REMARKS ON SOME CARDINAL INVARIANTS AND PARTITION RELATIONS

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Abstract. We answer some questions about two cardinal invariants associated with separating and almost disjoint families and a partition relation involving indecomposable countable linear orderings.

1. Introduction

The aim of this paper is to prove some results about two cardinals invariants and partition relations. These cardinal invariants (denoted $\mathfrak{ls}(\mathfrak{c})$ and $\mathfrak{la}(\mathfrak{c})$) were introduced by Higuchi, Lempp, Raghavan and Stephan in [2]. The primary motivation behind studying them was to shed some light on the order dimension of the Turing degrees.

Definition 1.1 ([2])**.** *Let κ be an infinite cardinal.*

(1) *$\mathfrak{ls}(\kappa)$ is the least cardinality of a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ that separates countable subsets of κ from points in the following sense: For every countable $A \subseteq \kappa$ and $\alpha \in \kappa \setminus A$, there exists $X \in \mathcal{F}$ such that $\alpha \in X$ and $A \cap X = \emptyset$.*
(2) *$\mathfrak{la}(\kappa)$ is the least cardinal λ such that $\mathsf{cf}(\lambda) \geq \omega_1$ and there exists an almost disjoint family $\mathcal{A} \subseteq [\lambda]^{\mathsf{cf}(\lambda)}$ with $|\mathcal{A}| \geq \kappa$. Here $X, Y \in \mathcal{A}$ are almost disjoint iff $|X \cap Y| < \mathsf{cf}(\lambda)$.*

In [2], the authors showed $\omega_1 \leq \mathfrak{ls}(\kappa) \leq \mathfrak{la}(\kappa)$ for every uncountable cardinal κ and asked if this inequality could be strict when κ is the successor of a cardinal of uncountable cardinality. This question also appears in [5] (Question 5.3) for the case $\kappa = \mathfrak{c} = \omega_3$. We positively answer it by showing the following.

Theorem 1.2. *Assume $V \models GCH$. Then there is a ccc forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models \mathfrak{ls}(\omega_3) = \omega_1 < \mathfrak{la}(\omega_3) = \omega_2 < \mathfrak{c} = \omega_3$.*

In the next section, we generalize our construction to separate $\mathfrak{la}(\kappa)$ and a stronger variant of $\mathfrak{ls}(\kappa)$ defined as follows.

Definition 1.3. *Let $\omega \leq \theta \leq \mu \leq \kappa$ be cardinals. $\mathfrak{ls}(\kappa, \mu, \theta)$ is the least cardinality of a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ that is a (μ, θ)-separating family on κ which means the following: For every $A \in [\kappa]^{<\mu}$ and $B \in [\kappa \setminus A]^{<\theta}$, there exists $X \in \mathcal{F}$ such that $B \subseteq X$ and $A \cap X = \emptyset$.*

It is easy to check that $\mathfrak{ls}(\kappa) = \mathfrak{ls}(\kappa, \omega_1, \omega)$ for every infinite κ.

Theorem 1.4. *Suppose $\mu < \kappa$, κ is Mahlo and $\lambda > 2^\kappa$. There is a $< \mu$-closed κ-cc forcing $\mathbb{P}$ such that the following hold in $V^{\mathbb{P}}$.*

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(1) $\kappa = \mu^+$.
(2) $\mathfrak{ls}(\lambda, \kappa, \mu) = \kappa$.
(3) $\mathfrak{la}(\lambda) \geq \kappa^+$.

An order-theoretic variant of $\mathfrak{ls}(\kappa)$ defined in [5] is $\mathfrak{los}(\kappa)$. It equals the order dimension of the Turing degrees when $\kappa = \mathfrak{c}$ (see Corollary 2.11 in [5]).

Definition 1.5. *Let κ be an infinite cardinal. Define $\mathfrak{los}(\kappa)$ to be the least cardinality of a family $\mathcal{F}$ of linear orders on κ that separates countable subsets of κ from points in the following sense: For every countable $A \subseteq \kappa$ and $\alpha \in \kappa \setminus A$, there exists $\prec$ in $\mathcal{F}$ such that for every $\beta \in A$, $\beta \prec \alpha$.*

Note that $\mathfrak{los}(\kappa) \leq \mathfrak{ls}(\kappa) \leq \mathfrak{la}(\kappa)$ and each of these two inequalities can be strict at $\kappa = \omega_3$ (by Theorem 1.2 above and Lemma 5.1 in [5]). So we ask the following.

Question 1.6. *Is it consistent to have $\mathfrak{los}(\kappa) < \mathfrak{ls}(\kappa) < \mathfrak{la}(\kappa)$ for some infinite cardinal κ? What if $\kappa = \omega_3$?*

1.1. Partition relations.

Definition 1.7. *An order type φ is unionwise indecomposable iff for every linear ordering $(L, \prec)$ of type φ and a partition $L = A \sqcup B$, at least one of $(A, \prec)$ and $(B, \prec)$ contains a subordering of type φ.*

Let $\varphi, \psi, \varphi_0, \varphi_1, \psi_0, \psi_1$ be order types. Recall that we write

$$\begin{pmatrix} \psi \\ \varphi \end{pmatrix} \longrightarrow \begin{pmatrix} \psi_0 & \psi_1 \\ \varphi_0 & \varphi_1 \end{pmatrix}$$

to denote the following statement: Whenever $(X, \prec_0)$ and $(Y, \prec_1)$ are linear orderings of type ψ and φ respectively and $c : X \times Y \to 2$, there exist $A \subseteq X$ and $B \subseteq Y$ such that one of the following holds.

(a) $(A, \prec_0)$ has type ψ_0, $(B, \prec_1)$ has type φ_0 and $c \restriction (A \times B)$ is constantly 0.
(b) $(A, \prec_0)$ has type ψ_1, $(B, \prec_1)$ has type φ_1 and $c \restriction (A \times B)$ is constantly 1.

The following questions were raised by Klausner and Weinert (Questions (C) and (D) in [4]).

Question 1.8 ([4])**.** *Does the following hold for all countable ordinals α and unionwise indecomposable countable order types φ?*

$$\begin{pmatrix} \omega_1 \\ \varphi \end{pmatrix} \longrightarrow \begin{pmatrix} \alpha & \alpha \\ \varphi & \varphi \end{pmatrix}$$

Question 1.9 ([4])**.** *Is it consistent to have the following for all countable ordinals α and unionwise indecomposable countable order types φ?*

$$\begin{pmatrix} \omega_1 \\ \varphi \end{pmatrix} \longrightarrow \begin{pmatrix} \omega_1 & \alpha \\ \varphi & \varphi \end{pmatrix}$$

In the final section, we will show that the answer to both of these questions is yes. In fact, we have the following.

