TORSION-FREE ABELIAN GROUPS ARE FAITHFULLY BOREL COMPLETE AND PURE EMBEDDABILITY IS A COMPLETE ANALYTIC QUASI-ORDER

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ABSTRACT. In [9] we proved that the space of countable torsion-free abelian groups is Borel complete. In this paper we show that our construction from [9] satisfies several additional properties of interest. We deduce from this that countable torsion-free abelian groups are faithfully Borel complete, in fact, more strongly, we can $\mathfrak{L}_{\omega_1,\omega}$-interpret countable graphs in them. Secondly, we show that the relation of pure embeddability (equiv., elementary embeddability) among countable models of $\mathrm{Th}(\mathbb{Z}^{(\omega)})$ is a complete analytic quasi-order.

1. Introduction

In [9][1] we showed that the Borel space of countable torsion-free abelian groups (TFAB_ω) is as complex as possible in terms of classification up to isomorphism, resolving a major conjecture of Friedman and Stanley from '89 (cf. [3]). The aim of this paper is to show that our construction from [9] satisfies several additional properties of interest, which imply stronger anti-classification results for the space TFAB_ω. In particular, we will prove the following (see what follows for a discussion):

Theorem 1.1. TFAB_ω *is a faithfully Borel complete class of structures. In fact, more strongly, we can* $\mathfrak{L}_{\omega_1,\omega}$*-interpret the space* Graphs_ω *into the space* TFAB_ω.

Theorem 1.2. *The pure embeddability relation on* TFAB_ω *is a complete analytic quasi-order. In fact, more strongly, elementary embeddability (equiv., pure embeddability) between countable models of* $\mathrm{Th}(\mathbb{Z}^{(\omega)})$ *is a complete analytic quasi-order.*

The property of Borel completeness for the space of countable models of a theory in $\mathfrak{L}_{\omega_1,\omega}$ is probably the most well-known anti-classification property in terms of classification up to isomorphism, as it literally says that the isomorphism relation on such a class reduces in a Borel way the isomorphism relation on countable models of *any* theory in $\mathfrak{L}_{\omega_1,\omega}$. But, actually, stronger forms of anti-classification are known in the literature, for example, the fact that countable groups can be first-order interpreted in countable graphs (cf. e.g. [7]) is widely agreed to be a much stronger result than the Borel completeness of the space of countable groups. This line of thought was already addressed by Friedman and Stanley in their seminal paper

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[1]As of 04.12.2023, [9] has been accepted for publication in Ann. of Math. (2), link here.

on Borel reducibility [3], in fact, abstracting from the model theoretic notion of interpretability, they[2] introduced the following strengthening of Borel completeness:

Definition 1.3. *Let $\mathbf{K}_\omega$ be the Borel space of models with domain ω of a $\mathfrak{L}_{\omega_1,\omega}$-theory. The space $\mathbf{K}_\omega$ is said to be* faithfully *Borel complete if there is a Borel reduction $\mathbf{F}$ from* Graph_ω *(graphs with domain ω) into $\mathbf{K}_\omega$ such that for any Borel subset X of $\mathbf{K}_\omega$ the closure under isomorphism of the image of X under $\mathbf{F}$ is Borel.*

Clearly, whenever we can interpret in a first-order manner countable graphs in our given space, said space is faithfully Borel complete, and so, in particular countable groups are faithfully Borel complete. On the other hand, any first-order theory of abelian groups is known to be stable and so we cannot expect to have a first-order interpretation of countable graphs in countable abelian groups. Our Theorem 1.1 is then the next best possible result in this respect; additionally the interpretation can also be taken to be with respect to very simple formulas, see 5.1 for details. In [3] one of the main motivations for the introduction of the notion of faithful Borel completeness was that whenever this property holds for T, then the full Vaught's conjecture reduces to the Vaught's conjecture for $\mathfrak{L}_{\omega_1,\omega}$-theories extending T, in particular we deduce from 1.1 the following (unexpected?) result:

Corollary 1.4. *Vaught's conjecture is equivalent to Vaught's conjecture for $\mathfrak{L}_{\omega_1,\omega}$-theories of torsion-free abelian groups of infinite rank, or, more suggestively, Vaught's conjecture can be considered to be a problem in countable abelian group theory.*

