there is (essentially) $\mathbf{q}_{\mathbf{m}}$ here, and so $\mathbb{P}_{\mathbf{m}\restriction L}$ here is dense in $\mathbb{P}^{\mathbf{q}}_{0,\alpha}$ there when $L = L_{\mathbf{m}} \restriction \alpha$,
(b) on [12, 1.9=Lz33, pag.7] where $\mathbb{P}^{\mathbf{q}}_{1,\alpha}$ defined there, is $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}} \restriction \alpha]$ here; see 2.4(3), page 29,
(c) on [12, 1.10=Lz35, pag.7], claim on the existence of generic; include changing the generic in $< \lambda$ places see here 1.13, 1.16, page 11, 13 respectively,
(d) on [12, 1.11=Lz38, pag.8] see 2.12 page 32 or 2.14, page 33,
(e) in $(*)_1$(A) in the proof of [12, 2.7 = La32, page 15], see (a)-(e) above,
(f) in $(*)_4$ in the proof of [12, 2.7 = La32, page 16], See 2.14.
(g) after $(*)_7$ in the proof of [12, 2.7 = La32, page 17] See 0.6(4).
(h) on $\boxplus_1$ inside the proof of Lemma [12, 2.7=La32, pag. 17], more details are in 2.12, that is: $\boxplus(a)(\alpha)$ by 2.12(A)(c); $\boxplus(a)(\beta)$ by 2.12(a)(h); $\boxplus(b)$ by 2.12(C); $\boxplus(c)$ by 2.12(A)(b); $\boxplus(d)$ by 2.12(B); $\boxplus(e)$ by 2.12(A)(e),
(i) on $\circledast_4'$ inside the proof of Lemma [12, 2.7=La32, pag.18-19], see [4.12-4.27 = Le53-Le70],
(j) In [12, 2.8 = La35, pg. 21] we use 4.26, page 69.

Note that even if $s \in M_{\mathbf{m}} \Rightarrow u_s \cap M_{\mathbf{m}} = \emptyset$ still: if $\mathbf{m} \in \mathbf{M}_{\text{ec}}$ then $M_{\mathbf{m}} \models s < t \Rightarrow \underset{\sim}{\eta}_s < \underset{\sim}{\eta}_t$ mod J^{bd}_λ, see 1.29.

Notation 0.1 We try to use standard notation. We use $\theta, \kappa, \lambda, \mu, \chi$, for cardinals and $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \xi$ for ordinals. We use also i and j as ordinals. We adopt the Cohen convention that $p \le q$ means that q gives more information, in forcing notions. The symbol $\triangleleft$ is preserved for "being a proper initial segment". Also recall ${}^B A = \{f : f$ a function from B to $A\}$ and let ${}^{\alpha>}A = \cup\{{}^\beta A : \beta < \alpha\}$, some prefer ${}^{<\alpha}A$, but ${}^{\alpha>}A$ is used systematically in the author's papers. Lastly, J^{bd}_λ denotes the ideal of the bounded subsets of λ.

Recall from [12]:

Definition 0.2 Let λ be inaccessible, $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$ be a sequence of regular cardinals $< \lambda$ satisfying $\theta_\varepsilon > \varepsilon$.

(1) We define the forcing notion $\mathbb{Q} = \mathbb{Q}_{\bar{\theta}}$ by:

(α) $p \in \mathbb{Q}$ iff:

(a) $p = (\eta, f) = (\eta^p, f^p)$,
(b) $\eta \in \prod_{\zeta<\varepsilon} \theta_\zeta$ for some $\varepsilon < \lambda$, (η is called the trunk of p),
(c) $f \in \prod_{\zeta<\lambda} \theta_\zeta$,
(d) $\eta \triangleleft f$.

(β) $p \leq_{\mathbb{Q}} q$ iff:
(a) $\eta^p \trianglelefteq \eta^q$,
(b) $f^p \leq f^q$, i.e. $(\forall \varepsilon < \lambda) f^p(\varepsilon) \leq f^q(\varepsilon)$,
(c) if $\ell g(\eta^p) \leq \varepsilon < \ell g(\eta^q)$ then $\eta^q(\varepsilon) \in [f^p(\varepsilon), \lambda)$, actually follows.

(2) The generic is $\underset{\sim}{\eta} = \cup\{\eta^p : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_{\bar\theta}}\}$.

The new forcing defined above is not λ-complete anymore. By fixing a trunk η one can define a short increasing sequence of conditions which goes up to some θ_ζ at the ζ-th coordinate and hence has no upper bound in $\prod_{\zeta<\varepsilon} \theta_\zeta$. However, this forcing is $(<\lambda)$-strategically complete since the COM (= completeness) player can increase the trunk at each move.

Remark 0.3 (0) The forcing parallel to the creature forcing from [8], [5] but they are ${}^\omega\omega$-bounding.

(1) The forcing is <u>parallel</u> to the creature forcing from [8, §1,§2], [5] though they are ${}^\omega\omega$-bounding and <u>not to Hechler</u> forcing, whose parallel for λ is $\mathbb{Q}_\lambda^{\text{dom}} = \mathbb{Q}_\lambda^{\text{Hechler}} = \{(\nu, f) : f \in {}^\lambda\lambda, \nu \triangleleft f\}$, ordered naturally. We can change the definition of order, saying $p < q$ iff $p = q$ or $p \leq q \wedge \text{tr}(p) \neq \text{tr}(q)$ and then all (strictly) increasing sequence of length $< \lambda$ have upper bound, but the gain is doubtful as we shall use only strategic completeness for some derived forcing notions.

(2) Closer to [8] we can use $\bar\theta = \langle \theta_{1,\varepsilon}, \theta_{0,\varepsilon} : \varepsilon < \lambda \rangle$ such that $\theta_{1,\varepsilon} \geq \theta_{0,\varepsilon} = \text{cf}(\theta_{\theta,\varepsilon}) > \varepsilon$ and $\lambda > \theta_{1,\varepsilon}$, and let $\mathbb{Q}$ be such that:

(a) $p = (n, f) = (n_p, f_p) \in \mathbb{Q}_{\bar{Q}}$ <u>iff</u>:
- $\eta \in \Pi_{\varepsilon<\zeta}\theta_{1,\varepsilon}, \zeta < \lambda$,
- $f \in \Pi_{\varepsilon\in[\zeta,\lambda)}[\theta_{1,\varepsilon}]^{<\theta_{0,\varepsilon}}$.

(b) $\mathbb{Q}_{\bar\theta} \models p \leq q$ <u>iff</u>:
- $p, q \in \theta_\theta$,
- $\eta_p \trianglelefteq \eta_q$,
- $\varepsilon \in [\lg(\eta_q), \lambda) \Rightarrow f_p(\varepsilon) \subseteq f_q(\varepsilon)$,
- $\varepsilon \in [\lg(\eta_p), \lg(\eta_q)) \Rightarrow \eta_q(\varepsilon) \in f_p(\varepsilon)$.

Does not matter.

Notation 0.4 (1) L, M, N are linear orders and r, s, t are members.

(2) If $\eta \in \Pi_{\varepsilon<\zeta}\theta_\varepsilon$ where $\zeta < \lambda$ then $(\Pi_{\varepsilon<\lambda}\theta_\varepsilon)^{[\eta]}$ will mean $\{\nu \in \Pi_{\varepsilon<\lambda}\theta_\varepsilon : \nu$ satisfies $\eta \trianglelefteq \nu\}$.

(3) For a cardinal λ by induction on ordinal α we define $\beth_\alpha(\lambda)$ as $\lambda + \Sigma_{\beta<\alpha} 2^{\beth_\beta(\lambda)}$ and $\beth_\alpha = \beth(\alpha) = \beth_\alpha(\aleph_0)$.

Discussion 0.5 (1) Fat λ^+-directed $\mathbf{m}$ are helpful when we like to have $\Vdash_{\mathbb{P}_\mathbf{m}}$ "$\{\underset{\sim}{\eta}_s : s \in M_\mathbf{m}\}$ is cofinal in $(\Pi_{\varepsilon<\lambda}\theta_\varepsilon, <_{J_\lambda^{\text{bd}}})$" as in [12], see Definition 1.5.

Recall,

Definition 0.6 (1) We say that a forcing notion $\mathbb{P}$ is α-strategically complete when for each $p \in \mathbb{P}$ in the following game $\Game_\alpha(p, \mathbb{P})$ between the players COM and INC, the player COM has a winning strategy.
A play lasts α moves; in the β-th move, first the player COM chooses $p_\beta \in \mathbb{P}$ such that $p \leq_{\mathbb{P}} p_\beta$ and $\gamma < \beta \Rightarrow q_\gamma \leq_{\mathbb{P}} p_\beta$ and second the player INC chooses $q_\beta \in \mathbb{P}$ such that $p_\beta \leq_{\mathbb{P}} q_\beta$.
The player COM wins a play if he has a legal move for every $\beta < \alpha$.
(2) We say that a forcing notion $\mathbb{P}$ is $(< \lambda)$-strategically complete when it is α-strategically complete for every $\alpha < \lambda$.
Basic properties of $\mathbb{Q}_{\bar{\theta}}$ are summarized and proved in [2, §2].

1 Iteration parameters

1.1 The frame

Hypothesis 1.1 (1) $\lambda = \lambda^{<\lambda}$ is strongly inaccessible.
(2) $\bar{\theta} = \langle \theta_\varepsilon : \varepsilon < \lambda \rangle$.
(3) θ_ε is an infinite regular cardinal $> \varepsilon$ and $< \lambda$.
(4) Assume $\lambda_2 \geq \lambda_1 \geq \lambda_0 = \mathrm{cf}(\lambda_0) > \lambda$ are such that[1] $(\lambda_1)^{\lambda_0} = \lambda_1$, so notations should have the parameter $\bar{\lambda} = (\lambda_2, \lambda_1, \lambda_0, \lambda)$ and even[2] $\bar{\lambda} = (\lambda_2, \lambda_1, \lambda_0, \lambda, \bar{\theta})$.

Notation 1.2 (1) L, M denote partial orders, well founded if not said otherwise.
(2) Below $\mathbf{m}, \mathbf{n}$ will be members of $\mathbf{M}$; we may write e.g. $L, \mathbf{q}$ instead $L_\mathbf{m}, \mathbf{q}_\mathbf{m}$ when $\mathbf{m}$ is clear from the context, see Def 1.5, 1.10.
(3) We may not pedantically distinguish the subset L_1 of L and the sub-partial order L_1 of L.

Remark 1.3 Here there is no harm in adding:

(a) $\theta_\varepsilon > \prod_{\zeta < \varepsilon} 2^{\theta_\zeta} + 2^{\aleph_0}$ for $\varepsilon < \lambda$, and/or,
(b) for $\mathbf{m} \in \mathbf{M}$ demanding $M_\mathbf{m}$ is a linear order, well founded (it suffices to assume even $M \cong (\kappa, <)$, κ regular from $[\lambda_0, \lambda_1)$).

Definition 1.4 (1) For a partial order L (not necessarily well founded) let:

(α) $\mathrm{dp}(L) = \cup\{\mathrm{dp}_L(t) + 1 : t \in L\}$, see below,
(β) $\mathrm{dp}_\mathrm{L}(\mathrm{t}) = \mathrm{dp}(\mathrm{t}, \mathrm{L}) \in \mathrm{Ord} \cup \{\infty\}$ be defined by $\mathrm{dp}_L(t) = \cup\{\mathrm{dp}_L(s) + 1 \colon s <_L t\}$.
(γ) $L_{<t} = L \restriction \{s \in L : s <_L t\}$,
(δ) $L_{\leq t} = L \restriction \{s \in L : s \leq_L t\}$.

(2) Let $L^+ = L(+)$ be $L \cup \{\infty\}$ with the natural order (but we may write $t <_L \infty$ instead of $t <_{L(+)} \infty$).
(3) We say the set L is an initial segment of the partial order L_*, when:

- $L \subseteq L_*$, i.e. $s \in L \Rightarrow s \in L_*$,
- $s <_{L_*} t \wedge t \in L \Rightarrow s \in L$.

[1] Usually $\lambda_2 = (\lambda_2)^\lambda \geq \lambda_1$ suffices but see 3.12, 3.22, however in §4A we add $\lambda_2 \geq \beth_{\lambda_1^+}$.

[2] We mainly can use $\lambda_0 = \lambda^+$, but when we restrict ourselves to lean $\mathbf{m}$-s, $\lambda_0 = \lambda$ seem to suffice, see mainly 1.13(f)(γ), §2, §3C but does not seem worthwhile to pursue.

The class $\mathbf{M}$ is central in this work, see explanation 1.9, in particular, $M_{\mathbf{m}}$ is our aim, the rest ($L_{\mathbf{m}}$ first of all) are the scaffoldings.

Definition 1.5 (1) Let $\mathbf{M}$ be the class of objects $\mathbf{m}$, called iteration parameters, of the following form (so really $\mathbf{M} = \mathbf{M}[\bar{\lambda}]$ and if we omit sub-clauses (θ), (ι) of clause (e) we may write $\mathbf{M}[*]$).

(a) L, a partial order,
(b) $M \subseteq L$, as partial orders, (in the main case M is linearly ordered),
(c) (α) $u = \langle u_t : t \in L\rangle$, $\bar{\mathscr{P}} = \langle \mathscr{P}_t : t \in L\rangle$, each $\mathscr{P}_t$ is closed under subsets and $\mathscr{P}_t \subseteq [u_t]^{\leq\lambda}$,
(β) $u_t \subseteq \{s \in L : s <_L t\}$,
(d) dp(L) $< \infty$, that is L is well founded,
(e)

(α) E' is a two-place relation (on L),
(β) $E'' := E' {\restriction} (L \setminus M)$ is an equivalence relation on $L \setminus M$,
(γ) the order $\leq_L$ is the transitive closure of $\bigcup\{\leq_L {\restriction}(s/E') : s \in L \setminus M\} \cup \{\leq_L {\restriction} M\}$, equivalently (using (δ)-(η) below):

- if $s, t \in L \setminus M$ are not E''-equivalent, then $s <_L$ t <u>iff</u> for some $r_1 \leq_{\mathbf{m}} r_2$, we have $s \leq_{\mathbf{m}} r_1$ from $\in s/E'_{\mathbf{m}}$, $r_2 \leq_{\mathbf{m}} t$, $r_2 \in t/E'_{\mathbf{m}}$,
- if $s \in L \setminus M$ and $t \in M$, then $s \leq_L t$ <u>iff</u> for some $r \in (s/E') \cap M$ we have $s < r \leq t$,
- if $s \in M$ and $t \in L \setminus M$, then $s <_L t$ <u>iff</u> for some $r \in (t/E') \cap M$ we have $s \leq r < t$.

(δ) if $sE't$ then $s \notin M \vee t \notin\in M$,
(ε) is $t \in L \setminus M$ then $\{s \in L : sE't\} = \{s \in L : tE's\}$; we call it t/E'; so E' is a symmetric relation,
(ζ) if $s, t \in L \setminus M$ are E''-equivalent then $s/E' = t/E'$,
(η) if $t \in L \setminus M$ then $u_t \subseteq t/E'$,
(θ) if $t \in L \setminus M$ then t/E' has cardinality $\leq \lambda_2$,
(ι) $\|M\| \leq \lambda_1$,

(f) disjoint subsets $M^{\text{fat}}_{\mathbf{m}}$, $M^{\text{lean}}_{\mathbf{m}}$ of $M_{\mathbf{m}}$ such that:

- if $s \in M^{\text{fat}}_{\mathbf{m}}$ then $\mathscr{P}_{\mathbf{m},s} = [u_t]^{\leq\lambda}$,
- if $s \in M^{\text{lean}}_{\mathbf{m}}$ then $u \in \mathscr{P}_{\mathbf{m},s} \Rightarrow (\exists t)(u \subseteq t/E_{\mathbf{m}})$
- we let $M^{\text{non}}_{\mathbf{m}} = M_{\mathbf{m}} \setminus (M^{\text{fat}}_{\mathbf{m}} \cup M^{\text{lean}}_{\mathbf{m}})$.

(2) Saying $\mathbf{m} \in M$ is lean means that $M_{\mathbf{m}} = M^{\text{lean}}_{\mathbf{m}}$. The lean context means that we restrict ourselves to lean $\mathbf{m}$: similarly for fat and neat, see below.
(3) We say $\mathbf{m} \in M$ is fat <u>when</u> $M_{\mathbf{m}} = M^{\text{fat}}_{\mathbf{m}}$ and moreover $t \in L_{\mathbf{m}} \Rightarrow \mathscr{P}_t = [u_t]^{\leq\lambda}$.
(4) $M_{\mathbf{m}}$ is neat <u>when</u> $M_{\mathbf{m}} = M^{\text{lean}}_{\mathbf{m}} \cup M^{\text{fat}}_{\mathbf{m}}$.

Remark 1.6 (1) We may demand $\mathbf{m}$ is strongly $(< \lambda)$-directed, see Definition 2.13(2) or even reasonable, see Definition 2.13(3); is harmless here and help [12].
(2) It may seem reasonable to demand:

$\boxplus$ is $s \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ and $s \in A \in \mathscr{P}_t$, then $(s/E') \cap u_t \in \mathscr{P}_t$.

However in the crucial claim 3.25, 3.26 this cause problems for $t \in M_{\mathbf{m}} \setminus M^{\text{fat}}_{\mathbf{m}}$.

Definition 1.7 For $\mathbf{m} \in \mathbf{M}$.

(0) In 1.5 we let $\mathbf{m} = (L_{\mathbf{m}}, M_{\mathbf{m}}, \bar{u}_{\mathbf{m}}, \bar{\mathscr{P}}_{\mathbf{m}}, E'_{\mathbf{m}}, M^{\text{lean}}_{\mathbf{m}}, M^{\text{fat}}_{\mathbf{m}})$, $\bar{u}_{\mathbf{m}} = \langle u_{\mathbf{m},t} : t \in L_{\mathbf{m}} \rangle$, $\bar{\mathscr{P}}_{\mathbf{m}} = \langle \mathscr{P}_{\mathbf{m},t} : t \in L_{\mathbf{m}} \rangle$, for $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ let $t/E_{\mathbf{m}} = (t/E'_{\mathbf{m}}) \cup M_{\mathbf{m}}$ and for $t \in M_{\mathbf{m}}$ let $t/E_{\mathbf{m}} = M_{\mathbf{m}}$; so there is no relation $E_{\mathbf{m}}$ but $t/E_{\mathbf{m}}$ for $t \in L_{\mathbf{m}}$, is well defined.

(1) In 1.5, let $\text{dp}_{\mathbf{m}}(t) = \text{dp}_{L_{\mathbf{m}}}(t)$, $\text{dp}_{\mathbf{m}} = \text{dp}(L_{\mathbf{m}})$ and $\leq_{\mathbf{m}} = \leq_{L_{\mathbf{m}}}$.

(2) For $L \subseteq L_{\mathbf{m}}$:

(a) let $\mathbf{n} = \mathbf{m} \restriction L$ mean $\mathbf{n} \in \mathbf{M}$, $L_{\mathbf{n}} = L$, $\leq_{\mathbf{n}} = \leq_{\mathbf{m}} \restriction L_{\mathbf{n}}$, $E'_{\mathbf{n}} = E'_{\mathbf{m}} \restriction L$, $u_{\mathbf{n},t} = u_{\mathbf{m},t} \cap L$ and $\mathscr{P}_{\mathbf{n},t} = \mathscr{P}_{\mathbf{m},t} \cap [L]^{\leq\lambda}$ for $t \in L$ and $M_{\mathbf{n}} = M_{\mathbf{m}} \cap L$, $M^{\text{lean}}_{\mathbf{n}} = M^{\text{lean}}_{\mathbf{m}} \cap L$, $M^{\text{fat}}_{\mathbf{n}} = M^{\text{fat}}_{\mathbf{m}} \cap L$,

(b) let $\text{dp}_{\mathbf{m}}(L) = \text{dp}(L_{\mathbf{m}} \restriction L)$ and we may write $\text{dp}(L)$ when $\mathbf{m}$ is clear from the context.

(3) For $t \in L^+_{\mathbf{m}}$, let $\mathbf{m}_{<t} = \mathbf{m}(< t) = \mathbf{m} \restriction L_{<t}$ where $L_{<t} = L_{\mathbf{m}(<t)} = L_{\mathbf{m},<t} = \{s : s <_{\mathbf{m}} t\}$ so $u_{\mathbf{m}(<t),s} = u_{\mathbf{m},s}$ for $s \in L_{<t}$, etc.

(3A) Also $\mathbf{m}_{\leq t} = \mathbf{m}(\leq t) = \mathbf{m} \restriction L_{\leq t}$ where $L_{\leq t} = L_{\mathbf{m}(\leq t)} = L_{<t} \cup \{t\}$; let $L_{<\infty} = L$, $L_{\leq\infty} = L^+$, etc.

(4) $\mathbf{M}_{<\mu}$ is the class of $\mathbf{m} \in \mathbf{M}$ such that $L_{\mathbf{m}}$ has cardinality $< \mu$. Similarly $\mathbf{M}_{\leq\mu}, \mathbf{M}_{=\mu}, \mathbf{M}_{>\mu}, \mathbf{M}_{\geq\mu}$; let $\mathbf{M}_{\mu} = \mathbf{M}_{=\mu}$.

(5) For $\mathbf{m}, \mathbf{n} \in \mathbf{M}$ let $\mathbf{m} \approx \mathbf{n}$, and we may say $\mathbf{m}, \mathbf{n}$ are equivalent meaning that $L_{\mathbf{m}} = L_{\mathbf{n}}$ (as partial orders) and $t \in L_{\mathbf{n}} \Rightarrow u_{\mathbf{m},t} = u_{\mathbf{n},t} \wedge \mathscr{P}_{\mathbf{m},t} = \mathscr{P}_{\mathbf{n},t}$; note that there are no demands on M and E'.

(6) We say f is an isomorphism from $\mathbf{m}_1 \in \mathbf{M}$ onto $\mathbf{m}_2 \in \mathbf{M}$ when:

(a) f is an isomorphism from the partial order $L_{\mathbf{m}_1}$ onto the partial order $L_{\mathbf{m}_2}$,

(b) for $s, t \in L_{\mathbf{m}_1}$ we have $s \in u_{\mathbf{m}_1,t} \Leftrightarrow f(s) \in u_{\mathbf{m}_2,f(t)}$ and $\mathscr{P}_{\mathbf{m}_2,f(t)} = \{\{f(s) : s \in u\} : u \in \mathscr{P}_{\mathbf{m}_1,t}\}$,

(c) for $s, t \in L_{\mathbf{m}_1}$ we have $s E'_{\mathbf{m}_1,} t \Leftrightarrow f(s) E'_{\mathbf{m}_2} f(t)$,

(d) $M_{\mathbf{m}_2} = \{f(s) : s \in M_{\mathbf{m}_1}\}$ and similarly for $M^{\text{lean}}_{m_l}, M^{\text{fat}}_{m_l}$.

(7) We define weak isomorphisms as in part (6) omitting clauses (c),(d).

(8) We say that $\mathbf{m}$ is ordinary when the set of elements of $L_{\mathbf{m}}$ is an ordinal $\alpha_{\mathbf{m}} = \alpha(\mathbf{m})$ satisfying $\beta <_{L_{\mathbf{m}}} \gamma \Rightarrow \beta < \gamma$.

(9) For a forcing notion $\mathbb{P}$ we say that $q \in \mathbb{P}$ is essentially above $p \in \mathbb{P}$ (inside $\mathbb{P}$) when $q \Vdash p \in \underset{\sim}{\mathbf{G}}$.

(10) We say $\mathbf{m} \in \mathbf{M}_{\text{bd}}$ or $\mathbf{m}$ is bounded, when: if $s \in L \setminus M$ then for some $t \in M$ we have $s/E' \subseteq L_{\leq t}$; we could have asked[3] there is $X \in [M]^{<\lambda_0}$ such that $s/E' \subseteq \bigcup_{t \in X} L_{\leq t}$.

(11) We say $\mathbf{m} \in \mathbf{M}_{\text{wbd}}$ or $\mathbf{m}$ is *weakly bounded* when $L_{\mathbf{m}} = \bigcup\{L_{\mathbf{m}(\leq t)} : t \in M_{\mathbf{m}}\}$.

Discussion 1.8 Concerning the aim of the choice to use u_t (and $\mathscr{P}_t$) in 1.5, note the following.

(1) By the partial order we already can get partial memory, so why not simply use only $u_t := \{s : s <_L t\}$? After all, the index set is only partially ordered, not necessarily linearly, so these sets can be independent of each other. The reason is that a partial order is transitive, so this simple definition would imply $s \in u_t \Rightarrow u_s \subseteq u_t$ which means (by definition) the memory is transitive, but we do not want that to hold in general, (this is central in [9]). Here $\bar{u}$ is not necessarily transitive, that is, $s \in u_t \not\Rightarrow u_s \subseteq u_t$. By a partial order we cannot get it.

[3] In the main case $M_{\mathbf{m}}$ is λ^+-directed, so this does not make a difference. Also no real case when we restrict ourselves to bounded $\mathbf{m}$'s.

(2) In [6, 11] we use $\mathscr{P}_t$'s which are ideals, but here not necessarily: this helps, but has a price; we are relying on "$\mathbb{Q}_{\bar\theta}$ is close to being λ-centered", i.e. any subset of $\{p \in \mathbb{Q}_{\bar\theta} : \mathrm{tr}(p) = \eta\}$ of cardinality $< \theta_{\ell g(\eta)}$ has a lub in this forcing. But for the fat context we get more than $(< \lambda)$-complete ideal.
(3) What is the point of "$\mathbf{m}$ being neat"? It tells us that in that case it is easy to be an automorphism of $\mathbf{m}$, see 1.16(2), we may forget to say we use it.

Explanation 1.9 *For* $\mathbf{m} \in \mathbf{M}$*:*

(a) We shall use $L_{\mathbf{m}}$ *as the index set for the iteration; always a well founded partial order.*
(b) $M_{\mathbf{m}}$ *is the part of the index set we are really interested in, it may be* $(\kappa, <)$ *as in* [12].
(c) The other part in the interesting case is "generic enough $\mathbf{m}$*", more accurately existentially closed enough so that the iteration restricted to* M *will be "stabilized" under further extensions. That is, for every* $\mathbf{m} \in \mathbf{M}$ *we define an iteration resulting in the forcing* $\mathbb{P}_{\mathbf{m}}$*, adding a generic* $\underset{\sim}{\eta}_s$ *for* $s \in L_{\mathbf{m}}$*, we are interested in the extension* $\mathbf{V}[\langle \underset{\sim}{\eta}_s : s \in M_{\mathbf{m}}\rangle]$*, it is the generic extension for the forcing we call* $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$*. But, in general, even if* $\mathbf{n} \in \mathbf{M}$ *extends* $\mathbf{m}$ *(see Definition 1.19 below of* $\leq_{\mathbf{M}}$*) maybe* $\mathbb{P}_{\mathbf{n}}[M_{\mathbf{m}}] \neq \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$*. Our aim is to define* $\mathbf{M}, \leq_{\mathbf{M}}$ *so that for a dense set of* $\mathbf{m}$*'s this holds; (done in the crucial claim 1.32). So our aim is having* $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$*, hence the* $s \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ *serves as scaffolding, (but see 2.17).*
Existentially closed structures are used in model theory, but this approach gives non-well founded structures, which is "bad" for us. So an essential point here is to prove (under suitable definitions) that "generic, existentially closed enough $\mathbf{m}$*" is well defined in spite of* $L_{\mathbf{m}}$ *being required to be well founded.*
(d) of course, the aim of $\mathbf{m} \in \mathbf{M}$ *is to be used to define the forcing, as in 1.10 below.*

Definition 1.10 (1) In the fat context, for $\mathbf{m} \in \mathbf{M}$ let $L = L_{\mathbf{m}}$ and we define the iteration $\mathbf{q}_{\mathbf{m}}$ to consist of:

(a) a forcing notion $\mathbb{P}_t = \mathbb{P}_{\mathbf{m},t}$ for $t \in L^+$; we let $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_\infty$,
(b) $\underset{\sim}{\mathbb{Q}}_t$ a $\mathbb{P}_t$-name of a sub-forcing of $\mathbb{Q}_{\bar\theta}$ in the universe $\mathbf{V}^{\mathbb{P}_t}$, even $\mathbb{Q}_t \leq_{\mathrm{ic}} \mathbb{Q}_{\bar\theta}$ (i.e. $\mathbb{Q}_t \subseteq \mathbb{Q}_{\bar\theta}$ as quasi orders and incompatibility and compatibility are preserved[4]),
(c) $p \in \mathbb{P}_t$ <u>iff</u>:

  (α) p is a function,
  (β) $\mathrm{dom}(p) \subseteq L_{<t}$ has cardinality $< \lambda$,
  (γ) if $s \in \mathrm{dom}(p)$ then $p(s)$ consists of $\mathrm{tr}(p(s)) \in \prod_{\varepsilon<\zeta(s)} \theta_\varepsilon$ for some $\zeta_s = \zeta(s) < \lambda$ and $\xi = \xi_{p(s)} = \xi(p(s)) \leq \lambda$ and $\mathbf{B}_{p(s)}$ and $\bar r = \bar r_{p(s)} = \langle r(\zeta) : \zeta < \xi_{p(s)}\rangle = \langle r_{p(s)}(\zeta) : \zeta < \xi_{p(s)}\rangle \in {}^{\xi}(u_s)$ lists the coordinates used in computing $p(s)$ and are such that:
    •$_1$ $\mathbf{B}_{p(s)}$ is a λ-Borel function[5], $\mathbf{B} = \mathbf{B}_{p(s)} : {}^{\xi}(\prod_{\varepsilon<\lambda} \theta_\varepsilon) \to \prod_{\varepsilon<\lambda} \theta_\varepsilon$ moreover into $(\prod_{\varepsilon<\lambda} \theta_\varepsilon)^{[\mathrm{tr}(p(s))]}$; and considering $(d)(\alpha)$ below less pedantically $p(s) = (\mathrm{tr}(p(s)), \underset{\sim}{f}_{p(s)})$, where
    $\underset{\sim}{f}_{p(s)} = \tilde{\mathbf{B}}_{p(s)}(\ldots, \underset{\sim}{\eta}_{r_{p(s)}(\zeta)}, \ldots)_{\zeta<\xi_{p(s)}}$ which means: absolutely, i.e. in every forcing extension $\mathbf{V}^{\mathbb{Q}}$ of $\mathbf{V}$ where $\mathbb{Q}$ is a $(< \lambda)$-strategically complete and is λ^+-c.c. forcing notion, still $\mathbf{B}_{p(s)}$ is such a (λ-Borel) function; we may write $\xi_{p,s}$ instead of $\xi_{p(s)}$, etc.,

[4] But maximal anti-chains - not necessarily. Recall that $\mathbb{Q}_{\bar\theta}$ is from 0.2, 0.3. What is $\underset{\sim}{\mathbb{Q}}_t$? It is implicitly defined in clause (c) and explicitly in 1.18).

[5] That is, a definition of one.

Sh:1126

(d) (α) $\underset{\sim}{\eta}_s$ is the $\mathbb{P}_t$-name, when $t \in L^+_{\mathbf{m}}, s \in L_{<t}$ defined by $\cup\{\mathrm{tr}(p(s)) : p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_t}\}$,

(β) For $p \in \mathbb{P}_t$ and $s \in \mathrm{dom}(p)$ we interpret $p(s)$ as a $\mathbb{P}_s$-name $(\mathrm{tr}(p(s)), \mathbf{B}_{p,s}(\ldots, \underset{\sim}{\eta}_{r_{p,s}(\zeta)}, \ldots)_{\zeta<\xi_{p,s}})$.

(e) $\mathbb{P}_t \models$ "$p \leq q$" iff:

(α) $p, q \in \mathbb{P}_t$,
(β) $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$,
(γ) if $t \in \mathrm{dom}(p)$ then $(q \restriction L_{<t}) \Vdash_{\mathbb{P}_{\mathbf{m}(<t)}}$ "$p(t) \leq_{\underset{\sim}{\mathbb{Q}}_{\bar{\theta}}} q(t)$".

(2) In the general context we replace clause (c)(γ) by: (so part (1) is a special case with $\iota_{p(s)} = 1, \bar{r}_{p(s),0} = \bar{r}_{p(s)}$).

(γ) if $s \in \mathrm{dom}(p)$ then $p(s)$ consists of $\mathrm{tr}(p(s)) \in \Pi_{\varepsilon<\zeta(s)}\theta_\varepsilon$ for some $\zeta_s = \zeta(s) < \lambda$ and $\varepsilon = \varepsilon_{p(s)} = \varepsilon(p(s)) \leq \lambda$ and $\mathbf{B}_{p(s)}$ and $\bar{r} = \bar{r}_{p(s)} = \langle r(\zeta) : \zeta < \varepsilon_{p(s)}\rangle = \langle r_{p(s)}(\zeta) : \zeta < \varepsilon_{p(s)} \in {}^{\varepsilon}(u_s)$ lists the coordinates used in computing $p(s)$ and[6] $\langle \mathbf{B}_{p(s),\iota}, \bar{r}_{p(s),\iota} : \iota < \iota(p(s))\rangle$ are such that:

- •1 $\mathbf{B}_{p(s)}$ is a λ-Borel function[7], $\mathbf{B} = \mathbf{B}_{p(s)} : {}^{\xi}(\prod_{\varepsilon<\lambda}\theta_\varepsilon) \to \prod_{\varepsilon<\lambda}\theta_\varepsilon$ moreover into $(\prod_{\varepsilon<\lambda}\theta_\varepsilon)^{[\mathrm{tr}(p(s))]}$; and considering $(d)(\alpha)$ below less pedantically $p(s) = (\mathrm{tr}(p(s)), \underset{\sim}{f}_{p(s)})$, where $\underset{\sim}{f}_{p(s)} = \mathbf{B}_{p(s)}(\ldots, \underset{\sim}{\eta}_{r_{p(s)}(\zeta)}, \ldots)_{\zeta<\xi_{p(s)}}$ which means: absolutely, i.e. in every forcing extension $\mathbf{V}^{\mathbb{Q}}$ of $\mathbf{V}$ where $\mathbb{Q}$ is a $(<\lambda)$-strategically complete and is λ^+-c.c. forcing notion, still $\mathbf{B}_{p(s)}$ is such a (λ-Borel) function; we may write $\xi_{p,s}$ instead of $\xi_{p(s)}$, etc.,
- •2 $\iota_{p(s)} = \iota(p(s)) < \lambda$ moreover[8] $< \theta_{\ell g(\mathrm{tr}(p(s))}$,
- •3 for $\iota < \iota_{p(s)}, \bar{r}_{p(s),\iota} = \bar{r}_{p(s)} \restriction w_{p(s),\iota}$ so $w_{p(s),\iota} = w(p(s),\iota) = \mathrm{dom}(\bar{r}_{p(s),\iota}) \subseteq \xi_{p(s)}$ and $\bar{r}_{p(s),\iota}$ is a subsequence of $\bar{r}_{p(s)}$,
- •4 $\mathbf{B}_{p(s),\iota}$ is a Borel function from ${}^{w(p(s),\iota)}(\prod_{\varepsilon<\lambda}\theta_\varepsilon)$ into $(\prod_{\varepsilon<\lambda}\theta_\varepsilon)^{[\mathrm{tr}(p(s))]}$,
- •5 $\mathbf{B}_{p(s)}(\langle \underset{\sim}{\eta}_{r_{p(s)}(\zeta)} : \zeta < \xi_{p(s)}\rangle) = \sup\{\mathbf{B}_{p(s),\iota}(\langle \underset{\sim}{\eta}_{r_{p(s)}(\zeta)} : \zeta \in w_{p(s),\iota}\rangle) : \iota < \iota(p(s))\}$ and naturally $\underset{\sim}{f}_{p(s)} = \sup\{\underset{\sim}{f}_{p(s),\iota} : \iota \leq \iota(p(s))\}, \underset{\sim}{f}_{p(s),\iota} = \mathbf{B}_{p(s),\iota}(\langle \underset{\sim}{\eta}_\zeta : \zeta \in w_{p(s),\iota}\rangle)$,
- •6 for each $\iota < \iota(p(s))$ for some $u \in \mathscr{P}_{\mathbf{m},s}$ we have $\{r_{p(s)}(\zeta) : \zeta \in w_{p(s),\iota}\} \subseteq u$ so is a subset of u_s,
- •7 (follows) when $\mathbf{m}$ is lean, if $\iota < \iota_{p(s)}$ and $\varepsilon \in w_{p(s),\iota}, r_{p(s)}(\varepsilon) \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ then $\{r_{p(s)}(\zeta) : \zeta \in w_{p(s),\iota}\} \subseteq r_{p(s)}(\varepsilon)/E_{\mathbf{m}}$,

[Why? As Definition 1.5(2) together with •6 implies $\{r_{p(s)}()\zeta : \zeta \in W_{p(s)},\iota\} \subseteq r_{p(s)}(\varepsilon)/E'_{\mathbf{m}}$].

- •8 we let $\mathscr{F}_{p(s)}$ be the set $\{f_{p(s),\iota} : \iota < \iota(p(s))\}$, so we may write $p(s) = (\mathrm{tr}(p(s)), \mathscr{F}_{p(s))})$.

The following matters only for [12].

Claim 1.11 *Assume $\mathbf{m} \in \mathbf{M}$ is, (see 1.7(8)) ordinary*[9]*, that is the set of elements of $L_{\mathbf{m}}$ is an ordinal $\alpha_{\mathbf{m}} = \alpha(\mathbf{m})$ satisfying $\beta <_{L_{\mathbf{m}}} \gamma \Rightarrow \beta < \gamma$.*

There is a unique object $\mathbf{q} = (\bar{u}, \bar{\mathbb{P}}, \bar{\mathbb{Q}}, \bar{\eta})$ such that:

---

[6] What is the point of "$\iota < \iota(p(s))$"? As the support is not just u_s but also $\mathscr{P}_s$ and $\mathscr{P}_s$ is a family of suitable subsets of u_s, $p(s)$ is $(\mathrm{tr}(p(s)), \underset{\sim}{f}_s)$, f_s is a name of a member of $\prod_{\varepsilon<\lambda}\theta_\varepsilon$ such that $\mathrm{tr}(p(s))$ is a (proper) initial segment. But how is f_s computed? As our memory is $\mathscr{P}_s \subseteq \mathscr{P}(u_s)$ and not just u_s (or even a $(<\lambda)$-complete ideal) $\underset{\sim}{f}_s$ is composed of $\iota_{p(s)}$ names each coming from $\langle \underset{\sim}{\eta}_t : t \in u_\iota\rangle, u_\iota \in \mathscr{P}_s$ for $\iota < \iota(p(s)$.

[7] That is, a definition of one.

[8] This and the rest of $(c)(\gamma)$ are used in the proof of 3.18. The aim is that defining $\mathbf{B}_{p(s)}$ from $\langle \mathbf{B}_{p(s),\iota} : \iota < \iota(p(s))\rangle$, the sup will not give in ε the value θ_ε.

[9] As $L_{\mathbf{m}}$ is well founded, this is not a real restriction.

(a) $\bar{u} = \bar{u}_{\mathbf{m}}$ *so* $\alpha_{\mathbf{m}} = \lg(\bar{u})$,
(b) $\langle \mathbb{P}^{\mathbf{q}}_{0,\alpha}, \mathbb{Q}^{\mathbf{q}}_{0,\beta} : \alpha \leq \alpha_{\mathbf{m}}, \beta < \alpha_{\mathbf{m}} \rangle$ *, the* $(< \lambda)$*-support iteration such that:* $\underset{\sim}{\mathbb{Q}}_\alpha$ *is essentially the forcing notion form from 1.10,*
(c) $\mathbf{q}$ *is as in [12, 1.8=Lz32, page 32].*

Proof 1.11 Follows from 1.18 below. □

Definition 1.12 (1) For $p \in \mathbb{P}_{\mathbf{m}}$ let,

(a) fsupp(p), the full support of p be $\cup\{\{r_{p(s)}(\zeta) : \zeta < \xi_{p,s}\} \cup \{\{s\} : s \in \text{dom}(p)\}$
(b) wsupp(p), the wide support of p be the set of $s \in L_{\mathbf{m}}$ such that for some t at least one of the following hold:
  $\bullet_1$ $s = t \in \text{fsupp}(p)$,
  $\bullet_2$ $t \in \text{fsupp}(p) \setminus M, s \in t/E'_{\mathbf{m}}$.

(2) For $\mathbf{m} \in \mathbf{M}$ let $\mathbb{P}^{\mathbf{m}}_t = \mathbb{P}_{\mathbf{m},t}$, etc., in Definition 1.10.
(3) For $L \subseteq L_{\mathbf{m}}$ let $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{m}} \restriction \{p \in \mathbb{P}_{\mathbf{m}} : \text{fsupp}(p) \subseteq L\}$, that is:

- $p \in \mathbb{P}_{\mathbf{m}}(L)$ iff $p \in \mathbb{P}_{\mathbf{m}}$ and fsupp$(p) \subseteq L$,
- $p \leq_{\mathbb{P}_{\mathbf{m}}(L)} q$ iff $p \in \mathbb{P}_{\mathbf{m}}(L) \wedge q \in \mathbb{P}_{\mathbf{m}}(L) \wedge p \leq_{\mathbb{P}_{\mathbf{m}}} q$,

(4) For $\mathbf{m} \in \mathbf{M}$ and $t \in L_{\mathbf{m}}$ let[10] $\underset{\sim}{\mathbb{Q}}_t = \underset{\sim}{\mathbb{Q}}_{\mathbf{m},t}$ be the $\mathbb{P}_t$-name of $\mathbb{Q}_{\bar{\theta}} \restriction \{(\nu, \underset{\sim}{f}[\mathbf{G}_{\mathbb{P}_{\mathbf{m}(<t)}}]) : (\nu, \underset{\sim}{f})$ as in Definition 1.10(c)(γ) with s there for t here$\}$.

Claim 1.13 *For* $\mathbf{m} \in \mathbf{M}$ *(so* $\mathbb{P}_t = \mathbb{P}_{\mathbf{m},t}$*, etc.):*

(a) *the iteration* $\mathbf{q}_{\mathbf{m}}$ *is well defined, i.e. exists and is unique,*
(b) (α) *if* $t \in L^+_{\mathbf{m}}$ *then* $\mathbb{P}_t$ *is indeed a forcing notion and is equal to* $\mathbb{P}_{\mathbf{m}(<t)}$,
  (β) *the* $\mathbb{P}_t$*-name* $\underset{\sim}{\eta}_s$ *does not depend on t as long as* $s <_{L_{\mathbf{m}}} t \in L^+_{\mathbf{m}}$,
  (γ) $\underset{\sim}{\eta}_t$ *is a* $\mathbb{P}_{\mathbf{m}(\leq t)}$*-name.*
(c) *if* $s <_L t$ *are from* $L^+_{\mathbf{m}}$ *then:*

  (α) $p \in \mathbb{P}_s \Rightarrow p \in \mathbb{P}_t \wedge p \restriction L_{<s} = p$,
  (β) *if* $p, q \in \mathbb{P}_s$ *then* $\mathbb{P}_t \models$ "$p \leq q$" $\Leftrightarrow \mathbb{P}_s \models$ "$p \leq q$",
  (γ) *if* $p \in \mathbb{P}_t$ *then* $p \restriction L_{<s} \in \mathbb{P}_s$ *and* $\mathbb{P}_t \models$ "$(p \restriction L_{\mathbf{m}(<s)}) \leq p$",
  (δ) $\mathbb{P}_t \models$ "$p \leq q$" $\Rightarrow \mathbb{P}_s \models$ "$p \restriction L_{\mathbf{m}(<s)} \leq q \restriction L_{\mathbf{m}(<s)}$",
  (ε) $\mathbb{P}_s \lessdot \mathbb{P}_t$*, moreover*
  (ζ) $p \in \mathbb{P}_t \wedge (p \restriction L_{\mathbf{m}(<s)}) \leq q \in \mathbb{P}_s \Rightarrow q \cup (p \restriction (L_{\mathbf{m}(<t)} \setminus L_{\mathbf{m}(<s)}) \in \mathbb{P}_t$ *is a* $\leq$*-lub of* p, q.

(d) *if L is an initial segment of* $L_{\mathbf{m}}$ *then* $\mathbb{P}_{\mathbf{m} \restriction L} = \mathbb{P}_{\mathbf{m}} \restriction \{p \in \mathbb{P}_{\mathbf{m}} : \text{dom}(p) \subseteq L$*, equivalently* fsupp$(p) \subseteq L\}$*; this holds in particular for* $L_{\mathbf{m}(\leq t)}$ *and for* $L_{\mathbf{m}(<t)}$.
(e) *if* $L_1 \subseteq L_2$ *are initial segments of* $L_{\mathbf{m}}$*, then the parallel of clause (b) holds replacing* $\mathbb{P}_{\mathbf{m},s}, \mathbb{P}_{\mathbf{m},t}$ *by* $\mathbb{P}_{\mathbf{m} \restriction L_1}, \mathbb{P}_{\mathbf{m} \restriction L_2}$*, respectively. Also the parallel of clause (c) holds.*
(f) *if* $p \in \mathbb{P}_{\mathbf{m}}$ *then:*

  (α) dom(p) *has cardinality* $< \lambda$,
  (β) fsupp(p) *has cardinality at most* λ,
  (γ) $\bullet_1$ wsupp(p) *is included in the union of* $\leq \lambda$ *sets of the form* $t/E_{\mathbf{m}}$ *or* $\{t\}$,
    $\bullet_2$ *if* $\mathbf{m}$ *is lean then the union is even of* $< \lambda$ *such sets.*

Proof 1.13 Straightforward. For $t \in L^+_{\mathbf{m}}$, by induction on $\text{dp}_{\mathbf{m}}(t)$, define $\mathbb{P}_t$ and prove the relevant parts of (a),(b),(c),(d),(e). □

[10] Not used, could have used it in 1.18.

Sh:1126

Note the following:

Observation 1.14 *If* $\mathbf{B}$ *is a* λ*-Borel function from* ${}^{\xi}(\Pi\bar{\theta})$ *to* $\mathscr{P}(\lambda)$ *or even* $\mathscr{H}(\lambda^+)$ *where* $\xi \leq \lambda$ *then there is a* λ*-Borel function* $\mathbf{B}'$ *from* ${}^{\xi}(\Pi\bar{\theta})$ *to* $\mathbb{Q}_{\bar{\theta}}$ *(so absolutely*[11] *to* $\mathbb{Q}_{\bar{\theta}}$*) such that for any* $\bar{\eta} \in {}^{\xi}(\Pi\bar{\theta})$ *we have, absolutely:*

- *if* $\mathbf{B}(\bar{\eta}) \in \mathbb{Q}_{\bar{\theta}}$ *then* $\mathbf{B}'(\bar{\eta}) = \mathbf{B}(\bar{\eta})$,
- *if* $\mathbf{B}(\bar{\eta}) \notin \mathbb{Q}_{\bar{\theta}}$ *then* $\mathbf{B}'(\bar{\eta}) = (\emptyset, 0_\lambda)$*, the minimal member of* $\mathbb{Q}_{\bar{\theta}}$.

Proof 1.14 Just define $\mathbf{B}'(\bar{\eta})$ as $\mathbf{B}(\bar{\eta})$ if $\mathbf{B}(\bar{\eta}) \in \mathbb{Q}_{\bar{\theta}}$ and the trivial condition $(\langle\rangle, 0_\lambda)$ otherwise. $\square$

Remark 1.15 (1) A reader may wonder, e.g.:

(∗) if $\langle \mathbf{B}_\alpha : \alpha < \alpha_* \leq \lambda \rangle$ is a sequence of λ-Borel subsets of $\Pi_{\varepsilon<\lambda}\theta_\varepsilon$ which form a partition (in $\mathbf{V}$), does they from a partition also in $\mathbf{V}^{\mathbb{P}}$.

In our case as $\mathbb{P}$ is λ-strategically complete (see 1.16(3A)) the answer is obviously yes.

(2) Note that in (∗) we cannot weaken the assumption too much because "if $\mathbb{P}$ add a new subset to $\theta < \lambda$ this certainly faill". Even $(< \lambda)$-strategically complete is not enough. Why? assume λ is a Mahlo cardinal $S \subseteq \{\theta < \lambda : \theta \text{ inaccessible}\}$ is stationary, such that (for transparency) $\diamond_S$ holds. We can find $\mathscr{T}$ such that:

⊞ (a) $\mathscr{T}$ a subtree of $({}^{\lambda>}2, \triangleleft)$,
(b) $\mathscr{T}$ with no $\triangleleft$-maximal nodes,
(c) if $\delta \in \lambda \setminus S$ a limit ordinal, $\eta \in {}^{\delta}2$ and $\alpha < \delta \Rightarrow \eta{\restriction}\alpha \in \mathscr{T}$, then $\eta \in \mathscr{T}$,
(d) $\mathscr{T}$ has no λ-branch.

