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Some variations on the splitting number [☆,☆☆]

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ABSTRACT

Variations on the splitting number $\mathfrak{s}$ are examined by localizing the splitting property to finite sets. To be more precise, rather than considering families of subsets of the integers that have the property that every infinite set is split into two infinite sets by some member of the family a stronger property is considered: Whenever an subset of the integers is represented as the disjoint union of a family of finite sets one can ask that each of the finite sets is split into two non-empty pieces by some member of the family. It will be shown that restricting the size of the finite sets can result in distinguishable properties. In §2 some inequalities will be established, while in §3 the main consistency result will be proved.

1. Introduction

While not included in the Cichon diagram, the cardinal invariant $\mathfrak{s}$, the splitting number, has been the source of considerable interest. Any of the surveys of cardinal invariants — such as [3], [8] or [2] — will provide ample justification for this assertion. Of course, $\mathfrak{s}$ is defined to be the least cardinal of a family $\mathcal{S}$ of infinite subsets of ω such that for any infinite $X \subseteq \omega$ there is $S \in \mathcal{S}$ such that $|S \cap X| = |X \setminus S|$. The article [6] introduces a modification of the splitting number obtained by what can be considered a localization of the concept. The authors of [6] define the pair splitting number $\mathfrak{s}_{\text{pair}}$ to be the least cardinal of a family $\mathcal{S}$ of subsets of ω such that for any infinite, pairwise disjoint family of pairs $X \subseteq [\omega]^2$ there is $S \in \mathcal{S}$ such that $|S \cap x| = |x \setminus S|$ for infinitely many $x \in X$. The authors establish connections between $\mathfrak{s}_{\text{pair}}$ and well known cardinal invariants of the continuum, as well as with the covering number of the finite chromatic ideal consisting of graphs, considered as sets of pairs of integers, with finite chromatic number.

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The present work will continue these explorations by expanding the definitions of [6] beyond pairs. It was already shown in [6] that generalizing $\mathfrak{s}_{\text{pair}}$ to, for example, $\mathfrak{s}_{\text{triple}}$ in the obvious way does not create a new concept. The goal of the research to be presented here is that generalizing from pairs to finite sets does introduce a new concept. Generalizing splitting to what may be called balanced splitting has been examined in [4]. Some connections between the research under consideration here and that of [4] will also be established.

2. Definitions and basic results

This section will introduce some cardinal invariants very similar to those introduced in [6]. Indeed, they are so similar that it will be shown they are, in fact, the same. It was mentioned in the introduction that it was shown in [6] that if $\mathfrak{s}_n$ is defined to be the least cardinal of a family $\mathcal{S}$ of subsets of ω such that for any infinite, pairwise disjoint family of pairs $X \subseteq [\omega]^n$ there is $S \in \mathcal{S}$ such that $S \cap x \neq \varnothing \neq x \setminus S$ for infinitely many $x \in X$ then $\mathfrak{s}_{\text{pair}} = \mathfrak{s}_{\text{n}}$. It will be shown that the same holds if one considers splitting n-sized sets into k pieces.

Notation 2.1. The reader unwilling to part with von Neumann's definition of ordinals is warned that if F is any function and x a subset of its domain then $F(x)$ will be used to denote the image of x under F; this will be used even when there is a slight danger that some confusion between an ordinals thought of as a point and a set may arise. For a function $f : \omega \to \omega$ define $\mathfrak{s}_{k,f}$ to be the least cardinal λ such that there is a family $\mathcal{F} \subseteq k^\omega$ of cardinality λ such that for each sequence of pairwise disjoint sets of integers $\{a_n\}_{n\in\omega}$ such that $|a_n| = f(n)$ there is $F \in \mathcal{F}$ such that $F(a_n) = k$ for infinitely many n. The notation $\mathfrak{s}_{k,m}$ will be used to denote $\mathfrak{s}_{k,f}$ when f is constant with value m.

Lemma 2.1. *If $f \leq^* g$ then $\mathfrak{s}_{k,g} \leq \mathfrak{s}_{k,f}$.*

Proof. Let $\mathcal{F} \subseteq k^\omega$ be such that $|\mathcal{F}| = \mathfrak{s}_{k,f}$ and for each sequence of pairwise disjoint sets $\{a_n\}_{n\in\omega}$ such that $|a_n| = f(n)$ there is $F \in \mathcal{F}$ such that $F(a_n) = k$ for infinitely many n. Given $\{b_n\}_{n\in\omega}$ such that $|b_n| = g(n)$ let $b_n^* \subseteq b_n$ be such that $|b_n^*| = f(n)$ and $F \in \mathcal{F}$ such that $F(b_n^*) = k$ for infinitely many n. Clearly it is also true that $F(b_n) = k$. □

Lemma 2.2. *If*

(a) f, g and h are functions from ω from ω
(b) h is increasing
(c) $g(n) = f(h(n))$ for all n

then $\mathfrak{s}_{k,f} \leq \mathfrak{s}_{k,g}$.

Proof. Suppose that $|\mathcal{F}| = \mathfrak{s}_{k,g}$ and for each sequence of pairwise disjoint sets $\{a_n\}_{n\in\omega}$ such that $|a_n| = g(n)$ there is $F \in \mathcal{F}$ such that $F(a_n) = k$ for infinitely many n. Then if $\{b_n\}_{n\in\omega}$ is sequence of pairwise disjoint sets such that $|b_n| = f(n)$ then $\{b_{h(n)}\}_{n\in\omega}$ is sequence of pairwise disjoint sets such that $|b_{h(n)}| = g(n)$ and so there is $F \in \mathcal{F}$ such that $F(b_{h(n)}) = k$ for infinitely many n and hence $F(b_n) = k$ for infinitely many n. □

Theorem 2.1. *If f and g are unbounded functions from ω to ω then $\mathfrak{s}_{k,g} = \mathfrak{s}_{k,f}$.*

Proof. Since f and g are both unbounded, it is possible to find an increasing h and a function e such that $f(n) < e(n)$ and such that $e(n) = g(h(n))$ for all n. By Lemma 2.1 it follows that $\mathfrak{s}_{k,e} \leq \mathfrak{s}_{k,f}$ and by

Lemma 2.2 it follows that $\mathfrak{s}_{k,g} \leq \mathfrak{s}_{k,e}$. Hence $\mathfrak{s}_{k,g} \leq \mathfrak{s}_{k,f}$. The symmetry of the hypothesis implies that $\mathfrak{s}_{k,g} = \mathfrak{s}_{k,f}$. $\square$

Theorem 2.1 justifies the following definition.

Definition 2.1. $\mathfrak{s}_{k,\infty}$ will be used to denote $\mathfrak{s}_{k,f}$ when f is any unbounded function.

Note that if f is any bounded function then a simple re-indexing argument shows that $\mathfrak{s}_{k,f} = \mathfrak{s}_{k,m}$ where $m = \limsup_n f(n)$. Hence the cardinals $\mathfrak{s}_{k,f}$ can be replaced by the cardinals $\mathfrak{s}_{k,\infty}$ and $\mathfrak{s}_{k,m}$ for $m \in \omega$. Some simple relationships between these cardinals are easily established.

Lemma 2.3. $\mathfrak{s}_{2,m} = \mathfrak{s}_{2,m^2}$ *for all* $m \geq 2$.

Proof. By Lemma 2.1 it follows that $\mathfrak{s}_{2,m} \geq \mathfrak{s}_{2,m^2}$ for all $m \geq 2$. Now suppose that $|\mathcal{F}| = \mathfrak{s}_{2,m^2}$ and for each sequence of pairwise disjoint sets $\{a_n\}_{n\in\omega}$ such that $|a_n| = m^2$ there is $F \in \mathcal{F}$ such that $F(a_n) = 2$ for infinitely many n. If $\mathfrak{s}_{2,m} > \mathfrak{s}_{2,m^2}$ then there is a sequence of pairwise disjoint sets $\{b_n\}_{n\in\omega}$ such that $|b_n| = m$ and for each $F \in \mathcal{F}$ there is $F^* : \omega \to 2$ such that for all but finitely many $n \in \omega$ the restriction of F to b_n has constant value $F^*(n)$.

Since $|\{F^* \mid F \in \mathcal{F}\}| < \mathfrak{s}_{2,m}$ there is a sequence of pairwise disjoint sets $\{c_n\}_{n\in\omega}$ such that $|c_n| = m$ for each n and such that for each $F \in \mathcal{F}$ for all but finitely many $n \in \omega$ the restriction of F^* to c_n has constant value. Then let $d_n = \bigcup_{m\in c_n} b_m$ and note that the d_n are pairwise disjoint elements of $[\omega]^{m^2}$ and F is eventually constant on each d_n. This contradicts the choice of $\mathcal{F}$. $\square$

Corollary 2.1. $\mathfrak{s}_{2,m} = \mathfrak{s}_{2,k}$ *for all* $m, k \geq 2$.

Lemma 2.4. $\mathfrak{s}_{2,2} \geq \mathfrak{s}_{m,m}$ *for all* $m \geq 2$.

Proof. Let $\mathcal{F} \subseteq 2^\omega$ be a family such that $|\mathcal{F}| = \mathfrak{s}_{2,2}$ and for each sequence of pairwise disjoint sets $\{a_n\}_{n\in\omega}$ such that $|a_n| = 2$ there is $F \in \mathcal{F}$ such that $F(a_n) = 2$ for infinitely many n. For any indexed family $\vec{F} = \{F_i\}_{i\in k} \subseteq \mathcal{F}$ and $\vec{\sigma} = \{\sigma_j\}_{j\in m}$ a family of distinct elements of 2^k define a partial function $H_{\vec{F},\vec{\sigma}} \in m^\omega$ by letting

$$H_{\vec{F},\vec{\sigma}}(n) = j \text{ if } (\forall i \in k)\ \sigma_j(i) = F_i(n)$$

and letting $H_{\vec{F},\vec{\sigma}}(n)$ be undefined otherwise. Then let

$$\mathcal{F}^* = \left\{ H_{\vec{F},\vec{\sigma}} \;\middle|\; \vec{F} = \{F_i\}_{i\in k} \subseteq \mathcal{F} \text{ and } \vec{\sigma} = \{\sigma_j\}_{j\in m} \text{ are distinct elements of } 2^k \right\}$$

and note that $|\mathcal{F}^*| = \mathfrak{s}_{2,2}$.

Now suppose that $\{a_n\}_{n\in\omega}$ are pairwise disjoint elements of $[\omega]^m$. Let $\vec{F} = \{F_i\}_{i\in k} \subseteq \mathcal{F}$ be such that if

$$\mathcal{S}_n = \left\{ a_n \cap \bigcap_{i\in k} F_i^{-1}\{\sigma(i)\} \right\}_{\sigma\in 2^k}$$

then $\limsup_n |\mathcal{S}_n|$ is maximal. Note that this means that $\lim_n |\mathcal{S}_n| = m$ because if there are infinitely many $b_n \in \mathcal{S}_n$ such that $|b_n| \geq 2$ then there is then $F \in \mathcal{F}$ such that $F(b_n) = 2$ for infinitely many n contradicting the maximality of $\limsup_n |\mathcal{S}_n|$. For all but finitely many n there are $\vec{\sigma}_n = \{\sigma_{n,i}\}_{i\in m} \subseteq 2^k$ such that for each $j \in a_n$ there is $i \in m$ such that

$$\{j\} = \bigcap_{\ell \in k} F_\ell^{-1}\{\sigma_{n,i}(\ell)\}.$$

Let $\vec{\sigma}$ be such that $\vec{\sigma}_n = \vec{\sigma}$ for infinitely many n. Then for each such n it follows that $H_{\vec{F},\vec{\sigma}}(a_n) = m$. □

The following is a refinement of Theorem 1.3 of [6] due to Kamo.

Corollary 2.2. *If* $2 \leq m \leq k < \omega$ *then* $\mathfrak{s}_{m,k} = \mathfrak{s}_{2,2}$.

Proof. From Lemma 2.4 and Corollary 2.1 it follows that $\mathfrak{s}_{2,2} \geq \mathfrak{s}_{m,m} \geq \mathfrak{s}_{m,k} \geq \mathfrak{s}_{2,k} = \mathfrak{s}_{2,2}$. □

Hence the only question that remains to be addressed is whether $\mathfrak{s}_{2,\infty} = \mathfrak{s}_{2,2}$. It will be shown in the next section that it is consistent for these cardinals to be different. But it is worth pointing out a connection to cardinals that have been studied elsewhere. The following is Definition 2.2 from [4].

Definition 2.2 *([4]).* If S and X are infinite subsets of ω say that S *bisects* X *in the limit* if

$$\lim_{n\to\infty} \frac{|S \cap X \cap n|}{|X \cap n|} = 1/2$$

and for ϵ such that $0 < \epsilon < 1/2$ say that S *ϵ-almost bisects* X if for all but finitely many $n \in \omega$

$$\frac{|S \cap X \cap n|}{|X \cap n|} \in (1/2 - \epsilon, 1/2 + \epsilon).$$

Then define $\mathfrak{s}_{1/2}$ to be the least cardinal of a family $\mathcal{S}$ such that for all $X \in [\omega]^{\aleph_0}$ there is an element of $\mathcal{S}$ that bisects X. Define $\mathfrak{s}_{1/2\pm\epsilon}$ to be the least cardinal of a family $\mathcal{S}$ such that for all $X \in [\omega]^{\aleph_0}$ there is an element of $\mathcal{S}$ that ϵ-almost bisects X.

Proposition 2.1. $\mathfrak{s}_{2,\infty} \leq \mathfrak{s}_{1/2\pm\epsilon}$ *if* $0 < \epsilon < 1/2$.

Proof. Let $\mathcal{S}$ be a family of cardinality $\mathfrak{s}_{1/2\pm\epsilon}$ such that for all $X \in [\omega]^{\aleph_0}$ there is an element of $\mathcal{S}$ that ϵ-almost bisects X. It suffices to show that if $\{a_n\}_{n\in\omega}$ is any family of pairwise disjoint finite sets such that $|a_n| > \sum_{i\in n} |a_i|(1/2 + \epsilon)$ then there is $S \in \mathcal{S}$ such that

$$(\exists^\infty n)\ a_n \cap S \neq \varnothing \neq a_n \setminus S. \tag{1}$$

Given such a family $\{a_n\}_{n\in\omega}$ let $A = \bigcup_n a_n$ and let $S \in \mathcal{S}$ be such that for all but finitely many $n \in \omega$

$$\frac{|S \cap A \cap n|}{|A \cap n|} \in (1/2 - \epsilon, 1/2 + \epsilon).$$

If (1) fails it can be assumed that there are infinitely many n such that $a_n \subseteq S$. But for any such n if $m = \max(a_n)$ then

$$\frac{|S \cap A \cap m|}{|A \cap n|} \geq \frac{|a_n|}{\sum_{i\in n} |a_i|} > 1/2 + \epsilon. \quad \square$$

Of course, $\mathfrak{s} \leq \mathfrak{s}_{2,\infty} \leq \mathfrak{s}_{1/2\pm\epsilon} \leq \mathfrak{s}_{1/2}$ and in Theorem 2.4 of [4] it is shown that $\mathfrak{s}_{1/2}$ is no greater than $\mathbf{non}(\mathcal{N})$, the least cardinal of a non-Lebesgue null set. A companion to this is the following, which is one of various inequalities established in Proposition 0.1 of [6].

Proposition 2.2. $\mathfrak{s}_{2,2} \leq \mathbf{non}(\mathcal{N})$.

The natural question about possible equality is easily answered by the following. The inequality $\mathfrak{s}_{2,\infty} < \mathfrak{s}_{2,2}$ is much harder and is the main result to be established in the current work.

Proposition 2.3. *It is consistent that* $\mathfrak{s}_{2,2} \neq \mathbf{non}(\mathcal{N})$.

