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A Borel maximal eventually different family [☆]

Haim Horowitz [a,*], Saharon Shelah [a,b]

[a] Einstein Institute of Mathematics, Edmond J. Safra campus, The Hebrew University of Jerusalem, Givat Ram, Jerusalem, 91904, Israel
[b] Department of Mathematics, Hill Center - Busch Campus, Rutgers, The State University of New Jersey, 110 Frelinghuysen road, Piscataway, NJ 08854-8019, USA

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ABSTRACT

We construct a Borel maximal eventually different family.

1. Introduction

Maximal almost disjoint families and their relatives have been studied by set theorists for decades. As the construction of such families is typically being done using the axiom of choice, questions about their definability naturally arise. The definability of mad families was investigated by Mathias who proved the following:

Theorem *([7]). There are no analytic mad families.*

The possibility of the non-existence of mad families was investigated by the authors in [3] where the following was proved (earlier such results were proven by Mathias and Toernquist using Mahlo and inaccessible cardinals, respectively):

Theorem *([3]). $ZF + DC +$ "There are no mad families" is equiconsistent with ZFC.*

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* Corresponding author.
E-mail addresses: haim.horowitz@mail.huji.ac.il (H. Horowitz), shelah@math.huji.ac.il (S. Shelah).

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In this paper we shall study maximal eventually different families in ω^ω. Recall that $f, g \in \omega^\omega$ are eventually different if $f(n) \neq g(n)$ for large enough n. A family $\mathcal{F} \subseteq \omega^\omega$ is a maximal eventually family if the members of $\mathcal{F}$ are pairwise eventually different, and $\mathcal{F}$ is maximal with respect to this property. Our main goal is to construct in ZF a Borel maximal eventually different family, thus answering a question asked by several set theorists (see for example [1], [6] and [15]) and showing that the analog for the above theorems is not true for maximal eventually different families.

Before embarking on the proof of our main result, we shall briefly overview some relevant previous results, as well as newer results that were established after the first appearance of this paper online. In a classical paper of Miller [8], it was shown that coanalytic mad families exist in L. This result was obtained together with Kunen, and the same argument establishes the existence of a coanalytic MED family in L. Miller's technique was further extended by Vidnyanszky in [16] establishing the existence of coanalytic instances in L of many other maximal sets of reals. Later work then established some combinatorial constraints that an analytic MED family must satisfy. An eventually different family $\mathcal{F} \subseteq \omega^\omega$ is called κ-maximal if for each $\{f_\alpha : \alpha < \kappa\} \subseteq \omega^\omega$, there exists $g \in \mathcal{F}$ such that, for every $\alpha < \kappa$, either $g \cap f_\alpha$ is infinite or f_α is in the ideal on $\omega \times \omega$ generated by $\mathcal{F}$. Kastermans, Steprans and Zhang showed in [6] that no analytic eventually different family can be $\aleph_0$-maximal. This was later improved by Raghavan in [10], who showed that analytic eventually different families can't be 2-maximal. It was also shown in the same paper that if $\mathcal{F} \subseteq \omega^\omega$ is an analytic eventually different family, then the set of $A \subseteq \omega$ for which there exists $f \in \omega^A$ such that f is eventually different to all members of $\mathcal{F}$ contains a copy of $\emptyset \times FIN$. Finally, following the proof of the main result in the current paper, Schrittesser improved our result obtaining a lightface Π^0_1 MED family ([11] and [12]). It should also be noted that the closely related line of research regarding the definability of maximal cofinitary groups has followed a quite parallel line of developments. Gao and Zhang showed in [2] that in L there exists a coanalytic set of permutations which generates a maximal cofinitary group. This was later improved by Kastermans in [5], where he established the existence of a coanalytic maximal cofinitary group in L. Analogously to the main result of this paper, a breakthrough was achieved in [4], where we established the existence of a Borel maximal cofinitary group in ZF. Our result was then improved by Schrittesser in [13] and Mejak-Schrittesser in [9], culminating in the construction of a Σ^0_2 maximal cofinitary group. For a detailed overview of this area of research, we refer the reader to [14].

2. The proof

Theorem 1 *(ZF). There exists a Borel MED family.*

Observe that the notion of a Borel MED family can be defined for A^B whenever $|A| = \aleph_0 = |B|$, and it's enough to prove that for some A and B of cardinality $\aleph_0$, there is a Borel MED family in A^B (with the natural Polish topology).

Definition and Claim 2. a. Let $T_* = 2^{<\omega}$.

b. $\mathcal{F}_* = \{f : f$ is a function from $2^{<\omega}$ to $H(\aleph_0)\}$.

c. For $n < \omega$ let $\mathcal{F}^*_n = \{f \restriction 2^{<n} : f \in \mathcal{F}_*\}$.

d. For $f, g \in \mathcal{F}_*$ let $eq(f, g) = \{\rho : f(\rho) = g(\rho)\}$ and $dif(f, g) = 2^{<\omega} \setminus eq(f, g)$.

e. Let $EDF = \{\mathcal{F} \subseteq \mathcal{F}_* : (\forall f \neq g \in \mathcal{F})(|eq(f, g)| < \aleph_0)\}$.