We now comment on Theorem 1.2. In recent years, descriptive set theorists have been paying attention to other equivalence relations or quasi-orders among classes of countable structures. In particular, among many other interesting results, in [6] it was shown that the embeddability relation between countable graphs is a complete analytic quasi-order, and so the relation of bi-embeddability among countable graphs is a complete analytic equivalence relation. Despite this, not much seems to be known in terms of analysis of the relation of elementary embeddability, apart from reference [10], where it is shown that this relation when considered between countable graphs is a complete analytic quasi-order. In particular, a careful analysis of the complexity of the relation of elementary embeddability between the countable models of familiar complete first-order theories does not seem to be addressed in the literature (notice that on the other hand in terms of complexity of isomorphism the situation is much different, as e.g. for any complete first-order theory T of Boolean algebras we know the exact complexity of the relation of isomorphism between the countable models of T, see [2]). In this respect our Theorem 1.2 seems to be particularly relevant, and we hope that it will inspire further research on the topic. Finally, we want to mention that in [1] it was proved that the embeddability relation between countable abelian groups is also a complete analytic quasi-order.

Some words of explanations on the structure of the paper seems to be in order. We only reproduce the relevant parts of the construction from [9], mostly without proofs, apart from the proofs which are necessary to understand the proofs of our Theorems 1.1 and 1.2. Despite this, the paper is self-contained, in the sense that all the definitions necessary to understand the construction from [9] are included in the present paper. Based on the structure of [9], in Section 3 we introduce a "combinatorial frame" which underlies our group theoretic construction, which is then introduced in Section 4. In Sections 5 and 6 we prove Theorems 1.1 and 1.2.

[2]The use of the term *faithful* to denote this property was introduced only later, cf. [4, pg. 300].

2. Notations and Preliminaries

For the readers of various backgrounds we try to make the paper self-contained.

2.1. General notations

Definition 2.1. *(1) Given a set X we write $Y \subseteq_\omega X$ for $\emptyset \neq Y \subseteq X$ and $|Y| < \aleph_0$.*

(2) Given a set X and $\bar{x}, \bar{y} \in X^{<\omega}$ we write $\bar{y} \triangleleft \bar{x}$ to mean that $\lg(\bar{y}) < \lg(\bar{x})$ and $\bar{x} \restriction \lg(\bar{y}) = \bar{y}$, where $\bar{x}$ is naturally considered as a function $\lg(\bar{x}) \to X$.

(3) Given a partial function $f : M \to M$, we denote by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ the domain and the range of f, respectively.

(4) For $\bar{a} \in B^n$ we write $\bar{a} \subseteq B$ to mean that $\mathrm{ran}(\bar{a}) \subseteq B$, where, as usual, $\bar{a}$ is considered as a function $\{0, ..., n-1\} \to B$.

(5) Given a sequence $\bar{f} = (f_i : i \in I)$ we write $f \in \bar{f}$ to mean that there exists $j \in I$ such that $f = f_j$.

2.2. Groups

Notation 2.2. *Let G and H be groups.*

(1) $H \leqslant G$ means that H is a subgroup of G.

(2) We let $G^+ = G \setminus \{e_G\}$, where e_G is the neutral element of G.

(3) If G is abelian we might denote the neutral element e_G simply as $0_G = 0$.

(4) We denote by $G^{(\omega)}$ the group $\bigoplus_{n<\omega} G$.

Definition 2.3. *Let $H \leqslant G$ be groups, we say that H is pure in G, denoted by $H \leqslant_* G$, when if $h \in H$, $0 < n < \omega$, $g \in G$ and (in additive notation) $G \models ng = h$, then there is $h' \in H$ s.t. $H \models nh' = h$. Given $S \subseteq G$ we denote by $\langle S \rangle^*_S$ the pure subgroup generated by S (the intersection of all the pure subgroups of G containing S).*

Observation 2.4. *$H \leqslant_* G \in \mathrm{TFAB}$, $h \in H$, $0 < n < \omega$, $G \models ng = h \Rightarrow g \in H$.*

Observation 2.5. *Let $G \in \mathrm{TFAB}$, p a prime and let:*

$$G_p = \{a \in G : a \text{ is divisible by } p^m, \text{ for every } 0 < m < \omega\},$$

then G_p is a pure subgroup of G.

