Journal of Mathematical Logic
Vol. 23, No. 2 (2023) 2250020 (26 pages)
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DOI: 10.1142/S0219061322500209

Exact saturation in pseudo-elementary classes for simple and stable theories

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Received 14 April 2021
Accepted 2 May 2022
Published 27 September 2022

We use exact saturation to study the complexity of unstable theories, showing that a variant of this notion called pseudo-elementary class (PC)-exact saturation meaningfully reflects combinatorial dividing lines. We study PC-exact saturation for stable and simple theories. Among other results, we show that PC-exact saturation characterizes the stability cardinals of size at least continuum of a countable stable theory and, additionally, that simple unstable theories have PC-exact saturation at singular cardinals satisfying mild set-theoretic hypotheses. This had previously been open even for the random graph. We characterize supersimplicity of countable theories in terms of having PC-exact saturation at singular cardinals of countable cofinality. We also consider the local analog of PC-exact saturation, showing that local PC-exact saturation for singular cardinals of countable cofinality characterizes supershort theories.

Keywords: Simple theories; stable theories; classification theory; exact saturation.

Mathematics Subject Classification 2020: 03C45, 03C55, 03C95

1. Introduction

One of the major goals of model theory is to develop techniques to quantify and compare the complexity of mathematical theories. This began with Morley's work on categoricity and the third-named author's work characterizing when models of a theory may be determined by an assignment of cardinal invariants. This work spawned a rich structure theory for stable theories, which enabled many applications. More recent work has pushed the boundaries of our understanding beyond stable theories, which has in turn required the development of new ways of measuring "complexity". In this paper, we introduce and develop the study of

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pseudo-elementary class (*PC*)*-exact saturation*, showing that this notion meaningfully measures how combinatorially complicated a theory can be. In addition to giving an essentially complete description of how PC-exact saturation behaves in stable theories, we prove the existence of PC-exact saturated models of simple theories. The constructions done here for simple theories are interesting in their own right — establishing techniques for model-construction which had been previously unknown even for the random graph — but, additionally, provide tantalizing new tools for possibly understanding further unstable classes beyond simplicity (e.g. NSOP_1 and NTP_1).

Our work here builds upon two distinct lines of model-theoretic research. A recurrent theme in model theory is the connection between the combinatorics of definable sets — measured by dividing lines like stability, simplicity and NSOP_2 — and the ability to construct saturated models of a given first-order theory. Recently, the notion of *exact saturation* has proved to be especially meaningful in relation to combinatorial dividing lines. A model is said be *exactly κ-saturated* if it is κ-saturated but not κ^+-saturated. Saturation is a kind of completeness condition and exact saturation names an *incompactness* phenomenon, in which completeness does not spill over. The study of this notion was begun in [13] which showed, for singular cardinals, that, modulo some natural set-theoretic hypotheses, among NIP theories, the existence of exactly saturated models is characterized by the non-distality of the theory and, additionally, that exactly saturated models of simple theories exist. Later, in [11], exact saturation motivated the discovery of a new dividing line, namely the *unshreddable theories*, containing the simple and NIP theories and giving a general setting in which exactly saturated models may be constructed. A second body of work in model theory concerns methods of directly comparing first-order theories by comparing how difficult it is to saturate them. The two main approaches to this are Keisler's order and the interpretability order; the former compares saturation of ultrapowers by the same ultrafilter, the latter describes when saturation of one theory may be transferred to another by means of a first-order theory that interprets both. These "orders" are, in fact, pre-orders which divide theories into classes. It was observed in [14, Corollary 9.29] that the existence of PC-exact saturated models, defined in the following paragraph, of a given theory depends only on its class in the interpretability order, and thus the tools developed here allow us to study in an *absolute* way part of what these orders are studying in a *relative* way.

PC-exact saturation allows us to give many new characterizations of dividing lines. We say that an L-theory T has *PC-exact saturation for κ* if, for any $T_1 \supseteq T$ with $|T_1| = |T|$, there is a model $M \models T_1$ such that the reduct $M \restriction L$ is κ-saturated but not κ^+-saturated. The PC version turns out to be much more tightly connected to the complexity of the theory T. For example, consider a two-sorted L-structure $M = (X^M, Y^M)$ where on one sort X, there is the structure of a countable model of Peano arithmetic, and on the second sort Y there is a countably infinite set with no structure, with no relations between the two sorts. From a model-theoretic point of view, M is as complicated as possible, because it interprets Peano Arithmetic (PA), but $\mathrm{Th}(M)$ has an exactly saturated model: interpreting X as any κ-saturated model of PA and Y as a set with exactly κ many elements yields a κ-saturated model of $\mathrm{Th}(M)$ which is not κ^+-saturated. Note, however, that $\mathrm{Th}(M)$ does *not* have PC-exact saturation for singular cardinals κ. In an expansion of M to M', in a language containing a function symbol for a bijection $f : X \to Y$, any model of $N \models \mathrm{Th}(M')$ whose reduct to L is κ-saturated will satisfy $|X^N| \geq \kappa^+$, as PA has no exactly κ-saturated model (see [11, Lemma 5.3 and the comment after]), hence $|Y^N| \geq \kappa^+$ and therefore $N \upharpoonright L$ is κ^+-saturated. It appears that, by looking at PC-exact saturation, one obtains a condition that more faithfully tracks the complicated combinatorics of the theory.

One of the motivations of the work here was a question [14, Question 9.31] of whether the random graph has PC-exact saturation. For an infinite cardinal κ, one may easily construct an exactly κ-saturated model of the random graph: in a κ^+-saturated random graph $\mathcal{G}$, one can choose an empty (induced) subgraph $X \subseteq \mathcal{G}$ with $|X| = \kappa$ and set $\mathcal{G}' = \{v \in \mathcal{G} \mid |N_{\mathcal{G}}(v) \cap X| < \kappa\}$, where $N_{\mathcal{G}}(v)$ denotes the neighbors of the vertex v. It is easily checked that, because $\mathcal{G}$ was chosen to be κ^+-saturated, $\mathcal{G}'$ is a κ-saturated random graph which omits the type $\{R(x, v) \mid v \in X\}$ and hence is not κ^+-saturated. Given an expansion of $\mathcal{G}$ to a larger language L, it is less clear that one can arrange for such a $\mathcal{G}'$ to be a reduct of a model of $\mathrm{Th}_L(\mathcal{G})$. We prove the existence of PC-exact saturated models of the random graph by proving a much more general result, showing that, modulo natural set-theoretic hypotheses, one may construct PC-exact saturated models of simple theories and, along the way, we obtain many precise equivalences between subclasses of the stable and simple theories.

In Sec. 3, we begin our study of PC-exact saturation by focusing on the stable theories. Here we show that, for a countable stable theory T and cardinal $\mu \geq 2^{\aleph_0}$, having PC-exact saturation at μ is equivalent to T being stable in μ; this is Corollary 3.11. One direction of this theorem is an easy consequence of the well-known fact that stable theories have saturated models in the cardinals in which they are stable and such models may be expanded to a model of any larger theory (of the same size). The other direction involves the construction of suitable expansions that allow one to find realizations of types over sets of size μ from realizations of types of smaller size. Although it does not appear in the statement, our proof relies on a division into cases based on whether or not the given stable theory has the finite cover property.

In Sec. 4, we consider simple unstable theories. In general, when T is unstable then it has PC-exact saturation at any large enough regular cardinal (Proposition 2.6), so we concentrate on singular cardinals here. We show that simple unstable theories have PC-exact saturation for singular cardinals κ, satisfying certain natural hypotheses. In particular, this implies that the random graph has singular PC-exact saturation, answering [14, Question 9.31]. In fact, we prove that for a theory $T_1 \supseteq T$ such that $|T_1| = |T|$, there exists $M \models T_1$ such that $M \upharpoonright L$ is κ-saturated but not even *locally* κ^+-saturated, that is, there is a partial type consisting of κ

many instances of a single formula (or its negation) which is not realized; this is Theorem 4.1. In the supersimple case we get a converse for singular cardinals of countable cofinality: an unstable L-theory T is supersimple if and only if T has PC-exact saturation at singular cardinals with countable cofinality (satisfying mild hypotheses); this is Corollary 4.9. Combined with the results described in the previous paragraph, we get that a countable theory T is supersimple if and only if for every $\kappa > 2^{\aleph_0}$ of countable cofinality satisfying mild set-theoretic assumptions, T has PC-exact saturation at κ; this is Corollary 4.10.

Finally, in Sec. 5, we elaborate on the case of *local* PC-exact saturation at singular cardinals. Here were are interested in determining, for a given L-theory T and singular cardinal κ of countable cofinality, if, for all $T_1 \supseteq T$ with $|T_1| = |T|$, there is $M \models T_1$ such that $M \restriction L$ is locally κ-saturated but not locally κ^+-saturated. We essentially characterize when this takes place in terms of the notion of a supershort theory, a class of theories introduced by Casanovas and Wagner to give a local analog of supersimplicity [5]. A theory is called *supershort* if every local type does not fork over some finite set. The supershort theories properly contain the supersimple theories. The main result of Sec. 5 shows that if κ is a singular cardinal of countable cofinality, satisfying natural hypotheses, then T has local PC-exact saturation for κ if and only T is supershort, giving the first "outside" characterization of this class of theories.

We conclude in Sec. 6 with some questions concerning possible extensions of the results of this paper.

2. Preliminaries

2.1. *PC classes and exact saturation*

Here, we give the basic definitions and facts about (PC-)exact saturation.

Definition 2.1. Suppose that T is a complete first-order L-theory, and let $L_1 \supseteq L$. Suppose that T_1 is an L_1-theory. Let $\mathrm{PC}(T_1, T)$ be the class of models M of T which have expansions M_1 to models of T_1.

Definition 2.2. Suppose T is a first-order theory and κ is a cardinal.

(1) Say that T *has exact saturation at* κ if T has a κ-saturated model M which is not κ^+-saturated.
(2) Say that a pseudo-elementary class $P = \mathrm{PC}(T_1, T)$ has *exact saturation at a cardinal* κ if there is a κ-saturated model $M \models T$ in P which is not κ^+-saturated.
(3) Say that T has *PC-exact saturation* at a cardinal κ if for every $T_1 \supseteq T$ of cardinality $|T|$, $\mathrm{PC}(T_1, T)$ has exact saturation at κ.

In general we expect that having exact saturation at κ should not depend on κ (up to some set-theoretic assumptions on κ). For example, in [13] the following facts were established.

Fact 2.3. Suppose T is a first-order theory.

(1) [13, Theorem 2.4] If T is stable then for all $\kappa > |T|$, T has exact saturation at κ.
(2) [13, Fact 2.5] If T is unstable then T has exact saturation at all regular $\kappa > |T|$.
(3) [13, Theorem 3.3] Suppose that T is simple, κ is singular with $|T| < \mu = \mathrm{cf}(\kappa)$, $\kappa^+ = 2^\kappa$ and $\square_\kappa$ holds (see Definition 4.6). Then T has exact saturation at κ.
(4) [13, Theorem 4.10] Suppose that κ is a singular cardinal such that $\kappa^+ = 2^\kappa$. An NIP theory T with $|T| < \kappa$ is distal (see e.g. [18]) if and only if it does not have exact saturation at κ.

2.2. *Local (PC) exact saturation*

Definition 2.4. Work in some complete theory T. Given a set Δ of formulas and a tuple of variables x, let $L_{x,\Delta}$ be the set of formulas of the form $\varphi(x)$ where φ is any formula in the Boolean algebra generated by Δ (where by $\varphi(x)$ we mean a substitution of the variables in φ by some variables from x). Similarly, given some set A, $L_{x,\Delta}(A)$ is the set of formulas of the form $\varphi(x, a)$ where φ is any formula from $L_{xy,\Delta}$ and a is some tuple from A (here y is a countable sequence of variables). A Δ-*type in variables x over a set A* is a maximal consistent collection of formulas from $L_{x,\Delta}(A)$. A *local type* over A is a Δ-type for some finite set of formulas Δ. The space of all Δ-types (in finitely many variables) over A is denoted by $S_\Delta(A)$. (Sometimes this notion is only defined when Δ is a partitioned set of formulas, $\Delta(x, y)$, but here we allow all partitions.)

We say that a structure M is κ-*locally saturated* if every local type (in finitely many variables) over a set $A \subseteq M$ of size $|A| < \kappa$ is realized. Say that M is *locally saturated* if it is $|M|$-locally saturated.

We define a local analog to Definition 2.2.

Definition 2.5. Suppose T is a first-order theory and κ is a cardinal.