Proof. Since $\mathbf{non}(\mathcal{N}) = \aleph_2$ after adding $\aleph_2$ Cohen reals, it suffices to show that if $\mathbb{C}$ is Cohen forcing and

$$1 \Vdash_{\mathbb{C}} \text{“}\{\dot{a}_n\}_{n\in\omega} \text{ are pairwise disjoint pairs ”}$$

then there is $F : \omega \to 2$ such that $1 \Vdash_{\mathbb{C}}$ “$(\exists^\infty n)\ F(\dot{a}_n) = 2$”. Now apply the argument that Cohen forcing does not add a dominating real. For the reader who would appreciate the details, let $\{p_n\}_{n\in\omega}$ enumerate $\mathbb{C}$. Construct inductively $\{b_{n,j}\}_{j\in n}$ such that

(a) $b_{n,j} \cap b_{m,i} = \varnothing$ unless $(n, j) = (m, i)$
(b) there is some $q_{n,j} \leq p_j$ such that $q_{n,j} \Vdash_{\mathbb{C}}$ “$\dot{a}_\ell = b_{n,j}$” for some ℓ.

To carry out the construction suppose that $\{b_{n,j}\}_{j\in n}$ and $\{b_{n+1,i}\}_{i\in k}$ have been constructed for some $k \in n + 1$. Let

$$B = \bigcup_{m\leq n} \bigcup_{j\in m} b_{m,j} \cup \bigcup_{m\leq k} b_{n+1,m}.$$

There is then some $q \leq p_k$ and $\ell > n + 1$ such that $q \Vdash_{\mathbb{C}}$ “$\dot{a}_\ell \cap B = \varnothing$”. Let $q_{n+1,k} \leq q$ and $b_{n+1,k}$ be such that $q_{n+1,k} \Vdash_{\mathbb{C}}$ “$\dot{a}_\ell = b_{n+1,k}$”.

Now let $F : \omega \to 2$ be such that $F(b_{n,j}) = 2$ for all n and j. To see that $1 \Vdash_{\mathbb{C}}$ “$(\exists^\infty n)\ F(\dot{a}_n) = 2$” suppose that there are p and k such that $p \Vdash_{\mathbb{C}}$ “$(\forall n \geq k)\ F$ is constant on $\dot{a}_n$”. If $p = p_j$ let ℓ be greater than both j and k. Then $F(b_{\ell,j}) = 2$ and $q_{\ell,j} \leq p$ and $q_{\ell,j} \Vdash_{\mathbb{C}}$ “$\dot{a}_\ell = b_{\ell,j}$” yielding a contradiction. □

3. Combinatorial content of consistency of $\mathfrak{s}_{2,\infty} < \mathfrak{s}_{2,2}$

The goal of this section to introduce the combinatorial results that will be used in the proof that it is consistent that $\mathfrak{s}_{2,2} = \aleph_2$ and $\mathfrak{s}_{2,\infty} = \aleph_1$. The forcing to be used is a countable support iteration of creature forcing partial orders about which the interested reader can find more in [7], although the reader familiar with [1] should have no trouble following the argument. The argument to be used will rely on a fusion over finite subsets of the support; so this section will look at the structures that result when obtaining finite approximations to the fusion argument. Although not logically necessary, it may be useful to some readers to jump ahead and look at Definition 4.1 before continuing to the results leading to Theorem 3.2 which will play a key role in establishing Theorem 4.1.

Definition 3.1. Define $\mathbf{Ramsey}_v(k) = r$ if r is the least integer such that $r \to (k)^4_v$. Let $\mathbf{Ramsey}^n_v$ be the n-fold iteration of $\mathbf{Ramsey}_v$ defined inductively by

$$\mathbf{Ramsey}^{n+1}_v(k) = \mathbf{Ramsey}_v(\mathbf{Ramsey}^n_v(k)).$$

The following obvious fact will often be used without further explanation.

Fact 3.1. $\mathbf{Ramsey}_v(m) \to (m)^2_v$.

Theorem 3.1 *(Canonical Ramsey Theorem [5]). If v is sufficiently large ($v \geq 6^6$ will do) then for any k, if $\mathbf{Ramsey}_v(k) = r$ then for any $Z : [r]^2 \to \omega$ there is $B \in [r]^k$ such that Z is canonical on $[B]^2$ in that one of the following four options holds:*

(1) Z is constant on $[B]^2$
(2) Z is one-to-one on $[B]^2$
(3) there is a one-to-one $Z^* : B \to \omega$ such that $Z(a) = Z^*(\min(a))$ for $a \in [B]^2$
(4) there is a one-to-one $Z^* : B \to \omega$ such that $Z(a) = Z^*(\max(a))$ for $a \in [B]^2$.

Definition 3.2. Construct an increasing sequence of integers $\{e_j\}_{j\in\omega}$ inductively. Let $e_0 = 0$ and $e_1 = 2$ and now suppose that e_k has been defined. First, define an interval of integers $I_j = [e_j, e_{j+1} - 1]$ and let

$$u_j = \prod_{i\in j} \binom{e_{i+1} - e_i}{2} = \left| \prod_{i\in j} [I_i]^2 \right|.$$

Note that if e_k has been defined then only I_{k-1} and u_k have been defined up to this point. Let M_k be so large that if W is a function from M_k to the family of partial functions from u_k^k to $4 \times k$ then there is $\mathcal{M} \in [M_k]^{2u_k}$ such that W is constant on $\mathcal{M}$. Let b_k be so large that

$$b_k > (4k)^{M_k} \tag{2}$$

and let $E_{k,0} = u_k$ and then define $E_{k,\ell+1}$ by

$$E_{k,\ell+1} = \mathbf{Ramsey}_{b_k}^{F^2_{k,\ell}}(2^{F_{k,\ell}} E_{k,\ell}) \tag{3}$$

where $F_{k,\ell}$ satisfies

$$F_{k,\ell} > 3u_k^{k+1} \prod_{i\leq\ell} E_{k,i} > b_k \tag{4}$$

Then let $e_{k+1} = e_k + E_{k,kM_k}$.

Definition 3.3. For $k \in \omega$ define $\mathbb{U}[k] = \prod_{j\in k}[I_j]^2$ and define $\mathbb{U} = \bigcup_{k\in\omega} \mathbb{U}[k]$. If $T \subseteq \mathbb{U}$ is a subtree and $t \in T$ let $\mathbf{succ}_T(t) = \left\{x \in [I_{|t|}]^2 \mid t^\frown x \in T\right\}$. Then let $\mathbb{P}$ consist of all trees $T \subseteq \mathbb{U}$ such that for all $t \in T$ there is $S \subseteq I_{|t|}$ such that $\mathbf{succ}_T(t) = [S]^2$ and either

- there is some j such that $|S| \geq E_{k,j}$ in which case $\|t\|_T$ will denote the greatest such integer, or,
- $|\mathbf{succ}_T(t)| = 1$

and, furthermore,

$$(\forall k \in \mathbb{N}) \left| \left\{ t \in T \;\middle|\; \|t\|_T < M^3_{|t|} k \right\} \right| < \aleph_0 \tag{5}$$

and order $\mathbb{P}$ by inclusion. Let $\mathbb{P}_\gamma$ be the countable support iteration of length γ of the partial order $\mathbb{P}$. (The iteration and Condition (5) will only play a role later in §4.) If $k \in \omega$ and $T \subseteq \mathbb{U}$ define $T[k] = T \cap \mathbb{U}[k]$.

Definition 3.4. Let J and K be positive integers. If $j \leq J$ and $k \leq K$ and $\theta \in \mathbb{U}[K]^J$ define

$$\theta \restriction (k, j) = (\theta(i) \restriction k)_{i\in j}$$

and for $\mathcal{U} \subseteq \mathbb{U}[K]^J$ define

$$\mathcal{U}[k,j] = \{\theta \restriction (k,j) \mid \theta \in \mathcal{U}\}.$$

For $\theta \in \mathcal{U}[k,j]$ define

$$\mathcal{U}\langle\theta\rangle = \{\theta^* \in \mathcal{U} \mid \theta = \theta^* \restriction (k,j)\}.$$

For $j \in J$ and $\theta \in \mathcal{U}$ define $\mathcal{U}\langle\theta, j\rangle = \{\tau(j) \restriction k \mid \tau \in \mathcal{U}\langle\theta \restriction (K,j)\rangle \ \ \& \ \ k \in K\}$ and note that this is a subtree of $\mathbb{U}$. If $j \leq J$ and $k \leq K$ and $\theta \in \mathcal{U}[k,j]$ then define

$$\mathcal{U}\langle\langle\theta\rangle\rangle = \{\tau \in \mathbb{U}[k, J-j] \mid \theta^\frown\tau \in \mathcal{U}[k,J]\}.$$

Finally, if $K_0 \leq K_1$ and $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$ then define $\mathcal{U}$ to be K_0-*stratified* if

$$(\forall\theta \in \mathcal{U})(\forall\theta^* \in \mathcal{U})(\forall k \leq K_0)(\forall j \leq J) \ \text{ if } \theta \restriction (k,j) = \theta^* \restriction (k,j) \text{ then } \mathcal{U}\langle\langle\theta\rangle\rangle[k, J-j] = \mathcal{U}\langle\langle\theta^*\rangle\rangle[k, J-j].$$

Define $\mathcal{S}[K_0, K_1, J] = \left\{\mathcal{U} \subseteq \mathbb{U}[K_1]^J \mid \mathcal{U} \text{ is } K_0\text{-stratified}\right\}$. If $K_0 = K_1$ then define $\mathcal{S}[K_0, K_1, J] = \mathcal{S}[K_0, J]$.

Fact 3.2. If $\mathcal{U} \subseteq \mathbb{U}[K]^J$ and $k \leq K$ and $j \leq J$ then $|\mathcal{U}[k,j]| \leq u^j_k$.

Definition 3.5. For θ and θ^* in $\mathbb{U}[K]^J$ and $k \in K$ and $j \in J$ define

$$\theta \sim_{k,j} \theta^*$$

if $\theta \restriction (k+1, j) = \theta^* \restriction (k+1, j)$ and $\theta \restriction (k, j+1) = \theta^* \restriction (k, j+1)$. For $\theta \in \mathcal{U} \in \mathcal{S}[K,J]$ and $k \in K$ and $j \in J$ define $B(k,j,\theta,\mathcal{U})$ to be any set guaranteed by Definition 3.3 to satisfy

$$[B(k,j,\theta,\mathcal{U})]^2 = \{\theta^*(j)(k) \mid \theta \sim_{k,j} \theta^* \ \ \& \ \ \theta^* \in \mathcal{U}\} = \mathbf{succ}_{\mathcal{U}\langle\theta,j\rangle}(\theta(j) \restriction k).$$

It is worth noting that the asymmetry of the definition of $\sim_{k,j}$ points to the fact that this plays a role in an iteration rather than a product of the $\mathbb{P}$. Indeed, from the forcing point of view, $B(k,j,\theta,\mathcal{U})$ will arise as follows: $p \in \mathbb{P}_\gamma$ will be a condition with $\{\sigma_i\}_{i=0}^j \subseteq \gamma$ enumerated in increasing order and satisfies that

$$p \restriction \sigma_i \Vdash_{\mathbb{P}_{\sigma_i}} \text{“}\theta(i) \in p(\sigma_i)[k+1]\text{”}$$

for each $i \in j$ and

$$p \restriction \sigma_j \Vdash_{\mathbb{P}_{\sigma_j}} \text{“}\theta(j) \in p(\sigma_j)[k]\text{”}$$

then

$$p_{\theta\restriction j} \Vdash_{\mathbb{P}_{\sigma_j}} \text{“}\mathbf{succ}_{p(\sigma_j)}(\theta(j)) = [B(k,j,\theta,\mathcal{U})]^2\text{”}.$$

The notation $p_{\theta\restriction j}$ will be explained and other details will be provided in §4; however, this interpretation will not play a role until that section. Applying fusion type arguments to conditions that are not stratified results in complications, so it is convenient to restrict attention to stratified $\mathcal{U}$. The following lemma, Lemma 3.1, makes this possible.

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Lemma 3.1. *If $J \leq K_0 \leq K_1$ then for any $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$ such that $\mathcal{U} \in \mathcal{S}[K_0, K_1, J]$ and*

$$(\forall \theta \in \overline{\mathcal{U}})(\forall j \in J)(\forall k \in [K_0, K_1))|B(k, j, \theta, \overline{\mathcal{U}})| \geq E_{k,1} \tag{6}$$

there is $\overline{\mathcal{U}} \subseteq \mathcal{U}$ such that

(1) $\overline{\mathcal{U}} \in \mathcal{S}[K_1, J]$
(2) $\mathbf{Ramsey}_{b_k}(|B(k, j, \theta, \overline{\mathcal{U}})|) \geq \left|\bigcup \mathbf{succ}_{\mathcal{U}\langle\theta,j\rangle}(\theta(j) \restriction k)\right|$ whenever $\theta \in \overline{\mathcal{U}}$ and $j \in J$ and $K_0 \leq k \in K_1$
(3) $\overline{\mathcal{U}}[K_0, J] = \mathcal{U}[K_0, J]$.
(4) if $\theta \in \overline{\mathcal{U}}[k, j]$ for some $k \leq K_1$ and $j \in J$ and $\mathcal{U}\langle\langle\theta\rangle\rangle \in \mathcal{S}[k, J - j]$ then $\overline{\mathcal{U}}\langle\langle\theta\rangle\rangle = \mathcal{U}\langle\langle\theta\rangle\rangle$.

Proof. Proceed by induction on J noting that in the case $J = 1$ it is possible to let $\mathcal{U}^* = \mathcal{U}$. Given the result for J, proceed by induction on $K_1 - K_0 = N$ to establish the result for $\mathcal{U} \in \mathcal{S}[K_0, K_1, J + 1]$. The case $K_1 - K_0 = N = 0$ is immediate so assume the result for N and let $\mathcal{U} \subseteq \mathbb{U}[K_1 + 1]^{J+1}$ be such that $K_1 - K_0 = N$. Use the induction hypothesis to find $\mathcal{U}^* \subseteq \mathcal{U}[K_1, J + 1]$ such that $\mathcal{U}^* \in \mathcal{S}[K_1, J + 1]$ and Conditions (2), (3) and (4) hold. Then define

$$\mathcal{U}^{**} = \{\theta \in \mathcal{U} \mid \theta \restriction (K_1, J + 1) \in \mathcal{U}^*\} = \bigcup_{\theta \in \mathcal{U}^*} \mathcal{U}\langle\theta\rangle$$

and note that Conditions (2), (3) and (4) hold for $\mathcal{U}^{**}$.

By the induction hypothesis, for each $\theta \in \mathcal{U}^{**}[K_1 + 1, 1]$ there is $\mathcal{U}^*_\theta \subseteq \mathcal{U}\langle\langle\theta\rangle\rangle$ satisfying Conditions (1), (2), (3) and (4). Now for each

$$\theta \in \mathcal{U}^{**}[K_1, 1] = \mathcal{U}^*[K_1, 1]$$

observe that $B(K_1, 0, \theta, \mathcal{U}^{**}) = B(K_1, 0, \theta, \mathcal{U})$ and define

$$P_\theta : [B(K_1, 0, \theta, \mathcal{U})]^2 \to u^J_{K_1}$$

by $P_\theta(a) = \mathcal{U}^*_{\theta^\frown a}[K_1, J]$ noting that $2^{u^J_{K_1}} < b_{K_1}$ by Inequality (2) of Definition 3.2. Let ℓ_θ be such that $|B(K_1, 0, \theta, \mathcal{U})| = E_{K_1,\ell_\theta}$. It is then possible to use Equation (3) of Definition 3.2 and the fact that

$$|B(K_1, 0, \theta, \mathcal{U})| = E_{K_1,\ell_\theta} \geq \mathbf{Ramsey}_{b_k}(E_{K_1,\ell_\theta - 1})$$

to find $B^*_\theta \subseteq B(K_1, 0, \theta, \mathcal{U})$ such that $|B^*_\theta| = E_{K_1,\ell_\theta - 1}$ and P_θ is constant on $[B^*_\theta]^2$. Let

$$\overline{\mathcal{U}} = \{\theta \in \mathcal{U}^{**} \mid \theta(0)(K_1) \in B^*_\theta\}$$

and note that the construction guarantees that Condition (1) holds. □

Lemma 3.1 will often be applied in the context that $J \leq K_0 \leq K_1$ and $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$ is such that $|\mathcal{U}[K_0, J]| = 1$. Observe that this automatically implies that $\mathcal{U}[K_0, J] \in \mathcal{S}[K_0, K_1, J]$ and hence the hypothesis of Lemma 3.1 is satisfied. As a result, the following fact will be used implicitly in various lemmas, starting with Lemma 3.4.