Theorem 1.10. *Let $c : \omega_1 \times L \to K$ where $K < \omega$ and $(L, \prec_L)$ is a unionwise indecomposable countable linear order. Then for each $\alpha < \omega_1$, there exist $A \in [\omega_1]^\alpha$ and $B \subseteq L$ such that $(B, \prec_L) \cong (L, \prec_L)$ and $c \restriction (A \times B)$ is constant.*

Theorem 1.11. *Assume Martin's axiom plus $\mathfrak{c} > \omega_1$. Let $c : \omega_1 \times L \to K$ where $K < \omega$ and $(L, \prec_L)$ is a unionwise indecomposable countable linear order. Then there exist $A \in [\omega_1]^{\omega_1}$ and $B \subseteq L$ such that $(B, \prec_L) \cong (L, \prec_L)$ and $c \restriction (A \times B)$ is constant.*

2. Consistency of $\mathfrak{ls}(\omega_3) < \mathfrak{la}(\omega_3)$

A natural attempt to get a model of $\mathfrak{ls}(\omega_3) = \omega_1 < \mathfrak{la}(\omega_3) = \omega_2$ would be to start with a model of GCH and add ω_3 subsets of ω_1 using countable or finite conditions. Both of these fail.

Fact 2.1. *Assume $V \models GCH$. Let $\mathbb{P}$ consist of all partial functions from ω_3 to 2 such that either $(\forall p \in \mathbb{P})(|dom(p)| < \omega)$ or $(\forall p \in \mathbb{P})(|dom(p)| < \omega_1)$. Then $V^{\mathbb{P}} \models \mathfrak{ls}(\omega_3) = \mathfrak{la}(\omega_3)$.*

Proof. Note that both of these forcings preserves all cofinalities (and hence cardinals). If $\mathbb{P}$ consists of all finite partial functions from ω_3 to 2, then Lemma 5.1 in [5] implies that $V^{\mathbb{P}} \models \mathfrak{la}(\omega_3) = \mathfrak{ls}(\omega_3) = \omega_2$. So assume that $\mathbb{P}$ consists of all countable partial functions from ω_3 to 2. Then $\mathbb{P}$ does not add any new countable set of ordinals. Since $V \models 2^\omega = \omega_1$, it follows that $V \cap 2^{<\omega_1} = V^{\mathbb{P}} \cap 2^{<\omega_1}$ has size ω_1 in $V^{\mathbb{P}}$. Furthermore, as $V^{\mathbb{P}} \models 2^{\omega_1} \geq \omega_3$ we can find a family $\mathcal{F} \in V^{\mathbb{P}}$ consisting of ω_3 distinct subsets of ω_1. For each $A \in \mathcal{F}$, define $S_A = \{1_A \restriction \alpha : \alpha < \omega_1\}$. Observe that $\{S_A : A \in \mathcal{F}\}$ is a mod countable almost disjoint family of subsets of $2^{<\omega_1}$. Since $V^{\mathbb{P}} \models |2^{<\omega_1}| = \omega_1$, it follows that $V^{\mathbb{P}} \models \mathfrak{la}(\omega_3) \leq \omega_1$. As $\omega_1 \leq \mathfrak{ls}(\omega_3) \leq \mathfrak{la}(\omega_3)$, it follows that $V^{\mathbb{P}} \models \mathfrak{ls}(\omega_3) = \mathfrak{la}(\omega_3) = \omega_1$. □

An infinite ordinal δ is (additively) indecomposable iff for each $X \subseteq \delta$, either $\mathsf{otp}(X) = \delta$ or $\mathsf{otp}(\delta \setminus X) = \delta$. Recall that the following are equivalent.

(i) δ is infinite and indecomposable.
(ii) Whenever A, B are sets of ordinals of order type $< \delta$, $\mathsf{otp}(A \cup B) < \delta$.
(iii) $\delta = \omega^\alpha$ (ordinal exponentiation) for some $\alpha \geq 1$.

Definition 2.2. *For an uncountable cardinal κ and an indecomposable ordinal $\delta < \kappa$, define the forcing $\mathbb{Q}_{\kappa,\delta}$ as follows. $p \in \mathbb{Q}_{\kappa,\delta}$ iff the following hold.*

(i) *p is a function, $dom(p) \subseteq \kappa$ and $range(p) \subseteq 2$.*
(ii) *$otp(dom(p)) < \delta$.*
(iii) *$\{\alpha \in dom(p) : p(\alpha) = 1\}$ is finite.*

For $p, q \in \mathbb{Q}_{\kappa,\delta}$, define $p \leq q$ iff $q \subseteq p$.

The following lemma shows that if $\delta \geq \omega_1$, then $\mathbb{Q}_{\kappa,\delta}$ collapses ω_1.

Lemma 2.3. *Let $\omega_1 \leq \delta < \kappa$ be indecomposable. Then $V^{\mathbb{Q}_{\kappa,\delta}} \models |\delta| = \omega$.*

Proof. Suppose $V \models |\delta| = \theta \geq \omega_1$. Choose $\alpha < \theta^+$ such that $\delta = \omega^{\theta+\alpha} = \theta \cdot \omega^\alpha = \theta \cdot \gamma$ where $\gamma = \omega^\alpha$. Let G be $\mathbb{Q}_{\kappa,\delta}$-generic over V and $F = \bigcup G$. Then $F : \kappa \to 2$. Define $W = \{\beta < \delta : F(\beta) = 1\}$. An easy density argument shows that $\mathsf{otp}(W) = \omega$. For each $k < \omega$, let α_k be the kth member of W. Choose $\xi_k < \theta$ and $j_k < \gamma$ such that $\alpha_k = \theta \cdot j_k + \xi_k$. Define $h : \omega \to \theta$ by $h(k) = \xi_k$. Another density argument shows that for every $X \in V \cap [\theta]^\theta$, $\mathsf{range}(h) \cap X \neq \emptyset$. It follows that $V[G] \models |\theta| = \omega$. □

Recall that a forcing $\mathbb{Q}$ has ω_1 as a precaliber iff for every uncountable $A \subseteq \mathbb{Q}$, there exists an uncountable $B \subseteq A$ such that every finite set of conditions in B has a common extension in $\mathbb{Q}$. It is easy to see that if $\mathbb{Q}$ has ω_1 as a precaliber, then it satisfies ccc.

Lemma 2.4. *Suppose κ is uncountable, $\delta < \omega_1$ is indecomposable and $\mathbb{Q}_{\kappa,\delta}$ is as in Definition 2.2. Then $\mathbb{Q}_{\kappa,\delta}$ has ω_1 as a precaliber.*

Proof. Let $\langle p_i : i < \omega_1 \rangle$ be a sequence of conditions in $\mathbb{Q}$. Put $D_i = \mathsf{dom}(p_i)$, $A_i = \{\alpha \in D_i : p_i(\alpha) = 0\}$, $B_i = \{\alpha \in D_i : p_i(\alpha) = 1\}$ and $D = \bigcup\{D_i : i < \omega_1\}$. Let $\gamma = \mathsf{otp}(D)$. Clearly, $\gamma < \omega_2$. Let $h : \gamma \to D$ be the order preserving bijection. By replacing each D_i with $h^{-1}[D_i]$, we can assume that $D_i \subseteq \gamma$. By induction on γ, we will show that there exists $X \in [\omega_1]^{\omega_1}$ such that for every $i, j \in X$, p_i and p_j are compatible (and hence $p_i \cup p_j \in \mathbb{Q}_{\kappa,\delta}$ as $\mathsf{otp}(D_i \cup D_j) < \delta$). This suffices since any finite set S of conditions in $\langle p_i : i \in X \rangle$ has a common extension (namely its union) in $\mathbb{Q}_{\kappa,\delta}$.