(1) Say that T *has local exact saturation at κ* if T has a κ-locally saturated model M which is not κ^+-locally saturated.
(2) Say that a PC-class $P = \mathrm{PC}(T_1, T)$ has *local exact saturation at a cardinal κ* if there is a κ-locally saturated model $M \models T$ in P which is not κ^+-locally saturated.
(3) Say that T has *local PC-exact saturation* at a cardinal κ if for every $T_1 \supseteq T$ of cardinality $|T|$, $\mathrm{PC}(T_1, T)$ has local exact saturation at κ.

2.3. *PC-exact saturation for unstable theories in regular cardinals*

For unstable theories and regular cardinals, the situation is as in Fact 2.3(2). Because of the following proposition, when we discuss unstable theories, we will subsequently concentrate on singular cardinals.

Proposition 2.6. *If T is not stable then T has PC-exact saturation at any regular $\kappa > |T|$. Moreover for any $T_1 \supseteq T$ of size $|T|$, there is a κ-saturated model of T_1 whose reduct to the language L of T is not κ^+-locally saturated.*

Proof. The proof is almost exactly the same as the proof of [13, Fact 2.5].

Let $M_0 \models T_1$ be of size $|T|$. For $i \leq \kappa$, define a continuous increasing sequence of models M_i where $|M_{i+1}| = 2^{|M_i|}$ and M_{i+1} is $|M_i|^+$-saturated. Hence, M_κ is κ-saturated and $|M_\kappa| = \beth_\kappa(|T|)$.

As T is unstable, and M_κ is $\beth_\kappa(|T|)^+$-universal $|S_L(M_\kappa \restriction L)| > \beth_\kappa(|T|)$ (for an explanation, see the proof of [13, Fact 2.5]).

However, as the number of L-types over M_κ which are invariant (i.e. which do not split) over M_i is $\leq 2^{2^{|M_i|}} \leq \beth_\kappa(|T|)$, there is $p(x) \in S_L(M_\kappa)$ which splits over every M_i. Hence for each $i < \kappa$, there is some L-formula $\varphi_i(x, y)$ and some $a_i, b_i \in M_\kappa$ such that $a_i \equiv_{M_i} b_i$ and $\varphi_i(x, a_i) \wedge \neg\varphi_i(x, b_i) \in p$. As $\kappa > |T|$, there is a cofinal subset $E \subseteq \kappa$ such that for $i \in E$, $\varphi_i = \varphi$ is constant. Let $q(x)$ be $\{\varphi(x, a_i) \wedge \neg\varphi(x, b_i) \mid i \in E\}$. Then the local type q is not realized in M_κ. $\square$

3. PC-Exact Saturation for Stable Theories

The goal of this section is Theorem 3.10: assuming that T is a strictly stable countable complete theory in the language L and μ is a cardinal in which T is *not* stable, we will find a countable $T_1 \supseteq T$ in the language $L_1 \supseteq L$ such that if $M \models T_1$ and $M \restriction L$ is $\aleph_1$-saturated and locally μ-saturated, then $M \restriction L$ is μ^+-saturated.

In this section and in the sections that follow, we will often work with trees (usually $\omega^{<\omega}$). We will write $\trianglelefteq$ to denote the tree partial order and $\perp$ to denote the relation of incomparability, i.e. $\eta \perp \nu$ if and only if $\neg(\eta \trianglelefteq \nu) \wedge \neg(\nu \trianglelefteq \eta)$.

3.1. *Description of the expansion*

We will define our desired theory T_1 by choosing a certain model of T, describing an expansion, and taking T_1 to be the theory of this structure in the expanded language. We will assume that L is disjoint from all symbols we are about to present. As T is not superstable, we can use the following fact.

Fact 3.1 ([6, Proposition 3.5]). If $\kappa(T) > \aleph_0$ (namely, T is not supersimple), then there is a sequence of formulas $\langle \psi_n(x, y_n) \mid n < \omega \rangle$ (where x is a single variable and the y_n's are variables of varying lengths) and a sequence $\langle a_\eta \mid \eta \in \omega^{<\omega} \rangle$ such that

— a_η is an $|y_{|\eta|}|$-tuple; for $\sigma \in \omega^\omega$, $\{\psi_n(x, a_{\sigma \restriction n}) \mid n < \omega\}$ is consistent; for every $\eta \in \omega^n, \nu \in \omega^m$ such that $\eta \perp \nu$, $\{\psi_n(x, a_\eta), \psi_m(x, a_\nu)\}$ is inconsistent.

Fix a countable model $M_0 \models T$ such that $\langle a_\eta \mid \eta \in \omega^{<\omega} \rangle$ is contained in M_0, and we will describe our expansion. Since M_0 is countable we may assume that its universe is $\omega \cup \omega^{<\omega} \cup \mathcal{P}_{\text{fin}}(L)$ where $\mathcal{P}_{\text{fin}}(L)$ is the set of all finite subsets of formulas from L in a fixed countable set of variables $\{v_i \mid i < \omega\}$. First, expand M_0 by adding a predicate $\mathcal{N}$ for ω and adding $+, \cdot, <$ on $\mathcal{N}$ and a bijection $e : \mathcal{N} \to M_0$. We also have a predicate $\mathcal{T}$ for $\omega^{<\omega}$ on which we add the order $\trianglelefteq$. Add two functions $l : \mathcal{T} \to \mathcal{N}$ and eval : $\mathcal{T} \times \mathcal{N} \to \mathcal{N}$ such that l is the length function and eval is the function defined by $\text{eval}(\eta, n) = \eta(n)$ for $n < l(\eta)$ and otherwise $\text{eval}(\eta, n) = 0$ (outside of their domain we define them in an arbitrary way). Note that, in terms of this structure, if $\eta \in \mathcal{T}$ and $n \in \mathcal{N}$, the concatenation $\eta \frown \langle n\rangle$ is defined as the unique element ν of $\mathcal{T}$ of length $l(\eta) + 1$ such that $\text{eval}(\nu, l(\eta)) = n$ and for $k < l(\eta)$, $\text{eval}(\nu, k) = \text{eval}(\eta, k)$. We similarly define $\langle n\rangle \frown \eta$. For notational simplicity, when $\eta \in \mathcal{T}$ and $k < l(\eta)$, we will write $\eta(k)$ instead of $\text{eval}(\eta, k)$. Additionally, we will always refer to $\mathcal{N}$ for the natural numbers predicate and use ω for the standard natural numbers. For convenience, we add a predicate $\mathcal{M}$ for the universe.

We will write n^n for the definable set of $\eta \in \mathcal{T}$ such that $l(\eta) = n$ and for all $k < n$, $\eta(k) < n$. Let i be a function with domain $\{(\eta, n) \mid \eta \in n^n, n \in \mathcal{N}\}$ and range $\mathcal{N}$ such that $i(-, n) : n^n \to \mathcal{N}$ is an injection onto an initial segment of $\mathcal{N}$.

We will add a predicate $\mathcal{L}$ for $\mathcal{P}_{\text{fin}}(L)$, together with $\subseteq$ giving containment and a truth predicate; formally, the set of formulas $\mathcal{L}_0$ is identified with atoms in the Boolean algebra $\mathcal{L}$ and the truth valuation is a function TV from $\mathcal{L}_0 \times \mathcal{T}$ to $\{0, 1\}$ such that $TV(\{\varphi\}, \eta) = 1$ if and only if φ holds in M_0 with the assignment $v_n \mapsto e(\eta(n))$ for $n < l(\eta)$ and $v_n \mapsto 0$ otherwise.

Add a function $d : \mathcal{N} \to \mathcal{L}_0$ mapping n to $\{\varphi_n(v_0; v_1, \ldots, v_{|y_n|})\}$ (where the formulas φ_n are as in Fact 3.1). Let $a : \mathcal{T} \to \mathcal{T}$ be a function such that if $\eta \in \omega^{<\omega}$ then $a(\eta) \in \omega^{|y_{l(\eta)}|}$ and there is some $c \in M_0$ such that $TV(d(i), \langle c\rangle \frown a(\eta|i)) = 1$ for all $i < l(\eta)$ (i.e. such that $\{\varphi_i(x; a(\eta|i)) : i < l(\eta)\}$ is consistent) and such that if $\eta, \nu \in \mathcal{T}$ are incomparable then there is no $c \in M_0$ such that both $TV(d(l(\eta)), \langle c\rangle \frown a(\eta)) = 1$ and $TV(d(l(\nu)), \langle c\rangle \frown a(\nu)) = 1$ (i.e. $\{\varphi_{l(\eta)}(x; a(\eta)), \varphi_{l(\nu)}(x; a(\nu))\}$ is inconsistent). This is a direct translation of the properties of the formulas φ_n described above.

We add a bijection $c : \mathcal{N} \to \mathcal{L}$ that associates to each natural number a finite set of formulas. We add a predicate $P \subseteq \mathcal{T} \times \mathcal{L}$ such that $(\eta, \Delta) \in P^{M_0}$ if and only if $\langle e(\eta(i)) \mid i < l(\eta)\rangle$ is a Δ-indiscernible sequence that extends to a Δ-indiscernible sequence of countable length. Now, we define a function $F : \mathcal{T} \times \mathcal{L} \times \mathcal{N} \to \mathcal{M}$ on M_0 so that, if $(\eta, \Delta) \in P^{M_0}$, then $F^{M_0}(\eta, \Delta, -) : \mathcal{N} \to M_0$ is a Δ-indiscernible sequence extending $\langle e(\eta(i)) \mid i < l(\eta)\rangle$. Otherwise, $F^{M_0}(\eta, \Delta, m)$ is defined arbitrarily.

We let $M_1 = (M_0, \mathcal{N}, \mathcal{T}, \mathcal{L})$ with this additional structure and constants for all elements (so that models are elementary extensions), and we set $T_1 = \text{Th}(M_1)$ and $L_1 = L(T_1)$.

3.2. *Properties of T_1*

Lemma 3.2. *In T_1, there is a definable function $H : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ such that if $M \models T_1$ and $M \restriction L$ is $\aleph_1$-saturated, then for any function $f : \omega \to \mathcal{N}^M$, there is* *some* $m_f \in \mathcal{N}^M$ *with the property that*

$$H^M(m_f, n) = f(n)$$

for all $n \in \omega$.

Proof. Define $H : \mathcal{N} \times \mathcal{N} \to \mathcal{N}$ as follows:

(1) If there is $k \in \mathcal{N}$ and $\eta \in \mathcal{T}$ with $l(\eta) = n$ and $\models TV(d(n+1), \langle g\rangle \frown a(\eta \frown \langle k\rangle))$, we set $H(g, n) = k$ (note that the second parameter of TV gets an element from $\mathcal{T}$ so g should be in $\mathcal{N}$ for this to be well defined).
(2) Else, we set $H(g, n) = 0$.

Note that if there are $\eta, \eta' \in \mathcal{T}$ of length n and $k' \in \mathcal{N}$ such that $\models TV(d(n+1), \langle g\rangle \frown a(\eta \frown \langle k\rangle)) \wedge TV(d(n+1), \langle g\rangle \frown a(\eta' \frown \langle k'\rangle))$ then $\eta \frown \langle k\rangle$ and $\eta' \frown \langle k'\rangle$ must be comparable hence the same (since this is true in M_0). It follows that $k = k'$, which shows that H is well defined. Also, H is clearly definable.

Let $M \models T_1$ be a model such that $M \restriction L$ is $\aleph_1$-saturated. Let $f : \omega \to \mathcal{N}^M$ be arbitrary and we will define a path $\langle \eta_i : i < \omega\rangle$ in $\mathcal{T}^M$ with $l(\eta_i) = i+1$ inductively, by setting $\eta_0 = \langle f(0)\rangle$ and $\eta_{i+1} = \eta_i \frown \langle f(i+1)\rangle$ (by this we mean that $\mathrm{eval}(\eta_0, 0^M) = f(0)$, etc.; this is possible to do in any model of T_1). By the choice of a, we know that $\{TV(d(i+1), \langle x\rangle \frown a(\eta_i)) \mid i < \omega\}$ is a partial type which in turn implies that $\{\varphi_{i+1}(x, e(a(\eta_i)(0)), \ldots, e(a(\eta_i)(|y_{i+1}| - 1))) \mid i < \omega\}$ is a partial type and, by $\aleph_1$-saturation, this is realized by some $m'_f \in M$. Unraveling definitions, we have $H^M(e^{-1}(m'_f), n) = f(n)$ for all $n \in \omega$, so let $m_f = e^{-1}(m'_f)$. □

Lemma 3.3. *Suppose that* T *has the finite cover property (fcp),* $M \models T_1$ *and* $M \restriction L$ *is locally* μ*-saturated. Then for any nonstandard* $n \in \mathcal{N}^M$, *we have* $|[0, n)| \geq \mu$.