Let $\mathbf{B}_0 = \{\eta \in {}^{\lambda}2 : \bigwedge_{\alpha<\lambda} \eta{\restriction}\alpha \in \mathscr{T}\}$ and $\mathbf{B}_1 = {}^{\lambda}2$.
In $\mathbf{V}$ those two λ-Borel sets form a partition: the first is empty and the second is all. The forcing notion $\mathscr{T}$ add a λ-branch to $\mathscr{T}$, hence $(\mathbf{B}_0, \mathbf{B}_1)$ are no longer disjoint so fail to form a partition of ${}^{\lambda}2$. Lastly, for $\alpha < \lambda$ the forcing notion $\mathscr{T}$ is α-strategically complete (just COM choose $p_0 \in \mathscr{T}$ of length $> \alpha$).

(3) Alternatively, if it suffice to us to have "for $\alpha < \lambda$, COM do not lose in the game of length α" let λ be inaccessible and S as above or just such that $\lambda \setminus S$ is fat i.e. for every club E of λ and $\alpha < \lambda$ there is an increasing continuous $h : \alpha \to E$ such that $S \cap \text{rang}(h) = \emptyset$. Let $\mathbb{Q} = \{\eta : \eta \in {}^{\lambda>}\lambda$ be increasing continuous with range disjoint to S and $\sup(\text{rang}(\eta_i))$ is not in $S\}$. Let the sequence $\langle \eta_i : i < \lambda \rangle$ of pairwise $\trianglelefteq$-incomparable be such that $\lg(\eta_i) \in S$ and $(\forall \alpha < \lg(\eta_i))[\eta_i{\restriction}\alpha \in \mathbb{Q}]$ and it is dense in $\mathbb{Q}$. For $i < \lambda$, let $\mathbf{B}_{1+i}$ be $\{\nu \in {}^{\lambda}\lambda : \eta_i \triangleleft \nu\}$, so closed and $\mathbf{B}_0 = \{\nu \in {}^{\lambda}\lambda : \nu \text{ is not increasing continuous}\}$, now $\langle \mathbf{B}_i : i < \lambda \rangle$ is as required.

(4) Another avenue is to assume $\aleph_0 < \theta = \text{cf}(\theta) < \lambda$, $S_0 \subseteq \{\delta < \lambda : \text{cf}(\lambda) < \theta\}$, $S \subseteq \{\delta < \lambda : \text{cf}(\delta) = \theta$ and $S_0 \cap \delta$ is a stationary subset of $\delta\}$. Now let $\mathbb{Q} = \{\eta : \eta \in {}^{\lambda>}2$ and for no $\delta \leq \lg(\eta)$ we have $\delta \in S$ and for some club E of δ do we have $\alpha \in E \cap S_0 \Rightarrow \eta(\alpha) = 1\}$. Continue as in 1.15(3).

(5) Note that if in 0.6(1) we let INC to choose first, then 1.15(a) does not work whereas in 1.15(2), (3) this does no matters.

(6) Anyhow in 1.14 this is not necessary; it is enough that being a member of $\mathbb{Q}_{\bar{\theta}}$ is a λ-Borel set.

[11] That is, for every forcing notion $\mathbb{P}$ which is λ-strategically complete, this property continue to hold in $\mathbf{V}^{\mathbb{P}}$; here the property is that the range is as indicated; parallely below. We could demand just preserving the regularity of λ and the θ_ε-s,

Claim 1.16 *Let* $\mathbf{m} \in \mathbf{M}$.

(1) *If* $L^+_{\mathbf{m}} \models$ "$s < t$" *then:*

(α) $\Vdash_{\mathbb{P}_{\mathbf{m},t}}$ "$\underset{\sim}{\eta}_s \in \prod\limits_{\varepsilon<\lambda} \theta_\varepsilon$",

(β) *if* $\mathbf{G} \subseteq \mathbb{P}_t$ *is generic over* $\mathbf{V}$, $\eta_r = \underset{\sim}{\eta}_r[\mathbf{G}]$ *for* $r \in L_{\mathbf{m},<t}$, $u \in \mathscr{P}_{\mathbf{m},t}$ *and* $\nu \in \Pi\bar{\theta}$ *is from* $\mathbf{V}[\langle \eta_r : r \in u\rangle] \subseteq \mathbf{V}[\mathbf{G}]$, *then* $\nu <_{J^{\mathrm{bd}}_\lambda} \eta_s$.

(2) $\mathbb{P}_{\mathbf{m}}$ *satisfies the* λ^+*-c.c., and even the* λ^+*-Knaster (and more).*

(3) $\mathbb{P}_{\mathbf{m}}$ *is* $(<\lambda)$*-strategically complete (even* λ*-strategically complete but not used*[12]*).*

(3A) *If* $\bar{p} = \langle p_i : i < \delta\rangle$ *is* $\leq_{\mathbb{P}_{\mathbf{m}}}$*-increasing,* $\delta < \lambda$ *and* $i < j < \delta \wedge t \in \mathrm{dom}(p_i) \Rightarrow \mathrm{tr}(p_i(t)) \lhd \mathrm{tr}(p_j(t))$ *then*[13] $\bar{p}$ *has a* $\leq_{\mathbb{P}_{\mathbf{m}}}$*-upper bound* p*. Moreover,* $\mathrm{dom}(p) = \cup\{\mathrm{dom}(p_i) : i < \delta\}$ *and* $s \in \mathrm{dom}(p_i) \Rightarrow \mathrm{tr}(p(s)) = \cup\{\mathrm{tr}(p_j(s)) : j \in [i,\delta)\}$*; in fact also* $\mathrm{fsupp}(p) = \cup\{\mathrm{fsupp}(p_i) : i < \delta\}$ *and* p *is a lub of* $\bar{p}$*. Also, we can weaken the demand above to* $i < \delta \wedge s \in \mathrm{dom}(p_i) \Rightarrow \delta < \theta_{\varepsilon(s)}$ *where we let* $\varepsilon(s) = \sup\{\ell g(\mathrm{tr}(p_j(s))) : j \in [i,\delta)\}$.

(3B) *If* $\zeta < \lambda$ *and* $L^+_{\mathbf{m}} \models$ "$s < t$", *then the following is a dense open subset of* $\mathbb{P}_t$*:* $\mathscr{I}_{s,t,\zeta} = \{p \in \mathbb{P}_t : s \in \mathrm{dom}(p)$ *and* $\mathrm{tr}(p(s))$ *has length* $\geq \zeta\}$.

(3C) *If* $p \in \mathbb{P}_{\mathbf{m}}$ *and* $\zeta < \lambda$ *then for some* $q \in \mathbb{P}_{\mathbf{m}}$ *we have* $p \leq q$ *and* $t \in \mathrm{dom}(p) \Rightarrow \mathrm{tr}(p(t)) \lhd \mathrm{tr}(q(t))$ *and* $t \in \mathrm{dom}(q) \Rightarrow \ell g(\mathrm{tr}(q(t))) > \zeta$.

(4) *If* $\underset{\sim}{x}$ *is a* $\mathbb{P}_{\mathbf{m}}$*-name of a member of* $\mathscr{H}(\lambda^+)$*, e.g. of* $\mathbb{Q}_{\bar{\theta}}$ *(in* $\mathbf{V}[\mathbb{P}_{\mathbf{m}}]$*) then for some* $\xi \leq \lambda$ *and* λ*-Borel function* $\mathbf{B} : {}^{\xi}(\Pi\bar{\theta}) \to \mathscr{H}(\lambda^+)$ *and a sequence* $\langle r_\zeta : \zeta < \xi\rangle$ *of members of* $L_{\mathbf{m}}$ *we have* $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$\underset{\sim}{x} = \mathbf{B}(\ldots, \underset{\sim}{\eta}_{r_\zeta}, \ldots)_{\zeta<\xi}$".

(4A) *If* $t \in L^+_{\mathbf{m}}$ *and* $u \subseteq L_{\mathbf{m}(<t)}$ *and* $\Vdash_{\mathbb{P}_t}$ "$\underset{\sim}{y}$ *is a member of* $\mathbb{Q}_{\bar{\theta}}$ *from* $\mathbf{V}[\langle \underset{\sim}{\eta}_s : s \in u\rangle]$", *then for some* $\xi \leq \lambda$ *and* λ*-Borel functions as in 1.10(c)* (γ), $\mathbf{B}_i : {}^{\xi}(\Pi\bar{\theta}) \to \mathbb{Q}_{\bar{\theta}}$ *for* $i < \xi$ *and sequence* $\langle r_\zeta : \zeta < \xi\rangle$ *of members of* u *we have* $\Vdash_{\mathbb{P}_t}$ *"for some* $i < \xi$ *we have* $\underset{\sim}{y} = \mathbf{B}_i(\ldots, \underset{\sim}{\eta}_{r_\zeta}, \ldots)_{\zeta<\xi}$".

(5) *If* $\mathbf{m}, \mathbf{n}$ *are equivalent then* $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_{\mathbf{n}}$ *and* $\mathbb{P}_{\mathbf{m},t} = \mathbb{P}_{\mathbf{n},t}$ *for* $t \in L^+_{\mathbf{m}} = L^+_{\mathbf{n}}$.

(6) *Assume that* $p, q \in \mathbb{P}_{\mathbf{m}}$ *are incompatible then there are* q_1 *and* s *such that:*

(a) $q_1 \in \mathbb{P}_{\mathbf{m},s}$,

(b) $s \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$,

(c) $(q \restriction L_{\mathbf{m},<s}) \leq_{\mathbb{P}_{\mathbf{m}}} q_1$,

(d) $(p \restriction L_{\mathbf{m},<s}) \leq_{\mathbb{P}_{\mathbf{m}}} q_1$,

(e) $q_1 \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p(s)$ *and* $q(s)$ *are incompatible in* $\mathbb{Q}_{\bar{\theta}}$ *which means* $\mathrm{tr}(p(s)) \perp \mathrm{tr}(q(s))$*, i.e. they are* $\trianglelefteq$*-incomparable or* $(\alpha) + (\beta) + (\gamma)$ *where:*

(α) $\ell g(\mathrm{tr}(q(s))) \neq \ell g(\mathrm{tr}(p(s)))$,

(β) *if* $\ell g(\mathrm{tr}(q(s))) < \ell g(\mathrm{tr}(p(s)))$ *then for some ordinal* ε, $\ell g(\mathrm{tr}(q(s))) \leq \varepsilon < \ell g(\mathrm{tr}(p(s)))$ *and* $q_1 \restriction L_{\mathbf{m}(<s)} \Vdash_{\mathbb{P}_{\mathbf{m}(<s)}}$ $\mathrm{tr}(p(s))(\varepsilon) < \underset{\sim}{f}_{q(s)}(\varepsilon)$",

(γ) *if* $\ell g(\mathrm{tr}(q(s))) > \ell g(\mathrm{tr}(p(s)))$ *then for some ordinal* ε, $\ell g(\mathrm{tr}(q(s))) > \varepsilon \geq \ell g(\mathrm{tr}(p(s)))$ *and* $q_1 \restriction L_{\mathbf{m}(<s)} \Vdash_{\mathbb{P}_{\mathbf{m}(<s)}}$ "$\mathrm{tr}(q(s))(\varepsilon) < \underset{\sim}{f}_{p(s)}(\varepsilon)$".

7) $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$\mathbf{V}[\langle \eta_s : s \in L_{\mathbf{m}}\rangle] = \mathbf{V}[\underset{\sim}{\mathbf{G}}]$".

8) *For* $t \in L^+_{\mathbf{m}}$ *the sequence* $\langle \underset{\sim}{\eta}_s : s \in L_{\mathbf{m},<t}\rangle$ *is generic for* $\mathbb{P}_{\mathbf{m},t}$*; that is:*

($*$) *if* $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{m},t}$ *is generic over* $\mathbf{V}$ *and* $\eta_s = \underset{\sim}{\eta}_s[\mathbf{G}]$ *for* $s \in L_{\mathbf{m},<t}$ *then* $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\langle \eta_s : s \in L_{\mathbf{m},<t}\rangle]$.

---

[12] Recall that being λ-strategically complete means that a play of the game lasts λ moves, and the COM player to win needs to have a legal choice in each move. So COM needs just to have a common upper bound to suitable increasing sequences of length $< \lambda$.

[13] But $\mathrm{tr}(p_i(t)) \trianglelefteq \mathrm{tr}(p_j(t))$ does not suffice, but if e.g. $\mathrm{cf}(\delta) < \theta_0$ it suffice.

9) *For* $\mathbf{m} \in \mathbf{M}$, π *is an automorphism of* $\mathbf{m}$ *when:*

(a) π *is a permutation of* $L_\mathbf{m}$,
(b) $\pi \restriction M_\mathbf{m}$ *is the identity,*
(c) *if for every* $s \in L_\mathbf{m} \backslash M_\mathbf{m}$, *for some* $t \in L_\mathbf{m} \backslash M_\mathbf{m}$ *we have* $\pi \restriction (s/E_\mathbf{m})$ *is an isomorphism from* $\mathbf{m} \restriction (s/E_\mathbf{m})$ *onto* $\mathbf{m} \restriction (t/E_\mathbf{m})$.

10) *In part (8), moreover, in* $\mathbf{V}[\mathbf{G}]$, *if* $\bar{\eta}' = \langle \eta'_s : s \in L_{\mathbf{m},t} \rangle$ *and* $\eta'_s \in \Pi_{\varepsilon<\lambda}\theta_\varepsilon$ *and the set* $\{(s, \varepsilon) : s \in L_{\mathbf{m},<t}, \varepsilon < \lambda$ *and* $\eta'_s(\varepsilon) \neq \eta_s(\varepsilon)\}$ *has cardinality* $< \lambda$ *then also* $\bar{\eta}'$ *is generic (for* $\mathbb{P}(L_{\mathbf{m},<t})$*) and* $\mathbf{V}[\bar{\eta}'] = \mathbf{V}[\mathbf{G}]$.

Remark 1.17 What is the use of e.g. (6), (6A)? See 2.12(A)(b) and 1.18.

Proof 1.16 We prove all parts simultaneously by induction on $\mathrm{dp}_\mathbf{m}$.

(1) For clause (α) for each $\mathbf{m}$, using the induction hypothesis and 1.13(e), the problem is only when $\mathrm{dp}_\mathbf{m}(t) = \mathrm{dp}_\mathbf{m} - 1$ and use part (5A) proved below (and 1.13(c)(ζ)). For clause (β) use also part (6A) for $\mathbb{P}_{\mathbf{m}(<t)}$ proved below in 1.13(c)(ζ). In both cases the proof of the parts quoted does not rely on part (1), (but may depend on the induction hypothesis).

(2) Recall that λ is strongly inaccessible. If $p_\varepsilon \in \mathbb{P}_\mathbf{m}$ for $\varepsilon < \lambda^+$ then we can find by the Δ-system lemma a set u and unbounded $S \subseteq \lambda^+$ such that $\varepsilon \neq \zeta \in S \Rightarrow \mathrm{dom}(p_\varepsilon) \cap \mathrm{dom}(p_\zeta) = u$ and $\langle \mathrm{tr}(p_\varepsilon(\beta)) : \beta \in u \rangle$ is the same for all $\varepsilon \in S$. Now p_ε, p_ζ has a common upper bound for every $\varepsilon, \zeta \in u$, i.e. we define r by:

- $\mathrm{dom}(r) = \mathrm{dom}(p_\varepsilon) \cup \mathrm{dom}(p_\zeta)$,
- $r(s) = p_\varepsilon(s)$ is $s \in \mathrm{dom}(p_\varepsilon) \backslash \mathrm{dom}(p_\zeta)$,
- $r(s) = p_\zeta(s)$ if $s \in \mathrm{dom}(p_\zeta) \backslash \mathrm{dom}(p_\varepsilon)$,
- if $s \in \mathrm{dom}(p_\varepsilon) \cap \mathrm{dom}(p_\zeta)$ then $r(s) = (\mathrm{tr}(p_\varepsilon(s)), \max\{\underset{\sim}{f}_{p_\varepsilon(s)}, \underset{\sim}{f}_{p_\zeta(s)}\})$.

(3) By (4), the second sentence + (4A) below which use only the induction hypothesis.

3A) We define p by:

- $\mathrm{dom}(p) = \cup\{\mathrm{dom}(p_i) : i < \delta\}$
- $\mathrm{tr}(p(s)) = \cup\{\mathrm{tr}(p_i(s)) : i < \delta$ satisfies $s \in \mathrm{dom}(p_i)\}$
- $\underset{\sim}{f}_{p(s)} = \sup\{\underset{\sim}{f}_{p_i(s)} : i < \delta$ satisfies $s \in \mathrm{dom}(p_i)\}$.

Note that here having to really start with $\langle \underset{\sim}{f}_{p_i(s),\iota} : \iota < \iota(p_i(s)) \rangle$ and get $\langle \underset{\sim}{f}_{p(s),\iota} : \iota < \iota(p(s)) \rangle$, see 1.10(c)($\gamma$) causes no problem, similarly in the proof of part (2) - just take the union.

3B) Obvious by the definition of $\mathbb{P}_\mathbf{m}$ and 1.13(c), recalling that $\mathbb{P}_{\mathbf{m}(<s)}$ is $(< \lambda)$-strategically complete, that is part (4) and (5B).

3C) The proof is by induction on $\mathrm{dp}_\mathbf{m}$ and is splitted in cases:

<u>Case 1</u>: $\mathrm{dp}_\mathbf{m}$ is zero:

So $L_\mathbf{m}$ is empty.

<u>Case 2</u>: $\mathrm{dp}_\mathbf{m} = \alpha + 1$:

Hence $L_2 = \{s \in L : \mathrm{dp}_\mathbf{m}(s) = \alpha\}$ is non-empty and letting $L_1 = L_\mathbf{m} \backslash L_2$; clearly $s \in L_1 \Rightarrow \mathrm{dp}_\mathbf{m}(s) < \alpha$, so $\mathrm{dp}_{\mathbf{m}\restriction L_1} \leq \alpha$. Let $\zeta_* = \sup(\{\ell g(\mathrm{tr}(p(s)) + 1 : s \in \mathrm{dom}(p)\} \cup \{\zeta + 1\})$. Hence applying parts (3) and (5B) to $\mathbf{m} \restriction L_1$, i.e. the induction hypothesis we can find q_1 such that $\mathbb{P}_{\mathbf{m}\restriction L_1} \models$ "$p \restriction L_1 \leq q_1$" and $[s \in \mathrm{dom}(q_1) \Rightarrow \ell g(\mathrm{tr}(q_1(s)) > \zeta_*]$ and q_1 forces a value to $\underset{\sim}{f}_{p(s)} \restriction \zeta_*$, call it ρ_s for $s \in \mathrm{dom}(p) \cap L_2$.

Define $q \in \mathbb{P}_\mathbf{m}$ by $\mathrm{dom}(q) = \mathrm{dom}(q_1) \cup (L_2 \cap \mathrm{dom}(p))$, $q \restriction L_1 = q_1$ and if $s \in L_2 \cap \mathrm{dom}(p)$ then $q(s) = (\rho_s, \rho_s\hat{\ }(\underset{\sim}{f}_{p(s)} \restriction [\zeta_*, \lambda))$, fully $\iota(q(s)) = \iota(p(s))$, $\bar{s}_{q(s),\iota} = \bar{s}_{p(s),\iota}$ and $\mathbf{B}_{q(s),\iota}$ is like $\mathbf{B}_{p(s),\iota}$ only restricting the range to $(\Pi_{\varepsilon<\lambda}\theta_\varepsilon)^{\mathrm{tr(q(s))}}$

Easily q is as required.

Case 3: $\delta = \mathrm{dp}_{\mathbf{m}}$ is a limit ordinal of cofinality $\geq \lambda$:

So $\alpha = \sup\{\mathrm{dp}_{\mathbf{m}}(s)+1 : s \in \mathrm{dom}(p)\}$ is an ordinal $< \delta$ and let $L = \{s \in L_{\mathbf{m}} : \mathrm{dp}_{\mathbf{m}}(s) < \alpha\}$, so L is an initial segment of $L_{\mathbf{m}}$ and applying the induction hypothesis to $\mathbf{m}{\restriction}L$, p we get q as required in $\mathbb{P}_{\mathbf{m}{\restriction}L}$ hence in $\mathbb{P}_{\mathbf{m}}$.

Case 4: $\delta = \mathrm{dp}_{\mathbf{m}}$ is a limit ordinal of cofinality $< \lambda$:

Let $\langle \alpha_i : i < \mathrm{cf}(\delta)\rangle$ be increasing continuous with limit δ, let $\alpha_{\mathrm{cf}(\delta)} = \delta$ and for $i \leq \mathrm{cf}(\delta)$ let $L_i := \{s \in L_{\mathbf{m}} : \mathrm{dp}_{\mathbf{m}}(s) < 1 + \alpha_i\}$.

Now we choose (p_i, ζ_i) by induction on $i < \mathrm{cf}(\delta)$ such that:

(a) $p_i \in \mathbb{P}_{\mathbf{m}{\restriction}L_i}$,
(b) $\mathbb{P}_{\mathbf{m}{\restriction}L_i} \models$ "$(p{\restriction}L_i) \leq p_i$ and $p_j \leq p_i$" when $j < i$,
(c) if i is a limit ordinal then p_i is gotten from $\langle p_j : j < i\rangle$ as in part (4),
(d) if $s \in \mathrm{dom}(p_i)$ then $\ell g(\mathrm{tr}(p_i(s))) \geq \zeta_i$,
(e) $\langle \zeta_j : j < i\rangle$ is an increasing continuous sequence of ordinals $< \lambda$ and if i is non-limit then ζ_i is $> \zeta$ and $\geq |\mathrm{dom}(p)|$ and $> \sup(\{\ell g(\mathrm{tr}(p_j(s))) : j < i$ and $s \in p_j\} \cup \{\ell g(\mathrm{tr}(p(s))) : s \in \mathrm{dom}(p)\})$.

Using 1.13 and the induction hypothesis this is easy.

4) For transparency assume $\Vdash$ "$\underset{\sim}{y} \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$" or just $\in {}^\lambda\mathbf{V}$. By parts (4) + (4A), i.e. part (3), for each $\zeta < \lambda$ the following subset of $\mathbb{P}_{\mathbf{m},t}$ is open and dense: $\mathscr{I}_\zeta = \{p \in \mathbb{P}_{\mathbf{m},t}$: for some $\nu \in \prod_{\varepsilon<\zeta} \theta_\varepsilon$ or $\in {}^\zeta\mathbf{V}$ (from $\mathbf{V}$!) we have $p \Vdash_{\mathbb{P}_{\mathbf{m},t}}$ "$\underset{\sim}{y}{\restriction}\zeta = \nu$"$\}$. Clearly there is a maximal antichain $\langle p_{\zeta,\varepsilon} : \varepsilon < \xi_\zeta\rangle$ of $\mathbb{P}_{\mathbf{m},t}$ included in $\mathscr{I}_\zeta$ and by part (2) without loss of generality $\xi_\zeta \leq \lambda$, the rest should be clear. In the general case we can code $\underset{\sim}{y}$ as a subset of λ, etc.

4A) This too should be clear as $\mathbb{P}_t$ satisfies the λ^+-c.c.

5) Look at the definitions.

6) Using parts (4) and (4A) and the definition this is easy.

7) Suppose toward contradiction that $\mathbf{G}_1 \neq \mathbf{G}_2$ are generic subsets of $\mathbb{P}_{\mathbf{m}}$ but $s \in L_{\mathbf{m}} \Rightarrow \underset{\sim}{\eta}_s[\mathbf{G}_1] = \eta_s = \underset{\sim}{\eta}_s[\mathbf{G}_2]$.

Let $p_1 \in \mathbf{G}_1 \backslash \mathbf{G}_2$ hence there is $p_2 \in \mathbf{G}_2$ such that $p_2 \Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$p_1 \notin \underset{\sim}{\mathbf{G}}_2$" hence p_1, p_2 are incompatible. Let $L_* = \{s \in L_{\mathbf{m}} : \mathbf{G}_1 \cap \mathbb{P}_{\leq s} = \mathbf{G}_2 \cap \mathbb{P}_{\leq s}\}$ so L_* is an initial segment of $L_{\mathbf{m}}$. If $L_* = L_{\mathbf{m}}$ we can easily get a contradiction, so $L_* \neq L_{\mathbf{m}}$ and let $r \in L_{\mathbf{m}} \backslash L_*$ be such that $L_{\mathbf{m}(<r)} \subseteq L_*$. Now as in part (8) we can get a contradiction having found a common upper bound to p_1, p_2.

Alternatively use part (6).

(8), (9), (10) Easy too. □

Conclusion 1.18 *Let* $\mathbf{m} \in \mathbf{M}$ *and for notational transparency is ordinary (see 1.7(8), which means that for some ordinal* $\beta(*)$, $t \in L_{\mathbf{m}} \Leftrightarrow t \in \beta(*)$ *and* $s <_{\mathbf{m}} t \Rightarrow s < t$.) *Then* $\mathbf{q}$ *is essentially*[14] *a* $(< \lambda)$*-support iteration of length* $\beta(*)$ *with* $\underset{\sim}{\mathbb{Q}}_\alpha = \{(\nu, f) \in \mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\langle \underset{\sim}{\eta}_\beta:\beta<\alpha\rangle]} : \nu \triangleleft f, f = \sup\{f_\iota : \iota < \iota(\alpha)\}, \iota(\alpha) < \lambda, \nu \triangleleft f_\iota$ *and* $\{f_\iota : \iota < \iota(\alpha)\} \subseteq \cup\{\mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\langle \underset{\sim}{\eta}_\alpha:\alpha\in u\rangle]} : u \in \mathscr{P}_{\mathbf{m},\alpha}\}\}$ *with the natural order, i.e. the order of* $\mathbb{Q}_{\bar{\theta}}^{\mathbf{V}[\mathbb{P}_\alpha]}$ *restricted to this set.*

Proof 1.18 Should be clear by 1.16. □

Till now $(E'_{\mathbf{m}}, M_{\mathbf{m}})$ have played no role and we could have omitted them.

[14] In particular - $\mathbb{P}_{\mathbf{m},\alpha}$ is a sub-forcing of the one we get by the iteration.

Definition 1.19 (1) We define the two-place relation $\le = \le_{\mathbf{M}}$ on $\mathbf{M}$ as follows: $\mathbf{m} \le \mathbf{n}$ <u>iff</u>:

(a) $L_{\mathbf{m}} \subseteq L_{\mathbf{n}}$, as partial orders of course,
(b) $M_{\mathbf{m}} = M_{\mathbf{n}}$, (yes! equal), and $M^{\text{fat}}_{\mathbf{m}} = M^{\text{fat}}_{\mathbf{n}}$, $M^{\text{lean}}_{\mathbf{m}} = M^{\text{lean}}_{\mathbf{n}}$,
(c) $u_{\mathbf{m},t} = u_{\mathbf{n},t} \cap L_{\mathbf{m}}$ and[15] $\mathscr{P}_{\mathbf{m},t} = \{u \cap L_{\mathbf{m}} : u \in \mathscr{P}_{\mathbf{n},t}\}$ for $t \in M_{\mathbf{m}}$,
(d) $u_{\mathbf{m},t} = u_{\mathbf{n},t}$ and $\mathscr{P}_{\mathbf{m},t} = \mathscr{P}_{\mathbf{n},t}$ for $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$,
(e) if $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ then $t/E'_{\mathbf{m},} = t/E'_{\mathbf{n},}$ hence $E'_{\mathbf{m}} = E'_{\mathbf{n}} \restriction L_{\mathbf{m}}$.
(f) Hence,
- $\bullet_1$ if $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ then $\mathscr{P}_{\mathbf{m},t} = \mathscr{P}_{\mathbf{n},t}$,
- $\bullet_2$ if $t \in M_{\mathbf{m}}$ and $s \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ then $\{u \in \mathscr{P}_{\mathbf{m},t} : u \subseteq s/E_{\mathbf{m}}\} = \{u \in \mathscr{P}_{\mathbf{n},t} : u \subseteq s/E_{\mathbf{n}}\}$,
- $\bullet_3$ if $t \in M_{\mathbf{m}}$ then $\{u \in \mathscr{P}_{\mathbf{m},t} : u \subseteq M_{\mathbf{m}}\} = \{u \in \mathscr{P}_{\mathbf{n},t} : u \subseteq M_{\mathbf{m}}\}$

(2) We define the two-place relation $\le_* = \le^*_{\mathbf{M}}$ as in part (1) omitting clauses (b),(e) and (f); natural but not used here.
(3) We define the two-place relation $\le^{\text{bd}}_{\mathbf{M}}$ by $\mathbf{m} \le^{\text{bd}}_{\mathbf{M}} \mathbf{n}$ iff $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ and both are bounded, see 1.7(10).

Claim 1.20 *(1) $\le_{\mathbf{M}}$ is a partial order or $\mathbf{M}$ and $\le^{\text{bd}}_{\mathbf{M}}$ a partial order on $\mathbf{M}_{\text{bd}}$ in fact is $\le_{\mathbf{M}} \restriction \mathbf{M}_{\text{bd}}$.*
(2) If $\langle \mathbf{m}_\alpha : \alpha < \delta \rangle$ is $\le_{\mathbf{M}}$-increasing, <u>then</u> its union $\mathbf{m}_\delta$ (naturally defined) is a $\le_{\mathbf{M}}$-lub and $|L_{\mathbf{m}_\delta}| \le \Sigma\{|L_{\mathbf{m}_\alpha}| : \alpha < \delta\}$.
(2A) Similarly for $\mathbf{M}_{\text{bd}}$.
(2B) We can restrict ourselves to any of the context (see 1.5)(2) including the fat context (there for $t \in M_{\mathbf{m}_0}$, $\mathscr{P}_t$ should be $[u_{\mathbf{m}_\delta},_t]^{\le\lambda}$ which may be different then $\bigcup\{[u_{\mathbf{m}_\alpha},_t]^{\le\lambda} < \delta\}$).
(3) If $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ and $L \subseteq L_{\mathbf{m}}$ <u>then</u> $p \in \mathbb{P}_{\mathbf{m}}(L) \Leftrightarrow p \in \mathbb{P}_{\mathbf{n}}(L)$ for every p.
(4) If $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ and $\mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{n}}$ and $L \subseteq L_{\mathbf{m}}$ <u>then</u> $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{n}}(L)$ as quasi orders.
(5) if $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$ <u>then</u>:

- *$\mathbf{m}$ is lean <u>iff</u> $\mathbf{n}$ is lean,*
- *$\mathbf{m}$ is fat <u>iff</u> $\mathbf{n}$ is fat,*
- *$\mathbf{m}$ is neat <u>iff</u> $\mathbf{n}$ is neat,*
- *$\mathbf{m}$ is bounded <u>if</u> $\mathbf{n}$ is.*

Proof 1.20 Easy.

(1) Obvious.
(2) Why is $L_{\mathbf{m}_\delta} := \cup\{L_{\mathbf{m}_\alpha} : \alpha < \delta\}$ well founded? Toward contradiction assume $\bar{t} = \langle t_n : n < \omega \rangle$ is $<_{L_{\mathbf{m}_\delta}}$-decreasing. We can replace $\bar{t}$ by any infinite sub-sequence. So without loss of generality:

(∗) either (α) or (β), where:
(α) for every $n < m$ there is $s_{n,m} \in M_{\mathbf{m}_0}$ such that $t_m <_{L_\delta} s_{n,m} <_{L_\delta} t_n$,
(β) for no $n < m$ this holds.

If clause (α) holds, then $\langle s_{n,n+1} : n < \omega \rangle$ is a $<_{M_{\mathbf{m}_0}}$-decreasing sequence contradiction. If clause (β) holds, then for $n < \omega$, let $\alpha(n) = \min\{\alpha : t_n \in L_{\mathbf{m}_\alpha}\}$; without loss of generality the sequence $\langle \alpha(n) : n < \omega \rangle$ is monotonically increasing or constant; so as $M_{\mathbf{m}_{\alpha(n)}} = M_{\mathbf{m}_0}$, by 1.19(1)(e) we get $t_n/E_{\mathbf{m}_{\alpha(n+1)}} = t_{n+1}/E_{\mathbf{m}_{\alpha(n+1)}}$ (recalling part (1)), hence $t_{n+1} \in L_{\mathbf{m}_{\alpha(n)}}$ hence $\alpha(n+1) \le \alpha(n)$. So $\{t_n : n < \omega\} \subseteq M_{\mathbf{m}_{\alpha(0)}}$ hence as $L_{\mathbf{m}_{\alpha(n)}}$ is well founded we are done.

[15] This is the parallel in clause (d) are covered by clause (f) but see part (2).

The proofs of (2A) and (2B) are easy too.

Finally for (3), (4) and (5), see the proof of $\boxplus_\alpha$ in the proof of 1.26.

$\square$

Claim 1.21 $(\mathbf{M}, \le_{\mathbf{m}})$ *has amalgamation. That is, if* $\mathbf{m}_0 \le_{\mathbf{M}} \mathbf{m}_1, \mathbf{m}_0 \le_{\mathbf{M}} \mathbf{m}_2$ *and* $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = L_{\mathbf{m}_0}$ *then there is* $\mathbf{m} \in \mathbf{M}$ *such that* $\mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}, \mathbf{m}_2 \le_{\mathbf{M}} \mathbf{m}$ *and* $L_{\mathbf{m}} = L_{\mathbf{m}_1} \cup L_{\mathbf{m}_2}$. *In fact,* $\mathbf{m}$ *is unique, so we call it* $\mathbf{m}_1 \oplus_{\mathbf{m}_0} \mathbf{m}_2$

Proof 1.21 Note that by clause $(e)(\gamma)$ of Definition 1.5 and clause (e) of Definition 1.19(1):

$(*)_1$ assume $(s_1 \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0}) \wedge (s_3 \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0})$ and $s_2 \in L_{\mathbf{m}_0}$;

- if $(s_1 <_{\mathbf{m}_1} s_2) \wedge (s_2 <_{\mathbf{m}_2} s_3)$, then for some $s_1', s_2' \in M_{\mathbf{m}_0}$ we have $s_1' \in (s/E_{\mathbf{m}}') \cap M_{\mathbf{m}}, s_3' \in t/E_{\mathbf{m}}' \cap M_{\mathbf{m}}, s_1 <_{\mathbf{m}_1} s_1' \le_{\mathbf{m}_0} s_2$ and $s_2 \le_{\mathbf{m}_1} s_2' <_{\mathbf{m}_2} s_3$,
- if $s_3 <_{\mathbf{m}_2} s_2 \wedge s_2 <_{\mathbf{m}_1} s_1$, then for some $s_1', s_2' \in M_{\mathbf{m}_0}$ we have $s_1' \in (s/E_{\mathbf{m}}') \cap M_{\mathbf{m}}, s_3' \in (t/E_{\mathbf{m}}') \cap M_{\mathbf{m}}, s_3 <_{\mathbf{m}_2} s_2' \le_{\mathbf{m}_1} s_2$ and $s_2 \le_{\mathbf{m}_2} s_1' <_{\mathbf{m}_1} s_1$.

We now define $\mathbf{m}$ by:

$(*)_2$ (a) (α) $t \in L_{\mathbf{m}}$ iff $t \in L_{\mathbf{m}_1} \vee t \in L_{\mathbf{m}_2}$,
(β) $M_{\mathbf{m}} = M_{\mathbf{m}_0}$ and $M_{\mathbf{m}}^{\text{fat}} = M_{\mathbf{m}_0}^{\text{fat}}, M_{\mathbf{m}}^{\text{lean}} = M_{\mathbf{m}_0}^{\text{lean}}$.
(b) $s <_{\mathbf{m}} t$ iff one of the following occurs:
(α) $s <_{\mathbf{m}_1} t$,
(β) $s <_{\mathbf{m}_2} t$,
(γ) $s \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0}$ and $t \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0}$ and for some $r \in M_{\mathbf{m}_0}, s \le_{\mathbf{m}_1} r \wedge r \le_{\mathbf{m}_2} t$,
(δ) $s \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0}$ and $t \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0}$ and for some $r \in M_{\mathbf{m}_0}, s \le_{\mathbf{m}_2} r \wedge r \le_{\mathbf{m}_1} t$.
(c) $u_{\mathbf{m},t}$ is:
(α) $u_{\mathbf{m}_1,t} \cup u_{\mathbf{m}_2,t}$ if[16] $t \in L_{\mathbf{m}_0}$,
(β) $u_{\mathbf{m}_1,t}$ if $t \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}_0}$.
(γ) $u_{\mathbf{m}_2,t}$ if $t \in L_{\mathbf{m}_2} \backslash L_{\mathbf{m}_0}$.
(d) $E_{\mathbf{m}}' = E_{\mathbf{m}_1}' \cup E_{\mathbf{m}_2}'$.
(e) $\mathscr{P}_{\mathbf{m},t}$ is:
(α) $\mathscr{P}_{\mathbf{m}_1,t}$ if $t \in L_{\mathbf{m}_1} \setminus L_{\mathbf{m}_0}$,
(β) $\mathscr{P}_{\mathbf{m}_2,t}$, if $t \in L_{\mathbf{m}_2} \setminus L_{\mathbf{m}_0}$,
(γ) $\mathscr{P}_{\mathbf{m}_1,t} \cup \mathscr{P}_{\mathbf{m}_1,t}$, if $t \in M_{\mathbf{m}_0}^{\text{lean}}$,
(δ) $\{u_1 \cup u_2 : u_1 \in \mathscr{P}_{\mathbf{m}_1,t}, u_2 \in \mathscr{P}_{\mathbf{m}_2,t}\}$ if $t \in M_{\mathbf{m}_0}^{\text{fat}}$,
(ε) $\mathscr{P}_{\mathbf{m}_1,t} \cup \mathscr{P}_{\mathbf{m}_1,t}$ if $t \in \mathbf{M}_{\mathbf{m}}^{\text{non}}$.

Clearly,

$\odot$ $\mathbf{m} \in \mathbf{M}$ and $\mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}$ and $\mathbf{m}_2 \le_{\mathbf{M}} \mathbf{m}$.

So we are done proving the existence of $\mathbf{m}$, the uniqueness is obvious. $\square$

Observation 1.22 *(1) For* $p, q \in \mathbb{P}_{\mathbf{m}}$ *we have:* $\mathbb{P}_{\mathbf{m}} \models$ *"*$p \le q$*" iff* $\text{dom}(p) \subseteq \text{dom}(q)$ *and* q *is essentially above* p *inside* $\mathbb{P}_{\mathbf{m}}$*, (see 1.7(9) or below).*

(2) For $p, q \in \mathbb{P}_{\mathbf{m}}$ *the following conditions are equivalent:*

((a) $q \Vdash$ *"*$p \in \mathop{\mathbf{G}}\limits_{\sim}{}_{\mathbb{P}_{\mathbf{m}}}$*", that is* q *is essentially above* p*, see 1.7(9),*
(b) if $s \in \text{dom}(p)$ *then either* $s \in \text{dom}(q)$ *and* $(q \restriction L_{\mathbf{m},<s}) \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ *"*$p(s) \le q(s)$*" or* $s \notin \text{dom}(q), \text{tr}(p(s)) = \emptyset$ *and* $q \restriction L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ *"*$p(s)$ *is trivial, i.e.* $\underset{\sim}{f}_{p(s)}$ *is constantly zero",*

16 But recall that for $\ell \in \{1, 2\}$ we have: $t \in L_{\mathbf{m}_0} \backslash M_{\mathbf{m}_0} \Rightarrow u_{\mathbf{m}_\ell,t} = u_{\mathbf{m}_0,t} \wedge \mathscr{P}_{\mathbf{m}_\ell,t} = \mathscr{P}_{\mathbf{m}_0,t}$.

(c) $\mathbb{P}_{\mathbf{m}} \models$ "$p \le q^+$" *where* $\mathrm{dom}(q^+) = \mathrm{dom}(q) \cup \mathrm{dom}(p)$ *and* $q^+(s)$ *is:*
   (α) $q(s)$ *if* $s \in \mathrm{dom}(q)$,
   (β) *the trivial condition if* $s \in \mathrm{dom}(p) \backslash \mathrm{dom}(q)$*; note that* $\mathrm{fsupp}(q^+) = \mathrm{fsupp}(q) \cup \mathrm{fsupp}(p)$.

Remark 1.23 We shall use this freely.

Proof 1.22 (1) Easy but we shall elaborate.

Let $p, q \in \mathbb{P}_{\mathbf{m}}$. If $p \le q$ then clearly $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and $q \Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$p \in \underset{\sim}{\mathbf{G}}$", that is q is essentially above p.

For the other direction assume $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ but $\mathbb{P}_{\mathbf{m}} \models \neg(p \le q)$ and we shall prove that q is not essentially above p, this suffices. By the present assumption there is $s \in \mathrm{dom}(p)$ (hence $s \in \mathrm{dom}(q)$) but $q \restriction L_{\mathbf{m}(<s)} \nVdash$ "$p(s) \le q(s)$".

Hence there is $q_1 \in \mathbb{P}_{\mathbf{m}(<s)}$ above $q \restriction L_{\mathbf{m}(<s)}$ such that $q_1 \Vdash_{\mathbb{P}_{\mathbf{m}(<s)}}$"$\neg(p(s) \le q(s))$". By the properties of $\mathbb{Q}_{\bar{\theta}}$ (and $\mathbb{Q}'_s$, 1.16(6)) there are q_2, q' such that:

$(*)_1$ (a) $q' \in \mathbb{P}_{\mathbf{m}}$, $\mathrm{dom}(q') = \{s\}$,
   (b) $q_1 \le q_2$ in $\mathbb{P}_{\mathbf{m}(<s)}$,
   (c) $q_2 \Vdash_{\mathbb{P}_{\mathbf{m}(s)}}$"$q(s) \le q'(s)$ but $q'(s)$, $p(s)$ are incompatible"

Lastly, choose the function q_3 by:

$(*)_2$ (a) $\mathrm{dom}(q_3) = \mathrm{dom}(q_2) \cup \mathrm{dom}(q)$,
   (b) $q_3 \restriction \mathrm{dom}(q_2) = q_2$,
   (c) $q_3(s) = q'(s)$,
   (d) $q_2(t) = q(t)$ if $t \in \mathrm{dom}(p) \setminus (\mathrm{dom}(q_2) \cup \{s\})$.

Clearly $q_3 \in \mathbb{P}_{\mathbf{m}}$, $q \le q_3$ and $q_3 \Vdash_{\mathbb{P}_{\mathbf{m}}}$"$p \notin \underset{\sim}{\mathbb{Q}}_{\mathbb{P}_{\mathbf{m}}}$" so we are done.

2) <u>(a) implies (c)</u>:

By the choice of q^+ we have $q \le q^+$, so clause (a) implies that q is essentially above p hence by part (1) in $\mathbb{P}_{\mathbf{m}}$ we have $p \le q^+$ so clearly clause (c) holds.

<u>(c) implies (a)</u>:

Easy.

<u>(c) iff (b)</u>:

Obvious recalling the properties of $\mathbb{Q}_{\bar{\theta}}$. □

1.2 Special sufficient conditions

Claim 1.24 *For* $\mathbf{m} \in \mathbf{M}$*, recalling 1.12(3), we have* $\mathbb{P}_{\mathbf{m}}(L_1) \lessdot \mathbb{P}_{\mathbf{m}}(L_3)$ <u>*when*</u>*:*

$(*)$ *(a)* $L_2 \subseteq L_3$ *are initial segments of* $L_{\mathbf{m}}$*,*
   (b) $L_1 \subseteq L_3$ *and* $L_0 = L_1 \cap L_2$*,*
   (c) L_0 *is an initial segment of* L_1*, (follows),*
   (d) $\mathbb{P}_{\mathbf{m}}(L_0) \lessdot \mathbb{P}_{\mathbf{m}}(L_2)$*,*
   (e) $L_1 \backslash L_0$ *is disjoint to* $M_{\mathbf{m}}$*,*
   (f) if $t \in L_1 \backslash L_0$ *then* $(t/E_{\mathbf{m}}) \cap L_{\mathbf{m},<t} \subseteq L_1$.

Remark 1.25 (1) We may phrase it differently. Recall that assuming $\mathbb{P}' \lessdot \mathbb{P}$, we say $p' \in \mathbb{P}'$ is a reduction of $p \in \mathbb{P}$ where every condition $r \in \mathbb{P}'$ stronger than p' (in $\mathbb{P}'$) is still compatible (in $\mathbb{P}$) with p. Let $\mathbb{P}_\ell = \mathbb{P}_{\mathbf{m}}(L_\ell)$. Now the statement is: to find a reduction of p_3 from $\mathbb{P}_3$ to $\mathbb{P}_1$ first consider $p_2 =$ the reduction of p_3 to $\mathbb{P}_2$, then let p_0 be a reduction of p_2 from $\mathbb{P}_2$

to $\mathbb{P}_0$ and finally extend p_0 to a condition p_1 by appending the information from p_3 on (L_1 minus L_0).

(2) Claim 1.24 is used only in the proof of 1.26 which is used only in the proof of 3.20 and 3.22.

Proof 1.24 As $\mathrm{dp}_{\mathbf{m}}(L_1) < \infty$ it suffices to prove by induction on the ordinal γ that:

$\boxplus_\gamma$ if $\langle L_\ell : \ell \leq 3\rangle$ satisfies $(*)$ of the claim and $\mathrm{dp}_{\mathbf{m}}(L_1) \leq \gamma$ then:

- $\bullet_1$ we have $p_1 \in \mathbb{P}_{\mathbf{m}}(L_1)$ and $p_1 \leq q_1 \in \mathbb{P}_{\mathbf{m}}(L_1) \Rightarrow p_3, q_1$ are compatible in $\mathbb{P}_{\mathbf{m}}(L_3)$ when:
  - (a) $p_3 \in \mathbb{P}_{\mathbf{m}}(L_3)$,
  - (b) $p_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$,
  - (c) if $p_0 \leq q_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$ then $p_2 := p_3{\restriction}L_2$ and q_0 are compatible in $P_{\mathbf{m}}(L_2)$,
  - (d) $p_1 = p_0 \cup (p_3{\restriction}(L_1\backslash L_0))$.
- $\bullet_2$ $\mathbb{P}_{\mathbf{m}}(L_1) \lessdot \mathbb{P}_{\mathbf{m}}(L_3)$.

Why this holds? Assume we have arrived to γ.

Clause $\bullet_1$: (notice that here we do not use the induction hypothesis): Recalling clause (f) of the assumption, indeed, $p_1 = p_0 \cup (p_3{\restriction}(L_1\backslash L_0)) \in \mathbb{P}_{\mathbf{m}}(L_1)$ by the definitions (clauses $\bullet_1(a)$), (b), (d) of $\boxplus_\gamma$), e.g. why $\mathrm{fsupp}(p_1) \subseteq L_1$? Note that if $s \in \mathrm{dom}(p_3{\restriction}(L_1\backslash L_0)$ then $s \in L_1\backslash L_0 \subseteq L_1$ and $\{r_{p_3(s)}(\zeta) : \zeta < \xi_{p(s)}\}$ is included in L_3 because $p \in \mathbb{P}_{\mathbf{m}}(L_3)$ and in $L_{<s}$ by Definition 1.10. As $s \in L_1\backslash L_0$ by $(*)(e)$ we have $s \notin M_{\mathbf{m}}$ hence by Definition 1.10 we have $\{r_{p_3(s)}(\zeta) : \zeta < \xi_{p(s)}\} \subseteq u_s \subseteq s/E_{\mathbf{m}}$. By $(*)(f)$ we have $(s/E_{\mathbf{m}}) \cap L_{\mathbf{m}<t} \subseteq L_1$ hence together $\{r_{p_3(s)}(\zeta) : \zeta < \xi_{p(s)}\} \subseteq L_1$, and we are done proving $\mathrm{fsupp}(p_1) \subseteq L_1$.

So the first statement in $\boxplus_\gamma \bullet_1$ holds; what about the second? Toward contradiction assume q_1 contradicts the desired conclusion. Then by 1.16(6) there are s and p_3^+ such that:

$\oplus$ (a) $s \in \mathrm{dom}(q_1) \cap \mathrm{dom}(p_3)$,
(b) $p_3^+ \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})$,
(c) p_3^+ is above $p_3{\restriction}L_{\mathbf{m},<s}$ and above $q_1{\restriction}L_{\mathbf{m},<s}$,
(d) $p_3^+ \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p_3(s), q_1(s) \in \mathbb{Q}_{\bar\theta}$ are incompatible (in $\mathbb{Q}_{\bar\theta}$)".

So $s \in \mathrm{dom}(q_1) \subseteq L_1$ and as L_2 is an initial segment of $L_{\mathbf{m}}$ and clause (c) of $\bullet_2$ (of $\boxplus_\gamma$), clearly $s \in L_0$ is impossible, so $s \in \mathrm{dom}(q_1)\backslash L_0 \subseteq L_1\backslash L_0$. As $\mathbb{P}_{\mathbf{m}} \models$ "$p_1 \leq q_1$", necessarily $q_1{\restriction}L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p_1(s) \leq q_1(s)$", so as $q_1{\restriction}L_{\mathbf{m},<s} \leq p_3^+{\restriction}L_{\mathbf{m},<s}$ (by $\oplus(c)$), also $p_3^+{\restriction}L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p_1(s) \leq q_1(s)$". As $s \notin L_0$ clearly $p_1(s) = p_3(s)$ by clauses $\boxplus_\gamma \bullet_2$ (b), (d), so $p_3^+{\restriction}L_{\mathbf{m},<s} \Vdash_{\mathbb{P}_{\mathbf{m},<s}}$ "$p_3(s) \leq q_1(s)$" and again easy contradiction to $\oplus(d)$.