f. Let $MEDF = \{\mathcal{F} \in EDF : \mathcal{F}$ be maximal$\}$.

g. Let $\mathbf{B} : \mathcal{F}_* \to 2^\omega$ be an injective continuous function.

h. Let $F_1 : \mathcal{F}_* \to \mathcal{F}_*$ be defined as $F_1(f)(\rho) = f \restriction 2^{<lg(\rho)}$.

i. Let $G_0 = \{F_1(f) : f \in \mathcal{F}_*\}$.

j. Let G_1 be the set of $g \in \mathcal{F}_*$ such that for some $f \in \mathcal{F}_*$, $dif(g, F_1(f))$ is infinite and satisfies:

1. $(\forall n)(|\{\rho : \mathbf{B}(f) \restriction n \not\leq \rho \wedge \rho \in dif(g, F_1(f))\}| < \aleph_0)$.

2. For every $\rho \in 2^{<\omega}$, if $\rho \leq \mathbf{B}(f)$ then there exists at most one ν such that $\rho \leq \nu \in dif(g, F_1(f))$ and $\nu \cap \mathbf{B}(f) = \rho$.

k. For $g \in G_1$, let f_g be the unique f as in clause (j). We shall prove that f_g is indeed unique, and can be Borel-computed from f.

l. For $g \in G_1$ and f_g as above, let $w_g = dif(g, F_1(f_g))$.

m. Let G_2 be the set of $g \in G_1$ satisfying (1) and (2) where:

1. $g \restriction w_g = f_g \restriction w_g$.

2. $(\forall \rho \in w_g)(g(\rho) \notin \mathcal{F}^*_{lg(\rho)})$ or $(\forall \rho \neq \nu \in w_g)(g(\rho) \in \mathcal{F}^*_{lg(\rho)} \wedge g(\rho) \not\subseteq g(\nu))$.

Proof (of clause (k)). Given $g \in G_1$, let $X_1(g) = \{\rho \in T_* : g(\rho) \in \mathcal{F}^*_{lg(\rho)}\}$. Let $X_2(g) = \{\rho \in T_* : (\forall \nu_1, \nu_2)(\rho \leq \nu_1 \leq \nu_2 \to \nu_1, \nu_2 \in X_1(g) \wedge g(\nu_1) \subseteq g(\nu_2))\}$, $X_3(g) = \{\rho \in T_* : |\{\nu : \rho \leq \nu \in T_*, \nu \notin X_2(g)\}| < \aleph_0\}$ and $X_4(g) = \{\rho \in X_3(g)$: there are no incompatible ν_1 and ν_2 such that $\rho \leq \nu_1, \nu_2 \in T_*$ and $\nu_l \notin X_2(g)$ $(l = 1, 2)\}$. As $g \in G_1$, there is f as in clause (j).

We shall now prove that if $\rho \not\leq \mathbf{B}(f)$ then $\rho \in X_3(g)$ and moreover, $\rho \in X_4(g)$: By the definition of G_1, $\Lambda_n := \{\nu \in T_* : \mathbf{B}(f) \restriction n \not\leq \nu, g(\nu) \neq F_1(f)(\nu)\}$ is finite for every $n < \omega$. Now let $\rho \in T_*$ such that $\rho \not\leq \mathbf{B}(f)$ and choose a minimal n such that $\mathbf{B}(f) \restriction n \not\leq \rho$. For every $\rho \leq \nu \in T_*$, if $\nu \notin \Lambda_n$ then $g(\nu) = F_1(f)(\nu)$, therefore, $\rho \leq \nu_1 \leq \nu_2 \in T_* \wedge \nu_1, \nu_2 \notin \Lambda_n \to g(\nu_1) = F_1(f)(\nu_1) \subseteq F_1(f)(\nu_2) = g(\nu_2)$. It follows that $\rho \in X_3(g)$, moreover, by 2(j)(2), $\rho \in X_4(g)$: There is at most one ν such that $\rho \leq \nu$ and $\nu \in dif(g, F_1(f))$. For every $\rho \leq \nu'$ which is not $\leq \nu$, $g(\nu') = f \restriction 2^{<lg(\nu')}$, hence $\nu' \in X_2(g)$. It follows that $\rho \in X_4(g)$.

Therefore, for every n, $|\{\rho \in T_* : lg(\rho) = n, \rho \notin X_4(g)\}| \leq 1$. Note that $X_i(g)$ $(i = 1, 2, 3, 4)$ can be simply computed.

Note that by 2(j)(2), for every $\rho \in 2^{<\omega}$ there exists $\rho' \in eq(g, F_1(f))$ above it, hence, if $\rho \in X_2(g)$ then $\rho \in eq(g, F_1(f))$. Now suppose that $\nu_1 \neq \nu_2 \in 2^n \cap dif(g, F_1(f))$. If $\nu_1 \cap \nu_2 \not\leq \mathbf{B}(f)$, then $\nu_1 \cap \nu_2 \in X_4(g)$, contradicting the fact that $\nu_1, \nu_2 \notin X_2(g)$ are incomparable. If $\nu_1 \cap \mathbf{B}(f) = \nu_2 \cap \mathbf{B}(f) = \nu_1 \cap \nu_2$, then we get a contradiction to 2(j)(2). The only possibility left is that $\nu_1 \cap \nu_2 \leq \mathbf{B}(f)$ but $\nu_1 \cap \mathbf{B}(f) \neq \nu(2) \cap \mathbf{B}(f)$, so wlog $\nu_1 \cap \mathbf{B}(f) < \nu_2 \cap \mathbf{B}(f)$. Therefore, there are at most n elements $\nu \in 2^n$ such that $\nu \in dif(g, F_1(f))$. As $2^{n-1} > n$ for $3 \leq n$, we have established the following:

$(*)$ If $3 \leq n$, then for most $\nu \in 2^n$, $g(\nu) = f \restriction 2^{<n}$.