Proof. As T has the finite cover property, there is a formula $\varphi(x, y)$ with x a singleton such that in any model of T, for all $k < \omega$, there are some $k' > k$ and $\langle a_i \mid i \leq k'\rangle$ such that $\{\varphi(x, a_i) \mid i < k'\}$ is inconsistent yet k-consistent. For simplicity assume that y is a singleton (otherwise replace η by a tuple of η's in what follows). In particular, for all $k < \omega$, M_1 satisfies $\chi(k)$ which asserts that there exists $k' > k$ such that $k' < n$ and $\eta \in \mathcal{T}$ with $l(\eta) = k'$ such that the following two conditions are satisfied:

(A) $\neg\exists x \in \mathcal{M}(\bigwedge_{i<k'} \varphi(x, e(\eta(i))))$.
(B) If $\nu \in \mathcal{T}$ satisfies $l(\nu) \leq k$ and $\mathrm{ran}(\nu) \subseteq \mathrm{ran}(\eta)$, then

$$\exists x \in \mathcal{M} \bigwedge_{i<k} \varphi(x, e(\nu(i))).$$

Note that this is expressible in L_1. By overspill, there is $k \in \mathcal{N}^M$ nonstandard such that $\chi(k)$ holds witnessed by some $k' < n$. Without loss of generality, we may assume $k' = n$, since it suffices to show $[0, k')$ has size μ. Consequently, in M there is an $\eta \in \mathcal{T}^M$ with $l(\eta) = n$ witnessing the above formula.

Note that, for each $N < \omega$, we have the following sentence is satisfied in M_0:

$$\forall m > N \forall \eta \in \mathcal{T}\left[l(\eta) = m \to (\forall x_0, \dots, x_{N-1} \in \operatorname{ran}(\eta))(\exists \xi \in \mathcal{T})\left[\bigwedge_{i<N} \xi(i) = x_i\right]\right].$$

It follows that M satisfies this sentence and so, by condition (B) and compactness, we have that $\{\varphi(x, e(\eta(i))) \mid i \in [0, n)\}$ is consistent.

However, by condition (A), $\{\varphi(x, e(\eta(i))) \mid i \in [0, n)\}$ is not realized and therefore has size $\geq \mu$ by the local μ-saturation of $M \restriction L$. $\square$

Lemma 3.4. *Suppose $M \models T_1$ is given such that $M \restriction L$ is $\aleph_1$-saturated.*

(1) *If $|M| \geq \mu$, then $|M| \geq \mu^{\aleph_0}$.*
(2) *If, moreover, M has the property that $[0, n) \geq \mu$ for all nonstandard $n \in \mathcal{N}^M$, then for any nonstandard $n \in \mathcal{N}^M$, $[0, n) \geq \mu^{\aleph_0}$.*

Proof. (1) Given any $f : \omega \to \mathcal{N}^M$, as $M \restriction L$ is $\aleph_1$-saturated, we know, by Lemma 3.2, there is some $m_f \in \mathcal{N}^M$ such that

$$H(m_f, i) = f(i),$$

for all $i < \omega$. As $e^M : \mathcal{N}^M \to \mathcal{M}^M$ is a bijection, $|\mathcal{M}^M| = |\mathcal{N}^M|$ so, as there are $|\mathcal{N}^M|^{\aleph_0} \geq \mu^{\aleph_0}$ functions $f : \omega \to \mathcal{N}^M$ and the map $f \mapsto m_f$ is clearly injective, we have $|M| \geq \mu^{\aleph_0}$, which proves (1).

Now, we prove (2). Recall that in $M_0 \models T_1$, we defined the function i such that, for all n, $i(-, n) : n^n \to \mathcal{N}$ is an injection onto an initial segment of $\mathcal{N}$.

Let $\log : \mathcal{N} \to \mathcal{N}$ be the (definable) function given by

$$\log(n) = \max\{k \mid \operatorname{ran}(i(-, k)) \subseteq [0, n)\}.$$

On the standard natural numbers, $\log(n)$ is the largest k such that $k^k \leq n$.

Given n nonstandard, let $k = \log^M(n)$. Because $i(-, k)$ gives an injection of $(k^k)^M$ into $[0, n)$, we are reduced to showing that $|(k^k)^M| \geq \mu^{\aleph_0}$. Note that k is also nonstandard, so by hypothesis, $|[0, k)| \geq \mu$. Let $f : \omega \to [0, k)$ be an arbitrary function. As in (1), by Lemma 3.2, there is some $m_f \in \mathcal{N}^M$ such that $H(m_f, i) = f(i)$ for all $i < \omega$.

Note that if $j < \omega$, and $m \in \mathcal{N}^{M_0}$, then there is an element $\eta \in j^j$ such that $\min\{H^{M_0}(m, i), j - 1\} = \eta(i)$ for all $i < j$, and hence

$$M_0 \models \forall j \in \mathcal{N} \forall m \in \mathcal{N} \exists \eta \in j^j [(\forall i < j)[\eta(i) = \min\{H(m, i), j - 1\}]],$$

and hence this sentence is contained in T_1. Applying this with $j = k$ and $m = m_f$, we know there is $\tilde{\eta}_f \in (k^k)^M$ such that

$$\tilde{\eta}_f(i) = H(m_f, i),$$

for all $i < \omega$ (using that $H(m_f, i) = f(i) < k$ for all $i < \omega$). Moreover, because $\tilde{\eta}_f(i) = f(i)$ for all standard i, we have $f \neq f'$ implies $\tilde{\eta}_f \neq \tilde{\eta}_{f'}$. This shows $|(k^k)^M| \geq \mu^{\aleph_0}$, which completes the proof. □

Fact 3.5 ([17, Theorem II.4.6]). Suppose T is nfcp, Δ is a finite set of formulas. There is an $n(\Delta) < \omega$ such that, if A is a set of parameters and $\{a_\gamma \mid \gamma < \alpha\}$ is a Δ-indiscernible set over A and $n(\Delta) \leq \alpha < \beta$, then there exist a_γ for $\alpha \leq \gamma < \beta$ such that $\{a_\gamma \mid \gamma < \beta\}$ is a Δ-indiscernible set over A.

Remark 3.6. In [17, Theorem II.4.6], the result is a bit more refined since it applies to Δ-n-indiscernible sequences and gives one $n(\Delta)$ which works for all n (note that being Δ-indiscernible is equivalent to being Δ-n-indiscernible for some large enough n), but this is not needed here.

Remark 3.7. If, in the context of the previous fact, M is a model and we are given that A is finite and $\alpha < \beta \leq \omega$ and $Aa_{<\alpha} \subseteq M$, then we can choose $a_\gamma \in M$ for $\alpha \leq \gamma < \beta$ (if $\beta = \omega$ we extend one by one). Likewise, if $Aa_{<\alpha} \subseteq M$ and M is $(|A| + |\beta|)^+$-saturated, we may choose $a_\gamma \in M$ for $\alpha \leq \gamma < \beta$.

Definition 3.8. The infinite indiscernible sequences I_1 and I_2 are *equivalent* if there is an infinite indiscernible sequence J such that $I_1 \frown J$ and $I_2 \frown J$ are both indiscernible. If M is a model and I is an infinite indiscernible sequence contained in M, we define $\dim(I, M)$ by

$$\dim(I, M) = \min\{|J| \mid J \text{ maximal indiscernible in } M \text{ equivalent to } I\}.$$

If $\dim(I, M) = |J|$ for all maximal indiscernible sequences J in M equivalent to I, then we say that the dimension is *true*.

Fact 3.9. Suppose T is a countable stable theory and $M \models T$.

(1) [17, Theorem III.3.9] For any infinite indiscernible I contained in M, the dimension $\dim(I, M)$ is true (there it is stated with the added condition that $\dim(I, M) \geq \kappa(T)$ but this condition is not necessary when T is countable as follows from the proof there; however since we will later assume $\aleph_1$-saturation we can just apply the reference as is).
(2) [17, Theorem III.3.10] Assume that M is $\aleph_1$-saturated. For an infinite cardinal λ, M is λ-saturated if and only if for every infinite indiscernible sequence I in M of single elements, we have $\dim(I, M) \geq \lambda$. (In the reference there is no restriction on the length of the tuples, but it is enough to consider sequences of elements as can be seen from the proof there and the fact that saturation is implied by realizing one-types.)[a]

[a]If the reader is not satisfied with this, they can alter the language L_1 by adding predicates P_n and functions F_n to deal with sequences of length n, for all $n < \omega$.

Theorem 3.10. *Suppose T is a strictly stable countable theory and μ is a cardinal such that T is not stable in μ. Then there is a countable theory $T_1 \supseteq T$ with the property that for all $M \models T_1$, if $M \restriction L$ is $\aleph_1$-saturated and locally μ-saturated, then $M \restriction L$ is μ^+-saturated.*

Proof. As T is stable in every cardinal κ satisfying $\kappa^{|T|} = \kappa$, we have $\mu^{\aleph_0} > \mu$. Let T_1 be the theory described above. Let $M \models T_1$ be arbitrary with the property that $M \restriction L$ is $\aleph_1$-saturated and locally μ-saturated. By Fact 3.9(1), the dimension of any infinite indiscernible sequence in M is true. Since every infinite indiscernible sequence is equivalent to all of its infinite initial segments, we know that, by Fact 3.9(2), to show $M \restriction L$ is μ^+-saturated, it suffices to show that every length ω indiscernible sequence (with respect to the language L) contained in M extends to one that has length $\geq \mu^+$. Note that $M \succ M_1$ so in particular contains all standard numbers and finite sets of formulas.

Fix an arbitrary indiscernible sequence $I = \langle a_i \mid i < \omega \rangle$ of elements in M and we will find an extension of length $\geq \mu^+$. For each $i < \omega$, there is some $\eta_i \in \mathcal{T}^M$ with $l(\eta_i) = i$ and $\langle e(\eta_i(j)) \mid j < i \rangle = \langle a_j \mid j < i \rangle$. We fix also an increasing sequence of finite sets of formulas Δ_n such that $L = \bigcup_{n<\omega} \Delta_n$.

Case 1. For all $n < \omega$, $M \models P(\eta_n, \Delta_n)$.

As $M \restriction L$ is, in particular $\aleph_1$-saturated, we may apply Lemma 3.2 to find $m_1, m_2 \in \mathcal{N}^M$ such that $H(m_1, n) = e^{-1}(a_n)$ and $H(m_2, n) = c^{-1}(\Delta_n)$ for all $n < \omega$. For $n \in \mathcal{N}^M$, let $\nu(n)$ be the element of $\mathcal{T}$ such that $l(\nu(n)) = n$ and $\mathrm{eval}(\nu(n), i) = H(m_1, i)$ for all $i < n$. Note that we have that the function $n \mapsto \nu(n)$ is definable in M and $\nu(n) = \eta_n$ for all (standard) $n < \omega$. Next, let $\Delta(n) = c(H(m_2, n))$ for all $n \in \mathcal{N}^M$. Likewise, we have that the function $n \mapsto \Delta(n)$ is M-definable and $\Delta(n) = \Delta_n$ for all $n < \omega$.

By our assumptions, we have that for all $n < \omega$,

$$M \models \forall k \leq n (\Delta(k) \subseteq \Delta(n) \wedge \nu(k) \trianglelefteq \nu(n)) \wedge P(\nu(n), \Delta(n)),$$

and, hence, by overspill (which we may apply since nonstandard elements exists by $\aleph_1$-saturation), there is some nonstandard $n_* \in \mathcal{N}^M$ such that

$$M \models \forall k \leq n_* (\Delta(k) \subseteq \Delta(n_*) \wedge \nu(k) \trianglelefteq \nu(n_*)) \wedge P(\nu(n_*), \Delta(n_*)).$$

Since n_* is nonstandard, we know that $\Delta(n_*)$ contains all standard formulas of L and $\nu(n_*)$ extends $\nu(k)$ for all $k < \omega$. Since, additionally, $M \models P(\nu(n_*), \Delta(n_*))$, it follows that $F^M(\nu(n_*), \Delta(n_*), -) : \mathcal{N}^M \to M$ enumerates an L-indiscernible sequence extending I. (Note that this was a property of F^{M_0} and using the truth predicate the fact that this is an indiscernible sequence is expressible in L_1 so this is also true in M; this also uses the fact that for standard formulas, the truth predicate gives the correct answer.)