Clause $\bullet_2$:

Clearly $\mathbb{P}_{\mathbf{m}}(L_1) \subseteq \mathbb{P}_{\mathbf{m}}(L_3)$ as quasi orders. Next we shall prove $\mathbb{P}_{\mathbf{m}}(L_1) \leq_{\mathrm{ic}} \mathbb{P}_{\mathbf{m}}(L_3)$, so assume $q_1, q_2 \in \mathbb{P}_{\mathbf{m}}(L_1)$ has a common upper bound p_3 in $\mathbb{P}_{\mathbf{m}}(L_3)$, and we should find one in $\mathbb{P}_{\mathbf{m}}(L_1)$. Hence (see 1.10(e)($\beta$)) we have $\mathrm{dom}(q_1) \cup \mathrm{dom}(q_2) \subseteq \mathrm{dom}(p_3)$.

As $p_3{\restriction}L_2 \in \mathbb{P}_{\mathbf{m}}(L_2)$ by $(*)(a)$ and we are assuming $\mathbb{P}_{\mathbf{m}}(L_0) \lessdot \mathbb{P}_{\mathbf{m}}(L_2)$, see $(*)(d)$ there is $p_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$ such that $p_0 \leq q \in \mathbb{P}_{\mathbf{m}}(L_0) \Rightarrow q, p_3{\restriction}L_2$ are compatible in $\mathbb{P}_{\mathbf{m}}(L_2)$ and let $p_1 = p_0 \cup (p_3{\restriction}(L_1\backslash L_0))$. By $\boxplus_\gamma(b)$, which we have proved noting that clauses (a)-(d) of $\boxplus_\gamma \bullet_2$ holds, we know that $p_1 \in \mathbb{P}_{\mathbf{m}}(L_1)$ and $p_1 \leq p_1' \in \mathbb{P}_{\mathbf{m}}(L_1) \Rightarrow p_3, p_1'$ are compatible in $\mathbb{P}_{\mathbf{m}}(L_3)$. It suffices to prove that p_1 is a common upper bound of q_1, q_2.

We could have replaced p_0 by p_0' whenever $p_0 \leq p_0' \in \mathbb{P}_{\mathbf{m}}(L_0)$. So without loss of generality for $\ell = 1, 2$ we have $\mathrm{dom}(q_\ell) \cap L_0 \subseteq \mathrm{dom}(p_0)$ hence $\subseteq \mathrm{dom}(p_1)$, also recall $\mathrm{dom}(q_\ell)\backslash L_0 \subseteq \mathrm{dom}(p_3) \cap L_1\backslash L_0$ and by the choice of p_1 we have $\mathrm{dom}(p_3) \cap L_1\backslash L_0 \subseteq \mathrm{dom}(p_1)\backslash L_0$.

So recalling $\mathrm{dom}(q_\ell) \subseteq L_1$ together $\mathrm{dom}(q_\ell) \subseteq \mathrm{dom}(p_1)$.

As we are assuming $\mathbb{P}_{\mathbf{m}}(L_0) \lessdot \mathbb{P}_{\mathbf{m}}(L_2)$ without loss of generality p_0 is above[17] $q_\ell \restriction L_0$. If toward contradiction we assume that $\ell \in \{1, 2\}$ and $q_\ell \not\leq p_1$ then for some $s \in \mathrm{dom}(q_\ell)$ we have $(q_\ell \restriction L_{\mathbf{m},<s}) \leq (p_1 \restriction L_{\mathbf{m},<s})$ but $p_1 \restriction L_{\mathbf{m},<s} \nVdash_{\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})}$ "$q_\ell(s) \leq p_1(s)$". Clearly, $s \in L_0$ is impossible so $s \in L_1 \backslash L_0$ hence $s \notin M_{\mathbf{m}}$ by clause $(*)(e)$.

Let $L'_0 = L_0$, $L'_1 = L_0 \cup (L_1 \cap L_{\mathbf{m},<s})$, $L'_2 = L_2$, $L'_3 = L_3$ so (L'_0, L'_1, L'_2, L'_3) satisfies the assumptions of the present claim and $\mathrm{dp}_{\mathbf{m}}(L'_1) < \gamma$, hence by the induction hypothesis, $\mathbb{P}_{\mathbf{m}}(L'_1) \lessdot \mathbb{P}_{\mathbf{m}}(L'_3)$.

Recall $s \in L_1 \backslash L_0$ hence $(s/E_{\mathbf{m}}) \cap L_{\mathbf{m},<s} \subseteq L_1$ by clause (f) of the assumption of the claim, so $\mathrm{fsupp}(p_1 \restriction \{s\}) \backslash \{s\}$, $\mathrm{fsupp}(q_\ell \restriction \{s\}) \backslash \{s\}$ are $\subseteq L'_1$ hence $p_1(s), q_\ell(s)$ are $\mathbb{P}_{\mathbf{m}}(L'_1)$-names. So recalling $p_1 \restriction L_{\mathbf{m},<s} \nVdash_{\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})}$ "$q_\ell(s) \leq p_1(s)$" and $\mathbb{P}_{\mathbf{m}}(L'_1) \lessdot \mathbb{P}_{\mathbf{m}}(L'_3)$ and $L_{\mathbf{m},<s} \subseteq L_3 = L'_3$ we have $p_1 \restriction L'_1 \nVdash_{\mathbb{P}_{\mathbf{m}}(L'_1)}$ "$q_\ell(s) \leq p_1(s)$". Hence there is p_1^+ such that $p_1 \restriction L'_1 \leq p_1^+ \in \mathbb{P}_{\mathbf{m}}(L'_1)$ such that $p_1^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_1)}$ "$q_\ell(s) \not\leq p_1(s)$" so recalling $\mathbb{P}_{\mathbf{m}}(L'_1) \lessdot \mathbb{P}(L'_3)$ we have $p_1^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_3)}$ "$q_\ell(s) \not\leq p_1(s)$".

But by $\boxplus_{\gamma_1} \bullet_2$ for $\gamma_1 = \mathrm{dp}_{\mathbf{m}}(L'_1)$, we know that p_1^+ and $p_3 \restriction L_{\mathbf{m},<s}$ are compatible (in $\mathbb{P}_{\mathbf{m}}$, equivalently $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})$) so let $p_3^+ \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})$ be a common upper bound of $p_1^+, p_3 \restriction L_{\mathbf{m},<s}$. Now $p_3^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_3)}$ "$q_\ell(s) \leq p_1(s)$" because: $q_\ell \leq p_3$ by the choice of p_3; $p_1(s) = p_3(s)$ by the choice of p_1 and $p_3 \leq p_3^+$, see above. However, $p_3^+ \Vdash_{\mathbb{P}_{\mathbf{m}}(L'_3)}$ "$q_\ell(s) \not\leq p_1(s)$" as $p_1^+ \leq p_3^+$, see above.

So we have proved $\mathbb{P}_{\mathbf{m}}(L_1) \leq_{\mathrm{ic}} \mathbb{P}_{\mathbf{m}}(L_3)$.

To finish proving clause $\boxplus_\gamma \bullet_1$, that is, $\mathbb{P}_{\mathbf{m}}(L_1) \lessdot \mathbb{P}_{\mathbf{m}}(L_3)$ note that clause $\boxplus_\gamma \bullet_1$ does this as for every $p_3 \in \mathbb{P}_{\mathbf{m}}(L_3)$ there is p_0 as in $\boxplus_\gamma \bullet_1$ $(b), (c)$ by clause (d) of the claim's assumption and let p_1 be as defined in $\boxplus_\gamma \bullet_1$ (d). □

Claim 1.26 *We have $\mathbb{P}_{\mathbf{m}_1}(L_1) = \mathbb{P}_{\mathbf{m}_2}(L_1)$ (i.e. as quasi orders) and $\mathbb{P}_{\mathbf{m}_\ell}(L_1) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$ for $\ell = 1, 2$ <u>when</u>:*

⊡ *(a)* $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$,
(b) $L_0 \subseteq L_1 \subseteq L_{\mathbf{m}_1}$,
(c) L_0 *is an initial segment of* L_1,
(d) $\mathbb{P}_{\mathbf{m}_1}(L_0) = \mathbb{P}_{\mathbf{m}_2}(L_0)$,
(e) $\mathbb{P}_{\mathbf{m}_\ell}(L_0) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$ *for* $\ell = 1, 2$,
(f) *if* $t \in L_1 \backslash L_0$ <u>*then*</u> $t \notin M_{\mathbf{m}_2}$ *and* $L_{\mathbf{m}_1,<t} \cap (t/E_{\mathbf{m}_1}) = L_{\mathbf{m}_2,<t} \cap (t/E_{\mathbf{m}_2}) \subseteq L_1$.

Remark 1.27 Used only in the proof of $\boxplus_{4.4}$ inside the proof of 3.20, so we could have used $M_\beta, \mathscr{E}$ from there.

Proof 1.26 For $\ell \in \{1, 2\}$ let $\bar{L}_\ell = \langle L_{\ell,i} : i < 4 \rangle$ be defined by:

$\oplus_1$ (a) $L_{\ell,0} = L_0$,
(b) $L_{\ell,1} = L_1$,
(c) $L_{\ell,2} = \{s \in L_{\mathbf{m}_\ell} : s \leq_{\mathbf{m}_\ell} t$ for some $t \in L_0\}$,
(d) $L_{\ell,3} = L_{\mathbf{m}_\ell}$.

Clearly,

[17] Why? It suffices to prove that there is $p'_0 \in \mathbb{P}_{\mathbf{m}}(L_0)$ above p_0 and above $q_\ell \restriction L_0$. So toward contradiction assume this fails hence there is $p_0^+ \in \mathbb{P}_{\mathbf{m}}(L_0)$ above p_0 incompatible with $q_\ell \restriction L_0$. By the choice of p_0 we know that $p_0^+, (p_3 \restriction L_2)$ are compatible, so let $p_3^+ \in \mathbb{P}_{\mathbf{m}}(L_2)$ be a common upper bound. Now L_2 is an initial segment of $L_{\mathbf{m}}$ by $(*)(a)$ and p_3 is above q_ℓ hence $p_3 \restriction L_2$ is above $q_\ell \restriction L_2$ and as $q_\ell \in \mathbb{P}_{\mathbf{m}}(L_1)$, $L_0 = L_1 \cap L_2$ we have $q_\ell \restriction L_2 = q_\ell \restriction L_0$, $p_3 \restriction L_2$ is above $q_\ell \restriction L_0$ but p_3^+ is above $p_3 \restriction L_2$ hence p_3^+ is above $q_\ell \restriction L_2$. Also p_3^+ is above p_0^+ which forces $q_\ell \restriction L_0 \notin \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}(L_0)}$, equivalently $q_\ell \restriction L_0 \notin \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}(L_2)}$, contradiction.

$\oplus_2$ (a) $(\mathbf{m}_\ell, \bar{L}_\ell)$ satisfies the assumptions of 1.24 hence,
(b) $\mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,1}) \lessdot \mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,3})$ which means $\mathbb{P}_{\mathbf{m}_\ell}(L_1) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$ for $\ell = 1, 2$.

Why $\oplus_2$? Clearly it suffices to prove clause (a), so we just have to check clauses $(*)(a)$ – (f) of 1.24.

Clause $(*)(a)$:

By $\oplus_1(d)$, $L_{\ell,3} = L_{\mathbf{m}_\ell}$ hence is an initial segment of $L_{\mathbf{m}_\ell}$ and by $\oplus_1(c)$, $L_{\ell,2}$ is an initial segment of $L_{\mathbf{m}_\ell}$ which is $L_{\ell,3}$ so $L_{\ell,2} \subseteq L_{\ell,3}$.

Clause $(*)(b)$:

For the first statement, $L_{\ell,1} \subseteq L_{\ell,3}$ is trivial by $\oplus_1(d) + \oplus_1(b) + \boxdot(a), (b)$. The second statement says $L_{\ell,0} = L_{\ell,1} \cap L_{\ell,2}$. Now $L_{\ell,0} \subseteq L_{\ell,1}$ by $\boxdot(a), (b)$ of the claim and $\oplus_1(a), (b)$. Also $L_{\ell,0} \subseteq L_{\ell,2}$ holds by $\oplus_1(c)$ (and $\oplus_1(a)$). Together $L_{\ell,0} \subseteq L_{\ell,1} \cap L_{\ell,2}$; to prove the inverse inclusion assume $s \in L_{\ell,2} \cap L_{\ell,1}$, so as $s \in L_{\ell,2}$ by $\oplus_1(c)$ there is $t \in L_0$ such that $s \leq_{\mathbf{m}_\ell} t$. But $s \in L_{\ell,1} = L_1$ so by $\boxdot(c)$ of the claim we have $s \in L_0 = L_{\ell,0}$ as promised.

Clause $(*)(c)$:

Holds by condition $\boxdot(c)$ of the claim.

Clause $(*)(d)$:

By clause $\boxdot(f)$ of the claim and $\oplus_1(c)$, $L_{\ell,2}$ is an initial segment of $L_{\mathbf{m}_\ell}$, hence by 1.13(e) we have $\mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,2}) \lessdot \mathbb{P}_{\mathbf{m}_\ell} = \mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,3})$. By $\boxdot(e)$ $\mathbb{P}_{\mathbf{m}_0}(L_{\ell,0}) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$; so together as $L_{\ell,0} \subseteq L_{\ell,2}$, we have $\mathbb{P}_{\mathbf{m}_\ell}(L_0) \lessdot \mathbb{P}_{\mathbf{m}_\ell}(L_{\ell,2})$.

Clauses $(*)(e), (f)$:

Hold by condition $\boxdot(f)$ of the claim.

So $\oplus_2$ holds indeed. So now we deal with the other half.

Proof of: $\mathbb{P}_{\mathbf{m}_1}(L_1) = \mathbb{P}_{\mathbf{m}_2}(L_1)$.

Let $\langle s_\alpha : \alpha < \alpha(*)\rangle$ list $L_1 \backslash L_0$ such that $s_\alpha \leq_{L_\mathbf{m}} s_\beta \Rightarrow \alpha \leq \beta$. This is possible as $L_{\mathbf{m}_2}$ is well founded.

Now,

$\oplus_3$ for $\ell = 1, 2$ and $\alpha \leq \alpha(*)$ let $\bar{L}^*_{\ell,\alpha} = \langle L^*_{\ell,\alpha,i} : i < 4\rangle$ be (but we can omit ℓ) where:

(a) $L^*_{\ell,\alpha,0} = L_0$,
(b) $L^*_{\ell,\alpha,1} = L_0 \cup \{s_\beta : \beta < \alpha\}$,
(c) $L^*_{\ell,\alpha,2} = \{s \in L_{\mathbf{m}_\ell} : s \leq_{\mathbf{m}_\ell} t$ for some $t \in L_0\}$,
(d) $L^*_{\ell,\alpha,3} = L_{\mathbf{m}_\ell}$,

$\oplus_4$ (a) $(\bar{\mathbf{m}}_\ell, \bar{L}^*_{\ell,\alpha})$ satisfies the assumption of 1.24,
(b) $\mathbb{P}_{\mathbf{m}_\ell}(L^*_{\ell,\alpha,1}) \lessdot \mathbb{P}_{\mathbf{m}_\ell}(L^*_{\ell,\alpha,3})$.

[Why? Note the $\mathbf{m}_\ell$, $\langle L^*_{\ell,\alpha,i} : i < 4\rangle$ satisfies the assumptions of 1.24, hence $\oplus_2$ holds for $\mathbf{m}_\ell, \bar{L}_{\ell,\alpha}$ for $\alpha \leq \alpha(*)$.]

Now by induction on $\alpha \leq \alpha(*)$ we prove that:

$\boxplus_\alpha$ $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) = \mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1})$.

Case 1: $\alpha = 0$:

As $L^*_{1,\alpha,1} = L_0 = L^*_{2,\alpha,1}$, clause $\boxdot(d)$ of the assumption gives $\boxplus_\alpha$ as promised.

Case 2: α a limit ordinal:

Easy by the definition of the iteration. That is, first, if $\mathrm{dom}(p) \in [L_{\mathbf{m}_2}]^{\leq\lambda}$ then we know $p \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \Leftrightarrow \bigwedge_{\beta<\alpha} [p{\restriction}L^*_{\beta,1} \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1})] \Leftrightarrow \bigwedge_{\beta<\alpha} [p{\restriction}L^*_{\beta,1} \in \mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1})] \Leftrightarrow p \in \mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1})$; second, for $p, q \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1})$ by the definition of the order and the induction hypothesis, $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \models$ "$p \le q$" iff $\bigwedge_{\beta<\alpha} [\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) \models$ "$p \restriction L^*_{\beta,1} \le q \restriction L^*_{\beta,1}$"] iff $\bigwedge_{\beta<\alpha} [\mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1}) \models$ "$p \restriction L^*_{\beta,1} \le q \restriction L^*_{\beta,1}$"] iff $\mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1}) \models$ "$p \le q$".

So $\boxplus_\alpha$ holds.

Case 3: $\alpha = \beta + 1$:

Clearly,

$(*)_1$ $p \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \Leftrightarrow p \in \mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1})$.

Next,

$(*)_2$ assume $p, q \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1})$ and we shall prove that $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \models$ "$p \le q$" implies $\mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1}) \models$ "$p \le q$".

[Why? If $s_\beta \notin \mathrm{dom}(p)$ this is obvious by the induction hypothesis. Hence we can assume $s_\beta \in \mathrm{dom}(p)$, so as we are assuming $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \models$ "$p \le q$", clearly $s_\beta \in \mathrm{dom}(q)$ hence $s_\beta \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$. First, similarly $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) \models$ "$(p \restriction L^*_{\beta,1}) \le (q \restriction L^*_{\beta,1})$" and $(q \restriction L^*_{\beta,1}) \Vdash_{\mathbb{P}_{\mathbf{m}_1(<s_\beta)}}$ "$p(s_\beta) \le_{\mathbb{Q}_{\bar\theta}} q(s_\beta)$" by the definition of $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1})$. Second, as $q \restriction L^*_{\beta,1} \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) = \mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1})$ and $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) \lessdot \mathbb{P}_{\mathbf{m}_1}$ by $\oplus_4$ and $\mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1}) \lessdot \mathbb{P}_{\mathbf{m}_2}$ by $\oplus_4$ and $p(s_\beta), q(s_\beta)$ are $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1})$-names (as $\mathrm{fsupp}(p(s_\beta)), \mathrm{fsupp}(q(s_\beta)) \subseteq L^*_{\beta,1}$) necessarily we have $q \restriction L^*_{\beta,1} \Vdash_{\mathbb{P}_{\mathbf{m}_2}}$ "$p(s_\beta) \le_{\mathbb{Q}_{\bar\theta}} q(s_\beta)$". Third, as $\mathbb{P}_{\mathbf{m}_1}(L^*_{\beta,1}) \models$ "$p \restriction L^*_{\beta,1} \le q \restriction L^*_{\beta,1}$", by the induction hypothesis $\mathbb{P}_{\mathbf{m}_2}(L^*_{\beta,1}) \models$ "$p \restriction L^*_{\beta,1} \le q \restriction L^*_{\beta,1}$. Fourth, by the last two sentence and the definition of the order in $\mathbb{P}_{\mathbf{m}_2}$ we have $\mathbb{P}_{\mathbf{m}_2} \models$ "$p \le q$" so the conclusion of $(*)_2$ holds also in this case.

Note that if $s_\beta \in \mathrm{dom}(p) \backslash \mathrm{dom}(q)$ then $p \nleq q$, so we are done proving $(*)_2$.]

$(*)_3$ if $p, q \in \mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1})$ and $\mathbb{P}_{\mathbf{m}_2}(L^*_{\alpha,1}) \models$ "$p \le q$" then $\mathbb{P}_{\mathbf{m}_1}(L^*_{\alpha,1}) \models$ "$p \le q$".

[Why? Similar to the proof of $(*)_2$.]

By $(*)_1$, $(*)_2$, $(*)_3$ clearly $\boxplus_\alpha$ holds. So we carried the induction so $\boxplus_\alpha$ holds for every $\alpha \le \alpha(*)$ and for $\alpha = \alpha(*)$ we get $\mathbb{P}_{\mathbf{m}_1}(L_1) = \mathbb{P}_{\mathbf{m}_2}(L_2)$. Together with $\oplus_2(b)$ in the beginning of the proof we are done. □

1.3 On existentially closed m's

Definition 1.28 (0) For $\mathbf{m} \in \mathbf{M}$ let:

(a) $\mathrm{dp}^*_{\mathbf{m}}(L) = \cup\{\mathrm{dp}_{M_{\mathbf{m}}}(t) + 1 : t \in L \cap M_{\mathbf{m}}\}$, for $L \subseteq L_{\mathbf{m}}$,

(b) $L^{\mathrm{dp}}_{\mathbf{m},\gamma} = \{t \in L_{\mathbf{m}} : t \in M_{\mathbf{m}} \Rightarrow \mathrm{dp}_{M_{\mathbf{m}}}(t) < \gamma$ and $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}} \Rightarrow \sup\{\mathrm{dp}_{M_{\mathbf{m}}}(s) : s \in M_{\mathbf{m}}$ and $s <_{L_{\mathbf{m}}} t\} < \gamma\}$. So,

- $L^{\mathrm{dp}}_{\mathbf{m},\gamma}$ is an initial segment of $L_{\mathbf{m}}$,
- $L^{\mathrm{dp}}_{\mathbf{m},\gamma}$ is $\subseteq$-increasing continuous with γ and is equal to $L_{\mathbf{m}}$ for $\gamma = \mathrm{dp}^*_{\mathbf{m}}(\mathrm{M}_{\mathbf{m}})$, or for $\gamma = \mathrm{dp}^*_{\mathbf{m}}(\mathrm{M}_{\mathbf{m}}) + 1$ (if $(\exists t \in L \setminus M)(\forall s \in M_{\mathbf{m}})(t > s)$).

(c) $L^{\mathrm{dq}}_{\mathbf{m},\gamma} = \{t \in L_{\mathbf{m}} : t \in M_{\mathbf{m}}, \mathrm{dp}_{M_{\mathbf{m}}}(t) < \gamma$ or $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ and $\min\{\mathrm{dp}_{M_{\mathbf{m}}}(s) : s \in M_{\mathbf{m}} \cup \{\infty\}, t < s\} \le \gamma\}$, note that (we mean):

- for $\gamma = 0$ this is $\{t \in L_{\mathbf{m}}$: if $(\exists s \in M_{\mathbf{m}})(t \le s)$ then for some $s \in M_{\mathbf{m}}$ we have $t < s$ and $\mathrm{dp}_{M_{\mathbf{m}}}(s) = 0\}$,
- each $L^{\mathrm{dq}}_{\mathbf{m},\gamma}$ is an initial segment of $L_{\mathbf{m}}$,
- the set $L^{\mathrm{dq}}_{\mathbf{m},\gamma}$ is $\subseteq$-increasing with γ, but not necessarily continuous,

- (meaningful only if we do not assume $\mathbf{m}$ is bounded, see 1.7(10)) if $t \in L_{\mathbf{m}}$ then we have: for no $s \in M_{\mathbf{m}}$ do we have $t \leq s$ iff $t \in L^{\rm dq}_{\mathbf{m},\gamma} \setminus \cup\{L^{\rm dq}_{M_{\mathbf{m}},\beta} : \beta < \gamma\}$ for $\gamma = {\rm dp}^*_{\mathbf{m}}(\mathrm{M}_{\mathbf{m}}) = \cup\{{\rm dp}_{\mathrm{M}_{\mathbf{m}}}(\mathrm{s}) + 1 : \mathrm{s} \in \mathbf{M}_{\mathbf{m}}\}$.

(1) (a) For an ordinal γ let $\mathbf{M}^{\rm bec}_\gamma$ (here bec stands for bounded existentially closed) be the class of $\mathbf{m} \in \mathbf{M}_{\rm bd}$ such that, recalling Definition 1.12(3):

(∗) if $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$ and $\mathbf{m}_1, \mathbf{m}_2$ are bounded, <u>then</u> $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_2}(L^{\rm dp}_{\mathbf{m}_2,\gamma})$ hence $L \subseteq L^{\rm dp}_{\mathbf{m}_1,\gamma}$ implies $\mathbb{P}_{\mathbf{m}_1}(L) = \mathbb{P}_{\mathbf{m}_2}(L)$ (by 1.20(4)).

(b) Let $\mathbf{M}^{\rm uec}_\gamma$ (where ueb stand for unbounded existentially closed) is defined similarly omitting ”bounded”.

(c) Let $\mathbf{M}^{\rm wec}_\gamma$ (where wec stand for weakly bounded existential closed) is defined similarly replacing “bounded” by “weakly bounded”.

(d) We may write $\mathbf{M}^{\rm ec}_\gamma$ for $\mathbf{M}^{\rm uec}_\gamma$.

(2) Let $\mathbf{M}_{\rm ec} = \mathbf{M}^{\rm ec}_\infty$ be the class of $\mathbf{m}$ which $\in \mathbf{M}^{\rm ec}_\gamma$ for every ordinal γ; similarly $\mathbf{M}_{\rm bec} = \mathbf{M}^{\rm bec}_\infty$.

(3) Let $\mathbf{M}^{\rm ec}_{\chi,\gamma} = \{\mathbf{m} \in \mathbf{M}^{\rm ec}_\gamma : |L_{\mathbf{m}}| \leq \chi\}$, similarly $\mathbf{M}^{\rm ec}_{\chi,\infty}$ and for bec.

Observation 1.29 *(1) Of course, $\mathbf{M}^{\rm ec}_{\gamma_2} \subseteq \mathbf{M}^{\rm ec}_{\gamma_1}$ and $L^{\rm dp}_{\mathbf{m},\gamma_1} \subseteq L^{\rm dp}_{\mathbf{m},\gamma_2}$ are initial segments of $L_{\mathbf{m}}$ when $\gamma_1 \leq \gamma_2$.*

(2) In 1.28(1), the following are equivalent:

(a) $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_2}(L^{\rm dp}_{\mathbf{m}_2,\gamma})$ for every γ,

(b) $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$.

(3) If $\mathbf{m} \in \mathbf{M}_{\rm ec}$ and $M_{\mathbf{m}} \models$“$s < t$” (in particular, $s, t \in M_{\mathbf{m}}$) then $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ “$\underset{\sim}{\eta}_s < \underset{\sim}{\eta}_t \mod J^{\rm bd}_\lambda$”. Moreover, if $M_{\mathbf{m}} \models s_i < t$ for $i < i_* \leq \lambda$ and $\mathbf{B}$ is an i_*-place λ-Borel function from $\Pi_\varepsilon\theta_\varepsilon$ into $\Pi_{\varepsilon<\lambda}\theta_\varepsilon$, then $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ “$\mathbf{B}(\ldots, \underset{\sim}{\eta}_{s_i}, \ldots)_{i<i_*} < \eta_i \mod J^{\rm bd}_\lambda$”,

(4) If for every $L \in [L_{\mathbf{m}}]^{\leq\lambda}$ for some $t \in M_{\mathbf{m}}$ we have $L \in \mathscr{P}_{\mathbf{m},t}$ <u>then</u> (see 2.13(3)) $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ “$\{\underset{\sim}{\eta}_t : t \in M_{\mathbf{m}}\}$ is cofinal in $(\Pi_{\varepsilon<\lambda}\theta_\varepsilon)$”.

Remark 1.30 Recall if $\mathbf{m}$ is fat, then $L \in \mathscr{P}_{\mathbf{m},t}$ means $L \subseteq u_{\mathbf{m},t}$.

Proof 1.29 (1) Easy.

(2) First, concerning $(a) \Rightarrow (b)$, note that for γ large enough we have $L^{\rm dp}_{\mathbf{m}_\ell,\gamma} = L_{\mathbf{m}_\ell}$ hence $\mathbb{P}_{\mathbf{m}_\ell}(L^{\rm dp}_{\mathbf{m}_\ell,\gamma}) = \mathbb{P}_{\mathbf{m}_\ell}$, so clear. Second, assume (b), note that $L^{\rm dp}_{\mathbf{m}_\ell,\gamma}$ is an initial segment of $L_{\mathbf{m}_\ell}$ hence $\mathbb{P}_{\mathbf{m}_\ell}(L^{\rm dp}_{\mathbf{m}_\ell,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_\ell}$ for $\ell = 1, 2$ by 1.13(c), hence we have $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$, but $\lessdot$ is transitive, hence $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_2}$. Also $\mathbb{P}_{\mathbf{m}_2}(L^{\rm dp}_{\mathbf{m}_2,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_2}$ and $L^{\rm dp}_{\mathbf{m}_1,\gamma} \subseteq L^{\rm dp}_{\mathbf{m}_2,\gamma}$ by the definition. Hence by the definition $p \in \mathbb{P}_{\mathbf{m}_1}(L^{\rm dp}_{\mathbf{m}_1,\gamma}) \Rightarrow p \in \mathbb{P}_{\mathbf{m}_2}(L^{\rm dp}_{\mathbf{m}_2,\gamma})$; but lastly $(\mathbb{Q}_1 \lessdot \mathbb{P} \wedge \mathbb{Q}_2 \lessdot \mathbb{P} \wedge (\forall p)(p \in \mathbb{Q}_1 \Rightarrow p \in \mathbb{Q}_2) \Rightarrow \mathbb{Q}_1 \lessdot \mathbb{Q}_2$ so we are done.

(3) Easy, as $\mathbf{m} \in \mathbf{M}_{\rm ec}$ its suffice to find $\mathbf{n}$ such that $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ and $\mathbf{n}$ satisfies the conclusion. So given i_*, t, s_i such that $s_i <_{\mathbf{m}} t$ (for $i < i_*$) we define $\mathbf{n} \in \mathbf{M}$ as follows:

(a) the set of elements of $L_{\mathbf{n}}$ are those of $L_{\mathbf{m}}$ and r_*, a new element,

(b) the order $<_{\mathbf{m}}$ is defined by: $r_1 <_{\mathbf{m}} r_2$ <u>iff</u> $r_1 <_{\mathbf{m}} r_2$ or $r_1 \leq_{\mathbf{m}} s_i \wedge r_2 = r_*$ for some $i < i_*$ or $r_1 = r_* \wedge t \leq_{\mathbf{m}} r_2$,

(c) $M_{\mathbf{n}} = M_{\mathbf{m}}$,

(d) $E'_{\mathbf{n}} = \{(r_1, r_2) : (r_1, r_2) \in E'_{\mathbf{m}}$ <u>or</u> $r_1 = r_* \wedge r_2 \in \{s_i : i < i_*\} \cup \{t\}$ <u>or</u> $r_2 = r_* \wedge r_1 \in \{s_i : i < i_*\} \cup \{t\}\}$,

(e) $u_{\mathbf{n},r}$ is:

- $u_{\mathbf{m},r}$ if $r \in L_{\mathbf{m}} \setminus \{t\}$,
- $u_{\mathbf{m},r} \cup \{r_*\}$ if $r = t$,
- $\{s_i : i < i_*\}$ if $r = r_*$.

(f) $\mathscr{P}_{\mathbf{n},r}$ is:

- $\mathscr{P}_{\mathbf{m},r}$ if $r \in L_{\mathbf{m}} \setminus \{t\}$,
- $\mathscr{P}_{\mathbf{m},r} \cup \{\{r_*\}\}$ if $r = t$, except when $t \in M^{\text{fat}}_{\mathbf{m}}$, in which case it us $\mathscr{P}(u_{\mathbf{n},t})$,
- $\mathscr{P}(\{s_i : i < i_*\})$ if $r = r_*$.

(4) Easy by 1.16(1)(β). □

Definition 1.31 Let $\mathbf{m} \in \mathbf{M}$.

(1) We say $\mathbf{m}$ is μ-wide[18] when $\mu \geq \lambda_0$ and for every $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ there are $t_\alpha \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ for $\alpha < \mu$ such that:

(a) $\mathbf{m}{\restriction}(t_\alpha/E_{\mathbf{m}})$ is isomorphic to $\mathbf{m}{\restriction}(t/E_{\mathbf{m}})$ over $M_{\mathbf{m}}$,
(b) $\beta < \gamma < \mu \Rightarrow t_\beta/E''_{\mathbf{m}} \neq t_\gamma/E''_{\mathbf{m}}$.

(1A) We say $\mathbf{m}$ is wide when it is λ_0-wide, see 1.1. We say $\mathbf{m}$ is very wide when it is $|L_{\mathbf{m}}|$-wide.

(2) We say $\mathbf{m}$ is *full* when: if $\mathbf{m}{\restriction}M_{\mathbf{m}} \leq_{\mathbf{M}} \mathbf{n}$ and $E''_{\mathbf{n}}$ has exactly one equivalence class then for some $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$, we have $\mathbf{n}$ is isomorphic to $\mathbf{m}{\restriction}(t/E_{\mathbf{m}})$ over $M_{\mathbf{m}}$. Similarly for $\mathbf{M}_{\text{wbd}}$.

(3) We say $\mathbf{m}$ is *μ-wide* or *full inside* $\mathbf{M}_{\text{bd}}$ when we restrict ourselves to $\mathbf{M}_{\text{bd}}$.

Crucial Claim 1.32 *(1) If $\chi = \chi^\lambda \geq 2^{\lambda_2}$ (see 1.1) and $\mathbf{m} \in \mathbf{M}_{\leq \chi}$ then for some $\mathbf{n}$ we have $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_\chi$ and $\mathbf{n} \in \mathbf{M}_{\text{uec}}$.*

(2) If in addition $\mathbf{m}$ is bounded, then for some $\mathbf{n}$ we have $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_\chi$ and $\mathbf{n} \in \mathbf{M}_{\text{bec}}$.

Proof 1.32 Let $x = u$ for part (1) and $x = b$ for part (2). Let $\mathscr{X} = \mathscr{X}_{\mathbf{m}} = \{\mathbf{n} : \mathbf{n}$ is bounded if $x = b$; and $(\mathbf{m}{\restriction}M_{\mathbf{m}}) \leq_{\mathbf{M}} \mathbf{n}$ and $L_{\mathbf{n}} \setminus M_{\mathbf{m}} = t/E''_{\mathbf{n}}$ for some t, hence $\|L_{\mathbf{n}}\| \leq \lambda_2\}$.

We define a two-place relation $\mathscr{E}$ on $\mathscr{X}$:

$(*)_0$ $\mathbf{n}_1 \mathscr{E} \mathbf{n}_2$ iff ($\mathbf{n}_1, \mathbf{n}_2 \in \mathscr{X}$ and) there is an isomorphism h from $\mathbf{n}_1$ onto $\mathbf{n}_2$ over $\mathbf{m}{\restriction}M_{\mathbf{m}}$, that is: an isomorphism from $L_{\mathbf{n}_1}$ onto $L_{\mathbf{n}_2}$ over $M_{\mathbf{m}}$ (as partial orders) such that:

(a) $t \in L_{\mathbf{n}_1} \Rightarrow u_{\mathbf{n}_2,h(t)} = \{h(s) : s \in u_{\mathbf{n}_1,t}\}$,
(b) $t \in L_{\mathbf{n}_1} \Rightarrow \mathscr{P}_{\mathbf{n}_2,h(t)} = \{\{h(s) : s \in u\} : u \in \mathscr{P}_{\mathbf{n}_1,t}\}$,
(c) $s, t \in L_{\mathbf{n}_1} \Rightarrow (s E'_{\mathbf{n}_1} t \Leftrightarrow h(s) E'_{\mathbf{n}_2} h(t))$.

Clearly $\mathscr{E}$ is an equivalence relation.

By our assumptions $\chi \geq 2^{\lambda_2}$ and $\mathbf{n} \in \mathscr{X} \Rightarrow |L_{\mathbf{n}}| \leq \lambda_2 \wedge (\forall t \in L_{\mathbf{n}})(\mathscr{P}_{\mathbf{n},t} \subseteq [L_{\mathbf{n},<t}]^{\leq \lambda})$, hence recalling $\lambda_2 = (\lambda_2)^\lambda$ clearly $\mathscr{E}$ has $\leq 2^{\lambda_2}$ equivalence classes and let $\langle \mathbf{n}_\alpha : \alpha < 2^{\lambda_2} \rangle$ be a set of representatives (not necessary, but no harm in allowing repetitions).

By 1.20(2) and 1.21 we can find $\mathbf{n}$ such that:

$(*)_1$ (a) $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_\chi$,
(b) for every $\alpha < 2^{\lambda_2}$ we can find $\langle t_{\alpha,i} : i < \chi \rangle$ such that:
(α) $t_{\alpha,i} \in L_{\mathbf{n}} \backslash L_{\mathbf{m}}$,
(β) $(\alpha \neq \beta) \vee (i \neq j) \Rightarrow t_{\alpha,i}/E_{\mathbf{n}} \neq t_{\beta,j}/E_{\mathbf{n}}$,
(γ) $\mathbf{n}{\restriction}(t_{\alpha,i}/E_{\mathbf{n}})$ is $\mathscr{E}$-equivalent to $\mathbf{n}_\alpha$, see 1.7(0) on $t_{\alpha,i}/E_{\mathbf{n}}$.

[18] No real harm if we demand $\mu \geq \lambda_0$ and use λ^+. in part (1A).

We shall now prove that $\mathbf{n}$ is as required. Let $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1 \leq_{\mathbf{M}} \mathbf{n}_2$, and $\mathbf{n}_1, \mathbf{n}_2$ are bounded when $x = b$ and define $\mathscr{F}$ as the set of functions f such that some L_1, L_2 satisfy:

$(*)_2$ (a) $L_\ell \subseteq L_{\mathbf{n}_\ell}$,
(b) $M_{\mathbf{m}} = M_{\mathbf{n}} \subseteq L_1 \cap L_2$,
(c) L_ℓ has cardinality $\leq \lambda_2$,
(d) L_ℓ is $E_{\mathbf{n}_\ell}$-closed, i.e. $M_{\mathbf{m}} \subseteq L_\ell$ and $t \in L_\ell \backslash M_{\mathbf{m}} \Rightarrow t/E_{\mathbf{n}_2} \subseteq L_\ell$,
(e) f is an isomorphism from $\mathbf{n}_2 \restriction L_1$ onto $\mathbf{n}_2 \restriction L_2$ over $M_{\mathbf{m}}$, i.e.:
$\bullet_1$ f is a one-to-one mapping from L_1 onto L_2,
$\bullet_2$ $f \restriction M_{\mathbf{m}}$ is the identity,
$\bullet_3$ f maps $\leq_{\mathbf{n}_2} \restriction L_1$ onto $\leq_{\mathbf{n}_2} \restriction L_2$,
$\bullet_4$ $s E'_{\mathbf{n}_1} t \Leftrightarrow f(s) E'_{\mathbf{n}_2} f(t)$,
$\bullet_5$ for $s, t \in L_1$ we have $s \in u_{\mathbf{n}_2, t} \Leftrightarrow f(s) \in u_{\mathbf{n}_2, f(t)}$.
$\bullet_6$ for $t \in L_1$ we have $\mathscr{P}_{\mathbf{n}_2, f(t)} = \{\{f(s) : s \in u\} : u \in \mathscr{P}_{\mathbf{n}_1, t}, u \subseteq L_1\}$.

Clearly,

$(*)_3$ if $f \in \mathscr{F}$ and $L' \subseteq L_{\mathbf{n}_1}, L'' \subseteq L_{\mathbf{n}_2}$ and $|L'| + |L''| \leq \lambda_2$ <u>then</u> for some $g \in \mathscr{F}$ extending f we have:

(a) $L' \subseteq \mathrm{dom}(g)$,
(b) $L'' \subseteq \mathrm{rang}(g)$,
(c) $\mathrm{rang}(g) \backslash (L'' \cup \mathrm{rang}(f)) \subseteq L_{\mathbf{n}_2}$,
(d) $\mathrm{dom}(g) \backslash (L' \cup \mathrm{dom}(f)) \subseteq L_{\mathbf{n}_1}$.

We can finish as in the parallel of the Tarski-Vaught criterion for $\mathbb{L}_{\infty, \lambda_2^+}$ but we shall elaborate. That is, first we can prove by induction on the ordinal $\gamma < |L_{\mathbf{n}_2}|^+$ (and in fact just $\gamma < \|M_{\mathbf{n}_2}\|^+$) that $(*)_4 - (*)_6$ below holds:

$(*)_4$ letting $L_\gamma = L^{\mathrm{dp}}_{\mathbf{n}_2, \gamma}$, if $g \in \mathscr{F}$ <u>then</u>:

(a) g maps $\mathrm{dom}(g) \cap L_\gamma$ onto $\mathrm{rang}(g) \cap L_\gamma$,
(b) g induces an isomorphism $\hat{g}$ from $\mathbb{P}_{\mathbf{n}_2}(\mathrm{dom}(g) \cap L_\gamma)$ onto $\mathbb{P}_{\mathbf{n}_2}(\mathrm{rang}(g) \cap L_\gamma)$, that is: $\hat{g}(p) = q$ <u>iff</u>:
(α) $p \in \mathbb{P}_{\mathbf{n}_2}(\mathrm{dom}(g) \cap L_\gamma)$,
(β) $q \in \mathbb{P}_{\mathbf{n}_2}(\mathrm{rang}(g) \cap L_\gamma)$,
(γ) g maps $\mathrm{dom}(p)$ onto $\mathrm{dom}(q)$ and $s \in \mathrm{dom}(p) \Rightarrow \mathrm{tr}(p(s)) = \mathrm{tr}(q(g(s)))$,
(δ) if $s \in \mathrm{dom}(g)$, $g(s) = t \in \mathrm{rang}(g)$ and $\underset{\sim}{f}_{p(s)} = \mathbf{B}_{p(s)}(\ldots, \underset{\sim}{\eta}_{r_{p(s)}(\zeta)}, \ldots)_{\zeta < \xi_{p(s)}}$ and $\underset{\sim}{f}_{q(t)} = \mathbf{B}_{q(t)}(\ldots, \underset{\sim}{\eta}_{r_{q(t)}(\zeta)}, \ldots)_{\zeta < \xi_{q(t)}}$ <u>then</u> $\xi_{q(t)} = \xi_{p(s)}$, $\mathbf{B}_{q(t)} = \mathbf{B}_{p(s)}$ and $\zeta < \xi_{p(s)} \Rightarrow r_{q(t)}(\zeta) = g(r_{p(s)}(\zeta))$,
(ε) moreover in (δ) we have $\iota(s, p) = \iota(t, q)$ and if $\iota < \iota(s, p)$ then $w_{p,s,\iota} = w_{q,t,\iota}$, $\mathbf{B}_{p(s),\iota} = \mathbf{B}_{q(t),\iota}$.

[Why? We use freely 1.16(9). Let χ_* be such that $\gamma, g, \mathbf{n}, \mathbf{n}_1, \mathbf{n}_2 \in \mathscr{H}(\chi_*)$. Let $\mathfrak{A} \prec (\mathscr{H}(\chi_*), \in)$ be such that $\gamma, g, \mathbf{n}, \mathbf{n}_1, \mathbf{n}_2 \in \mathfrak{A}$, $\|\mathfrak{A}\| = \chi$, $\chi + 1 \subseteq \mathfrak{A}$ and $[\mathfrak{A}]^{\leq \lambda} \subseteq \mathfrak{A}$, (hence $\mathfrak{A} \prec_{\mathbb{L}_{\lambda^+, \lambda^+}} (\mathscr{H}(\chi_*), \in)$).
For $\ell = 1, 2$ let $L^*_\ell = L_{\mathbf{n}_\ell} \cap \mathfrak{A}$ and $\mathbf{n}^*_\ell = \mathbf{n}_\ell \restriction L^*_\ell$, so by absoluteness $\mathbb{P}_{\mathbf{n}^*_\ell}(L_{\mathbf{n}^*_\ell}) = \mathbb{P}_{\mathbf{n}_\ell}(L_{\mathbf{n}^*_\ell})$ hence $\mathbb{P}_{\mathbf{n}^*_\ell}(L_{\mathbf{n}^*_\ell}) \lessdot \mathbb{P}_{\mathbf{n}_\ell}(L_{\mathbf{n}_\ell})$. By the choice of $\mathbf{n}$ as very wide and full (see Definition 1.31), also $\mathbf{n} \restriction (\mathfrak{A} \cap L_{\mathbf{n}})$ is very wide and full of cardinality χ. But we have $\mathbf{n} \restriction (\mathfrak{A} \cap L_{\mathbf{n}}) \leq \mathbf{n}_2 \restriction (\mathfrak{A} \cap L_{\mathbf{n}_2})$ both of cardinality χ hence also $\mathbf{n}^*_2$ is very wide and full (see Definition 1.5) of cardinality χ. Now easily g can be extended to an automorphism of $\mathbf{n}^*_2$. The promised statement now follows.]
Second,

$(*)_5$ $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1}) \lessdot \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$.

[Why? By[19] the definitions and the induction hypothesis $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1}) \subseteq \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$ as quasi orders.

Also if $p_1, p_2 \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ are compatible in $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$ let $q \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$ be a common upper bound there. We can find an $E_{\mathbf{n}_2}$-closed $L' \subseteq L_{\mathbf{n}_1}$ of cardinality $\leq \lambda_2$ (recalling $\mathbf{n} \in \mathscr{X} \Rightarrow |L_\mathbf{n}| \leq \lambda_2$) such that $p_1, p_2 \in \mathbb{P}_{\mathbf{n}_1}(L')$ and $E_{\mathbf{n}_2}$-closed $L'' \subseteq L_{\mathbf{n}_1}$ of cardinality $\leq \lambda_2$ such that $L' \subseteq L''$ and $q \in \mathbb{P}_{\mathbf{n}_2}(L'')$. Now we can find $f_1 \in \mathscr{F}$ such that $\mathrm{dom}(f_1) = \cup\{t/E_{\mathbf{n}_2} : t \in L'\} \cup M_\mathbf{m}$ recalling that $t/E_\mathbf{m} \supseteq M_\mathbf{m}$, see 1.7(0) and f_1 is the identity. Then by $(*)_3$ we can find $f_2 \in \mathscr{F}$ extending f_1 with $\mathrm{dom}(f_2) = \cup\{t/E_{\mathbf{n}_2} : t \in L''\}$ and $\mathrm{rang}(f_2)\backslash\mathrm{rang}(f_1) \subseteq L_{\mathbf{n}_1}$. So recalling $(*)_4(b)$ applied to f we have $\mathbb{P}_{\mathbf{n}_2} \models$ "$(p_1 \leq \hat{f}_2(q)) \wedge (p_2 \leq \hat{f}_2(q))$" and $\hat{f}_2(q) \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ recalling $(*)_4$. So p_1, p_2 are compatible also in $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$. Obviously, if $p_1, p_2 \in \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ are compatible in $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$, say, q witnesses, then q is a common upper bound of p_1, p_2 in $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$.

So every antichain of $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ is an antichain of $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$. Similarly to the above every maximal antichain of $\mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1})$ is a maximal antichain of $\mathbb{P}_{\mathbf{n}_2}(L_\gamma)$; similarly for the other direction. So we are done.]

$(*)_6$ $\mathbb{P}_{\mathbf{n}_1}(L_\gamma \cap L_{\mathbf{n}_1}) = \mathbb{P}_{\mathbf{n}_2}(L_\gamma \cap L_{\mathbf{n}_1}) \lessdot \mathbb{P}_{\mathbf{n}_2}(L_\gamma)$.

[Why? We prove this by induction on γ, as in proving the Tarski-Vaught criterion is sufficient (we shall elaborate later in the proof of 3.20, more specifically $\boxplus_4$ proves a similar statement in detail with weaker assumptions).]

Hence (using $\gamma = |L_{\mathbf{n}_2}|^+$),

$(*)_7$ $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{n}_2}$.

Hence for every $L \subseteq L_{\mathbf{n}_1}$ by 1.20(4) we have $\mathbb{P}_{\mathbf{n}_1}(L) = \mathbb{P}_{\mathbf{n}_2}(L)$ as required for $\mathbf{n} \in \mathbf{M}_{\mathrm{ec}}$, see Definition 1.28. □

Definition 1.33 (1) For $\mathbf{m} \in M$, let $\mathbf{n} = \mathbf{m}^{[\mathrm{bd}]}$ be $\mathbf{m}{\restriction}L^{\mathrm{bd}}_\mathbf{m}$, where $L^{\mathrm{bd}}_\mathbf{m} = \{s \in L_\mathbf{m}$: for some $t \in M_\mathbf{m}$ we have $s/E'_\mathbf{m} \subseteq L_{\mathbf{m}(\leq t)}$ or just for some $\mathscr{X} \in [M_\mathbf{m}]^{\leq\lambda}$ we have $s/E'_\mathbf{m} \subseteq \bigcup\{L_{\mathbf{m}(\leq t)} : t \in X\}\}$.

(1A) For $\mathbf{m} \in \mathbf{M}$, let $\mathbf{n} := \mathbf{m}^{[\mathrm{wbd}]}$ be $\mathbf{m}{\restriction}L^{\mathrm{wbd}}_\mathbf{m}$, where $L^{\mathrm{wbd}}_\mathbf{m} := \bigcup\{L_{\mathbf{m}(\leq t)} : t \in M_\mathbf{m}\}$.