Definition 2.6. *Let p be a prime. We let:*

$$\mathbb{Q}_p = \{\frac{m_1}{m_2} : m_1 \in \mathbb{Z}, m_2 \in \mathbb{Z}^+, p \text{ and } m_2 \text{ are coprime}\}.$$

3. The Combinatorial Frame

Notation 3.1. *For Z a set and $0 < n < \omega$, we let $\mathrm{seq}_n(Z) = \{\bar{x} \in Z^n : \bar{x} \text{ injective}\}$.*

Hypothesis 3.2. *(1) $\mathbf{K}^{\mathrm{eq}}$ is the class of models M in a vocabulary $\{\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2\}$ such that each $\mathfrak{E}_i^M$ is an equivalence relation and $\mathfrak{E}_2^M$ is the equality relation. We use the symbol $\mathfrak{E}_i$ to avoid confusions, as the symbol E_i appears in 3.4.*

(2) M is the countable homogeneous universal model in $\mathbf{K}^{\mathrm{eq}}$.

(3) $\mathcal{G}$ is essentially the set of finite non-empty partial automorphisms g of M but for technical reasons[3] it is the set of objects $g = (\mathbf{h}_g, \iota_g)$ where:

(A) (a) $\mathbf{h}_g$ is a finite non-empty partial automorphism of M;

(b) $\iota_g \in \{0, 1\}$;

(B) for $g \in \mathcal{G}$ we let:

[3] The reason is that we want to force that $g \neq g^{-1}$.

(a) $g^{-1} = (\mathbf{h}_g^{-1}, 1 - \iota_g)$;
(b) for $a \in M$, $g(a) = \mathbf{h}_g(a)$;
(c) for $\mathcal{U} \subseteq M$, $g[\mathcal{U}] = \{\mathbf{h}_g(a) : a \in \mathcal{U}\}$;
(d) $g_1 \subseteq g_2$ means $\mathbf{h}_{g_1} \subseteq \mathbf{h}_{g_2}$ and $\iota_{g_1} = \iota_{g_2}$;
(e) $g_1 \subsetneq g_2$ means $g_1 \subseteq g_2$ and $g_1 \neq g_2$;
(f) $\mathrm{dom}(g) = \mathrm{dom}(\mathbf{h}_g)$ and $\mathrm{ran}(g) = \mathrm{ran}(\mathbf{h}_g)$;
(g) for $\mathcal{U} \subseteq M$, $g \restriction \mathcal{U} = (\mathbf{h}_g \restriction \mathcal{U}, \iota_g)$.

(4) For $m < \omega$, $\mathcal{G}_*^m = \{(g_0, ..., g_{m-1}) \in \mathcal{G}^m : g_0 \subsetneq \cdots \subsetneq g_{m-1}\}$.
(5) $\mathcal{G}_* = \bigcup\{\mathcal{G}_*^m : m < \omega\}$ (notice that the empty sequence belongs to $\mathcal{G}_*$).

Notation 3.3. *(1) We use $s, t, ...$ to denote finite non-empty subsets of M and $\mathcal{U}, \mathcal{V}, ...$ to denote arbitrary subsets of M. Recall from 2.1 that $\subseteq_\omega$ means finite subset.*
(2) For A a set, we let $s \subseteq_1 A$ mean $s \subseteq A$ and $|s| = 1$.
(3) For $\bar{g} = (g_0, ..., g_{\mathrm{lg}(\bar{g})-1}) \in \mathcal{G}_*^{\mathrm{lg}(\bar{g})}$ and $s, t \subseteq_\omega M$, we let:
(a) for $a, b \in M$, $\bar{g}(a) = b$ mean that $g_{\mathrm{lg}(\bar{g})-1}(a) = b$;
(b) $\bar{g}[s] = t$ mean that $g_{\mathrm{lg}(\bar{g})-1}[s] = t$;
(c) $\mathrm{dom}(\bar{g}) = \mathrm{dom}(g_{\mathrm{lg}(\bar{g})-1})$, and $\emptyset$ if $\mathrm{lg}(\bar{g}) = 0$;
(d) $\mathrm{ran}(\bar{g}) = \mathrm{ran}(g_{\mathrm{lg}(\bar{g})-1})$, and $\emptyset$ if $\mathrm{lg}(\bar{g}) = 0$;
(e) $\bar{g}^{-1} = (g_i^{-1} : i < \mathrm{lg}(\bar{g}))$;
(f) $\bar{g}((x_\ell : \ell < n)) = (\bar{g}(x_\ell) : \ell < n)$.