The local μ-saturation of $M \restriction L$ implies $|M| \geq \mu$, so, by Lemma 3.4(1), we have $|\mathcal{N}^M| = |M| \geq \mu^{\aleph_0} \geq \mu^+$, we have shown that I extends to an indiscernible sequence of length μ^+.

Case 2. There is an $N < \omega$ such that $M \models \neg P(\eta_N, \Delta_N)$.

Note that it follows that for all $\omega > r \geq N$, $M \models \neg P(\eta_r, \Delta_N)$ (since η_r extends η_N and this implication is true in M_1). If T was nfcp, then in particular this is true for any $r \geq N, n(\Delta_N)$, where $n(\Delta_N)$ is from Fact 3.5. Thus by elementarity,

$$M_1 \models \exists \eta \in \mathcal{T}[l(\eta) = r \wedge \neg P(\eta, \Delta_N) \wedge \langle e(\eta(i)) \mid i < r\rangle \text{ is } \Delta_N\text{-indiscernible}]$$

hence T has the finite cover property by choice of r and Remark 3.7. However, because I is L-indiscernible, we have $\nu(n)$ is Δ_n-indiscernible for all $n < \omega$. Therefore, we have that for all $n < \omega$, M satisfies the conjunction of the following sentences (which can be expressed using the truth predicate):

$\forall\, k \leq n((\Delta(k)) \subseteq \Delta(n) \wedge \nu(k) \trianglelefteq \nu(n))$.

$\forall\, k \leq n[\langle e(\nu(n)(i)) \mid i < n\rangle \text{is } \Delta(n)\text{-indiscernible}]$.

Thus, by overspill, there is some nonstandard $n_* \in \mathcal{N}^M$ such that $\langle e(\nu(n_*)(i)) \mid i < n_*\rangle$ is L-indiscernible and extends I. By Lemmas 3.3 and 3.4(2), we know $|[0, n_*)| \geq \mu^{\aleph_0} > \mu$, so we have shown I extends to an indiscernible sequence of length $\geq \mu^+$.

This completes the proof. □

Corollary 3.11. *For a countable stable theory T and $\mu \geq 2^{\aleph_0}$, the following are equivalent*:

(1) *T has PC-exact saturation in μ.*
(2) *T is stable in μ.*

Proof. (1) implies (2) if T is superstable then (2) holds automatically. Otherwise, we are done by Theorem 3.10.

(2) implies (1) if T is stable in μ, then by [17, Theorem 4.7, Chap. VIII] T has a saturated model M of size μ. Since saturated models are resplendent [15, Theorem 9.17] we can expand M to a model M' of T_1. □

4. PC-exact Saturation for Simple and Supersimple Theories

4.1. *Simple theories*

This section is devoted to the proof of the following theorem, which in particular answers [14, Question 9.31] (about the random graph).

Theorem 4.1. *Assume that T is a complete simple unstable L-theory, $T_1 \supseteq T$ is a theory in $L_1 \supseteq L$, $|T_1| \leq |T|$. Also, assume that κ is a singular cardinal such that $\kappa(T) \leq \mu = \mathrm{cf}(\kappa)$, $|T| < \kappa$, $\kappa^+ = 2^\kappa$ and $\square_\kappa$ holds (see Definition 4.6). Then* $\mathrm{PC}(T_1, T)$ *has exact saturation at κ. Moreover, there is a model $M \models T_1$ whose reduct to L is κ-saturated but not κ^+-locally saturated.*

The proof is somewhat similar to the proof of the parallel theorem from [13] (Fact 2.3(3)), with some important differences. The class $\mathcal{M}$ from there is similar to the class $\mathcal{C}$ here, and the overall structure of the proof is similar, but the proof of the main lemma (Lemma 4.3) is quite different.

We may assume that T_1 has built-in Skolem functions. Since T is unstable and simple it has the independence property, as witnessed by some formula $\varphi(x,y)$ from L. Suppose that $\kappa = \sum_{i<\mu} \lambda_i$ where the sequence $\langle \lambda_i \mid i < \mu \rangle$ is continuous, increasing, and each λ_i is regular for $i < \mu$ a successor. Also, assume that $|T| \leq \lambda_0$ (here we use the assumption that $|T| < \kappa$). We work in a monster model $\mathfrak{C}_1$ of T_1, and denote its L-reduct by $\mathfrak{C}$. Let $I = \langle a_\alpha \mid \alpha < \kappa \rangle$ be an L_1-indiscernible sequence of y-tuples of order type κ, which witnesses that φ has IP. For $i < \mu$, let $I_i = \langle a_\alpha \mid \alpha < \lambda_i \rangle$. Also, for $\alpha < \kappa$, let $\bar{a}_\alpha$ be the sequence $\langle a_{\omega\alpha+k} \mid k < \omega \rangle$.

Definition 4.2. Let $\mathcal{C}$ be the class of sequences $\langle A_i \mid i < \kappa \rangle$ such that

(1) For all $i < \kappa$, $A_i \prec \mathfrak{C}_1$; $|A_i| \leq \lambda_i$; $I_i \subseteq A_i$.
(2) The sequence $\langle A_i \mid i < \kappa \rangle$ is increasing continuous.
(3) For all $i < \kappa$ and every finite tuple $c \in A_{i+1}$, there is a club $E \subseteq \lambda_{i+1}$ such that for all $\alpha \in E$, $\bar{a}_\alpha$ is L_1-indiscernible over c, where $\overline{a}_\alpha = (a_{\omega\alpha+k})_{k<\omega}$.

For $\bar{A}, \bar{B} \in \mathcal{C}$, write $\bar{A} \leq \bar{B}$ for: for every $i < \mu$, $A_i \subseteq B_i$.

For example, letting $A_i = \mathrm{Sk}(I_i)$ (the Skolem hulls of I_i) for $i < \mu$, $\bar{A} = \langle A_i \mid i < \mu \rangle \in \mathcal{C}$. Then $\bar{A}$ is $\leq$-minimal.

Main Lemma 4.3. *Suppose that $\bar{A} \in \mathcal{C}$ and let $A = \bigcup\{A_i \mid i < \mu\}$. Suppose that $C \subseteq A$ has size $< \kappa$ and $p(x) \in S_L(C)$. Then there is $\bar{B} \in \mathcal{C}$ such that $\bar{A} \leq \bar{B}$ and $B = \bigcup\{B_i \mid i < \mu\}$ realizes p.*

Proof. We want to make the following assumptions first. By maybe increasing C by a set of size $\leq |T| + |C| + \mu$ we may assume that

($\star$) $C \mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise0.6ex\hbox{$|$}}_{A_i \cap C} A_i$ and $A_i \cap C \prec \mathfrak{C}_1$ for all $i < \mu$. ($\mathrel{\raise0.2ex\hbox{$\smile$}\kern-0.75em\raise0.6ex\hbox{$|$}}$ denotes non-forking independence.)

(This is straightforward, but for a proof see the beginning of the proof in [13, Main Lemma 3.11].)

Without loss of generality assume that p does not fork over $A_* = C \cap A_0$ (this uses the assumption that $\mathrm{cf}(\kappa) = \mu \geq \kappa(T)$). (Let $i_* < \mu$ be minimal successor or 0 such that p does not fork over $C \cap A_i$, and let $\lambda'_i = \lambda_{i_*+i}$, for $i < \mu$. If the lemma is true for λ'_i (and $A'_i = A_{i_*+i}$) instead of λ_i, and $\bar{B}' = \langle B'_i \mid i < \mu \rangle$ witnesses this (so $|B'_i| \leq \lambda'_i$, etc.), then let $B_i = A_i$ for $i < i_*$ and for $i \geq i_*$, let $j < \mu$ be such that $i_* + j = i$ and $B_i = B'_j$.)

Fix an enumeration of A as $\langle b_\alpha \mid \alpha < \kappa \rangle$ such that, for $i < \mu$, A_i is enumerated by $\langle b_\alpha \mid \alpha < \lambda_i \rangle$.

For each $i < \mu$, let $E_i \subseteq \lambda_{i+1}$ be a club such that

– If $\alpha \in E_i$ then $\alpha = \omega\alpha$; $\bar{a}_\alpha = \langle a_{\omega\alpha+k} \mid k < \omega\rangle$ is indiscernible over $A_{<\alpha} = \{b_\beta \mid \beta < \alpha\}$ (which equals $A_{<\omega\alpha}$); $A_{<\alpha} \prec \mathfrak{C}_1$; if $\alpha \in E_i$ then $\bar{a}_\beta \subseteq A_{<\alpha}$ for all $\beta < \alpha$ and $A_{<\alpha} \supseteq C \cap A_i$.

Its existence is proved as follows. First note that the set of $\alpha < \lambda_{i+1}$ for which $\alpha = \omega\alpha$ forms a club. Next, since the club filter on λ_{i+1} is λ_{i+1}-complete, for every set $D \subseteq A_{i+1}$ of size $< \lambda_{i+1}$, there is a club $E_D \subseteq \lambda_{i+1}$ such that if $\alpha \in E_D$ then $\bar{a}_\alpha$ is indiscernible over D. Now, let E'_i be the diagonal intersection $\triangle_{\beta<\lambda_{i+1}} E_{A_{<\beta}} = \{\alpha < \lambda_{i+1} \mid \alpha \in \bigcap_{\beta<\alpha} E_{A_{<\beta}}\}$ and let $E_i = \{\alpha \in E'_i \mid \omega\alpha = \alpha\}$. Now, if $\alpha \in E_i$, then $\bar{a}_\alpha$ is indiscernible over $A_{<\beta}$ for all $\beta < \alpha$, and since α is a limit (as $\omega\alpha = \alpha$), $\bar{a}_\alpha$ is indiscernible over $\bigcup\{A_{<\beta} \mid \beta < \alpha\} = A_{<\alpha}$. This takes care of the requirement that $\bar{a}_\alpha$ is indiscernible over $A_{<\alpha}$. The other requirements follow since $\{\alpha < \lambda_{i+1} \mid A_{<\alpha} \prec \mathfrak{C}_1\}$, $\{\alpha < \lambda_{i+1} \mid \forall \beta < \alpha(\bar{a}_\beta \subseteq A_{<\alpha})\}$ and $\{\alpha < \lambda_{i+1} \mid C \cap A_i \subseteq A_{<\alpha}\}$ are clubs.

Let $E = \bigcup\{E_i \mid i < \mu\}$. Let $\Gamma(x)$ be the set of formulas saying that $p(x)$ holds and that for all $\alpha \in E$, $\bar{a}_\alpha$ is L_1-indiscernible over $A_{<\alpha} \cup x$.

Claim 4.4. *It is enough to show that $\Gamma(x)$ is consistent.*

Proof. Let $d \models \Gamma(x)$. Let $B'_i = A_i \cup d$ for all $i < \mu$. Note that for each $i < \mu$ there is a club $E_i \subseteq \lambda_{i+1}$ such that if $\alpha \in E_i$ then $\bar{a}_\alpha$ is indiscernible over $A_{<\alpha}d$. Now, let $c \subseteq B'_i$ be any finite set. Then $c \subseteq A_{<\alpha}d$ for some $\alpha < \lambda_{i+1}$, so $E' = E_i \cap [\alpha+1, \lambda_i)$ is such that for any $\alpha \in E'$, $\bar{a}_\alpha$ is indiscernible over c. Finally, let B_i be $\mathrm{Sk}(B'_i)$. Since the indiscernibility was in L_1, $\bar{B} = \langle B_i \mid i < \mu\rangle \in \mathcal{C}$. $\square$

So fix finite sets $F_0 \subseteq E$, $v \subseteq \kappa$, and a finite set of L_1-formulas Δ, and let $\Gamma_{F_0,v,\Delta}(x)$ say that $p(x)$ holds and for all $\alpha \in F_0$, $\bar{a}_\alpha$ is Δ-indiscernible over $\{b_\beta \mid \beta \in v \cap \alpha\} \cup \{x\}$, and we want to show that $\Gamma_{F_0,v,\Delta}$ is consistent.

Let $n = |F_0|$, and write $F_0 = \{\alpha_i \mid i < n\}$ where $\alpha_0 < \cdots < \alpha_{n-1}$. Let $\mathcal{T}$ be the tree of all functions $([\omega]^{\aleph_0})^{\leq n}$ ($[\omega]^{\aleph_0}$ is the set of countably infinite subsets of ω). For every $i < n$ and every infinite $s \subseteq \omega$, let $f_{i,s}$ be a partial elementary map taking $\bar{a}_{\alpha_i}$ to $\bar{a}_{\alpha_i} \restriction s$ fixing $A_{<\alpha_i}$ (i.e. mapping $a_{\omega\alpha_i+k}$ to $a_{\omega\alpha_i+k'}$ where k' is the kth element in s). Note that $A_* = A_0 \cap C$ is fixed by all the $f_{i,s}$ since $A_* \subseteq A_{<\alpha}$ for every $\alpha \in E$ (by the last requirement on E_i in the bullet above).