Case 1: γ is a successor ordinal. Let $\gamma = \xi + 1$. Applying the inductive hypothesis to the sequence $\langle p_i \restriction \xi : i < \omega_1 \rangle$, we can find $Y \in [\omega_1]^{\omega_1}$ such that for every $i, j \in Y$, $p_i \restriction \xi$ and $p_j \restriction \xi$ are compatible. Choose $X \in [Y]^{\omega_1}$ and $k < 2$ such that either $(\forall i \in X)(\xi \notin D_i)$ or $(\forall i \in X)(\xi \in D_i \wedge p_i(\xi) = k)$. Then X is as required.

Case 2: $\mathsf{cf}(\gamma) = \omega$. Since each B_i is a finite subset of γ and $\mathsf{cf}(\gamma) = \omega$, we can choose $Y \in [\omega_1]^{\omega_1}$ and $\gamma' < \gamma$ such that for every $i \in Y$, $B_i \subseteq \gamma'$. Applying the inductive hypothesis to $\langle p_i \restriction \gamma' : i \in Y \rangle$, we can find $X \in [Y]^{\omega_1}$ such that for every $i, j \in X$, $p_i \restriction \gamma'$ and $p_j \restriction \gamma'$ are compatible. Since for every $i \in X$, $p_i \restriction [\gamma', \gamma)$ is constantly 0, it follows that for every $i, j \in X$, p_i and p_j are compatible.

Case 3: $\mathsf{cf}(\gamma) = \omega_1$. Let $\langle \gamma_\xi : \xi < \omega_1 \rangle$ be a continuously increasing cofinal sequence in γ. Since D_i's are countable subsets of γ, we can choose a club $E \subseteq \omega_1$ consisting of limit ordinals such that for every $\xi \in E$ and $i < \xi$, $D_i \subseteq \gamma_\xi$.

Let $F = \{\xi \in E : (\forall i > \xi)(D_i \cap \gamma_\xi \text{ is unbounded in } \gamma_\xi)\}$. We claim that F is countable. Suppose not and fix a strictly increasing sequence $\langle \xi(i) : i < \omega_1 \rangle$ in F. Choose j such that $\xi(\delta) < j < \omega_1$. Then $\sup(D_j \cap \gamma_{\xi(i)}) = \gamma_{\xi(i)}$ for every $i < \delta$. Define $f : \delta \to D_j$ by $f(i) = \min([\gamma_{\xi(i)}, \gamma_{\xi(i+1)}) \cap D_j$. Then f is strictly increasing and hence $\mathsf{otp}(\mathsf{range}(f)) = \delta$. But this implies that $\mathsf{otp}(D_j) \geq \mathsf{otp}(\mathsf{range}(f)) = \delta$ which is impossible. So F must be countable.

Next fix a club $C \subseteq E \setminus F$ and a function $h : C \to \omega_1$ such that for every $\xi \in C$, $h(\xi) > \xi$ and $\sup(D_{h(\xi)} \cap \gamma_\xi) < \gamma_\xi$. It follows that the function $g : C \to \omega_1$ defined by $g(\xi) = \min(\{\xi' < \xi : \sup(D_{h(\xi)} \cap \gamma_\xi) < \gamma_{\xi'}\})$ is regressive on C. By Fodor's lemma, we can find a stationary $S \subseteq C$ and $\xi_\star < \omega_1$ such that $\min(S) > \xi_\star$ and for every $\xi \in S$, $D_{h(\xi)} \cap \gamma_\xi \subseteq \gamma_\star = \gamma_{\xi_\star}$. Let $T \in [S]^{\omega_1}$ be such that for every $\xi_1 < \xi_2$ in T, $\xi_1 < h(\xi_1) < \xi_2 < h(\xi_2)$. Put $Y = h[T]$ and note that for every $i < j$ in Y, $D_i \cap D_j \subseteq \gamma_\star$. Applying the inductive hypothesis to $\langle p_i \restriction \gamma_\star : i \in Y \rangle$, choose $X \in [Y]^{\omega_1}$ such that for every $i, j \in X$, $p_i \restriction \gamma_\star$ and $p_j \restriction \gamma_\star$ are compatible. Since for every $i < j$ in X, $D_i \cap D_j \cap [\gamma_\star, \gamma) = \emptyset$, it follows that $\langle p_i : i \in X \rangle$ has pairwise compatible functions.

As $\gamma < \omega_2$, there are no more cases and we are done. $\square$

Lemma 2.5. *Let κ be an uncountable cardinal. Let $\mathbb{P}$ be the finite support product of $\mathbb{Q}_{\kappa,\delta}$'s where δ runs over the set of indecomposable ordinals $< \omega_1$. Then the following hold.*

(1) *$\mathbb{P}$ has ω_1 as a precaliber.*
(3) *$V^{\mathbb{P}} \models \mathfrak{ls}(\kappa) = \omega_1$.*

Proof. (1) Let $\langle p_i : i < \omega_1 \rangle$ be a sequence in $\mathbb{P}$. By the Δ-system lemma, we can find $Y \in [\omega_1]^{\omega_1}$ and a finite set R of indecomposable countable ordinals such that for every $i < j$ in Y, $\mathsf{dom}(p_i) \cap \mathsf{dom}(p_j) = R$. Using Lemma 4.4, we can choose $X \in [Y]^{\omega_1}$ such that for every $i, j \in X$ and $\delta \in R$, $p_i(\delta)$ and $p_j(\delta)$ are compatible in $\mathbb{Q}_{\kappa,\delta}$. It follows any finite set of conditions in $\langle p_i : i \in X \rangle$ has a common extension in $\mathbb{P}$. Therefore $\mathbb{P}$ has ω_1 as a precaliber.

(2) Let G be $\mathbb{P}$-generic over V. By Clause (1), all cofinalities (and hence cardinals) from V are preserved in $V[G]$. Since $\kappa \geq \omega_1$, it is easy to see that $\mathfrak{ls}(\kappa) \leq \omega_1$ – For any countable $\mathcal{F} \subseteq \mathcal{P}(\kappa)$, consider $A = \{\min(X) : X \in \mathcal{F}\}$ and $\alpha \in \kappa \setminus A$. For the other inequality, we'll show that in $V[G]$, there is a family $\mathcal{F}$ of size ω_1 that separates countable subsets of κ from points. For each indecomposable $\delta < \omega_1$, define $X_\delta = \{\alpha < \kappa : (\exists p \in G)(\delta \in \mathsf{dom}(p) \wedge \alpha \in \mathsf{dom}(p(\delta)) \wedge p(\delta)(\alpha) = 1)\}$. Put $\mathcal{F} = \{X_\delta : \delta < \omega_1 \text{ is indecomposable}\}$. We claim that $\mathcal{F}$ separates countable subsets of κ from points. For suppose $A \subseteq \kappa$ is countable and $\alpha \in \kappa \setminus A$. Since $\mathbb{P}$ satisfies ccc, there exists $B \in V \cap [\kappa]^{<\omega_1}$ such that $A \subseteq B$ and $\alpha \notin B$. Now a simple density argument shows that there exists $p \in G$ and $\delta \in \mathsf{dom}(p)$ with $\delta > \mathsf{otp}(B)$ such that $B \cup \{\alpha\} \subseteq \mathsf{dom}(p(\delta))$, $(\forall \beta \in B)(p(\delta)(\beta) = 0)$ and $p(\delta)(\alpha) = 1$. This means that $\alpha \in X_\delta$ and $X_\delta \cap A = \emptyset$. Hence $\mathcal{F}$ separates countable subsets of κ from points. □