Definition 3.4. *In the context of Hyp. 3.2, let $\mathrm{K}_2^{\mathrm{bo}}(M)$ be the class of objects (called* systems*) $\mathfrak{m}(M) = \mathfrak{m} = (X^{\mathfrak{m}}, \bar{X}^{\mathfrak{m}}, \bar{f}^{\mathfrak{m}}, \bar{E}^{\mathfrak{m}}) = (X, \bar{X}, \bar{f}, \bar{E})$ such that:*

(1) X is an infinite countable set and $X \subseteq \omega$;
(2) (a) $(X'_s : s \subseteq_1 M)$ is a partition of X into infinite sets;
(b) for $s \subseteq_\omega M$, let $X_s = \bigcup_{t \subseteq_1 s} X'_t$;
(c) $\bar{X} = (X_s : s \subseteq_\omega M)$ and so $s \subseteq t \subseteq_\omega M$ implies $X_s \subseteq X_t$;
(3) for $\mathcal{U} \subseteq M$ let $X_{\mathcal{U}} = \bigcup\{X_s : s \subseteq_1 \mathcal{U}\}$ and so $X = X_M = \bigcup\{X_s : s \subseteq_1 M\}$;
(4) $\bar{f} = (f_{\bar{g}} : \bar{g} \in \mathcal{G}_*)$ (recall the definition of $\mathcal{G}_*$ from 3.2(5)) and:
(a) $f_{\bar{g}}$ is a finite partial bijection of X and $f_{\bar{g}}$ is the empty function iff $\mathrm{lg}(\bar{g}) = 0$;
(b) $\mathrm{dom}(f_{\bar{g}}) \subseteq X_{\mathrm{dom}(\bar{g})}$ and $\mathrm{ran}(f_{\bar{g}}) \subseteq X_{\mathrm{ran}(\bar{g})}$ (cf. 3.3(3c)(3d)), so $\mathrm{dom}(f_{()}) = \emptyset$;
(c) for $s, t \subseteq_1 M$ and $\bar{g}[s] = t$ we have:

$$f_{\bar{g}}(x) = y \text{ implies } (x \in X'_s \text{ iff } y \in X'_t).$$

(d) for $s, t \subseteq_1 M$, $(f_{\bar{g}}(x) = y, x \in X'_s, y \in X'_t)$ implies $(\bar{g}[s] = t)$;
(e) $f_{\bar{g}^{-1}} = f_{\bar{g}}^{-1}$ (recall that $\bar{g}^{-1} \neq \bar{g}$, when $\mathrm{dom}(\bar{g}) \neq \emptyset$);
(5) $\bar{g}, \bar{g}' \in \mathcal{G}_*$, $\bar{g} \triangleleft \bar{g}' \Rightarrow f_{\bar{g}} \subsetneq f_{\bar{g}'}$;
(6) we define the graph $(\mathrm{seq}_n(X), R_n^{\mathfrak{m}})$ as $(\bar{x}, \bar{y}) \in R_n^{\mathfrak{m}} = R_n$ when $\bar{x} \neq \bar{y}$ and:

$$\text{for some } \bar{g} \in \mathcal{G}_* \text{ we have } f_{\bar{g}}(\bar{x}) = \bar{y},$$