(2) Assume $\mathbf{n}_1 \leq_\mathbf{M} \mathbf{m}_1$, $\mathbf{n}_1 \leq_\mathbf{M} \mathbf{n}_2$ and $L_{\mathbf{n}_2} \cap L_{\mathbf{m}_1} = L_{\mathbf{n}_1}$. Then let $\mathbf{m}_2 = \mathbf{n}_2 \oplus_{\mathbf{n}_1} \mathbf{m}_1$ be defined by:

(a) the set of elements of $L_{\mathbf{m}_2}$ is $L_{\mathbf{m}_1} \cup L_{\mathbf{m}_2}$,
(b) $<_{\mathbf{m}_2}$ is the transitive closure of $<_{\mathbf{n}_2} \cup <_{\mathbf{m}_1}$,
(c) $E'_{\mathbf{m}_2} = E'_{\mathbf{n}_2} \cup E'_{\mathbf{m}_1}$, $M_{\mathbf{m}_2} = M_{\mathbf{n}_1}$ and $M^{\mathrm{fat}}_{\mathbf{m}_2} = M^{\mathrm{fat}}_{\mathbf{n}_1}$, $M^{\mathrm{lean}}_{\mathbf{m}_2} = M^{\mathrm{lean}}_{\mathbf{n}_1}$
(d) $u_{\mathbf{m}_2,t}$ is:

- $u_{\mathbf{m}_2,t}$ if $t \in L_{\mathbf{n}_2} \setminus L_{\mathbf{n}_1}$,
- $u_{\mathbf{m}_1,t}$ if $t \in L_{\mathbf{m}_1} \setminus L_{\mathbf{n}_1}$,
- $u_{\mathbf{n}_2} \cup u_{\mathbf{m}_1,t}$ if $t \in L_{\mathbf{m}_1}$ (so in $u_{\mathbf{n}_1,t}$ if $L \in L_{\mathbf{m}_1} \setminus M_{\mathbf{m}_0}$).

(e) $\mathscr{P}_{\mathbf{m},t}$ is defined naturally, that is:

- $\mathscr{P}_{\mathbf{m}_2,t}$ if $t \in L_{\mathbf{n}_2} \setminus L_{\mathbf{n}_1}$,
- $\mathscr{P}_{\mathbf{n}_2,t}$ if $t \in L_{\mathbf{n}_2} \setminus L_{\mathbf{n}_1}$,
- $\mathscr{P}_{\mathbf{n}_2} \cup \mathscr{P}_{\mathbf{m}_1,t}$ if $t \in L_{\mathbf{m}_1}$ except when $t \in M^{\mathrm{fat}}_{\mathbf{m}_1}$ (so in $\mathscr{P}_{\mathbf{n}_1,t}$ if $L \in L_{\mathbf{m}_1} \setminus M_{\mathbf{m}_0}$),

[19] Can repeat the proof of $(*)_4$ but for variety we give another proof.

- $[u_{\mathbf{m}_2,t}]^{\leq\lambda}$ if $t \in M^{\rm fat}_{\mathbf{m}_1}$.

Claim 1.34 *(1) In 1.33(1) indeed* $\mathbf{m}^{[\rm bd]} \in \mathbf{M}$ *and moreover it is bounded.*

(2) If $\mathbf{m} \in \mathbf{M}$, $\mathbf{m}$ *is bounded* <u>*iff*</u> $\mathbf{m} = \mathbf{m}^{[\rm bd]}$.

(3) In 1.33(2) indeed $\mathbf{m}_2 = \mathbf{n}_2 \underset{\mathbf{n}_1}{\oplus} \mathbf{m}_1$ *belongs to* M, $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$ *and* $\mathbf{n}_1 = \mathbf{m}_1^{[\rm bd]} \leq_{\mathbf{M}} \mathbf{n}_2^{[\rm bd]} \Rightarrow \mathbf{m}_2^{[\rm bd]} = \mathbf{n}_2^{[\rm bd]}$.

(4) In 1.33(1) we can add $\mathbf{n}^{[\rm bd]} \in M_{\rm bec}$.

Proof 1.34 Easy, e.g.

For part (3) we are given $\mathbf{m} \in \mathbf{M}$ and let $\mathbf{n}$ be as constructed above for $x = u$. Clearly $\mathbf{n}^{[\rm bd]}$ is as constructed above for $x = b$, so we are done. □

2 The Corrected $\mathbb{P}_{\mathbf{m}}$

Discussion 2.1 Here for $L \subseteq L_{\mathbf{m}}$, we define $\mathbb{P}_{\mathbf{m}}[L]$, the complete subforcing of the completion of $\mathbb{P}_{\mathbf{m}}$ generated by $\langle \underset{\sim}{\eta}_s : s \in L\rangle$, the central case is $L = M_{\mathbf{m}}$, of course.

Definition 2.2 Let $\mathbb{P}$ be a forcing notion and $Y \subseteq \mathbb{P}$ and χ a regular cardinal.

(1) Let $\mathbb{L}_\chi(Y)$ be the set of sentences formed from $\{p : p \in \mathbb{P}\}$ closing under the operations $\neg p$ and $\bigwedge_{i<\alpha} p_i$, for $\alpha < \chi$; so (infinitary) propositional logic.

(2) For a directed $\mathbf{G} \subseteq \mathbb{P}$ and $\psi \in \mathbb{L}_\chi(Y)$ we define the truth value $\psi[\mathbf{G}]$ naturally (by induction on ψ starting with $p[\mathbf{G}] = $ true $\Leftrightarrow p \in \mathbf{G}$).

(3) Let $\mathbb{L}^+_\chi(Y, \mathbb{P})$, the $\mathbb{L}_\chi$-closure of Y for $\mathbb{P}$, (where $Y \subseteq \mathbb{P}$; if $Y = \mathbb{P}$ we may omit Y) be the following partial order:
   - set of elements $\{\psi \in \mathbb{L}_\chi(Y, \mathbb{P}) : \nVdash_{\mathbb{P}}$ "$\psi[\underset{\sim}{\mathbf{G}}] = $ false"$\}$,
   - the order $\psi_1 \leq \psi_2$ iff $\Vdash_{\mathbb{P}}$ "if $\psi_2[\underset{\sim}{\mathbf{G}}] = $ true then $\psi_1[\underset{\sim}{\mathbf{G}}] = $ true".

(4) The completion of $\mathbb{P}$ is the $\mathbb{L}_\chi$-closure of $\mathbb{P}$ which is $\mathbb{L}^+_\chi(\mathbb{P}) = \mathbb{L}^+_\chi(\mathbb{P}, \mathbb{P})$ where χ is minimal such that $\mathbb{P}$ satisfies the χ-c.c.

Claim 2.3 *For a cardinal* χ *and forcing notion* $\mathbb{P}$ *and* $Y \subseteq \mathbb{P}$ *we have:*

(a) $\mathbb{L}^+_\chi(Y, \mathbb{P})$ *is a forcing notion,*

(b) $\mathbb{P} \lessdot \mathbb{L}^+_\chi(\mathbb{P})$ *under the natural identification*[20],

(c) $\mathbb{L}^+_\chi(Y, \mathbb{P}) \lessdot \mathbb{L}^+_\chi(\mathbb{P})$,

(d) $\mathbb{L}^+_{\chi_1}(Y, \mathbb{P}) \lessdot \mathbb{L}^+_{\chi_2}(Y, \mathbb{P})$ *when* $\chi_1 \leq \chi_2$ *are regular,*

(e) if $\mathbb{P}$ *satisfies the* χ_1*-c.c. and* $\chi_1 < \chi_2$ *are regular,* <u>*then*</u> $\mathbb{L}^+_{\chi_1}(Y, \mathbb{P})$ *is essentially equal to* $\mathbb{L}^+_{\chi_2}(Y, \mathbb{P})$*, i.e. up to the natural equivalence of elements in a quasi order,*

(f) if $Y = \mathbb{P}$ *then* $\mathbb{P}$ *is a dense subset of* $\mathbb{L}^+_\chi(\mathbb{P})$.

Proof 2.3 Easy. □

Definition 2.4 Let $\mathbf{m} \in \mathbf{M}$.

(1) For $t \in L_{\mathbf{m}}$, $\varepsilon < \lambda$ and $\eta \in \prod_{i<\varepsilon} \theta_i$ let $p = p^*_{t,\eta} \in \mathbb{P}_{\mathbf{m}}$ be the function with domain $\{t\}$ such that $p(t) = (\eta, \eta\hat{}0_\lambda)$, i.e. $f_{p(t)} \in \prod_{i<\lambda} \theta_i$ is defined by $f_{p(t)}(\varepsilon)$ is $\eta(\varepsilon)$ if $\varepsilon < \ell g(\eta)$ and is zero otherwise.

[20] Pedantically $\mathbb{P} \lessdot' \mathbb{L}^+_{\chi_1}[\mathbb{P}]$, see 2.4(8), because $\mathbb{L}^+_\chi[\mathbb{P}] \models$ "$p \leq q$" <u>iff</u> $q \Vdash_{\mathbb{P}}$ "$p \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}}$".

(2) For $L \subseteq L_{\mathbf{m}}$ let $Y_L = Y_{\mathbf{m},L} = \{p^*_{t,\eta} : t \in L$ and $\eta \in \prod_{\varepsilon<\zeta} \theta_\varepsilon$ for some $\zeta < \lambda\}$.
(3) For $L \subseteq L_{\mathbf{m}}$ let $\mathbb{P}_{\mathbf{m}}[L]$ be $\mathbb{L}^+_{\lambda_0}(Y_L, \mathbb{P}_{\mathbf{m}})$, see Definition 2.2(3) and Hypothesis 1.4(4) on λ_0.
(4) For $L \subseteq L_{\mathbf{m}}$ let $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{m}} \restriction \{p \in \mathbb{P}_{\mathbf{m}} : \mathrm{fsupp}(p) \subseteq L\}$, see Definition 1.12(1), recalling 1.12(2),(3).
(5) $\mathbb{P}'_{\mathbf{m}}$ is the partial order with the same set of elements as $\mathbb{P}_{\mathbf{m}}$ and $\leq_{\mathbb{P}'_{\mathbf{m}}} = \{(p,q) : p, q \in \mathbb{P}_{\mathbf{m}}$ and no r above q is incompatible with $p\}$ and $\mathbb{P}'_{\mathbf{m}}(L) = \mathbb{P}'_{\mathbf{m}} \restriction \{p \in \mathbb{P}_{\mathbf{m}} : \mathrm{fsupp}(p) \subseteq L\}$, we may "forget" the distinction[21].
(6) For quasi orders $\mathbb{Q}_1, \mathbb{Q}_2$ let $\mathbb{Q}_1 \subseteq' \mathbb{Q}_2$ mean that:

  (a) $s \in \mathbb{Q}_1 \Rightarrow s \in \mathbb{Q}_2$
  (b) $s \leq_{\mathbb{Q}_1} t \Rightarrow s \leq_{\mathbb{Q}_2} t$.

(7) For quasi orders $\mathbb{Q}_1, \mathbb{Q}_2$ let $\mathbb{Q}_1 \subseteq'_{\mathrm{ic}} \mathbb{Q}_2$ means that $\mathbb{Q}_1 \subseteq' \mathbb{Q}_2$ and

  (c) if $s, t \in \mathbb{Q}_1$ are incompatible in $\mathbb{Q}_1$ then they are incompatible in $\mathbb{Q}_2$.

(8) We define $\lessdot'$ similarly, that is $\mathbb{Q}_1 \subseteq'_{\mathrm{ic}} \mathbb{Q}_2$ and every maximal antichain of $\mathbb{Q}_1$ is a maximal antichain of $\mathbb{Q}_2$.
(9) Let $\mathbb{Q}_1 \subseteq'_{\mathrm{eq}} \mathbb{Q}_2$ means that $\mathbb{Q}_1 \lessdot' \mathbb{Q}_2$ and for every $p \in \mathbb{Q}_2$ there is $q \in \mathbb{Q}_1$ equivalent to it which means $\Vdash_{\mathbb{Q}_2}$ "$p \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_2}$ <u>iff</u> $q \in \underset{\sim}{\mathbf{G}}_{\mathbb{Q}_1}$.

Claim 2.5 *Let $\mathbf{m} \in \mathbf{M}$ and $L \subseteq L_{\mathbf{m}}$.*

(1) $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ is equivalent to $\mathbb{P}_{\mathbf{m}}$ as forcing notions, in fact, $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}}) \lessdot \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ and is a dense subset of it under the natural identification (see 2.2(1)), but we should pedantically use $\mathbb{P}'_{\mathbf{m}}(L_{\mathbf{m}})$ or use $\lessdot'$.
(2) $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ is $(< \lambda)$-strategically complete and is λ^+-c.c.
(3) $\mathbb{P}_{\mathbf{m}}(L) \subseteq \mathbb{P}_{\mathbf{m}}[L]$ as sets and $\mathbb{P}_{\mathbf{m}}[L] \lessdot \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ and $\mathbb{P}_{\mathbf{m}}(L) \subseteq' \mathbb{P}_{\mathbf{m}}[L]$.
(4) If $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{m}}$ is generic over $\mathbf{V}$ and $\eta_t = \underset{\sim}{\eta}_t[\mathbf{G}]$ for $t \in L_{\mathbf{m}}$ and $\mathbf{G}^+ = \{\psi \in \mathbb{L}_{\lambda^+}(Y_{L_{\mathbf{m}}}, \mathbb{P}_{\mathbf{m}}) : \psi[\mathbf{G}] = \mathrm{true}\}$, see 2.2(2)(3), <u>then</u> $\mathbf{V}[\tilde{\mathbf{G}}] = \mathbf{V}[\mathbf{G}^+] = \mathbf{V}[\langle \eta_t : t \in L_{\mathbf{m}}\rangle]$.
(5) In part (4), moreover $\mathbf{G}^+$ is a subset of $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ generic over $\mathbf{V}$.
(6) $\mathbb{P}_{\mathbf{m}}(L_1) \subseteq \mathbb{P}_{\mathbf{m}}(L_2)$ and $\mathbb{P}_{\mathbf{m}}[L_1] \lessdot \mathbb{P}_{\mathbf{m}}[L_2]$ <u>when</u> $L_1 \subseteq L_2 \subseteq L_{\mathbf{m}}$.
(7) If $\mathbf{m}, \mathbf{n} \in \mathbf{M}$ are equivalent <u>then</u> $\mathbb{P}_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$ and $\mathbb{P}_{\mathbf{m}}(L) = \mathbb{P}_{\mathbf{n}}(L)$ for $L \subseteq L_{\mathbf{m}}$.
8) $[(> \lambda)$-continuity$]$ Assume I_* to be a λ^+-directed partial order and $\bar{L} = \langle L_r : r \in I_*\rangle$ be such that $r \in I_* \Rightarrow L_r \subseteq L_{\mathbf{m}}$ and $r <_{I_*} s \Rightarrow L_r \subseteq L_s$ and $L = \cup\{L_r : r \in I_*\}$. <u>Then</u>, as sets and moreover as partial orders $\mathbb{P}_{\mathbf{m}}[L] = \cup\{\mathbb{P}_{\mathbf{m}}[L_r] : r \in I_*\}$ and $\mathbb{P}_{\mathbf{m}}(L) = \cup\{\mathbb{P}_{\mathbf{m}}(L_r) : r \in I_*\}$.
(9) If $\mathbf{m} \in \mathbf{M}_{\mathrm{ec}}$ and $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$ <u>then</u> $\mathbb{P}_{\mathbf{m}_1}[L_{\mathbf{m}}] = \mathbb{P}_{\mathbf{m}_2}[L_{\mathbf{m}}]$.
(10) The sequence $\bar{\eta}_L = \langle \eta_s : s \in L\rangle$ is a generic for $\mathbb{P}_{\mathbf{m}}[L]$, that is: if $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{m}}[L]$ is generic over $\mathbf{V}$ and $\nu_s = \underset{\sim}{\eta}_s[\tilde{\mathbf{G}}]$ for $s \in L$ then:

  (a) $\mathbf{V}[\mathbf{G}] = \mathbf{V}[\langle \nu_s : s \in L\rangle]$,
  (b) $\bar{\nu} = \langle \nu_s : s \in L\rangle$ determines $\mathbf{G}$ uniquely.

Remark 2.6 What about $\mathbb{P}_{\mathbf{m}}(L) \subseteq'_{\mathrm{ic}} \mathbb{P}_{\mathbf{m}}[L]$ and $\mathbb{P}_{\mathbf{m}}(L) \lessdot' \mathbb{P}_{\mathbf{m}}[L]$?

Concerning the second, there may be a maximal antichain $\langle p_i : i < i_*\rangle$ of $\mathbb{P}(L)$, but some $q \in \mathbb{P}_{\mathbf{m}}$ is incompatible with p_i for $i < i_*$. This witness $\neg(\mathbb{P}_{\mathbf{m}}(L) \lessdot \mathbb{P}_{\mathbf{m}})$ hence $\neg(\mathbb{P}_{\mathbf{m}}(L) \lessdot \mathbb{P}_{\mathbf{m}}[L])$. Concerning the first $(\mathbb{P}_{\mathbf{m}}(L) \subseteq'_{\mathrm{ic}} \mathbb{P}_{\mathbf{m}}[L])$ easily it holds. Note that $(\mathbb{P}_{\mathbf{m}}(L) \subseteq \mathbb{P}_{\mathbf{m}}[L])$ may fail as explained earlier as maybe $q \Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$p \in \underset{\sim}{\mathbf{G}}$" but $\nleq_{\mathbb{P}_{\mathbf{m}}} q$, see 1.7(9) and 1.22.

[21] Really the only difference is the possibility that $\mathrm{dom}(p) \nsubseteq \mathrm{dom}(q)$, see 1.22.

Proof 2.5

(1) Easy.

(2) Follows by part (1) and 1.16.

(3) The first statement by their definitions, the second statement by part (1).

For the third clause, "$\mathbb{P}_{\mathbf{m}}[L] \subseteq' \mathbb{P}_{\mathbf{m}}(L)$", note that:

$(*)_1$ if $p, q \in \mathbb{P}_{\mathbf{m}}(L)$, then $\mathbb{P}_{\mathbf{m}}(L) \models$ "$p \leq q$" iff $\mathbb{P}_{\mathbf{m}} \models$"$p \leq q$" which implies $\mathbb{P}_{\mathbf{m}}[L] \models$ "$p \leq q$" by the definition of $\mathbb{P}_{\mathbf{m}}[L]$.

$(*)_2$ if $p, q \in \mathbb{P}_{\mathbf{m}}(L)$ and $\text{dom}(p) \subseteq \text{dom}(q)$, then $\mathbb{P}_{\mathbf{m}}(L) \models$"$p \leq q$" iff $\mathbb{P}_{\mathbf{m}} \models$"$p \leq q$" iff $\mathbb{P}_{\mathbf{m}}[L] \models$"$p \leq q$".

[The first "iff" by the definition of $\mathbb{P}_{\mathbf{m}}(L)$, the second "iff" by 1.22.]

4), 5), 6) Should be clear recalling 1.16(7).

7) Easy, recalling 1.16(5).

(8), 9) Easy.

(10) By the definition of $\mathbb{P}_{\mathbf{m}}[L]$. □

The Uniqueness Claim 2.7 *There is an isomorphism from $\mathbb{P}_{\mathbf{m}_1}[M_1]$ onto $\mathbb{P}_{\mathbf{m}_2}[M_2]$ which (recalling Definition 2.4(1)) maps $p^*_{t,\eta}$ to $p^*_{h(t),\eta}$ for $t \in M_1, \eta \in \cup\{\prod_{\varepsilon<\zeta} \theta_\varepsilon : \zeta < \lambda\}$ when:*

⊞ (a) $\mathbf{m}_\ell \in \mathbf{M}^{\text{ec}}_\infty$ *for* $\ell = 1, 2$,
  (b) $M_\ell = M_{\mathbf{m}_\ell}$ *for* $\ell = 1, 2$,
  (c) h *is an isomorphism from* $\mathbf{m}_1 \restriction M_1$ *onto* $\mathbf{m}_2 \restriction M_2$.

Proof 2.7 By renaming without loss of generality $M_1 = M_2$ call it M and h is the identity and $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = M$. Let $\mathbf{m}_0 = \mathbf{m}_1 \restriction M = \mathbf{m}_2 \restriction M$ so $\mathbf{m}_0 \leq_{\mathbf{M}} \mathbf{m}_\ell$ for $\ell = 1, 2$ and $L_{\mathbf{m}_0} = L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2}$.

By 1.21, there is $\mathbf{m}$ such that $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}$ and $\mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{m}$. As $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}^{\text{ec}}_\infty$ by 2.5(9) we have $\mathbb{P}_{\mathbf{m}_1}[M] = \mathbb{P}_{\mathbf{m}}[M]$ and $\mathbb{P}_{\mathbf{m}_2}[M] = \mathbb{P}_{\mathbf{m}}[M]$ so together we get the desired conclusion. □

Definition 2.8 (1) We call $\mathbf{m} \in \mathbf{M}$ reduced when $L_{\mathbf{m}} = M_{\mathbf{m}}$. We call $\mathbf{m}$ unary when the equivalence relation $E''_{\mathbf{m}}$ has exactly one equivalence class.

(2) For $\mathbf{m} \in \mathbf{M}$ let $\mathbb{P}^{\text{cor}}_{\mathbf{m}}$ be $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{m}}]$ and $\mathbb{P}^{\text{cor}}_{\mathbf{m}}[L]$ be $\mathbb{P}_{\mathbf{n}}[L]$ for $L \subseteq L_{\mathbf{m}}$ <u>when</u> $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_{\text{ec}}$.

Remark 2.9 (1) Why is $\mathbb{P}^{\text{cor}}_{\mathbf{m}}[L]$ well defined? see below.

(2) Here "cor" stands for corrected.

The interest in the definition is because:

Claim 2.10 *(1) If $\mathbf{m} \in \mathbf{M}$ and $L \subseteq L_{\mathbf{m}}$ <u>then</u> $\mathbb{P}^{\text{cor}}_{\mathbf{m}}[L]$ is well defined.*

(2) $\mathbb{P}^{\text{cor}}_{\mathbf{m}}[M_{\mathbf{m}}]$ is well defined and depends only on $\mathbf{m} \restriction M_{\mathbf{m}}$.

(3) If $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ and $L_1 \subseteq L_2 \subseteq L_{\mathbf{m}}$ <u>then</u> $\mathbb{P}^{\text{cor}}_{\mathbf{m}}[L_1] = \mathbb{P}^{\text{cor}}_{\mathbf{n}}[L_1] \lessdot \mathbb{P}^{\text{cor}}_{\mathbf{n}}[L_2] \lessdot \mathbb{P}^{\text{cor}}_{\mathbf{n}}$.

(4) Assume $\mathbf{m}$ is bounded and $\mathbf{m} \leq_M \mathbf{n} \in \mathbf{M}_{\text{bec}}$. If $L \subseteq L_{\mathbf{m}}$ then $\mathbb{P}^{\text{cor}}_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$.

(5) Assume $\mathbf{m}$ is weakly bounded and $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_{\text{wec}}$. If $L \subseteq L_{\mathbf{m}}$ then $\mathbb{P}^{\text{cor}}_{\mathbf{m}}[L] = \mathbb{P}_{\mathbf{n}}[L]$.

(6) If $\mathbf{n} \in \mathbf{M}_{\text{wec}}$ then $\mathbf{m} \in \mathbf{M}_{\text{ec}}$.

Proof 2.10, each L_a is an initial segment of $L_{\mathbf{m}}$

(1) By 1.32, $\mathbb{P}^{\text{cor}}_{\mathbf{m}}[L]$ has at least one definition so it suffices to prove uniqueness. So assume $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_\ell \in \mathbf{M}_{\text{ec}}$ for $\ell = 1, 2$ and we should prove that $\mathbb{P}_{\mathbf{m}_1}[L] = \mathbb{P}_{\mathbf{m}_2}[L]$. Without loss of generality $L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2} = L_{\mathbf{m}}$. Now by 1.21 we can find $\mathbf{n} \in \mathbf{M}$ such that $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{n}$ and $\mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{n}$; as $\mathbf{m}_\ell \in \mathbf{M}_{\text{ec}}$ see Definition 1.28 we have $\mathbb{P}_{\mathbf{m}_\ell} \lessdot \mathbb{P}_{\mathbf{n}}$ for $\ell = 1, 2$. As in the end of the proof of 2.7 we are done.

Sh:1126

(2) By 2.7.
(3) Follows from Definition 1.28(2) and 2.8(2).
(4) On the one hand, we can find $\mathbf{m}_1 \in \mathbf{M}_{\rm bec}$ such that $\mathbf{n} \le_{\mathbf{M}} \mathbf{m}_1$ by 1.32(2). On the other hand, can find $\mathbf{m}_3 \in \mathbf{M}_{\rm ec}$ such that $\mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}_3$ by 1.32(1). Let $\mathbf{m}_2 = \mathbf{m}_3^{[\rm bd]}$ and let $\mathbf{m}_0 = \mathbf{m}$ so $\mathbf{m}_0 \le_{\mathbf{M}} \mathbf{m}_1 \le_{\mathbf{M}} \mathbf{m}_2 \le_{\mathbf{M}} \mathbf{m}_3$. By the choice of $\mathbf{m}_1$ we have

- $\mathbb{P}_{\mathbf{m}_1}[L] = \mathbb{P}_{\mathbf{m}_2}[L] \lessdot \mathbb{P}_{\mathbf{m}_2}$.

As $L_{\mathbf{m}_2}$ is an initial segment of $L_{\mathbf{m}_3}$, clearly,

- $\mathbb{P}_{\mathbf{m}_2} \lessdot \mathbb{P}_{\mathbf{m}_3}$ so $\mathbb{P}_{\mathbf{m}_2}[L] = \mathbb{P}_{\mathbf{m}_3}[L]$.

Lastly as $\mathbf{m}_3 \in \mathbf{M}_{\rm ec}$, $\mathbb{P}_{\mathbf{m}_3}[L] = \mathbb{P}^{\rm cor}_{\mathbf{m}}[L]$. Together we are done.
(5) Similarly to part (4).
(6) Easy.

□

Discussion 2.11 (1) But we like to prove for reduced $\mathbf{m} \in \mathbf{M}$ and $M \subseteq M_{\mathbf{m}}$ that $\mathbb{P}^{\rm cor}_{\mathbf{m}\restriction M} \lessdot \mathbb{P}^{\rm cor}_{\mathbf{m}}$, this is the whole point of the corrected iteration. This is delayed to 3.27. We now prove that this suffices.
(2) Conclusion 2.12 below is the desired conclusion but it relies on §3, specifically on 3.27 (or §4A).
(3) The reader may understand 2.12 without reading the rest of §2, §3 by ignoring clause (A)(d), or reading 2.2, 2.3.
(4) By 2.10(4) we may restrict ourselves to $\mathbf{M}_{\rm bd}$. We use it freely.

Conclusion 2.12 *For every ordinal δ_* there is $\mathbf{q} = \langle \mathbb{P}_\alpha, \underset{\sim}{\eta}_\beta : \alpha \le \delta_*, \beta < \delta_* \rangle$ such that:*

(A) (a) $\langle \mathbb{P}_\alpha : \alpha \le \delta_* \rangle$ is $\lessdot$-increasing sequence of forcing notions,
(b) $\underset{\sim}{\eta}_\alpha$ is a $\mathbb{P}_{\alpha+1}$-name of a member of $\prod_{\varepsilon<\lambda} \theta_\varepsilon$ which dominates $(\prod_{\varepsilon<\lambda} \theta_\varepsilon)^{\mathbf{V}[\mathbb{P}_\alpha]}$,
(c) $\underset{\sim}{\eta}_\alpha$ is a generic for $\mathbb{P}_{\alpha+1}/\mathbb{P}_\alpha$, moreover $\langle \underset{\sim}{\eta}_\beta : \beta < \alpha \rangle$ is a generic for $\mathbb{P}_\alpha$,
(d) $\mathbb{P}_\alpha \lessdot' \mathbb{L}^+_{\lambda_0}(Y_\alpha, \mathbb{P}_\alpha)$ in fact $\mathbb{P}_\alpha$ is dense in $\mathbb{L}^+_{\lambda_0}(Y_\alpha, \mathbb{P}_\alpha)$ where Y_α is defined as in 2.4(2) with α here standing for L there and see 2.2,
(e) $\mathbb{P}_\alpha$ is $(<\lambda)$-strategically complete and λ^+-c.c.,
(f) if $\delta \le \delta_*$ has cofinality $> \lambda$ then $\mathbb{P}_\delta = \cup\{\mathbb{P}_\alpha : \alpha < \delta\}$, if $\mathrm{cf}(\delta) = \lambda$ then the union is just a dense subset of $\mathbb{P}_\delta$,
(g) $\mathbb{P}_{\delta_*}$ has cardinality $|\delta_*|^\lambda$ if $\delta_* \ge 2$.
(B) if $\mathscr{U} \subseteq \delta_*$ then the complete sub-forcing generated by $\langle \underset{\sim}{\eta}_\alpha : \alpha \in \mathscr{U} \rangle$ is isomorphic to $\mathbb{P}_{\mathrm{otp}(\mathscr{U})}$,
(C) if $\mathbf{G} \subseteq \mathbb{P}_{\delta_*}$ is generic over $\mathbf{V}$ and $\eta_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}]$ for $\alpha < \delta_*$ and $\eta'_\alpha \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ for $\alpha < \delta_*$ and $\{(\alpha, \varepsilon) : \alpha < \delta_*, \varepsilon < \lambda$ and $\eta'_\alpha(\varepsilon) \ne \eta_\alpha(\varepsilon)\}$ has cardinality $< \lambda$ then also $\langle \eta'_\alpha : \alpha < \delta_* \rangle$ is a generic for $\mathbb{P}_{\delta_*}$, determining a possibly different $\mathbf{G}'$ but $\mathbf{V}[\mathbf{G}'] = \mathbf{V}[\mathbf{G}]$,
(D) in clause (B), moreover if $\mathscr{U} \subseteq \delta_*$ and $\langle \alpha_i : i < \mathrm{otp}(\mathscr{U}) \rangle$ list $\mathscr{U}$ in increasing order then for some unique $\mathbf{G}'' \subseteq \mathbb{P}_{\mathrm{otp}(\mathscr{U})}$ generic over $\mathbf{V}$, $i < \mathrm{otp}(\mathscr{U}) \Rightarrow \eta'_{\alpha_i} = \underset{\sim}{\eta}_i[\mathbf{G}'']$.

Proof 2.12 Without loss of generality $\lambda_1 \ge |\delta_*|$; we can use only $\mathbf{m} \in \mathbf{M}_{\rm bd}$ (by 2.10(4)). We define $\mathbf{m} \in \mathbf{M}$ by:

$(*)$ (a) $L_{\mathbf{m}} = \delta_*$,
(b) $M_{\mathbf{m}} = \delta_*$ and[22] $M^{\rm fat}_{\mathbf{m}} = \delta_*$,

[22] Other reasonable choice is $M^{\rm fat}_{\mathbf{m}} = \emptyset$, $M^{\rm lean}_{\mathbf{m}} = \delta_*$ and $M^{\rm fat}_{\mathbf{m}} = \emptyset = M^{\rm lean}_{\mathbf{m}}$.

(c) $u_{\mathbf{m},\alpha} = \alpha$ and $\mathscr{P}_{\mathbf{m},\alpha} = [\alpha]^{\leq\lambda}$ for $\alpha < \delta_*$,
(d) $E'_{\mathbf{m}} = \emptyset$.

It is easy to check that indeed $\mathbf{m} \in \mathbf{M}$ and let $\mathbf{n} \in \mathbf{M}_{\mathrm{ec}}$ be such that $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$, exists by the Crucial Claim 1.32 and let $\mathbb{P}_\alpha = \mathbb{P}_{\mathbf{n}}[\{i : i < \alpha\}]$ for $\alpha \leq \delta_*$.

Now clearly clause (A) holds and $\mathbb{P}_\delta = \mathbb{P}^{\mathrm{cor}}_{\mathbf{m}}$ by 2.8(2), 2.10(1) and e.g. clause (A)(b) holds by 1.16(4A).

As for clause (B), first note that for every $L \subseteq \delta_*$, the sequence $\bar{\eta}_L = \langle \underset{\sim}{\eta}_\alpha : \alpha \in L \rangle$ is generic for $\mathbb{P}_{\mathbf{m}}[L]$ by Definition 2.4.

Second, for $M \subseteq \delta_*$ let $\alpha = \mathrm{otp}(M)$ and $h : M \to \alpha$ be $h(i) = \mathrm{otp}(i \cap M)$ so h is an isomorphism from $\mathbf{m}{\restriction}M$ onto $\mathbf{m}{\restriction}\alpha$ hence by 3.27(2) below, with $\mathbf{m}, \mathbf{m}{\restriction}\alpha, M, \alpha$ here standing for $\mathbf{m}_1, \mathbf{m}_2, M_1, M_2$ there we have h induces an isomorphism from $\mathbb{P}^{\mathrm{cor}}_{\mathbf{m}}[M]$ onto $\mathbb{P}^{\mathrm{cor}}_{\mathbf{m}{\restriction}\alpha}[L_{\mathbf{m}{\restriction}\alpha}]$. In particular, id_α induces an isomorphism from $\mathbb{P}^{\mathrm{cor}}_{\mathbf{m}{\restriction}\alpha}$ onto $\mathbb{P}^{\mathrm{cor}}_{\mathbf{m}}[\alpha]$.

Together we get clause (B). Also Clause (C) holds by 1.16(8) and clause (D) follows so we are done. □

Definition 2.13 (1) We say $\mathbf{m}$ is essentially $(< \mu)$−directed (if $\mu = \aleph_0$ we may omit it) when: if $L \subseteq M$, $|L| < \mu$ then for some $t \in M_{\mathbf{m}}$, we have:

- $s \in L \Rightarrow s <_{\mathbf{m}} t \wedge s \in u_t$ so $M_{\mathbf{m}}$ is directed[23]).

[Note that it follows because $\mathbf{m}$ is bounded and $M_{\mathbf{m}}$ is $\mathrm{cf}(\mu)$-directed.]

(2) We say $\mathbf{m}$ is strongly μ-directed (or $(< \mu)$-directed; if $\mu = \aleph_0$ we may omit it) when: for every $L \subseteq L_{\mathbf{m}}$ of cardinality $< \mu$ there is $t \in M_{\mathbf{m}}$ such that $L \in \mathscr{P}_{\mathbf{m},t}$ (the condition implies "$\mathbf{m}$ is weakly bounded" and "$\mathbf{m}$ is not lean, $t \notin M^{\mathrm{lean}}_{\mathbf{m}}$ when $E''_{\mathbf{m}}$ has at least two equivalence classes").

(3) We say $\mathbf{m}$ is reasonable when:

(α) $\mathbf{m}$ is strongly λ^+-directed and $M^{\mathrm{fat}}_{\mathbf{m}}$ is cofinal in $M_{\mathbf{m}}$,
(β) $\mathbf{m}(\leq t) \in \mathbf{M}_{\mathrm{ec}}$ for every $t \in M_{\mathbf{m}}$,
(γ) $\mathbf{m}$ is wide and bounded (see Definition 1.7(10) and Definition 1.31(1A)).

Similarly we can deal with such iterations with partial memory and spell out how $\mathbb{P}^{\mathrm{cor}}_{\mathbf{m}}[L]$ is defined from a $(< \lambda)$-support iteration with partial memory. This is used in [12], but we need more: see §3.

Conclusion 2.14 *Assume M is a well founded partial order and $\bar{u}' = \langle u'_t : t \in M \rangle$, $u'_t \subseteq M_{<t}$ and $\bar{\mathscr{P}}' = \langle \mathscr{P}'_t : t \in M \rangle$ with $\mathscr{P}'_t \subseteq [u'_t]^{\leq\lambda}$ is closed under subsets. Then we can find $\beta(*), h, \mathbb{P}_\beta = \mathbb{P}_{0,\beta}, \underset{\sim}{\mathbb{Q}}_\beta = \underset{\sim}{\mathbb{Q}}_{0,\beta}, \mathbb{P}_{1,\beta}, \underset{\sim}{\mathbb{Q}}_{1,\alpha}, \underset{\sim}{\eta}_\alpha, \underset{\sim}{\eta}'_s$ and $\mathbb{P}_{1,v}, \mathbb{P}'_u$ (for $\beta \leq \beta(*), \alpha < \beta(*), s \in M$ and $v \subseteq \beta(*), u \subseteq M$) and $h, \bar{u}, \bar{\mathscr{P}}$ such that:*

(A) (a) $\langle \mathbb{P}_\beta, \underset{\sim}{\mathbb{Q}}_\alpha : \beta \leq \beta(*), \alpha < \beta(*) \rangle$ is $(< \lambda)$-support[24] iteration,
(b) (α) $\bar{\bar{u}} = \langle u_\beta : \beta < \beta(*) \rangle$ such that $u_\beta \subseteq \beta$,
(β) $\bar{\mathscr{P}} = \langle \mathscr{P}_\beta : \beta < \beta(*) \rangle$ such that $\mathscr{P}_\beta \subseteq [u_\beta]^{\leq\lambda}$ is closed under subsets,
(c) $\underset{\sim}{\eta}_\alpha$ is a $\mathbb{P}_{\alpha+1}$-name of a member of $\prod_{\varepsilon<\lambda} \theta_\varepsilon$,
(d) $\langle \eta_\alpha : \alpha < \beta \rangle$ is generic for $\mathbb{P}_\beta$,
(e) $\tilde{\mathbb{Q}}_\alpha$ is defined as in Definition 1.12(4),
(f) $\Vdash_{\mathbb{P}_{\beta(*)}}$ "$\underset{\sim}{\eta}_\beta \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ dominates every $\nu \in \prod_{\varepsilon<\lambda} \theta_\varepsilon$ from $\mathbf{V}[\langle \underset{\sim}{\eta}_\alpha : \alpha \in u \rangle]$ when $u \in \mathscr{P}_\beta$.

23 Why not add $\{s\} \in \mathscr{P}_{\mathbf{m},t}$? See 1.29(13).

24 This will be $\mathbf{q}_{\mathbf{m}}$, well up to equivalence, see §1.

(B) (a) h is a one-to-one function from M into[25] $\beta(*)$; stipulate $h(\infty) = \beta(*)$,
(b) $s <_M t \Rightarrow h(s) < h(t)$,
(c) $u_{h(t)} \cap \mathrm{rang}(h) = \{h(s) : s \in u'_t\}$,
(d) $\mathscr{P}_{h(t)} \cap [\mathrm{rang}(h)]^{\leq\lambda} = \{\{h(s) : s \in u\} : u \in \mathscr{P}'_t\}$,

(C) (a) $\mathbb{P}_{1,\beta} = \mathbb{L}^+_{\lambda^+}(Y_\beta, \mathbb{P}_\beta)$ where we let $Y_\beta = \{p^*_{\alpha,\nu} : \alpha < \beta, \nu \in \prod_{\varepsilon<\zeta} \theta_\varepsilon$ for some $\zeta < \lambda\}$, see 2.2, 2.4(1),
(b) $\mathbb{P}_{1,u} = \mathbb{L}^+_{\lambda_0}(Y_u, \mathbb{P}_\beta)$, where Y_u is defined similarly when $u \subseteq \beta(*)$,
(c) $\mathbb{P}'_u$ is a forcing notion for $u \subseteq M$ and $\underset{\sim}{\eta}'_s$ is a $\mathbb{P}'_{\{s\}}$-name for $s \in M$,
(d) h induces an isomorphism from $\mathbb{P}'_u$ onto $\mathbb{P}_{1,\{h(s):s\in u\}}$ for $u \subseteq M$ and $\underset{\sim}{\eta}'_s$ to $\underset{\sim}{\eta}_{h(s)}$ for $s \in M$,
(e) $\langle \underset{\sim}{\eta}_{h(s)} : s \in u\rangle$ is generic for $\mathbb{P}'_u$ for $u \subseteq M$,

(D) (a) $\mathbb{P}'_u \lessdot \mathbb{P}'_v$ when $u \subseteq v \subseteq M$,
(b) $\mathbb{P}_\beta, \mathbb{P}_{1,u}, \mathbb{P}'_{1,u}$ are $(<\lambda)$-strategically complete and λ^+-c.c.,
(c) if $M_1, M_2 \subseteq M$ and f is an isomorphism from M_1 onto M_2 as partial orders such that $t \in M_1 \Rightarrow u'_{f(t)} \cap M_2 = \{f(s) : s \in u'_t \cap M_1\}$ and $t \in M_1 \Rightarrow \mathscr{P}'_{f(t)} \cap [M_2]^{\leq\lambda} = \{\{f(s) : s \in u \cap M_1\} : u \in \mathscr{P}'_t\}$ <u>then</u> the mapping $h(s) \mapsto h(f(s))$ induces an isomorphism from the forcing notion $\mathbb{P}'_{1,M_1}$ onto $\mathbb{P}'_{1,M_2}$.

(E) if M is $(<\lambda^+)$-directed and the set $Y \subseteq M$ is cofinal in M, <u>then</u> the set $\{\underset{\sim}{\eta}_{h(s)} : s \in Y\}$ is cofinal in $\{\underset{\sim}{\eta}_\beta : \beta < \beta(*)\}$ and even in $\Pi_{\varepsilon<\lambda}\theta_\varepsilon$ in $\mathbf{V}^{\mathbb{P}_{\beta(*)}}$ (see 1.29(3)).

Proof 2.14 Easy. We can assume $\lambda_1 \geq |M|$. Similarly to the proof of 2.12, the proof of clause (E) is easy by 3.22 □

Claim 2.15 *If* $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{n}$ *and* $\mathbb{P}_{\mathbf{m}_\ell} \lessdot \mathbb{P}_{\mathbf{n}}$ *for* $\ell = 1, 2$ *then* $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$

Proof 2.15 Easy. □

The following will be used in 2.17.

Claim 2.16 *1) If (A) then (B), where:*

(A) (a) $\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}$,
(b) L_* is an initial segment of $L_{\mathbf{m}_1}$,
(c) $L_* = L_{\mathbf{m}_1} \cap L_{\mathbf{m}_2}$,
(d) $\mathbf{m}_0 = \mathbf{m}_1 \restriction L_* \leq_{\mathbf{M}} = \mathbf{m}_2$,

(B) there is $\mathbf{m} \in \mathbf{M}$ such that:

(a) $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}$,
(b) $\mathbf{m}_2 = \mathbf{m} \restriction L_{\mathbf{m}_2}$.

2) If $L_1 \subseteq L_2$ are initial segments of $L_{\mathbf{m}}$ and $\mathbf{m}\restriction L_2 \in \mathbf{M}_{\mathrm{ec}}$ then $\mathbf{m}\restriction L_1 \in \mathbf{M}_{\mathrm{ec}}$.
3) In part (1) we may add (e) to clause (A) and (c), (d) to clause (B), where:

(A) (e) $L_* \subseteq L_{\mathbf{m}_1(<t_*)}$, where $t_* \in M_{\mathbf{m}_1}$,
(B) (c) if $s \in L_{\mathbf{m}} \setminus L_{\mathbf{m}_1}$ then $s <_{\mathbf{m}_1} t_*$,
(d) if $s \in M_{\mathbf{m}_1} \setminus M_{\mathbf{m}_0}$ and $t_* \leq_{\mathbf{m}_1} s$ then $u_{\mathbf{m},s} = u_{\mathbf{m}_1,s} \cup ((L_{\mathbf{m}} \setminus L_{\mathbf{m}_1}) \cap L_{\mathbf{m}(<s)})$.

Proof 2.16 1) Easy but we elaborate. We define $\mathbf{m}$ as follows:

$(*)_1$ (a) $L_{\mathbf{m}}$ as a set is $L_{\mathbf{m}_1} \cup L_{\mathbf{m}_2}$,

[25] In general not onto!

(b) $\leq_{\mathbf{m}}$ is the transitive closure of $\{(s,t) : L_{\mathbf{m}_1} \models s < t \text{ or } L_{\mathbf{m}_2} \models s < t\}$,
(c) $M_{\mathbf{m}} = M_{\mathbf{m}_1}$, $M^{\text{lean}}_{\mathbf{m}} = M^{\text{lean}}_{\mathbf{m}_1}$, $M^{\text{fat}}_{\mathbf{m}} = M^{\text{fat}}_{\mathbf{m}_1}$,
(d) $u_{\mathbf{m},t}$ is:
  (α) $u_{\mathbf{m}_1,t}$ when $t \in L_{\mathbf{m}_1} \setminus L_*$, and,
  (β) $u_{\mathbf{m}_2,t}$ when $t \in L_{\mathbf{m}_2} \setminus M_{\mathbf{m}_0}$,
  (γ) $u_{\mathbf{m}_1,t} \cup u_{\mathbf{m}_2,t}$ if $t \in M_{\mathbf{m}_0}$.
(e) $\mathscr{P}_{\mathbf{m},t}$ is:
  (α) $\mathscr{P}_{\mathbf{m}_1,t}$ when $t \in L_{\mathbf{m}_1} \setminus L_*$ and,
  (β) $\mathscr{P}_{\mathbf{m}_2,t}$ when $t \in L_{\mathbf{m}_2} \setminus M_{\mathbf{m}_0}$,
  (γ) $[u_{\mathbf{m},t}]^{\leq\lambda}$ if $t \in M^{\text{fat}}_{\mathbf{m}_0}$,
  (δ) $\mathscr{P}_{\mathbf{m}_1,t} \cup \mathscr{P}_{\mathbf{m}_2,t}$ if $t \in M_{\mathbf{m}_0} \setminus M^{\text{fat}}_{\mathbf{m}_\gamma}$.
(f) We define $E'_{\mathbf{m}}$ by: for $s,t \in L_{\mathbf{m}}$, we have $sE'_{\mathbf{m}}t$ iff $sE'_{\mathbf{m}_1}t$ or $sE'_{\mathbf{m}_2}t$.

As L_* is an initial segment of $L_{\mathbf{m}_1}$ we have:

$(*)_2$ $L_{\mathbf{m}} \models$ "$s \leq t$" iff $L_{\mathbf{m}_2} \models$ "$s \leq t$" or $s \in L_{\mathbf{m}_2}$, $t \in L_{\mathbf{m}_1} \setminus L_*$ and for some $r \in L_*$ we have $L_{\mathbf{m}_2} \models$ "$s \leq r$" and $L_{\mathbf{m}_1} \models$ "$r \leq t$".
$(*)_3$ $L_{\mathbf{m}_2}$ is an initial segment of $L_{\mathbf{m}}$.

Now check that $\mathbf{m}$ is as required.
2) Follows.
3) Easy (changing $(*)_1$ above naturally). □

Sometime we would like to have in addition to being in $\mathbf{M}_{\text{ec}}$ that $\{\underset{\sim}{\eta}_s : s \in M\}$ be cofinal in $(\Pi_{\varepsilon<\lambda}\theta_\varepsilon, \leq_{J^{\text{bd}}_\lambda})$ in $\mathbf{V}^{\mathbb{P}_{\mathbf{m}}}$. Toward this we use the following claim:

Claim 2.17 *Assume* $\mathbf{m} \in \mathbf{M}$.
1) A sufficient condition for $\mathbf{m} \in \mathbf{M}_{\text{bec}}$ *is:*

$(*)_{\mathbf{m}}$ *For some* $\delta, \bar{L}, \bar{c}$ *we have:*

(a) $\bar{c} = \langle c_\alpha : \alpha < \delta\rangle \in {}^{\delta}(M_{\mathbf{m}})$, *each* L_a *is an initial segment*
(b) $\bar{L} = \langle L_\alpha : \alpha < \delta\rangle$,
(c) $\mathbf{m}\restriction L_\alpha$ *belongs to* $\mathbf{M}_{\text{ec}}$ *for every* $\alpha < \delta$,
(d) $L_\alpha \subseteq L_{\mathbf{m},<c_\alpha}$, $L_\alpha \subseteq u_{\mathbf{m},c_\alpha}$, $M_{\mathbf{m}(<c_\alpha)} \subseteq L_\alpha$ *and if* $t \in L_\alpha \setminus M_{\mathbf{m}}$ *then* $L_\alpha \cap (t/E_{\mathbf{m}})$ *is an initial segment of* $t/E_{\mathbf{m}}$,
(e) δ *has cofinality* $> \lambda$,
(f) $\bar{c}$ *is increasing and cofinal in* $L_{\mathbf{m}}$,
(g) $\bar{L}$ *is* $\subseteq$*-increasing with union* $L_{\mathbf{m}}$.