Claim 4.5. *To show that $\Gamma_{F_0,v,\Delta}$ is consistent, it is enough to prove the following*:

(†) *There is an assignment, assigning each $\eta \in \mathcal{T}$ of height $i+1 \leq n$ an automorphism σ_η of $\mathfrak{C}_1$ extending $f_{i,\eta(i)}$ in such a way that, letting $\tau_\eta = \sigma_{\eta\restriction 1} \circ \cdots \circ \sigma_{\eta\restriction i+1}$, $\Theta(x) = \bigcup\{\tau_\eta(p) \mid \eta \in \mathcal{T}\}$ is consistent.*

Proof. Suppose $d_* \models \Theta(x)$. By Ramsey, there is some infinite $s_0 \subseteq \omega$ such that $f_{0,s_0}(\bar{a}_{\alpha_0}) = \bar{a}_{\alpha_0} \restriction s_0$ is Δ-indiscernible over $\{b_\beta \mid \beta \in v \cap \alpha_0\} \cup \{d_*\}$. Let $\eta(0) = s_0$.

Suppose we chose $\eta \restriction i$ for some $1 \leq i < n$. Again by Ramsey there is some infinite $s_i \subseteq \omega$ such that, letting $\eta(i) = s_i$ we have that

$$\sigma_{\eta\restriction 1} \circ \cdots \circ \sigma_{\eta\restriction i+1}(\bar{a}_{\alpha_i}) = \sigma_{\eta\restriction 1} \circ \cdots \circ \sigma_{\eta\restriction i}(\bar{a}_{\alpha_i} \restriction s_i) = [\sigma_{\eta\restriction 1} \circ \cdots \circ \sigma_{\eta\restriction i}(\bar{a}_{\alpha_i})] \restriction s_i$$

is Δ-indiscernible over $\sigma_{\eta\restriction 1} \circ \cdots \circ \sigma_{\eta\restriction i}(\{b_\beta \mid \beta \in v \cap \alpha_i\}) \cup \{d_*\}$. This procedure defines $\eta \in \mathcal{T}$ of height n. Then $\tau_\eta^{-1}(d_*) \models p$ and $\bar{a}_\alpha$ is Δ-indiscernible over $\{\tau_\eta^{-1}(d_*)\} \cup \{b_\beta \mid \beta \in v \cap \alpha\}$ for each $\alpha \in F_0$ as we wanted. Indeed, $\tau_\eta^{-1}(d_*) \models p$ obviously. Also, for each $i < n$, $\tau_\eta^{-1}(\sigma_{\eta\restriction 1} \circ \cdots \circ \sigma_{\eta\restriction i+1}(\bar{a}_{\alpha_i})) = \sigma_{\eta\restriction n}^{-1} \circ \cdots \circ \sigma_{\eta\restriction 1}^{-1}(\sigma_{\eta\restriction 1} \circ \cdots \circ \sigma_{\eta\restriction i+1}(\bar{a}_{\alpha_i})) = \bar{a}_{\alpha_i}$ because for $j > i+1$, $\sigma_{\eta\restriction j}$ fixes $A_{<\alpha_{i+1}}$ which contains $\bar{a}_{\alpha_i}$. Hence, $\bar{a}_{\alpha_i}$ is indiscernible over the union of $\sigma_\eta^{-1}(d_*)$ and $\tau_\eta^{-1} \circ \sigma_{\eta\restriction 1} \circ \cdots \circ \sigma_{\eta\restriction i}\{b_\beta \mid \beta \in v \cap \alpha_i\}$ which is just $\{b_\beta \mid \beta \in v \cap \alpha_i\}$ because $\sigma_{\eta\restriction j}$ fixes $A_{<\alpha_{i+1}}$ for $j > i$. $\square$

Fix some enumeration of $\mathcal{T}$, $\langle \eta_\varepsilon \mid \varepsilon < \varepsilon_* \rangle$ such that $\eta_0 = \emptyset$, ε_* is a limit and if $\eta_\varepsilon \trianglelefteq \eta_\zeta$ then $\varepsilon \leq \zeta$.

Let $\sigma_0 = \tau_0 = \mathrm{id}$ and for $0 < \varepsilon < \varepsilon_*$ define $\sigma_\varepsilon = \sigma_{\eta_\varepsilon}$ as in (†) and consequently $\tau_\varepsilon = \tau_{\eta_\varepsilon} = \sigma_{\eta_\varepsilon\restriction 1} \circ \cdots \circ \sigma_{\eta_\varepsilon\restriction|\eta_\varepsilon|}$ by induction on ε, in such a way that

$(\dagger_\epsilon)$ $\Theta_\varepsilon(x) = \bigcup\{\tau_\zeta(p) \mid \zeta < \varepsilon\}$ is consistent and does not fork over A_*.

This will obviously suffice in order to prove (†), and holds trivially for $\varepsilon = 0$ by choice of A_*.

Suppose we have already defined σ_ζ for all $0 \leq \zeta < \varepsilon$, and let $\eta = \eta_\varepsilon$. By assumption on the order, we already defined $\sigma_{\eta'}$ for the predecessor η' of η (recall that $\eta \neq \emptyset$ because $0 < \varepsilon$). Assume $i+1 = |\eta| \leq n$. First, let σ be any automorphism extending $f_{i,\eta(i)}$ and let q be a global coheir extending $\mathrm{tp}(\tau_{\eta'}(\sigma(C))/N)$ where $N = \tau_{\eta'} \circ \sigma(\mathrm{Sk}(A_{<_{\alpha_i}}\bar{a}_{\alpha_i}))$. Now, let $C' \models q|_{N\tau_{<\varepsilon}(C)}$ where $\tau_{<\varepsilon}(C) = \bigcup\{\tau_\zeta(C) \mid \zeta < \varepsilon\}$. Note that

$$\sigma(C)\sigma(\mathrm{Sk}(A_{<\alpha_i}\bar{a}_{\alpha_i})) \equiv \tau_{\eta'}(\sigma(C))N \equiv C'N \equiv \tau_{\eta'}^{-1}(C')\sigma(\mathrm{Sk}(A_{<\alpha_i}a_{\alpha_i})).$$

Let σ' be an automorphism mapping $\sigma(C)$ to $\tau_{\eta'}^{-1}(C')$ fixing $\sigma(\mathrm{Sk}(A_{<_{\alpha_i}}\bar{a}_{\alpha_i}))$. Now, let $\sigma_\eta = \sigma' \circ \sigma$. By construction, σ_η extends $f_{i,\eta(i)}$. Note that $C' = \tau_\eta(C) \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{u}_{N} \tau_{<\varepsilon}(C)$ (where $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{u}$ means coheir independence) and that $N = \tau_\eta(\mathrm{Sk}(A_{<_{\alpha_i}}\bar{a}_{\alpha_i}))$ (because $\sigma' \circ \sigma(\mathrm{Sk}(A_{<_{\alpha_i}}\bar{a}_{\alpha_i})) = \sigma(\mathrm{Sk}(A_{<_{\alpha_i}}\bar{a}_{\alpha_i}))$).

Now, we check that $(\dagger)_{\varepsilon+1}$ holds.

By $(\star)$ above, we know that $C \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{A_{<\alpha_i}} \mathrm{Sk}(A_{<\alpha_i}\bar{a}_{\alpha_i})$ since $A_{<\alpha_i}\bar{a}_{\alpha_i}$ is contained in the appropriate A_{i_0} (i.e. for an $i_0 < \mu$ for which $\alpha_i \in E_{i_0}$) which is a model, and $A_{<\alpha_i}$ contains $C \cap A_{i_0}$. Hence, applying τ_η, we get $\tau_\eta(C) \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\tau_\eta(A_{<\alpha_i})} N$. By transitivity, we get that $\tau_\eta(C) \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{\tau_\eta(A_{<\alpha_i})} \tau_{<\varepsilon}(C)$. Note that σ_η fixes $A_{<\alpha_i}$, so $\tau_\eta(A_{<\alpha_i}) = \tau_{\eta'}(A_{<\alpha_i})$.

By induction, there is some $d \models \Theta_\varepsilon(x)$ such that $d \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{A_*} \tau_{<\varepsilon}(C)$. Let $d_0 \equiv_{A_*\tau_{<\varepsilon}(C)} d$ be such that $d_0 \mathop{\smile\!\!\!\!\!\!\mid}\nolimits_{A_*} \tau_{\eta'}(A_{<\alpha_i})\tau_{<\varepsilon}(C)$. Let $d_1 = \tau_\eta(\tau_{\eta'}^{-1}(d_0))$.

Since η' comes before η in the enumeration, we have that $d_1 \models \tau_\eta(\tau_{\eta'}^{-1}(\tau_{\eta'}(p))) = \tau_\eta(p)$ and $d_1 \mathop{\smile\!\!\!\!\!\!\,|}_{A_*} \tau_\eta(A_{<\alpha_i})\tau_\eta(C)$ (recall that A_* is fixed by σ_ζ for all $\zeta \leq \varepsilon$). Recalling that $\tau_\eta(A_{<\alpha_i}) = \tau_{\eta'}(A_{<\alpha_i})$, by base monotonicity we have that $d_0 \mathop{\smile\!\!\!\!\!\!\,|}_{\tau_{\eta'}(A_{<\alpha_i})} \tau_{<\varepsilon}(C)$, $d_1 \mathop{\smile\!\!\!\!\!\!\,|}_{\tau_{\eta'}(A_{<\alpha_i})} \tau_\eta(C)$, $\tau_\eta(C) \mathop{\smile\!\!\!\!\!\!\,|}_{\tau_{\eta'}(A_{<\alpha_i})} \tau_{<\varepsilon}(C)$ and $d_0 \equiv_{\tau_{\eta'}(A_{<\alpha_i})} d_1$ (the last equivalence is because $\tau_\eta \circ \tau_{\eta'}^{-1}$ fixes $\tau_{\eta'}(A_{<\alpha_i})$, because τ_η and $\tau_{\eta'}$ agree on $A_{<\alpha_i}$, since $\sigma_\eta \restriction A_{<\alpha_i} = \mathrm{id}$). By the independence theorem for simple theories (see [19, Theorem 7.3.11]), we can find some $d_2 \equiv_{\tau_{\eta'}(A_{<\alpha_i})\tau_\eta(C)} d_1$ (so $d_2 \models \tau_\eta(p)$), $d_2 \equiv_{\tau_{\eta'}(A_{<\alpha_i})\tau_{<\varepsilon}(C)} d_0$ (so $d_2 \models \Theta_\varepsilon(x)$) and $d_2 \mathop{\smile\!\!\!\!\!\!\,|}_{\tau_{\eta'}(A_{<\alpha_i})} \tau_{\leq\varepsilon}(C)$. Finally, since $d_2 \mathop{\smile\!\!\!\!\!\!\,|}_{A_*} \tau_{\eta'}(A_{<\alpha_i})$, by transitivity we have that $d_2 \mathop{\smile\!\!\!\!\!\!\,|}_{A_*} \bigcup \tau_{\leq\varepsilon}(C)$. This finishes the proof of the lemma.

Now, the proof of Theorem 4.1 continues precisely as the proof of [13, Theorem 3.3] with small changes.

First, we recall the definition of $\square_\kappa$.

Definition 4.6 (Jensen's Square principle [7, p. 443]). Let κ be an uncountable cardinal; $\square_\kappa$ (square-κ) is the following condition.

There exists a sequence $\langle C_\alpha \mid \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ such that

(1) C_α is a closed unbounded subset of α.
(2) If $\beta \in \mathrm{Lim}(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$ (where for a set of ordinals X, $\mathrm{Lim}(X)$ is the set of limit ordinals in X).
(3) If $\mathrm{cf}(\alpha) < \kappa$, then $|C_\alpha| < \kappa$.