It follows that if $g \in G_1$ then f_g is uniquely determined, and there exists a Borel function $\mathbf{B}' : \mathcal{F}_* \to \mathcal{F}_*$ such that $g \in G_1 \to \mathbf{B}'(g) = f_g$. □

Claim 3. *1. If $g_1, g_2 \in G_2$ and $f_{g_1} \neq f_{g_2}$, then:*

a. $eq(g_1, g_2)$ is finite.

b. $w_{g_1} \cap w_{g_2}$ is finite.

c. $eq(g_2, F_1(f_{g_1}))$ is finite.

2. If $g_1 \in G_2$, $f_0 \in \mathcal{F}_*$ and $f_{g_1} \neq f_0$, then $eq(g_1, F_1(f_0))$ is finite.

Proof. 1. As $\mathbf{B}$ is injective, $\mathbf{B}(f_{g_1}) \neq \mathbf{B}(f_{g_2})$, therefore $\rho := \mathbf{B}(f_{g_1}) \cap \mathbf{B}(f_{g_2}) \in 2^{<\omega}$ and WLOG $\hat{\rho}(l) \leq \mathbf{B}(f_{g_l})$. By the definition of G_1, $\{\nu \in w_{g_l} : \hat{\rho}(l) \not\leq \nu\}$ is finite for $l = 1, 2$, therefore $w_{f_{g_1}} \cap w_{f_{g_2}}$ is finite, which proves clause (b). Now let n_* be such that $f_{g_1} \restriction 2^{<n_*} \neq f_{g_2} \restriction 2^{<n_*}$. If $\nu \in 2^{<\omega} \setminus w_{g_1} \setminus w_{g_2} \setminus 2^{\leq n_*}$, then $g_l(\nu) = F_1(f_{g_l})(\nu)$ $(l = 1, 2)$ by the definition of w_{g_l}. By the choice of n_* and the definition of F_1, $F_1(f_{g_1})(\nu) \neq F_2(f_{g_2})(\nu)$, so $g_1(\nu) \neq g_2(\nu)$. Note that $|\{\nu \in w_{g_2} : g_2(\nu) = F_1(f_{g_1})(\nu)\}| \leq 1$: By the definition of G_2, either $g_2(\nu) \notin \mathcal{F}^*_{lg(\nu)}$ for every $\nu \in w_{g_2}$ (in this case, the above set is empty by the definition of F_1) or $\{g_2(\nu) : \nu \in w_{g_2}\}$ are pairwise incomparable with respect to inclusion, and then as $\{F_1(f_{g_1})(\nu) : \nu \in w_{g_2}\}$ form a chain, the above set has cardinality ≤ 1. Suppose now that $\nu \in w_{g_2} \setminus w_{g_1}$, then $g_1(\nu) = F_1(f_{g_1})(\nu)$, and by the above claim, there is at most one $\nu \in w_{g_2} \setminus w_{g_1}$ such that $g_1(\nu) = g_2(\nu)$. Similarly, there is at most one $\nu \in w_{g_1} \setminus w_{g_2}$ such that $g_1(\nu) = g_2(\nu)$. Therefore, $eq(g_1, g_2)$ is finite, which proves clause (a). Clause (c) follows from (2).

2. By the definition of G_2, either $g_1(\nu) \notin \mathcal{F}^*_{lg(\nu)}$ for every $\nu \in w_{g_1}$ (and therefore $w_{g_1} \cap eq(g_1, F_1(f_0)) = \emptyset$), or $\{g_1(\nu) : \nu \in w_{g_1}\}$ are pairwise incomparable (and then $|w_{g_1} \cap eq(g_1, F_1(f_0))| \leq 1$). If $\nu \notin w_{g_1}$ is long enough, then $g_1(\nu) = F_1(f_{g_1})(\nu) = f_{g_1} \restriction 2^{<lg(\nu)} \neq f_0 \restriction 2^{<lg(\nu)} = F_1(f_0)(\nu)$. Together we get the desired conclusion. □

Definition 4. Let $H_3 = \{f \in \mathcal{F}_*$: there is $g \in G_2$ such that $f_g = f\}$.

Definition 5. Given a formula $\psi(x)$, we say that the truth value $TV(\psi(f))$ $(f \in \mathcal{F}_*)$ is Borel-computable if there exists a Borel function $F : \mathcal{F}_* \to \{0, 1\}$ such that $TV(\psi(f)) = true$ iff $F(f) = 1$.