2) A sufficient condition for $\mathbf{m} \in \mathbf{M}_{\text{bec}}$ *is:*

$(*)'_{\mathbf{m}}$ *For some* $\bar{c}, \bar{L}$ *we have:*

(a)-(e) as above,
(f) if $L \subseteq L_{\mathbf{m}}$ *has cardinality* $\leq \lambda$ *then for some* $\alpha < \delta$ *we have* $L \subseteq L_\alpha$,

3) For $L_* \subseteq L_{\mathbf{m}}$ *we have* $(A)_{L_*} \Rightarrow (B)_{L_*}$, *where:*

$(A)_{L_*}$ *if* $L \subseteq L_*$ *has cardinality* $\leq \lambda$ *and* $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ *then* $\mathbb{P}_{\mathbf{n}}[L] = \mathbb{P}_{\mathbf{m}}[L]$,
$(B)_{L_*}$ *if* $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$ *then* $\mathbb{P}_{\mathbf{n}}[L_*] = \mathbb{P}_{\mathbf{m}}[L_*]$,

4) If $c \in L_{\mathbf{m}}$, $L_* \subseteq u_{\mathbf{m},c}$, $\mathbf{m}\restriction L_* \in \mathbf{M}_{\text{ec}}$, $M_{\mathbf{m}(<c)} \subseteq L_*$ *and* $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ *implies* $L_* \cap (t/E_{\mathbf{m}})$ *is an initial segment of* $t/E_{\mathbf{m}}$ *then clause* $(B)_{L_*}$ *above holds,*
5) We have $(a) \Rightarrow (b)$, *when:*

(a) *we have:*

(α) $\mathbf{m}$ *is strongly* $(< \lambda^+)$*-directed,*
(β) *for every* $t \in M_{\mathbf{m}}$ *(or just for cofinally many* $t \in M_{\mathbf{m}}$*) we have* $\mathbf{m}(\leq t) \in \mathbf{M}_{\text{ec}}$.

(b) $\mathbf{m} \in \mathbf{M}_{\text{ec}}$.

5A) Similarly for $\mathbf{M}_{\text{bec}}$.

6) If M *is a* $< \lambda^+$*-directed well founded partial order of cardinality* $\leq \lambda_1$*, for example,* $M = (\kappa, <), \kappa = \text{cf}(\kappa) \in (\lambda, \lambda_1]$*, our main case,* then *there is a strongly* λ^+*-directed* $\mathbf{m} \in \mathbf{M}$ *such that* $M_{\mathbf{m}} = M$ *and* $(*)'_{\mathbf{m}}$ *from part (2) holds, (hence* $\mathbf{m} \in \mathbf{M}_{\text{bec}}$ *and* $\{\underset{\sim}{\eta}_s : s \in M_{\mathbf{m}}\}$ *is cofinal in* $\left(\Pi_{\varepsilon<\lambda}\theta_\varepsilon, <_{J^{\text{bd}}_\lambda}\right)$ *in the universe* $\mathbf{V}^{\mathbb{P}_{\mathbf{m}}}$.

Proof 2.17 Straightforward (recalling 2.10(4)), i.e.
1) By (2).
2) By (3) and (4).
3) Obvious, see 2.5(8).
4) Clear.
5) Easy.
6) Choose $\bar{c}$ such that:

$(*)_1$ (a) $\bar{c} \in {}^\delta(M_{\mathbf{m}})$ for some ordinal δ,
(b) if $\alpha < \beta < \delta$ then $c_\beta \nleq_M c_\alpha$,
(c) $\bar{c}$ lists $M_{\mathbf{m}}$,
(d) (follows), if $L \subseteq L_{\mathbf{m}}$ has cardinality $\leq \lambda$ then for some $\alpha < \delta$ the element c_α is an upper bound of L, moreover $L \in \mathscr{P}_{\mathbf{m},<c_\alpha}$.

Now we choose fat bounded $\mathbf{m}_\alpha$ by induction on $\alpha \leq \delta$ such that:

$(*)_2$ (a) $\langle \mathbf{m}_\beta : \beta \leq \alpha \rangle$ is $\leq_{\mathbf{M}}$-increasing continuous,
(b) $L_{\mathbf{m}_0} = M$ and $u_{\mathbf{m}_0,s} = M_{<s}$, (hence $\mathscr{P}_{\mathbf{m}_0,s} = [u_{\mathbf{m}_0,s}]^{\leq\lambda}$ recalling $\mathbf{m}_0$ being fat) for $s \in M$,
(c) for every $s \in L_{\mathbf{m}_\alpha} \setminus M$ for some $\beta < \alpha$ we have $L_{\mathbf{m}_\alpha} \models s \leq c_\beta$,
(d) if $\gamma \in [\alpha, \delta)$ then $u_{\mathbf{m}_\alpha, c_\gamma} = L_{\mathbf{m}_\alpha, <c_\gamma}$,
(e) if $\alpha = \beta + 1$ then $\mathbf{m}_\alpha(< c_\beta) \in \mathbf{M}_{\text{ec}}$,
(f) $L_{\mathbf{m}_\alpha}$ has cardinality at most 2^{λ_2} or even λ_2, but this does not matter,
(g) if $t \in L_{\mathbf{m}_\alpha}$ then for some $\beta < \alpha$ we have $t/E''_{\mathbf{m}_\alpha} \subseteq L_{\mathbf{m}_{\beta+1}} \setminus L_{\mathbf{m}_\beta}$.

There is no problem to carry the definition; as:
For $\alpha = 0$ we have defined $\mathbf{m}_0$ in clause (b) of $(*)_2$ above.
For α a limit ordinal use 1.20(1), so in particular $L_{\mathbf{m}} = \cup\{L_{\mathbf{m}_\beta} : \beta < \alpha\}$.
For $\alpha = \beta + 1$ by 1.32 there is $\mathbf{n}_\beta \in \mathbf{M}_{\text{ec}}$ such that $\mathbf{m}_\beta(< c_\beta) \leq_{\mathbf{M}} \mathbf{n}_\beta$, without loss of generality we have $L_{\mathbf{m}_\beta} \cap L_{\mathbf{n}_\beta} = L_{\mathbf{m}(<c_\beta)}$.

By 3.22 below without loss of generality the cardinality of $L_{\mathbf{n}_\beta}$ is at most λ_2. Now apply 2.16(3) with $\mathbf{m}_\beta, L_{\mathbf{m}_\beta,<c_\beta}, \mathbf{n}_\beta$ here standing for $\mathbf{m}_2, L_*, \mathbf{m}_1$ there.

So we have carried the induction. Now clearly $\mathbf{m}_\delta$ is as promised, That is, $(*)_{\mathbf{m}_\delta}$ from part (2) of the claim holds, hence $\mathbf{m} \in \mathbf{M}_{\text{ec}}$ by part (2) being cofinal holds by 1.29; so we are done. □

3 The main conclusion

3.1 Wider m's

Recall that in this section our main interest is in restricting ourselves to lean $\mathbf{m}$, but in §3C we do not assume this and in §3A, §3B, §3D we rely on §1, §2, in particular §1B

In §3B, §3D we restrict ourselves to lean $\mathbf{m}$, but not in §3A, however the projection defined in 3.1(1) are helpful only in the lean case.

Note that here we fulfil the promises from §2, Now in §4A we rely on §3A, §3C, but we do not rely on §3B, §3D. Lastly, §4A gives alternative proof of the promises from §2 proved in §3D, it relies on §3A, §3C but not on §3B, §3D (except Def 3.25). In §4B and in 2.17 we fulfil additional promises from [12].

We have a debt from §2, i.e. see discussion 2.11. Toward this we explicate what appear in the proof of 1.32. We use mainly the notions of wide, full and "being in $\mathbf{M}_{\rm ec}$".

Note that 3.1(2), (4) and 3.2(3), (4) are of interest exceptionally only for the neat context.

Definition 3.1 Let $\mathbf{m} \in \mathbf{M}$.

(1) For $L \subseteq L_{\mathbf{m}}$ we say $p \in \mathbb{P}_{\mathbf{m}}(L)$ is the projection (to L) of $q \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ and write $p = q \upharpoonright L$ <u>when</u>:

(a) $\mathrm{dom}(p) = \mathrm{dom}(q) \cap L$,
(b) if $s \in \mathrm{dom}(p)$ then:

- (α) $\mathrm{tr}(p(s)) = \mathrm{tr}(q(s))$,
- (β) $\{\underset{\sim}{f}_{p(s),\iota} : \iota < \iota(p(s))\} = \{\underset{\sim}{f}_{q(s),\iota} : \iota < \iota(q(s))$ and $\bar{r}_{p(s),\iota}$ is a sequence of members of $L\}$, see Definition 1.10(2).

(2) Let $\mathscr{F}_{\mathbf{m},\mu}$ be the set of the functions f such that for some L_1, L_2:

(a) f is an isomorphism from $\mathbf{m}{\upharpoonright}L_1$ onto $\mathbf{m}{\upharpoonright}L_2$,
(b) L_ℓ is a subset of $L_{\mathbf{m}}$ for $\ell = 1, 2$,
(c) $M_{\mathbf{m}} \subseteq L_\ell$ for $\ell = 1, 2$ and $f{\upharpoonright}M_{\mathbf{m}}$ is the identity,
(d) L_ℓ is $E_{\mathbf{m}}$-closed, i.e. $M_{\mathbf{m}} \subseteq L_\ell$ and if $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ and $t \in L_\ell$ then $t/E_{\mathbf{m}} \subseteq L_\ell$ for $\ell = 1, 2$,
(e) $\{t/E''_{\mathbf{m}} : t \in L_\ell \backslash M_{\mathbf{m}}\}$ has cardinality $< \mu$.

2A) Let $\mathscr{F}_{\mathbf{m}} = \mathscr{F}_{\mathbf{m},\lambda_0}$.

(3) If $L_1, L_2 \subseteq L_{\mathbf{m}}$ and f is an isomorphism from $\mathbf{m}{\upharpoonright}L_1$ onto $\mathbf{m}{\upharpoonright}L_2$ <u>then</u> we let $\hat{f}$ be the one-to-one mapping[26] from $\mathbb{P}_{\mathbf{m}}(L_1)$ onto $\mathbb{P}_{\mathbf{m}}(L_2)$ as in $(*)_4(b)$ of the proof of 1.32.

(4) Let $\mathbb{P}^-_{\mathbf{m}}(L)$ be $\{p \in \mathbb{P}_{\mathbf{m}}(L) : \mathrm{fsupp}(p) \subseteq L$ and $\iota(p(\alpha)) \leq 1$ for every $\alpha \in \mathrm{dom}(p)\}$ with the order inherited from $\mathbb{P}_{\mathbf{m}}$.

Observation 3.2 *Let* $\mathbf{m} \in \mathbf{M}$ *and* $L \subseteq L_{\mathbf{m}}$.

(1) The projection of $q \in \mathbb{P}_{\mathbf{m}}$ *to* L *is well defined and* $\in \mathbb{P}_{\mathbf{m}}(L)$.

(2) Moreover, it is unique.

(3) If $p \in \mathbb{P}_{\mathbf{m}}(L)$ *is the projection of* $q \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ *to* L <u>*then*</u> $p \leq q$ *in* $\mathbb{P}_{\mathbf{m}}$.

(4) Each $p \in \mathbb{P}_{\mathbf{m}}$ *is equivalent to* $\mathscr{S} := \{(p{\upharpoonright}\{t\}) \upharpoonright L : t \in \mathrm{dom}(p) \wedge L \in \mathscr{P}_{\mathbf{m},\leq t}\} \cup \{p \upharpoonright M_{\mathbf{m}}\}$; *the equivalence means* $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$p \in \underset{\sim}{G}_{\mathbb{P}_{\mathbf{m}}}$ <u>*iff*</u> $\mathscr{S}_p \subseteq \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}$". *More specifically it is*

[26] We have not said "order preserving"! still it is a function from $\mathbb{P}_{\mathbf{m}}(L_1)$ onto $\mathbb{P}_{\mathbf{m}}(L_1)$ by the way we have defined the $\mathbb{P}_{\mathbf{m}}(L)$-s and because of 1.5(e)(χ).

Sh:1126

equivalent to $\mathscr{S}_p = \{(p{\restriction}\{t\}) \upharpoonright L : t \in \mathrm{dom}(p) \wedge L \in \mathscr{L}_t\}$ *when* $\mathscr{L}_t$ *satisfies: if* $\iota < \iota_{p(s)}$ *then for some* $L \in \mathscr{L}_t$*, (recalling 1.10) we have* $\mathrm{rang}(\bar{r}_{p(t),\iota}) \subseteq L$*.*

(5) For every $p \in \mathbb{P}_{\mathbf{m}}$*,* p *is equivalent to* $\mathscr{S}'_p := \{p^{[t]} : t \in \mathrm{dom}(p)\}$ *where* $p^{[t]} \in \mathbb{P}_{\mathbf{m}}$ *has domain* $\{t\}$ *and* $p(t) = (\mathrm{tr}(p_t), \mathbf{B}_{p(t)}(\langle \eta_{r_{p(t)}(\zeta)} : \zeta \in w_{p(t)}\rangle)$*; recall Definition 1.10 for the meaning of* $\mathbf{B}_{p(t)}$*, etc.*

Remark 3.3 (1) Note that the choice in Definition 1.10$(c)(\gamma)$ to require such $\langle \underset{\sim}{f}_{p(t),\iota} : \iota < \iota(p_t)\rangle$ exists, is necessary for 3.2(4), which is crucial in the proof of 3.27.

(2) In Definition 1.31(1A) we choose "wide means λ-wide" as when applying it, if $X = \mathrm{fsupp}(p)$ then for some $Y \subseteq L_{\mathbf{m}}$ of cardinality $< \lambda$, $X \subseteq \cup\{t/E_{\mathbf{m}} : t \in Y\}$.

Proof 3.2 Easy e.g.

(4) Now if $q \in \mathscr{S}_p$ then q has the form $(p{\restriction}\{t\}) \upharpoonright L$ where $L \in \mathscr{P}_{\mathbf{m},t}$ hence $\Vdash$ "$p \in \underset{\sim}{\mathbf{G}}$ implies $q \in \underset{\sim}{\mathbf{G}}$", hence $\Vdash$"$p \in \mathbf{G}$ implies $\mathscr{S}_p \subseteq \underset{\sim}{\mathbf{G}}$".

For the other direction assume $q \in \mathbb{P}_{\mathbf{m}}$ forces $\mathscr{S}_p \subseteq \underset{\sim}{\mathbf{G}} \subseteq \mathbb{P}_{\mathbf{m}}$ and we shall prove that q is compatible with p, this suffices, so toward contradiction assume q, p are incompatible.

Without loss of generality $\mathrm{dom}(p) \subseteq \mathrm{dom}(q)$ and recalling $t \in \mathrm{dom}(p) \Rightarrow q \Vdash$ "$p \upharpoonright (t/E_{\mathbf{m}}) \in \underset{\sim}{\mathbf{G}}$" clearly $s \in \mathrm{dom}(p) \Rightarrow q \Vdash$ "$\mathrm{tr}(p(s)) \subseteq \underset{\sim}{\eta}_s$" so necessarily $s \in \mathrm{dom}(p) \Rightarrow \mathrm{tr}(p(s)) \trianglelefteq \mathrm{tr}(q(s))$. Recalling 1.16(6), as p, q are incompatible there are $s \in \mathrm{dom}(p) \cap \mathrm{dom}(q)$ and q_1 such that $q{\restriction}L_{\mathbf{m},<s} \le q_1 \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m},<s})$ and $q_1 \Vdash$ "$q(s)$, $p(s)$ are incompatible in $\mathbb{Q}_{\bar{\theta}}$".

As $\mathrm{tr}(p(s)) \trianglelefteq \mathrm{tr}(q(s))$ this implies $q_1 \Vdash$ "$\mathrm{tr}(q(s))$, $p(s)$ are incompatible, so recalling $q \Vdash$"$\mathrm{tr}(p(s)) \subseteq \eta_s$" this implies $f_{p(s)}{\restriction}\ell g(\mathrm{tr}(q(s))) \ntrianglelefteq \mathrm{tr}(q(s))$". Recalling Definition 1.10(2)$(c)(\gamma)$, $q_1 \Vdash_{\mathbb{P}_{\mathbf{m},s}}$ "there is $\iota < \iota(s, p)$ such that $f_{p(s),\iota}$, $\mathrm{tr}(q(s))$ are incompatible". Possibly increasing q_1, we can fix ι. But letting $u \in \mathscr{P}_{\mathbf{m},s}$ be such that $\bar{r}_{p(s),\iota} \subseteq u$ this implies that $q_1 \Vdash$ "$(p{\restriction}\{s\}) \upharpoonright u \notin \underset{\sim}{\mathbf{G}}$ or $\mathrm{tr}(q(s)) \nsubseteq \underset{\sim}{\eta}_s$". However, q_1, q are compatible and this contradicts the choice of q. □

Claim 3.4 *(1) For* $\chi \ge 2^{\lambda_2}$ *the* $\mathbf{n} \in \mathbf{M}_\chi$ *constructed in 1.32 satisfies: if* $\mathbf{n} \le_{\mathbf{M}} \mathbf{n}_1$ *then* $\mathbf{n}_1$ *is full and wide, even* λ_2*-wide and if* $\mathbf{n}_1 \in \mathbf{M}_\chi$ *even very wide.*

(2) If $\mathbf{n} \in \mathbf{M}_{\mathrm{ec}}$ *and* $\mathbf{n} \le_{\mathbf{M}} \mathbf{n}_1$ *then* $\mathbf{n}_1 \in \mathbf{M}_{\mathrm{ec}}$*.*

(3) If $\mathbf{m} \in \mathbf{M}_\chi$ *is full and very wide (or just* λ_2*-wide and even* λ_0*-wide), then* $\mathbf{m} \in \mathbf{M}_{\mathrm{ec}}$*.*

(4) If $\mathbf{m} \in \mathbf{M}$*, then there is a very wide full* $\mathbf{n} \in \mathbf{M}$ *such that* $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$*.*

Proof 3.4

(1) Holds by the proof of 1.32.

(2) Holds by Definition 1.28(1), (2).

(3),4) By the proof of 1.32. □

Claim 3.5 *Assume* $\mathbf{m}$ *is* μ*-wide where* $\mu \ge \lambda_0$*.*

1) If $f \in \mathscr{F}_{\mathbf{m},\mu}$ *and* $X \subseteq L_{\mathbf{m}}$ *has cardinality* $< \mu$*, then there is* g *such that:*

(a) $g \in \mathscr{F}_{\mathbf{m},\infty}$ *and even belongs to* $\mathscr{F}_{\mathbf{m},\mu}$*,*

(b) $f \subseteq g$*,*

(c) $\mathrm{dom}(g) = \mathrm{rang}(g)$*,*

(d) $X \subseteq \mathrm{dom}(g)$*.*

(2) If $g \in \mathscr{F}_{\mathbf{m},\mu}$ *and* $\mathrm{dom}(g) = \mathrm{rang}(g)$ *then* $g^{+\mathbf{m}} = g \cup \mathrm{id}_{L_{\mathbf{m}} \setminus \mathrm{dom}(g)}$ *is an automorphism of* $\mathbf{m}$*.*

(3) If f *is an automorphism of* $\mathbf{m}$ *then it naturally induces an automorphism* $\hat{f}$ *of* $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ *similarly to* $\hat{f}$ *from* $(*)_4(b)$ *of the proof of 1.32 and it induces an automorphism of* $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ *as well; abusing our notation we denote both by* $\hat{f}$*.*

(4) If $f \in \mathscr{F}_{\mathbf{m},\mu}$ then it induces an isomorphism $\hat{f}$ from $\mathbb{P}_{\mathbf{m}}[\mathrm{dom}(f)]$ onto $\mathbb{P}_{\mathbf{m}}[\mathrm{rang}(f)]$ hence (as above) from $\mathbb{P}_{\mathbf{m}}(\mathrm{dom}(f))$ onto $\mathbb{P}_{\mathbf{m}}(\mathrm{rang}(f))$.

(5) If $p \in \mathbb{P}_{\mathbf{m}}$ then the set $\{t/E_{\mathbf{m}} : t \in \mathrm{wsupp}(p)\}$ has cardinality $< \lambda$.

Proof 3.5 1) Easy by the definition of wide in 1.31(1) and of $\mathscr{F}_{\mathbf{m}}$ in 3.1(2), in particular clause (e).

(2) Just read the definition of $\mathbf{m} \in \mathbf{M}$ and of $f \in \mathscr{F}_{\mathbf{m}}$, in particular:

(a) if $t_1, t_2 \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ are not $E'_{\mathbf{m}}$-equivalent then $(t_1/E_{\mathbf{m}}) \cap (t_2/E_{\mathbf{m}}) = M_{\mathbf{m}}$ and $\leq_{\mathbf{m}} \restriction (t_1/E_{\mathbf{m}} \cup t_2/E_{\mathbf{m}})$ is determined by $\leq_{\mathbf{m}} \restriction (t_1/E_{\mathbf{m}}), \leq_{\mathbf{m}} \restriction (t_2/E_{\mathbf{m}})$,

(b) $g \restriction M_{\mathbf{m}} = \mathrm{id}_{M_{\mathbf{m}}}$.

(3) Naturally by the definition.

(4) Let $g \in \mathscr{F}$ be as in part (1) and let $h = g^{+\mathbf{m}}$ so an automorphism of $\mathbf{m}$ which extends g as in part (2). So $\hat{h}$ is an automorphism of $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ and clearly $\hat{f} = \hat{h} \restriction \mathbb{P}_{\mathbf{m}}(\mathrm{dom}(f))$ is as required.

(5) Is clear, see 1.13(f). □

Claim 3.6 *Let $\mathbf{m} \in \mathbf{M}$ and $\mu \geq \lambda_0$.*

If $f_1, f_2 \in \mathscr{F}_{\mathbf{m},\mu}$ then:

(a) $f_1 \subseteq f_2 \Rightarrow \hat{f}_1 \subseteq \hat{f}_2$,

(b) $f_1 = f_2^{-1} \Rightarrow \hat{f}_1 = (\hat{f}_2)^{-1}$.

Proof 3.6 Just consider the definition, see 3.1(3) and $(*)_4(b)$ of the proof of 1.32. □

3.2 Ordinal equivalence

Context 3.7 All $\mathbf{m}$-s are lean[27].

Observation 3.8 *(1) $\mathbb{P}^-_{\mathbf{m}}(L) \subseteq \mathbb{P}_{\mathbf{m}}(L)$, see Definition 3.1(4).*

(2) For every $p \in \mathbb{P}_{\mathbf{m}}$ there is a sequence $\langle p_i : i < i(*)\rangle$ of $< \lambda$ members of $\mathbb{P}^-_{\mathbf{m}}$, (see 3.1(6)) such that $\Vdash_{\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})}$ "$p \in \underset{\sim}{\mathbf{G}} \iff \{p_i : i < i(*)\} \subseteq \underset{\sim}{\mathbf{G}}$".

Proof 3.8 (1) By their definitions.

(2) Should be clear, see Definition 3.1(4) and 3.2(3). □

Remark 3.9 (1) Observation 3.8 is not used.

(2) Probably we can avoid using "wide" and prove earlier the density of $\mathbf{M}_{\mathrm{ec}}$ with smaller cardinality but the present way seems more transparent.

Definition 3.10 Assume $\mathbf{m} \in \mathbf{M}$.

(1) Let $\mathscr{Y}_{\mathbf{m}}$ be the set of pairs $(t, \bar{s})$ such that $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ and $\bar{s} \in {}^{\zeta}(t/E''_{\mathbf{m}})$ for some $\zeta < \lambda^+$; we may write $\bar{s}$ instead of $(t, \bar{s})$ as usually $\bar{s}$ determines $t/E''_{\mathbf{m}}$, but this is the only information about t that matter. We could have used instead pairs $(t/E''_{\mathbf{m}}, \bar{s})$.

(2) By induction on the ordinal γ we define when $(t_1, \bar{s}_1), (t_2, \bar{s}_2)$ are γ-equivalent in $\mathbf{m}$ or are $(\mathbf{m}, \gamma)$-equivalent:

(a) if $\gamma = 0$, then letting $L_\ell = (M_{\mathbf{m}} \cup \mathrm{rang}(\bar{s}_\ell))$ for $\ell = 1, 2$ there is h such that:

---

[27] So maybe we can use $\lambda_0 = \lambda$.

(α) h is an isomorphism from $\mathbf{m}\restriction L_1$ onto $\mathbf{m}\restriction L_2$,
(β) h maps $\bar{s}_1$ to $\bar{s}_2$,
(γ) $h\restriction M_{\mathbf{m}}$ is the identity,
(δ) h induces an isomorphism from $\mathbb{P}_{\mathbf{m}}(L_1)$ onto $\mathbb{P}_{\mathbf{m}}(L_2)$ (as defined in 1.5$(*)_4(b)$),
(ε) moreover, h induces an isomorphism from $\mathbb{P}_{\mathbf{m}}[L_1]$ onto $\mathbb{P}_{\mathbf{m}}[L_2]$, as defined in 2.7, so $p^*_{t,\eta} \mapsto p^*_{h(t),\eta}$, see 2.4(3),

(b) if $\gamma = \beta + 1$ then for every $\ell \in \{1, 2\}$ for every $\varepsilon < \lambda^+$ and $\bar{s}'_\ell \in {}^{\varepsilon}(t_\ell/E''_{\mathbf{m}})$ there is $\bar{s}'_{3-\ell} \in {}^{\varepsilon}(t_{3-\ell}/E''_{\mathbf{m}})$ such that $(t_1, \bar{s}_1{}^\frown\bar{s}'_1)$, $(t_2, \bar{s}_2{}^\frown\bar{s}'_2)$ are β-equivalent,
(c) if γ is a limit ordinal then $(t_1, \bar{s}_1)$, $(t_2, \bar{s}_2)$ are β-equivalent for every $\beta < \gamma$.

Remark 3.11 (1) Note above that if $\bar{s}_\ell$ is the empty sequence then t_ℓ would not be determined by $\bar{s}_\ell$, still in those cases the equivalence just means $\bar{s}_1 = \bar{s}_2$.

(2) We can use $t/E_{\mathbf{m}}$ or $t/E'_{\mathbf{m}}$ instead of $t/E''_{\mathbf{m}}$ as everything is over $M_{\mathbf{m}}$.

Claim 3.12 *For $\mathbf{m} \in \mathbf{M}$ and ordinal α the number of equivalence classes of "being $(\mathbf{m}, \alpha)$-equivalent" is $\leq \beth_{1+\alpha+1}(\lambda_1)$.*

Proof 3.12 By induction on α.

Case 1: $\alpha = 0$:

Note that the set of elements of $\mathbb{P}_{\mathbf{m}}(M_{\mathbf{m}} \cup \text{rang}(\bar{s}))$ has cardinality $\leq 2^{\lambda_1}$ (and even $\leq (\lambda_1)^{\lambda}$) and depends just on $\mathbf{m}\restriction(M_{\mathbf{m}} \cup \text{rang}(\bar{s}))$ but there are $\beth_2(\lambda_1)$ possibilities for the quasi order on $\mathbb{P}_{\mathbf{m}}(L_1)$ and even for $\mathbb{P}_{\mathbf{m}}[L_1]$.

Case 2: α is a limit ordinal:

By clause (c) of Definition 3.10, the number of α-equivalence classes is $\leq \prod_{\beta<\alpha}$ (the number of β-equivalence classes) $\leq \prod_{\beta<\alpha} \beth_{1+\beta+1}(\lambda_1) \leq (\beth_{1+\alpha+1}(\lambda_1))^{\beth_{1+\alpha}} = \beth_{1+\alpha+1}(\lambda_1)$.

Case 3: $\alpha = \beta + 1$:

Clearly every α-equivalence class can be coded as a set of β-equivalence classes hence the number of α-equivalence classes is $\leq 2^{\beth_{1+\beta+1}(\lambda_1)} = \beth_{1+\beta+2}(\lambda_1) = \beth_{1+\alpha+1}(\lambda_1)$, as promised. □

Definition 3.13 For an ordinal β, let $\mathscr{G}_{\mathbf{m},\beta}$ be the set of functions f such that for some $t^\ell_i, \bar{s}^\ell_i$ for $i < i(*)$ and $\ell \in \{1, 2\}$ we have:

(a) $i(*) < \lambda^+$,
(b) $\langle t^\ell_i : i < i(*)\rangle$ is a sequence of pairwise non-$E''_{\mathbf{m}}$-equivalent members of $L_{\mathbf{m}} \setminus M_{\mathbf{m}}$,
(c) $\bar{s}^\ell_i \in {}^{\zeta(i)}(t^\ell_i/E''_{\mathbf{m}})$ where $\zeta(i) < \lambda^+$,
(d) $(t^1_i, \bar{s}^1_i)$, $(t^2_i, \bar{s}^2_i)$ are β-equivalent (members of $\mathscr{Y}_{\mathbf{m}}$),
(e) f is an isomorphism from $\mathbf{m}\restriction L_1$ onto $\mathbf{m}\restriction L_2$ when $L_\ell = \cup\{\text{rang}(\bar{s}^\ell_i) : i < i(*)\} \cup M_{\mathbf{m}}$,
(f) $f\restriction M_{\mathbf{m}} =$ the identity,
(g) f maps $\bar{s}^1_i$ to $\bar{s}^2_i$ for $i < i(*)$.

(2) For $f \in \mathscr{G}_{\mathbf{m},0}$ we define $\hat{f}$ as the mapping from $\mathbb{P}_{\mathbf{m}}(\text{dom}(f))$ onto $\mathbb{P}_{\mathbf{m}}(\text{rang}(f))$ induced by f; see clause 3.10(2)(a)(ε); (clearly well defined 1-to-1 function, but does it preserve the order? we shall return to this in 3.18).

3.3 Representing $p \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$

Applying this subsection in §3D we may assume all **m**-s are lean and so maybe $\lambda_0 = \lambda$ is O.K., but certainly not applying it in §4.

Claim 3.14 *Assume* **m** *is μ-wide and $\mu \geq \lambda_0$.*

(1) The conditions $p, q \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ are compatible when for some ψ the following condition holds:

(stt)$_{p,q,\psi}$ (a) $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$,
(b) $p, q \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ *and* $\mathrm{wsupp}(p) \cap \mathrm{wsupp}(q) \subseteq M_{\mathbf{m}}$, *see Definition 1.12(1)(b), equivalently* $\big(s \in \mathrm{fsupp}(p) \backslash M_{\mathbf{m}}\big) \wedge \big(t \in \mathrm{fsupp}(q) \backslash M_{\mathbf{m}}\big) \Rightarrow \neg(s E''_{\mathbf{m}} t)$,
(c) *if $\psi \leq \varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ then φ, p are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$,*
(d) *ψ, q are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, equivalently $q \nVdash_{\mathbb{P}_{\mathbf{m}}}$ "$\psi[\underset{\sim}{\mathbf{G}}] = false$".*

(2) For a dense set of $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ there are $\bar{L}, \bar{p}$ such that:

(a) $\bar{p} = \langle p_\varepsilon : \varepsilon < \mu \rangle \in {}^{\mu}(\mathbb{P}_{\mathbf{m}})$,
(b) $\bar{L} = \langle L_\varepsilon : \varepsilon < \mu \rangle$ *where* $\mathrm{fsupp}(p_\varepsilon) \subseteq L_\varepsilon$,
(c) $\mathbf{m} \restriction L_\varepsilon \leq_{\mathbf{M}} \mathbf{m}$ *so in particular* $t \in L_\varepsilon \backslash M_{\mathbf{m}} \Rightarrow t/E_{\mathbf{m}} \subseteq L_\varepsilon$,
(d) $\langle L_\varepsilon \backslash M_{\mathbf{m}} : \varepsilon < \mu \rangle$ *are pairwise disjoint,*
(e) $(L_\varepsilon \backslash M_{\mathbf{m}})/E''_{\mathbf{m}}$ *has cardinality* $< \lambda_0$, *(see 1.4(4) and 1.13(f)(γ)),*
(f) for every permutation π of μ there is an automorphism $\hat{f}$ of **m** *over $M_{\mathbf{m}}$ mapping $(L_\varepsilon, p_\varepsilon)$ to $(L_{\pi(\varepsilon)}, p_{\pi(\varepsilon)})$ for $\varepsilon < \mu$,*
(g) if $u \subseteq \mu$ has cardinality λ then ψ, $\bigvee_{\varepsilon \in u} p_\varepsilon$ are equivalent in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, i.e. $\psi \leq \bigcup_{\varepsilon \in u} p_\varepsilon \leq \psi$.

(3) Assume that L is a μ-wide initial segment of $L_{\mathbf{m}}$ and $\psi_0 \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}} \cap L]$. Then there is a pair $(\psi, \bar{p})$ satisfying $\psi_0 \leq \psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}} \cap L]$ and clauses (a)-(g) above hold and:

(h) if $\varepsilon < \mu$ then $p_\varepsilon \in \mathbb{P}_{\mathbf{m}}(L)$.

Also we can add:

(i) the sequence $\langle \eta_s : s \in L \cap M_{\mathbf{m}} \rangle$ is a generic for $\mathbb{P}_{\mathbf{m}}[L \cap M_{\mathbf{m}}]$, that is it determines $\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}[L \cap M_{\mathbf{m}}]$.

Remark 3.15 (1) In 3.14(1) instead of stt$_{p,q,\psi}$ we can use the stronger statement:

(stt)$'_{p,q,\psi}$ as there but omit clause (d) and add to clause (c): also φ, q are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$,

But the present choice is more convenient in the proof of 3.14(1).

(2) We use $\lambda > \aleph_0$ in the proof, to eliminate it we can imitate the completeness theorem[28] for $\mathbb{L}_{\aleph_1,\aleph_0}$.

[28] but we give details. First as a warm up notice that (for $\lambda = \aleph_0$):

($*$) if $r \in \mathbb{P}_{\mathbf{m}}$ then we can find $\mathscr{T}$ and $\bar{r}, \bar{s}$ such that:

(a) (α) $\mathscr{T}$ is a sub-tree of ${}^{\omega>}\omega$ which is well founded,
(β) if $\eta \in \mathscr{T}$, then $\mathrm{suc}_{\mathscr{T}}(\eta)$ is empty or is ω.
(b) $\bar{r} = \langle r_\eta : \eta \in \mathscr{T} \rangle$ and $r_{\langle\rangle} = r$,
(c) $r_\eta \in \mathbb{P}_{\mathbf{m}}$ and $r_\eta \subseteq r_\nu$ for $\eta \trianglelefteq \nu \in \mathscr{T}$,
(d) $\bar{s} = \langle s_\eta : \eta \in \mathscr{T} \setminus \max(\mathscr{T}) \rangle$ such that $\eta \triangleleft \nu \Rightarrow s_\nu \nleq s_\eta$,
(e) if $\eta = \nu^\frown\langle k \rangle \in \mathscr{T}$, then $s_\nu \in \mathrm{dom}(r_\nu) \cap M_{\mathbf{m}}$ and $r_\nu \restriction (\mathrm{dom}(r_\nu) \setminus L_{\mathbf{m}(\leq s_\nu)}) = r_\eta \restriction (\mathrm{dom}(r_\nu) \setminus L_{\mathbf{m}(s_\nu)})$,
(f) if $\eta \in \max(\mathscr{T})$, then $\mathrm{dom}(r_\eta) \cap M_{\mathbf{m}} = \{s_{\eta \restriction \ell} : 0 < \ell \leq \lg(\eta)\}$,

Proof 3.14 (1) We choose (p_n, q_n, ψ_n) by induction on n such that:

$\boxplus_n$ (a) (α) $(\mathrm{stt})_{p_n,q_n,\psi_n}$ holds if n is even,

(β) $(\mathrm{stt})_{q_n,p_n,\psi_n}$ holds if n is odd,

(b) $(p_0, q_0, \psi_0) = (p, q, \psi)$,

(c) if $n = 2m + 1$ and $s \in \mathrm{dom}(p_{2m}) \cap M_{\mathbf{m}}$, then $s \in \mathrm{dom}(q_{2m+1})$, and $\mathrm{tr}(p_{2m}(s)) \lhd \mathrm{tr}(q_{2m+1}(s))$,

(d) if $n = 2m+2$ and $s \in \mathrm{dom}(q_{2m+1}) \cap M_{\mathbf{m}}$, then $s \in \mathrm{dom}(p_{2m+2})$ and $\mathrm{tr}(q_{2m+1}(s)) \lhd \mathrm{tr}(p_{2m+2}(s))$,

(e) if $n = m + 1$ then $p_m \le p_n$, $q_m \le q_n$.

Case 1: For $n = 0$ use clause (b).

Case 2: $n = 2m + 1$.

So the triple $(p_{2m}, q_{2m}, \psi_{2m})$ is well defined, let $u_{2m} = \mathrm{dom}(p_{2m}) \cap M_{\mathbf{m}}$ and let $\bar{\nu} = \langle \nu_s : s \in u_{2m} \rangle$ be defined by $\nu_s = \mathrm{tr}(p_{2m}(s))$.

Clearly,

$(*)_1$ $\psi_{2m} \Vdash p^*_{s,\nu_s}$ for $s \in u_{2m}$.

[Why? Clearly $p_{2m} \Vdash_{\mathbb{P}_{\mathbf{m}}} p^*_{s,\nu_s}$, i.e. $p^*_{s,\nu_s} \le p_{2m}$ in $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$, hence in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ and therefore, if $\psi_{2m} \nVdash p^*_{s,\nu_s}$, then $\psi' = \psi_{2m} \wedge \neg p^*_{s,\nu_s} \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ is $\ge \psi_{2m}$, hence compatible with p_{2m}, contradiction, see clause (c) in $(\mathrm{stt})_{p,q,\psi}$ which holds by $\boxplus_{2m}(a)(\alpha)$.]

$(*)_2$ there is $q'_{2m} \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ which is above q_{2m} and above ψ_{2m} and naturally $u_{2m} \subseteq \mathrm{dom}(q'_{2m})$ hence $s \in u_{2m}$ implies $\nu_s \subseteq \mathrm{tr}(q'_{2m}(s))$.

[Why? By clause (d) of $(\mathrm{stt})_{p_{2m},q_{2m},\psi_{2m}}$ which holds by $\boxplus_{2m}(a)(\alpha)$ recalling $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ is dense $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$; the "hence" by $(*)_1$.]

$(*)_3$ there is $\psi'_{2m} \in \mathbb{P}_{\mathbf{m}}[M_m]$ such that:

(a) if $\psi'_{2m} \le \varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ then φ, q'_{2m} are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$,

(b) if $s \in u_{2m}$ then $\psi'_{2m} \Vdash p^*_{s,\nu_s}$,

(c) $\psi_{2m} \le \psi'_{2m}$.

[Why? Obvious using the λ^+-c.c., i.e. $\psi'_{2m} = \psi_{2m} \wedge \neg(\vee\{\varphi : \varphi \in \mathscr{I}\})$ where $\mathscr{I}$ is a maximal anti-chain of members $\varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ satisfying $\varphi \perp q'_{2m}$ in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$; see more in 3.14.]

(g) if $\eta \in \mathscr{T} \setminus \max(\mathscr{T})$, then for some k we have:

- if $\ell \ge k$, then $\mathrm{tr}(r_{\eta^\frown\langle i\rangle}(s_\eta))$ has length ℓ,
- if $\ell \ge k$, $\varrho = \mathrm{tr}(r_{\eta^\frown\langle i\rangle}(s_\eta))$ for some $\varrho \in \Pi_{\varepsilon<\ell}\theta_\varepsilon$, then for every $\rho \in \Pi_{\varepsilon<\ell}\theta_\varepsilon$ satisfying $\varrho \le \ell$ and $\mathrm{tr}(r_\eta(s_\eta)) \trianglelefteq \rho$ for some $j < \omega$ we have $\rho = \mathrm{tr}(r_{\eta^\frown\langle j\rangle}(s_\eta))$.

Footnote 29 continued

This can be proved by induction on $\sup\{\mathrm{rk}(\mathrm{M}_{\mathbf{m}}(\mathrm{s})) + 1 : \mathrm{s} \in \mathrm{dom}(\mathrm{r}) \cap \mathrm{M}_{\mathbf{m}}\}$.

Let $\langle s_i : i < i_* \rangle$ lists $M_{\mathbf{m}}$ such that $s_i <_{\mathbf{m}} s_j \Rightarrow i < j$, and let $s_{i_*} = \infty$. For $i \le i_*$ let $L_i = \bigcup\{L_{\mathbf{m}(\le s_j)} : j < i\}$, it is as an initial segment of $L_{\mathbf{m}}$. We prove by induction on $i \le i_*$ that the statement holds when $p, q \in \mathbb{P}_{\mathbf{m}}(L_i)$. For $i = 0$ this is trivial and limit i it is. So assume $i = j + 1$, now if $s_i \notin \mathrm{dom}(p) \cup \mathrm{dom}(q)$ this is trivial and if $s_j \in \mathrm{dom}(p) \setminus \mathrm{dom}(q)$ this is obvious. Similarly if $s_j \in \mathrm{dom}(q) \setminus \mathrm{dom}(p)$. So assume $s_j = \mathrm{dom}(p) \cap \mathrm{dom}(q)$. as in the proof of 3.14(1), without loss of generality $\mathrm{tr}(q(s_j)) \trianglelefteq \mathrm{tr}(p(s_j))$. As in the proof of 3.14(1), for some $q_1 \in \mathbb{P}_{\mathbf{q}}$, we have $(*)_{q_1,p,\psi}$ and $q \le q_1$ and $\lg(\mathrm{tr}(q_1(s))) > \lg(\mathrm{tr}(p(s)))$, hence $\mathrm{tr}(p(s)) \lhd \mathrm{tr}(q_1(s))$.

Clearly $(*)_{q_1\restriction L_j, p\restriction L_j, \psi}$ holds, therefore $q_1\restriction L_s, p\restriction L_s$ are compatible in $\mathbb{P}_{\mathbf{m}}$, hence in $\mathbb{P}_{\mathbf{m}}(L_j)$, and let $r \in \mathbb{P}_{\mathbf{m}}(L_j)$ be a common upper bound. Now, r forces (i.e. $\Vdash_{\mathbb{P}_{\mathbf{m}}(L_j)}$) then $\underset{\sim}{f}_{q(s)}\restriction(\lg(\mathrm{tr}(q(s))), \lg(\mathrm{tr}(p(s)))) \le \mathrm{tr}(q_1(s))$, hence $r \Vdash_{\mathbb{P}_{\mathbf{m}}(L_j)}$"$p(s), q(s)$ are compatible in $\underset{\sim}{\mathbb{Q}}_{s_j}$", therefore r, p, q have a common upper bound. So we are done.

$(*)_4$ without loss of generality $\mathrm{wsupp}(q'_{2m}) \cap \mathrm{wsupp}(p_{2m}) \subseteq M_{\mathbf{m}}$.

[Why? As $\mathbf{m}$ is μ- wide using an automorphism of $\mathbf{m}$ which is the identity on $\mathrm{wsupp}(q_{2m})$, i.e. by 3.5. Even if $\mathbf{m}$ is fat this is fine.]

Lastly, let $p_n = p'_{2m}, q_n = q'_{2m}, \psi_n = \psi'_{2m}$ and check.

Case 3: $n = 2m + 2$.

Similar to case 2 with the roles of the p's and the q's interchanged.

Having carried the induction we can define p_* as the upper bound of, in fact the union of $\{p_n : n < \omega\}$ as in 1.16(3A), in particular:

$(*)_7$ (a) $(\mathrm{dom}(p_*) = \bigcup\limits_n \mathrm{dom}(p_n)$; in fact, also $\mathrm{fsupp}(p_*) = \bigcup\limits_n \mathrm{fsupp}(p_n)$ and $\mathrm{wsupp}(p_*) = \bigcup\limits_n \mathrm{wsupp}(p_n)$,

(b) if $s \in \mathrm{dom}(p_*)$ and n is minimal such that $s \in \mathrm{dom}(p_n)$ then $\mathrm{tr}(p_*(s)) = \bigcup\limits_{k \geq n} \mathrm{tr}(p_k(s))$ and $\{f_{p_*,\iota} : \iota < \iota(p_*)(s)\}$ is equal to $\{\mathrm{tr}(p_*(s)) \cup f_{p_k,\iota}\restriction[\lg(\mathrm{tr}(p_*(s))), \lambda) : \iota < \iota(p_k(s))$ for some $k \in [n, \omega)\}$.

Similarly let q_* be the upper bound of, in fact the union of $\{q_n : n < \omega\}$ as in 1.16(3A), so again, in particular:

$(*)_8$ (a) $\mathrm{dom}(q_*) = \bigcup\limits_n \mathrm{dom}(q_n)$, and also $\mathrm{fsupp}(q_*) = \bigcup\limits_n \mathrm{fsupp}(q_n)$ and $\mathrm{wsupp}(q_*) = \cup_n \mathrm{wsupp}(q_n)$,

(b) if $s \in \mathrm{dom}(p_*)$ and n is minimal such that $s \in \mathrm{dom}(q_n)$ then:

- $\bullet_1$ $\mathrm{tr}(q_*(s)) = \bigcup\limits_{k \geq n} \mathrm{tr}(q_k(s))$,
- $\bullet_2$ $\{f_{q_*,\iota} : \iota < \iota(q_*)(s)\}$ is equal to $\{\mathrm{tr}(p_*(s)) \cup f_{p_k,\iota}\restriction[\lg(\mathrm{tr}(p_*(s))), \lambda) : \iota < \iota(q_k(s))$ for some $k \in [n, \omega)\}$.

Hence,

$(*)_9$ (a) $p_*, q_* \in \mathbb{P}_{\mathbf{m}}$,

(b) $\mathrm{dom}(p_*) \cap \mathrm{dom}(q_*) \subseteq M_{\mathbf{m}}$, moreover, $\mathrm{wsupp}(p_*) \cap \mathrm{wsupp}(q_*) \subseteq M_{\mathbf{m}}$,

(c) $\mathrm{dom}(p_*) \cap M_{\mathbf{m}} = \mathrm{dom}(q_*) \cap M_{\mathbf{m}}$,

(d) if $s \in \mathrm{dom}(p_*) \cap M_{\mathbf{m}}$, equivalently, $s \in \mathrm{dom}(p_*) \cap \mathrm{dom}(q_*)$ then: $\mathrm{tr}(p_*(s)) = \mathrm{tr}(q_*(s))$.

[Why? Clause (a) by properties of $\mathbb{P}_{\mathbf{m}}$ and $p_n \leq p_{n+1}, q_n \leq q_{n+1}$ see above, clause (b) as $\mathrm{dom}(p_{2m}) \cap \mathrm{dom}(q_{2m}) \subseteq M_{\mathbf{m}}$ as $(\mathrm{stt})_{p_{2m},q_{2m},\psi_{2m}}$. Clause (c) by $\boxplus_n(c), (d)$, the first conclusion and clause (d) by $\boxplus_n(c), (d)$, the second conclusion.]

It follows that p_*, q_* are compatible in $\mathbb{P}_{\mathbf{m}}$ but $p = p_0 \leq p_*, q = q_0 \leq q_*$, so p, q are compatible as promised.

(2) Let $\psi_0 \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ be given. Let $p \in \mathbb{P}_{\mathbf{m}}$ be such that $p \Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$\psi_0[\underset{\sim}{\mathbf{G}}] = \text{true}$".

Let $\mathscr{I}_0 = \{\varphi : \varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ and φ, p are incompatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]\}$ and let $\mathscr{I}_1$ be a maximal set of pairwise incompatible members of $\mathscr{I}_0$. As $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ satisfies the λ^+-c.c., clearly $\mathscr{I}_1$ has cardinality at most λ and let $\psi = \wedge\{\neg\varphi : \varphi \in \mathscr{I}_1\}$. Clearly we have:

$(*)_1$ $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ and:

(a) if $\psi \leq \varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$, then p, φ are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$,

(b) $\psi_0 \leq \psi$ in $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$,

(c) $\psi \leq p$ in $\mathbb{P}_{\mathbf{m}}[L]$.

Let $L_0 = \cup\{t/E_{\mathbf{m}} : t \in \mathrm{fsupp}(p)\} \cup M_{\mathbf{m}}$, so $(L_0 \setminus M_{\mathbf{m}})/E''_{\mathbf{m}}$ has cardinality $< \lambda_0$ and as $\mathbf{m}$ is μ-wide, we can find L_ε, $(\varepsilon \in [1, \mu))$ as required, that is, choose an automorphism π_ε of $\mathbf{m}$ for $\varepsilon < \mu$ such that $\pi_\varepsilon \restriction M_{\mathbf{m}}$ is the identity, $\langle \pi_\varepsilon(L_0) \backslash M_{\mathbf{m}} : \varepsilon < \mu \rangle$ are pairwise disjoint where we let π_0 be the identity and so $L_\varepsilon = \pi_\varepsilon(L)$, and let $p_\varepsilon = \hat{\pi}(p)$ for $\varepsilon < \mu$. Note:

$(*)_2$ if $\varphi_1 \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, and $\mathbb{P}_{\mathbf{m}}[L] \models$ "$\psi \leq \varphi_1$" then for for all but $< \lambda$ ordinals $\varepsilon < \mu$, the conditions p_ε, φ_1 are compatible.

[Why? Let $q \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ be above φ_1 in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, so the set $\{t/E_{\mathbf{m}} : t \in \mathrm{fsupp}(q)\}$ has cardinality $< \lambda_0$.

So for every $\varepsilon < \mu$ except $< \lambda_0$ many, the sets $\mathrm{wsupp}(q) = \cup\{t/E_{\mathbf{m}} : t \in \mathrm{fsupp}(q)\}$ and $L_\varepsilon \backslash M_{\mathbf{m}}$ are disjoint. Now for every such ε, the triple (p_ε, q, ψ) satisfies the assumptions of part (1), hence p_ε, q are compatible hence p_ε, φ_1 are compatible, so $(*)_2$ holds indeed].

Now clearly $\langle (p_\varepsilon, L_\varepsilon) : \varepsilon < \mu \rangle$ satisfies clauses (a)-(f) of part (2), so we are left with clause (g), that is:

- if $u \in [\mu]^\lambda$ then $\psi, \bigvee_{\varepsilon \in u} p_\varepsilon$ are equivalent in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, i.e. $\psi \leq \bigvee_{\varepsilon \in u} p_\varepsilon \leq \psi$.

