ADDING HIGHLY UNDEFINABLE SETS OVER L

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ABSTRACT. We sow that there exists a generic extension of the Gödel's constructible universe in which diamond holds and there exists a subset $Y \subseteq \omega_1$ such that for stationary many $\delta < \omega_1$, the set $Y \cap \delta$ is not definable in the structure $(L_{F(\delta)}, \in)$, where $F(\delta) > \delta$ is the least ordinal such that $L_{F(\delta)} \models$"δ is countable".

§ 1. INTRODUCTION

In this short note we introduce a Σ^2_2 sentence ϕ which is false in L, the Gödel's constructible universe. Furthermore, we force a proper generic extension of L in which both ϕ and diamond hold. More precisely, let ϕ be the following sentence.

Definition 1.1. Let ϕ be the sentence

$$(\exists R, S_1, S_2, F, Y) \bigwedge_{i=1}^{3} \varphi_i,$$

where

(1) $\varphi_1 := \varphi_1(R, F)$ is the conjunction of the following statements:
  (a) (ω_1, R) is isomorphic to $(L_{\omega_1}, \in)$,
  (b) Let $\mathrm{Lim}(\omega_1)$ be the set of countable limit ordinals. Then $(\mathrm{Lim}(\omega_1), R)$ is isomorphic to $(\mathrm{Lim}(\omega_1), \in)$,
  (c) F is a function defined on $\mathrm{Lim}(\omega_1)$, such that for every limit ordinal δ,
  $$(\omega_1, R) \models \ulcorner F(\delta) \text{ is the minimal limit ordinal such that } L_{F(\delta)} \models \text{“}|\delta| = \aleph_0\text{”} \urcorner.$$
(2) $\varphi_2 := \varphi_2(S_1, S_2)$ is the statement: S_1 and S_2 form a partition of $\mathrm{Lim}(\omega_1)$ into disjoint stationary sets.

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(3) $\varphi_3 := \varphi_3(R, S_2, F, Y)$ is the statement:
$$(\forall \delta \in S_2)\big(Y \cap \delta \text{ is not definable in } (L_{F(\delta)})^{(\omega_1, R)}\big).$$

It is clear that ϕ is a Σ^2_2 statement. We prove the following theorem.

Theorem 1.2. (a) *The statement ϕ fails in L,*
(b) *There exists a proper forcing notion $\mathbb{P} \in L$ which forces $\diamondsuit + \phi$.*

We assume familiarity with proper forcing notions. Given a forcing notion $\mathbb{P}$ and conditions $p, q \in \mathbb{P}$, by $p \leq q$ we mean q is stronger than p.

§ 2. Some preliminaries

In this paper, we are interested in forcing notions which preserve diamond at ω_1. Let us recall the definition of a diamond sequence.

Definition 2.1. ([1]) Assume $S \subseteq \omega_1$ is stationary. Then $\diamondsuit_S$ asserts the existence of a sequence $\langle s_\alpha : \alpha \in S\rangle$ such that $s_\alpha \subseteq \alpha$, for $\alpha \in S$, and for every $X \subseteq \omega_1$, the set $\{\alpha \in S : X \cap \alpha = s_\alpha\}$ is stationary in ω_1. By $\diamondsuit$ we mean $\diamondsuit_{\omega_1}$.

By the work of Jensen [1], $\diamondsuit_S$ holds in the Gödel's constructible universe, for all stationary subsets S of ω_1. We now introduce a property of forcing notions which is sufficient to guarantee that $\diamondsuit$ is preserved, see Lemma 2.3.

Definition 2.2. ([3, Ch. V, Definition 1.1]) Suppose $S \subseteq \omega_1$ is stationary, $\mathbb{P}$ is a forcing notion and N is a countable model with $\mathbb{P} \in N$.

1. The sequence $\langle p_n : n < \omega\rangle$ is a generic sequence for $(N, \mathbb{P})$ if it is an increasing sequence from $\mathbb{P} \cap N$ and for every dense open subset D of $\mathbb{P}$ in N, $D \cap \{p_n : n < \omega\} \neq \emptyset$.
2. The pair $(N, \mathbb{P})$ is complete if every generic sequence $\langle p_n : n < \omega\rangle$ for $(N, \mathbb{P})$ has an upper bound in $\mathbb{P}$.
3. We say $\mathbb{P}$ is $\{S\}$-complete if for every large enough regular χ and countable model $N \prec (\mathscr{H}(\chi), \in)$, if $S, \mathbb{P} \in N$ and $N \cap \omega_1 \in S$, then the pair $(N, \mathbb{P})$ is complete.