Theorem 2.6. *Suppose $\omega_1 \leq \theta$ and $2^\theta < \kappa = \kappa^\omega$. Let $\mathbb{P}$ be the finite support product of $\mathbb{Q}_{\kappa,\delta}$'s where δ runs over the set of indecomposable ordinals $< \omega_1$. Then $V^{\mathbb{P}} \models \mathfrak{c} = \kappa$, $\mathfrak{ls}(\kappa) = \omega_1$ and $\theta < \mathfrak{la}(\kappa)$.*

Proof. Since $\mathbb{P}$ satisfies ccc and $|\mathbb{P}| = \kappa^\omega = \kappa$, an easy name counting argument shows that $V^{\mathbb{P}} \models \mathfrak{c} = |\mathcal{P}(\omega)| \leq \kappa$. To see that $V^{\mathbb{P}} \models \mathfrak{c} \geq \kappa$, just note that $\mathbb{Q}_{\kappa,\omega} \lessdot \mathbb{P}$ and $\mathbb{Q}_{\kappa,\omega}$ is the forcing for adding κ Cohen reals. Furthermore, Lemma 2.5 implies that $V^{\mathbb{P}} \models \mathfrak{ls}(\kappa) = \omega_1$. So we only need to check that $V^{\mathbb{P}} \models \theta < \mathfrak{la}(\kappa)$.

Towards a contradiction, assume $V^{\mathbb{P}} \models \mathfrak{la}(\kappa) \leq \theta$. Then we can find $p \in \mathbb{P}$, $\lambda \leq \theta$ and $\langle \mathring{A}_i : i < \kappa \rangle$ such that the following hold.

(i) $\mathsf{cf}(\lambda) = \mu \geq \omega_1$.
(ii) For every $i < \kappa$, $p \Vdash \mathring{A}_i \in [\lambda]^\mu$.
(iii) For all $i < j < \kappa$, $p \Vdash |\mathring{A}_i \cap \mathring{A}_j| < \mu$.

For each $i < \kappa$, define $B_i = \{\xi < \lambda : (\exists q \leq p)(q \Vdash \xi \in \mathring{A}_i)\}$. Since $\mathbb{P}$ satisfies ccc, it is easy to see that for every $i < \kappa$, $|B_i| = \mu$. Furthermore, each $B_i \in V \cap \mathcal{P}(\lambda)$ and $p \Vdash \mathring{A}_i \subseteq B_i$. As $V \models |\mathcal{P}(\lambda)| = 2^\lambda \leq 2^\theta < \kappa$, we can find $X \in [\kappa]^\kappa$ and $B_\star \subseteq \lambda$ such that for every $i \in X$, $B_i = B_\star$. Fix a bijection $h : B_\star \to \mu$. Since $\mathbb{P}$ satisfies ccc and μ is regular uncountable, for every $i < j$ in X, we can choose $\xi(i,j) < \mu$ such that $p \Vdash h[\mathring{A}_i \cap \mathring{A}_j] < \xi(i,j)$. As $V \models \kappa > 2^\theta \geq 2^\mu$, by the Erdős-Rado theorem, we can find $Y \in [X]^{\mu^+}$ and $\xi_\star < \mu$ such that for every $i < j$ in Y, $\xi(i,j) = \xi_\star$. Choose $\alpha \in B_\star$ such that $\xi_\star < h(\alpha) < \mu$. For each $i \in Y$, choose $q_i \leq p$ such that $q_i \Vdash \alpha \in \mathring{A}_i$. Now note that no two conditions in $\{q_i : i \in Y\}$ are compatible. Since $|Y| = \mu^+ > \mu \geq \omega_1$, this contradicts the fact that $\mathbb{P}$ satisfies ccc. Hence $V^{\mathbb{P}} \models \mathfrak{la}(\kappa) > \theta$. □

Corollary 2.7. *Assume $V \models 2^{\omega_k} = \omega_{k+1}$ for $k < 3$. Let $\mathbb{P}$ be the finite support product of $\mathbb{Q}_{\omega_3,\delta}$'s where δ runs over the set of indecomposable ordinals $< \omega_1$. Then $V^{\mathbb{P}} \models \mathfrak{c} = \omega_3$, $\mathfrak{ls}(\omega_3) = \omega_1$ and $\mathfrak{la}(\omega_3) = \omega_2$.*

Proof. By Theorem 2.6, $V^{\mathbb{P}} \models \mathfrak{c} = \omega_3$, $\mathfrak{ls}(\omega_3) = \omega_1$ and $\mathfrak{la}(\omega_3) > \omega_1$. Since there is an almost disjoint family in $[\omega_2]^{\omega_2}$ of size ω_3, we must have $V^{\mathbb{P}} \models \mathfrak{la}(\omega_3) = \omega_2$. $\square$

3. Stronger separating families

Definition 3.1. *Let $\mu < \delta \leq \lambda$ be infinite cardinals such that μ, δ are regular and $\delta = \delta^{<\mu}$. Define a forcing $\mathbb{Q}_{\lambda,\delta,\mu}$ as follows. $p \in \mathbb{Q}_{\lambda,\delta,\mu}$ iff the following hold.*

(i) *p is a function, $\mathsf{dom}(p) \subseteq \lambda$ and $\mathsf{range}(p) \subseteq 2$.*
(ii) *$|\mathsf{dom}(p)| < \delta$.*
(iii) *$|\{\xi \in \mathsf{dom}(p) : p(\xi) = 1\}| < \mu$.*

For $p, q \in \mathbb{Q}_{\lambda,\delta,\mu}$, define $p \leq q$ iff $q \subseteq p$.

Lemma 3.2. *Let $\mathbb{Q} = \mathbb{Q}_{\lambda,\delta,\mu}$ be as in Definition 3.1. Then the following hold.*

(1) *$\mathbb{Q}$ is $< \mu$-closed.*
(2) *For every $X \in [\mathbb{Q}]^{\delta^+}$, there exists $Y \in [X]^{\delta^+}$ such that for any $F \in [Y]^{<\mu}$, there exists $p \in \mathbb{Q}$ such that $(\forall q \in F)(p \leq q)$. So $\mathbb{Q}$ satisfies δ^+-cc.*
(3) *$V^{\mathbb{Q}} \models |\delta| = \mu$.*
(4) *Forcing with $\mathbb{Q}$ preserves all cardinals $\leq \mu$ and $\geq \delta^+$ and collapses every cardinal in (μ, δ^+) to μ.*

Proof. That $\mathbb{Q}$ is $< \mu$-closed is easy to see. This implies that all cardinals $\leq \mu$ are preserved.