Why this holds? First by the choice of ψ, that is by $(*)_1$ clearly $p \Vdash_{\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]}$ "$\psi \in \underset{\sim}{\mathbf{G}}$" hence for $\varepsilon < \mu$ by the choice of p_ε also $p_\varepsilon \Vdash_{\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]}$ "$\psi \in \underset{\sim}{\mathbf{G}}$" hence $\psi \leq p_\varepsilon$ in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ hence $\psi \leq \vee_{\varepsilon \in u} p_\varepsilon\ \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$.

Second, for the other inequality, just note that:

$(*)_3$ if $q \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ and $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models$ "$\psi \leq q$" then q is compatible with p_ε for every $\varepsilon < \mu$ except $< \lambda$ many.

[Why does $(*)_3$ holds? as in the proof of $(*)_2$.]

(3) We use part (2) on $\mathbf{n} = \mathbf{m} \restriction L$; so find $\psi \in \mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]$ above ψ_0 satisfying clauses (a)-(g), but $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}] = \mathbb{P}_{\mathbf{m}}[L_{\mathbf{n}}] = \mathbb{P}_{\mathbf{m}}[L]$, and so clause (h) is obvious and clause (i) holds by the definition of $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$. □

Claim 3.16 *The set $\{\psi_i : i < i(*)\} \cup \{\psi_*\}$ has a common upper bound in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ <u>when</u>:*

$(*)$ *(a)* $\mathbf{m} \in \mathbf{M}$ *is μ-wide and $\mu \geq \lambda_0$,*
(b) $i(*) < \lambda$ *or just* $i_* < \lambda_0$,
(c) $L_i \subseteq L_{\mathbf{m}}$ *for* $i < i(*)$,
(d) $L_i \cap L_j = M_{\mathbf{m}}$ *for* $i \neq j < i(*)$,
(e) $\psi_* \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$,
(f) $t \in L_i \Rightarrow (t/E_{\mathbf{m}}) \subseteq L_i$,
(g) $\psi_i \in \mathbb{P}_{\mathbf{m}}[L_i]$,
(h) if $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}] \models$ "$\psi_* \leq \varphi$" *and* $i < i(*)$ *then* ψ_i, φ *are compatible in* $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, *equivalently in* $\mathbb{P}_{\mathbf{m}}[L_i]$.

Proof 3.16 We can for $i < i(*)$ replace L_i by L'_i when $M_{\mathbf{m}} \subseteq L'_i \subseteq L_i$ and the parallel of clauses (f), (g) of $(*)$ hold. Hence without loss of generality:

$(*)_1$ the set $\{t/E''_{\mathbf{m}} : t \in L_i \setminus M_{\mathbf{m}}\}$ has cardinality $< \lambda_0$.

As $\psi_* \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$, there is $p \in \mathbb{P}_{\mathbf{m}}$ such that $p \Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$\psi_*[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}] =$ true". As $\mathbf{m}$ is μ-wide, by 3.5 there is an automorphism f of $\mathbf{m}$ over $M_{\mathbf{m}}$ such that $i < i(*) \Rightarrow f''(\mathrm{wsupp}(p)) \cap L_i \subseteq M_{\mathbf{m}}$, hence without loss of generality $i < i(*) \Rightarrow \mathrm{wsupp}(p) \cap L_i \subseteq M_{\mathbf{m}}$. Now we choose p_i by induction on $i \leq i(*)$ such that:

⊞ (a) $p_i \in \mathbb{P}_{\mathbf{m}}$,
(b) $\langle p_j : j \leq i \rangle$ is increasing,

(c) if $s \in \mathrm{dom}(p_i), i < i(*)$ then $\ell g(\mathrm{tr}(p_{i+1}(s)) > i(*)$,
(d) $p_0 = p$,
(e) if $i = j + 1$ then $p_i \Vdash$ "$\psi_j[\mathbf{G}_{\mathbb{P}_\mathbf{m}}] = \mathrm{true}$",
(f) $\mathrm{wsupp}(p_i)$ hence also $\mathrm{fsupp}(p_i)$ is disjoint to $\cup\{L_j \backslash M_\mathbf{m} : j \in [i, i(*))\}$.

This is sufficient for the claim as $p_{i(*)}$ is as required. So let us carry the induction. For $i = 0$ use clause (d), for i limit by 1.16(3A) we know that $\langle p_j : j < i\rangle$ has a $\le_{\mathbb{P}_\mathbf{m}}$-upper bound p_i with domain $\cup\{\mathrm{dom}(p_j) : j < i\}$ satisfying $\mathrm{wsupp}(p_i) \subseteq \cup\{\mathrm{wsupp}(p_j) : j < i\}$ by 1.16(3A), hence p_i is as required, in particular as in clause (f).

Recall p_j is above $p_0 = p$ hence above ψ_* (in $\mathbb{P}_\mathbf{m}[L_\mathbf{m}]$). As in the proof of $(*)_3$ inside 3.14(1) (or see 4.11(1) below) there is $\varphi_j \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}]$ such that:

- $\bullet_1$ $\psi_* \le \varphi_j$,
- $\bullet_2$ if $\varphi_j \le \varphi \in \mathbb{P}_\mathbf{m}[M_\mathbf{m}]$ then p_j, φ are compatible.

Lastly, assume $i = j + 1$, by $(*)$(h) there is $q_j \in \mathbb{P}_\mathbf{m}$ above $\varphi_j \wedge \psi_j$. Because $\mathbf{m}$ is μ-wide there is an automorphism π of $\mathbf{m}$ over $M_\mathbf{m}$ satisfying $\pi{\restriction}L_j$ is the identity, so $\pi''(\mathrm{dom}(q_j) \setminus M_\mathbf{m}$ is disjoint to $\mathrm{wsupp}(p_\epsilon)$ and to L_ε for $\varepsilon \in i_* \setminus i$. So without loss of generality:

$(*)_2$ q_j itself satisfies this.

Now the statement $(\mathrm{stt})_{p_j, q_j, \varphi_j}$ holds.

[Why? because $\mathrm{wsupp}(p_j) \cap \mathrm{wsupp}(q_j) \subseteq M_\mathbf{m}$ by $(*)_2$, the choice of φ_j and q_j above.]

Hence by 3.14 p_j, q_j has a common upper bound called p_i. As $\mathbf{m}$ is wide, for some automorphism π of $\mathbf{m}$ over $M_\mathbf{m}$ such that $\pi{\restriction}\mathrm{wsupp}(p_j)$ is the identity and π"$\mathrm{wsupp}(p_j)$ is disjoint to $\cup\{L_\varepsilon : \varepsilon \in [i, i_*)\}$, hence by renaming without loss of generality:

$(*)_3$ $\mathrm{wsupp}(p_i) \setminus M_\mathbf{m}$ is disjoint to $\cup\{L_\varepsilon : \varepsilon \in [i, i_*)\}$,

Clearly p_i is as required so we have finished proving $(*)_3$.

So we have finished proving the last case in the the induction.

So we are done. □

3.4 The main result

Here we continue §3A, §3B, and in particular prove the main result, it does not rely on 3.3. Concerning §1B, we rely on it only in one point: quoting 1.26 while proving $\boxplus_{4.4}$ and the beginning of Case 3 inside the proof of 3.20, this can be avoided using §4A. We have not work out if e.g. §3D works for the fat context.

Hypothesis 3.17 We are in the lean context (for this subsection).

Conclusion 3.18 *If $\beta \ge 0$ and $\mathbf{m}$ is wide and $f \in \mathscr{G}_{\mathbf{m},\beta}$ and L_1, L_2 its domain and range respectively <u>then</u> f induces an isomorphism $\hat{f}$ from $\mathbb{P}_\mathbf{m}(L_1)$ onto $\mathbb{P}_\mathbf{m}(L_2)$.*

Remark 3.19 (1) See Definition 3.1(3); note that this claim is not covered by Definition 3.1(2).

(2) Here we use 3.2(4), so the choice in Definition 1.10(c)(γ) is justified (see Remark 3.3(1) used below in the proof).

(3) We could have separated the definition of "analyze" and its properties.

(4) Note that in Definition 3.10, we deal only with $L_1 \subseteq t/E_\mathbf{m}$ for some t.

(5) How come even $\beta = 0$ is suitable for 3.18? The point is clause $(a)(\varepsilon)$ of Definition 3.10(2). But there is no real harm using larger β.

Proof 3.18 By the definitions, clearly $\hat{f}$ is a one-to-one function from $\mathbb{P}_{\mathbf{m}}(L_1)$ onto $\mathbb{P}_{\mathbf{m}}(L_2)$. Next assume $p_1, q_1 \in \mathbb{P}_{\mathbf{m}}(L_1)$, $\mathrm{dom}(p_1) \subseteq \mathrm{dom}(q_1)$ and let $p_2 := \hat{f}(p_1), q_2 := \hat{f}(q_1)$; clearly they belong to $\mathbb{P}_{\mathbf{m}}(L_2)$. We shall prove that $\mathbb{P}_{\mathbf{m}} \models$ "$p_1 \leq q_1$" iff $\mathbb{P}_{\mathbf{m}} \models$ "$p_2 \leq q_2$".

Let $i(*) < \lambda$ and $\bar{t}_1 = \langle t^1_i : i < i(*)\rangle$ be such that:

$\oplus_1$ (a) $t^1_i \in \mathrm{fsupp}(q_1) \backslash M_{\mathbf{m}} \subseteq L_1$ such that $\mathrm{fsupp}(q_1)$ is included in $\cup\{t^1_i/E_{\mathbf{m}} : i < i(*)\}$,
(b) $\langle t^1_i : i < i(*)\rangle$ are pairwise non-$E''_{\mathbf{m}}$-equivalent.

Next let,

$\oplus_2$ (c) let $t^2_i = f(t^1_i)$ for $i < i(*)$ and let $\bar{t}_2 = \langle t^2_i : i < i(*)\rangle$,
(d) $\mathrm{fsupp}(p_\ell) \subseteq \cup\{t^\ell_i/E''_{\mathbf{m}} : i < j(*)\} \cup M_{\mathbf{m}}$, so $j(*) \leq i(*)$, for $\ell = 1, 2$.

For $i < i(*)$ let $\psi^*_{1,i} \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ be such that: $\vartheta \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ is compatible with $q_{1,i} := q_1 \upharpoonright (t^1_i/E_{\mathbf{m}})$ (the projection!) <u>iff</u> $\vartheta \wedge \psi^*_{1,i} \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$; clearly exists as $\mathbb{P}_{\mathbf{m}}$ satisfies the λ^+-c.c. Clearly $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models$ "$\psi^*_{1,i} \leq q_{1.i} \leq q_1$ for $i < i(*)$ and let $\psi^*_1 = \wedge\{\psi^*_{1,i} : i < i(*)\}$.

Now $\psi^*_1 \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ as $q_1 \Vdash$ "$\psi^*_1[\underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}}] = \mathrm{true}$". We will say "$\psi^*_1, \bar{\psi}^*_1 = \langle \psi^*_{1,i}, q_{1,i} : i < i(*)\rangle$ analyze q_1 or $(q_1, \bar{t}_1)$" <u>when</u> the above holds.

Next choose $\varphi^*_1, \langle \varphi^*_{1,i}, p_{1,i} : i < j(*)\rangle$ which analyze $p_1, \langle t^1_i : i < j(*)\rangle$ where without loss of generality $j(*) \leq i(*)$. Why possible? As above recalling $p_1 \leq q_1 \Rightarrow \mathrm{fsupp}(p_1) \subseteq \mathrm{fsupp}(q_1)$.

Lastly, let $\psi^*_{2,i} = \check{f}(\psi^*_{1,i}), p_{2,i} = \hat{f}(p_{1,i}), \psi^*_2 = \check{f}(\psi^*_1), \varphi^*_{2,i} = \check{f}(\varphi^*_{1,i}), q_{2,i} = \hat{f}(q_{1,i}), \varphi^*_2 = \check{f}(\varphi^*_1)$ where $\check{f}$ is the function from $\mathbb{L}_{\lambda_0}(Y_{L_1}, \mathbb{P}_{\mathbf{m}})$ onto $\mathbb{L}_{\lambda_0}(Y_{L_2}, \mathbb{P}_{\mathbf{m}})$ induced by f, i.e. where $\check{f}$ is the one-to-one function with domain $\mathbb{L}_{\lambda^+}[Y_{L_1}]$ defined by $p^*_{t,\eta} \mapsto p^*_{f(t),\eta}$. Now,

$(*)$ for $\ell = 1, 2$ the sequence $(p_\ell, q_\ell, \bar{\psi}^*_\ell, \bar{\psi}^*_\ell, \varphi^*_\ell, \bar{\varphi}^*_\ell)$ where $\bar{\psi}^*_\ell = \langle \psi^*_{\ell,i}, q_{\ell,i} : i < i_\ell(*)\rangle, \bar{\varphi}^*_\ell = \langle \varphi^*_{\ell,i}, p_{\ell,i} : i < i(*)\rangle$ satisfy the same demands as listed above for $\ell = 1, 2$, that is

(a) $(\psi^*_\ell, \bar{\psi}^*_\ell)$ analyze $(q_\ell, \bar{t}_\ell)$ for $\ell = 1, 2$
(b) $(\varphi^*_\ell, \bar{\varphi}^*_\ell)$ analyze $(p_\ell, \bar{t}_\ell \upharpoonright j(*))$ for $\ell = 1, 2$.

[Why? Think, recalling $f \upharpoonright (t^1_i/E_{\mathbf{m}})$ is an isomorphism from $\mathbf{m} \upharpoonright ((t^1_i/E_{\mathbf{m}}) \cap L_1)$ onto $\mathbf{m} \upharpoonright ((t^2_i/E_{\mathbf{m}}) \cap L_2)$, inducing an isomorphism between $\mathbb{P}_{\mathbf{m}}[(t^1_i/E_{\mathbf{m}}) \cap L_1$ and $\mathbb{P}_{\mathbf{m}}[(t^2_i/E_{\mathbf{m}}) \cap L_2]$ by 3.10(a)(δ) and $\psi^*_2 = \wedge\{\psi^*_{2,i} : i < i(*)\}$ is because each function $f \upharpoonright (t^1_i/E_{\mathbf{m}})$ induces the identity mapping on $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$.]

Next,

$\boxplus$ for $\ell = 1, 2$ we have $(A)_\ell \Leftrightarrow (B)_\ell$ where:

$(A)_\ell$ $\mathbb{P}_{\mathbf{m}} \models$ "$p_\ell \leq q_\ell$",
$(B)_\ell$ for every $i < j(*)$ we have $\mathbb{P}_{\mathbf{m}}[t^\ell_i/E_{\mathbf{m}}] \models$ "$(\varphi^*_\ell \wedge p_{\ell,i}) \leq (\psi^*_\ell \wedge q_{\ell,i})$".

Why? First, assume that the condition $(B)_\ell$ fails, say for i, hence there is $\vartheta \in \mathbb{P}_{\mathbf{m}}[t^\ell_i/E_{\mathbf{m}}]$ such that $\mathbb{P}_{\mathbf{m}}[t^\ell_i/E_{\mathbf{m}}] \models$ "$(\psi^*_\ell \wedge q_{\ell,i}) \leq \vartheta$", and $\varphi^*_\ell \wedge p_{\ell,i} \wedge \vartheta \notin \mathbb{P}_{\mathbf{m}}[t^\ell_i/E_{\mathbf{m}}]$. So by claim 3.16 there is $q^+_\ell \in \mathbb{P}_{\mathbf{m}}$ such that $q^+_\ell \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ is above ϑ, hence above ψ^*_ℓ and above $q_{\ell,j} = q_\ell \upharpoonright (t^\ell_j/E_{\mathbf{m}})$ for $j < i(*)$. That is, first get $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ such that $\psi \geq \psi^*_\ell$ and $[\psi \leq \psi' \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}] \Rightarrow \psi', \vartheta$ are compatible] (using $\vartheta \geq \psi^*_\ell$). Then apply 3.16 to $(\{q_{\ell,j} : j < i(*), j \neq i\} \cup \{\vartheta\}) \cup \{\psi\}$ to get q^+_ℓ. We have used $i(*) < \lambda$.

Hence by 3.2(4) the condition q^+_ℓ is above q_ℓ, but $q^+_\ell \Vdash$ "$\varphi^*_\ell \wedge p_{\ell,i}[\underset{\sim}{\mathbf{G}}] = \mathrm{false}$" as q^+_ℓ is above ϑ. However, $p_\ell \Vdash_{\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]}$ "$p_{\ell,i} \in \underset{\sim}{\mathbf{G}}$ and $\varphi^*_\ell \in \mathbf{G}$". By the last two sentences q^+_ℓ, p_ℓ are incompatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ equivalently in $\mathbb{P}_{\mathbf{m}}$. So indeed $\neg(B)_\ell \Rightarrow \neg(A)_\ell$.

For the other direction assume condition $(B)_\ell$ holds, but condition $(A)_\ell$ fails and we shall get a contradiction. So there is $q_\ell^+ \in \mathbb{P}_{\mathbf{m}}$ above q_ℓ incompatible with p_ℓ.

For each $i < i(*)$ as $(\psi_\ell^*, \langle \psi_{\ell,j}^*, q_{\ell,j} : j < i(*)\rangle)$ analyze q_ℓ, clearly $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models$ "$(\psi_\ell^* \wedge q_{\ell,i}) \leq q_\ell$" but $q_\ell \leq q_\ell^+$ hence $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models$ "$(\psi_\ell^* \wedge q_{\ell,i}) \leq q_\ell^+$", and as we are assuming clause $(B)_\ell$ we have $j < j(*) \Rightarrow \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models$ "$(\varphi_\ell^* \wedge p_{\ell,j}) \leq q_\ell^+$". Hence by 3.2(4), q_ℓ^+ is above p_ℓ in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ hence they are compatible in $\mathbb{P}_{\mathbf{m}}$, contradiction. So indeed $(B)_\ell \Rightarrow (A)_\ell$. Together, $\boxplus$ holds].

Now clearly $(B)_1 \Leftrightarrow (B)_2$, see Definition 3.10, 3.13; so by $\boxplus$ we have $(A)_1 \Leftrightarrow (A)_2$ which is the desired conclusion. $\square$

Claim 3.20 *We have $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}}$ <u>when</u>:*

(a) $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}$,
(b) *if $t \in L_{\mathbf{m}} \backslash M_{\mathbf{m}_1}$ and $\bar{s} \in {}^\zeta(t/E''_{\mathbf{m}})$, $\zeta < \lambda^+$ <u>then</u> we can find $t_i, \bar{s}_i$ for $i < \lambda^+$ such that:*

  (α) $t_i \in L_{\mathbf{m}_1} \backslash M_{\mathbf{m}_1}$,
  (β) $t_i/E''_{\mathbf{m}_1} \neq t_j/E''_{\mathbf{m}_1}$ *when* $i \neq j < \lambda^+$,
  (γ) $\bar{s}_i \in {}^\zeta(t_i/E''_{\mathbf{m}_1})$,
  (δ) $(t_i, \bar{s}_i)$ *is ξ-equivalent to* $(t, \bar{s})$ *in* $\mathbf{m}$ *where*[29] $\xi = 1$.

(c) $\mathbf{m}$ *is wide.*

Remark 3.21 In the proof we use conclusion 3.18 but not clause $(a)(\varepsilon)$ of Definition 3.10(2).

Proof 3.20

$\boxplus_1$ for $\beta \geq 0$ and $f \in \mathscr{G}_{\mathbf{m},\beta}$,

  (a) $\hat{f}$ preserves "p_2 is above p_1 in $\mathbb{P}_{\mathbf{m}}$", and its negations,
  (b) if $\beta > 0$ then $\hat{f}$ preserves also incompatibility in $\mathbb{P}_{\mathbf{m}}$.

[Why? Clause (a) holds by 3.18. For clause (b) use clause (a) and Definitions 3.10 and 3.13 or see the proof of $\boxplus_2$.]

$\boxplus_2$ if $p_i \in \mathbb{P}_{\mathbf{m}_1}$ for $i < i(*) < \lambda^+$ and $p \in \mathbb{P}_{\mathbf{m}}$ <u>then</u> there is p^* such that:

  (a) $p^* \in \mathbb{P}_{\mathbf{m}_1}$, equivalently $p^* \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}_1})$,
  (b) $\mathbb{P}_{\mathbf{m}_1} \models$ "$p_i \leq p^*$" iff $\mathbb{P}_{\mathbf{m}} \models$ "$p_i \leq p$",
  (c) $\mathbb{P}_{\mathbf{m}_1} \models$ "p_i, p^* are compatible" iff $\mathbb{P}_{\mathbf{m}} \models$ "p_i, p are compatible".

[Why? Let $q_i \in \mathbb{P}_{\mathbf{m}}$ be such that: if p_i, p are compatible in $\mathbb{P}_{\mathbf{m}}$ then $p_i \leq q_i \wedge p \leq q_i$. We can find $L_1 \subseteq L_2$ such that

- $M_{\mathbf{m}} \subseteq L_1 \subseteq L_{\mathbf{m}_1}$, $|L_1 \backslash M_{\mathbf{m}}| \leq \lambda$,
- $\{p_i : i < i(*)\} \subseteq \mathbb{P}_{\mathbf{m}}(L_1)$,
- $L_1 \subseteq L_2 \subseteq L_{\mathbf{m}}$, $|L_2 \backslash M_{\mathbf{m}}| \leq \lambda$ and $p, q_i \in \mathbb{P}_{\mathbf{m}}(L_2)$ for $i < i(*)$.

By the assumption of the claim there is $f \in \mathscr{G}_{\mathbf{m},1}$ such that:

- $\mathrm{dom}(f) \subseteq \cup\{(t/E''_{\mathbf{m}}) \cap L_2 : t \in L_2\} \cup M_{\mathbf{m}}$,
- $t \in L_1 \Rightarrow f{\restriction}(t/E_{\mathbf{m}} \cap L_2) = \mathrm{id}_{(t/E_{\mathbf{m}}) \cap L_2}$,
- if $q \in \{q_i : i < i(*)\} \cup \{p\} \cup \{p_i : i < i(*)\}$ and $t \in \mathrm{dom}(q) \backslash M_{\mathbf{m}}$ then $\mathrm{fsupp}(q(t)) \subseteq \mathrm{dom}(f)$,

29 no real harm in using larger ξ.

- $\mathrm{rang}(f) \subseteq L_{\mathbf{m}_1}$.

Let $p^* = \hat{f}(p)$: by $\boxplus_1(a)$ clearly clauses (a),(b) of $\boxplus_2$ holds; and the choice of the q_i's (and as $p \le q_2 \Rightarrow \hat{f}(p) \le \hat{f}(q_i)$) also the implication "if" of clause (c). The "only if" of clause (c) holds by $\boxplus_1(b)$ so we are done.]

$\boxplus_3$ if $p \in \mathbb{P}_{\mathbf{m}}$ then $p \in \mathbb{P}_{\mathbf{m}_1}$ iff $\mathrm{fsupp}(p) \subseteq L_{\mathbf{m}_1}$.

[Why? Obvious.]
Recalling Definition 1.28(0)(c):

$\boxplus_4$ for every ordinal γ, we have $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dq}}_{\mathbf{m},\gamma})$.

[Why? We shall prove this by induction on γ using $\boxplus_2 + \boxplus_3$.
Note that:

$\boxplus_{4.1}$ (a) $L^{\mathrm{dq}}_{\mathbf{m},\gamma} \cap L_{\mathbf{m}_1} = L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma}$,
(b) if $f \in \mathscr{G}_{\mathbf{m},\beta}$, $s \in \mathrm{dom}(f)$ and β is an ordinal then:
- $s \in L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma} \Leftrightarrow f(s) \in L^{\mathrm{dq}}_{\mathbf{m},\gamma}$,

(c) the parallel of $\boxplus_2$ holds replacing the pair $(\mathbb{P}_{\mathbf{m}_1}, \mathbb{P}_{\mathbf{m}})$ by the pair $(\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma}), \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dq}}_{\mathbf{m},\gamma}))$; so e.g. $p^* \in \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma})$,
(d) $L^{\mathrm{dq}}_{\mathbf{m},\gamma}$ is an initial segment of $L_{\mathbf{m}}$,
(e) $L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma}$ is an initial segment of $L_{\mathbf{m}_1}$,
(f) $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma}) \lessdot \mathbb{P}_{\mathbf{m}_1}(L_{\mathbf{m}_1})$, similarly for $\mathbf{m}$.

We shall use this freely. The inductive proof on γ splits to three cases.
Case 1: $\gamma = 0$.
So,

- $E = E''_{\mathbf{m}} \restriction L^{\mathrm{dq}}_{\mathbf{m},\gamma}$ is an equivalence relation on $L^{\mathrm{dq}}_{\mathbf{m},\gamma}$,
- $E \restriction L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma} = E''_{\mathbf{m}_1} \restriction L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma}$,
- if $t \in L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma}$ then $t \notin M_{\mathbf{m}_1}$, $t/E'_{\mathbf{m}_1} = t/E'_{\mathbf{m}}$, $(t/E'_{\mathbf{m}_1}) \cap L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma} = (t/E_{\mathbf{m}_1}) \cap L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma} = (t/E'_{\mathbf{m}}) \cap L^{\mathrm{dq}}_{\mathbf{m},\gamma}$ initial segment of $L_{\mathbf{m}_1}$ and of $L_{\mathbf{m}}$ and $\mathbb{P}_{\mathbf{m}}((t/E_{\mathbf{m}_1}) \cap L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma}) = \mathbb{P}_{\mathbf{m}_1}((t/E_{\mathbf{m}_1}) \cap L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma})$,
- $\mathbb{P}_{\mathbf{m}}(L^{\mathrm{dq}}_{\mathbf{m},\gamma})$ is the product with $(< \lambda)$-support of $\{\mathbb{P}_{\mathbf{m}}((t/E_{\mathbf{m}_1}) \cap L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma}) : t \in L^{\mathrm{dq}}_{\mathbf{m},\gamma}\}$,
- similarly for $\mathbf{m}_1$.

So the result should be clear.
Case 2: $\gamma = \beta + 1$
Let $M_\beta = \{s \in M_{\mathbf{m}} : \mathrm{dp}_{\mathbf{m}}(s) = \beta\}$, clearly:

$\boxplus_{4.2}$ (a) M_β is a set of pairwise incomparable elements,
(b) $s \in M_\beta \Rightarrow L_{\mathbf{m}_1,<s} \subseteq L^{\mathrm{dq}}_{\mathbf{m}_1,\beta} \wedge L_{\mathbf{m},<s} \subseteq L^{\mathrm{dq}}_{\mathbf{m}_1,\beta}$,
(c) M_β is disjoint to $L^{\mathrm{dq}}_{\mathbf{m}_1,\beta}, L^{\mathrm{dq}}_{\mathbf{m},\beta}$,
(d) $M_\beta \subseteq L^{\mathrm{dq}}_{\mathbf{m}_1,\gamma}$,
(e) $L^{\mathrm{dq}}_{\mathbf{m},\beta} \cup M_\beta$ is an initial segment of $L_{\mathbf{m}}$,
(f) $L^{\mathrm{dq}}_{\mathbf{m}_1,\beta} \cup M_\beta$ is an initial segment of $L_{\mathbf{m}_1}$.

As first half we prove:

$\boxplus_{4.3}$ $\mathbb{P}_{\mathbf{m}_1}(L^{\mathrm{dq}}_{\mathbf{m}_1,\beta} \cup M_\beta) \lessdot \mathbb{P}_{\mathbf{m}}(L^{\mathrm{dq}}_{\mathbf{m},\beta} \cup M_\beta)$.

Why? Recalling $\boxplus_{4.1}(a)$, note

$(a)^\dagger$ for $p, q \in \mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,\beta} \cup M_\beta)$ we have $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,\beta} \cup M_\beta) \models$ "$p \le q$" iff $\mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m},\gamma} \cup M_\beta) \models$ "$p \le q$".

[Why? Immediate by the definition of the order and the induction hypothesis.]

$(b)^\dagger$ if $p_1, p_2 \in \mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,\beta} \cup M_\beta)$ <u>then</u> p_1, p_2 are compatible in $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,\beta} \cup M_\beta)$ iff they are compatible in $\mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m},\beta} \cup M_\beta)$.

[Why? The implication $\Rightarrow$ holds by clause $(a)^\dagger$.

So assume $p_3 \in \mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m},\beta} \cup M_\beta)$ is a common upper bound of p_1, p_2 in $\mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m},\beta} \cup M_\beta)$ equivalently in $\mathbb{P}_{\mathbf{m}}$.

Now (by clause (b) of the claim assumption) there is $f \in \mathscr{G}_{\mathbf{m},1}$ (actually $\mathscr{G}_{\mathbf{m},0}$ suffices here) such that:

- $f{\restriction}(\mathrm{fsupp}(p_1) \cup \mathrm{fsupp}(p_2))$ is the identity, moreover
- $s \in \mathrm{wsupp}(p_1) \cup \mathrm{wsupp}(p_2) \wedge s \in \mathrm{dom}(f) \Rightarrow f(s) = s$,
- $\mathrm{dom}(f) = \cup\{\mathrm{fsupp}(p_\ell) : \ell = 1, 2, 3\}$
- $\mathrm{rang}(f) \subseteq L_{\mathbf{m}_1}$.

Hence clearly $f{\restriction}M_\beta = \mathrm{id}_{M_\beta}$ so by $\boxplus_{4.1}(b)$ we have $\mathrm{rang}(f) \subseteq L^{\rm dq}_{\mathbf{m}_1,\beta} \cup M_\beta$ so $\hat{f}(p_3) \in \mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m}_1,\beta} \cup M_\beta)$.

By $\boxplus_1$ the condition $\hat{f}(p_3)$ is a common upper bound of p_1, p_2 in $\mathbb{P}_{\mathbf{m}}$ and by the previous sentence also in $\mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m}_1,\beta} \cup M_\beta)$, so by clause $(a)^\dagger$ the conclusion of $(b)^\dagger$ holds.]

$(c)^\dagger$ If $\mathscr{I}$ is a maximal antichain in $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,\beta} \cup M_\beta)$ <u>then</u> $\mathscr{I}$ is a maximal antichain of $\mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m},\beta} \cup M_\beta)$.

[Why? As in the proof of clause $(b)^\dagger$ and of $\boxplus_2$.]

So we are done proving $\boxplus_{4.3}$.

Now we return to proving $\boxplus_4$, note

$\boxplus_{4.4}$ let $\mathscr{E} = \{(s_1, s_2) : s_1, s_2 \in L_*$ and $s_1/E_{\mathbf{m}} = s_2/E_{\mathbf{m}}\}$ where $L_* = L^{\rm dq}_{\mathbf{m},\gamma} \setminus (L^{\rm dq}_{\mathbf{m},\beta} \cup M_\beta)\}$, <u>then</u>:

(a) $\mathscr{E}$ is an equivalence relation on L_*,
(b) if $s_1, s_2 \in L_*$ and $s_1 \le_{L_{\mathbf{m}}} s_2$ then $s_1 \mathscr{E} s_2$,
(c) if $s_1, s_2 \in L_*$ and $s_1 \mathscr{E} s_2$ then $s_1 \in L^{\rm dq}_{\mathbf{m}_1,\gamma} \Leftrightarrow s_2 \in L^{\rm dq}_{\mathbf{m}_1,\gamma}$ (and both $\notin M_\beta$),
(d) if $s \in L_*$ then $L_{\mathbf{m},<s} \subseteq L^{\rm dq}_{\mathbf{m},\beta} \cup M_\beta \cup (s/\mathscr{E})$,
(e) if $s \in L_* \cap L_{\mathbf{m}_1}$ then $L_{\mathbf{m}_1,<s} \subseteq L^{\rm dq}_{\mathbf{m}_1,\beta} \cup M_\beta \cup (s/\mathscr{E})$.

Hence let $L_0 = L^{\rm dq}_{\mathbf{m}_1,\beta} \cup M_\beta$ and $L_1 = L^{\rm dq}_{\mathbf{m}_1,\gamma} = L^{\rm dq}_{\mathbf{m}_1} \cup M_\beta$ they satisfy all the assumptions of 1.26 hence its conclusion, so we are done easily proving Case 2 of $\boxplus_4$.

<u>Case 3</u>: γ is a limit ordinal

Note that in this case the set $L^{\rm dq}_{\mathbf{m},\gamma} \setminus \cup\{L^{\rm dq}_{\mathbf{m},\beta} : \beta < \gamma\}$ consist of $s \in L^{\rm dq}_{\mathbf{m},\gamma} \setminus M_{\mathbf{m}}$ which are not below any elements from $L^{\rm dq}_{\mathbf{m},<\gamma} = \cup\{L^{\rm dq}_{\mathbf{m},\beta} : \beta < \gamma\}$ hence as in case 2 we can treat them as in the proof of $\boxplus_{4.4}$, citing 1.26, so we shall ignore them below.

Clearly $p \in \mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,<\gamma})$ iff $p \in \mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m}_1,<\gamma})$; also each of them implies $p \in \mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m},<\gamma})$. Also for $p, q \in \mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,<\gamma})$ we have $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,<\gamma}) \models$ "$p \le q$" <u>iff</u>

$\mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m},<\gamma}) \models$ "$p \le q$" by the definition of the order and the induction hypothesis. Together $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,<\gamma}) \subseteq \mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m},<\gamma})$, (as partial orders).

Next assume that $q_1, q_2 \in \mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,<\gamma})$ and p_3 is a common upper bound of q_1, q_2 in $\mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m},<\gamma})$.

We shall find $p_1 \in \mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,<\gamma})$ such that:

$(*)_1$ (a) p_1 is above q_1, q_2 (in $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,<\gamma})$ or equivalently in $\mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m}_1,<\gamma})$),
(b) if $p_1 \le p'_1 \in \mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m}_1,<\gamma})$ then p'_1, p_3 are compatible in $\mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m},<\gamma})$.

This clearly suffices; why? e.g. if $\{r_i : i < i(*)\} \subseteq \mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,<\gamma})$ is a maximal antichain of $\mathbb{P}_{\mathbf{m}_1}(L^{\rm dq}_{\mathbf{m}_1,<\gamma})$ but not of $\mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m},<\gamma})$, let $q_1 = q_2 = \emptyset$ and $p_3 \in \mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m},<\gamma})$ be incompatible with every r_i; let p_1 be as in $(*)_1$, it gives a contradiction.

If $\mathrm{cf}(\gamma) \ge \lambda$ then for some $\gamma_1 < \gamma$ we have $q_1, q_2 \in \mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m}_1,\gamma_1})$ and $\mathrm{fsupp}(p_3) \cap L^{\rm dq}_{\mathbf{m},<\gamma} \subseteq L^{\rm dq}_{\mathbf{m},\gamma_1}$ and use the induction hypothesis on γ_1 for clause (a) of $(*)_1$; for clause (b) of $(*)_1$ we also recall 1.16(6); (alternatively imitate the case $\mathrm{cf}(\gamma) < \lambda$, choosing "changing our minds" $\gamma_\varepsilon < \gamma$ with the induction). So assume $\aleph_0 \le \mathrm{cf}(\gamma) < \lambda$ and let $\langle \gamma_\varepsilon : \varepsilon < \mathrm{cf}(\gamma)\rangle$ be increasing continuous with limit γ.

Now we choose $p_{1,\varepsilon}$ by induction on $\varepsilon \le \mathrm{cf}(\gamma)$ such that:

$(*)_2$ (a) $p_{1,\varepsilon} \in \mathbb{P}_{\mathbf{m}}(L^{\rm dq}_{\mathbf{m}_1,\gamma_\varepsilon})$,
(b) $(\gamma_\varepsilon, q_1 \restriction L^{\rm dq}_{\mathbf{m},\gamma_\varepsilon}, q_2 \restriction L^{\rm dq}_{\mathbf{m},\gamma_\varepsilon}, p_3 \restriction L^{\rm dq}_{\mathbf{m},\gamma_\varepsilon}, p_{1,\varepsilon})$ are like $(\gamma, q_1, q_2, p_3, p_1)$ in $(*)_1$,
(c) $p_{1,\zeta} \le p_{1,\varepsilon}$ for $\zeta < \varepsilon$,
(d) if $\varepsilon = \zeta + 1$ and $s \in \mathrm{dom}(p_{1,\zeta})$ then $\ell g(\mathrm{tr}(p_\varepsilon(s)) > \mathrm{cf}(\gamma)$.

So we are done proving $\boxplus_4$.]

$\boxplus_5$ $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}}$.

[Why? By $\boxplus_4$ for γ large enough.]
So we are done. □

Claim 3.22 *If* $\mathbf{m} \in \mathbf{M}$ *is reduced or just* $L_{\mathbf{m}}$ *has cardinality* $\le \lambda_2$ *<u>then</u> there is* $\mathbf{n} \in \mathbf{M}_{\rm ec}$ *of cardinality* $\le \lambda_2$ *such that* $\mathbf{m} \le_{\mathbf{M}} \mathbf{n}$.

Remark 3.23 By this we may restrict ourselves to $\mathbf{M}_{\le\lambda_2}$ (but then similarly in the end of §2).

Proof 3.22 We choose χ large enough and $\mathbf{m}_* \in \mathbf{M}_\chi$ which is wide, belongs to $\mathbf{M}_{\rm ec}$ and $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_*$; moreover is full and very wide (see 3.1(1), as constructed in 1.32).

We can choose $\mathbf{n}$ such that:

$(*)$ (a) $\mathbf{n} \in \mathbf{M}$ and $\mathbf{n}$ is wide and $|L_{\mathbf{n}}| = \lambda_2$,
(b) $\mathbf{m} \le_{\mathbf{M}} \mathbf{n} \le_{\mathbf{M}} \mathbf{m}_*$,
(c) $(\mathbf{n}, \mathbf{m}_*)$ satisfies the criterion from 3.20, with $\mathbf{m}_1, \mathbf{m}$ there standing for $\mathbf{n}, \mathbf{m}_*$ here.

[Why? Let $\xi = 1$ and recalling Definition 3.10(1) choose $\langle (t_\alpha, \bar{s}_\alpha) : \alpha < \lambda_2)\rangle$ such that $(t_\alpha, \bar{s}_\alpha) \in \mathscr{Y}_{\mathbf{m}_*}$, $t_\alpha \in L_{\mathbf{m}_*} \backslash M_{\mathbf{m}_*}$, $\langle t_\alpha / E_{\mathbf{m}} : \alpha < \lambda_2\rangle$ are pairwise distinct and for every $(t, \bar{s}) \in \mathscr{Y}_{\mathbf{m}_*}$ there are λ^+ ordinals $\alpha < \lambda_2$ such that $(t, \bar{s}), (t_\alpha, \bar{s}_\alpha)$ are ξ-equivalent, possible by 3.12 recalling $\lambda_2 \ge \beth_3(\lambda_1)$. Let $L' = \cup\{t_\alpha / E_{\mathbf{m}_*} : \alpha < \lambda_2\} \cup L_{\mathbf{m}}$ and for each $t \in L' \backslash M_{\mathbf{m}_*}$ let $\langle s_{t,\alpha} : \alpha < \lambda^+\rangle$ be such that $s_{t,\alpha} \in L_{\mathbf{m}_*} \backslash M_{\mathbf{m}_*}$ and $\mathbf{m}_* \restriction (s_{t,\alpha} / E_{\mathbf{m}_*})$ is isomorphic to $\mathbf{m}_* \restriction (t / E_{\mathbf{m}_*})$ over $M_{\mathbf{m}}$. Let $L = L' \cup \{s_{t,\alpha} : \alpha < \lambda^+, t \in L' \backslash M_{\mathbf{m}_*}\}$ and $\mathbf{n} = \mathbf{m}_* \restriction L$. Now it is easy to check that $\mathbf{n}$ is as required.]

It suffices to prove that $\mathbf{n}$ belongs to $\mathbf{M}_{ec}$, let $\mathbf{n} \le_{\mathbf{M}} \mathbf{n}_1 \le_{\mathbf{M}} \mathbf{n}_2$.

Without loss of generality $L_{\mathbf{n}_2}$ has cardinality $\le 2^{\lambda_2}$, by the LST argument; (what is the LST argument here? let χ_* be large enough such that $\lambda, \mathbf{m}, \mathbf{m}_*, \mathbf{n}_1, \mathbf{n}_2$ all belong to $\mathscr{H}(\chi_*)$ and let $\mathfrak{A} \prec (\mathscr{H}(\chi_*), \in)$ be of cardinality 2_2^{λ} such that all the above belong to it and $u \subseteq \mathfrak{A} \wedge |u| \le \lambda_2 \Rightarrow u \in \mathfrak{A}$. Now replace $\mathbf{n}_1, \mathbf{n}_2$ by their restriction to $\mathfrak{A}$).

Now as $\mathbf{m}_*$ is very wide and full without loss of generality $\mathbf{n}_2 \le_{\mathbf{M}} \mathbf{m}_*$. Now $(\mathbf{n}_1, \mathbf{m}_*)$ satisfies the criterion from 3.20 hence $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{m}_*}$.

Also the pair $(\mathbf{n}_2, \mathbf{m}_*)$ satisfies the criterion from 3.20 looking at the criterion. Hence by 3.20 we have $\mathbb{P}_{\mathbf{n}_2} \lessdot \mathbb{P}_{\mathbf{m}_*}$.

As $\mathbf{n}_1 \le_{\mathbf{M}} \mathbf{n}_2 \le_{\mathbf{M}} \mathbf{m}_*$ from the last two sentences it easily follows that $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{n}_2}$, so we are done. □

Discussion 3.24 In what way does this proof help? Will it not be simpler to omit in Definition 1.10 clause (c) the $\iota_{p(s)}, \mathbf{B}_{p(s),\iota}$, etc.?

In this case in 3.1 we cannot define the projection directly hence we should look for projection as in general forcing, but then we run into problems of absoluteness. More specifically, 3.20 seems to be problematic; anyhow this does not matter.

Definition 3.25 For $\mathbf{m} \in \mathbf{M}$ and M is a subset of $M_{\mathbf{m}}$ so of cardinality $\le \lambda_1$ we define $\mathbf{n} := \mathbf{m}\langle M \rangle \in \mathbf{M}$ as follows:

(a) $L_{\mathbf{n}} = L_{\mathbf{m}}$ even as a partial order,
(b) $\bar{u}_{\mathbf{n}} = \bar{u}_{\mathbf{m}}$ and $\bar{\mathscr{P}}_{\mathbf{n}} = \bar{\mathscr{P}}_{\mathbf{m}}$,
(c) $M_{\mathbf{n}} = M$; yes M not $M_{\mathbf{m}}$!
(d) $E'_{\mathbf{n}} = \{(s,t) : s, t \in L_{\mathbf{m}}$, and $\{s,t\} \nsubseteq M\}$.

Claim 3.26 *Assume* $\mathbf{m} \in \mathbf{M}_{\le\lambda_2}$ *and* M *is a subset of* $M_{\mathbf{m}}$.

1) $\mathbf{n} := \mathbf{m}\langle M \rangle$ *indeed belongs to* $\mathbf{M}$ *and is equivalent to* $\mathbf{m}$ *hence* $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}}) = \mathbb{P}_{\mathbf{n}}(L_{\mathbf{m}})$ *i.e.* $\mathbb{P}_{\mathbf{m}} = \mathbb{P}_{\mathbf{n}}$.

2) If $\mathbf{n} = \mathbf{m}\langle M \rangle \le_{\mathbf{M}} \mathbf{n}_1$ *then for some* $\mathbf{m}_1$ *we have* $\mathbf{m} \le_{\mathbf{M}} \mathbf{m}_1$ *and* $\mathbf{m}_1, \mathbf{n}_1$ *are equivalent.*

3) If $\mathbf{m} \in \mathbf{M}_{ec}$ *and* $\mathbf{n} = \mathbf{m}\langle M \rangle$ *then* $\mathbf{n} \in \mathbf{M}_{ec}$.

4) If $\mathbf{m} \in \mathbf{M}_{wec}$ *and* $\mathbf{n} = \mathbf{m}\langle M \rangle$ *then* $\mathbf{n} \in \mathbf{M}_{wec}$.

Proof 3.26 1) Check, noting that $t \in L_{\mathbf{n}} \backslash M_{\mathbf{n}} \Rightarrow t \in L_{\mathbf{m}} \backslash M \Rightarrow |t/E'_{\mathbf{n}}| \le |L_{\mathbf{n}}| = |L_{\mathbf{m}}| \le \lambda_2$ and $|M_{\mathbf{m}}| = |M| \le |M_{\mathbf{m}}| \le \lambda_1$, (in fact, here $M \subseteq M_{\mathbf{m}}$ is not necessary, only "M has cardinality $\le \lambda_1$").

2) Given such $\mathbf{n}_1$ we now define $\mathbf{m}_1 \in \mathbf{M}$ by:

$(*)_1$ (a) $L_{\mathbf{m}_1} = L_{\mathbf{n}_1}$,
(b) $\bar{u}_{\mathbf{m}_1} = \bar{u}_{\mathbf{n}_1}$ and $\bar{\mathscr{P}}_{\mathbf{m}_1} = \bar{\mathscr{P}}_{\mathbf{n}_1}$,
(c) $M_{\mathbf{m}_1} = M_{\mathbf{m}}$,
(d) $E'_{\mathbf{m}_1} = \{(s,t) : s E'_{\mathbf{m}} t$ or $\{s,t\} \nsubseteq L_{\mathbf{m}}$ but $\{s,t\} \subseteq L_{\mathbf{n}_1}$ and $s E'_{\mathbf{n}_1} t\}$.

Clearly:

$(*)_2$ (a) $\langle M_{\mathbf{m}}\rangle \char`^ \langle s/E''_{\mathbf{m}} : s \in L_{\mathbf{m}_1} \backslash M_{\mathbf{m}}\rangle \char`^ \langle t/E''_{\mathbf{n}_1} : t \in L_{\mathbf{n}_1} \backslash L_{\mathbf{n}}\rangle$ is a partition of $L_{\mathbf{m}_1} = L_{\mathbf{n}_1}$,
(b) $E''_{\mathbf{m}_1} = E'_{\mathbf{m}_1} \restriction \{(s,t) \in E'_{\mathbf{m}_1}$ and $s, t \notin M_{\mathbf{m}}\}$ is an equivalence relation, its equivalence classes being the sets listed in clause (a) except $M_{\mathbf{m}}$,
(c) $\mathbf{m}_1$ satisfies clause $(e)(\gamma)$ of Definition 1.5.

$(*)_3$ (a) if $s \in L_{\mathbf{m}} \backslash M_{\mathbf{m}}$ then:
(α) $s \in L_{\mathbf{m}_1} \backslash M_{\mathbf{m}_1}$,
(β) $s/E'_{\mathbf{m}_1} = s/E'_{\mathbf{m}}$,

(γ) $u_{\mathbf{m}_1,s} = u_{\mathbf{n}_1,s} = u_{\mathbf{n},s} = u_{\mathbf{m},s}$,
(δ) $\mathscr{P}_{\mathbf{m}_1,s} = \mathscr{P}_{\mathbf{m},s} = \mathscr{P}_{\mathbf{n}_1,s} = \mathscr{P}_{\mathbf{n},s}$.

(b) if $s \in L_{\mathbf{m}_1} \backslash L_{\mathbf{m}}$ then:
(α) $s \in L_{\mathbf{n}_1} \backslash L_{\mathbf{n}}$,
(β) $s/E'_{\mathbf{m}_1} = s/E'_{\mathbf{n}_1}$,
(γ) $u_{\mathbf{m}_1,s} = u_{\mathbf{n}_1}, s$,
(δ) $\mathscr{P}_{\mathbf{m}_1,s} = \mathscr{P}_{\mathbf{n}_1,s}$.

(c) if $s \in M_{\mathbf{m}_1}$, equivalently $s \in M_{\mathbf{m}}$ then
(α) $u_{\mathbf{m}_1,s} = u_{\mathbf{n}_1,s}$
(β) $\mathscr{P}_{\mathbf{m}_1,s} = \mathscr{P}_{\mathbf{n}_1,s} = \mathscr{P}_{\mathbf{n},s} \cup (\mathscr{P}_{\mathbf{n}_1,s} \setminus \mathscr{P}_{\mathbf{n},s})$

and easily,

$(*)_4$ (a) indeed $\mathbf{m}_1 \in \mathbf{M}$,
(b) $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_1$,
(c) $\mathbf{m}_1, \mathbf{n}_1$ are equivalent.

So we are done.

3) Assume $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_1 \leq_{\mathbf{M}} \mathbf{n}_2$, as in the proof of part (2) there are $\mathbf{m}_1, \mathbf{m}_2$ such that $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$ and $\mathbf{m}_\ell, \mathbf{n}_\ell$ are equivalent for $\ell = 1, 2$. As $\mathbf{m} \in \mathbf{M}_{\text{ec}}$ we have $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$ but this means $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{n}_2}$, as required.