Lemma 2.3. *([3, Chapter V, Claim 1.9]) Suppose $S \subseteq \omega_1$ is stationary. Assume $\mathbb{P}$ is $\{S\}$-complete. If $\diamondsuit_S$ holds in V, then it holds in $V^{\mathbb{P}}$ as well.*

§ 3. Proof of the main theorem

In this section we prove Theorem 1.2.

§ 3(A). ϕ **fails in** L. In this subsection we show that the statement ϕ is false in the constructible universe L. This follows from the following lemma.

Lemma 3.1. *Assume $V = L$. Let $S \subseteq \omega_1$ be stationary. Then for every $X \subseteq \omega_1$, there exists $\delta \in S$ such that $X \cap \delta$ is definable in $(L_{F(\delta)}, \in)$.*

Proof. Let us recall the construction of a $\diamondsuit_S$-sequence. By induction on δ we define a sequence $\langle (s_\delta, c_\delta) : \delta \in S \rangle$ as follows. Suppose $\delta \in S$ and we have defined (s_γ, c_γ), for $\gamma \in S \cap \delta$ such that for each γ, $s_\gamma \subseteq \gamma$ and $c_\gamma \subseteq \gamma$ is a club. If there exists a pair (s, c) such that:

(1) $s \subseteq \delta$,
(2) $c \subseteq \delta$ is a club,
(3) for all $\gamma \in c \cap S, s \cap \gamma \neq s_\gamma$,

then let (s_δ, c_δ) be the $<_L$-least such pair. Otherwise set $s_\delta = \emptyset$ and $c_\delta = \delta$.

Claim 3.2. *There exists a club C of ω_1 such that*

$$C \cap S \subseteq \{\delta \in S : (s_\delta, c_\delta) \textit{ is definable in } (L_{F(\delta)}, \in)\}.$$

Proof. To see this, suppose by the way of contradiction there is no such club C. It then follows that the set

$$S_* = \{\delta \in S : (s_\delta, c_\delta) \text{ is not definable in } (L_{F(\delta)}, \in)\}$$

is stationary. Let M be a countable elementary submodel of $(L_{\omega_2}, \in)$, such that M contains all relevant information and $M \cap \omega_1 = \mu \in S_*$. Let also $\pi : M \simeq L_\delta$ be the transitive collapse map. Then

- $\pi(\omega_1) = \mu$
- $\pi(S) = S \cap \mu$, and $\pi(S_*) = S_* \cap \mu$,
- $\pi(\langle (s_\delta, c_\delta) : \delta \in S \rangle) = \langle (s_\gamma, c_\gamma) : \gamma \in S \cap \mu \rangle$.

As $\mu \in S_*$, it follows from the definition of S_* that (s_μ, c_μ) is not definable in $(L_{F(\mu)}, \in)$.

On the other hand, (s_μ, c_μ) is uniformly definable using the sequence $\langle (s_\gamma, c_\gamma) : \gamma \in S \cap \mu \rangle$, hence (s_μ, c_μ) is definable in L_δ. Now note that $(L_\delta, \in) \models$"μ is uncountable", hence $F(\mu) > \delta$, and thus (s_μ, c_μ) is definable in $(L_{F(\mu)}, \in)$, a contradiction. □

Now suppose that $X \subseteq \omega_1$ is given. It follows that the set

$$T = \{\delta \in S : X \cap \delta = s_\delta\}$$

is stationary in ω_1. Let $\delta \in C \cap T$. It then follows that $X \cap \delta = s_\delta$ and s_δ is definable in $(L_{F(\delta)}, \in)$. Thus $X \cap \delta$ is definable in $(L_{F(\delta)}, \in)$, as requested. □

§ 3(B). **Consistency of $\diamondsuit + \phi$.** We now show that in some forcing extension of L, $\diamondsuit + \phi$ holds.