Next, suppose $\langle p_i : i < \delta^+ \rangle$ is a sequence of conditions in $\mathbb{Q}$. Put $A_i = \mathsf{dom}(p_i)$, $B_i = \{\xi \in \mathsf{dom}(p_i) : p_i(\xi) = 1\}$ and $A = \bigcup\{A_i : i < \delta^+\}$. Then $|A| \leq \delta^+$. WLOG, we can assume $A \subseteq \delta^+$. Fix a club $E \subseteq \delta^+$ such that for each $\gamma \in E$ and $i < \gamma$, $A_i \subseteq \gamma$. Let $S = \{\gamma \in E : \mathsf{cf}(\gamma) = \delta\}$. Then S is stationary in δ^+ and the function $h : S \to \delta^+$ defined by $h(\gamma) = \sup(A_\gamma \cap \gamma)$ is regressive. By Fodor's lemma, we can find $T \subseteq S$ and $\gamma_\star < \delta^+$ such that T is stationary in δ^+ and $h \restriction T$ is constantly $\gamma_\star$. Observe that, as $T \subseteq E$, for every $i < j$ in T, $A_i \cap A_j \subseteq \gamma_\star$. Since $|\gamma_\star| \leq \delta$ and $\delta^{<\mu} = \delta$, we can find $B_\star \in [\gamma_\star]^{<\mu}$ and $W \subseteq T$ such that W is stationary in δ^+ and for every $i \in W$, $B_i \cap \gamma_\star = B_\star$. It follows that $\langle p_i : i \in W \rangle$ consists of pairwise compatible functions. Clause (2) follows.

To see (3), suppose G is $\mathbb{Q}$-generic over V. Put $F = \bigcup G$. Then $F : \lambda \to 2$. Let $W = \{\xi < \delta : F(\xi) = 1\}$. Fix a partition $\delta = \bigsqcup\{W_i : i < \delta\}$ in V such that each $W_i \in [\delta]^\delta$. An easy density argument shows that $\mathsf{otp}(W) = \mu$ and for every $i < \delta$, $W \cap W_i \neq \emptyset$. It follows that $V[G] \models |\delta| = \mu$.

By Clause (1) all cardinals $\geq \delta^+$ are preserved and by Clause (1) all cardinals $\leq \mu$ are preserved. Hence Clause (4) follows from Clauses (1)-(3). $\square$

Definition 3.3. *Suppose $\omega \leq \mu = \mathsf{cf}(\mu) < \kappa < \lambda$ and $S_\star = \{\delta : \mu < \delta < \kappa$ and δ is inaccessible$\}$ is stationary in κ (so κ is Mahlo). Let $\mathbb{P}_{\lambda,\kappa,\mu}$ be the Easton-support product of $\langle \mathbb{Q}_{\lambda,\delta,\mu} : \delta \in S_\star \rangle$. So $p \in \mathbb{P}_{\lambda,\kappa,\mu}$ iff*

(a) *p is a function with $\mathsf{dom}(p) \subseteq S_\star$,*
(b) *for every $\delta \in S_\star \cup \{\kappa\}$, $\sup(\mathsf{dom}(p) \cap \delta) < \delta$ and*
(c) *for every $\delta \in \mathsf{dom}(p)$, $p(\delta) \in \mathbb{Q}_{\lambda,\delta,\mu}$.*

For $p, q \in \mathbb{P}_{\lambda,\kappa,\mu}$, define $p \leq q$ iff $\mathsf{dom}(q) \subseteq \mathsf{dom}(p)$ and for every $\delta \in \mathsf{dom}(q)$, $p(\delta) \leq_{\mathbb{Q}_{\lambda,\delta,\mu}} q(\delta)$.

Lemma 3.4. *Let $\mu, \kappa, \lambda, S_\star$ and $\mathbb{P} = \mathbb{P}_{\lambda,\kappa,\mu}$ be as above. Then the following hold.*

(1) *Forcing with $\mathbb{P}$ collapses all cardinals in the interval (μ, κ) to μ.*

(2) *$\mathbb{P}$ is $< \mu$-closed and κ-cc. So all cardinals $\leq \mu$ are preserved and $V^{\mathbb{P}} \models \kappa = \mu^+$.*

Proof. For each $\delta \in S_\star$, $\mathbb{Q}_{\lambda,\delta,\mu} \lessdot \mathbb{P}$. Therefore Clause (1) follows from Lemma 3.2. It is also clear that $\mathbb{P}$ is $< \mu$-closed.

Let us check that $\mathbb{P}$ satisfies the κ-cc. Towards a contradiction, suppose $\langle p_i : i < \kappa \rangle$ is a sequence of pairwise incompatible conditions in $\mathbb{P}$. Choose a club $E \subseteq \kappa$ such that for every $\gamma \in E$ and $i < \gamma$, $\sup(\mathsf{dom}(p_i)) < \gamma$. Since the function $h : E \cap S_\star \to \kappa$ defined by $h(\delta) = \sup(\mathsf{dom}(p_\delta) \cap \delta)$ is regressive, by Fodor's lemma, we can find a stationary subset $T \subseteq E \cap S_\star$ and $\gamma_\star < \kappa$ such that for every $\delta \in T$, $h(\delta) < \gamma_\star$. Note that for any $\delta_1 < \delta_2$ in T, $\mathsf{dom}(p_{\delta_1}) \cap \mathsf{dom}(p_{\delta_2}) \subseteq \gamma_\star$. Define a coloring $c : [T]^2 \to \gamma_\star$ by $c(\{\delta_1, \delta_2\})$ is the least $\gamma \in \mathsf{dom}(p_{\delta_1}) \cap \mathsf{dom}(p_{\delta_2})$ such that $p_{\delta_1}(\gamma)$ and $p_{\delta_2}(\gamma)$ are incompatible in $\mathbb{Q}_{\lambda,\gamma,\mu}$. Put $\theta = |\gamma_\star|^{++}$. Since $|T| = \kappa$ is inaccessible and $\theta < \kappa$, using Erdős-Rado theorem, we can find $X \in [T]^\theta$ and $\gamma < \gamma_\star$ such that $c \restriction [X]^2$ takes the constant value γ. But this means that $\{p_\delta(\gamma) : \delta \in X\}$ is an antichain of size $\theta > \gamma^+$ in $\mathbb{Q}_{\lambda,\gamma,\mu}$ which is impossible by Lemma 3.2. □

Lemma 3.5. *Let $\mu, \kappa, \lambda, S_\star$ and $\mathbb{P} = \mathbb{P}_{\lambda,\kappa,\mu}$ be as above. Then the following hold in $V^{\mathbb{P}}$.*

(1) *There is a family $\mathcal{F} \subseteq \mathcal{P}(\lambda)$ such that $|\mathcal{F}| = \kappa$ and for any $A \in [\lambda]^{<\kappa}$ and $B \in [\lambda]^{<\mu}$, if $A \cap B = \emptyset$, then there exists $X \in \mathcal{F}$ such that $B \subseteq X$ and $A \cap X = \emptyset$.*
(2) *If $\mu = \omega$, then $\mathfrak{ls}(\lambda) = \omega_1$. If $\mu \geq \omega_1$, the $\mathfrak{ls}(\lambda) = \mu$.*
(3) *If $\lambda > 2^\kappa$, then there is no family $\mathcal{A} \subseteq [\kappa]^\kappa$ such that $|\mathcal{A}| = \lambda$ and for every $X \neq Y$ in $\mathcal{A}$, $|X \cap Y| < \kappa$.*

Proof. (1) Let G be $\mathbb{P}$-generic over V. For each $\delta \in S_\star$, define

$$X_\delta = \{\xi < \lambda : (\exists p \in G)(p(\delta)(\xi) = 1)\}.$$

Put $\mathcal{F} = \{X_\delta : \delta \in S_\star\}$. Then $|\mathcal{F}| = \kappa$. We claim that $\mathcal{F}$ is as required. For suppose $A \in [\lambda]^{<\kappa}$ and $B \in [\lambda]^{<\mu}$. Since $\mathbb{P}$ is $< \mu$-closed, $B \in V$. Since $\mathbb{P}$ satisfies κ-cc, we can find $C \subseteq V \cap [\lambda \setminus B]^{<\kappa}$ such that $A \subseteq C$. Now observe that the set of conditions $p \in \mathbb{P}$ satisfying the following is dense in $\mathbb{P}$: There exists $\delta \in dom(p)$ such that (a)-(c) below hold.