4) Similarly because $\mathbf{m} \in \mathbf{M}_{\text{wbd}} \Rightarrow \mathbf{m}\langle M \rangle \in \mathbf{M}_{\text{wbd}}$. □

Conclusion 3.27 *1) If $\mathbf{m} \in \mathbf{M}$, M is a subset of $M_{\mathbf{m}}$ and $\mathbf{n} = \mathbf{m} \restriction M$ then $\mathbb{P}^{\text{cor}}_{\mathbf{n}} \lessdot \mathbb{P}^{\text{cor}}_{\mathbf{m}}$.*

2) If $\mathbf{m}_\ell \in \mathbf{M}$ and M_ℓ is a subset of $M_{\mathbf{m}_\ell}$ for $\ell = 1, 2$ and h is an isomorphism from $\mathbf{m}_1 \restriction M_1$ onto $\mathbf{m}_2 \restriction M_2$ then h induces an isomorphism from $\mathbb{P}^{\text{cor}}_{\mathbf{m}_1}[M_1]$ onto $\mathbb{P}^{\text{cor}}_{\mathbf{m}_2}[M_2]$.

Proof 3.27 (1) Without loss of generality $\mathbf{m} \in \mathbf{M}_{\leq \lambda_2}$; (why? because trivially $\mathbf{n} \in \mathbf{M}_{\leq \lambda_1}$ and letting $\mathbf{m}_1 = \mathbf{m} \restriction M_{\mathbf{m}}$ we have $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}$ and $\mathbb{P}^{\text{cor}}_{\mathbf{m}_1} = \mathbb{P}^{\text{cor}}_{\mathbf{m}}[M_{\mathbf{m}}]$ and $\mathbf{n} = \mathbf{m}_1 \restriction M_{\mathbf{m}}$). By 3.22 there is $\mathbf{m}_* \in \mathbf{M}^{\text{ec}}_{\lambda_2}$ such that $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_*$ hence by 2.10(2) $\mathbb{P}^{\text{cor}}_{\mathbf{m}} = \mathbb{P}_{\mathbf{m}_*}[M_{\mathbf{m}}]$.

Let $\mathbf{n}_* = \mathbf{m}_*\langle M \rangle$, see Definition 3.25, so $\mathbf{n}_* \restriction M = \mathbf{n}$ and by 3.26(3) we have $\mathbf{n}_* \in \mathbf{M}_{\text{ec}}$, hence $\mathbb{P}_{\mathbf{n}_*}[M_{\mathbf{n}}] = \mathbb{P}^{\text{cor}}_{\mathbf{n}}$. But $\mathbf{n}_*, \mathbf{m}_*$ are equivalent, hence $\mathbb{P}_{\mathbf{n}_*} = \mathbb{P}_{\mathbf{m}_*}$ hence $\mathbb{P}_{\mathbf{n}_*}[L] = \mathbb{P}_{\mathbf{m}_*}[L]$ for every $L \subseteq L_{\mathbf{m}_*}$ hence by 2.10(3) $\mathbb{P}^{\text{cor}}_{\mathbf{n}} = \mathbb{P}_{\mathbf{n}_*}[M_{\mathbf{n}}] \lessdot \mathbb{P}_{\mathbf{n}_*}[M_{\mathbf{m}}] = \mathbb{P}_{\mathbf{m}_*}[M_{\mathbf{m}}] = \mathbb{P}^{\text{cor}}_{\mathbf{m}}$. So the conclusion holds.

(2) Easy, too. □

4 General m's

This section depend on §1A, §1C, §2, §3A, §3C but not on §1B, §3B, §3D.

4.1 Alternative proof

Hypothesis 4.1 We are in the general context.

This sub-section plays a double role. First, we give an alternative proof of the main results, they may be simpler but we lose some information and we are assuming $\lambda_2 \geq \beth_{\lambda_1^+}$. Second, it give proof which works also for the fat context and even the neat and general contexts not just the lean context (as in §3D). Specifically,

⊞ in this section:

(a) we ignore §1B, that is 1.24, 1.26,
(b) we ignore §3B that is 3.8–3.13
(c) we ignore or replace almost all §3D, that is:
(α) we ignore Claims 3.18, 3.19, 3.20,
(β) we replace Claim 3.22 by 4.2(2),
(γ) Def 3.25, 3.26(1),(2), (3) remains,
(δ) Claim 3.27 is replaced by 4.9 (whose proof just say "repeat the proof of 3.27").

Definition 4.2 1) Let $\Omega^1_{\mathbf{m}} := \bigcup\{\Omega^1_{\mathbf{m},t} : t \in L_{\mathbf{m}}\}$, where for $t \in L_{\mathbf{m}}$, $\Omega^1_{\mathbf{m},t}$ is the set of $\mathbf{b} = \langle t, \mathbf{B}, \bar{c}, d, c, \iota, g\rangle$ such that:

(a) $\bar{c} = \langle c_i : i < i_{\mathbf{b}} \le \lambda\rangle$,
(b) $d \subseteq c = \bigcup\{c_i : i < i_{\mathbf{b}}\} \subseteq \lambda$,
(c) $\mathbf{B}$ is a Borel function from ${}^c\mathscr{H}(\lambda)$ into $\mathscr{H}_{<\lambda(+)}(\mathscr{B}_t)$, so if $\rho \in ({}^c\mathscr{H}(\lambda))^{\mathbf{V}[\mathbb{R}]}$, then $\mathbf{B}(\rho)$ belongs to $({}^c\mathscr{H}(\lambda))^{\mathbf{V}[\mathbb{R}]}$ but not necessarily to $\mathbf{V}$,
(d) $\iota < i_{\mathbf{b}}, t \in L_{\mathbf{m}}$
(e) g is a function from c into L such that $\varepsilon \in c \Rightarrow [g(\varepsilon) \in M_{\mathbf{m}} \equiv (\varepsilon \in d)]$ and rang(g) is included in some $L \in \mathscr{P}_t$.

(2) Let $\Omega^2_{\mathbf{m}} := \bigcup\{\Omega^2_{\mathbf{m},t} : t \in L_,\}$, where for $t \in L_{\mathbf{m}}$, $\Omega^2_{\mathbf{m},t}$ is the set of $\bar{\mathbf{b}}$ such that:

(a) $\bar{\mathbf{b}} = \langle \mathbf{b}_j : j < \lg(\bar{\mathbf{b}}) \le \lambda\rangle$,
(b) $\mathbf{b}_j \in \Omega^1_{\mathbf{m},t}$,
(c) $t_{\mathbf{b}_j} = t, \iota_{\mathbf{b}_j} = j$ for $j < \lg(\bar{\mathbf{b}})$,
(d) $\langle t_{\mathbf{b}_j}, \bar{c}_{\mathbf{b}_j}, d_{\mathbf{b}_j}\rangle$ is the same for all $j < \lg(\bar{\mathbf{b}})$,

(2A) For $t \in L_{\mathbf{m}}$, we say $\bar{\mathbf{b}} \in \Omega^2_{\mathbf{m},t}$ *strictly represent* $p(t)$ <u>when</u>:

(a) $p = p\restriction\{t\} \in \mathbb{P}_{\mathbf{m}}$ and in Definition 1.10(2) we have $\iota_p(s) = 1$,
(b) $p(s)$ is $\mathbf{B}(\ldots, g_{\mathbf{b}_j(\zeta)}, \ldots)_{j \in \lg(\bar{\mathbf{b}}), \zeta \in c^i_{\mathbf{b}[j],i}}$.

(2B) We let $\Omega^3_{\mathbf{m}} := \bigcup\{\Omega^3_{\mathbf{m},t} : t \in L_{\mathbf{m}}\}$, where for $t \in L_{\mathbf{m}}$ we let $\Omega^3_{\mathbf{m},t}$ be the family of $\mathfrak{b}$ such that $\mathfrak{b}$ is a subset of $\Omega^2_{\mathbf{m},t}$ of cardinality $< \lambda$.

(2C) We say $\mathfrak{b} \in \Omega^3_{\mathbf{m}}$ represents $p(s)$ <u>when</u>:

(a) $p \in \mathbb{P}$,
(b) $s \in \mathrm{dom}(p)$,
(c) $p(s) = \sup_{\varepsilon < \varepsilon_*}(\eta, \underset{\sim}{f}_\varepsilon)$, where $\mathfrak{b} = \{\bar{\mathbf{b}}_\varepsilon : \varepsilon < \varepsilon_*\}$, each $\bar{\mathbf{b}}_\varepsilon \in \Omega^2_{\mathbf{m},*}$ and $(\eta, \underset{\sim}{f}_\varepsilon)$ is strictly represented by $\bar{\mathbf{b}}_\varepsilon$.

(3) For $\mathbf{m} \in \mathbf{M}$ we shall define a model md$(\mathbf{m})$, pedantically it is md$_{\bar{t}}(\mathbf{m})$, where[30] $\bar{t} = \langle t_\alpha = t_{\mathbf{m},\alpha} : \alpha \le \alpha_{\mathbf{m}} = \alpha(\mathbf{m})\rangle$, $t_{\alpha(\mathbf{m})}$ is a fix member of $M_{\mathbf{m}}$ $\bar{t}\restriction\alpha_{\mathbf{m}}$ is a maximal sequence of pairwise non-$E''_{\mathbf{m}}$-equivalent members of $L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ (so below $A^\ell_\alpha = A^\ell_{\mathbf{m},\alpha}$ for $\alpha \le \alpha_{\mathbf{m}}$), $t_{\alpha(\mathbf{m})} \in M_{\mathbf{m}}$ and stipulate $M_{\mathbf{m}} = t_{\alpha(\mathbf{m})}/E''_{\alpha(\mathbf{m})}$ ignoring the case $M_{\mathbf{m}} = \emptyset$:

(A) The set of elements of md$(\mathbf{m})$ is the disjoint union of the following sets; below $\alpha < \alpha(\mathbf{m})$:

(a) $A^1_\alpha = \{(1, t_\alpha, s) : s \in t_\alpha/E'_{\mathbf{m}}\}$, see 1.5(e)($\varepsilon$), ($\zeta$),
(b) $A^2_\alpha = \{(2, t_\alpha, p) : p \in \mathbb{P}_{\mathbf{m}}(t_\alpha/E_{\mathbf{m}})\} \cup \{(2, t_\alpha, p, q) : \mathbb{P}_{\mathbf{m}}(t_\alpha/E_{\mathbf{m}}) \models p \le q\}$, see Definition 1.12, central for the lean context,
(c) $A^3_\alpha = \{(3, t_\alpha, s, \mathbf{b}) : s \in t_\alpha/E''_{\mathbf{m}}, \mathbf{b} \in \Omega^1_{\mathbf{m}}$ and rang$(g_{\mathbf{b}}) \subseteq t_\alpha/E_{\mathbf{m}}\}$,

[30] So instead we can use $\langle t_\alpha\restriction E''_{\mathbf{m}} : \alpha < \alpha_*\rangle$.

(d) $A^4_\alpha = \{(4, t_\alpha, \psi) : \psi \in \mathbb{P}_{\mathbf{m}}[t_\alpha/E_{\mathbf{m}}]\} \cup \{(4, t_\alpha, \psi, \varphi) : \mathbb{P}_{\mathbf{m}}[t_\alpha/E_{\mathbf{m}}] \models \psi \le \varphi\}$,
(e) $A^1_{\alpha(\mathbf{m})} = \{(1, \alpha_{\mathbf{m}}, s, \ell) : s \in M_{\mathbf{m}}$ and $\ell = 1 \Rightarrow s \in M^{\text{lean}}_{\mathbf{n}}, \ell = 2 \Rightarrow s \in M^{\text{fat}}_{\mathbf{m}}, \ell = 0 \Rightarrow s \in M^{\text{non}}_{\mathbf{m}}\}$,
(f) $A^2_{\alpha(\mathbf{m})} = \{(2, t_{\alpha_{\mathbf{m}}}, p) : p \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})\} \cup \{(2, \alpha_{\mathbf{m}}, p, q) : \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}}) \models p \le q\}$,
(g) $A^3_{\alpha(\mathbf{m})} = \{(3, t_{\alpha_{\mathbf{m}}}, s, \mathbf{b}) : s \in M_{\mathbf{m}}, \mathbf{b} \in \Omega^1_{\mathbf{m}},$ and $\text{rang}(g_{\mathbf{b}}) \subseteq M_{\mathbf{m}}\}$,
(h) $A^4_\alpha(\mathbf{m}) = \{(4, \alpha, \psi) : \psi \in \mathbb{P}_{\mathbf{m}}[t_\alpha/E_{\mathbf{m}}]$ and $\alpha = \alpha_{\mathbf{m}}\} \cup \{(4, \alpha_{\mathbf{m}}, \psi, \varphi) : \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}] \models \psi \le \varphi\}$,
(i) notation: for $\alpha < \alpha(\mathbf{m})$, $A_\alpha = A^1_\alpha \cup A^2_\alpha \cup A^3_\alpha \cup A^4_\alpha \cup A^1_{\alpha(\mathbf{m})} \cup A^2_{\alpha(\mathbf{m})} \cup A^3_{\alpha(\mathbf{m})} \cup A^4_{\alpha(\mathbf{m})}$.

(B) The relations of $\text{md}(\mathbf{m})$ are the relations R on $\text{md}(\mathbf{m})$ such that:

(a) $R = \bigcup\{R \restriction A_\alpha : \alpha < \alpha_{\mathbf{m}}\}$,
(b) (an overkill) R is first order definable in $(\mathscr{H}(\chi_{\mathbf{m}}), \in, <^*_{\chi[\mathbf{m}]}, \mathbf{m})$, where $<^*_{\chi[\mathbf{m}]}$ is a well ordering of $\mathscr{H}(\chi_{\mathbf{m}})$.

(C) In particular there is an individual constant for each $c \in A^1_{\alpha(\mathbf{m})} \cup A^2_{\alpha(\mathbf{m})} \cup A^3_{\alpha(\mathbf{m})} \cup A^4_{\alpha(\mathbf{m})}$, or code them by unary relations.

(4) For $s \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$, the model $\text{md}(\mathbf{m}) \restriction (s/E_{\mathbf{m}})$ is naturally defined as (when $s E'_{\mathbf{m}} t_\alpha$) the restriction of the model $\text{md}(\mathbf{m})$ to $\bigcup\{A^\ell_\alpha : \ell = 1, 2, 3, 4\} \cup \{A^\ell_{\alpha(\mathbf{m})} : \ell = 1, 2, 3, 4\}$.

Definition 4.3 (1) We say that $\bar{a} \in {}^\lambda(\text{md}(\mathbf{m}))$ represents $p \in \mathbb{P}_{\mathbf{m}}$ <u>when</u> for some $\bar{\alpha}$ we have:

(a) $\bar{\alpha}$ is an sequence of ordinals $\le \alpha(\mathbf{m})$ of length $\zeta_p < \lambda$,
(b) we let $\bar{\alpha} = \langle \alpha(\varepsilon) = \langle \alpha_\varepsilon : \varepsilon < \zeta_p \rangle$,
(c) $\text{wsupp}(p)$ is equal to $\bigcup\{t_{\mathbf{m},\alpha_\varepsilon}/E_{\mathbf{m}} : \varepsilon < \zeta_p\} \cup M_{\mathbf{m}}$,
(d) if $s \in \text{dom}(p)$, then the following set $\mathfrak{b}$ represents $p(s)$, where $\mathfrak{b}$ is the set of $\bar{\mathbf{b}} \in \Omega^2_{\mathbf{m},s}$ such that for each $i < \lg(\bar{\mathbf{b}})$ we have $t_{\mathbf{b}_i} = s$ and there is an $\varepsilon < \zeta_p$ such that:

- $a_{2\varepsilon} = (1, t_{\alpha_\varepsilon}, s) \in A^1_\alpha$ and $\alpha_\varepsilon < \alpha_{\mathbf{m}}$,
- $a_{2\varepsilon} = (1, t_{\alpha_\varepsilon}, s, \ell) \in A^1_{\alpha(\mathbf{m})}$ and $\alpha_\varepsilon = \alpha_{\mathbf{m}}$,
- $a_{2\varepsilon+1} = (3, t_{\alpha_\varepsilon}, s, \mathbf{b}_i, \varepsilon) \in A^3_{\alpha_\varepsilon}$.

(e) if $\varepsilon < \zeta_p$ then one of the cases above occurs.
(f) if $\varepsilon \in [2\zeta_p, \lambda)$ then a_ε is the triple $(2, \alpha_{\mathbf{m}}, \psi) \in A^2_{\alpha(\mathbf{m})}$ where $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ is witnessed by p.

(2) We say $\bar{a}$ is a formal representative for $\mathbb{P}_{\mathbf{m}}$ <u>when</u> for some $\bar{\alpha}$ the demands above holds (ignoring the existence of p).

(3) We say $\bar{a} \in {}^\lambda(\text{md}(\mathbf{m}))$ is a formal representation of a member of $\mathbb{P}_{\mathbf{n}}[M_{\mathbf{m}}]$ similarly using $A^4_\alpha, A^4_{\alpha(\mathbf{m})}$.

Claim 4.4 *Here,*

(a) every $p \in \mathbb{P}$ is represented by some $\bar{a} \in {}^\lambda\text{md}(\mathbf{m})$,
(b) every formal representative represent some member of $\mathbb{P}_{\mathbf{m}}$,
(c) there is a formula $\psi_{\text{rep}}(\bar{x}_{[\lambda]})$ in the logic $\mathbb{L}_{\lambda_1^+,\lambda_1^+}$ in the vocabulary of $\text{md}(\mathbf{m})$ defining the set of formal representative,
(d) similarly for $p \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ more accurately $\psi \in \mathbb{L}_{\lambda^+}[Y_{\mathbf{m}}]$ not excluding contradictory ones.

Proof 4.4 Easy. □

Definition 4.5 We say L is *good* <u>when</u>:

(a) L is a initial segment of $L_{\mathbf{m}}$,
(b) L is $\mathbb{L}_{\lambda_1^+,\lambda_1^+}$-definable in $\mathrm{md}_{\bar{t}}(\mathbf{m})$ (without parameters),
(c) the following are definable in $\mathrm{md}_{\bar{t}}(\mathbf{m})$ by a formula (without parameters) in $\mathbb{L}_{\lambda_1^+,\lambda_1^+}$:

- $\bar{a}$ represent some $p \in \mathbb{P}_{\mathbf{m}}[L]$,
- $\bar{a}$ represent some $p \in \mathbb{P}_{\mathbf{m}}(L)$,
- $\bar{a}_1, \bar{a}_2$ represent $p_1, p_2 \in \mathbb{P}_{\mathbf{m}}(L)$ respectively, and $p_1 \leq_{\mathbb{P}_{\mathbf{m}(L)}} p_2$,
- $\bar{a}_1, \bar{a}_2$ represent $p_1, p_2 \in \mathbb{P}_{\mathbf{m}}[L]$ respectively and $p_1 \leq_{\mathbb{P}_{\mathbf{m}}[L]} p_2$.

Claim 4.6 *(1) The set $L = \{s \in L_{\mathbf{m}}$: for no $t \in M_{\mathbf{m}}$ do we have $t \leq_{\mathbf{m}} s\}$ is good.*
(2) If L is good and $t_* \in M_{\mathbf{m}} \setminus L$ but $L_{\mathbf{m}(<t_*)} \subseteq L$, then $L \cup \{t_*\}$ is good.
(3) If $\langle L_\alpha : \alpha < \delta\rangle$ is an $\subseteq$-increasing sequence of good sets and $\delta < \lambda_1^+$ then so is $\bigcup_{\alpha<\delta} L_\alpha$.
(4) If L is good then $L^+ = \{s$: there is no $t \in M_{\mathbf{m}} \setminus L$ such that $t \leq_{\mathbf{m}} s\}$ is good.

Proof 4.6 Notice that 1), 2) and 3) are straightforward. Concerning part (2) the reader may wonder: how do you define $\mathbb{P}_{\mathbf{m}}(L)$ not using parameters if, say, multiple such t_*'s exists?. The answer is by 4.2(3) clause (C); that is, each $t \in M_{\mathbf{m}}$ is definable without parameters.

(4) The point is that we do not like to induct on $\mathrm{dp}(s, L_{\mathbf{m}})$ just on $\mathrm{dp}(t, M_{\mathbf{m}})$. Note that the clauses on $\mathbb{P}_{\mathbf{m}}[L^+]$ follows by those on $\mathbb{P}_{\mathbf{m}}(L^+)$. What we do is noting:

$\oplus$ for $p, q \in \mathbb{P}_{\mathbf{m}}(L^+)$, $p \leq q$ iff:

(a) $p\restriction L \leq_{\mathbb{P}_{\mathbf{m}}(L)} q\restriction L$,
(b) if $s \in \mathrm{dom}(p) \setminus L$ then necessarily $s \in L^+ \setminus L$ and $s/E_{\mathbf{m}}$ appears in $\langle t_\alpha/E_{\mathbf{m}} : \alpha < \alpha_{\mathbf{m}}\rangle$ and,

- $\mathbb{P}_{\mathbf{m}}[L \cup (s/E_{\mathbf{m}})] \models p\restriction(L \cup (L^+ \cap s/E_{\mathbf{m}})) \subseteq q\restriction(L \cup (L^+ \cap s/E_{\mathbf{m}}))$.

[Why? Just think.]
Recalling 4.2(3)(A)(d) it suffice to prove that:

$(*)$ Assume $s \in L^+ \setminus L$, $p, q \in \mathbb{P}_{\mathbf{m}}$ and $\mathrm{dom}(p) \subseteq \mathrm{dom}(q) \subseteq L \cup (s/E'_{\mathbf{m}})$ then $\mathbb{P}_{\mathbf{m}} \models$ "$p \leq q$ iff (a) + (b)", where:

(a) $(p\restriction L) \leq_{\mathbb{P}_{\mathbf{m}}(L)} (q\restriction L)$,
(b) for some $\psi \in \mathbb{L}_{\lambda^+}[Y_{s/E'_{\mathbf{m}}\cap L}]$ we have:
 $\bullet_1$ $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models \psi \subseteq q$,
 $\bullet_2$ $\psi \wedge q \wedge \neg p \notin \mathbb{P}_{\mathbf{m}}[L^+ \cap (s/E'_{\mathbf{m}})]$.

[Why $(*)$ holds? As in §3C.] □

Claim 4.7 *(1) $\mathbb{P}_{\mathbf{m}}$ is $\mathbb{L}_{\lambda_1^+,\lambda_1^+}$-interpretable in* $\mathrm{md}(\mathbf{m})$.
(2) We have $\mathbb{P}_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{n}}$ when:

(a) $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$,
(b) for every $\zeta < \lambda_1^+$ and $t \in L_{\mathbf{n}} \setminus L_{\mathbf{m}}$ there are at least λ_1^+ elements $s \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ such that, recalling 4.2(3), the models $\mathrm{md}(\mathbf{n})\restriction(t/E_{\mathbf{n}})$, $\mathrm{md}(\mathbf{n})\restriction(s/E_n)$ *are $\mathbb{L}_{\beth_\zeta^+,\beth_\zeta^+}$-equivalent.*

(3) If $\mathbf{n} \in \mathbf{M}_{\mathrm{ec}}$ is wide and full, then there is $\mathbf{m}$ such that:

(a) $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}$,
(b) $L_{\mathbf{m}}$ has cardinality $\leq \beth_{\lambda_1^+}$,
(c) $\mathbf{m} \in \mathbf{M}_{\mathrm{ec}}$.

(4) Similarly to part (3) for $\mathbf{M}_{\rm bec}$.

Proof 4.7 (1) Let $\langle s_\zeta : \zeta < \zeta_* \rangle$ lists the elements of $M_{\mathbf{m}}$ such that $s_\varepsilon <_{M_{\mathbf{m}}} s_\zeta \Rightarrow \varepsilon < \zeta$: exists as $L_{\mathbf{m}}$ is a (possibly partial) well order. Clearly $\zeta_* < \|M_{\mathbf{m}}\|^+ \leq \lambda_1^+$. We define L_ζ for $\zeta \leq 2\zeta_* + 1$ as follows:

- if $\varepsilon \leq \zeta_*$, then $L_{2\varepsilon} = \{t \in L_{\mathbf{m}} :$ for some $\zeta < \varepsilon$ we have $t \leq_{L_{\mathbf{m}}} s_\zeta\}$,
- for $\varepsilon \leq \zeta_*$ we let $L_{2\varepsilon+1} = L_{2\varepsilon} \cup \{t \in L_{\mathbf{m}}$: if $\zeta \in [\varepsilon, \zeta_*)$ then $s_\zeta \nleq t$ or for some $\xi < \varepsilon$ we have $(\forall \zeta \in [\xi, \zeta_*))(s_\zeta \nleq t)\}$.

Pedantically, the models $\langle \mathrm{md}(\mathbf{m}) \restriction A_{\mathbf{m},\alpha} : \alpha < \alpha_{\mathbf{m}} \rangle$ are not pairwise disjoint but the common part consists of λ_1 individual constants hence this does not matter.

Clearly $L_{2\zeta_*+1} = L_{\mathbf{m}}$, L_ε is a definable initial segment of $L_{\mathbf{m}}$. By Definition 4.5 it suffice to prove that $L_{\mathbf{m}}$ is good.

Now we prove by induction on ε that L_ε is good, so for $\varepsilon = 2\zeta_* + 1$ we get the desired conclusion.

For $\zeta = 0$, this holds by 4.6(1).

For $\zeta = 2\varepsilon + 1$ we have $L_\zeta \setminus L_{2\varepsilon} = \{s_\varepsilon\}$ and $L_{\mathbf{m}(<s_\alpha)} \subseteq L_{2\varepsilon}$; hence by 4.6(2) we are done.

For $\zeta = 2\varepsilon + 2$ we apply 4.6(4).

Lastly, for ζ limit we apply 4.6(3). Together we are done.

(2) By part (1) and the addition theorem, (best formulated for the intermediate logic $\mathbb{L}_{\infty,\lambda_1^+,\zeta}$ for $\zeta < \lambda_1^+$), see [1]).

(3), (4) As in the proof of 4.8 below. □

Claim 4.8 *Assume* $\lambda_2 \geq \beth_{\lambda_1^+}$.

(1) If $\mathbf{n} \in \mathbf{M}$ *satisfy* $L_{\mathbf{n}} = M_{\mathbf{n}}$ *(e.g. it is isomorphic to* $(\gamma, <), \gamma < \lambda_1^+$*) then there is* $\mathbf{m} \in \mathbf{M}_{\rm ec}$ *of cardinality* λ_2 *above* $\mathbf{n}$.

(1A) Similarly for $\mathbf{M}_{\rm bec}$.

(1B) Moreover if $\mathbf{n}$ *is strongly* $(< \lambda^+)$*-directed (see 2.13(2), if* $L_{\mathbf{n}} = M_{\mathbf{n}} = (\gamma, <)$ *for some* $\gamma < \lambda_1^+$*, this mean* $\mathrm{cf}(\gamma) = \lambda$*) then (in part (1A))* $\mathbf{m}$ *is strongly* $(< \lambda^+)$*-directed, so* $\{\underset{\sim}{\eta}_r : r \in M_{\mathbf{m}}\}$ *is cofinal in* $\Pi_{\varepsilon<\lambda}\theta_\varepsilon$ *in* $\mathbf{V}^{\mathbb{P}_{\mathbf{m}}}$*, so* $\mathbf{m} \in \mathbf{M}_{\rm bec}$.

(2) If $\mathbf{m}_1 \in \mathbf{M}$ *has cardinality* $\leq \lambda_2$ *then we can demand in part (1)* $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}$.

(2A) If $\mathbf{m}_1 \in \mathbf{M}_{\rm bd}$ *has cardinality* $\leq \lambda_1$*, then in part (1A) we can demand* $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}$.

Proof 4.8 (1) Let $\mathbf{n}_\bullet$ be very wide full of cardinality 2^{λ_2} such that $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}_\bullet$ and let $\mathbf{n}_* = \mathbf{n}_\bullet^{[\rm bd]}$. We can find $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n}_*$ of cardinality λ_2 as in 4.7(2), because for every $\zeta < \lambda_1^+$ there are $< \beth_{\lambda_1^+}$ theories in the relevant vocabulary and logic. So $L_{\mathbf{m}}$ has cardinality $\leq \lambda_2$ and $\mathbf{n} \leq_{\mathbf{M}} \mathbf{m}$ but why does it belong to $\mathbf{M}_{\rm bec}$? Toward contradiction let $\mathbf{m}_1, \mathbf{m}_2 \in \mathbf{M}_{\rm bec}$ be such that $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$ but $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$ fail. By the L.S.T. argument, (see the proof of 3.22 third paragraph), without loss of generality $\mathbf{m}_2$ has cardinality $\leq 2^{\lambda_2}$, Hence by the choice of $\mathbf{m}, \mathbf{n}$ without loss of generality $\mathbf{m}_2 \leq_{\mathbf{M}} \mathbf{n}_*$. Now for $\ell = 1, 2$, by 4.7(2) applied to $(\mathbf{m}_\ell, \mathbf{n}_*)$ we have $\mathbb{P}_{\mathbf{m}_\ell} \lessdot \mathbb{P}_{\mathbf{n}_*}$. But this implies $\mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$ so we are done.

(1A) Similarly.

(1B) In the proof of part (1) we restrict ourselves to strongly $(< \lambda^+)$-directed $\mathbf{m}$-s (see 1.7(10)) so we use the relevant criterion for being in $\mathbf{M}_{\rm bec}$, see 2.17(7) i.e. consider bounded $\mathbf{m}$-s only: $\mathbf{m} \leq \mathbf{m}_1 \leq \mathbf{m}_2$, $\mathbf{m}_1, \mathbf{m}_2$ strongly λ^+-directed $\Rightarrow \mathbb{P}_{\mathbf{m}_1} \lessdot \mathbb{P}_{\mathbf{m}_2}$. The cofinality is by 1.29(3).

(2), 2A) Similarly. □

Conclusion 4.9 *(1) If* $\mathbf{m} \in \mathbf{M}$, $M \subseteq M_{\mathbf{m}}$ *and* $\mathbf{n} = \mathbf{m}{\restriction}M$ *then* $\mathbb{P}^{\rm cor}_{\mathbf{n}} \lessdot \mathbb{P}^{\rm cor}_{\mathbf{m}}$.

(2) If $\mathbf{m}_\ell \in \mathbf{M}$ *and* $M_\ell \subseteq M_{\mathbf{m}_\ell}$ *for* $\ell = 1, 2$ *and h is an isomorphism from* $\mathbf{m}_1{\restriction}M_1$ *onto* $\mathbf{m}_2{\restriction}M_2$ *then h induces an isomorphism from* $\mathbb{P}^{\rm cor}_{\mathbf{m}_1}[M_1]$ *onto* $\mathbb{P}^{\rm cor}_{\mathbf{m}_2}[M_2]$.

(3) If $\mathbf{m} \in \mathbf{M}_{\rm bec}$ *is strongly* λ^+*-directed,* $M \subseteq M_{\mathbf{m}}$ *is cofinal in* $M_{\mathbf{m}}$ *then* $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ *"*$\{\underset{\sim}{\eta}_s : s \in M\}$ *is cofinal in* $(\Pi_{\varepsilon<\lambda}\theta_\varepsilon, <_{J^{\rm bd}_\lambda})$*".*

Proof 4.9 For 1) and 2) it suffices to proceed exactly as the proof 3.27, replacing quoting 3.22 by quoting 4.8(2). Also, 3) is easy by now. □

4.2 General m's

See Discussion 4.17 for our aim and 4.24 on the connection to [12].

Definition 4.10 Assume $\mathbf{m}$ is λ_0-wide. Let $\mathbb{P}^\dagger_{\mathbf{m}} = \mathbb{P}^\dagger_{\mathbf{m}}[M_{\mathbf{m}}]$ be the forcing notion $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ restricted to the set of $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ such that there is $p \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ witnessing it, which means that (it is the projection of p into $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$, that is):

- $\bullet_1$ the condition ψ is smaller or equal to p in the forcing notion $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$,
- $\bullet_2$ if $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}] \models$ "$\psi \le \varphi$" then φ, p are compatible in the forcing notion $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$.

Claim 4.11 *Assume* $\mathbf{m}$ *to be* λ_0*-wide.*

1) $\mathbb{P}^\dagger_{\mathbf{m}}$ *is a dense subset of* $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$*, hence* $\mathbb{P}^\dagger_{\mathbf{m}} \lessdot \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$.

2) If L is an initial segment of $L_{\mathbf{m}}$ *and* $\mathbf{n} = \mathbf{m}{\restriction}L$*, then* $\mathbb{P}^\dagger_{\mathbf{n}} = \mathbb{P}^\dagger_{\mathbf{m}} \cap \mathbb{P}_{\mathbf{n}}[M_{\mathbf{n}}]$.

3) If L is a λ_0*-wide initial segment of* $L_{\mathbf{m}}$*, and* $\mathbf{n} = \mathbf{m} \restriction L$*, then:*

(a) $\mathbb{P}_{\mathbf{n}}[M_{\mathbf{n}}] \lessdot \mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]$ *and* $\mathbb{P}_{\mathbf{n}}[M_{\mathbf{n}}] = \mathbb{P}_{\mathbf{m}}[M_{\mathbf{n}}] \lessdot \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$,
(b) if $p_1 \in \mathbb{P}_{\mathbf{n}}(L_{\mathbf{n}})$ *then there is* $\psi \in \mathbb{P}_{\mathbf{n}}[M_{\mathbf{n}}]$ *satisfying:*

  (α) $\psi \le p_1 \in \mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]$,
  (β) if $\psi \le \varphi \in \mathbb{P}_{\mathbf{n}}[M_{\mathbf{n}}]$ *then* p_1, φ *are compatible in* $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]$,
  (γ) ψ *being witnessed by* p_1*, (see Definition 4.10 this follows).*

Proof 4.11 1) Let $\varphi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ and we should find $\psi \in \mathbb{P}^\dagger_{\mathbf{m}}$ above it, this suffice. Clearly there is $p_1 \in \mathbb{P}_{\mathbf{m}}$ such that $\varphi \le p_1$, that is $p_1 \Vdash$"$\varphi \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]}$". Now let $\langle \psi_i : i < i_* \rangle$ be a maximal anti-chain of members of $\mathbb{P}_{\mathbf{n}}[M_{\mathbf{n}}]$ which are incompatible with p_1 in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$. Clearly $i_* < \lambda^+$ hence without loss of generality $i_* \le \lambda$ and let $\psi = \bigwedge_{i<i_*} \neg\psi_i$. Clearly p_1 witnesses $\psi \in \mathbb{P}^\dagger_{\mathbf{m}}$ hence $\varphi \le \psi$, see more details in the proof of 4.11(3).

2) Trivially $\mathbb{P}^\dagger_{\mathbf{n}} \subseteq \mathbb{P}_{\mathbf{n}}[M_{\mathbf{n}}]$, so it suffice assume $\psi \in \mathbb{P}_{\mathbf{n}}[M_{\mathbf{n}}]$ and prove $\psi \in \mathbb{P}^\dagger_{\mathbf{n}} \Leftrightarrow \psi \in \mathbb{P}^\dagger_{\mathbf{m}}$.

First assume $\psi \in \mathbb{P}^\dagger_{\mathbf{n}}$ is witnessed by $p \in \mathbb{P}_{\mathbf{n}}$ and we shall prove that p witness $\psi \in \mathbb{P}^\dagger_{\mathbf{m}}$; we have to check the two conditions $\bullet_1 + \bullet_2$ of Definition 4.10. Now clearly $p \in \mathbb{P}_{\mathbf{m}}$ and $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ (the second because $\psi \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ and $\mathbb{P}_{\mathbf{n}} \lessdot \mathbb{P}_{\mathbf{m}}$ and $M_{\mathbf{n}} \subseteq M_{\mathbf{m}}$, hence $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{n}}] \lessdot \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$). Also $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}] \models \psi \le p$ but $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}] \lessdot \mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ hence $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}] \models \psi \le p$. So in Definition 4.10 condition $\bullet_1$ holds; for proving condition $\bullet_2$, assume that $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}] \models$"$\psi \le \varphi$" hence, by part (1), we can find $q \in \mathbb{P}_{\mathbf{m}}$ and $\vartheta \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ which is witnessed by q such that ϑ is above φ. Without loss of generality, $\mathrm{dom}(q) \cap \mathrm{dom}(p) \subseteq M_{\mathbf{m}}$ and let $q_1 = q{\restriction}L$, now q_1, ψ are compatible in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, hence in $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]$, also $\mathrm{dom}(q_1) \cap \mathrm{dom}(p)$ is included in $L = L_{\mathbf{n}}$ and is included in $\mathrm{dom}(q) \cap \mathrm{dom}(p)$ which is included in $M_{\mathbf{m}}$; together $\mathrm{dom}(q_1) \cap \mathrm{dom}(p) \subseteq L_{\mathbf{n}} \cap M_{\mathbf{m}} = M_{\mathbf{n}}$. Therefore by 3.14(1), p, q_1 are compatible in $\mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$, hence in $\mathbb{P}_{\mathbf{n}}(L_{\mathbf{n}})$, so let $r \in \mathbb{P}_{\mathbf{n}}$ be a common upper bound. As $q{\restriction}L \le_{\mathbb{P}_{\mathbf{n}}} r$, clearly r, q has a common upper bound r_1 (in $\mathbb{P}_{\mathbf{m}}$) and so r_1 is a common upper bound of φ, p.

So we are done proving one implication (the "first" above) and the second is easier: if $p \in \mathbb{P}_{\mathbf{m}}$ witness $\psi \in \mathbb{P}^{\dagger}_{\mathbf{m}}$, then $p{\restriction}L$ witness $\psi \in \mathbb{P}^{\dagger}_{\mathbf{n}}$.

3) Why?

Clause (a):

The first clause is obvious, the second recalling $\mathbb{P}_{\mathbf{n}} \lessdot \mathbb{P}_{\mathbf{m}}$ it is clear.

Clause (b):

In $\mathbb{P}_{\mathbf{n}}[M_{\mathbf{n}}]$ let $\bar{\psi} = \langle \psi_i : i < i_* \rangle, \psi$ be as in the proof of part (1) for p_1 (and $\mathbf{n}$).

Now,

(∗) $\bar{\psi}$ is maximal also for $\mathbf{m}, p_1$.

Why (∗) holds? It means: if $\vartheta \in \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ is incompatible with p_1 in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ then ϑ is compatible with some ψ_i $(i < i_*)$ in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$. But if ϑ is a counterexample then there is $p_2 \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{m}})$ above ϑ, so p_2 is incompatible with ψ_i for $i < i_*$ and with p_1. Let $q_2 \in \mathbb{P}_{\mathbf{m}}(L_{\mathbf{n}})$ be above $p_2{\restriction}L_{\mathbf{n}}$ and decide "does $\psi \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]}$". As q_2, p_2 are compatible necessarily $q_2 \Vdash$"$\psi \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{m}}[L_{\mathbf{n}}]}$" hence $q_2 \Vdash_{\mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]}$"$\psi \in \underset{\sim}{\mathbf{G}}_{\mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]}$". Let $\vartheta_* \in \mathbb{P}_{\mathbf{n}}[M_{\mathbf{n}}]$ be witnessed by q_2 (exists by part (1)) so $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}] \models$"$\psi \le \vartheta_*$". Also without loss of generality $\mathrm{dom}(q_2) \cap \mathrm{dom}(p_1) \subseteq M_{\mathbf{n}}$ (by 3.12) and p_1, q_2 are incompatible in $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]$, (otherwise p_1, q_1 would be compatible).

So by 3.12, ϑ_*, p_1 are incompatible in $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]$ so ϑ_* contradicts the maximality of $\bar{\psi}$. □

Definition 4.12 1) Let $\mathbf{R}$ be the class of objects $\mathbf{r}$ consisting of (so $N = N_{\mathbf{r}}, \mathbf{m} = \mathbf{m}_{\mathbf{r}}$, but we may omit the subscript $\mathbf{r}$ when its identity is clear from the context, also in other parts):

(a) $\mathbf{m} \in \mathbf{M}$ which is λ_2^+-wide (actually λ_1^+ suffices),
(b) a cardinal χ such that $\mathbf{m} \in \mathscr{H}(\chi)$ and $2^{|L_{\mathbf{m}}|+\lambda_2} < \chi$,
(c) $N \prec (\mathscr{H}(\chi), \in)$ such that $\mathbf{m} \in N$, and $N \cap \mathrm{Ord} = N \cap \chi$ has order type $\chi_{\mathbf{r}}$ (a cardinal $< \lambda$),
(d) $N \cap \lambda$ is an inaccessible cardinal $< \lambda$ called $\lambda_{\mathbf{r}} = \lambda(\mathbf{r}) = \lambda_N = \lambda(N)$,
(e) $\|N\| < \theta_{\lambda(\mathbf{r})}$ and[31] $[N]^{<\lambda(\mathbf{r})} \subseteq N$,
(f) $M_{\mathbf{m}}$ is listed in non-decreasing order $\bar{s}_{\mathbf{r}} = \langle s_i = s(i) : i < i(\mathbf{m}) = i_{\mathbf{m}} \rangle$ and let $s_{i(\mathbf{m})}$ be $\infty (\in L^+_{\mathbf{m}})$, (so $s_{\mathbf{r},i} = s_{\mathbf{m},i} = s_i$); let $\mathbf{U}_{\mathbf{r}} := \{j < i_{\mathbf{m}} : s_j \in N\}$ and $\mathbf{U}_{\mathbf{r},i} := \{j < i : s_j \in N\}$, and $\mathbf{U}^+_{\mathbf{r}} = \mathbf{U}_{\mathbf{r}} \cup \{i_{\mathbf{m}}\}$,
(g) for $i \in \mathbf{U}^+_{\mathbf{r}}$ let $L_{\mathbf{r},i} = \cup\{L_{\mathbf{m}(\le s_j)} : j < i\} \cap N$, and $L_{\mathbf{r}} = L_{\mathbf{r},i(\mathbf{m})} \subseteq N$ so if s_i is $<_{\mathbf{m}}$-increasing, then $i = j + 1 \Rightarrow L_{\mathbf{r},i} = L_{\mathbf{m}(\le s_j)} \cap N$,
(h) $\Xi^+_{\mathbf{r}} \neq \emptyset$, see (2B) below.

2) For $\mathbf{r} \in \mathbf{R}$ and $i \in \mathbf{U}^+_{\mathbf{r}}$ let $\Xi^{\dagger}_i = \Xi^{\dagger}_{\mathbf{r},i}$ be[32] the set of sequences $\bar{\nu}$ such that:

(a) $\bar{\nu} = \langle \nu_j : j \in \mathbf{U}_{\mathbf{r},i} \rangle$,
(b) $\nu_j \in \Pi_{\varepsilon < \lambda(\mathbf{r})} \theta_\varepsilon$,
(c) there is $\mathbf{G}$ weakly witnessing $\bar{\nu}$ which means:
  (α) $\mathbf{G} \subseteq N \cap \mathbb{P}^{\dagger}_{\mathbf{m}{\restriction}L_{\mathbf{r},i}}$ is generic over N;
  (β) if $j \in \mathbf{U}_{\mathbf{r},i}$ then $\nu_j = \underset{\sim}{\eta}_{s(j)}[\mathbf{G}]$, that is for every $\xi < \lambda_{\mathbf{r}}$, for some $\psi \in \mathbf{G}$ we have $\psi \Vdash_{\mathbb{P}_{\mathbf{m}(<s(i))}[M_{\mathbf{m}(<s(i))}]}$ "$(\underset{\sim}{\eta}_{s(j)}{\restriction}\xi) = (\nu_j{\restriction}\xi)$".

[31] We shall use just $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ has cardinality $\le \lambda_1$ because $\lambda_1 = \lambda_1^{<\lambda_0}$ in the proof $(*)_3$ in 1.32(1).

[32] Justified when $\mathbf{r}$ is clear from the context.

Note that, if $M_{\mathbf{m}} \cap N$ is not linearly ordered, then maybe $j < i$ and $s(j) \notin L_{\mathbf{m}(<s(i))}$ but $s(j) \in L_{\mathbf{r},i}$ so these two may not coincide.

2A) We have:

(a) For $\mathbf{r} \in \mathbf{R}$, $i \in \mathbf{U}^+_{\mathbf{r}}$ and $\bar\nu \in \Xi^\dagger_{\mathbf{r},i}$, let $\mathbf{G}^\dagger_{\bar\nu} = \mathbf{G}^\dagger_{\mathbf{r},\bar\nu}$ weakly witness $\bar\nu$, see 4.12(2)(c) above, so (by 4.14 below) uniquely determined (by $\bar\nu$ and $\mathbf{r}$), unlike in (2B) below.
(b) let $\Xi^\dagger = \Xi^\dagger_{\mathbf{r}}$ be $\Xi^\dagger_{\mathbf{r},i(\mathbf{m})}$[33].

2B) For $\mathbf{r} \in \mathbf{R}$ and $i \in \mathbf{U}^+_{\mathbf{r}}$ let $\Xi^+_i = \Xi^+_{\mathbf{r},i}$ be[34] the set of sequences $\bar\nu$ such that:

(a) $\bar\nu = \langle \nu_j : j < i \rangle$,
(b) $\nu_j \in \Pi_{\varepsilon<\lambda(\mathbf{r})}\theta_\varepsilon$,
(c) there is $\mathbf{G}$ strongly witnessing $\bar\nu$ which means:

  (α) $\mathbf{G} \subseteq N \cap \mathbb{P}_{\mathbf{m}}(L_{\mathbf{r},i})$ is generic over N; (<u>but</u> $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ is not sufficient),
  (β) if $j < i$ <u>then</u> $\nu_j = \underset{\sim}{\eta}_{s(j)}[\mathbf{G}]$, that is for every $\xi < \lambda_{\mathbf{r}}$, for some $\psi \in \mathbf{G}$ we have $\psi \Vdash_{\mathbb{P}_{\mathbf{m}(L_j)}[M_{\mathbf{m}(L_j)}]}$ "$(\underset{\sim}{\eta}_{s(j)}{\restriction}\xi) = (\nu_j{\restriction}\xi)$",
  (γ) $N[\mathbf{G}]$ is isomorphic to $\mathscr{H}(\chi')$ for some χ' in fact $\chi' = \chi_{\mathbf{r}}$; see clause (c) of part (1).

2C) We have:

(a) for $\bar\nu \in \Xi^+_{\mathbf{r}}$, let $\mathbf{G}^+_{\bar\nu} = \mathbf{G}^+_{\mathbf{r},\bar\nu}$ strongly witness $\bar\nu$, see 4.12(2B)(c) above, so not necessarily uniquely determined,
(b) let $\Xi^+ = \Xi^+_{\mathbf{r}}$ be $\Xi^+_{\mathbf{s},i(\mathbf{r})}$[35].

Remark 4.13 Assume $\mathbf{m}$ is λ-wide. The following Claim 4.14 justifies 4.12(2A)(a).

Claim 4.14 *Let $\mathbf{r} \in \mathbf{R}$ and $i \in \mathbf{U}^+_{\mathbf{r}}$, and $N = N_{\mathbf{r}}$.*

1) If $\mathbf{G} \subseteq \mathbb{P}_{\mathbf{m}}[L_{\mathbf{r},i} \cap M_{\mathbf{m}}] \cap N$ is generic over N <u>then</u> there is one and only one $\bar\nu \in {}^i(\Pi_{\varepsilon<\lambda_{\mathbf{r}}}\theta_\varepsilon)$ such that for every $j \in \mathbf{U}_{\mathbf{r},i}$ we have $\nu_j = \cup\{\varrho :$ there is $\psi \in \mathbf{G}$ satisfying ψ forces $\varrho \trianglelefteq \underset{\sim}{\eta}_{s(j)}\}$.

1A) We can use $\mathbb{P}^\dagger_{\mathbf{m}}$ instead of $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$.

2) If $\mathbf{G}_1, \mathbf{G}_2 \subseteq \mathbb{P}_{\mathbf{m}}[L_{\mathbf{r},i} \cap M_{\mathbf{m}}] \cap N$ are generic over N, $i \in \mathbf{U}^+_{\mathbf{r}}$ and $\bar\nu = \langle \nu_j : j < i \rangle$, and the pair $(\mathbf{G}_\ell, \bar\nu)$ is as above for $\ell = 1, 2$ <u>then</u> $\mathbf{G}_1 = \mathbf{G}_2$, (not essentially used).

3) In part (1), we have $\mathbf{G} \cap \mathbb{P}^\dagger_{\mathbf{m}} = \mathbf{G}^\dagger_{\langle \nu_j : j \in \mathbf{U}_{\mathbf{r},i}\rangle} = \mathbf{G}^\dagger_{\mathbf{r},\langle \nu_j : j \in \mathbf{U}_{\mathbf{r},i}\rangle}$, see 4.12(2A)(a).

4) Similarly for $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, $\langle \underset{\sim}{\eta}_s : s \in L_{\mathbf{m}} \cap N\rangle$ instead $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$, $\langle \underset{\sim}{\eta}_s : s \in M_{\mathbf{m}} \cap N\rangle$.

Proof 4.14 1) For $\psi \in \mathbb{P}_{\mathbf{m}}[L_{\mathbf{r},i} \cap M_{\mathbf{m}}] \cap N$ and $j < i$ let $\varrho_{\psi,j}$ be the $\trianglelefteq$-maximal ϱ such that $\psi \Vdash_{\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}(<s_j)}]}$ "$\varrho \trianglelefteq \underset{\sim}{\eta}_{s(j)}$".