Lemma 3.3. *Assume $V = L$, and let S_1, S_2 be a partition of* $\mathrm{Lim}(\omega_1)$ *into disjoint stationary sets. Suppose $\varphi_1(R, F)$ holds. Then there exists a $\{S_1\}$-complete proper forcing notion $\mathbb{P}$, which adds a set Y witnessing $\varphi_3(R, S_2, F, Y)$ holds. In particular, $\diamondsuit + \phi$ holds in $L[G_{\mathbb{P}}]$.*

Proof. Let $\mathbb{P}$ be the set of all conditions p where:

- $(*)_1$ p is a countable subset of ω_1,
- $(*)_2$ $\max(p)$ exists,
- $(*)_3$ for all $\delta \in S_2 \cap (\max(p)+1)$, $p \cap \delta$ is not definable in $(L_{F(\delta)})^{(\omega_1, R)}$.

Given two conditions p, q let us say that $p \leq q$ (q is stronger than p), iff $p = q \cap (\max(p) + 1)$.

Claim 3.4. $\mathbb{P}$ *is proper.*

Proof. Suppose χ is large enough regular and $N \prec (\mathscr{H}(\chi), \in)$ is countable such that:

- $N = \bigcup_{n<\omega} N_n$, where $\langle N_n : n < \omega \rangle$ is a $\prec$-increasing sequence of elementary submodels of N with $\{N_n : n < \omega\} \subseteq N$,
- $R, F, S_1, S_2, \mathbb{P} \in N_0$.[1]

Let also $p \in \mathbb{P} \cap N$. We have to find $q \geq p$ which is an $(N, \mathbb{P})$-generic condition. We may assume that $p \in N_0$.

Let $\delta = N \cap \omega_1$ and for each $n < \omega$ set $\delta_n = N_n \cap \omega_1$. Choose an increasing ω-sequence $\eta_\delta = \langle \eta_\delta(n) : n < \omega \rangle$, definable in $(L_{F(\delta)})^{(\omega_1, R)}$, coding a cofinal sequence in δ with $\eta_\delta(0) > \max(p)$. Such a sequence exists by the choice of $F(\delta)$. Let also $c_\delta = \langle c_\delta(n) : n < \omega \rangle$ be a real, not definable in $(L_{F(\delta)})^{(\omega_1, R)}$.

Now let $\langle \mathcal{D}_n : n < \omega \rangle$ be an enumeration of dense open subsets of $\mathbb{P}$ in N. Following ideas from [2], we define an increasing sequence $\langle p_m : m < \omega \rangle$ of conditions such that:

1. $p_0 = p$,
2. For all $n < \omega$, there exists $m < \omega$ such that $p_m \in \mathcal{D}_n$,
3. For all $n < \omega$,

$$\eta_\delta(n) \in \bigcup_{m<\omega} p_m \iff c_\delta(n) = 1.$$

To start set $p_0 = p$. Note that $p \cap \{\eta_\delta(n) : n < \omega\} = \emptyset$. Let us define p_1. Let $k_1 < \omega$ be such that $\{\eta_\delta(n) : n < \omega\} \cap N_1 = \{\eta_\delta(n) : n < k_1\}$, and

[1] Note that the class of all countable models $N \in [\mathscr{H}(\chi)]^{\aleph_0}$ as above forms a club of $[\mathscr{H}(\chi)]^{\aleph_0}$, thus it suffices to check properness with respect to such models.

let $\mu_1 > \delta_0$ be such that $\sup\{\eta_\delta(n) : n < k_1\} < \mu_1 < \delta_1$ with $\mu_1 \notin S_2$. Set

$$q_1 = p_0 \cup \{\eta_\delta(n) : n < k_1 \text{ and } c_\delta(n) = 1\} \cup \{\mu_1\}.$$

Note that $q_1 \in \mathbb{P} \cap N_1$. Now let p_1 be such that:

(4) $p_1 \in N_1$,
(5) $p_1 \geq q_1$,
(6) $p_1 \in \bigcap\{\mathcal{D}_n : n < 1 \text{ and } \mathcal{D}_n \in N_1\}$. If $\mathcal{D}_0 \notin N_1$, set $p_1 = q_1$.