(a) $|C| < \delta$.
(b) $(\forall \xi \in B)(p(\delta)(\xi) = 1)$.
(c) $(\forall \xi \in C)(p(\delta)(\xi) = 0)$.

Choose such a $p \in G$ and a witnessing $\delta \in \mathsf{dom}(p)$. It follows that $B \subseteq X_\delta$ and $C \cap X_\delta = A \cap X_\delta = \emptyset$.

(2) Recall that forcing with $\mathbb{P}$ preserves all cardinals $\leq \mu$ and $\geq \kappa$ and collapses all cardinals in the interval (μ, κ) to μ. So $V^{\mathbb{P}} \models \kappa = \mu^+$. First suppose $\mu = \omega$. Then by (1), $V^{\mathbb{P}} \models \mathfrak{ls}(\lambda) \leq \kappa = \mu^+ = \omega_1$. Since λ is uncountable, we also have $\mathfrak{ls}(\lambda) \geq \omega_1$. Therefore $V^{\mathbb{P}} \models \mathfrak{ls}(\lambda) = \omega_1$.

Next assume $\mu \geq \omega_1$. Fix $\delta \in S_\star$ and a bijection $h : \lambda \times \delta \to \lambda$ such that $h \in V$. Let G be $\mathbb{P}$-generic over V. For each $i < \delta$, define

$$X_i = \{\xi < \lambda : (\exists p \in G)(p(\delta)(h(\xi, i)) = 1)\}.$$

Put $\mathcal{F} = \{X_i : i < \delta\}$. An easy density argument shows that for every $A \in [\lambda]^{<\delta} \cap V$ and $B \in [\lambda \setminus A]^{<\mu} \cap V$, there exists $i < \delta$ such that $A \cap X_i = \emptyset$ and $B \subseteq X_i$. Since $\mathbb{P}$ is $< \mu$-closed, forcing with $\mathbb{P}$ does not add new countable subsets of λ. Hence $V[G] \models \mathcal{F}$ separates countable subsets of λ from points. Since $V[G] \models |\mathcal{F}| = |\delta| = \mu$, it follows that $\mathfrak{ls}(\lambda) \leq \mu$.

Finally, to see that $V^{\mathbb{P}} \models \mathfrak{ls}(\lambda) \geq \mu$, towards a contradiction, fix $p \in \mathbb{P}$, $\theta < \mu$ and $\langle \mathring{A}_i : i < \theta \rangle$ such that $(\forall i < \theta)(p \Vdash \mathring{A}_i \in \mathcal{P}(\lambda))$ and $p \Vdash \{\mathring{A}_i : i < \theta\}$ separates countable subsets of λ from points. Define $\mathring{B}_\xi = \{i < \theta : \xi \in \mathring{A}_i\}$. As $\mathbb{P}$ is $< \mu$-closed, $p \Vdash \mathring{B}_\xi \in \mathcal{P}(\theta) \cap V$. Since $V \models 2^\theta < \kappa < \lambda$, we can choose $\mathring{X} \in [\lambda]^\lambda \cap V^{\mathbb{P}}$, $q \in \mathbb{P}$ and $B_\star \in \mathcal{P}(\theta) \cap V$ such that $q \leq p$ and $q \Vdash (\forall \xi \in \mathring{X})(\mathring{B}_\xi = B_\star)$. Choose $q \leq p$ and $\xi_1 < \xi_2 < \lambda$ such that $q \Vdash \{\xi_1, \xi_2\} \subseteq \mathring{X}$. Then $q \Vdash \mathring{B}_{\xi_1} = \mathring{B}_{\xi_2} = B_\star$. Now observe that for every $i < \theta$,

$$q \Vdash (\xi_1 \in \mathring{A}_i \iff i \in \mathring{B}_{\xi_1} \iff i \in B_\star \iff i \in \mathring{B}_{\xi_2} \iff \xi_2 \in \mathring{A}_i).$$

Therefore $q \Vdash \{\mathring{A}_i : i < \theta\}$ does not separate countable subsets of λ from points. Hence $V^{\mathbb{P}} \models \mathfrak{ls}(\lambda) = \mu$.

(3) Towards a contradiction, fix $p \in \mathbb{P}$ and $\langle \mathring{A}_i : i < \lambda \rangle$ such that

$$p \Vdash (\forall i < \lambda)(\mathring{A}_i \in [\kappa]^\kappa) \text{ and } (\forall i < j < \lambda)(|\mathring{A}_i \cap \mathring{A}_j| < \kappa).$$

Since $\mathbb{P}$ satisfies the κ-cc, we can find $c : [\lambda]^2 \to \kappa$ in V such that for every $i < j < \lambda$, $p \Vdash \sup(\mathring{A}_i \cap \mathring{A}_j) < c(\{i,j\})$. Since $V \models \lambda > 2^\kappa$, by the Erdős-Rado theorem, there are $H \in [\lambda]^{\kappa^+}$ and $\gamma < \kappa$ such that for every $i < j$ in H, $c(\{i,j\}) = \gamma$. It now follows that $p \Vdash \{\mathring{A}_i \setminus \gamma : i \in H\}$ is a family of κ^+ pairwise disjoint sets in $[\kappa]^\kappa$ which is impossible since all cardinals $\geq \kappa$ are preserved in $V^{\mathbb{P}}$. $\square$

Theorem 3.6. *Suppose $\mu < \kappa$, κ is Mahlo and $\lambda > 2^\kappa$. There is a $< \mu$-closed κ-cc forcing $\mathbb{P}$ such that*

(1) $V^{\mathbb{P}} \models \kappa = \mu^+$.
(2) $V^{\mathbb{P}} \models \mathfrak{ls}(\lambda, \kappa, \mu) = \kappa$.
(3) $V^{\mathbb{P}} \models \mathfrak{la}(\lambda) \geq \kappa^+$.

Proof. Readily follows from Lemmas 3.4 and 3.5. $\square$

4. Partition relations

Let $(L, \prec)$ be a linear ordering. Throughout this section, we will assume that $|L| \geq 2$. Recall that (C_0, C_1) is a cut in $(L, \prec)$ iff C_0 is downward closed in L, C_1 is upward closed in L and $L = C_0 \sqcup C_1$.

Definition 4.1. *A linear ordering $(L, \prec)$ is additively indecomposable iff for every cut (C_0, C_1) in $(L, \prec)$, at least one of $(C_0, \prec)$ and $(C_1, \prec)$ contains an isomorphic copy of $(L, \prec)$.*

It is easy to see that unionwise indecomposable linear orderings are also additively indecomposable but the converse is false. Recall that a linear ordering is scattered iff it does not contain a copy of the rationals $(\mathbb{Q}, <)$. The following fact appears in [3] (also Exercise 10.4.1 in [7]).