Fact 3.3. If $J \leq K_0 \leq K_1$ and $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$ and $|\mathcal{U}[K_0]| = 1$ then $\mathcal{U} \in \mathcal{S}[K_0, K_1, J]$.

Definition 3.6. For $\mathcal{U} \in \mathcal{S}[K, J]$ and $k \in K$ define

$$\|\mathcal{U}\|_k = \min_{\theta \in \mathcal{U}, j \in J} \|\theta(j) \restriction k\|_{\mathcal{U}\langle \theta, j\rangle}.$$

For $k \in K$ define $\mathcal{U}$ to be k-organized if

$$|B(k, j, \theta, \mathcal{U})| = E_{k, \|\mathcal{U}\|_k + j}$$

whenever $\theta \in \mathcal{U}$ and $j \in J$.

Lemma 3.2. *If $\mathcal{U} \in \mathcal{S}[K, J]$ and $\mathcal{U}$ is k-organized and $j \leq J$ then the number of $\sim_{k,j}$ equivalence classes in $\mathcal{U}$ is bounded by $u_k^{j+1} \prod_{i \in j} E_{k, \|\mathcal{U}\|_k + i}$.*

Proof. Each $\sim_{k,j}$ equivalence classes corresponds to some $\theta \in \mathbb{U}[k]^{j+1}$ and some $\vec{\sigma} \in \prod_{i \in j} B(k, i, \theta, \mathcal{U})$. From Fact 3.2 it follows that u_k^{j+1} bounds the number of such θ. From the k-organized hypothesis it follows that $|B(k, i, \theta, \mathcal{U})| = E_{k, \|\mathcal{U}\|_k + i}$ for $i \in j$ and so $\prod_{i \in j} E_{k, \|\mathcal{U}\|_k + i}$ bounds the number of possible $\vec{\sigma}$. Hence the number of equivalence classes is bounded by

$$u_k^{j+1} \prod_{i \in j} E_{k, \|\mathcal{U}\|_k + i}. \quad \square \tag{7}$$

Definition 3.7. If $\mathcal{U} \in \mathcal{S}[K, J]$ and $k < K$ define $\mathcal{U}^* \sqsubseteq_k \mathcal{U}$ if $\mathcal{U}^* \subseteq \mathcal{U}$, $\mathcal{U}^* \in \mathcal{S}[K, J]$ and for all $j \in J$ and $\theta \in \mathcal{U}^*$

$$\mathbf{Ramsey}_{b_k}^{F_{k,j-1}}(2|B(k, j, \theta, \mathcal{U}^*)|) \geq |B(k, j, \theta, \mathcal{U})|.$$

For $K_0 \leq K_1$ define $\mathcal{U}^* \sqsubseteq_{K_0, K_1} \mathcal{U}$ if $\mathcal{U}^* \sqsubseteq_k \mathcal{U}$ holds provided that $K_0 \leq k < K_1$.

It should be noted that the superscripts in $\mathbf{Ramsey}_{b_k}^{F_{k,j}}$ are correct and are not intended to be $\mathbf{Ramsey}_{b_k}^{F_{k,j}^2}$ as in Definition 3.2. The reason for this will become clear in Lemma 3.15.

Lemma 3.3. *If $J \leq K_0 \leq k < K_1$ and $M \leq F_{K_0, 0}$ and $\{\mathcal{U}_i\}_{i=0}^M$ satisfy that:*

- $\{\mathcal{U}_i\}_{i=0}^M \subseteq \mathcal{S}[K_1, J]$
- $\mathcal{U}_{i+1} \sqsubseteq_k \mathcal{U}_i$
- $\|\mathcal{U}_0\|_k \geq 1$

then $\|\mathcal{U}_M\|_k \geq \|\mathcal{U}_0\|_k - 1$.

Proof. It must be shown that if $\ell = \|\mathcal{U}_M\|_k$ then $|B(k, j, \theta, \mathcal{U}_M)| \geq E_{k, \ell-1}$ for each $j \in J$ and $\theta \in \mathcal{U}_M$. To this end, fix j and θ. Then $M \leq F_{K_0, 0} \leq F_{k, \ell-1}$ and so

$$\begin{gathered}
\mathbf{Ramsey}_{b_k}^{F_{k,\ell-1}^2}\left(2^{F_{k,\ell-1}}|B(k, j, \theta, \mathcal{U}_M)|\right) \geq \mathbf{Ramsey}_{b_k}^{M F_{k,\ell-1}}\left({}^{M}|B(k, j, \theta, \mathcal{U}_M)|\right) = \\
\mathbf{Ramsey}_{b_k}^{(M-1)F_{k,\ell-1}}\left(\mathbf{Ramsey}_{b_k}^{F_{k,\ell-1}}\left(2^{M-1}|B(k, j, \theta, \mathcal{U}_M)|\right)\right) \geq \\
\mathbf{Ramsey}_{b_k}^{(M-1)F_{k,\ell-1}}\left(2^{M-1}\,\mathbf{Ramsey}_{b_k}^{F_{k,\ell-1}}\left(2|B(k, j, \theta, \mathcal{U}_M)|\right)\right) \geq \\
\mathbf{Ramsey}_{b_k}^{(M-1)F_{k,\ell-1}}\left(2^{M-1}|B(k, j, \theta, \mathcal{U}_{M-1})|\right) \geq
\end{gathered}$$

$$\mathbf{Ramsey}_{b_k}^{(M-2)F_{k,\ell-1}}\left(2^{M-2}\mathbf{Ramsey}_{b_k}^{F_{k,\ell-1}}(2|B(k,j,\theta,\mathcal{U}_{M-1})|)\right)\geq$$
$$\mathbf{Ramsey}_{b_k}^{(M-2)F_{k,\ell-1}}\left(2^{M-2}|B(k,j,\theta,\mathcal{U}_{M-2}|\right)\geq\ldots\geq|B(k,j,\theta,\mathcal{U}_0)|\geq E_{k,\ell}$$

and this is as required, recalling that $\mathbf{Ramsey}_{b_k}^{F_{k,\ell-1}^2}\left(2^{F_{k,\ell-1}}E_{k,\ell-1}\right)=E_{k,\ell}$ by Equation (3) of Definition 3.2. □

Lemma 3.4. *If*

- $J\leq K_0\leq K_1$
- $\mathcal{U}\in\mathcal{S}[K_1,J]$
- $\|\mathcal{U}\|_k\geq 1$ *if* $K_0\leq k<K_1$
- $|\mathcal{U}[K_0,J]|=1$
- $P:\mathcal{U}\to b_{K_0}$

then there is a non-empty $\mathcal{U}^*\sqsubseteq_{K_0,K_1}\mathcal{U}$ *such that* P *is constant on* $\mathcal{U}^*$.

Proof. Proceed by induction on J to prove the following stronger statement: Under the hypotheses of the lemma there is a non-empty $\mathcal{U}^*\subseteq\mathcal{U}$ such that

(1) P is constant on $\mathcal{U}^*$
(2) $\mathcal{U}^*\in\mathcal{S}[K_1,J]$
(3) $\mathbf{Ramsey}_{b_k}^2(|B(k,j,\theta,\mathcal{U}^*)|)\geq|B(k,j,\theta,\mathcal{U})|$ whenever $K_0\leq k<K_1$, $j\in J$ and $\theta\in\mathcal{U}^*[k+1,J]$.

For the case $J=1$ proceed by induction on K_1-K_0 to prove the even stronger statement where Inequality (3) is replaced by

$$(\forall k\in[K_0,K_1)(\forall j\in J)(\forall\theta\in\mathcal{U}^*[k,J])\ \mathbf{Ramsey}_{b_k}(|B(k,j,\theta,\mathcal{U}^*)|)\geq|B(k,j,\theta,\mathcal{U})|. \tag{8}$$

The case that $K_1-K_0=0$ is trivial. If $K_1>K_0$ and the result is true for K_1-K_0-1 then let ρ be the unique element of $\mathcal{U}[K_0,1]$. Then $\mathcal{U}\langle\rho^\frown a\rangle$ and $P\restriction\mathcal{U}\langle\rho^\frown a\rangle$ satisfy the inductive hypothesis for each $a\in[B(K_0,0,\rho,\mathcal{U})]^2$. Hence there is $P^*(a)\in b_{K_0}<b_{K_0+1}$ and $\mathcal{U}_a^*\subseteq\mathcal{U}$ such that Conditions (8) and (3) hold and P has constant value $P^*(a)$ on $\mathcal{U}_a^*$. Then there is $B^*\subseteq B(K_0,0,\rho,\mathcal{U})$ such that $\mathbf{Ramsey}_{b_{K_0}}(|B^*|)\geq|B(K_0,0,\rho,\mathcal{U})|$ and P^* is constant on B^*. Then $\bigcup_{a\in B^*}\mathcal{U}_a^*$ is as desired.

Assuming the result holds for $J-1$, let $\mathcal{U}\in\mathcal{S}[K_1,J]$ be such that $|\mathcal{U}[K_0,J]|=1$ and suppose that $P:\mathcal{U}\to b_{K_0}$. Using the induction hypothesis, for each $\theta\in\mathcal{U}[K_1,1]$ find $\mathcal{U}_\theta\subseteq\mathcal{U}\langle\langle\theta\rangle\rangle$ such that Conditions (2) and (3) hold and such that P is constant on $\{\theta^\frown\tau\mid\tau\in\mathcal{U}_\theta\}$ with constant value $Q(\theta)$. Now apply the case $J=1$ to find $\mathcal{U}^{**}\subseteq\mathcal{U}[K_1,1]$ such that Conditions (8) and (3) hold and such that Q is constant on $\mathcal{U}^{**}$. Let $\overline{\mathcal{U}}=\{\theta^\frown\tau\mid\theta\in\mathcal{U}^{**}\ \ \&\ \ \tau\in\mathcal{U}_\theta\}$ and use Lemma 3.1 to find $\mathcal{U}^*\subseteq\overline{\mathcal{U}}$ such that

(1) $\mathcal{U}^*\in\mathcal{S}[K_1,J]$
(2) $\mathbf{Ramsey}_{b_k}(|B(k,j,\theta,\mathcal{U}^*)|)\geq\left|\bigcup\mathbf{succ}_{\overline{\mathcal{U}}\langle\theta,j\rangle}(\theta(j)\restriction k)\right|$ whenever $\theta\in\mathcal{U}^*$, $j\in J$ and $K_0\leq k<K_1$
(3) $\overline{\mathcal{U}}[K_0,J]=\mathcal{U}^*[K_0,J]$
(4) if $\theta\in\mathcal{U}^*[k,j]$ for some $k\in K_1$ and $j\in J$ and $\overline{\mathcal{U}}\langle\langle\theta\rangle\rangle\in\mathcal{S}[k,J-j]$ then $\overline{\mathcal{U}}\langle\langle\theta\rangle\rangle=\mathcal{U}^*\langle\langle\theta\rangle\rangle$.

It follows that if $j>1$ and $\theta\in\mathcal{U}^*[k,j]$ for some $k\in K_1$ then

$$\mathbf{Ramsey}_{b_k}^2(|B(k,j,\theta,\mathcal{U}^*)|)=\mathbf{Ramsey}_{b_k}^2(|B(k,j,\theta,\mathcal{U}_{\theta\restriction(k,1)})|)\geq|B(k,j,\theta,\mathcal{U})|$$

by the induction hypothesis. When $j = 1$ then

$$\mathbf{Ramsey}^2_{b_k}(|B(k,j,\theta,\mathcal{U}^*)|) \geq \mathbf{Ramsey}_{b_k}(|B(k,1,\theta,\mathcal{U}^{**})|) \geq |B(k,j,\theta,\mathcal{U})|. \quad \square$$

Definition 3.8. If $\mathcal{U} \in \mathcal{S}[K,J]$ and $Z : \mathcal{U} \to \omega$ define $C(\mathcal{U},Z)$ to be the set of all 4-tuples (k,j,θ,ℓ) such that $k \in K$, $j \in J$, $\theta \in \mathcal{U}$ and $\ell \in 3$ and there is $Z_{k,j,\theta} : [B(k,j,\theta,\mathcal{U})]^2 \to \omega$ such that if $\theta^* \sim_{k,j} \theta$ and $\theta^*(k,j) = a$ then $Z_{k,j,\theta}(a) = Z(\theta^*)$ and:

(1) if $\ell = 0$ then $Z_{k,j,\theta}$ is one-to-one
(2) if $\ell = 1$ then there is a one-to-one $Z^*_{k,j,\theta} : B(k,j,\theta,\mathcal{U}) \to \omega$ such that $Z^*_{k,j,\theta}(\min(a)) = Z_{k,j,\theta}(a)$
(3) if $\ell = 2$ then there is a one-to-one $Z^*_{k,j,\theta} : B(k,j,\theta,\mathcal{U}) \to \omega$ such that $Z^*_{k,j,\theta}(\max(a)) = Z_{k,j,\theta}(a)$

and $|B(k,j,\theta,\mathcal{U})| \geq E_{k,1}$. Let

$$R(Z,k,j,\theta,\mathcal{U}) = \begin{cases} Z_{k,j,\theta}\left([B(k,j,\theta,\mathcal{U})]^2\right) & \text{if } (\exists \ell \in 3)\ (k,j,\theta,\ell) \in C(\mathcal{U},Z) \\ \varnothing & \text{if } (\forall \ell \in 3)\ (k,j,\theta,\ell) \notin C(\mathcal{U},Z). \end{cases}$$

Say that $C(\mathcal{U},Z)$ is a front in $\mathcal{U}$ if for every $\theta \in \mathcal{U}$ there are $k_\theta \leq K$, $j_\theta \in J$ and $\ell_\theta \in 3$ such that $(k_\theta, j_\theta, \theta, \ell_\theta) \in C(\mathcal{U},Z)$.

Fact 3.4. $Z_{k,j,\theta}$ and ℓ in Definition 3.8 are invariant under the $\sim_{k,j}$ equivalence relation and $R(Z,k,j,\theta,\mathcal{U})$ depends only on the $\sim_{k,j}$ equivalence class of θ.

Lemma 3.5. *If*

- $\mathcal{U} \in \mathcal{S}[k+1,J]$
- $\|\mathcal{U}\|_k \geq 2$
- $|\mathcal{U}[k,J]| = 1$
- $Z : \mathcal{U} \to \omega$

then there is $\mathcal{U}^* \sqsubseteq_k \mathcal{U}$ such that either Z is constant on $\mathcal{U}^*$ or $C(\mathcal{U}^*,Z)$ is a front in $\mathcal{U}^*$.