The theorem will follow from the following claim together with Claim 8:

Claim 6. *There is a Borel function F^*_3 such that $Dom(F^*_3) = \mathcal{F}_*$, $f \in H_3 \iff F^*_3(f) \in G_2$ and $f_{F^*_3(f)} = f$ when $f \in H_3$. As a consequence, H_3 is Borel.*

Definition 7. Let $G_4 := \{F^*_3(f) : f \in H_3\} \cup \{F_1(f) : f \in \mathcal{F}_* \setminus H_3\}$.

Claim 8. *a. G_4 is Borel and $G_4 \subseteq G_0 \cup G_2$ (and $G_2 \subseteq G_1$).*
b. $G_4 \in EDF$.
c. $G_4 \in MEDF$.

Proof of Claim 8. a. The second part of the claim is obvious. As for the first part, first observe that $f \in G_4$ iff $TV_1(f) = true$ or $TV_2(f) = true$ where:

1. $TV_1(f) = true$ iff $f \in G_0$ and $F_1^{-1}(f) \notin H_3$ (where G_0 was defined in 2(i)).
2. $TV_2(f) = true$ iff $\mathbf{B}'(f) \in H_3$ and $f = F^*_3(\mathbf{B}'(f))$ (where $\mathbf{B}'$ is the Borel function from Claim 2(k), which is defined in the end of the proof of the claim).

Next observe that $TV_1(f)$ is Borel-computable: It's easy to see that G_0 is closed and F_1^{-1} is continuous on G_0. As H_3 is Borel, we're done.

$TV_2(f)$ is Borel-computable as well, as H_3 and all of the functions involved are Borel. It follows that G_4 is Borel.

b. Suppose that $g_1 \neq g_2 \in G_4$ as witnessed by $f_{g_1} = f_1$ and $f_{g_2} = f_2$. Clearly, $f_1 = f_2$ is impossible, as then, if $f_1 \in H_3$ then $f_2 \in H_3$, hence $g_1 = F^*_3(f_1) = F^*_3(f_2) = g_2$, and similarly, if $f_1, f_2 \notin H_3$, then $g_1 = F_1(f_1) = F_1(f_2) = g_2$. Therefore, $f_1 \neq f_2$. If $f_1, f_2 \in H_3$ then $g_1, g_2 \in G_2$ and by Claim 3(1), $eq(g_1, g_2)$ is finite. If $f_1, f_2 \notin H_3$, then $g_1 = F_1(f_1), g_2 = F_1(f_2)$, and by the definition of F_1, $eq(g_1, g_2)$ is finite. If $f_1 \in H_3$ and $f_2 \notin H_3$ or vice versa, then $eq(g_1, g_2)$ is finite by 3(2).

c. Let $f \in \mathcal{F}_*$, we shall find $g \in G_4$ such that $eq(f, g)$ is infinite. Denote $\mathbf{B}(f)$ (from 2(g)) by η_f. If $f \in H_3$ then $g = F^*_3(f) \in G_4$ is well-defined. By the definition of G_2 and F^*_3, $g \restriction w_g = f \restriction w_g$. By the definition of G_2, w_g is infinite. Therefore, we may assume that $f \notin H_3$.

Case I: For every n there is ν such that $\eta_f \restriction n \leq \nu \in 2^{<\omega}$ and $f(\nu) \notin \mathcal{F}^*_{lg(\nu)}$. In this case, choose the $<_*$-least witness ν_n for every n. There is an infinite set $A \subseteq \omega$ such that $(lg(\nu_n \cap \eta_f) : n \in A)$ is strictly increasing. Let $g = (f \restriction \{\nu_n : n \in A\}) \cup (F_1(f) \restriction (2^{<\omega} \setminus A))$, it's straightforward to verify that $g \in G_2$ (by the first possibility in Definition 2(m)(2)) and $f = f_g$, which is a contradiction.

Case II: Case I fails, but there are $A \in [\omega]^\omega$ and $\bar{\nu} = (\nu_n : n \in A)$ such that $\eta_f \restriction n \leq \nu_n$, $lg(\nu_n \cap \eta_f) = n$ and $(f(\nu_n) : n \in A)$ are pairwise incomparable. In this case, we shall derive a contradiction as in the previous case (using the second possibility in Definition 2(m)(2)). Note that if n exemplifies the failure of case I, then as $(f(\nu_m) : n \leq m \in A)$ are pairwise incomparable, there is at most one $n \leq m \in A$ such that $f(\nu_m) = F_1(f)(\nu_m)$. If $n \leq n_*$ and $f(\nu_m) \neq F_1(f)(\nu_m)$ for every $n_* \leq m \in A$, then we define g as in the previous case, with $\{\nu_m : n_* \leq m \in A\}$ here instead of $\{\nu_n : n \in A\}$ there, and we get a contradiction similarly.

Case III: ¬Case I∧¬Case II. We shall prove the following statement:

$(*)$ There are n_*, k^* and $f_0, ..., f_{k_*} \in \mathcal{F}_*$ such that $\eta_f \restriction n_* \leq \nu \to f(\nu) \in \{f_0 \restriction 2^{<lg(\nu)}, ..., f_{k_*-1} \restriction 2^{<lg(\nu)}\}$.