Fact 4.2. *Let $(L, \prec_L)$ be a scattered additively indecomposable linear ordering. Then one of the following holds.*

(a) *For every cut (C_0, C_1) in L, if $C_0 \neq \emptyset$, then L embeds into C_0. In this case, we say that $(L, \prec_L)$ is indecomposable to the left.*
(b) *For every cut (C_0, C_1) in L, if $C_1 \neq \emptyset$, then L embeds into C_1. In this case, we say that $(L, \prec_L)$ is indecomposable to the right.*

Definition 4.3. *Let $(L, \prec)$ be a unionwise indecomposable linear ordering. An ultrafilter $\mathcal{U}$ on L is uniform iff for every $A \in \mathcal{U}$, L embeds into A.*

The following lemma is a straightforward generalization of the results of Section 13.3 in [8].

Lemma 4.4. *Assume $\mathsf{MA}+\mathfrak{c} > \omega_1$. Let $(L, \prec_L)$ be a countable scattered unionwise indecomposable linear ordering. Then there exists a uniform ultrafilter $\mathcal{U}_L$ on L such that for every family $\mathcal{F} \subseteq \mathcal{U}_L$, if $|\mathcal{F}| \leq \omega_1$, then there exists $X \in \mathcal{U}_L$ such that for every $A \in \mathcal{F}$, $X \setminus A$ is indecomposably bounded in $(L, \prec)$ which means the following.*

(a) *Either $(L, \prec_L)$ is indecomposable to the left and for every cut (C_0, C_1) in L with $C_0 \neq \emptyset$, $C_0 \in \mathcal{U}_L$ and $X \setminus A$ is bounded from below in $(L, \prec_L)$ or*

(b) *$(L, \prec_L)$ is indecomposable to the right and for every cut (C_0, C_1) in L with $C_1 \neq \emptyset$, $C_1 \in \mathcal{U}_L$ and $X \setminus A$ is bounded from above in $(L, \prec_L)$.*

Proof. Let $\mathcal{C}$ be the class of all countable scattered indecomposable linear orders. Laver [6] showed that there is a rank function $r : \mathcal{C} \to \omega_1$ such that the following hold.

(i) For every $L \in \mathcal{C}$, either $r(L) = 1$ and $L \in \{\omega, \omega^\star\}$ or L is the sum of an ω or $\omega^\star$ sequence of members of $\mathcal{C}$ of strictly smaller ranks.

(ii) If $L_1, L_2 \in \mathcal{C}$ and L_1 embeds into L_2, then $r(L_1) \leq r(L_2)$.

We construct $\mathcal{U}_L$ by induction on the rank of L. If $r(L) = 1$, this is clear: Say $(L, \prec_L) = (\omega, <)$. Using MA, build a sequence $\langle A_i : i < \mathfrak{c} \rangle$ of $\subseteq^\star$-descending sequence in $[\omega]^\omega$ such that for every $X \subseteq \omega$, there exists $i < \mathfrak{c}$ such that either $A_i \subseteq X$ or $X \cap A_i = \emptyset$. Take $\mathcal{U}_\omega$ to be the filter generated by $\{A_i : i < \mathfrak{c}\}$.

Now suppose $r(L) > 1$. Let us consider the case when L is the sum of an ω-sequence of members of $\mathcal{C}$ of smaller ranks. The case when L is the sum of an $\omega^\star$-sequence is similar. Fix $\{(L_n, \prec_n) : n < \omega\}$ such that L_n's are pairwise disjoint, $r(L_n) < r(L)$, $L = \bigcup\{L_n : n < \omega\}$ and for every $a, b \in L$, $a \prec_L b$ iff $a \in L_m$, $b \in L_n$ and $(m < n$ or $(m = n \wedge a \prec_n b))$.

We can assume that $(L, \prec_L)$ is indecomposable to the right. Otherwise, since L is indecomposable, it would embed into some L_n which is impossible since $r(L_n) < r(L)$. Fix $\mathcal{U}_\omega$ as above. For each $n < \omega$, let $\mathcal{U}_n$ be a uniform ultrafilter on L_n such that for every $\mathcal{A} \subseteq \mathcal{U}_n$, if $|\mathcal{A}| \leq \omega_1$, then there exists $X \in \mathcal{U}_n$ such that for every $A \in \mathcal{A}$, $X \setminus A$ is indecomposably bounded in $(L_n, \prec_n)$.

Define $\mathcal{U}_L = \{X \subseteq L : \{n : X \cap L_n \in \mathcal{U}_n\} \in \mathcal{U}_\omega\}$. We claim that $\mathcal{U}_L$ is as required. To see this, suppose $\mathcal{F} \subseteq \mathcal{U}_L$ and $|\mathcal{F}| \leq \omega_1$. We can assume that $L \in \mathcal{F}$. For $A \in \mathcal{F}$ and $n < \omega$, define $S_A = \{n : A \cap L_n \in \mathcal{U}_n\}$. Choose $B \in \mathcal{U}_\omega$ such that $B \setminus S_A$ is finite for every $A \in \mathcal{F}$. Let $\mathcal{A}_n = \{A \cap L_n : A \in \mathcal{F}\} \cap \mathcal{U}_n$. Choose $X_n \in \mathcal{U}_n$ such that $X_n \setminus W$ is indecomposably bounded in L_n for every $W \in \mathcal{A}_n$. Put $Y = \bigcup\{X_n : n \in B\}$. It is clear that $Y \in \mathcal{U}_L$.

Choose $\langle x_n(k) : k < \omega \rangle$ such that for every n, one of $(\star)_n$, $(\star\star)_n$ below holds.

$(\star)_n$ Either L_n is indecomposable to the right and the following hold.

(1) $X_n \setminus W$ is bounded from above in L_n for every $W \in \mathcal{A}_n$.
(2) $\langle x_n(k) : k < \omega \rangle$ is increasing and right-cofinal in $(L_n, \prec_n)$.

$(\star\star)_n$ Or L_n is indecomposable to the left and the following hold.

(2) $X_n \setminus W$ is bounded from below in L_n for every $W \in \mathcal{A}_n$.
(3) $\langle x_n(k) : k < \omega \rangle$ is decreasing and left-cofinal in $(L_n, \prec_n)$.