Proof. Proceed by induction on J, the case $J = 1$ following from Theorem 3.1. Now assume the result true for J and suppose that $\mathcal{U} \subseteq \mathbb{U}[k+1]^{J+1}$. Let θ be the unique member of $\mathcal{U}[k,1]$. For $a \in [B(k,0,\theta,\mathcal{U})]^2$ apply the induction hypothesis to each $\mathcal{U}\langle\langle\theta^\frown a\rangle\rangle$ and $Z_a : \mathcal{U}\langle\langle\theta^\frown a\rangle\rangle \to \omega$ defined by $Z_a(\tau) = Z((\theta^\frown a)^\frown\tau)$. This yields $\mathcal{U}^*_a \sqsubseteq_k U\langle\langle\theta^\frown a\rangle\rangle$ such that either Z_a is constant on $\mathcal{U}^*_a$ or $C(\mathcal{U}^*_a, Z_a)$ is a front in $\mathcal{U}^*_a$.

Then define $Q : [B(k,0,\theta,\mathcal{U})]^2 \to 2$ by $Q(a) = 0$ if and only if Z_a is constant on $\mathcal{U}^*_a$. By Lemma 3.1 there is $B^* \subseteq B(k,0,\theta,\mathcal{U})$ such that $\mathbf{Ramsey}_{b_k}(|B^*|) \geq |B(k,0,\Theta,\mathcal{U})|$ and B^* is homogeneous for Q. If B^* is 1-homogeneous then let

$$\mathcal{U}^* = \left\{(\Theta^\frown a)^\frown\tau \;\middle|\; a \in [B^*]^2 \;\&\; \tau \in \mathcal{U}^*_a\right\}$$

and note that $\bigcup_{a\in[B^*]^2} C(\mathcal{U}^*_a, Z_a)$ is a front in $\mathcal{U}^*$. To see this, let $\tau \in \mathcal{U}^*$. Then there is some $a \in [B^*]^2$ such that $\tau(0)(k) = a$ and so if $\tau = \tau(0)^\frown\tau^*$ with $\tau^* \in \mathcal{U}_a$ then there are $j_\tau \in J$ and $\ell_\tau \in 3$ such that $(k,j_\tau,\tau^*,\ell_\tau) \in C(\mathcal{U}_a,Z_a)$. But then k, j_τ and ℓ_τ witness that $(k,j_\tau,\tau,\ell_\tau) \in C(\mathcal{U}^*,Z)$.

On the other hand, if B^* is 0-homogeneous then let $Z^*(a)$ be the constant value of Z_a on $\mathcal{U}^*_a$ for each $a \in [B^*]^2$. By Theorem 3.1 it is then possible to find $B^{**} \subseteq B^*$ such that $\mathbf{Ramsey}_{b_k}(|B^{**}|) \geq |B^*|$ (and hence $\mathbf{Ramsey}^2_{b_k}(|B^{**}|) \geq |B|$) such that Z^* is either constant on B^{**} or one of the three alternatives of Definition 3.8 holds. Now let

$$\mathcal{U}^* = \{(\theta^\frown a)^\frown\tau \mid a \in [B^{**}]^2 \ \&\ \tau \in \mathcal{U}^*_a\}$$

and note that either Z is constant on $\mathcal{U}^*$ or there is $\ell \in 3$ such that $(k, 0, \tau, \ell) \in C(\mathcal{U}^*, Z)$ for all $\tau \in \mathcal{U}^*$. □

Lemma 3.6. *If*

- $J \leq K_0 \leq K_1$
- $\mathcal{U} \in \mathcal{S}[K_1, J]$
- $\|\mathcal{U}\|_k \geq 2$ *for all* k *such that* $K_0 \leq k < K_1$
- $|\mathcal{U}[K_0, J]| = 1$
- $Z : \mathcal{U} \to \omega$

then there is $\mathcal{U}^* \sqsubseteq_{K_0,K_1} \mathcal{U}$ *such that either* Z *is constant on* $\mathcal{U}^*$ *or* $C(\mathcal{U}^*, Z)$ *is a front in* $\mathcal{U}^*$.

Proof. Proceed by induction on $K_1 - K_0$ using Lemma 3.5. □

Lemma 3.7. *If*

- $K_0 \leq K_1$
- $\mathcal{U} \in \mathcal{S}[K_1, J]$
- $B^*(k, j, \theta) \in [B(k, j, \theta, \mathcal{U})]^{\geq 2}$ *provided that* $K_0 \leq k < K_1$, $j < J$ *and* $\theta \in \mathcal{U}[k, j]$

then there is $\mathcal{U}^* \subseteq \mathcal{U}$ *such that*

- $\mathcal{U}^* \in \mathcal{S}[K_1, J]$
- $B(k, j, \theta, \mathcal{U}^*) = B^*(k, j, \theta)$ *provided that* $K_0 \leq k < K_1$, $j < J$ *and* $\theta \in \mathcal{U}^*[k, j]$
- $\mathcal{U}^*[K_0, J] = \mathcal{U}[K_0, J]$.

Proof. Proceed by induction on J, the case $J = 1$ being easy. Given $\mathcal{U} \in \mathcal{S}[K_1, J + 1]$ use the induction hypothesis to find $\overline{\mathcal{U}} \subseteq \mathcal{U}[K, J]$ such that

- $\overline{\mathcal{U}} \in \mathcal{S}[K_1, J]$
- $B(k, j, \theta, \overline{\mathcal{U}}) = B^*(k, j, \theta)$ provided that $K_0 \leq k < K_1$, $j < J$ and $\theta \in \overline{\mathcal{U}}$
- $\overline{\mathcal{U}}[K_0, J] = \mathcal{U}[K_0, J]$.

Let $\mathcal{U}^{**} = \{\tau \in \mathcal{U} \mid \tau[K, J] \in \overline{\mathcal{U}}\}$ and note that $\mathcal{U}^{**} \in \mathcal{S}[K_1, J + 1]$. It follows that

$$B^*(k, J, \theta) \subseteq B(k, J, \theta, \mathcal{U}) = B(k, J, \theta, \mathcal{U}^{**})$$

for each $k \in K$ and $\theta \in \mathcal{U}^{**}[k, J]$. Therefore, let

$$\mathcal{U}^* = \{\theta \in \mathcal{U}^{**} \mid (\forall k \in K)\ \theta(J)(k) \in B^*(k, J, \theta \restriction (k, J + 1))\}. \quad \square$$

Lemma 3.8. *Suppose that*

- $J \leq K_0 \leq K_1$
- $\mathcal{U} \in \mathcal{S}[K_1, J]$
- $\|\mathcal{U}\|_k \geq J$ *for* $k \geq K_0$.

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ *such that:*

- $\mathcal{U}^* \in \mathcal{S}[K_1, J]$
- $\mathcal{U}^*[K_0, J] = \mathcal{U}[K_0, J]$
- $\mathcal{U}^*$ *is* k*-organized for all* k *such that* $K_0 \leq k < K_1$
- $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - J$ *for* $k \geq K_0$*.*

Proof. Let $n(k)$ be maximal such that $\|\mathcal{U}\|_k \geq n(k) + J$ for all $k \geq K_0$. Then simply choose $B^*(k, j, \theta) \subseteq B(k, j, \theta, \mathcal{U})$ such that $|B^*(k, j, \theta)| = E_{k,n(k)+j}$ and apply Lemma 3.7. □

Lemma 3.9. *If* $J \leq K_0 < K_1$ *and*

- $\mathcal{U} \in \mathcal{S}[K_1, J]$
- $\theta \in \mathcal{U}[K_0, J]$
- $\mathcal{V} \subseteq \mathcal{U}\langle\theta\rangle$

then there is $\mathcal{W}$ *such that*

- $\mathcal{W}[K_0, J] = \mathcal{U}[K_0, J]$
- $\mathcal{V} = \mathcal{W}\langle\theta\rangle$
- *if* $\tau \in \mathcal{W}[K_0, j]$ *and* $\tau \neq \theta \restriction (K_0, j)$ *then* $\mathcal{W}\langle\tau\rangle = \mathcal{U}\langle\tau\rangle$*.*

Proof. This is a routine argument by induction on J. □

Lemma 3.10. *Suppose that*

- $J \leq K_0 \leq K_1$
- $\mathcal{D} \subseteq \mathcal{S}[K_1, J]$
- *if* $\mathcal{U} \subseteq \mathcal{V} \in \mathcal{D}$ *and* $\mathcal{U} \in \mathcal{S}[K_1, J]$ *then* $\mathcal{U} \in \mathcal{D}$
- *for all* $\mathcal{U} \in \mathcal{S}[K_1, J]$ *and* $\theta \in \mathcal{U}[K_0, J]$ *there is* $\mathcal{U}^* \sqsubseteq_{K_0,K_1} \mathcal{U}\langle\theta\rangle$ *such that* $\mathcal{U}^* \in \mathcal{D}$*.*

Then for any $\mathcal{U} \in \mathcal{S}[K_1, J]$ *such that* $\|\mathcal{U}\|_k \geq 1$ *for all* k *such that* $K_0 \leq k < K_1$ *there is* $\bar{\mathcal{U}} \subseteq \mathcal{U}$ *such that*

- $\bar{\mathcal{U}}[K_0, J] = \mathcal{U}[K_0, J]$
- $\bar{\mathcal{U}}\langle\theta\rangle \in \mathcal{D}$ *for each* $\theta \in \mathcal{U}[K_0, J]$
- $\|\bar{\mathcal{U}}\|_k \geq \|\mathcal{U}\|_k - 1$ *for all* $k \geq K_0$*.*

Proof. Let $\{\theta_i\}_{i=0}^{u}$ enumerate $\mathcal{U}[K_0, J]$ where $u \leq u^J_{K_0} \leq u^{K_0}_{K_0} \leq F_{K_0,0}$ by Inequality (4) of Definition 3.2. Then construct inductively $\mathcal{U}_i$ such that

- $\mathcal{U} = \mathcal{U}_0$
- $\mathcal{U}_{i+1}\langle\theta_i\rangle \in \mathcal{D}$ for each i
- $\mathcal{U}_i[K_0, J] = \mathcal{U}_{i+1}[K_0, J]$
- $\mathcal{U}_{i+1} \sqsubseteq_{K_0,K_1} \mathcal{U}_i$.

Letting $\bar{\mathcal{U}} = \mathcal{U}_u$ it follows from Lemma 3.3 and consideration of the sequence

$$\mathcal{U}_0 \sqsupseteq_{K_0,K_1} \mathcal{U}_1 \sqsupseteq_{K_0,K_1} \mathcal{U}_2 \sqsupseteq_{K_0,K_1} \ldots \sqsupseteq_{K_0,K_1} \mathcal{U}_u$$

that $\|\bar{\mathcal{U}}\|_k \geq \|\mathcal{U}\|_k - 1$ for all $k \geq K_0$ and hence $\bar{\mathcal{U}}$ satisfies the lemma.

To see that the induction can be completed suppose that $\mathcal{U}_i$ is given. Use the hypothesis to find

$$\mathcal{V}_i \sqsubseteq_{K_0,K_1} \mathcal{U}_i\langle\theta_i\rangle$$

such that $\mathcal{V}_i \in \mathcal{D}$. The use Lemma 3.9 to find $\mathcal{U}_{i+1}$ such that $\mathcal{U}_i[K_0, J] = \mathcal{U}_{i+1}[K_0, J]$ and $\mathcal{U}_{i+1}\langle\theta_i\rangle = \mathcal{V}_i$ and, moreover, such that if $\tau \in \mathcal{U}_{i+1}[K_0, j]$ and $\tau \neq \theta_i \restriction (K_0, j)$ then $\mathcal{U}_{i+1}\langle\tau\rangle = \mathcal{U}_i\langle\tau\rangle$. From this it immediately follows that $\mathcal{U}_{i+1} \sqsubseteq_{K_0,K_1} \mathcal{U}_i$. Note that $\bar{\mathcal{U}}\langle\theta\rangle \in \mathcal{D}$ for each $\theta \in \mathcal{U}[K_0, J]$ by the closure of $\mathcal{D}$ under subset. □

Lemma 3.11. *Suppose that*

(a) $J \leq K_0 \leq K_1$
(b) $\mathcal{U} \in \mathcal{S}[K_1, J]$
(c) $Z : \mathcal{U} \to \omega$.

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ *such that*

(d) $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - 1$ *for* $k \geq K_0$
(e) $\mathcal{U}^*[K_0, J] = \mathcal{U}[K_0, J]$
(f) for each $\theta \in \mathcal{U}[K_0, J]$ *either* Z *is constant on* $\mathcal{U}^*\langle\theta\rangle$ *or* $C(\mathcal{U}^*\langle\theta\rangle, Z)$ *is a front in* $\mathcal{U}^*\langle\theta\rangle$.

Proof. Let $\mathcal{D}$ consist of all $\mathcal{V} \subseteq \mathcal{U}$ such that either Z is constant on $\mathcal{V}$ or $C(\mathcal{V}, Z)$ is a front in $\mathcal{V}$. Inspection of Definition 3.8 reveals that $\mathcal{D}$ is closed under subsets. From this and Lemma 3.6 it follows that $\mathcal{D}$ satisfies the hypotheses of Lemma 3.10 and, hence, there is $\bar{\mathcal{U}} \subseteq \mathcal{U}$ such that

- $\bar{\mathcal{U}}[K_0, J] = \mathcal{U}[K_0, J]$
- $\bar{\mathcal{U}}\langle\theta\rangle \in \mathcal{D}$ for each $\theta \in \mathcal{U}[K_0, J]$
- $\|\bar{\mathcal{U}}\|_k \geq \|\mathcal{U}\|_k - 1$ for $k \geq K_0$,

as required. □

Lemma 3.12. *Suppose that* $|S_0| = |S_1| = \mathbf{Ramsey}_2(2m)$ *and* $m \geq 3$ *and* $Z_i : [S_i]^2 \to \omega$ *for* $i \in 2$. *Suppose also that* $Z_i(a) \neq Z_i(b)$ *if* $a \cap b = \varnothing$ *for each* $i \in 2$. *There are then* $S_i^* \subseteq S_i$ *such that* $|S_0^*| = |S_1^*| = m$ *and* $Z_0([S_0^*]^2) \cap Z_1([S_1^*]^2) = \varnothing$.

Proof. For $x \subseteq \omega$ let $\{x[i]\}_{i\in|x|}$ enumerate x in increasing order. Let $\Psi : S_0 \to S_1$ be an order preserving bijection and define $P : [S_0]^4 \to 2$ by $P(x) = 0$ if and only if $Z_0(\{x[0], x[1]\}) = Z_1(\{\Psi(x[2]), \Psi(x[3])\})$. It is easy to see that the hypothesis on the Z_i rules out the possibility that there is a 0-homogeneous set for P of cardinality greater than 5. Let S^* be homogenous for P of cardinality $2m$ and let $S_0^* = \{S^*[i]\}_{i\in m}$ and $S_1^* = \Psi(S^* \setminus S_0^*)$. □

Corollary 3.1. *Suppose that*

(1) $Z_i^\ell : [S_i]^2 \to \omega$ *for* $i \in L$ *and* $\ell \in 2$
(2) $Z_i^\ell(a) \neq Z_i^\ell(b)$ *if* $a \cap b = \varnothing$.
(3) $|S_i| \geq \mathbf{Ramsey}_2^L(2^L m)$ *for each* $i \in L$.

There are then $S_i^* \subseteq S_i$ *such that*

- $|S_i^*| = m$ *for each* i
- $Z_i^0([S_i^*]^2) \cap Z_j^1([S_j^*]^2) = \varnothing$ *if* $i < j < L$.

Proof. Apply Lemma 3.12 iteratively for each pair $\{i, j\} \in [L]^2$ noting that for each i Lemma 3.12 is not applied more than L times. □

Lemma 3.13. *Suppose that* $m \geq 8$ *and* $|S| = \mathbf{Ramsey}_2(m)$ *and* $Z_i : [S]^2 \to \omega$ *for* $i \in 2$. *Suppose also that* $Z_0(a) \neq Z_1(a)$ *for all* $a \in [S]^2$ *and that one of the following three options holds:*

- *each* Z_i *is one-to-one*
- $Z_i(x) = Z_i(y)$ *if and only if* $\min(x) = \min(y)$ *for each* i
- $Z_i(x) = Z_i(y)$ *if and only if* $\max(x) = \max(y)$ *for each* i.