In order to prove $(*)$, assume that it fails and we shall derive a contradiction to the assumptions of case III.

Let n_1 witness the failure of case I, we choose by induction on k a triple $(\bar{\eta_k}, A_k, f_k)$ such that:

a. $\bar{\eta_k} = (\eta_{k,n} : n_1 \leq n \in A_k)$.

b. $\eta_f \restriction n \leq \eta_{k,n}$ but $\eta_f \restriction (n+1) \nleq \eta_{k,n}$.

c. $f(\eta_{k,n}) \notin \{f_l \restriction 2^{<lg(\eta_{k,n})} : l < k\}$.

d. $A_k \subseteq \omega$ is infinite and $(f(\eta_{k,n}) : n_1 \leq n \in A_k)$ is $\subseteq$-increasing.

e. $f_k \in \mathcal{F}_*$ and $f_k = \bigcup_{n_1 \leq n} f(\eta_{k,n})$.

Why can we carry the induction? At stage k, let $A^1_k = \{n : n_1 \leq n$ and there is $\eta_{k,n}$ satisfying (b)+(c)$\}$. If A^1_k is finite, then letting $n_* = max(A^1_k) + 1$, $(n_*, k-1, f_0, ..., f_{k-1})$ are as required in the above statement $(*)$, contradicting the assumption that $(*)$ fails. If A^1_k is infinite, we can choose for every $n \in A^1_k$ an $\eta_{k,n}$ satisfying (b)+(c) (for example, by taking the $<_*$-minimal such sequence), by Ramsey's theorem there is an infinite $A_k \subseteq A^1_k$ such that $(f(\eta_{k,n}) : n \in A_k)$ is either $\subseteq$-increasing, $\subseteq$-decreasing or pairwise incomparable (note that we don't need any form of the axiom of choice here, as we can carry the argument in a model of the form $L[X]$). If the elements of $\{f(\eta_{k,n}) : k \in A_n\}$ are pairwise incomparable, let $w = \{\eta_{k,n} : n \in A_k\}$ and $g = (f \restriction w) \cup (F_1(f) \restriction (2^{<\omega} \setminus w))$. It's straightforward to verify that $g \in G_2$ and $f_g = f$ (note that by the pairwise incomparability of the $f(\eta_{k,n})$s, there is at most one $\eta_{k,n}$ for which $f(\eta_{k,n}) = F_1(f)(\eta_{k,n})$). Therefore, $f \in H_3$, contradicting our assumption. By the choice of n_1, the sequence $(f(\eta_{k,n}) : n \in A_k)$ can't be $\subseteq$-decreasing, therefore, it's $\subseteq$-increasing. Let $f_k = \cup\{f(\eta_{k,n}) : n \in A_k\}$, then $f_k \in \mathcal{F}_*$ and $n \in A_k \to f(\eta_{k,n}) = F_1(f_k)(\eta_{k,n})$, so we've carried the induction.

We shall now get a contradiction by showing that the assumptions of case II hold: Note that $k_1 \neq k_2 \to f_{k_1} \neq f_{k_2}$ (by clauses (c) and (e)). Let $B_0 = \omega$, choose l_0 such that $f_0 \restriction 2^{\leq l_0} \neq f_1 \restriction 2^{\leq l_0}$. Therefore, there are $h_0 \in \{0,1\}$ and an infinite set $B_1 \subseteq \omega \setminus \{0,1\}$ such that $\bigwedge_{k \in B_1} f_k \restriction 2^{\leq l_0} \neq f_{h_0} \restriction 2^{\leq l_0}$. Now choose $i_{1,0} \neq i_{1,1} \in B_1$ and l_1 such that $f_{i_{1,0}} \restriction 2^{\leq l_1} \neq f_{i_{1,1}} \restriction 2^{\leq l_1}$. As before, there are $h_1 \in \{0,1\}$ and an infinite set $B_2 \subseteq B_1 \setminus (i_{1,0} + i_{1,1})$ such that $\bigwedge_{k \in B_2} f_k \restriction 2^{\leq l_1} \neq f_{i_{1,h_1}} \restriction 2^{\leq l_1}$. We continue as above and obtain the sets $B = \{h_0 < i_{1,h_1} < i_{2,h_2} < ...\}$, $(B_n : n < \omega)$, $((i_{m,0}, i_{m,1}) : m < \omega)$ and $(l_m : m < \omega)$. For every $k \in B$, if $k = i_{m,h_m}$, choose $n_k \in A_k$ such that $max\{l_m, n_{k-1}\} < n_k$ and let $\nu_{n_k} = \eta_{k,n_k}$ and $A = \{n_k : k \in B\}$. It's now easy to verify that A and $(\nu_{n_k} : k \in B)$ satisfy the assumptions of case II, but we shall elaborate: We shall prove that $f(\nu_{k_1}) = f(\eta_{k_1,n_{k_1}}) = f_{k_1} \restriction 2^{\leq lg(\eta_{k_1,n_{k_1}})}$ and $f(\nu_{k_2}) = f(\eta_{k_2,n_{k_2}}) = f_{k_2} \restriction 2^{\leq lg(\eta_{k_2,n_{k_2}})}$ are incomparable for $k_1 \neq k_2 \in B$. Suppose that $k_1 = i_{m,h_m}$ and $k_2 = i_{j,h_j}$ and wlog $m < j$, then $f_{k_1} \restriction 2^{\leq l_m} \neq f_{k_2} \restriction 2^{\leq l_m}$, therefore $f_{k_1} \restriction 2^{\leq lg(\eta_{k_1,n_{k_1}})} \neq f_{k_2} \restriction 2^{\leq lg(\eta_{k_1,n_{k_1}})}$ and $f_{k_1} \restriction 2^{\leq lg(\eta_{k_2,n_{k_2}})} \neq f_{k_2} \restriction 2^{\leq lg(\eta_{k_2,n_{k_2}})}$, and therefore $f(\nu_{k_1})$ and $f(\nu_{k_2})$ are incomparable. This completes the proof of $(*)$.