For each $A \in \mathcal{F}$, fix $N_A < \omega$ such that $B \setminus N \subseteq S_A$. Choose $f_A : \omega \to \omega$ such that for every $n \in B \setminus N_A$, if $(\star)_n$ holds, then $(Y \setminus A) \cap L_n$ is $\prec_n$-bounded from above by $x_n(f_A(n))$ and if $(\star\star)_n$ holds, then then $(Y \setminus A) \cap L_n$ is $\prec_n$-bounded from below by $x_n(f_A(n))$. Using MA_{ω_1}, fix $f_\star : \omega \to \omega$ dominating every function in $\{f_A : A \in \mathcal{F}\}$. Put $Z = \bigcup\{Y \cap W_n : n < \omega\}$ where

$$W_n = \begin{cases} (x_n(f_\star(n)), \infty)_{L_n} & \text{if } (\star)_n \text{ holds} \\ (-\infty, x_n(f_\star(n)))_{L_n} & \text{otherwise} \end{cases}$$

Then $Z \in \mathcal{U}_L$ (as $Y \setminus Z \notin \mathcal{U}_L$) and for every $A \in \mathcal{F}$, $Z \setminus A$ is indecomposably bounded in $(L, \prec_L)$. $\square$

Proof of Theorem 1.11: If $(L, \prec_L)$ contains a copy of rationals, then this is Theorem 6.7 in [4] (For a slightly stronger result see Lemma 4.5 below). So assume it is scattered. Fix $\mathcal{U}_L$ as in Lemma 4.4 and suppose $c : \omega_1 \times L \to K$. For each $\alpha < \omega_1$, fix $A_\alpha \in \mathcal{U}_L$ and $k_\alpha < K$ such that for every $x \in A_\alpha$, $c(\alpha, x) = k_\alpha$. Fix $X \in [\omega_1]^{\omega_1}$ such that $k_\alpha = k_\star$ does not depend on $\alpha < X$. Now apply Lemma 4.4 to the family $\{A_\alpha : \alpha \in X\}$ to get $B \in \mathcal{U}_L$ such that for each $\alpha \in X$, $B \setminus A_\alpha$ is indecomposably bounded in $(L, \prec_\alpha)$ by $y_\alpha \in L$. Choose $Y \in [X]^{\omega_1}$ such that $y_\alpha = y_\star$ does not depend on $\alpha \in Y$. Choose $D \subseteq B$ such that $(\forall \alpha \in Y)(D \subseteq A_\alpha)$ and $(D, \prec_L) \cong (L, \prec_L)$. Then $c \restriction (Y \times D)$ is constant. $\square$

Proof of Theorem 1.10: We use an absoluteness argument like the one in [1]. Let W be a ccc extension of V satisfying $\mathsf{MA} + \mathfrak{c} > \omega_1$. Let $\alpha < \omega_1$. Fix linear orders $\prec_1$ and $\prec_2$ on ω such that $\mathsf{otp}(\omega, \prec_1) = (\alpha, <)$ and $\mathsf{otp}(\omega, \prec_2) = (L, \prec_L)$. Let T be the set of all pairs (s, t) such that

- s, t are functions, $\mathsf{dom}(s) = \mathsf{dom}(t) = N < \omega$,
- $\mathsf{range}(s) \subseteq \omega_1$, $\mathsf{range}(t) \subseteq L$,
- $c \restriction (\mathsf{range}(s) \times \mathsf{range}(t))$ is constant and
- for every $m, n < N$, $(m \prec_1 n \iff s(m) < s(n))$ and $(m \prec_2 n \iff t(m) \prec_L t(n))$.

Define $(s, t) \preceq_T (s', t')$ iff $s \subseteq s'$ and $t \subseteq t'$ and note that $(T, \preceq_T)$ is well-founded iff there is no c-homogeneous set of type $\alpha \times L$. But this is absolute between V and W since a tree is well-founded iff there is a rank function on it. So it suffices to construct such a homogeneous set in W. But this was already done above. $\square$

Lemma 4.5. *Let $f : \omega_1 \times \mathbb{Q} \to K$ where $K < \omega$.*

(1) *Assume MA_{ω_1} (or just $\mathfrak{p} > \omega_1$). Then there exist $X \in [\omega_1]^{\omega_1}$ and $Y \subseteq \mathbb{Q}$ such that Y is somewhere dense in $\mathbb{Q}$ and $f \restriction (X \times Y)$ is constant.*
(2) *For each $\alpha < \omega_1$, there exist $X \subseteq \omega_1$ and $Y \subseteq \mathbb{Q}$ such that $\mathsf{otp}(X) = \alpha$, Y is somewhere dense in $\mathbb{Q}$ and $f \restriction (X \times Y)$ is constant.*

Proof. (1) It is enough to show this for $K = 2$ for then we can argue by induction on K. For each $i < \omega_1$, put $A_i = \{x \in \mathbb{Q} : f(i, x) = 1\}$. The following is Lemma 6.11 in [4].

Fact 4.6 ([4]). *Let $\langle A_i : i < \omega_1 \rangle$ be a sequence of subsets of $\mathbb{Q}$. There exist $W \in [\omega_1]^{\omega_1}$, $c < 2$ and a rational interval J such that for every finite $F \subseteq W$, $\bigcap_{i \in F} A_i^c$ is dense in J. Here, $A_i^c = A_i$ if $c = 0$ and $\mathbb{Q} \setminus A_i$ otherwise.*

Using Fact 4.6, we can find a rational interval J and $W \in [\omega_1]^{\omega_1}$ such that either the intersection of any finite subfamily of $\{A_i : i \in W\}$ is dense in J or the intersection of any finite subfamily of $\{\mathbb{Q} \setminus A_i : i \in W\}$ is dense in J. WLOG, let us assume that the former situation holds. Define a forcing $\mathbb{P}$ as follows: $p \in \mathbb{P}$ iff $p = (u_p, v_p, I_p)$ where

(i) $u_p \in [\mathbb{Q}]^{<\omega}$ and $v_p \in [W]^{<\omega}$.
(ii) I_p is a finite family of rational subintervals of J.
(iii) For each $I \in I_p$, $u_p \cap I \neq \emptyset$.
(iv) For $p, q \in \mathbb{P}$, define $p \leq q$ iff
  (a) $u_q \subseteq u_p$, $v_q \subseteq v_p$, $I_q \subseteq I_p$.
  (b) If $x \in u_p \setminus u_q$ and $i \in v_q$, then $x \in A_i$.

$\mathbb{P}$ is σ-centered (as there are countably many u_p's) and if $G \subseteq \mathbb{P}$ is sufficiently generic (use $\mathfrak{p} > \omega_1$) then $X = W = \bigcup\{v_p : p \in G\}$ and $Y = \bigcup\{u_p : p \in G\}$ are as claimed in (1).

(2) Fix a linear order $\prec_\alpha$ on ω such that $\mathsf{otp}(\omega, \prec_\alpha) = (\alpha, <)$. For each rational interval J, fix a computable enumeration $\langle J_n : n < \omega\rangle$ of all rational subintervals of J and define $X = X_J$ to be the set of all finite sequences $s = \langle (x_n, i_n) : n < N\rangle$ such that the following hold.

- For every $n < N$, $x_n \in J_n$.
- $\langle i_n : n < N\rangle$ is an injective sequence of countable ordinals.
- For every $m, n < N$, $(i_m < i_n \iff m \prec_\alpha n)$.
- $f \restriction (\{i_n : n < N\} \times \{x_n : n < N\})$ is constant.

Define a relation $R = R_J$ on X by sRt iff $t \subsetneq s$. Note that (X_J, R_J) is not well-founded iff there exist $X \subseteq \omega_1$ and $Y \subseteq \mathbb{Q}$ such that $\mathsf{otp}(X) = \alpha$, Y is dense in J and $f \restriction (X \times Y)$ is constant.

Now we can start repeating the proof of part (1). Choose a rational interval J and the forcing $\mathbb{P}$ as there and get a $\mathbb{P}$-generic filter G over V. In $V[G]$, (X_J, R_J) is not well-founded. By absoluteness, the same holds in V and we are done. □

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