There is then $S^* \subseteq S$ *such that* $|S^*| = m$ *and* $Z_0([S^*]^2) \cap Z_1([S^*]^2) = \varnothing$.

Proof. Define $P : [S]^4 \to 2$ by $P(x) = 0$ if and only if there are a and b in $[x]^2$ such that $Z_0(a) = Z_1(b)$. It suffices to show that no 0-homogeneous subset of P can have cardinality 5. Three cases need to be considered.

If each Z_i is one-to-one and $w \in [S]^5$ is 0-homogeneous. Let a and b be distinct elements of $[w]^2$ such that $Z_0(a) = Z_1(b)$ and let $x \in [w]^4$ be such that $a \cup b \subseteq x \subseteq w$. There is then $x' \in [w]^4$ such the isomorphism taking x to x' moves precisely one of a or b. This yields a contradiction to the assumption that each Z_i is one-to-one.

If $Z_i(x) = Z_i(y)$ if and only if $\min(x) = \min(y)$ for each i let $Z_i^* : S \to \omega$ be such that $Z_i(a) = Z_i^*(\min(a))$ and note that $Z_0^*(s) \neq Z_1^*(s)$ for all $s \in S$ and each Z_i^* is one-to-one. However, if w is 0-homogenous for P and $|w| \geq 8$ then it is possible to find $x \in [w]^4$ such that $Z_0^*(x) \cap Z_1^*(x) = \varnothing$ contradicting that w is 0-homogenous.

The case that $Z_i(x) = Z_i(y)$ if and only if $\max(x) = \max(y)$ for each i is handled similarly. □

Note that in the last part of the proof of Lemma 3.13 if it were the case that $Z_0(x) = Z_0^*(\min(x))$ and $Z_1(x) = Z_1^*(\max(x))$ then the argument to get x from w would fail since it might be possible that $Z_0^*(s) = Z_1^*(s)$ without violating the hypothesis that $Z_0(a) \neq Z_1(a)$ for all $a \in [S]^2$.

Corollary 3.2. *Suppose that:*

- $J \leq K_0 \leq K_1$
- $\mathcal{U} \in \mathcal{S}[K_1, J]$
- $\|\mathcal{U}\|_k \geq 1$ *if* $K_0 \leq k < K_1$
- $\mathcal{U}$ *is* k*-organized if* $K_0 \leq k < K_1$
- $Z_i : \mathcal{U} \to \omega$ *for* $i \in 2$ *are such that* $Z_0(\theta) \neq Z_1(\theta)$ *for all* $\theta \in \mathcal{U}$.

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ *such that*

- $\mathcal{U}^* \in \mathcal{S}[K_1, J]$
- $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - 1$ *for all* $k \geq K_0$
- $\mathcal{U}^*[K_0, J] = \mathcal{U}[K_0, J]$
- *if* $\ell \in 3$ *and* $(k, j, \theta, \ell) \in C(\mathcal{U}^*, Z_i)$ *for* $i \in 2$ *then* $R(Z_0, k, j, \theta, \mathcal{U}^*) \cap R(Z_1, k, j, \theta, \mathcal{U}^*) = \varnothing$ *provided that* $k \geq K_0$.

Proof. By Lemma 3.10 and Lemma 3.7 it suffices to show that if:

- $\theta \in \mathcal{U}[K_0, J]$
- $\ell \in 3$
- $K_0 \leq k < K_1$
- $j < J$
- $\theta^* \in \mathcal{U}\langle\theta\rangle$
- $(k, j, \theta^*, \ell) \in C(\mathcal{U}, Z_i)$ for each $i \in 2$

then there is $B^* \subseteq B(k, j, \theta^*, \mathcal{U})$ such that $\mathbf{Ramsey}_2(|B^*|) \geq |B(k, j, \theta, \mathcal{U})|$ and

$$R(Z_0, k, j, \theta, \mathcal{U}^*) \cap R(Z_1, k, j, \theta, \mathcal{U}^*) = \varnothing$$

provided that $B(k, j, \theta, \mathcal{U}^*) = B^*$. This follows directly from Lemma 3.13. □

Lemma 3.14. *Suppose that*

- $J \leq K_0 \leq K_1$
- $\mathcal{U} \in \mathcal{S}[K_1, J]$
- $\|\mathcal{U}\|_k \geq 1$ *for all* $k \geq K_0$
- $\mathcal{U}$ *is* k*-organized if* $K_0 \leq k < K_1$
- $Z_i : \mathcal{U} \to \omega$ *for* $i \in 2$.

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ *such that*

(1) $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - (J + 1)$ *for* $k \geq K_0$
(2) $\mathcal{U}^*[K_0, J] = \mathcal{U}[K_0, J]$
(3) if the following hold

  - (k_0, j_0) *is lexicographically less than* (k_1, j_1)
  - $(k_i, j_i, \theta_i, \ell_i) \in C(\mathcal{U}^*, Z_i)$ *for* $i \in 2$

  then $R(Z_0, k_0, j_0, \theta_0, \mathcal{U}^*) \cap R(Z_1, k_1, j_1, \theta_1, \mathcal{U}^*) = \varnothing$
(4) if $\theta \in \mathcal{U}^*[K_0, J]$ *and* Z_0 *has constant value* v *on* $\mathcal{U}^*\langle\theta\rangle$ *then* $v \notin R(Z_1, k_1, j_1, \theta, \mathcal{U}^*)$.

Proof. Begin by using Lemma 3.8 to find $\overline{\mathcal{U}} \subseteq \mathcal{U}$ such that

(a) $\overline{\mathcal{U}} \in \mathcal{S}[K_1, J]$
(b) $\|\overline{\mathcal{U}}\|_k \geq \|\mathcal{U}\|_k - J$ if $K_0 \leq k < K_1$
(c) $\overline{\mathcal{U}}[K_0, J] = \mathcal{U}[K_0, J]$
(d) $\overline{\mathcal{U}}$ is k-organized if $K_0 \leq k < K_1$.

Let V be the set of all v such that there is some $\theta \in \overline{\mathcal{U}}[K_0, J]$ such that Z_0 has constant value v on $\overline{\mathcal{U}}\langle\theta\rangle$. Then let

$$R_{k,j} = \bigcup_{\theta \in \overline{\mathcal{U}}} \left(\bigcup_{K_0 \leq k^* < k} \bigcup_{j^* \in J} R(Z_0, k^*, j^*, \theta, \overline{\mathcal{U}}) \cup \bigcup_{j^* \in j} R(Z_0, k, j^*, \theta, \overline{\mathcal{U}}) \right) \cup V$$

and note that if $K_0 \leq k^* < k$ then using Inequality (4) of Definition 3.2 and the fact that $\|\mathcal{U}\|_k \geq 1$

$$|R(Z_0, k^*, j^*, \theta, \overline{\mathcal{U}})| \leq |[I_{k^*}]^2| \leq u_k \tag{9}$$

and hence

$$\left|\bigcup_{K_0\leq k^*<k}\left(\bigcup_{j^*\in J} R(Z_0,k^*,j^*,\theta,\overline{\mathcal{U}})\right)\right| \leq kJu_k \tag{10}$$

while if $j^* \in j$ then from the fact that $\overline{\mathcal{U}}$ is k-organized it follows that

$$|R(Z_0,k,j^*,\theta,\overline{\mathcal{U}})| \leq |[B(k,j^*,\theta,\overline{\mathcal{U}})]^2| = E_{k,\|\overline{\mathcal{U}}\|_k+j^*} \leq E_{k,\|\overline{\mathcal{U}}\|_k+j-1}. \tag{11}$$

Combining Inequalities (10) and (11) it follows that

$$\left|\bigcup_{K_0\leq k^*<k}\bigcup_{j^*\in J} R(Z_0,k^*,j^*,\theta,\overline{\mathcal{U}}) \cup \bigcup_{j^*\in j} R(Z_0,k,j^*,\theta,\overline{\mathcal{U}})\right| < kJu_k + jE_{k,\|\overline{\mathcal{U}}\|_k+j-1} \tag{12}$$

Using Lemma 3.2, Fact 3.4 and Property (d) of $\overline{\mathcal{U}}$ the number of $\sim_{k,j}$ equivalence classes in $\overline{\mathcal{U}}$ is bounded by

$$u_k^{j+1}\prod_{i\in j} E_{k,\|\overline{\mathcal{U}}\|_k+i} \leq u_k^{j+1}E^j_{k,\|\overline{\mathcal{U}}\|_k+j-1}$$

and it follows from Inequality (12) that

$$\left|\bigcup_{\theta\in\overline{\mathcal{U}}}\left(\bigcup_{K_0\leq k^*<k}\bigcup_{j^*\in J} R(Z_0,k^*,j^*,\theta,\overline{\mathcal{U}}) \cup \bigcup_{j^*\in j} R(Z_0,k,j^*,\theta,\overline{\mathcal{U}})\right)\right| < u_k^{j+1}E^j_{k,\|\overline{\mathcal{U}}\|_k+j-1}(kJu_k + jE_{k,\|\overline{\mathcal{U}}\|_k+j-1}). \tag{13}$$

Since Fact 3.2 and Inequality (4) of Definition 3.2 imply that

$$|V| \leq |\overline{\mathcal{U}}[K_0,J]| \leq u^J_{K_0} \leq u^J_k$$

and since $j < J \leq K_0 \leq k$ it follows that

$$|R_{k,j}| < u_k^{j+1}E^j_{k,\|\overline{\mathcal{U}}\|_k+j-1}\left(kJu_k + jE_{k,\|\overline{\mathcal{U}}\|_k+j-1}\right) + u^J_k \leq u_k^{J+1}(2kJ+1)E^J_{k,\|\overline{\mathcal{U}}\|_k+j-1} \leq u_k^{k+1}3k^2E^k_{k,\|\overline{\mathcal{U}}\|_k+j-1}. \tag{14}$$

Fix k, j and θ. Let $(Z_1)_{k,j,\theta}$ be as defined in Definition 3.8 and consider two cases.

Case 1. If there is $\ell \in 3$ such that $(k,j,\theta,\ell) \in C(\overline{\mathcal{U}}, Z_1)$ then let

$$B^*(k,j,\theta) = B(k,j,\theta,\overline{\mathcal{U}}) \setminus \bigcup (Z_1)^{-1}_{k,j,\theta}(R_{k,j}).$$

Case 2. If there is no ℓ such that $(k,j,\theta,\ell) \in C(\overline{\mathcal{U}}, Z_1)$ then let

$$B^*(k,j,\theta) = B(k,j,\theta,\overline{\mathcal{U}}).$$

Notice that it easily follows from Equation (3) of Definition 3.2 that $E_{k,\ell+1} \geq u_k^{k+1}3k^2E^k_{k,\ell} + E_{k,\ell}$. Using this with $\ell = \|\overline{\mathcal{U}}\|_k + j - 1$ it follows that

$$E_{k,\|\overline{\mathcal{U}}\|_k+j} \geq u_k^{k+1}3k^2E^k_{k,\|\overline{\mathcal{U}}\|_k+j-1} + E_{k,\|\overline{\mathcal{U}}\|_k+j-1}$$

and combining this with Inequality (14) and keeping in mind that Z_1 is one-to-one, it follows that

$$|B^*(k,j,\theta)| \geq |B(k,j,\theta,\overline{\mathcal{U}})| - |R_{k,j}| \geq E_{k,\|\overline{\mathcal{U}}\|_k+j} - u_k^{k+1} 3k^2 E^k_{k,\|\overline{\mathcal{U}}\|_k+j-1} \geq E_{k,\|\overline{\mathcal{U}}\|_k+j-1}.$$

Now apply Lemma 3.7 to find $\mathcal{U}^* \subseteq \overline{\mathcal{U}}$ such that

- $\mathcal{U}^* \in \mathcal{S}[K_1, J]$
- $B(k,j,\theta,\mathcal{U}^*) = B^*(k,j,\theta)$ provided that $K_0 \leq k \leq K_1$, $j < J$ and $\theta \in \mathcal{U}^*[k,j]$
- $\mathcal{U}^*[K_0,J] = \mathcal{U}[K_0,J]$.

It follows that $\|\mathcal{U}^*\|_k \geq \|\overline{\mathcal{U}}\|_k - 1 = \|\mathcal{U}\|_k - (J+1)$ provided that $K_0 \leq k < K_1$ or, in other words, Conclusion (1) holds. The fact that (4) holds follows directly from the choice of V and the construction of $\mathcal{U}^*$.

To see that (3) holds suppose that

- $K_0 \leq k_i < K_1$ for $i \in 2$
- $j_i < J$ for $i \in 2$
- $(k_i, j_i, \theta_i, \ell_i) \in C(\mathcal{U}^*, Z_i)$ for $i \in 2$
- (k_0, j_0) is lexicographically less than (k_1, j_1).

Then to show that

$$R(Z_0,k_0,j_0,\theta_0,\mathcal{U}^*) \cap R(Z_1,k_1,j_1,\theta_1,\mathcal{U}^*) = \varnothing$$

it must be shown that, letting $(Z_1)_{k_1,j_1,\theta_1} = Z$,

$$R(Z_0,k_0,j_0,\theta_0,\mathcal{U}^*) \cap Z(B(k_1,j_1,\theta_1,\mathcal{U}^*)) = R(Z_0,k_0,j_0,\theta_0,\mathcal{U}^*) \cap Z(B^*(k_1,j_1,\theta_1)) = \varnothing.$$

Consider first the case that $k_0 < k_1$. It follows from the construction that

$$R(Z_0,k_0,j_0,\theta_0,\mathcal{U}^*) \subseteq R(Z_0,k_0,j_0,\theta_0,\overline{\mathcal{U}}) \subseteq R_{k_1,j_1}$$

and so either

$$B(k_1,j_1,\theta_1,\mathcal{U}^*) \cap Z^{-1}(R(Z_0,k_0,j_0,\theta_0,\mathcal{U}^*)) = \varnothing$$

or

$$B(k_1,j_1,\theta_1,\mathcal{U}^*) \cap \bigcup Z^{-1}(R(Z_0,k_0,j_0,\theta_0,\mathcal{U}^*)) = \varnothing.$$

The final case to consider is that $k_0 = k_1$ and $j_0 < j_1$ and a similar argument works here. □

Lemma 3.15. *Suppose that*

(a) $J \leq K_0 \leq K_1$
(b) $\mathcal{U} \in \mathcal{S}[K_1, J]$
(c) $\mathcal{U}$ *is* k*-organized if* $K_0 \leq k < K_1$
(d) $Z^i : \mathcal{U} \to \omega$ *for* $i \in 2$

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ such that

(e) $\mathcal{U}^* \in \mathcal{S}[K_1, J]$
(f) $\mathcal{U}^*[K_0, J] = \mathcal{U}[K_0, J]$
(g) if $K_0 \leq k < K_1$ *and* $j < J$ *and the following two conditions hold*

$$(\forall i \in 2)\ (k, j, \theta_i, \ell_i) \in C(\mathcal{U}^*, Z^i) \tag{15}$$

$$\theta_0 \nsim_{k,j} \theta_1 \tag{16}$$

then $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - 2$ *for* $k \geq K_0$ $R(Z^0, k, j, \theta_0, \mathcal{U}^*) \cap R(Z^1, k, j, \theta_1, \mathcal{U}^*) = \varnothing$.