Now let $n_*, k_*, f_0, ..., f_{k_*-1}$ be as in $(*)$, then for every $n \geq n_1$, there is $l_n < k_*$ such that the set $Y_n = \{\rho \in 2^{<\omega} : \eta_f \restriction n \leq \rho, \eta_f(n) \neq \rho(n)$ and $f(\rho) = F_1(f_{l_n})(\rho)\}$ is infinite. Choose $l_* < k_*$ such that $B = \{n : n_1 \leq n, l_n = l_*\}$ is infinite.

Subcase I: $f_{l_*} \notin H_3$. If $n \in B$ and $\rho \in Y_n$, then $f(\rho) = F_1(f_{l_n})(\rho) = F_1(f_{l_*})(\rho)$, therefore, $eq(f, F_1(f_{l_*}))$ is infinite. As $f_{l_*} \notin H_3$, $F_1(f_{l_*}) \in G_4$ (by the definition of G_4). Therefore, we've found $g \in G_4$ such that $eq(f,g)$ is infinite and we're done.

Subcase II: $f_{l_*} \in H_3$. For each $n \in B$, Y_n is infinite, therefore we can find $\rho_n \in Y_n \setminus w_{F^*_3(f_{l_*})}$ (by the definition of G_2, $\{\rho \in w_{F^*_3(f_{l_*})} : \rho \cap \eta_f = \eta_f \restriction n\}$ is finite, and as $Y_n \subseteq \{\rho \in 2^{<\omega} : \rho \cap \eta_f = \eta_f \restriction n\}$ is infinite, there is ρ_n as required).

As $f_{l_*} \in H_3$, $f_{l_*} = f_g$ for some $g \in G_2$, and $F_3^*(f_{l_*}) = g$, hence $F_3^*(f_{l_*})(\rho_n) = g(\rho_n) = F_1(f_{l_*})(\rho_n) = f(\rho_n)$ (the equalities follow from the definitions of F_1, F_3^* and Y_n, and the assumption that $\rho_n \notin w_{F_3^*(f_{l_*})}$). Therefore, $eq(F_3^*(f_{l_*}), f)$ is infinite, and by the definition of G_4, $F_3^*(f_{l_*}) \in G_4$ so we're done. $\square$

Proof of Claim 6. For $f \in \mathcal{F}_*$, let $\eta_f = \mathbf{B}(f)$ and let $TV_*(f)$ be the truth value of the statement:

$(*)$ For every $n < \omega$ there exists $\nu \in 2^{<\omega}$ such that $\eta_f \restriction n \leq \nu$ and $f(\nu) \notin \mathcal{F}^*_{lg(\nu)}$.

Note that $TV_*(f)$ is Borel-computable and so are the truth values $TV_{2,k,i}(f)$ and $TV_{3,j}(f)$ (to be defined later), therefore, it suffices to define F_3^* separately for each combination of truth values.

Case I: $TV_*(f) = true$**.** In this case, we shall prove that $f \in H_3$ and define $F_3^*(f)$:

Let A_f be the set of n for which there is $\nu \in 2^{<\omega}$ such that $\nu \cap \eta_f = \eta_f \restriction n$ and $f(\nu) \notin \mathcal{F}^*_{lg(\nu)}$. By the assumption, A_f is infinite.

For each $n \in A_f$, let $\nu_{f,n}$ be a sequence for which $(*)$ is true, such that:

1. $lg(\nu_{f,n})$ is minimal.
2. $\nu_{f,n}$ is $<_*$-minimal among the sequence satisfying (1) (where $<_*$ is the lexicographic ordering).

Let $w_f = \{\nu_{f,n} : n \in A_f\}$ and let $F_3^*(f) = f \restriction w_f \cup F_1(f) \restriction (2^{<\omega} \setminus w_f)$. It's straightforward to verify that $F_3^*(f) \in G_2$ and that $f_{F_3^*(f)} = f$, therefore $f \in H_3$.

Case II: $TV_*(f) = false$**.** We can compute $m(f) = min\{m$: If $\eta_f \restriction m \leq \nu \in 2^{<\omega}$ then $f(\nu) \in \mathcal{F}^*_{lg(\nu)}\}$. Let $TV_{2,k,i}(f)$ be the truth value of the following statement:

$(*)_{2,k,i}$ There exist k and $f_0, ..., f_{k-1} \in \mathcal{F}_*$ such that for every $\nu \in 2^{<\omega}$, $\eta_f \restriction i \leq \nu \to f(\nu) \in \{F_1(f_l)(\nu) : l < k\}$.