Remark 4.7. Suppose that $\langle C_\alpha \mid \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ witnesses $\square_\kappa$. Let $C'_\alpha = \mathrm{Lim}(C_\alpha)$. Then the following holds for $\alpha \in \mathrm{Lim}(\kappa^+)$:

(1) If $C'_\alpha \neq \emptyset$, then either $\sup(C'_\alpha) = \alpha$, or C'_α has a last element $< \alpha$ in which case $\mathrm{cf}(\alpha) = \omega$. If $C'_\alpha = \emptyset$ then $\mathrm{cf}(\alpha) = \omega$ as well.
(2) $C'_\alpha \subseteq \mathrm{Lim}(\alpha)$ and for all $\beta \in C'_\alpha$, $C'_\alpha \cap \beta = C'_\beta$.
(3) If $\mathrm{cf}(\alpha) < \kappa$, then $|C'_\alpha| < \kappa$.

Define $\bar{A} \leq_i \bar{B}$ for $\bar{A}, \bar{B} \in \mathcal{C}$ and $i < \mu$ as $A_j \subseteq B_j$ for all $i \leq j$ (so $\leq = \leq_0$ on $\mathcal{C}$). Write $\bar{A} \leq_* \bar{B}$ for: there is some $i < \mu$, such that $\bar{A} \leq_i \bar{B}$.

Proof of Theorem 4.1. Let $\langle C_\alpha \mid \alpha \in \mathrm{Lim}(\kappa^+)\rangle$ be a sequence as in Remark 4.7. Note that $|C_\alpha| < \kappa$ for all $\alpha < \kappa^+$ as κ is singular. Let $\{S_\alpha \mid \alpha < \kappa^+\}$ be a partition of κ^+ to sets of size κ^+. We construct a sequence $\langle(\bar{A}_\alpha, \bar{p}_\alpha) \mid \alpha < \kappa^+\rangle$ such that

(1) $\bar{A}_\alpha = \langle A_{\alpha,i} \mid i < \mu\rangle \in \mathcal{C}$ (recall that $\mu = \mathrm{cf}(\kappa)$);
(2) $\bar{p}_\alpha$ is an enumeration $\langle p_{\alpha,\beta} \mid \beta \in S_\alpha \backslash \alpha\rangle$ of all complete types over subsets of $\bigcup_i A_{\alpha,i}$ of size $< \kappa$ (this uses $\kappa^+ = 2^\kappa$ and $|T| \leq \kappa$);
(3) if $\beta < \alpha$ then $\bar{A}_\beta \leq_* \bar{A}_\alpha$;
(4) if $\alpha \in S_\gamma$ and $\gamma \leq \alpha$, then $\bar{A}_{\alpha+1}$ contains a realization of $p_{\gamma,\alpha}$;
(5) if α is a limit ordinal, then for all $i < \mu$ such that $|C_\alpha| < \lambda_i$ and for all $\beta \in C_\alpha$, $\bar{A}_\beta \leq_i \bar{A}_\alpha$.

The construction is done almost precisely as in [13, Proof of Theorem 3.3], but we explain.

Put $A_{0,i} = \mathrm{Sk}(I_i)$. For α successor use Main Lemma 4.3. For α limit, we divide into two cases.

Case 1. $\sup(C_\alpha) = \alpha$. Let $i_0 = \min\{i < \mu \mid |C_\alpha| < \lambda_i\}$ (which is a successor). For $i < i_0$, let $A_{\alpha,i} = A_{0,i}$. For $i \geq i_0$ successor, let $A_{\alpha,i} = \bigcup_{\beta \in C_\alpha} A_{\beta,i}$. Note that $|A_{\alpha,i}| \leq \lambda_i$ for all $i < \mu$. We have to show that $\bar{A}_\alpha$ satisfies (1), (3) and (5). The latter is by construction.

For (1), suppose $s \subseteq A_{\alpha,i}$ is a finite set where $i_0 \leq i \in \mathrm{Succ}(\kappa)$. For every element $e \in s$, there is some $\beta_e \in C_\alpha$ such that $e \in A_{\beta_e,i}$. Let $\beta = \max\{\beta_e \mid e \in s\}$. Then β is a limit ordinal and $C_\alpha \cap \beta = C_\beta$. As $|C_\beta| < \lambda_{i_0}$, it follows by the induction hypothesis that $s \subseteq A_{\beta,i}$. Hence for some club E of λ_{i+1}, $\bar{a}_\alpha$ is L_1-indiscernible over s for all $\alpha \in E$. We also have to check that $A_{\alpha,i} \prec \mathfrak{C}_1$, but this is immediate as $A_{\beta,i} \prec \mathfrak{C}_1$ for all $\beta \in C_\alpha$ by induction.

Lastly, (3) is easy by assumption of the case and transitivity of $\leq_*$.

Case 2. $\sup(C_\alpha) < \alpha$. If $C_\alpha = \emptyset$, let $\gamma = 0$ and otherwise let $\gamma = \max C_\alpha$ (recall that it exists). Let $\langle \beta_n \mid n < \omega \rangle$ be a cofinal increasing sequence in α starting with $\beta_0 = \gamma$ (which exists since $\mathrm{cf}(\alpha) = \omega$). For every $n < \omega$, there is some $i_n < \mu$ such that $\bar{A}_{\beta_n} \leq_{i_n} \bar{A}_{\beta_{n+1}}$. Without loss of generality assume that $i_n < i_{n+1}$ for all $n < \omega$. Letting $i_{-1} = 0$, for all successor $i \geq i_{n-1}$ such that $i < i_n$ define $A_{\alpha,i} = A_{\beta_n,i}$, and for all successor $i \geq \sup\{i_n \mid n < \omega\}$, let $A_{\alpha,i} = \bigcup\{A_{\beta_n,i} \mid n < \omega\}$. Note that $\bar{A}_{\beta_n} \leq_{i_{n-1}} \bar{A}_\alpha$ for all $n < \omega$. This easily satisfies all the requirements. For example (5): if $C_\alpha = \emptyset$, then there is nothing to check, so assume $C_\alpha \neq \emptyset$. Let $i < \mu$ be such that $|C_\alpha| < \lambda_i$ and fix some $\beta \in C_\alpha$. Hence $\beta \leq \gamma = \max C_\alpha$. Note that $\bar{A}_\gamma \leq \bar{A}_\alpha$ (so also $\bar{A}_\gamma \leq_i \bar{A}_\alpha$), so we may assume $\beta < \gamma$. In this case, since $C_\gamma = C_\alpha \cap \gamma$, $\beta \in C_\gamma$, and since $|C_\gamma| = |C_\alpha \cap \gamma| < \lambda_i$, by induction it follows that $\bar{A}_\beta \leq_i \bar{A}_\gamma \leq \bar{A}_\alpha$ so we are done.

Finally, let $M = \bigcup_{\alpha < \kappa^+, i < \mu} A_{\alpha,i}$. Then M is a κ-saturated model of T by (4). However, it is not κ^+-locally saturated because the local type $\{\varphi(x, a_j) \mid j \in I \text{ even}\} \cup \{\neg\varphi(x, a_j) \mid j \in I \text{ odd}\}$ is not realized in M. To see this, suppose towards contradiction that b realizes it. Since $\bar{A}_\alpha$ is an increasing continuous sequence for all $\alpha < \mu^+$, there must be some $\alpha < \mu^+$ and $i \in \mathrm{Succ}(\kappa)$ such that $b \in A_{\alpha,i}$. But then by (1), for some $\alpha < \lambda_i$, $\bar{a}_\alpha$ is indiscernible over b — contradiction. □

4.2. *Supersimple theories*

From the previous section it follows that if T is countable, unstable and supersimple ($\kappa(T) = \aleph_0$), and κ is singular with cofinality ω such that $\kappa^+ = 2^\kappa$ and $\square_\kappa$ holds, then T has PC-exact saturation at κ. In this section, we will show that this property identifies supersimplicity among unstable theories: assuming that T is not supersimple, we will find an expansion T_1 (of the same size) such that if $M \models T_1$ and $M \restriction L$ is κ-saturated, then M is κ^+-saturated.

Theorem 4.8. *Suppose that T is unstable and not supersimple, and that κ is singular with $\mathrm{cf}(\kappa) = \omega$ (in particular uncountable) and $\kappa \geq |T|$. Then there is an expansion $T_1 \supseteq T$ with $|T_1| = |T|$ and such that if $M \models T_1$ is such that $M \restriction L$ is κ-saturated, then $M \restriction L$ is κ^+-saturated.*

Proof. If T is countable we can use essentially the same expansion T_1 as in Sec. 3.1 (adding to it the function k from below), but since we do not assume that T is countable we give details. Let $\lambda = |T| + \aleph_0$. As T is not supersimple, we can use Fact 3.1 and compactness to find a sequence of formulas $\langle \psi_n(x, y_n) \mid n < \omega \rangle$ and a tree $\langle a_\eta \mid \eta \in \lambda^{<\omega} \rangle$ as there. In addition, since T is not stable, there is a formula $\varphi(x, y)$ where x is a singleton and a sequence $\langle b_n, c_n \mid n < \omega \rangle$ such that $\varphi(b_n, c_m)$ holds if and only if $n < m$.

Let $M_0 \models T$ be of size λ containing $\langle a_\eta \mid \eta \in \lambda^{<\omega} \rangle$ and $\langle b_n \mid n < \omega \rangle$. Without loss of generality we may assume that the universe of M_0 is $\lambda \cup \lambda^{<\omega} \cup L$ where L is the set of all formulas from L in a fixed countable set of variables $\{v_i \mid i < \omega\}$. We put a predicate $\mathcal{K}$ for λ and a predicate $\mathcal{T}$ for $\lambda^{<\omega}$ on which we add the order $\trianglelefteq$. We put a predicate $\mathcal{N}$ on $\omega \subseteq \lambda$ and enrich it with $+, \cdot, <$. Add two functions $l : \mathcal{T} \to \mathcal{N}$ and $\mathrm{eval} : \mathcal{T} \times \mathcal{N} \to \mathcal{K}$ as in Sec. 3.1: $\mathrm{eval}(\eta, n) = \eta(n)$ for $n < l(\eta)$ and otherwise $\mathrm{eval}(\eta, n) = 0$. Also, add a bijection $e : \mathcal{K} \to M_0$. Add a predicate $\mathcal{L}$ for L and a truth valuation $TV : \mathcal{L} \times \mathcal{T} \to \{0, 1\}$ as in Sec. 3.1 (this time there is no need to add finite subsets of formulas). Add a function $d : \mathcal{N} \to \mathcal{L}$ mapping n to $\{\psi_n(v_0; v_1, \ldots, v_{|y_n|})\}$ and a function $a : \mathcal{T} \to \mathcal{T}$ such that if $\eta \in \lambda^{<\omega}$ then $a(\eta) \in \lambda^{|y_{l(\eta)}|}$, $\{TV(d(i), \langle x \rangle \frown a(\eta|i)) \mid i < l(\eta)\}$ is consistent and such that if $\eta, \nu \in \mathcal{T}$ are incomparable then $\{TV(d(l(\eta)), \langle x \rangle \frown a(\eta)), TV(d(l(\nu)), \langle x \rangle \frown a(\nu))\}$ is inconsistent. Also, add a map $k : \omega \to M_0$ such that $k(i) = b_i$. Let $M_1 = (M_0, \mathcal{N}, \mathcal{K}, \mathcal{T}, \mathcal{L})$ with this additional structure, and set $T_1 = \mathrm{Th}(M_1)$ and $L_1 = L(T_1)$. In models of T_1, we will denote by ω the (interpretations of the) standard natural numbers, while elements in $\mathcal{N}$ which are not from ω are nonstandard.

(*) Note that Lemma 3.2 still holds, with some minor adjustments to the proof (replacing $\mathcal{N}$ by $\mathcal{K}$), giving us a definable function $H : \mathcal{K} \times \mathcal{N} \to \mathcal{K}$ such that if $M \models T_1$ and $M \restriction L$ is $\aleph_1$-saturated, then for any function $f : \omega \to \mathcal{K}^M$, there is some $m_f \in \mathcal{K}^M$ with the property that $H^M(m_f, n) = f(n)$ for all $n \in \omega$.