Proof. The first step will be to show that for all k and j and $L \leq F_{k,\|\mathcal{U}\|_k+j-1}$ and $\{[\theta_\ell]_{\sim_{k,j}}\}_{\ell\in L}$ enumerating the $\sim_{k,j}$ equivalence classes there are $B_{k,j,\ell} \subseteq B(k, j, \theta_\ell, \mathcal{U})$ such that

(a) $\mathbf{Ramsey}^L_{b_k}\left(2^L|B_{k,j,\ell}|\right) \geq |B(k, j, \theta_\ell, \mathcal{U})|$
(b) if $\theta_{\ell_0} \nsim_{k,j} \theta_{\ell_1}$ and $(k, j, \theta_{\ell_i}, m) \in C(\mathcal{U}, Z^i)$ for each $i \in 2$ then $Z^0_{k,j,\theta_{\ell_0}}(B_{k,j,\ell_0}) \cap Z^1_{k,j,\theta_{\ell_1}}(B_{k,j,\ell_1}) = \varnothing$

where $Z^i_{k,j,\theta^*} : [B(m, j, \theta^*, \mathcal{U})]^2 \to \omega$ is as defined in Definition 3.8 for $C(\mathcal{U}, Z^i)$. By Lemma 3.7 it then follows that there is some $\mathcal{U}^* \subseteq \mathcal{U}$ such that

- $\mathcal{U}^* \in \mathcal{S}[K_1, J]$
- $B(k, j, \theta, \mathcal{U}^*) = B_{k,j,\ell}$ provided that $K_0 \leq k < K_1$, $j < J$ and $\theta \in \mathcal{U}^*[k, j]$ and $\theta \sim_{k,j} \theta_\ell$
- $\mathcal{U}^*[K_0, J] = \mathcal{U}[K_0, J]$.

From (a) it will then follow that $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - 1$ for $k \geq K_0$

To see that it is possible to obtain Conditions (a) and (b) begin by fixing k and j. Using Inequality (4) of Definition 3.2 it follows that

$$F_{k,\|\mathcal{U}\|_k+j-1} > u_k^{j+1} \prod_{i\leq j-1} E_{k,\|\mathcal{U}\|_k+i}$$

and so it is possible to apply Lemma 3.2 to find $L \leq F_{k,\|\mathcal{U}\|_k+j-1}$ and an enumeration $\{\theta_n\}_{n\in L} \subseteq \mathcal{U}\langle\theta\rangle$ such that each $\sim_{k,j}$ equivalence classes of $\mathcal{U}\langle\theta\rangle$ is represented in the enumeration.

Then apply Corollary 3.1 with $S_n = B(k, j, \theta_n, \mathcal{U})$ and

$$Z^i_n = Z^i_{k,j,\theta_n} \tag{17}$$

for $n \in L$ and $i \in 2$ noting that Hypothesis (2) of Corollary 3.1 is satisfied since $(k, j, \theta_n, \ell_n) \in C(\mathcal{U}^*, Z^i)$. To see that Hypothesis (3) is satisfied let $m = E_{k,\|\mathcal{U}\|_k+j-1}$. Then, using Inequality (3) in Definition 3.2,

$$|B(k, j, \theta_\ell, \mathcal{U})| = E_{k,\|\mathcal{U}\|_k+j} = \mathbf{Ramsey}^{F^2_{k,\|\mathcal{U}\|_k+j-1}}_{b_k}\left(2^{F_{k,\|\mathcal{U}\|_k+j-1}} E_{k,\|\mathcal{U}\|_k+j-1}\right) \geq \mathbf{Ramsey}^L_2\left(2^L E_{k,\|\mathcal{U}\|_k+j-1}\right)$$

and this yields $B_{k,j,\ell} \subseteq B(k, j, \theta_\ell, \mathcal{U})$ such that

(c) $|B_{k,j,\ell}| = E_{k,\|\mathcal{U}\|_k+j-1}$
(d) $Z^0_n([B_{k,j,n}]^2) \cap Z^1_m([B_{k,j,m}]^2) = \varnothing$ for $n < m < L$.

Therefore $\mathbf{Ramsey}^L_{b_k}(2^L|B_{k,j,\ell}|) \geq |B(k, j, \theta_n, \mathcal{U})|$ and so Condition (a) holds. Note that by Equation (c) it follows that $\mathcal{U}^*$ is k-organized. □

Lemma 3.16. *Suppose that*

(1) $J \leq K_0 \leq K_1$
(2) $\mathcal{U} \in \mathcal{S}[K_1, J]$
(3) $Z_\ell : \mathcal{U} \to \omega$ *for* $\ell \in 2$ *are such that* $Z_0(\theta) \neq Z_1(\theta)$ *for all* $\theta \in \mathcal{U}$.

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ *such that*

(4) $\mathcal{U}^* \in \mathcal{S}[K_1, J]$
(5) $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - (6J + 5)$ *for* $k \geq K_0$
(6) $\mathcal{U}^*[K_0, J] = \mathcal{U}[K_0, J]$
(7) *if* $(k_i, j_i, \theta_i, \ell_i) \in C(\mathcal{U}, Z_i)$ *for each* $i \in 2$ *and one of following three options holds:*
   (a) $k_0 = k_1 = k$ *and* $j_0 = j_1 = j$ *and* $\theta_0 \nsim_{k,j} \theta_1$
   (b) $(k_0, j_0) \neq (k_1, j_1)$
   (c) $\theta_0 \sim_{k,j} \theta_1$ *and* $\ell_0 = \ell_1$

   then

$$R(Z_0, k_0, j_0, \theta_0, \mathcal{U}^*) \cap R(Z_1, k_1, j_1, \theta_1, \mathcal{U}^*) = \varnothing$$

(8) *if* $i_0 \neq i_1$ *and* $\theta \in \mathcal{U}[K_0, J]$ *and* Z_{i_0} *has constant value* v *on* $\mathcal{U}^*\langle\theta\rangle$ *then* $v \notin R(Z_{i_1}, k_{i_1}, j_{i_1}, \theta, \mathcal{U}^*)$.

Proof. Using Lemma 3.8 find $\mathcal{U}_1^* \subseteq \mathcal{U}$ such that Conclusion (6) holds and

(a) $\mathcal{U}_1^* \in \mathcal{S}[K_1, J]$
(b) $\mathcal{U}_1^*$ is k-organized for all k such that $K_0 \leq k < K_1$
(c) $\|\mathcal{U}_1^*\|_k \geq \|\mathcal{U}\|_k - J$ for $k \geq K_0$.

Then apply Lemma 3.15 to get $\mathcal{U}_2^* \subseteq \mathcal{U}_1^*$ satisfying the conclusion of (7) under hypothesis (7a) such that $\|\mathcal{U}_2^*\|_k \geq \|\mathcal{U}_1^*\|_k - 2$ for $k \geq K_0$. Then apply Lemma 3.8 again to find $\mathcal{U}_3^* \subseteq \mathcal{U}_2^*$ such that Conclusion (6) and Conditions (a), (b) and (c) hold with $\mathcal{U}_3^*$ in place of $\mathcal{U}_1^*$. Then use Corollary 3.2 to get $\mathcal{U}_4^* \subseteq \mathcal{U}_3^*$ such that $\|\mathcal{U}_4^*\|_k \geq \|\mathcal{U}_3^*\|_k - 1$ the conclusion of (7) follows from hypothesis (7 c). Then apply Lemma 3.8 again and then Lemma 3.14 twice, once for the pair (Z_0, Z_1) and again for (Z_1, Z_0), to get $\mathcal{U}^* \subseteq \mathcal{U}_4^*$ such that the conclusion of (7) holds under hypothesis (7b) and such that $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}_4^*\|_k - 2J - 2(J+1)$. This also gives (8) and (5). □

Corollary 3.3. *Suppose that*

(a) $J \leq K_0 \leq K_1$
(b) $\mathcal{U} \in \mathcal{S}[K_1, J]$
(c) $Z : \mathcal{U} \to [\omega]^M$ *and* $Z_m(\theta)$ *be the* m^{th} *element of* $Z(\theta)$.

There is then $\mathcal{U}^* \subseteq \mathcal{U}$ *such that*

- $\mathcal{U}^* \in \mathcal{S}[K_1, J]$
- $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - M^2(6J + 5)$ *for* $k \geq K_0$
- $\mathcal{U}[K_0, J] = \mathcal{U}^*[K_0, J]$

- *if* $m_i \in M$ *and* $(k_i, j_i, \theta_i, \ell_i) \in C(\mathcal{U}, Z_{m_i})$ *and one of following three options holds:*

  (1) $k_0 = k_1 = k$ *and* $j_0 = j_1 = j$ *and* $\theta_0 \nsim_{k,j} \theta_1$
  (2) $(k_0, j_0) \neq (k_1, j_1)$
  (3) $\theta_0 \sim_{k,j} \theta_1$ *and* $\ell_0 = \ell_1$

  then

$$R(Z_{m_0}, k_0, j_0, \theta_0, \mathcal{U}^*) \cap R(Z_{m_1}, k_1, j_1, \theta_1, \mathcal{U}^*) = \varnothing$$

- *if* $m_0 \neq m_1$ *are in* M *and* $\theta \in \mathcal{U}[K_0, J]$ *and* Z_{m_0} *has constant value* v *on* $\mathcal{U}^*\langle\theta\rangle$ *then* $v \notin R(Z_{m_1}, k_{m_1}, j_{m_1}, \theta, \mathcal{U}^*)$.

Proof. For each pair $(m, m') \in M^2$ apply Lemma 3.16. It needs to be noted that $Z_m(\theta) \neq Z_{m'}(\theta)$ for all $\theta \in \mathcal{U}$ and $m \neq m'$ by the definition of the Z_m and hence the hypotheses of Lemma 3.16 are satisfied. □

Theorem 3.2. *Suppose that*

- $J < K_0 \leq K_1$
- $\mathcal{U}[K_0, J] \in \mathcal{S}[K_0, J]$
- $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$
- $Z : \mathcal{U} \to [\omega]^{M_{K_0}}$.

There are then $\mathcal{U}^* \subseteq \mathcal{U}$ *and disjoint* A *and* B *such that*

- $\|\mathcal{U}^*\|_k \geq \|\mathcal{U}\|_k - (M_{K_0}^2(6J + 5) + M + 2)$ *for* $k \geq K_0$
- $\mathcal{U}[K_0] = \mathcal{U}^*[K_0]$
- $A \cap Z(\theta) \neq \varnothing \neq B \cap Z(\theta)$ *for all* $\theta \in \mathcal{U}^*$.

Proof. Let $M = M_{K_0}$. Let $Z_m(\theta)$ be the m^{th} element of $Z(\theta)$. To begin use Lemma 3.1 to find $\tilde{\mathcal{U}} \subseteq \mathcal{U}$ such that

(1) $\tilde{\mathcal{U}} \in \mathcal{S}[K_1, J]$
(2) $\mathbf{Ramsey}_{b_k}(|B(k, j, \theta, \tilde{\mathcal{U}})| \geq \left|\bigcup \mathbf{succ}_{\mathcal{U}\langle\theta,j\rangle}(\theta(j) \restriction k)\right|$ whenever $\theta \in \tilde{\mathcal{U}}$ and $j \in J$ and $K_0 \leq k \in K_1$
(3) $\tilde{\mathcal{U}}[K_0, J] = \mathcal{U}[K_0, J]$.

Using Lemma 3.11 inductively, find $\mathcal{U}_m$ for each $m \leq M$ such that

- $\mathcal{U}_0 = \tilde{\mathcal{U}}$
- $\|\mathcal{U}_m\|_k \geq \|\mathcal{U}_{m-1}\|_k - 1$
- $\mathcal{U}_m[K_0, J] = \mathcal{U}_{m-1}[K_0, J]$
- for each $\theta^* \in \mathcal{U}_m[K_0]$ either Z_m is constant on $\mathcal{U}_{m+1}\langle\theta^*\rangle$ or $C(\mathcal{U}_{m+1}, Z_m)$ is a front in $\mathcal{U}_{m+1}$.

Hence $\|\mathcal{U}_M\|_k \geq \|\tilde{\mathcal{U}}\|_k - M \geq \|\mathcal{U}\|_k - M - 1$.

For each $\theta \in \mathcal{U}_M$ let $\Psi_\theta : M \to (J + 1) \times 4$ be the mapping defined by

$$\Psi_\theta(m) = \begin{cases} (j_{\theta,m}, \ell_{\theta,m}) & \text{if this is defined as in Definition 3.8 with } (k_{\theta,m}, j_{\theta,m}, \theta, \ell_{\theta,m}) \in C(\mathcal{U}, Z_m) \\ (J, 3) & \text{otherwise.} \end{cases}$$

Note that $4(J+1)^M \leq 4K_0^M < b_{K_0}$. Hence, it is possible to use Lemma 3.4 in conjunction with Lemma 3.10 to find $\bar{\mathcal{U}} \subseteq \mathcal{U}_M$ such that for each $\theta \in \bar{\mathcal{U}}[K_0, J]$ and $m \in M$ there are $j^*_\theta(m)$ and $\ell^*_\theta(m)$ such that $\Psi_{\theta^*}(m) = (j^*_\theta(m), \ell^*_\theta(m))$ for each m and $\theta^* \in \bar{\mathcal{U}}\langle\theta\rangle$ and such that $\|\bar{\mathcal{U}}\|_k \geq \|\mathcal{U}\|_k - (M+2)$ for $k \geq K_0$. Then use Corollary 3.3 to get the conclusion of that corollary to hold on $\mathcal{U}^* \subseteq \bar{\mathcal{U}}$ and $\|\bar{\mathcal{U}}\|_k \geq \|\mathcal{U}^*\|_k - (M^2(6J + 5) + M + 2)$.

Let W be the function defined on M such that $W(m)$ is the function from $\mathcal{U}^*[K_0, J]$ to $4 \times K_0$ (noting that $J + 1 \leq K_0$) defined by $W(m)(\theta) = (\ell^*_\theta(m), j^*_\theta(m))$. Referring to Definition 3.2, let $\mathcal{M} \subseteq M$ and $W^* : \mathcal{U}^*[K_0, J] \to 4 \times K_0$ be such that $W(m) = W^*$ for all $m \in \mathcal{M}$ and note that

$$|\mathcal{M}| \geq 2u_{K_0-1} \geq |\mathbb{U}[K_0]|^J \geq 2|\mathcal{U}^*[K_0, J]|.$$

Let the coordinate functions of W^* be given by $W^* = (W^*_\ell, W^*_j)$.

Then for $\theta \in \mathcal{U}^*[K_0, J]$ and $m \in \mathcal{M}$ define

$$R^*_m(\theta) = \bigcup_{\theta^* \in \mathcal{U}^*\langle\theta\rangle} R(Z_m, k_{\theta^*}(Z_m), W^*_j(\theta), \theta^*, \mathcal{U}^*).$$

Let $\mathcal{Y} = \left\{\theta \in \mathcal{U}^*[K_0, J] \;\middle|\; W^*_j(\theta) = J\right\}$ or, in other words, $\mathcal{Y}$ consists of those $\theta \in \mathcal{U}^*[K_0, J]$ such that Z_m is constant on $\mathcal{U}_M\langle\theta\rangle$ for each $m \in \mathcal{M}$. Let $Y(\theta, m)$ be this constant value and define $Y_\theta = \{Y(\theta, m) \mid m \in \mathcal{M}\}$. Observing that $|Y_\theta| = |\mathcal{M}| \geq 2|\mathcal{U}^*[K_0, J]|$ it is easy to find disjoint A_0 and B_0 such that $A_0 \cap Y_\theta \neq \varnothing \neq B_0 \cap Y_\theta$ for $\theta \in \mathcal{Y}$.