By compactness, $(*)_{2,k,i}$ holds iff for every finite $u \subseteq \{\nu : \eta_f \restriction i \leq \nu \in 2^{<\omega}\}$ there exist $f_0, ..., f_{k-1}$ as above with domain $2^{<lg(u)+1}$ where $lg(u) = max\{lg(\nu) : \nu \in u\}$. Therefore, $TV_{2,k,i}(f)$ is Borel-computable. Note that there is no essential use of the axiom of choice in the compactness argument, as we can argue in an appropriate $L[X]$.

Note that if $TV_{2,k,i}(f) = true$ for some k and i, then $f \notin H_3$: Let $f_0, ..., f_{k-1}$ be as in $(*)_{2,k,i}$ and suppose towards contradiction that there exists $g \in G_2$ such that $f = f_g$. Let $(\nu_n : n \in A)$ list w_g, then one of the two possibilities in 2(m)(2) holds. As $TV_*(f) = false$, the first possibility of 2(m)(2) fails. Suppose that the second possibility holds. By 2(j)(1), for every $n \in A \setminus i$ there is $m(n) \in A \setminus i$ such that $\mathbf{B}(f) \restriction n \leq \nu_{m(n)}$. As $TV_{2,k,i}(f) = true$, for every such $n \in A \setminus i$, there exists $l < k$ such that $f(\nu_{m(n)}) = F_1(f_l)(\nu_{m(n)})$. Therefore, for some $l_* < k$, the set $B := \{n \in A \setminus i : f(\nu_{m(n)}) = F_1(f_{l_*})(\nu_{m(n)})\}$ is infinite. It follows that the elements of $(f(\nu_{m(n)}) : n \in B) = (g(\nu_{m(n)}) : n \in B)$ are pairwise comparable, contradicting the second possibility of 2(m)(2). It follows that $f \notin H_3$. If $TV_{2,k,i}(f) = true$ for some k, i, we let $F_3^*(f) = f_0$ where $f_0(\nu)$ is defined as the constant function 0 whose domain is $2^{<lg(\nu)}$. It's easy to see that $f_0 \notin G_2$.

From now on, we assume that $TV_*(f) = false$ and $\underset{k,i}{\wedge} TV_{2,k,i}(f) = false$. As in the proof of Claim 8(c) (case III), we shall choose by induction A_k, h_k and $\bar{\eta}_k = (\eta_{k,n} : n_1 \leq n \in A_k)$ (where $n_1 = m(f)$ is witnessing the failure of the statement of case I in the proof of 8(c), and h_k here stand for f_k there) such that:

a. $A_k \subseteq \omega$ is infinite.

b. $\eta_f \restriction n \leq \eta_{k,n}$ and $\eta_f \restriction (n+1) \nleq \eta_{k,n}$.

c. $(f(\eta_{k,n}) : n_1 \leq n \in A_k)$ is $\subseteq$-increasing.

d. $h_k = \underset{n \in A_k}{\cup} f(\eta_{k,n}) \in \mathcal{F}_*$.

e. $f(\eta_{k,n}) \notin \{F_1(h_l)(\eta_{k,n}) : l < k\}$.

Moreover, the objects will be computed in a Borel way, The only non-trivial point is the application of Ramsey's theorem in the construction of A_k from A_k^1 (i.e. why can we Borel-compute an infinite homogeneous set?): Given a function $R : [\omega]^2 \to \{0, 1\}$, we shall Borel-compute an infinite homogeneous set (we shall write $R(m, k)$ for $R(\{m, k\})$ where $m < k$). Define $\rho_n \in 2^n$ by induction on n such that:

a. $\rho_n \leq \rho_{n+1}$.

b. For infinitely many $k < \omega$, $R(m, k) = \rho_{n+1}(n)$ for every $m < n + 1$. Let A_n be the set of these k's.

c. $A_{n+1} \subseteq A_n$.

d. $\rho_{n+1}(n) = 0$ if possible (i.e. if the above requirements are satisfied).

The sequence $(\rho_n : n < \omega)$ can be Borel-computed. Now choose $n_i \in \omega$ by induction such that:

a. $n_i < n_{i+1}$.

b. n_i is the minimal $k \in A_{n_{i-1}}$ such that $\bigwedge\limits_{j<i} n_j < k$ and $R(n_j, k) = \rho_{n_j+1}(n_j)$ (this is possible by the choice of the ρ_ns).

So $(n_i : i < \omega)$ is Borel-computable as well. If $i_1 < i_2 < i_3$ then $R(n_{i_1}, n_{i_2}) = \rho_{n_{i_1}+1}(n_{i_1}) = R(n_{i_1}, n_{i_3})$. Let $i(*) \in \{0, 1\}$ be the minimal such that $\{i : R(n_i, n_{i+1}) = i(*)\}$ is infinite (this is Borel-computable as well). Finally, the set $\{n_i : R(n_i, n_{i+1}) = i(*)\}$ is a Borel-computable infinite homogeneous set. This completes the argument on the induction.