Now suppose that $1 \leq m < \omega$ and we have defined p_m. We define p_{m+1}. Let $k_{m+1} \geq k_m$ be such that $\{\eta_\delta(n) : n < \omega\} \cap (N_{m+1} \setminus N_m) = \{\eta_\delta(n) : k_m \leq n < k_{m+1}\}$, and let $\mu_{m+1} > \delta_m$ be such that $\sup\{\eta_\delta(n) : n < k_{m+1}\} < \mu_{m+1} < \delta_{m+1}$ with $\mu_{m+1} \notin S_2$. Set

$$q_{m+1} = p_m \cup \{\eta_\delta(n) : k_m \leq n < k_{m+1} \text{ and } c_\delta(n) = 1\} \cup \{\mu_{m+1}\}.$$

Note that $q_{m+1} \in \mathbb{P} \cap N_{m+1}$. Now let p_{m+1} be such that:

(7) $p_{m+1} \in N_{m+1}$,
(8) $p_{m+1} \geq q_{m+1}$,
(9) $p_{m+1} \in \bigcap\{\mathcal{D}_n : n < m+1 \text{ and } \mathcal{D}_n \in N_{m+1}\}$. If there are no such $\mathcal{D}_n$'s, set $p_{m+1} = q_{m+1}$.

Set $p = \bigcup_{m<\omega} p_m \cup \{\delta\}$. We claim that $p \in \mathbb{P}$. To show p is a condition, it suffices to show that $p \cap \delta = \bigcup_{m<\omega} p_m$ is not definable in $(L_{F(\delta)})^{(\omega_1, R)}$. Suppose by the way of contradiction that $p \cap \delta$ is definable in $(L_{F(\delta)})^{(\omega_1, R)}$. As the sequence η_δ is definable in $(L_{F(\delta)})^{(\omega_1, R)}$, it follows from clause (3) that $\langle c_\delta(n) : n < \omega\rangle$ is definable in $(L_{F(\delta)})^{(\omega_1, R)}$, which is a contradiction. It is clear from our construction that p is an $(N, \mathbb{P})$-generic condition. $\square$

The above proof implies the following.

Claim 3.5. *Assume* $p \in \mathbb{P}$ *and* $\gamma > \max(p)$. *Then there exists a condition* $q \geq p$ *such that* $\max(q) \geq \gamma$.

The next claim guarantees that $\diamondsuit_{S_1}$ is preserved by $\mathbb{P}$.

Claim 3.6. $\mathbb{P}$ *is* $\{S_1\}$*-complete.*

Proof. Suppose χ is large enough regular and $N \prec (\mathscr{H}(\chi), \in)$ is countable such that $R, S_1, S_2, F, \mathbb{P} \in N$ and $\delta = N \cap \omega_1 \in S_1$. We show that the pair $(N, \mathbb{P})$ is complete. Thus let $\langle p_n : n < \omega\rangle$ be a generic sequence for $(N, \mathbb{P})$. Set $p = \bigcup_{n<\omega} p_n \cup \{\delta\}$. Note that, by Claim 3.5, $\sup \bigcup_{n<\omega} p_n = \delta$, and since $\delta \notin S_2$, $p \in \mathbb{P}$. Then p is an upper bound in $\mathbb{P}$ for the sequence $\langle p_n : n < \omega\rangle$. $\square$

Now let G be $\mathbb{P}$-generic over V and let $Y = \bigcup\{p : p \in G\}$.

Claim 3.7. *The set Y witnesses that $\varphi_3(R, S_2, F, Y)$ holds in $V[G]$.*

Proof. This is clear. □

It follows from Lemma 2.3 and Claim 3.6 that $\diamondsuit_{S_1}$ holds in $L[G]$. Finally note that ϕ holds in $V[G]$:

- $\varphi_1(R, F)$ holds by the choice of R and F.
- By Claim 3.4, S_1 and S_2 remain stationary in $L[G]$. It then follows that $\varphi_2(S_1, S_2)$ holds in $L[G]$ as well.
- $\varphi_3(R, S_2, F, Y)$ holds, by Claim 3.7.

The lemma follows. □

Proof of Theorem 1.2. It follows from Lemmas 3.1 and 3.3. □

References

[1] Jensen, R. Björn; The fine structure of the constructible hierarchy. With a section by Jack Silver. Ann. Math. Logic 4 (1972), 229-308; erratum, ibid. 4 (1972), 443.

[2] Shelah, Saharon Whitehead groups may be not free, even assuming CH. I. Israel J. Math. 28 (1977), no. 3, 193-204.

[3] Shelah, Saharon, Proper and improper forcing. Second edition. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1998. xlviii+1020 pp. ISBN: 3-540-51700-6

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