Finally, let $\mathcal{Y}^* = \mathcal{U}^*[K_0, J] \setminus \mathcal{Y}$ let $m_a \in \mathcal{M}$ and $m_b \in \mathcal{M}$ be distinct and define

$$A = A_0 \cup \left(\bigcup_{\theta \in \mathcal{Y}^*} R^*_{m_a}(\theta)\right) \;\&\; B = B_0 \cup \left(\bigcup_{\theta \in \mathcal{Y}^*} R^*_{m_b}(\theta)\right).$$

To see that $\mathcal{U}^*$ and A and B satisfy the conclusion, two points need to be verified. The first is that if $\theta^* \in \mathcal{U}^*$ then $Z(\theta^*) \cap A \neq \varnothing \neq Z(\theta^*) \cap B$. Let $\theta = \theta^* \restriction (K_0, J)$. If $\theta \in \mathcal{Y}$ then it follows that $A_0 \cap Y_\theta \neq \varnothing \neq B_0 \cap Y_\theta$ and hence $A_0 \cap Z(\theta^*) \neq \varnothing \neq B_0 \cap Z(\theta^*)$. On the other hand, if $\theta \in \mathcal{Y}^*$ then

$$Z_{m_a}(\theta^*) \in R(Z_{m_a}, k_{\theta^*}(Z_{m_a}), W^*_j(\theta), \theta^*, \mathcal{U}^*) \subseteq R^*_{m_a}(\theta) \subseteq A$$

and the result follows. The same argument works for B.

The final point that needs to be checked is that $A \cap B = \varnothing$. The fact that $A_0 \cap B_0 = \varnothing$ follows from the construction. The fact that

$$\left(\bigcup_{\theta \in \mathcal{Y}^*} R^*_{m_a}(\theta)\right) \cap (A_0 \cup B_0) = \varnothing = \left(\bigcup_{\theta \in \mathcal{Y}^*} R^*_{m_b}(\theta)\right) \cap (A_0 \cup B_0)$$

follows immediately from the last clause of Corollary 3.3. The fact that $R^*_{m_a}(\theta) \cap R^*_{m_b}(\theta') = \varnothing$ for all θ and θ' in $\mathcal{Y}^*$ will be shown to follow from the first part of Corollary 3.3.

To see this it has to be shown that if $\theta_i \in \mathcal{Y}^*$ for $i \in 2$ and $\theta^*_i \in \mathcal{U}^*\langle\theta_i\rangle$ then

$$R(Z_{m_a}, k_{\theta^*_0}(Z_{m_a}), W^*_j(\theta_0), \theta^*_0, \mathcal{U}^*) \cap R(Z_{m_b}, k_{\theta^*_1}(Z_{m_b}), W^*_j(\theta_1), \theta^*_1, \mathcal{U}^*) = \varnothing.$$

If $k_{\theta^*_0}(Z_{m_a}) \neq k_{\theta^*_0}(Z_{m_b})$ or if $W^*_j(\theta_0) \neq W^*_j(\theta_1)$ then Corollary 3.3 can be directly applied, so assume that $k_{\theta^*_0}(Z_{m_a}) = k_{\theta^*_0}(Z_{m_b}) = k$ and $W^*_j(\theta_0) = W^*_j(\theta_1) = w$. If $\theta^*_0 \nsim_{k,w} \theta^*_1$ then again Corollary 3.3 can be directly applied, so it may be assumed that $\theta^*_0 \sim_{k,w} \theta^*_1$. In this case it must be verified that $\ell^*_{\theta^*_0}(Z_{m_a}) = \ell^*_{\theta^*_1}(Z_{m_b})$ and for this it suffices to show that $W^*_\ell(\theta_0) = W^*_\ell(\theta_1)$.

But this follows from the fact that $\theta_0^* \sim_{k,w} \theta_1^*$. To see this, note that $W(m_a) = W(m_b) = W^*$. Hence $\ell^*_{\theta_0^*}(Z_{m_a}) = \ell^*_{\theta_0^*}(Z_{m_b})$ and $\ell^*_{\theta_1^*}(Z_{m_a}) = \ell^*_{\theta_1^*}(Z_{m_b})$, as required. Moreover, since $\theta_0^* \sim_{k,w} \theta_1^*$ it follows from Definition 3.8 applied to Z_{m_a} that $\ell^*_{\theta_0^*}(Z_{m_a}) = \ell^*_{\theta_1^*}(Z_{m_a})$ and hence $\ell^*_{\theta_0^*}(Z_{m_a}) = \ell^*_{\theta_1^*}(Z_{m_b})$. □

4. The iteration

Definition 4.1. Let Γ be a finite subset of ω_2 of cardinality J enumerated in the ordinal ordering as $\Gamma = \{\gamma_j\}_{j\in J}$ and $p \in \mathbb{P}_{\omega_2}$. Recalling Definition 3.3, for $j \le J$ and $\sigma \in \mathbb{U}^j$ define $p_{\sigma,\Gamma}$ (which will be denoted by p_σ if the dependence on Γ is clear) by induction on j as follows:

(a) $p_\varnothing = p \restriction \gamma_0$
(b) if $i = |\sigma|$ and p_σ is defined and $t \in \mathbb{U}$ then

  (i) $p_{\sigma^\frown t} \restriction \gamma_i = p_\sigma$
  (ii) $p_{\sigma^\frown t}(\gamma_i) = \{s \in p(\gamma_i) \mid s \subseteq t \text{ or } t \subseteq s\}$
  (iii) $p_{\sigma^\frown t}(\gamma) = p(\gamma)$ if $\gamma_i < \gamma < \gamma_{i+1}$ where γ_J is defined to be ω_2.

The $p(\gamma)$ are, of course, $\mathbb{P}_\gamma$ names, but the reader will not be reminded of this by dots in forcing statements. Note that it may well be the case that $p_\sigma \notin \mathbb{P}_{\omega_2}$. Indeed, $p_\sigma \in \mathbb{P}_{\omega_2}$ precisely if

$$(\forall j \in \mathbf{domain}(\sigma))\ p_{\sigma\restriction j} \in \mathbb{P}_{\omega_2} \text{ and } p_{\sigma\restriction j} \Vdash_{\mathbb{P}_{\gamma_j}} \text{“}\sigma(j) \in p(\gamma_j)\text{”} \tag{18}$$

for every j in the domain of σ. Let $\mathcal{U}_{p,\Gamma,K} = \left\{\sigma \in \mathbb{U}[K]^J \mid p_\sigma \in \mathbb{P}_{\omega_2}\right\}$.

A condition p will be called (Γ, K)-determined if for each $\sigma \in \mathbb{U}[K]^J \setminus \mathcal{U}_{p,\Gamma,K}$ there is some $j \in J$ such that $p_{\sigma\restriction j} \in \mathbb{P}_{\omega_2}$ and $p_{\sigma\restriction j} \Vdash_{\mathbb{P}_{\gamma_j}}$ “$\sigma(j) \notin p(\gamma_j)$” and p will be called (Γ, K, N)-determined if, in addition,

$$(\forall\sigma \in \mathcal{U}_{p,\Gamma,K})(\forall j \in J)\ p_\sigma \Vdash_{\mathbb{P}_{\gamma_j}} \text{“}(\forall t \in p(\gamma_j)) \text{ if } |t| \ge K \text{ then } \|t\|_{p(\gamma_j)} \ge NM^3_{|t|}\text{”}. \tag{19}$$

Definition 4.2. If $\Gamma = \{\gamma_j\}_{j\in J}$ and $\mathcal{V} \subseteq \mathcal{U}_{p,\Gamma,K}$ define the $p^{\mathcal{V}}$ by defining $p^{\mathcal{V}} \restriction \beta$ by induction on β. For $\beta = 0$ there is nothing to do. For β a limit let $\bar\beta < \beta$ be so large that $\Gamma \subseteq \bar\beta$ and let

$$p^{\mathcal{V}} = (p \restriction \bar\beta)^{\mathcal{V}\frown}(p \restriction [\bar\beta, \beta)).$$

Given $p^{\mathcal{V}} \restriction \beta$ and $\beta \notin \Gamma$ define $p^{\mathcal{V}} \restriction \beta + 1$ by letting $p^{\mathcal{V}}(\beta) = p(\beta)$. If $\beta = \gamma_j$ then define $p^{\mathcal{V}}(\beta)$ by

$$(\forall\sigma \in \mathcal{U}_{p,\Gamma,K}[K, j])\ p_\sigma \restriction \beta \Vdash_{\mathbb{P}_\gamma} \text{“}p^{\mathcal{V}}(\beta) = \{\theta \in p(\gamma) \mid \sigma^\frown(\theta \restriction K) \in \mathcal{V}[K, j+1]\}\text{”}.$$

Lemma 4.1. *If p is (Γ, K)-determined and $\mathcal{V} \subseteq \mathcal{U}_{p,\Gamma,K}$ then $p^{\mathcal{V}} \in \mathbb{P}_{\omega_2}$.*

Proof. This is immediate from Definition 4.2 and induction on $|\Gamma|$. □

Lemma 4.2. *If the following hold:*

- *p is (Γ, K_0)-determined*
- $\mathcal{U}_{p,\Gamma,K_0} \in \mathcal{S}[K_0, |\Gamma|]$
- $q \le p$
- $\mathcal{U}_{p,\Gamma,K_0} = \mathcal{U}_{q,\Gamma,K_0}$

then $\mathcal{U}_{q,\Gamma,K_1} \in \mathcal{S}[K_0, K_1, |\Gamma|]$.

Proof. Suppose that $\theta \in \mathcal{U}_{q,\Gamma,K_1}$ and γ is the j^{th} element of Γ. Then $q_\theta \restriction \gamma \leq p_{\theta\restriction(K_0,j)} \restriction \gamma$ and $p_{\theta\restriction(K_0,j)} \restriction \gamma$ decides $p(\gamma)[K_0]$. Hence, $q_\theta \restriction \gamma$ makes the same decision. Since this is true for any $\theta^* \in \mathcal{U}_{p,\Gamma,K_0}\langle \theta \restriction (K_0, j)\rangle$ the result follows. □

Lemma 4.3. *If*

- $p \in \mathbb{P}_{\omega_2}$
- Γ *is a finite subset of* ω_2
- K_0 *and* N *are in* ω
- $p \Vdash_{\mathbb{P}_{\omega_2}}$ "$\dot{x} \in 2 \;\&\; \dot{y} \in \omega$"
- $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$|p(\gamma)[K_0]| = 1$" *for all* $\gamma \in \Gamma$

then there are $K_1 \in \omega$, $q \leq p$ *and* $n \in 2$ *and* $N \in \omega$ *such that* $q \Vdash_{\mathbb{P}_{\omega_2}}$ "$\dot{x} = \check{n} \;\&\; \dot{y} < \check{N}$" *and for all* $\gamma \in \Gamma$, *using the notation of Definition 3.3,*

$$q \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“ if } K_0 \leq |t| < K_1 \text{ then } \mathbf{Ramsey}_2\left(\left|\bigcup \mathbf{succ}_{q(\gamma)}(t)\right|\right) \geq \left|\bigcup \mathbf{succ}_{p(\gamma)}(t)\right| \text{”} \tag{20}$$

$$q \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“ if } K_1 \leq |t| \text{ then } \|t\|_{q(\gamma)} \geq NM^3_{|t|} \text{”.} \tag{21}$$

Proof. A standard rank argument, such as can be found in §7.3 of [1], can be used. For $T \in \mathbb{U}$ and $t \in T$ define $\mathbf{rank}(t) = 0$ if $\|t\|_T \geq NM^3_{|t|}$ and there is $T^* \subseteq \{s \in T \mid s \subseteq t \text{ or } t \subseteq s\}$ such that $T^* \Vdash_{\mathbb{P}}$ "$\dot{x} = \check{n}$ & $\dot{y} = \check{m}$" for some $n \in 2$ and $m \in \omega$. Then define $\mathbf{rank}(t) \leq r + 1$ if there is $S \subseteq I_{|t|}$ such that

$$[S]^2 \subseteq \{a \in \mathbf{succ}_T(t) \mid \mathbf{rank}(t^\frown a) \leq r\}$$

and

$$\mathbf{Ramsey}_2(|S|) \geq |\bigcup \mathbf{succ}_T(t)|.$$

This shows that if $\|t\|_T \geq 1$ for $t \supseteq t^*$ then $\mathbf{rank}(t^*)$ is defined. A standard induction yields the result for the iteration. □

Lemma 4.4. *Suppose that*

(a) $p \in \mathbb{P}_{\omega_2}$
(b) $\Gamma \in [\omega_2]^J$
(c) $\dot{z}$ *is a* $\mathbb{P}_{\omega_2}$ *name for a finite set of integers*
(d) $K_0 > J = |\Gamma|$ *and* N *are in* ω.

There are then q, K_1 *and* Z *such that:*

(e) q *is* (Γ, K_1, N)*-determined*
(f) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$q(\gamma)[K_0] = p(\gamma)[K_0]$" *for all* $\gamma \in \Gamma$
(g) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$(\forall t \in q(\gamma))$ *if* $|t| \geq K_0$ *then* $\|t\|_{q(\gamma)} \geq \|t\|_{p(\gamma)} - 1$" *for all* $\gamma \in \Gamma$
(h) $Z : \mathcal{U}_{q,\Gamma,K_1} \to [\omega]^{<\aleph_0}$ *and* $q_\sigma \Vdash_{\mathbb{P}_{\omega_2}}$ "$\dot{z} = Z(\sigma)$" *for all* $\sigma \in \mathcal{U}_{q,\Gamma,K_1}$.

Proof. Proceed by induction on $J = |\Gamma|$. In order to prove the general case a stronger induction hypothesis is required: There is $\bar{K} \in \omega$ such that for all $K_1 \geq \bar{K}$ there is $q \leq p$ and $\mathcal{U} \subseteq \mathbb{U}[K_1]^J$ such that conditions (e), (f), (g) and (h) are all satisfied for $\dot{z}$, q, K_0, K_1, $\mathcal{U}$ and Γ. If $\Gamma = \varnothing$ there is nothing to do and if $|\Gamma| = 1$, Lemma 4.3 can be applied.

Now suppose that the general result has been established if $|\Gamma| = J$ and that Γ is given such that $|\Gamma| = J+1$. Let γ be the minimum element of Γ and let $\tilde{\Gamma} = \Gamma \setminus \{\gamma\}$. Then find $\tilde{q} \leq p \restriction \gamma$ such that:

(i) there is $\tilde{K}$ such that $\tilde{q} \Vdash_{\mathbb{P}_\gamma}$ " the induction hypothesis holds for $p/\mathbb{P}_\gamma, \tilde{\Gamma}, \dot{z}/\mathbb{P}_\gamma$ and $\tilde{K}$"
(ii) there is K' such that $\tilde{q} \Vdash_{\mathbb{P}_\gamma}$ "$\|t\|_{p(\gamma)} > (N+1)M^3_{|t|}$ if $|t| \geq K'$".

Let $\bar{K} = \max(\tilde{K}, K')$ and suppose that $K_1 \geq \bar{K}$.