Suppose that $M \models T_1$ and $M \restriction L$ is κ-saturated. Note that $\kappa \geq \aleph_1$ and hence $M \restriction L$ is $\aleph_1$-saturated. Let $p(x) \in S_L(A)$ be some complete type where $A \subseteq M$ is of size $|A| = \kappa$. Since $\mathrm{cf}(\kappa) = \omega$ we can write A as an increasing union $A = \bigcup_{i<\omega} A_i$ where $|A_i| < \kappa$. For $i < \omega$, let $c_i \models p|_{A_i}$. By (*), there is some $m \in \mathcal{K}^M$ such that $H(m, i) = e^{-1}(c_i)$ for all $i < \omega$. For every $\varphi(x) \in p$, there is $k_\varphi < \omega$ such that the set $D_\varphi = \{i \in \mathcal{N}^M \mid \forall j \in [k_\varphi, i] M \models \varphi(e(H(m, j)))\}$ contains $[k_\varphi, \omega)$. Note that D_φ is convex, and that it is A-definable in L_1. By overspill, D_φ contains some nonstandard element $d_\varphi \in \mathcal{N}^M$. Let $C = \{d_\varphi \mid \varphi \in p\} \subseteq \mathcal{N}^M$. Note that if $r \in \mathcal{N}^M$ is nonstandard and $r \leq d_\varphi$ then $r \in D_\varphi$, so that if $r \leq d_\varphi$ for all $\varphi \in p$ then $e(H(m, r)) \models p$ so that p is realized.

Since $|A| \leq \kappa$, $|C| \leq \kappa$, and since κ is singular, the cofinality of C (going down) is $< \kappa$. Let $C' \subseteq C$ be a coinitial subset of size $< \kappa$. To conclude, we show that the set

$$\Gamma = \{x \in \mathcal{N}\} \cup \{n < x \mid n < \omega\} \cup \{x \leq d \mid d \in C'\}$$

is realized in M. Recall the choice of φ and the function k above, and consider the set $\Pi = \{\varphi(k(i), y) \mid i < \omega\} \cup \{\neg\varphi(k(d), y) \mid d \in C'\}$. Then Π is consistent by choice of φ, since all elements in C' are nonstandard. By κ-saturation Π is realized, say by $f \in M^{|y|}$. Let $g \in \mathcal{N}^M$ be minimal such that $M \models \neg\varphi(k(g), f)$. Then g is nonstandard and $g \leq d$ for all $d \in C'$ (since $\neg\varphi(k(d), f)$ holds for all $d \in C'$), so $g \models \Gamma$ and we are done. □

Combining Theorem 4.1 with Theorem 4.8, we get the following.

Corollary 4.9. *Let T be unstable theory. Suppose that κ is singular with cofinality ω such that $|T| < \kappa$, $\kappa^+ = 2^\kappa$ and $\square_\kappa$ holds. Then, T is supersimple if and only if T has PC-exact saturation at κ.*

Corollary 4.10. *Suppose that T is a countable theory. Suppose that κ is singular with cofinality ω such that $2^{\aleph_0} < \kappa$, $\kappa^+ = 2^\kappa$ and $\square_\kappa$ holds. Then T is supersimple if and only if T has PC-exact saturation at κ.*

Proof. If T is unstable this follows from Corollary 4.9. If T is stable and supersimple then T is superstable so T is κ-stable and hence has PC-exact saturation by Corollary 3.11. On the other hand, if T has PC-exact saturation at κ then T is κ-stable. But if T is not superstable then by the stability spectrum theorem [17, III], $\kappa^{\aleph_0} = \kappa$, contradicting the cofinality assumption. □

5. On Local Exact Saturation and Cofinality ω

In this section, we will see that for κ of cofinality ω, having local PC-exact saturation defines the class of *supershort* simple theories: the class of theories for which every local type does not fork over a finite set. Before doing that, we discuss stable theories.

5.1. *Stable theories*

Here, we will prove that contrary to the situation with PC-exact saturation (i.e. to Corollary 3.11), stable theories always have *local* PC-exact saturation. The proof is similar to the proof that λ-stable theories have saturated models of size λ [17, Theorem III.3.12], but a bit simpler.

Definition 5.1. For any theory T, $\kappa_{\text{loc}}(T)$ is the smallest cardinal κ such that any local type $p \in S_\Delta(A)$ (see Definition 2.4) does not fork over a set of size $< \kappa$. If no such cardinal exists, then $\kappa_{\text{loc}}(T) = \infty$.

In other words, the definition is the same as that of $\kappa(T)$, where types are replaced with local types.

For stable theories it is always $\aleph_0$.

Proposition 5.2. *Suppose that T is stable. Then $\kappa_{\mathrm{loc}}(T) = \aleph_0$: every local type $p \in S_\Delta(A)$ does not fork over a finite subset $A_0 \subseteq A$.*

Proof. Let $q \in S(\mathfrak{C})$ be a global non-forking extension of p over A. Then q is definable over $\mathrm{acl}^{\mathrm{eq}}(A)$. In particular, $q \restriction \Delta$ is definable over $\mathrm{acl}^{\mathrm{eq}}(A_0)$ for some finite subset $A_0 \subseteq A$, so $q \restriction \Delta$ does not fork over A_0. □

Lemma 5.3. *Suppose that T is a stable L-theory. Suppose that $\langle M_i \mid i < \mu\rangle$ is an increasing continuous sequence of λ-locally saturated models. Then $M_\mu = \bigcup_{i<\mu} M_i$ is also λ-locally saturated.*

Proof. If $\lambda = \aleph_0$, this is clear so assume that $\lambda > \aleph_0$. We are given $p(x) \in S_\Delta(A)$, $A \subseteq M_\mu$, $|A| < \lambda$, and we want to realize p in M_μ (x is any finite tuple of variables). Let L' be a countable sublanguage of L containing Δ. The models $M_i \restriction L'$ are still λ-locally saturated, so we may assume that $L = L'$ and in particular it is countable. By Proposition 5.2, p does not fork over a finite set $B \subseteq A$. In particular, $B \subseteq M_j$ for some $j < \mu$. Find a countable model $M' \prec M_j$ containing B. Let q be a global extension of p which does not fork over B. By stability, q is definable and finitely satisfiable over M'.

By stability, if $\langle a_i \mid i < \omega\rangle$ is any indiscernible sequence and $\varphi(x, y)$ is some formula then for any b, either for almost all $i < \omega$ (i.e. all but finitely many) $\varphi(a_i, b)$ holds or for almost all $i < \omega$, $\neg\varphi(a_i, b)$ holds. By compactness, there is some finite set of formulas Δ_0 and $N < \omega$ such that if $\langle a_i \mid i < 2N\rangle$ is any Δ_0-indiscernible sequence then for any (partition of any) formula $\varphi(x, y)$ from Δ and any b, it cannot be that $\varphi(a_i, b)$ for $i < N$ and $\neg\varphi(a_i, b)$ for $N \le i < 2N$.

As M_j is λ-locally saturated (and $\lambda > \aleph_0$), we can realize in M_j a Δ_0-Morley sequence generated by q over M': $a_0 \models (q \restriction \Delta_0)|_{M'}$, $a_1 \models (q \restriction \Delta_0)|_{M'a_0}$, etc. It is not hard to see that in fact the sequence $\langle a_n \mid n < \omega\rangle$ realizes $(q^{(\omega)}|_{M'}) \restriction \Delta_0$ (the latter is just the restriction of $q^{(\omega)}|_{M'}$ to the set of formulas from Δ_0 in the variables $\langle x_n \mid n < \omega\rangle$ over M' where $|x_n| = |x|$). For a proof, see [12, Claim 4.11]. In particular, $\langle a_n \mid n < \omega\rangle$ is a Δ_0-indiscernible sequence over M'. We can continue and realize in M_j a Δ_0-Morley sequence $\langle a_i \mid i < \lambda\rangle$ generated by q over M'.

We claim that for some $i < \lambda$, $a_i \models p$. Indeed, suppose not. This means that for every $i < \lambda$, for some $\varphi_i(x, y)$ from Δ and some $b_i \in A$, $\neg\varphi_i(a_i, b_i)$ holds while $\varphi_i(x, b_i) \in p$. As Δ is finite and $|A| < \lambda$, there are some φ and $b \in A$ such that $\neg\varphi(a_i, b)$ holds for all $i \in I_0$ where $I_0 \subseteq \lambda$ is infinite and $\varphi(x, b) \in p$. However, we can now realize $a'_0 \models q \restriction \Delta|_{M'b\cup\{a_i \mid i \in I_0\}}$, $a'_1 \models (q \restriction \Delta)|_{M'b\cup\{a_i \mid i\in I_0\}\cup\{a'_0\}}$, etc. (where $a'_i \in \mathfrak{C}$). Since q extends p, $\varphi(a'_i, b)$ holds for all $i < \omega$, so we have a contradiction to the choice of Δ_0. □

Lemma 5.4. *Suppose that T is stable. Then for any $M \models T$ there is an extension $M' \succ M$ such that M' is locally saturated and $|M'| \leq |M| + |T|$.*

Proof. We may assume that $|T| \leq |M|$. Let $\kappa = |M|$.

For every finite set of formulas Δ, the number of Δ-types over M is bounded by κ (by stability). Hence, there is some $M_1 \succ M$ such that M_1 realizes every local type in $S_\Delta(M)$ and $|M_1| = \kappa$. This allows us to construct a continuous increasing elementary chain $\langle M_\alpha \mid \alpha < \kappa \rangle$ starting with $M_0 = M$ with the properties that $|M_\alpha| = \kappa$ and for each $\alpha < \kappa$, $M_{\alpha+1}$ realizes every local type in $S_\Delta(M_\alpha)$. Let $M' = \bigcup_{\alpha<\kappa} M_\alpha$.

If κ is regular then M' is as requested.

Otherwise, for every regular $\lambda < \kappa$, $M_{\alpha+\lambda}$ is λ-locally saturated. Since $M' = \bigcup\{M_{\alpha+\lambda} \mid \alpha < \kappa\}$ then by Lemma 5.3, M' is λ-locally saturated. Since this is true for every such λ, M' is κ-locally saturated. $\square$

Theorem 5.5. *Suppose that T is stable. Then for every cardinal $\kappa \geq |T|$, T has local PC-exact saturation at κ.*

Proof. Suppose that $T_1 \supseteq T$ has cardinality $|T|$. Without loss of generality assume that T_1 has Skolem functions. We construct a sequence of T-models $\langle M_i \mid i < \omega \rangle$ and T_1-models $\langle N_i \mid i < \omega \rangle$ such that

– $N_i = \mathrm{Sk}(M_i)$; $N_i \restriction L \prec M_{i+1}$; M_i is locally saturated and $|M_i| = \kappa$.

For the construction use Lemma 5.4. By Lemma 5.3, $M = \bigcup_{i<\omega} M_i$ is locally saturated, and by construction it is in $\mathrm{PC}(T_1, T)$. It is not κ^+-locally saturated since $|M| = \kappa$ (so does not realize the local type $\{x \neq a \mid a \in M\}$). $\square$

5.2. *Simple theories*

In this section, we will analyze local PC-exact saturation in the context of simple theories. We start with a positive result.

Theorem 5.6. *Assume that T is a complete simple L-theory, $T_1 \supseteq T$ is a theory in $L_1 \supseteq L$, $|T_1| \leq |T|$. Also, assume that κ is a singular cardinal such that $\kappa_{\mathrm{loc}}(T) \leq \mu = \mathrm{cf}(\kappa)$, $|T| < \kappa$, $\kappa^+ = 2^\kappa$ and $\square_\kappa$ holds. Then $\mathrm{PC}(T_1, T)$ has local exact saturation at κ.*

Proof. The proof is almost exactly the same as the proof of Theorem 4.1 (where we also assumed instability).

By Theorem 5.5, we may assume that T is unstable, so there is an L-formula $\varphi(x, y)$ with the independence property. We find an L_1-indiscernible sequence I witnessing this and define λ_i, I_i for $i < \mu$ as in the proof of Theorem 4.1. We also define the class $\mathcal{C}$ in exactly the same way.

The proof of the parallel to Main Lemma 4.3 is similar, but the first step is to say that given a local type $p(x) \in S_\Delta(C)$, without loss of generality it does not fork over $A_* = A_0 \cap C$, so we may extend it to a complete type $p'(x) \in S(C)$ which also does not fork over A_*. Then we continue with the same proof.

Note also that x may not be a single variable but we never needed that assumption in the proof of Main Lemma 4.3. $\square$

We will now discuss $\kappa_{\mathrm{loc}}(T)$ (see Definition 5.1), which will lead us to our next result.

Claim 5.7. *For any complete theory T with infinite models, $\kappa_{\mathrm{loc}}(T)$ (see Definition 5.1) can be either $\aleph_0$, $\aleph_1$ or ∞. In the first two cases T is simple, and in the last case T is not simple.*

Proof. The proof is standard, but we give details.