notice that $f_{\bar{g}}^{-1} = f_{\bar{g}^{-1}} \in \bar{f}$, as $\bar{g} \in \mathcal{G}_*$ implies $\bar{g}^{-1} \in \mathcal{G}_*$;
(7) $\bar{E}^{\mathfrak{m}} = \bar{E} = (E_n : 0 < n < \omega) = (E_n^{\mathfrak{m}} : 0 < n < \omega)$, and, for $0 < n < \omega$, E_n is the equivalence relation corresponding to the partition of $\mathrm{seq}_n(X)$ given by the connected components of the graph $(\mathrm{seq}_n(X), R_n)$;
(8) if p is a prime, $k \geqslant 2$, $\bar{x} \in \mathrm{seq}_k(X)$, $\mathbf{y} = (\bar{y}^i : i < i_*) \in (\bar{x}/E_k^{\mathfrak{m}})^{i_*}$, with the $\bar{y}^i$'s pairwise distinct, $\bar{r} \in \mathbb{Q}^{\mathbf{y}}$, $q_\ell \in \mathbb{Q}_p$, for $\ell < k$, and:

$$a_{(\mathbf{y}, \bar{r})}(y) = a_{(\mathbf{y}, \bar{r}, y)} = \sum\{r_{\bar{y}} q_\ell : \ell < k, \bar{y} = \bar{y}^i, i < i_*, y = y_\ell^i\},$$

for $y \in \mathrm{set}(\mathbf{y}) = \bigcup\{\mathrm{ran}(\bar{y}^i) : i < i_*\}$*, then we have the following:*

$$|\{y \in \mathrm{set}(\mathbf{y}) : a_{(\mathbf{y},\bar{r})}(y) \notin \mathbb{Q}_p\}| \neq 1,$$

where we recall that $\mathbb{Q}_p$ *was defined in Definition 2.6;*

(9) if for every $n < \omega$*,* $g_n \in \mathcal{G}$ *and* $g_n \subsetneq g_{n+1}$*,* $\mathcal{U} = \bigcup_{n<\omega} \mathrm{dom}(g_n) \subseteq M$ *and* $\mathcal{V} = \bigcup_{n<\omega} \mathrm{ran}(g_n) \subseteq M$*, then we have the following:*

$$\bigcup_{n<\omega} \mathrm{dom}(f_{(g_\ell\,:\,\ell<n)}) = X_{\mathcal{U}} \text{ and } \bigcup_{n<\omega} \mathrm{ran}(f_{(g_\ell\,:\,\ell<n)}) = X_{\mathcal{V}}.$$

4. Borel Completeness of Torsion-Free Abelian Groups

4.1. The Definition of the Groups $G_{(1,\mathcal{U})}$

Definition 4.1. *Let* $\mathrm{K}_3^{\mathrm{bo}}(M)$ *be the class of* $\mathfrak{m} \in \mathrm{K}_2^{\mathrm{bo}}(M)$ *expanded with a sequence* $\bar{p} = \bar{p}^{\mathfrak{m}}$ *of prime numbers without repetitions such that we have the following:*

(1) $\bar{p} = (p_{(e,\bar{q})} : e \in \mathrm{seq}_n(X)/E_n^{\mathfrak{m}}$ *for some* $0 < n < \omega$ *and* $\bar{q} \in (\mathbb{Z}^+)^n)$*;*

(2) for every $\ell < n$*,* $p \not| \, q_\ell$*.*

Fact 4.2. *Clearly every element of* $\mathfrak{m} \in \mathrm{K}_2^{\mathrm{bo}}(M)$ *can be expanded to an element of* $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$*, and, as* $\mathrm{K}_2^{\mathrm{bo}}(M) \neq \emptyset$ *we have* $\mathrm{K}_3^{\mathrm{bo}}(M) \neq \emptyset$*.*

We try to give some intuition on the group $G_1 = G_1[\mathfrak{m}]$ which we are about to introduce in 4.3. This group will be some sort of universal domain for our construction, and in fact all the TFAB_ω's which will be in the range of our Borel reduction from $\mathbf{K}_2^{\mathrm{eq}}$ (cf. 3.2) to TFAB_ω will be pure subgroups of this group G_1. The group G_1 naturally interpolates between $G_0 = \bigoplus\{\mathbb{Z}x : x \in X\}$ and $G_2 = \bigoplus\{\mathbb{Q}x : x \in X\}$, which have respectively the minimal and the maximal amount of divisibility possible. Clearly, the groups G_0 and G_2 do not code anything of the universal countable model $M \in \mathbf{K}_2^{\mathrm{eq}}$ (cf. 3.2). Thus, we want to find a subgroup $G_0 \leqslant G_1 \leqslant G_2$ which does encode M. We do this adding divisibility conditions to G_0 which depend on the relation $E_n^{\mathfrak{m}}$ from 3.4. So the first step is that for every $a \in G_0^+$ we choose a prime p_a and require the following condition:

$$G_0 \models a = \sum_{\ell<k} q_\ell x_\ell \neq 0 \;\Rightarrow\; G_1 \models p_a^\infty | \, a.$$