Using the stronger induction hypothesis and Lemma 4.3 it is possible to find and $q^* \leq \tilde{q}$ and $T \subseteq \mathbb{U} \restriction K_1$ such that

$$q^* \Vdash_{\mathbb{P}_\gamma} \text{"}T \subseteq p(\gamma) \text{ and } (\forall s \in T) \ \text{ if } K_0 \leq |s| < K_1 \text{ then } \|s\|_T \geq \|s\|_{p(\gamma)} - 1\text{"} \tag{22}$$

and, moreover, for each $t \in T[K_1]$ there are $\dot{T}_t$, $\dot{q}_t$, Z_t and $\mathcal{U}_t$ such that for each $t \in T[K_1]$

$$q^* \Vdash_{\mathbb{P}_\gamma} \text{"}p(\gamma)\langle t\rangle \supseteq \dot{T}_t\text{"} \tag{23}$$

$$q^* \Vdash_{\mathbb{P}_\gamma} \text{"}(\forall s \in \dot{T}_t) \ \text{ if } |s| \geq k \text{ then } \|s\|_{\dot{T}_t} \geq NM^3_{|s|}\text{"} \tag{24}$$

$$q^* \Vdash_{\mathbb{P}_\gamma} \text{"}\mathcal{U}_t = \dot{\mathcal{U}}_{\dot{q}_t, \tilde{\Gamma}, K_1}\text{"} \tag{25}$$

$$q^* * \dot{T}_t \Vdash_{\mathbb{P}_{\omega_2}} \text{"}\dot{q}_t, K_1, Z_t \text{ witness that conditions (e), (f), (g) and (h) of Lemma 4.4 hold."} \tag{26}$$

Let q be defined by letting $q(\gamma) = \bigcup_{t \in T[K_1]} \dot{T}_t$ and having $q^* * \dot{T}_t \Vdash_{\mathbb{P}_{\gamma+1}}$ "$q \restriction [\gamma+1, \omega_2) = \dot{q}_t$". Observe that (24), (22) and (ii) together imply that $q \Vdash_{\mathbb{P}_\gamma}$ "$(\forall s \in q(\gamma))$ if $K_1 \leq |s|$ then $\|s\|_T \geq NM^3_{|s|}$". Hence (19) of Definition 4.1 holds. Then let $Z(t^\frown\sigma) = Z_t(\sigma)$ for $t \in T^*[K_1]$ and $\sigma \in \mathcal{U}_t$. It follows easily that q, K_1 and Z are as required. □

Lemma 4.5. *If p is (Γ, K, N)-determined and $\gamma^* \notin \Gamma$ then there is $q \leq p$ such that:*

(1) q is $(\Gamma \cup \{\gamma^*\}, K, N)$-determined
(2) $\mathcal{U}_{p,\Gamma,K} \in \mathcal{S}[K, |\Gamma|]$
(3) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$p(\gamma)[K] = q(\gamma)[K]$" for all $\gamma \in \Gamma$
(4) $\mathcal{U}_{q,\Gamma\cup\{\gamma^*\},K} \in \mathcal{S}[K, |\Gamma|+1]$
(5) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$(\forall t \in q(\gamma))$ if $|t| > K$ then $\|t\|_{p(\gamma)} - 1 \leq \|t\|_{q(\gamma)}$" for all $\gamma \in \Gamma$.

Proof. Proceed by induction on $|\Gamma|$ the case that $\Gamma = \varnothing$ being immediate. For the general case proceed as in Lemma 4.4. □

Theorem 4.1. *It is consistent that $\mathfrak{s}_{2,2} = \aleph_2$ and $\mathfrak{s}_{2,\infty} = \aleph_1$.*

Proof. It should be clear that $\mathbb{P}$ is proper and ω^ω-bounding by Lemma 4.3. Moreover, the choice of the $E_{k,j}$ guarantees (see Definition 3.2) that if $P \subseteq I_k$ and $|P| \geq E_{k,j+1}$ and $X \subseteq I_k$ then either $|P \cap X| \geq E_{k,j}$ or $|P \setminus X| \geq E_{k,j}$ and this implies that if $G_\xi \in \prod_{n\in\omega}[I_n]^2$ is the generic sequence added by $\mathbb{P}_{\omega_2}$ at stage ξ then for each $X \subseteq \omega$ such that $X \in V^{\mathbb{P}_\xi}$ there are only finitely many n such that $|G_\xi(n) \cap X| = 1$. This shows that

$$1 \Vdash_{\mathbb{P}_{\omega_2}} \text{"}\mathfrak{s}_{2,2} = \aleph_2\text{"}. \tag{27}$$

Using Observation (27) it suffices to show that $1 \Vdash_{\mathbb{P}_{\omega_2}}$ "$\mathfrak{s}_{2,\infty} = \aleph_1$" so suppose that

$$p \Vdash_{\mathbb{P}_{\omega_2}} \text{"}\dot{\mathcal{Z}} \in [\omega]^{<\aleph_0} \ \& \ \limsup_{z \in \dot{\mathcal{Z}}} |z| = \infty\text{"}.$$

Using the countable support, it suffices to show that there are $p_n \in \mathbb{P}_{\omega_2}$, $\Gamma_n \in [\omega_2]^n$, positive integers K_n, finite sets A_n and B_n and names $\dot{z}_n$ such that:

(a) $p_0 = p$
(b) $A_n \cap B_n = \varnothing$
(c) $\min(A_{n+1} \cup B_{n+1}) > \max(A_n \cup B_n)$
(d) $p_n \Vdash_{\mathbb{P}_{\omega_2}}$ "$\dot{z}_n \in \dot{\mathcal{Z}}$ & $\min(\dot{z}_n) > n$"
(e) $p_n \Vdash_{\mathbb{P}_{\omega_2}}$ "$\dot{z}_n \cap A_n \neq \varnothing \neq \dot{z}_n \cap B_n$"
(f) $K_{n+1} > K_n$
(g) p_n is $(K_n, \Gamma_n, n+3)$-determined
(h) $p_{n+1} \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$(\forall t \in p_{n+1}(\gamma))$ if $|t| \geq K_n$ then $\|t\|_{p_{n+1}(\gamma)} \geq nM^3_{K_n}$" for all $\gamma \in \Gamma_n$
(i) $\mathcal{U}_{p_{n+1},\Gamma_n,K_n} = \mathcal{U}_{p_n,\Gamma_n,K_n}$
(j) $\mathcal{U}_{p_n,\Gamma_n,K_n} \in \mathcal{S}[K_n, |\Gamma_n|] = \mathcal{S}[K_n, n]$
(k) $\Gamma_{n+1} \supseteq \Gamma_n$
(l) $\bigcup_n \Gamma_n = \bigcup_n \mathbf{domain}(p_n)$.

Given this, it follows from Inductive Hypotheses (g), (h), (i) and (k) that a standard fusion argument establishes that there is $q \in \mathbb{P}_{\omega_2}$ such that $q \leq p_n$ for all n. Using (c) and (b) it is possible to define $A = \bigcup_{n\in\omega} A_n$ and $B = \bigcup_{n\in\omega} B_n$ such that $A \cap B = \varnothing$ and, using (e), such that

$$q \Vdash_{\mathbb{P}_{\omega_2}} \text{"}(\forall n)(\exists z \in \dot{\mathcal{Z}})\ \min(z) \geq n\ \&\ z \cap A \neq \varnothing \neq z \cap B\text{"} \tag{28}$$

thus establishing that $\mathfrak{s}_{2,\infty} = \aleph_1$ after forcing with $\mathbb{P}_{\omega_2}$ over a model of $2^{\aleph_0} = \aleph_1$.

To carry out the inductive construction, suppose that $p_n \in \mathbb{P}_{\omega_2}$, $\Gamma_n \in [\omega_2]^n$, K_n, A_n, B_n and z_n have been constructed. Let $\Gamma_{n+1} = \Gamma_n \cup \{\gamma^*\}$ where γ^* has been chosen according to some scheme that will guarantee that (l) will be satisfied and, of course, $|\Gamma_{n+1}| = n+1$. Let $\dot{z}_{n+1}$ be a name such that

$$1 \Vdash_{\mathbb{P}_{\omega_2}} \text{"}\dot{z}_{n+1} \in \dot{\mathcal{Z}}\ \&\ \&\ \min(\dot{z}_{n+1}) > n+1\ \&\ |\dot{z}_{n+1}| \geq M_{K_n}\text{"}.$$

Using Lemma 4.5 and Hypothesis (g) to find $q \leq p_n$ such that:

(m) q is $(\Gamma_{n+1}, K_n, n+2)$-determined
(n) $\mathcal{U}_{q,\Gamma_{n+1},K_n} \in \mathcal{S}[K_n, n+1]$
(o) $q \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$(\forall t \in q(\gamma))$ if $|t| > K_n$ then $\|t\|_{p_n(\gamma)} - 1 \leq \|t\|_{q(\gamma)}$" for all $\gamma \in \Gamma_n$
(p) $\mathcal{U}_{q,\Gamma_n,K_n} = \mathcal{U}_{p_n,\Gamma_n,K_n}$.

Then use Lemma 4.4 to find $\bar{q} \leq q$, $K_{n+1} \in \omega$ and Z such that:

(q) $\bar{q}$ is $(\Gamma_{n+1}, K_{n+1}, n+4)$-determined
(r) $\mathcal{U}_{q,\Gamma_{n+1},K_n} = \mathcal{U}_{\bar{q},\Gamma_{n+1},K_n}$
(s) $\bar{q} \restriction \gamma \Vdash_{\mathbb{P}_\gamma}$ "$(\forall t \in \bar{q}(\gamma))$ if $|t| \geq K_n$ then $\|t\|_{\bar{q}(\gamma)} \geq \|t\|_{q(\gamma)} - 1$" for all $\gamma \in \Gamma_{n+1}$
(t) $\bar{q} \Vdash_{\mathbb{P}_{\omega_2}}$ "$\dot{z}_{n+1} \in \dot{\mathcal{Z}}$ & $\min(\dot{z}_{n+1}) \geq n+1$ & $|\dot{z}_{n+1}| \geq M_{K_n}$"
(u) there is $Z : \mathcal{U}_{\bar{q},\Gamma_{n+1},K_{n+1}} \to [\omega]^{M_{K_n}}$ such that $\bar{q}_\sigma \Vdash_{\mathbb{P}_{\omega_2}}$ "$\dot{z}_{n+1} = Z(\sigma)$" for all $\sigma \in \mathcal{U}_{\bar{q},\Gamma_{n+1},K_{n+1}}$.

Observe that it follows from Induction Hypotheses (g) and Conditions (s) and (o) that

$$\bar{q} \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{"}(\forall t \in \bar{q}(\gamma))\ \text{ if } |t| \geq K_n \text{ then } \|t\|_{\bar{q}(\gamma)} \geq (n+3)M^3_{|t|} - 2 \geq (n+2)M^3_{K_n}\text{"}$$

and hence

$$\|\mathcal{U}_{\bar{q},\Gamma_{n+1},K_{n+1}}\|_k \geq (n+2)M^3_{K_n} \tag{29}$$

for all $k \geq K_n$. From Lemma 4.2 it follows that $\mathcal{U}_{\bar{q},\Gamma_{n+1},K_{n+1}} \in \mathcal{S}[K_0, K_1, n+1]$ so it is possible to use Lemma 3.1 to find $\mathcal{U} \subseteq \mathcal{U}_{\bar{q},\Gamma_{n+1},K_n}$ such that

(v) $\mathcal{U} \in \mathcal{S}[K_{n+1}, n+1]$
(w) $\|\mathcal{U}\|_k \geq \|\mathcal{U}_{\bar{q},\Gamma_{n+1},K_n}\|_k - 1$ for $k \geq K_n$
(x) $\mathcal{U}[K_n, n+1] = \mathcal{U}_{\bar{q},\Gamma_{n+1},K_n}[K_n, n+1]$.

Then use Theorem 3.2 and Condition (t) to find $\tilde{\mathcal{U}} \subseteq \mathcal{U}$ and disjoint A_{n+1} and B_{n+1} such that

$$(\forall k \in [K_n, K_{n+1}))\ \|\tilde{\mathcal{U}}\|_k \geq \|\mathcal{U}\|_k - (M^2_{K_n}(6n+5) + M_{K_n} + 2) \tag{30}$$

$$\tilde{\mathcal{U}}[K_n] = \mathcal{U}[K_n] = \mathcal{U}_{\bar{q},\Gamma_{n+1},K_{n+1}}[K_n] \tag{31}$$

$$(\forall \theta \in \mathcal{U})\ A_{n+1} \cap Z(\theta) \neq \varnothing \neq B_{n+1} \cap Z(\theta). \tag{32}$$

Note that from Condition (30) it follows that if $K_n \leq k < K_{n+1}$ that then

$$\|\tilde{\mathcal{U}}\|_k \geq \|\mathcal{U}_{\bar{q},\Gamma_{n+1},K_n}\|_k - (M^2_{K_n}(6n+5) + M_{K_n} + 2) - 2 \geq M^3_{K_n}(n+2) - M^3_{K_n} \geq (n+1)M^3_{K_n}. \tag{33}$$

Then let $p_{n+1} = \bar{q}^{\mathcal{U}}$. It then follows from (o) and (s) that Induction Hypothesis (h) holds. It follows from Condition (r) and Equation (31) that Induction Hypothesis (i) holds. Condition (t) guarantees that Induction Hypothesis (d) holds. Condition (q) ensures that Induction Hypothesis (g) will be satisfied by p_{n+1}. Of course, Induction Hypothesis (e) follows from Condition (32). □

5. Some more cardinal invariants

Those readers who have followed the proof of Theorem 4.1 may well be asking themselves whether better results are possible. In order to formulate precise questions along these lines it is worth introducing some new cardinal invariants that incorporate ideas already found in the definitions of $\mathfrak{s}_{1/2\pm\epsilon}$ and $\mathfrak{s}_{1/2\pm\epsilon}$.

Definition 5.1. For $\epsilon > 0$ define $\mathfrak{s}_{k,\epsilon}$ to be the least cardinal of a family $\mathcal{F} \subseteq k^\omega$ such that for each infinite, pairwise disjoint family $\mathcal{A} \subseteq [\omega]^{<\aleph_0}$ whose elements have unbounded cardinality there is $F \in \mathcal{F}$ such that for infinitely many $a \in \mathcal{A}$

$$\frac{1-\epsilon}{k} < \frac{|a \cap F^{-1}(j)|}{|a|} < \frac{1+\epsilon}{k}$$

for all $j \in k$. Define $\mathfrak{s}_{k,0}$ to be the least cardinal of a family $\mathcal{F} \subseteq k^\omega$ such that for each infinite, pairwise disjoint family $\mathcal{A} \subseteq [\omega]^{<\aleph_0}$ whose elements have unbounded cardinality there is $F \in \mathcal{F}$ such that

$$\liminf_{a\in\mathcal{A}} \left(\max_{j\in k} \left(\frac{|a \cap F^{-1}(j)|}{|a|} - 1/k \right) \right) = 0.$$

Other variations of the splitting cardinals also come to mind.

Definition 5.2. Let $2 \leq m \leq k$ and let $\mathfrak{s}^*_{m,k}$ be the least cardinal of a family $\mathcal{F} \subseteq m^\omega$ such that for any infinite, pairwise disjoint family $\mathcal{A} \subseteq [\omega]^k$ there is $F \in \mathcal{F}$ such that for any non-empty $x \subseteq m$ there are infinitely many $a \in \mathcal{A}$ such that $F[a] = x$.

Finally, recall that $\mathfrak{s}_{\omega_1}$ is the following stronger version of the statement $\mathfrak{s} = \aleph_1$: There is a family $\{S_\xi\}_{\xi\in\omega_1}$ such that for each infinite $X \subseteq \omega$ there is $\beta \in \omega_1$ such that $|S_\alpha \cap X| = \aleph_0 = |X \setminus S_\alpha|$ for all $\alpha > \beta$. The following definition extends this to the current context.

Definition 5.3. Define $\mathfrak{s}^{\omega_1}_{k,m}$ to be the assertion that there is family $\{f_\eta\}_{\eta\in\omega_1}$ such that $f_\eta : \omega \to k$ and for each infinite, pairwise disjoint family $\mathcal{A} \subseteq [\omega]^m$ there is $\beta \in \omega_1$ such that $f_\eta[a] = k$ for infinitely many $a \in \mathcal{A}$ and each $\eta > \beta$.

It can easily be checked that the proof of Corollary 2.1 shows that $\mathfrak{s}^{\omega_1}_{2,m}$ holds for some m if and only if $\mathfrak{s}^{\omega_1}_{2,2}$ holds. However, the proof of Lemma 2.4 does not seem to extend to show that $\mathfrak{s}^{\omega_1}_{k,m}$ holds for some k and m if and only if $\mathfrak{s}^{\omega_1}_{2,2}$ holds. These questions will be considered in a forthcoming paper.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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