If T is not simple, then T has the tree property (see [19, Definition 7.2.1]) as witnessed by some formula $\varphi(x, y)$ and some $k < \omega$: there is a sequence $\langle a_s \mid s \in \omega^{<\omega}\rangle$ such that for every $s \in \omega^{<\omega}$, $\{\varphi(x, a_{s\frown\langle i\rangle}) \mid i < \omega\}$ is k-inconsistent while for any $\eta \in \omega^\omega$, $\Gamma_\eta = \{\varphi(x, a_{\eta\restriction n}) \mid n < \omega\}$ is consistent. Let μ be any regular cardinal. By compactness, we may extend the tree to have width $\lambda = (2^\mu)^+$ and height μ (so that s ranges over $\lambda^{<\mu}$). We correspondingly extend the definition of Γ_η to all $\eta \in \lambda^\mu$. For $\alpha < \mu$, find an increasing continuous sequence $s_\alpha \in \lambda^\alpha$ such that $s_{\alpha+1}$ extends s_α and $\varphi(x, a_{s_{\alpha+1}})$ divides (and even k-divides) over $\{\varphi(x, a_{s_\beta}) \mid \beta \le \alpha\}$ (the construction uses the fact that for infinitely many $i < \lambda$, $a_{s_\alpha\frown\langle i\rangle}$ will have the same type over $a_{s_{<\alpha}}$). Letting $\eta = \bigcup_{\alpha<\mu} s_\alpha$, any complete φ-type extending Γ_η over $\{a_{s_\alpha} \mid \alpha < \mu\}$ divides over any subset of size $< \mu$ its domain. Since μ was arbitrary, this show that $\kappa_{\mathrm{loc}}(T) = \infty$.

Now, if $\kappa_{\mathrm{loc}}(T) > \aleph_1$, then there is a local type $p \in S_\Delta(A)$ which forks over any countable subset of A. Let L' be a countable subset of the language L of T containing all the symbols appearing in Δ. Then any completion $q \in S_{L'}(A)$ of p forks over any countable subset of A, so $T \restriction L'$ does not satisfy local character for non-forking, so it is not simple, and so is T, thus $\kappa_{\mathrm{loc}}(T) = \infty$.

Finally, $\kappa_{\mathrm{loc}}(T)$ cannot be any $n < \omega$, since given $a_0, \ldots, a_{n-1}$ with $a_i \notin \mathrm{acl}(a_{\neq i})$ (e.g. a_i come from an infinite indiscernible sequence), $\mathrm{tp}_=(a_0, \ldots, a_{n-1}/a_0, \ldots, a_{n-1})$ divides over any proper subset of $\{a_i \mid i < n\}$. $\square$

Definition 5.8 ([5, Definition 8]). A theory is called *supershort* if $\kappa_{\mathrm{loc}}(T) = \aleph_0$.

Remark 5.9. This is not the precise definition given in [5] which is given in terms of dividing chains, but it is equivalent to it: given an infinite dividing chain of conjunctions of a single formula $\varphi(x, y)$ as in the definition there, the partial φ-type containing them divides over any finite subset of its domain. On the other hand, if $\kappa_{\mathrm{loc}}(T) > \aleph_0$ and $p \in S_\Delta(A)$ witnesses this (i.e. divides over every finite $A_0 \subseteq A$) for some finite Δ, then by coding finitely many formulas as one formula (see [17, Proof of Theorem II.2.12(1)]), we can recover a dividing chain as in the definition in [5].

Recall that a theory T is *low* if whenever $\varphi(x, y)$ is a formula then there is some $n < \omega$ such that if $\langle a_i \mid i < \omega \rangle$ is an indiscernible sequence such that $\{\varphi(x, a_i) \mid i < \omega\}$ is inconsistent, then it is already n-inconsistent. This is not the original definition from [1, 16], which is given using local ranks, but it is equivalent to it when T is simple, see [3, Proposition 18.19].

The following proposition gives a simple criterion for supershortness.

Proposition 5.10. *Suppose that T is a simple theory such that if Δ is a finite set of formulas and $p(x) \in S_\Delta(A)$ then there is a finite set of formulas Δ' such that whenever p divides over $A_0 \subseteq A$, there is some formula $\varphi(x, y)$ from Δ' and some $a \in \mathfrak{C}$ such that $p \vdash \varphi(x, a)$ and $\varphi(x, a)$ divides over A_0. Then if T is low then $\kappa_{\text{loc}}(T) = \aleph_0$.*

Proof. If $\kappa_{\text{loc}}(T) > \aleph_0$ then there is a local type $p(x) \in S_\Delta(A)$ (for Δ finite) such that p divides over any finite subset of A, in particular A is infinite. Let Δ' be as above. Thus, we can construct an increasing chain $\langle A_i \mid i < \omega \rangle$ such that $A_i \subseteq A$ are finite and $p|_{A_{i+1}}$ divides over A_i. As Δ' is finite we can find a single formula $\varphi(x, y) \in \Delta'$ and $a_i \in \mathfrak{C}$ such that $\varphi(x, a_i)$ divides over A_i and $p|_{A_{i+1}} \vdash \varphi(x, a_i)$. If J_i is an indiscernible sequence over A_i witnessing that $\varphi(x, a_i)$ divides over A_i, by Ramsey and compactness and applying an automorphism we can assume that J_i is indiscernible over $a_{<i}$ (perhaps changing $a_{<i}$). By compactness we can assume that $\{\varphi(x, a_i) \mid i < \omega\}$ is consistent and $\varphi(x, a_i)$ divides over $a_{<i}$ (we need compactness since we changed $a_{<i}$ in every stage). As T is low, there is some k such that $\varphi(x, a_i)$ k-divides over $a_{<i}$. From this we can recover the tree property. Alternatively, this also follows from [3, Proposition 18.19(5)]. $\square$

However, in general there is no connection between being low and being supershort. We found the following table useful.

	Supershort	Not supershort
Low	Any stable theory	[5, Sec. 4]
Not low	[4] (even supersimple)	[2, Sec. 5]

Our goal is to show that when $\text{cf}(\kappa) = \aleph_0$, T has local PC-exact saturation at κ if and only if T is supershort.

Proposition 5.11. *If $\kappa_{\text{loc}}(T) > \aleph_0$ then there is a formula $\varphi(x, y)$ and a sequence of formulas $\langle \psi_n(x, y_n) \mid n < \omega \rangle$ (where x is a finite tuple of variables and the y_n's are tuples of variables of varying lengths) and a sequence $\langle a_\eta \mid \eta \in \omega^{<\omega} \rangle$*

such that

– *Each formula* ψ_n *has the form* $\bigwedge_{j<l}\varphi(x,y_j)$ *for some* l; a_η *is an* $|y_{|\eta|}|$*-tuple; For* $\sigma\in\omega^\omega$, $\{\psi_n(x,a_{\sigma\restriction n})\mid n<\omega\}$ *is consistent; For every* $\eta\in\omega^n,\nu\in\omega^m$ *such that* $\eta\perp\nu$, $\{\psi_n(x,a_\eta),\psi_m(x,a_\nu)\}$ *is two-inconsistent.*

Proof. This is a local version of Fact 3.1. Since $\kappa_{\mathrm{loc}}(T)>\aleph_0$, there is a finite set Δ and a type $p(x)\in S_\Delta(A)$ (for some infinite set A) that forks over every finite subset $A_0\subseteq A$. By Remark 5.9, there is a formula $\varphi(x,y)$ and a sequence of formulas $\langle\psi_n(x,y_n)\mid n<\omega\rangle$ where each ψ_n is a conjunction of the form $\bigwedge_{i<k_n}\varphi(x,y_i)$, and a sequence $\langle a_n\mid n<\omega\rangle$ such that $\{\psi_n(x,a_n)\mid n<\omega\}$ is consistent and $\psi_n(x,a_n)$ divides over $a_{<n}$. Thus, there are $\langle k_n<\omega\mid n<\omega\rangle$ and a tree $\langle a_\eta\mid\eta\in\omega^{<\omega}\rangle$ such that for every $\sigma\in\omega^\omega$, $\{\psi_n(x,a_{\sigma\restriction n})\mid n<\omega\}$ is consistent and such that for every $n<\omega$ and $\eta\in\omega^n$, $\{\psi_{n+1}(x,a_{\eta\frown\langle i\rangle})\mid i<\omega\}$ is k_n-inconsistent. By applying the same proof as in [6, Proposition 3.5] to this tree, we are done. □

Theorem 5.12. *Let* T *be any complete theory. Suppose that* κ *is singular with cofinality* ω *such that* $|T|<\kappa$, $\kappa^+=2^\kappa$ *and* $\square_\kappa$ *holds. Then,* T *is supershort if and only if* T *has local PC-exact saturation at* κ.

Proof. Right to left follows from Theorem 5.6. For the other direction, use the same proof as in Theorem 4.8, noting that the proof goes through, because the only use of actual κ-saturation as opposed to local κ-saturation was the use of (*) (i.e. the use of the suitable version of Lemma 3.2). Here, all the formulas ψ_n are conjunctions of instances of φ so the types

$$\{\psi_{i+1}(x,e(a(\eta_i)(0)),\ldots,e(a(\eta_i)(|y_{i+1}|-1)))\mid i<\omega\}$$

(using the notation from the proof of Lemma 3.2) are still consistent. Of course, since the x may now be a tuple of length >1, the function H has domain $\mathcal{K}^{|x|}\times\mathcal{N}$ (where x is from $\psi_n(x,y_n)$).

One more difference is that now the type $p(z)$ we wish to realize is in possibly more than one variable. However, this is easy to overcome by taking a tuple of "codes" for the function $n\mapsto c_n$. □

6. Final Thoughts and Questions

6.1. *NSOP*$_1$

We would like to extend Theorem 4 to NSOP$_1$-theories, but we do not even know the situation with elementary classes (i.e. not PC-exact saturation, just exact saturation). The approach of mimicking the proof or the proof of [13, Theorem 3.3] using Kim-independence and all its properties (see [8–10]) does not seem to work without new ideas. Both proofs use base monotonicity and hence are not applicable.

Question 6.1. Is Theorem 4 true for NSOP$_1$-theories?

Note that [14, Theorem 9.30] states that if T has SOP_2 then it has *PC-singular compactness* (the negation of PC-exact saturation): for some T_1 containing T of cardinality $\leq |T|$ and every singular $\kappa > |T|$, if $M \in \mathrm{PC}(T, T_1)$ is κ-saturated then it is κ^+-saturated. Thus, a positive answer to Question 6.1 will help to "close the gap".

6.2. *NIP*

In [13, Theorem 4.10], it is proved that if T is NIP, and $|T| < \kappa$ is singular such that $2^\kappa = \kappa^+$, then T has exact saturation at κ if and only if T is not distal. While the situation for PC-exact saturation seems less clear, one can ask about local exact saturation (without PC). The proof of the direction that if T is distal then T does not have exact saturation at κ goes through in the local case: if T is distal, $|T| < \kappa$ is singular, then every κ-locally saturated model is κ^+-locally saturated. This is proposition [13, Proposition 4.12]. The proof has to be adjusted. Following the notation there, we elaborate a bit. Given a finite set Δ and a type $p(x) \in S_\Delta(A)$, let p' be an extension of p to $S_{\Delta'}(A)$ where $\Delta' = \Delta \cup \{\theta^\varphi \mid \varphi \in \Delta\}$ (we also consider all possible partitions of formulas in Δ). We let $b_i \models p'|_{A_i}$ for $i < \mu$ and find d_i^φ as there for any $\varphi \in \Delta$. Letting $q_i = \{\theta^\varphi(x, d_i^\varphi) \mid \varphi \in \Delta\}$ (so a finite set), the proof of [13, Proposition 4.13] goes through because p' is a complete Δ'-type. We then find e_i realizing the Δ''-type of the finite tuple $d_i = \langle d_i^\varphi \mid \varphi \in \Delta\rangle$ over $A_i \cup \{b_i \mid i < \mu\}$ where Δ'' contains Δ' and the formulas $(\forall x)(\theta^\varphi(x, z) \to \varphi(x, y))$ and $(\forall x)(\theta^\varphi(x, z) \to \neg\varphi(x, y))$ for $\varphi \in \Delta$. The rest goes through.

However, the other direction, namely that if T is not distal then T has local exact saturation at κ for κ as above seems less clear. The main issue is that the model constructed omits a type of an element over an indiscernible sequence, and this type is not local. For example if I is an indiscernible set, then the type omitted is that of a new element in the sequence.

Question 6.2. Which NIP theories have local exact saturation at singular cardinals as above?

Acknowledgments

Itay Kaplan would like to thank the Israel Science Foundation for partial support of this research (Grants Nos. 1533/14 and 1254/18). Saharon Shelah would like to thank the Israel Science Foundation Grant No. 1838/19 and the European Research Council Grant 338821. Paper no. 1197 in Saharon Shelah's publication list.

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