Let $TV_{3,j}(f)$ be the truth value of the following statement (which is Borel-computable):

$(*)_{3,j}$ The set $\{h_k \restriction 2^{<j} : k < \omega\}$ is infinite.

Case I: $\bigvee\limits_{j<\omega} TV_{3,j}(f) = true$**.** In this case we can Borel compute $F_3^*(f)$ witnessing that $f \in H_3$. Let j_f be the minimal j such that $TV_{3,j}(f) = true$ (this is Borel-computable) and let $B = \{k : j_f < k, h_k \restriction 2^{<j_f} \notin \{h_l \restriction 2^{<j_f} : l < k\}\}$, this set is infinite by our assumption. We choose $n_k \in A_k \setminus j_f$ by induction on $k \in B$ such that $k < k' \to n_k < n_{k'}$. Let $w_f = \{\eta_{k,n_k} : k \in B\}$ and let $F_3^*(f) = f \restriction w_f \cup (F_1(f) \restriction 2^{<\omega} \setminus w_f)$. It's easy to check that $F_3^*(f) \in G_2$ is witnessed by f (hence $f \in H_3$): If there are $k \neq k' \in B$ such that $F_3^*(f)(\eta_{k,n_k}) = F_1(f)(\eta_{k,n_k})$ and $F_3^*(f)(\eta_{k',n_{k'}}) = F_1(f)(\eta_{k',n_{k'}})$, then $f(\eta_{k,n_k}) = h_k \restriction 2^{<lg(\eta_{k,n_k})}$ and $f(\eta_{k',n_{k'}}) = h_{k'} \restriction 2^{<lg(\eta_{k',n_{k'}})}$ are comparable, contradicting the definition of B. It's easy to check that the other requirements in the definition of G_2 are satisfied as well.

Case II: $\bigwedge\limits_{j<\omega} TV_{3,j}(f) = false$**.** We can Borel-compute a set $B \in [\omega]^\omega$ such that $(h_j : k \in B)$ converges to some $h^* \in \mathcal{F}_*$: Let $B_0 = B_1 = \omega$. As $TV_{3,2}(f) = false$, there exists $k(2) > 2$ such that for infinitely many k, $h_{k(2)} \restriction 2^{<2} = h_k \restriction 2^{<2}$. Choose $k(2)$ to be the minimal number with the above property and let $B_2 = \{k \in B_1 : k > k(2)$ and $h_{k(2)} \restriction 2^{<2} = h_k \restriction 2^{<2}\}$. As $TV_{3,3}(f) = false$, there is a minimal $k(3) \in B_2$ such that $h_{k(3)} \restriction 2^{<3} = h_k \restriction 2^{<3}$ for infinitely many $k \in B_2$, let $B_3 = \{k \in B_2 : k > k(3)$ and $h_{k(3)} \restriction 2^{<3} = h_k \restriction 2^{<3}\}$. We now continue the construction by induction, and obtain the set $B = \{k(2) < k(3) < k(4) < ...\}$. Now let $h^* = \bigcup\limits_{n<\omega} h_{k(n)} \restriction 2^{<n}$, it's easy to see that B and h^* are as required. Note that as $l \neq k \to h_l \neq h_k$ (by the definition of the h_ks), there is at most one k such that $h_k = h^*$.

We can Borel-compute (k_i, n_i, m_i) by induction on i such that:

1. $k_i \in B$ is increasing with i.
2. $m(f) \leq n_i \in A_{k_i}$ is increasing with i.
3. $m_i = lg(\eta_{k_i,n_i})$.
4. $f(\eta_{k_i,n_i}) \nsubseteq h^*$.
5. If $j < i$ then $f(\eta_{k_i,n_i}) \restriction 2^{<m_j} \subseteq h^*$.

The induction step: Suppose that we've carried the induction up to i and let $j = i - 1$. Let $m_{i(*)} = max\{m_l : l < i\}$. As $\lim\limits_{n\in B} h_n = h^*$, for every $n \in B$ large enough (say, $n_* \leq n$ for some n_*) we have $h_n \restriction 2^{<m_{i(*)}} \subseteq h^*$. Let $k_i \in B$ be the first such n above k_j such that, in addition, $h_{k_i} \neq h^*$ (recall that there is at most one n for which $h_n = h^*$). Recall that $h_{k_i} = \bigcup\limits_{m(f)\leq n\in A_{k_i}} f(\eta_{k_i,n})$, and for $n_1 = m(f) \leq n \in A_{k_i}$ large enough, $h_{k_i} \restriction 2^{<n} \nsubseteq h^*$ (otherwise, $h_{k_i} = h^*$, which is a contradiction).

Let $n_i \in A_{k_i} \setminus n_1$ be the first such n above n_j, and let $m_i = lg(\eta_{k_i,n_i})$, so we have carried the induction successfully. Now let $w_f = \{\eta_{k_i,n_i} : i < \omega\}$ and let $F_3^*(f) = f \restriction w_f \cup (F_1(f) \restriction 2^{<\omega} \setminus w_f)$. It's easy to check that $F_3^*(f) \in G_2$ as witnessed by f, which belongs to H_3. $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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