Clearly,

$(*)_1$ for ψ, j as above, $\varrho_{\psi,j}$is well defined and belongs to $\cup\{\Pi_{\varepsilon<\xi}\theta_\varepsilon : \xi < \lambda_{\mathbf{r}}\}$.

[Easy, e.g. why $\lg(\varrho_{\psi,j}) < \lambda_{\mathbf{r}}$? because $\Vdash$ "$\underset{\sim}{\eta}_{s(j)} \notin \mathbf{V}$" and $\psi \in N_{\mathbf{r}}$.]

$(*)_2$ for $j < i$ and $\xi < \lambda_{\mathbf{r}}$ for some $\psi \in \mathbf{G}_{\bar\nu}$ we have $\lg(\nu_{\psi,j}) \geq \xi$.

[Why? by genericity and the definition of $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{r},i} \cap M_{\mathbf{m}}]$].

$(*)_3$ if $j < i$ and $\psi_1 \leq \psi_2$ are from $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{r},i} \cap M_{\mathbf{m}}]$ then $\varrho_{\psi_1,j} \trianglelefteq \varrho_{\psi_2,j}$.

33 Justified when $\mathbf{r}$ is clear from the context.

34 Justified when $\mathbf{r}$ is clear from the context.

35 Justified when $\mathbf{r}$ is clear from the context.

[Why? Obvious].

$(*)_4$ if $j < i$ and $\psi_1, \psi_2 \in \mathbf{G}$ where $\mathbf{G}$ is a subset of $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{r},i} \cap M_{\mathbf{m}}]$ generic over N then $\varrho_{\psi_1,j}, \varrho_{\psi_2,j}$ are $\trianglelefteq$-comparable.

[Why? as $\mathbf{G}$ is directed and $(*)_3$].

Together we are done proving part (1).

1A) Easy.

2) Toward contradiction $\mathbf{G}_1 \neq \mathbf{G}_2$ so we can assume $\psi_1 \in \mathbf{G}_1 \backslash \mathbf{G}_2$, hence there is $\psi_2 \in \mathbf{G}_2$ which is incompatible with ψ_1. Without loss of generality $\psi_1, \psi_2 \in \mathbb{P}^{\dagger}[M_{\mathbf{m}}]$. So there is $p_l \in \mathbb{P}_{\mathbf{m}}$ witnessing ψ_l (for $l = 1, 2$), and without loss of generality $\mathrm{dom}(p_1) \cap \mathrm{dom}(p_2) \subseteq M_{\mathbf{m}}$.

Now by induction on n we choose $(\psi_{1,n}, p_{1,n}, \psi_{2,n}, p_{2,n})$ such that:

$(*)_n$ for $l = 1, 2$:

(a) $p_{l,n} \in \mathbb{P}_{\mathbf{m}} \cap N_{\mathbf{r}}$ and $m < n \Rightarrow p_l \leq p_{l,m} \leq p_{l,n}$,
(b) $\psi_{l,n} \in \mathbf{G}_l$ is witnessed by $p_{l,n}$,
(c) $m < n \Rightarrow \mathbb{P}[M_{\mathbf{m}}] \models \psi_{l,m} \leq \psi_{l,n}$,
(d) $\mathrm{dom}(p_{1,n}) \cap \mathrm{dom}(p_{2,n}) \subseteq M_{\mathbf{m}}$,
(e) if $n = m+1$, $s \in \mathrm{dom}(p_{1,m}) \cap \mathrm{dom}(p_{2,m})$ then $\max\{\mathrm{lg}(\mathrm{tr}(\mathrm{p}_{1,\mathrm{m}}(\mathrm{s}))), \mathrm{lg}(\mathrm{tr}(\mathrm{p}_{2,\mathrm{m}}(\mathrm{s})))\} < \min\{\mathrm{lg}(\mathrm{tr}(\mathrm{p}_{1,\mathrm{n}}(\mathrm{s}))), \mathrm{lg}(\mathrm{tr}(\mathrm{p}_{2,\mathrm{n}}(\mathrm{s})))\}$,
(f) if $s \in \mathrm{dom}(p_{l,n}) \cap M_{\mathbf{m}}$ then $\eta^{p_{2,n}(s)} \triangleleft \nu_s$.

Why it is enough to carry the induction? Because for $l = 1, 2$ we can let p_l be the lub of the increasing sequence $\langle p_{l,n} : n < \omega \rangle$, and now p_1, p_2 are compatible (as $s \in \mathrm{dom}(p_1) \cap \mathrm{dom}(p_2)$) implies $s \in \mathrm{dom}(p_{1,n}) \cap \mathrm{dom}(p_{2,n}) \cap M_{\mathbf{m}}$ for some $n \in \omega$ which implies $\mathrm{tr}(p_1(s)) = \mathrm{tr}(p_2(s))$.

Now if q is a common upper bound p_1, p_2 in $\mathbb{P}_{\mathbf{m}}$, then it is a common upper bound of ψ_1, ψ_2 in $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$, contradicting the choice of ψ_2.

Why can we carry the induction?

In the induction step we use having enough automorphisms and (reflecting to $N_{\mathbf{r}}$).

$(*)$ if $q_1 \in \mathbb{P}_{\mathbf{m}} \cap N_{\mathbf{r}}$ witnesses $\vartheta \in G_l$ and $\zeta < \lambda_{\mathbf{r}}$ then there are $q_2 \in \mathbb{P}_{\mathbf{m}}$ and $\vartheta_2 \in \mathbb{P}^{\dagger}_{\mathbf{m}}$ such that $\vartheta_1 \leq \vartheta_2, q_1 \leq q_2, q_2$ witnesses ϑ_2 and $s \in \mathrm{dom}(q_1) \cap M_{\mathbf{m}} \Rightarrow \mathrm{lg}(\mathrm{tr}(q_2)) \geq \zeta$.

[Why? let $\mathscr{I} = \{\varphi \in \mathbb{P}^{\dagger}_{\mathbf{m}}$: <u>either</u> φ, ϑ_1 are incompatible in $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ <u>or</u> $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}] \models$ "$\psi_2 \leq \varphi$" and there is $q_2 \in \mathbb{P}_{\mathbf{m}}$ above $q_1, s \in \mathrm{dom}(q_1) \cap M_{\mathbf{m}} \Rightarrow \mathrm{lg}(\mathrm{tr}(q_2(s))) \geq \zeta$ and $q_2$witnessing $\varphi\}$. By 3.14(1), $\mathscr{I}$ is a dense subset of $\mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ and it belongs to N, so necessary $\mathscr{I} \cap \mathbf{G}_l \neq \emptyset$ and we can finish.]

3) Follows.

4) Similarly. □

We may note:

Definition 4.15 Assume that $\bar{p} = \langle p_i : i < i_* \rangle$ where $p_i \in \mathbb{P}_{\mathbf{m}}$ for $i < i_*$ and $i_* < \theta_0$ (or just $i_* < \lambda$ and $i_* < \theta_{\mathrm{lg}(\mathrm{tr}(p_i(s)))}$ for every $i < i_*, s \in \mathrm{dom}(p_i)$).

We define $q = \oplus(\bar{p})$ as the following function q:

(A) q is a function with domain $\cup\{\mathrm{dom}(p_i) : i < i_*\}$,
(B) if $s \in \mathrm{dom}(q)$ then $q(s)$ is defined as in Definition 1.10, as follows (see (c) on j_s):

(a) $\mathrm{tr}(q(s)) = \cup\langle \mathrm{tr}(p_i(s) : i < j_s$ satisfies $s \in \mathrm{dom}(p_i)\rangle$, on j_s see below,
(b) for $\varepsilon \in [\mathrm{lg}(\mathrm{tr}(q(s)), \lambda)$ we let $f_q(\varepsilon) = \sup\{p_i(s)(\varepsilon) : i < j_s$ satisfies $s \in \mathrm{dom}(p_i)\rangle\}$; pedantically we consider each of the "components" of f_q; see Definition. 1.10; <u>where</u>:

(c) $j_s = \sup\{j : j \leq i_*$ and the set $\{\mathrm{tr}(p_i(s)) : i < j$ and $s \in \mathrm{dom}(p_i)\}$ is a set of pairwise $\trianglelefteq$-comparable sequences$\}$.

Claim 4.16 *1) If (A) then (B) where:*

(A) (a) $p_i \in \mathbb{P}_{\mathbf{m}}$ *for* $i < i_*$,
(b) $i_* < \theta_0$ *at least* $i_* < \theta_{\lg(\mathrm{tr}(p_i(s)))}$ *whenever* $i < i_*, s \in \mathrm{dom}(p_i)$.
(B) (a) $q = \oplus(\bar{p})$ *is a member of* $\mathbb{P}_{\mathbf{m}}$,
(b) *if* $r \in \mathbb{P}_{\mathbf{m}}$ *is a common upper bound of* $\{p_i : i < i_*\}$ *then* $q \leq r \in \mathbb{P}_{\mathbf{m}}$,
(c) q *is a common upper bound of* $\{p_i : i < i_*\}$ *<u>when</u> in addition to (A) and (B)(a):*
(*) *if* $i_1, i_2 < i_*, s \in \mathrm{dom}(p_{i_1}) \cap \mathrm{dom}(p_{i_2})$ *and* $\lg(\mathrm{tr}(p_{i_1})) \leq \epsilon < \lg(\mathrm{tr}(p_{i_2}))$ *<u>then</u> for some* $i_3 < i_*$ *we have* $p_{i_\ell} \leq p_{i_3}$ *for* $\ell = 1, 2$.

2) If $(A)^+$ *then* $(B)^+$ *where:*

$(A)^+$ *as in (A) above adding:*

(c) p_i *witnesses* $\psi_i \in \mathbb{P}^{\dagger}_{\mathbf{m}}[M_{\mathbf{m}}]$,
(d) $\psi = \bigwedge_{i<i_*} \psi_i \in \mathbb{P}^{\dagger}_{\mathbf{m}}[M_{\mathbf{m}}]$,

$(B)^+$ *as in (B) above adding:*

(d) *if* q *is a common upper bound of* $\{p_i : i < i_*\}$ *<u>then</u>* q *witnesses* $\psi \in \mathbb{P}^{\dagger}_{\mathbf{m}}[M_{\mathbf{m}}]$.

Proof 4.16 1) Clearly $q \in \mathbb{P}_{\mathbf{m}}$. Also if $r \in \mathbb{P}_{\mathbf{m}}$ is a common upper bound of $\{p_i : i < i_*\}$ clearly $q \leq r \in \mathbb{P}_{\mathbf{m}}$.

2) Easy and will not be used. □

4.3 Nicely existentially closed

Discussion 4.17 (1) In the main case we have $M \subseteq M_{\mathbf{m}}$ cofinal in $M_{\mathbf{m}}$ and $\mathbf{m}{\restriction}M \cong \mathbf{m}{\restriction}M_{\mathbf{m}}$. In §4A we proved that if $\mathbf{m} \in M_{\mathrm{ec}}$ then $\mathbb{P}_{\mathbf{m}}[M] \cong \mathbb{P}_{\mathbf{m}}[M_{\mathbf{m}}]$ even in the general case.

Our main aim is to prove more; e.g.

(∗) (a) for $\mathbf{m}$, M as above, there is $\mathbf{n} \in \mathbf{M}$ such that $\mathbf{n} = \mathbf{m}{\restriction}L_{\mathbf{n}}$, $M_{\mathbf{n}} = M$ and $\mathbb{P}_{\mathbf{n}} \lessdot \mathbb{P}_{\mathbf{m}}$,
(b) moreover there is $\mathbf{n}$ such that $M_{\mathbf{n}} = M$, $L_{\mathbf{n}} \subseteq L_{\mathbf{m}}$, $\mathbb{P}_{\mathbf{n}}[M] = \mathbb{P}_{\mathbf{m}}[M]$ and $\mathbb{P}_{\mathbf{m}}[L_{\mathbf{m}}]$ is isomorphic to $\mathbb{P}_{\mathbf{n}}[L_{\mathbf{n}}]$ over $\mathbb{P}_{\mathbf{m}}[M]$.

(2) In the main case ($\mathbf{m} \in \mathbf{M}_{\mathrm{ec}}$ is strongly by λ^+-directed even in the general case),

- $\Vdash_{\mathbb{P}_{\mathbf{m}}}$ "$\{\underset{\sim}{\eta}_s : s \in M\}$ is cofinal in $(\Pi_{\varepsilon<\lambda}\theta_\varepsilon, <_{J^{\mathrm{bd}}_\lambda})$".

From Definition 4.18 we shall use $L' = \bigcup \mathrm{cmp}(\mathrm{M}, \mathbf{m})$ defined below.

Earlier in §3D we doctored $\mathbf{m} \in \mathbf{M}_{\leq\lambda_2}$ to an equivalent $\mathbf{n}$ such that $M_{\mathbf{n}} = M_{\mathbf{m}}$, $E''_{\mathbf{n}}$ has one equivalent class "glueing" together all $t/E_{\mathbf{n}}, t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$. Here things are more elaborated. First, in 4.18 we define the set cmp(M, $\mathbf{m}$) of the t/E'_m for which M is enough and then in 4.19 doctor $\mathbf{m}$ to an equivalent $\mathbf{n}$ with $M_{\mathbf{n}} = M$, but glueing together all $t/E'_{\mathbf{n}}$ not in cmp(M, $\mathbf{m}$). Later we can treat $\mathbf{n}$ as earlier.

Definition 4.18 Assume $\mathbf{m} \in \mathbf{M}$ and $M \subseteq M_{\mathbf{m}}$.

(1) Let cmp(M, $\mathbf{m}$) be the set of L such that for some $t \in L_{\mathbf{m}} \setminus M_{\mathbf{m}}$, we have:

(a) $L = t/E'_{\mathbf{m}} = \{s \in L_{\mathbf{m}} : sE'_{\mathbf{m}}t\}$,

Sh:1126

(b) $L \cap M_{\mathbf{m}} \subseteq M$.

(2) Let $\mathrm{cmp}^+(\mathrm{M}, \mathbf{m}) = \mathrm{cmp}(\mathrm{M}, \mathbf{m}) \cup \{\mathrm{M}\}$.

(3) For $t_* \in M$ let $\mathrm{cmp}(\mathrm{t}_*, \mathrm{M}, \mathbf{m})$ be the set of $L \in \mathrm{cmp}(\mathrm{M}, \mathbf{m})$ such that $L \subseteq L_{\mathbf{m}(\leq t_*)}$ and similarly $\mathrm{cmp}^+(\mathrm{t}_*, \mathrm{M}, \mathbf{m}) = \mathrm{cmp}(\mathrm{t}_*, \mathrm{M}, \mathbf{m}) \cup (\mathrm{M}_{\mathbf{m}(\leq \mathrm{t}_*)} \cap \mathrm{M})$.

Definition 4.19 Assume $\mathbf{m} \in \mathbf{M}$, $M \subseteq M_{\mathbf{m}}$ and $\mathscr{L} \subseteq \mathrm{cmp}(\mathrm{M}, \mathbf{m})$.

(1) We define $\mathrm{rest}(\mathscr{L}, \mathrm{M}, \mathbf{m})$ as the object $\mathbf{n}$ consisting of (intended to be in $\mathbf{M}$):

(a) the set of elements of $L_{\mathbf{n}}$ is $\bigcup\{L : L \in \mathscr{L}\} \cup M$,
(b) the order on $L_{\mathbf{m}}$ is $<_{\mathbf{m}} \restriction L_{\mathbf{n}}$,
(c) $M_{\mathbf{n}} = M$,
(d) $E'_{\mathbf{n}} = E'_{\mathbf{m}} \restriction L_{\mathbf{n}}$,
(e) $u_{\mathbf{n},s} = u_{\mathbf{m},s} \cap L_{\mathbf{n}}$,
(f) $\mathscr{P}_{\mathbf{n},s} = \mathscr{P}_{\mathbf{m},s} \cap [u_{\mathbf{n},s}]^{\leq\lambda}$.

(2) Further assume $\mathbf{m} \in \mathbf{M}_{\leq\lambda_2}$. We define $\mathrm{Rest}(\mathscr{L}, \mathrm{M}, \mathbf{m})$ as the object $\mathbf{n}$ consisting of (intended to be in $\mathbf{M}$):

(a) $L_{\mathbf{n}} = L_{\mathbf{m}}$,
(b) $M_{\mathbf{n}} = M$,
(c) $E'_{\mathbf{n}} = \{(s_1, s_2) :$ for some $L \in \mathscr{L}$, $(s_1, s_2) \in E'_{\mathbf{m}} \restriction L$ <u>or</u> $(\forall L \in \mathscr{L})(s_1, s_2 \notin L \setminus M)$, $\{s_1, s_2\} \subseteq L_{\mathbf{n}}$, $\{s_1, s_2\} \nsubseteq M_{\mathbf{n}}\}$,
(d) $u_{\mathbf{n},s} = u_{\mathbf{m},s}$ for $s \in L_{\mathbf{n}}$,
(e) $\mathscr{P}_{\mathbf{n},s} = \mathscr{P}_{\mathbf{m},s}$.

(3) We may omit $\mathscr{L}$ when $\mathscr{L} = \mathrm{cmp}(\mathrm{M}, \mathbf{m})$.

(4) For $\mathbf{r} \in \mathbf{R}$, let $\Xi^\bullet_{\mathbf{r}}$ be the set of $\bar{\nu}$ such that some $\mathbf{G}^\bullet$ witness it, which means:

(a) $\mathbf{G}^\bullet \subseteq \mathbb{P}_{\mathbf{m}} \cap N_{\mathbf{r}}$ is generic over $N_{\mathbf{r}}$,
(b) $\bar{\nu} = \langle \nu_s : s \in M_{\mathbf{m}} \cap N_{\mathbf{r}} \rangle$,
(c) $\nu_s = \underset{\sim}{\eta}_s[\mathbf{G}^\bullet]$ for $s \in M_{\mathbf{m}} \cap N_{\mathbf{r}}$.

(So compared to Definition 4.12(2B) clause (c)(γ) is not required here).

(5) For $\mathbf{r} \in \mathbf{R}$, $M' \subseteq M_{\mathbf{m}}$ such that $M' \in N_{\mathbf{r}}$ and $M = M' \cap N_{\mathbf{r}}$, let $\Xi^\bullet_M = \Xi^\bullet_{\mathbf{r},M}$ be the set of $\bar{\nu} = \langle \nu_s : s \in M \rangle$ such that some pair $(\mathbf{n}, \mathbf{G})$ witnesses $\bar{\nu}$ which means:

(α) $\mathbf{n} \leq_{\mathbf{M}} \mathrm{rest}(\mathrm{M}', \mathbf{m})$ and $\mathbf{n} \in N_{\mathbf{r}}$ and $\mathbb{P}_{\mathbf{n}} \lessdot \mathbb{P}_{\mathbf{m}}$,
(β) $\mathbf{G}$ is a subset of $\mathbb{P}_{\mathbf{n}} \cap N_{\mathbf{r}}$ generic over $N_{\mathbf{r}}$,
(γ) $\nu_s = \underset{\sim}{\eta}_s[\mathbf{G}]$ for $s \in M$.

Claim 4.20 *Assume* $\mathbf{m} \in \mathbf{M}$, $M \subseteq M_{\mathbf{m}}$, $\mathscr{L} \subseteq \mathrm{cmp}(\mathrm{M}, \mathbf{m})$, $\mathbf{n}_2 := \mathrm{Rest}(\mathscr{L}, \mathrm{M}, \mathbf{m})$ *(see 4.19(2)) and* $\mathbf{n}_1 = \mathrm{rest}(\mathscr{L}, \mathrm{M}, \mathbf{m})$, *(see 4.19(1)):*

(1) $\mathbf{n}_1 \leq_{\mathbf{M}} \mathbf{n}_2$,

(2) $L_{\mathbf{n}_2} = L_{\mathbf{m}}$, $\mathbb{P}_{\mathbf{n}_2} = \mathbb{P}_{\mathbf{m}}$ *and* $\mathrm{cmp}(\mathrm{M}, \mathbf{n}_1) = \mathscr{L} \subseteq \mathrm{cmp}(\mathrm{M}, \mathbf{n}_2)$ *and* $\mathrm{cmp}(M, \mathbf{n}_2) \setminus \mathscr{L}$ *is empty or a singleton; note that this is the single* $E'_{\mathbf{n}_2}$*-class that is not in* $\mathscr{L}_{\mathbf{n}_1}$.

(3) If $M = M_{\mathbf{m}}$ *and* $\mathscr{L} = \mathrm{cmp}(M, \mathbf{m})$ *then* $\mathbf{n}_1 = \mathbf{n}_2 = \mathbf{m}$.

(4) if $\iota \in \{1, 2\}$ *and* $\mathbf{n}_\iota \leq_{\mathbf{M}} \mathbf{n}_*$ *and* $L_{\mathbf{n}_*} \cap L_{\mathbf{m}} = L_{\mathbf{n}_\iota}$, <u>*then*</u> *we can find* $\mathbf{m}_*$ *such that:*

(a) $\mathbf{m} \leq_{\mathbf{M}} \mathbf{m}_*$ *and* $L_{\mathbf{m}_*} = L_{\mathbf{n}_*} \cup L_{\mathbf{m}}$,
(b) $\mathscr{L} \subseteq \mathrm{cmp}(\mathrm{M}, \mathbf{m}_*)$,
(c) if $\iota = 2$ *then* $\mathbf{n}_* = \mathrm{Rest}(\mathscr{L}, \mathrm{M}, \mathbf{m}_*)$,
(d) if $\iota = 1$ *letting* $\mathscr{L}_1 = \mathrm{cmp}(\mathrm{M}, \mathbf{n}_*)$ *we have* $\mathbf{n}_* = \mathrm{rest}(\mathscr{L}_1, \mathrm{M}, \mathbf{m}_*)$,
(e) (choosing minimal u) if $t \in M_{\mathbf{m}} \setminus M$ *and* $\iota = 1$, *then* $u_{\mathbf{m}_*,t} = u_{\mathbf{m},t} \cup \{s : s \in L_{\mathbf{n}_*} \setminus L_{\mathbf{m}}\ s \in u_{\mathbf{n},t}\}$.

Proof 4.20 Straightforward, as in earlier proofs (in particular 2.16) in particular for part (1), check the clause (e)(γ) of 3.1. □

Claim 4.21 *Assume* $\mathbf{m} \in \mathbf{M}$.

(1) If $t \in M_{\mathbf{m}}$ *and* $\mathbf{n} = \mathbf{m}(\leq t)$ *then* $t = \max(M_{\mathbf{n}}) = \max(L_{\mathbf{n}})$.

(2) If for every $t \in M_{\mathbf{m}}$, $\mathrm{rest}(\mathrm{M}_{\mathbf{m}(\leq t)}, \mathbf{m}) \in \mathbf{M}_{\mathrm{ec}}$ *then* $\mathbf{m} \in \mathbf{M}_{\mathrm{ec}}$ *provided that is strongly* λ^+*-directed.*

(3) If $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}_2$ *and* $t \in M_{\mathbf{m}_1}$ *then* $\mathrm{rest}(\mathrm{M}_{\mathbf{m}(\leq t),\mathbf{m}}) \leq_{\mathbf{M}} \mathrm{rest}(\mathrm{M}_{\mathbf{m}(\leq t)}, \mathbf{m})$.

Proof 4.21 Easy. □

Observation 4.22 *(1) If* $L_{\mathbf{m}} = M_{\mathbf{m}}$ *then* $\mathbf{m}$ *is essentially* λ^+*-directed (see Definition 2.13(1)) if* $\bigcup\{u_s : s \in M_{\mathbf{m}}\} = M_{\mathbf{m}} = M_{\mathbf{m}}$ *and* $(\{u_s : s \in M_{\mathbf{m}}\}, \subseteq)$ *is* λ^+*−directed.*

(2) Assume $\mathbf{m}$ *is strongly* μ*-directed and* $\mathbf{m}_1 \leq \mathbf{m}$, *then* $\mathbf{m}_1$ *is strongly* μ*-directed.*

(3) if $\mathbf{m}_1 \leq_{\mathbf{M}} \mathbf{m}$ *and* $\mathbf{m}$ *is essentially* μ*-directed then so is* $\mathbf{m}_1$.

Claim 4.23 *1) Assume* $\mathbf{m} \in \mathbf{M}_{\leq\lambda_2}$ *is strongly* λ^+*-directed (hence bounded). There are* $\mathbf{n}$ *and* $\bar{\mathscr{L}}$ *such that:*

⊞ *(a)* $\mathbf{m} \leq_{\mathbf{M}} \mathbf{n} \in \mathbf{M}_{\lambda_2}$,

(b) $\mathbf{n} \in \mathbf{M}_{\mathrm{bec}}$,

(c) $\bar{\mathscr{L}} = \langle \mathscr{L}_M : M \in \mathscr{P}^-(M_{\mathbf{m}})\rangle$, *recalling that for a set* X, $\mathscr{P}(X)^- := \{Y \subseteq X : Y \neq \emptyset\}$,

(d) $\mathscr{L}_M \subseteq \mathrm{cmp}(\mathrm{M}, \mathbf{n})$ *for* $M \in \mathscr{P}^-(M_{\mathbf{m}})$ *and* $\mathscr{L}_{M_{\mathbf{m}}} = \mathrm{cmp}(M_{\mathbf{m}}, \mathbf{m})$,

(e) $\mathbf{n}_M = \mathbf{n}[M] := \mathrm{rest}(\mathscr{L}_{\mathrm{M}}, \mathrm{M}, \mathbf{m}) \in \mathbf{M}_{\mathrm{bec}}$ *for* $M \in \mathscr{P}^-(M_{\mathbf{m}})$, *so* $\mathbf{n}_{M_{\mathbf{m}}} = \mathbf{m}$,

(f) $\mathbb{P}_{\mathbf{n}_M} \lessdot \mathbb{P}_{\mathbf{n}}$ *for* $M \in \mathscr{P}^-(M_{\mathbf{m}})$,

(g) if $t_2 <_{\mathbf{m}} t_3$ *are from* $M_{\mathbf{m}}$ *and* $t_1 \in L_{\mathbf{n}} \setminus L_{\mathbf{m}}$ *and* $(\forall s \in t_1/E''_{\mathbf{n}})$ $(s <_{\mathbf{m}} t_2)$ *then* $t_1/E''_{\mathbf{n}} \subseteq u_{\mathbf{n},t_3}$ *(yes! not* $t_1/E'_{\mathbf{n}}$*)*

(h) $\mathbf{n}$ *is strongly* λ^+*-directed (see 2.13(2)),*

(i) if $M_1 \subseteq M_2$ *are from* $\mathscr{P}^-(M_{\mathbf{m}})$ *then* $\mathscr{L}_{M_1} \subseteq \mathscr{L}_{M_2}$.

2) We can add:

(j) if $M_1, M_2 \subseteq M_{\mathbf{m}}$ *and* h *is an isomorphism from* $\mathbf{m}\restriction M_1$ *onto* $\mathbf{m}\restriction M_2$ *then* h *can be extended to* $\hat{h}$, *an isomorphism from* $\mathbf{n}_{M_1}$ *onto* $\mathbf{n}_{M_2}$,

(k) if $M \in \mathscr{P}^-(M_{\mathbf{m}})$ *and* $L \in \mathscr{L}_M$ *then* $L \cap L_{\mathbf{m}} \subseteq M_{\mathbf{m}}$.

(l) If $M \in \mathscr{P}^-(M_{\mathbf{m}})$ *and* h *is an isomorphism from* $\mathbf{m}\restriction M$ *onto* $\mathbf{m}\restriction M_{\mathbf{m}}$, *then:*

- *there is an isomorphism* $\hat{h}$ *from* $\mathbf{n}_M$ *onto* $\mathbf{n}_{M_{\mathbf{m}}} = \mathbf{m}$ *embedding* h,
- *if* $\mathbf{G}^+$ *is a subset of* $\mathbb{P}_{\mathbf{m}}$ *generic over* $\mathbf{V}$, *then there is* $\mathbf{G} \in \mathbf{V}[\mathbf{G}^+]$, *a generic subset of* $\mathbb{P}_{\mathbf{m}}$ *over* $\mathbf{V}$ *such that* $s \in M \Rightarrow \underset{\sim}{\eta}_s[\mathbf{G}^+] = \eta_{h(s)}[\mathbf{G}]$.

3) Above $\mathbf{n}$ *is reasonable (see Definition 3.21).*

Proof 4.23 1) Let $\langle M_\alpha : \alpha < \alpha_* < \lambda_1^+\rangle$ list $\mathscr{P}^-(M_{\mathbf{m}})$ so $\alpha_* < \lambda_2$ such that $t_\alpha <_{\mathbf{m}} t_\beta \Rightarrow \alpha < \beta$.

We choose by induction on $\alpha \leq \alpha_*$, $\mathbf{m}_\alpha$ and if $\alpha < \alpha_*$, also $\mathbf{n}^0_\alpha, \mathbf{n}^1_\alpha, \mathscr{L}_\alpha$ such that:

$(*)_0$ (a) $\mathbf{m}_\alpha \in \mathbf{M}_{\leq\lambda_2}$,

(b) $\mathbf{m}_\alpha$ is $\leq_{\mathbf{M}}$-increasing continuous,

(c) $\mathbf{m}_0 = \mathbf{m}$,

(d) if $\alpha = \beta + 1$, then:

(α) $\mathbf{n}^0_\beta = \mathbf{m}\restriction M_\beta$,

(β) $\mathbf{n}^0_\beta \leq_{\mathbf{M}} \mathbf{n}^1_\beta \in \mathbf{M}_{\rm bec}$,
(γ) $\mathbf{n}^1_\beta \in \mathbf{M}_{\leq\lambda_2}$ and $L_{\mathbf{n}^1_\alpha} \cap L_{\mathbf{m}_\beta} = L_{\mathbf{n}^0_\beta}$,
(δ) $\mathbf{n}^1_\beta = \text{rest}(\mathscr{L}_{\rm fi}, \text{M}_{\rm fi}, \mathbf{m})$,
(ε) if $t \in M_{\mathbf{m}} \setminus \cup\{M_\gamma : \gamma \leq \beta\}$, then $u_{\mathbf{m}_\alpha,t} = u_{\mathbf{m}_\beta,t} \cup \{s \in L_{\mathbf{m}_\alpha} : s \notin L_{\mathbf{m}_\beta}, s{\restriction}E'_{\mathbf{m}_\alpha} \subseteq L_{\mathbf{m}_{\alpha(\leq t)}}\}$.

Why can we carry the induction?
First, arriving to α we choose $\mathbf{m}_\alpha$ as follows:

- if $\alpha = 0$ let $\mathbf{m}_\alpha = \mathbf{m}$ (so $(*)(c)$ holds).
- if α is a limit ordinal then $\mathbf{m}_\alpha = \cup\{\mathbf{m}_\beta : \beta < \alpha\}$, see §1A, noting $\mathbf{m}_\alpha \in \mathbf{M}_{\leq\lambda_2}$ because $\alpha \leq \alpha_* < \lambda_2$.
- if $\alpha = \beta + 1$ so $\mathbf{n}^0_\beta, \mathbf{n}^1_\beta$ have been chosen then choose $\mathbf{m}_\alpha$ by 4.20(4) and $(*)(d)(\varepsilon)$.

Second assuming $\alpha < \alpha_*$, and $\mathbf{m}_\alpha$ has been defined we choose $\mathbf{n}^0_\alpha$ as rest(M, $\mathbf{m}$) so $\mathbf{n}^0_\alpha \in \mathbf{M}_{\leq\lambda_2}$ by 4.20(1), so $(*)(d)(\alpha)$ holds. Then choose $\mathbf{n}^1_\alpha \in M_{\leq\lambda_2}$ such that $\mathbf{n}^0_\alpha \leq_{\mathbf{M}} \mathbf{n}^1_\alpha \in \mathbf{M}_{\rm ec}$ (by claim 3.22), without loss of generality $L_{\mathbf{n}^1_\alpha} \cap L_{\mathbf{m}_\alpha} = L_{\mathbf{n}^0_\alpha}$ so clause $(*)(d)(\beta), (\gamma)$ holds.
So we have carried the induction.

$(*)_1$ Let $\mathbf{n} = \mathbf{m}_{\alpha_*}$ so $\mathbf{n} \in \mathbf{M}_{\leq\lambda_2}$ and $\mathbf{m} = \mathbf{n}_0 \leq_{\mathbf{M}} \mathbf{n}$.

So clause $\boxplus(a)$ holds. Why $\mathbf{n} \in \mathbf{M}_{\rm bec}$? (i.e. clause $\boxplus(b)$). As clearly $M_{\mathbf{n}}$ is strongly $(<\lambda^+)$-directed, by 2.17(5) it suffices to prove that $M_{\mathbf{n}(\leq t)} \in \mathbf{M}_{\rm bec}$ for every $t \in M_{\mathbf{m}}$. But $M_{\mathbf{n}(\leq t)} \in \{M_\alpha : \alpha < \alpha_*\}$ and if $M = M_{\mathbf{m}(\leq t_\alpha)}$ then rest(M, $\mathbf{n}$) $\leq_{\mathbf{M}}$ $\mathbf{m}(\leq$ t) so this follows if we prove clauses $\boxplus(d), (e)$.

Why clauses $\boxplus(d), (e)$ holds? So assume $M \in \mathscr{P}^-(M_{\mathbf{m}})$ then for some $\alpha = \alpha(M) < \alpha_*$ we have $M = M_\alpha$ and let $\mathscr{L}_M = \text{cmp}(\text{M}, \mathbf{n}^1)$ (so $\bigcup\{L : L \in \mathscr{L}_M\} = L_{\mathbf{n}^1_\alpha}$). Assume $\text{rest}(\mathscr{L}_M, M, \mathbf{n}) = \mathbf{n}_0 \leq_{\mathbf{M}} \mathbf{n}_1 \leq_{\mathbf{M}} \mathbf{n}_2$ and we shall prove that $\mathbb{P}_{\mathbf{n}_1} \lessdot \mathbb{P}_{\mathbf{n}_2}$, this suffices for $\boxplus(e)$.

Let $\alpha < \alpha_*$ be such that $M_\alpha = M$, so clearly.

$(*)_2$ $\bullet_1$ $\text{rest}(\mathscr{L}_M, M, \mathbf{n}) = \mathbf{n}^1_\alpha$,
$\bullet_2$ $\mathbf{n}^1_\alpha = \text{rest}(\text{M}, \mathbf{m})$,
$\bullet_3$ $\text{rest}(\text{M}, \mathbf{m}) \leq_{\mathbf{M}} \text{rest}(\text{M}, \mathbf{n})$,
$\bullet_4$ $\text{rest}(\text{M}, \mathbf{n}) \leq_{\mathbf{M}} \text{Rest}(\text{M}, \mathbf{n}) \leq_{\mathbf{M}} \text{Rest}(\text{M}, \mathbf{n}_1) \leq_{\mathbf{M}} \text{Rest}(\text{M}, \mathbf{n}_2)$.

As $\mathbf{n}^1_\alpha \in \mathbf{M}_{\rm bec}$ it follows that $\text{Rest}(\text{M}, \mathbf{n}_1) \lessdot \text{Rest}(\text{M}, \mathbf{n}_2)$, but by 4.20(1),

$(*)_3$ $\mathbb{P}_{\mathbf{n}_l} = \mathbb{P}_{\text{Rest}(M_\alpha, \mathbf{n}_l)}$ for $l = 1, 2$.

So we are done proving $\boxplus(e)$.
Now $\boxplus(c)$ is just a choice, so we are left with $\boxplus(f)$ which says $\mathbb{P}_{\mathbf{n}_M} \lessdot \mathbb{P}_{\mathbf{n}}$ but it follows by $(*)_2$ and $(*)_3$.
2) We can find $\mathbf{n}^+$ such that:

$(*)$ (a) $\mathbf{n} \leq_{\mathbf{M}} \mathbf{n}^+ \in \mathbf{M}_{\leq\lambda_2}$, ($\mathbf{n}$ is from part (1)),
(b) if $M_1, M_2 \in \mathscr{P}^-(M_{\mathbf{m}})$ and h is an isomorphism from $\mathbf{m}{\restriction}M_1$ onto $\mathbf{m}{\restriction}M_2$ and $L \in \text{cmp}(\text{M}_1, \mathbf{n})$ then there is $\langle (L_i, h_i) : i < \lambda_2\rangle$ such that:
(α) $L_i \in \text{cmp}(\text{M}_2, \mathbf{m})$,
(β) h_i is an isomorphism from $\mathbf{m}{\restriction}L$ onto $\mathbf{m}{\restriction}L_i$ which extends $h{\restriction}(L \cap M_i)$,
(γ) $L_i \cap L_{\mathbf{n}} \subseteq M_{\mathbf{m}}$.
(c) if $t \in L_{\mathbf{n}}(t) \setminus L_{\mathbf{n}}$ then for some $s \in L_{\mathbf{n}} \setminus M_{\mathbf{m}}$ we have $\mathbf{n}{\restriction}(t/E_{\mathbf{n}(t),*}), \mathbf{n}{\restriction}(s/E_{\mathbf{n},t})$ are isomorphic,

(d) if $s \in L_{\mathbf{n}} \setminus M_{\mathbf{n}}$, $M_1 = s/E'$, $M_2 \in \mathscr{P}^-(M_{\mathbf{m}})$ and h is an isomorphism from $\mathbf{m}{\restriction}M_1$ onto $\mathbf{m}{\restriction}M_2$, then there is a sequence $\langle t_\varepsilon : \varepsilon < \lambda_2\rangle$ of pairwise non-$E''_{\mathbf{n}(t)}$-equivalent members of $L_{\mathbf{m}} \setminus M_{\mathbf{m}}$ such that $(t_\zeta/E'_{\mathbf{m}}) \cap M_{\mathbf{m}} = M_2$ and there is an isomorphism from $\mathbf{n}{\restriction}(s/E'_{\mathbf{n}})$ onto $\mathbf{n}^\dagger{\restriction}(t_\zeta/E'_{\mathbf{n}(+)})$

Now we can easily find the isomorphism promised in clause (j). Lastly, clause (k) holds because $\mathbf{n}^0_\alpha = \mathbf{m}{\restriction}M_\alpha$ above. So $\mathbf{n}^+$ is as required recalling $t \in L_{\mathbf{n}^+} \setminus M_{\mathbf{n}^+} \Rightarrow t/E'_{\mathbf{n}^+} = \cup\{L : L \in \mathrm{cmp}(M_{\mathbf{m}}, \mathbf{m})\}$ as $\mathbf{m}$ is bounded.

3) Easy. □

Remark 4.24 How does this subsection help [12]?

(1) Note in the family $\mathbf{R}$ of $\mathbf{r}$'s see Definition 4.12 we demand that there is $\mathbf{G}^+ \subseteq N_{\mathbf{r}} \cap \mathbb{P}_{\mathbf{m}}$ generic over N such that: (so $N \prec (\mathscr{H}(\chi), \in)$, $(\mathbf{n}, \lambda, \ldots) \in N$ and $\mathbf{j}_N$ is the Mostowski collapse)

(∗) $\mathbf{j}''_N(N)[\mathbf{G}^+] = \mathscr{H}(\chi_\varepsilon)$, $\chi_\varepsilon = \mathrm{otp}(N \cap \chi)$.

You can think of it as: in the preliminary forcing to get Laver diamond, in stage $\lambda_N = N \cap \lambda$ we force by $\mathbf{j}_\varepsilon(\mathbb{P}_{\mathbf{m}} \cap N)$.

(2) The present 4.26 tells us to use $\Xi^\bullet_{\bar\nu}$ (defined in 4.19(4)) that instead of using $\Xi^\dagger_{\bar\nu} = \{\mathbf{G} : \mathbf{G} \subseteq N_s \cap \mathbb{P}^\dagger_{\mathbf{m}}$ is generic over N such that $\nu_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}]$ for $\alpha \in M\}$ which gives too may candidates or $\Xi^+_{\bar\nu} = \{\mathbf{G} : \mathbf{G} \subseteq N \cap \mathbb{P}_{\mathbf{m}}$ is generic over N such that $\mathbf{j}''(N)[\mathbf{j}''_N(\mathbf{G})] = \mathscr{H}(\chi_N)$ and $\nu_\alpha = \underset{\sim}{\eta}_\alpha[G]\}$ which seems too restrictive.

Enough to use the middle ground $\Xi^\bullet_{\bar\nu} = \{\mathbf{G} : \mathbf{G} \subseteq N \cap \mathbb{P}_{\mathbf{m}}$ is generic over N and $\nu_\alpha = \underset{\sim}{\eta}_\alpha[\mathbf{G}]$ for $\alpha \in M\}$.

(3) Now the original idea was that $\mathbf{G} \in \Xi^\dagger_{\bar\nu}$ is enough in [12] but not so, however $\mathbf{G} \in \Xi^\bullet_{\bar\nu}$ is sufficient.

(4) Also we need that $\mathbf{m}$ is reasonable (see 3.21(3)) so if $M \subseteq M_{\mathbf{m}}$ is cofinal then $\langle \underset{\sim}{\nu}_\alpha : \alpha \in M\rangle$ is cofinal for $\mathbf{m}$.

(5) The point is that for $M \nsubseteq M_{\mathbf{m}}$ (or with $M \cap N$, $M_{\mathbf{m}} \cap N$ the reflection) we need stronger homogeneity of $\mathbb{P}_{\mathbf{m}}$, which is the aim of 4.17-4.25 relying on 4.23.

Conclusion 4.25 *If $\mathbf{m}_0 \in \mathbf{M}$ is strongly λ^+-directed, (so bounded) of cardinality $\leq \lambda_2$, then there is $\mathbf{m}$ such that:*

(a) $\mathbf{m} \in \mathbf{M}$ *of cardinality* λ_2,
(b) $\mathbf{m}_0 \leq_{\mathbf{M}} \mathbf{m}$,
(c) $\mathbf{m} \in \mathbf{M}_{\mathrm{bec}}$,
(d) $(\alpha) \Rightarrow (\beta)$, *where:*

(α) $\bullet_1$ $\mathbf{r} \in \mathbf{R}$ *and* $\mathbf{m}_{\mathbf{r}} = \mathbf{m}$ *and* $\langle \eta_s : s \in M_{\mathbf{m}}\rangle \in \Xi^+_{\mathbf{r}}$.
$\bullet_2$ $M' \subseteq M_{\mathbf{m}}$, *(in the main case is* $M_{\mathbf{m}} \cong (\kappa, <)$, M_* *a cofinal subset of* $M_{\mathbf{m}}$*),* $M_* = M' \cap N_{\mathbf{r}}$, $M' \in N_{\mathbf{r}}$,
$\bullet_3$ *h is an isomorphism from* $\mathbf{m}{\restriction}M_*$ *onto* $\mathbf{m}{\restriction}M_{\mathbf{m}}$,
$\bullet_4$ *so there is* $\mathbf{G}^+ \subseteq \mathbb{P}_{\mathbf{m}} \cap N_{\mathbf{r}}$ *generic over* $N_{\mathbf{r}}$ *such that* $s \in M_{\mathbf{m}} \Rightarrow \eta_s = \underset{\sim}{\eta}_s[\mathbf{G}^+]$ *and* $\mathscr{H}(\chi_{\mathbf{r}}) = \mathbf{j}''_N(N_{\mathbf{r}})[\mathbf{j}''_N[\mathbf{G}^+]]$ *(by the definition of* $\mathbf{R}$*).*

(β) *there is* $\mathbf{G} \subseteq N \cap \mathbb{P}_{\mathbf{m}}$ *generic over N such that* $s \in M_* \Rightarrow \underset{\sim}{\eta}_{h(s)}[\mathbf{G}] = \eta_s$.

Proof 4.25 Let $(\mathbf{n}, \bar{\mathscr{L}})$ be as in 4.23 for $\mathbf{m}_0$ and we shall show that $\mathbf{n}$ can serve as $\mathbf{m}$.

Clauses (a), (b) of 4.25 holds by clause (a) of 4.23(1).

Clause (c) of 4.25 holds by clause (b) of 4.23(1).

To prove clause (d) of 4.25 assume (α) there. Let $\mathbf{n} = \mathbf{n}_M$ from 4.23(1)(e) and use 4.23(1),(2).

2) Use 4.23(2). □

Claim 4.26 *(1) Assume* $\mathbf{m}$ *is as in 4.25,* $\mathbf{r} \in \mathbf{R}$ *(and* $\mathbf{m} = \mathbf{m}_\mathbf{r}$*). If* $M \subsetneq M_\mathbf{m} \cap N_\mathbf{r}$, $M = M' \cap N_\mathbf{r}$, $M' \in N_\mathbf{r}$ *and* $\bar{\nu} = \bar{\eta}_\mathbf{r} \restriction M$ *and* $h \in N$ *is an isomorphism from* $\mathbf{m} \restriction M'$ *onto* $\mathbf{m} \restriction M_\mathbf{m}$, *<u>then</u> there is* $\mathbf{G}$ *witnessing* $\bar{\eta}_h = \langle \eta_{h^{-1}(s)} : s \in M \rangle$ *belongs to* $\Xi^\bullet_\mathbf{r}$ *(see Definition 4.19(4)).*

(2) Above, $\mathbf{G}$ *has (in* $\mathbb{P}_\mathbf{m}$*) an upper bound* p^+ *which satisfies* $s \in M_\mathbf{n} \Rightarrow \mathrm{tr}(p(s)) = \eta_s[\mathbf{G}]$.

(3) If $p^+ \in \mathbb{P}_\mathbf{m}$ *is as above <u>then</u>* p^+ *is also an upper bound of* $\mathbf{G}' = \mathbf{G} \cap \mathbb{P}_\mathbf{m}[M]$ *in* $\tilde{\mathbb{P}}_\mathbf{m}[L_\mathbf{m}]$

(4) If $\mathbf{m}$ *is strongly* $(< \lambda^+)$*-directed, then* $\bar{\eta}_h$ *is cofinal in* $\left(\Pi_{\zeta < \lambda_\mathbf{r}} \theta_\zeta, <_{J^{\mathrm{bd}}_{\lambda_\mathbf{r}}}\right)^{\mathbf{V}[\mathbb{P}_\mathbf{n}]}$.

(5) If $\mathbf{m}$ *is strongly* $(< \lambda^+)$*-directed (or just essentially directed, see 3.20) <u>then</u> for every* $p \in \mathbb{P}_\mathbf{m}$ *and* $s \in \mathrm{dom}(p) \cap M_\mathbf{m}$ *for every large enough* $t \in M_\mathbf{m}$ *we have* $p \Vdash_{\mathbb{P}_\mathbf{m}}$ "$\underset{\sim}{f}_{p(s)} \leq \underset{\sim}{\eta}_t \mod J^{\mathrm{bd}}_\lambda$"

Proof 4.26

(1) Let $\mathbf{n}$ from 4.25(1)(d)(β) for our M.
(2) This is because $\|N_2\| < \theta_{\lambda(\mathbf{r})}$ 4.12(1)(e) and 4.16.
(3) As $\mathbb{P}_\mathbf{n}[M_\mathbf{n}] = \mathbb{P}_\mathbf{m}[M]$ because $\mathbb{P}_\mathbf{n} \lessdot \mathbb{P}_\mathbf{m}$.
(4) Easy recalling 4.23(2)(j) and 4.22(4)).
(5) Just check the definition (and) or see 1.29.

□

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Dickman, M.A.: Larger infinitary languages. In: Barwise, J., Feferman, S. (eds.) Model Theoretic Logics. Perspectives in Mathematical Logic, chapter IX, pp. 317–364. Springer, New York (1985)
2. Garti, S., Shelah, S.: A strong polarized relation. J. Symbol. Log **77**(3), 766–776 (2012). arXiv: 1103.0350
3. Haim, H., Saharon, S.: Abstract corrected iterations
4. Judah, H.I., Shelah, S.: Souslin forcing. J. Symbol. Log **53**(4), 1188–1207 (1988)
5. Kellner, J., Shelah, S.: Creature forcing and large continuum: the joy of halving. Arch. Math. Logic **51**(1–2), 49–70 (2012). arXiv: 1003.3425
6. Saharon, S.: Are a and $\mathfrak{d}$ your cup of tea? Revisited. Revised version of [Sh:700]. arXiv:2108.03666
7. Saharon, S.: Corrected iteration revisited
8. Saharon, S.: Vive la différence. I. Nonisomorphism of ultrapowers of countable models. In *Set theory of the continuum (Berkeley, CA, 1989)*, volume 26 of Math. Sci. Res. Inst. Publ., pages 357–405. Springer, New York (1992). arXiv:math/9201245
9. Shelah, S.: Covering of the null ideal may have countable cofinality. Fund. Math. **166**(1–2), 109–136 (2000). arXiv: math/9810181
10. Shelah, S.: Properness without elementaricity. J. Appl. Anal. **10**(2), 169–289 (2004). arXiv: math/9712283
11. Saharon, S.: Two cardinal invariants of the continuum ($\mathfrak{d}<a$) and FS linearly ordered iterated forcing. Acta Math. **192**(2), 187–223 (2004). Previous title "Are a and $\mathfrak{d}$ your cup of tea?" arXiv:math/0012170
12. Shelah, S.: On $con(\mathfrak{d}_\lambda > cov_\lambda(\text{meagre}))$. Trans. Am. Math. Soc. **373**(8), 5351–5369 (2020). arXiv: 0904.0817v7 [math.LO]