However, we want the partial permutations $f_{\bar{g}}$ of X from 3.4 to induce partial automorphisms $\hat{f}^1_{\bar{g}}$ of our desired group G_1, and so we naturally demand:

$$\iota \in \{1,2\},\; a_\iota = \sum_{\ell<k} q_\ell x^\iota_\ell,\; \bigwedge_{\ell<k} f_{\bar{g}}(x^1_\ell) = x^2_\ell \;\Rightarrow\; p_{a_1} = p_{a_2}.$$

Formally, this translates into a choice of $p_{(e,\bar{q})}$ as in 4.1, where condition 4.1(2) is simply a useful technical requirement. We finally define our "universal" group G_1.

Definition 4.3. *Let* $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$*.*

(1) Let $G_2 = G_2[\mathfrak{m}]$ *be* $\bigoplus\{\mathbb{Q}x : x \in X\}$*.*

(2) Let $G_0 = G_0[\mathfrak{m}]$ *be the subgroup of* G_2 *generated by* X*, i.e.* $\bigoplus\{\mathbb{Z}x : x \in X\}$*.*

(3) Let $G_1 = G_1[\mathfrak{m}]$ *be the subgroup of* G_2 *generated by:*
   - *(a)* G_0*;*
   - *(b)* $p^{-m}(\sum_{\ell<n} q_\ell x_\ell)$*, where:*
     - *(i)* $0 < m < \omega$*;*
     - *(ii)* $\bar{x} = (x_\ell : \ell < n) \in \mathrm{seq}_n(X)$*,* $e = \bar{x}/E_n^{\mathfrak{m}}$*,* $n > 0$*;*

(iii) $\bar{q}$ *is as in 4.1;*

(iv) $p = p_{(e,\bar{q})}$ *(so a prime, recalling Definition 4.1);*

(c) *[follows] for every* $a \in G_1$ *there are* $i_* < \omega$ *and, for* $i < i_*$, k_i, $\bar{x}_i \in \mathrm{seq}_{k_i}(X)$, $\bar{q}_i \in (\mathbb{Z}^+)^{k(i)}$, $e_i = \bar{x}_i/E^{\mathfrak{m}}_{k_i}$, $p_i = p_{(e_i,\bar{q}_i)}$ *(hence* $\bar{q}_i$ *is as in 4.1),* $m(i) \geqslant 0$ *and* $r^i \in \mathbb{Z}^+$ *such that the following condition holds:*

$$a = \sum\{p_i^{-m(i)} r^i q_{(i,\ell)} x_{(i,\ell)} : i < i_*, \ell < k_i\}.$$

(4) *For a prime* p, *let* $G_{(1,p)} = \{a \in G_1 : a$ *is divisible by* p^m, *for every* $0 < m < \omega\}$ *(notice that, by Observation 2.5,* $G_{(1,p)}$ *is always a pure subgroup of* G_1*).*

(5) *For* $\mathcal{U} \subseteq M$, *we let:*

$$G_{(1,\mathcal{U})}[\mathfrak{m}] = G_{(1,\mathcal{U})}[\mathfrak{m}(M)] = G_{(1,\mathcal{U})} = \langle y : y \in X_u, u \subseteq_1 \mathcal{U}\rangle^*_{G_1} = \langle X_{\mathcal{U}}\rangle^*_{G_1}.$$

The notation $\mathfrak{m}(M)$ *is from the second line of Def. 3.4 and* $X_{\mathcal{U}}$ *is from 3.4(3).*

(6) *For* $f_{\bar{g}} \in \bar{f}^{\mathfrak{m}}$ *(cf. Definition 3.4(4)), let* $\hat{f}^2_{\bar{g}}$ *be the unique partial automorphism of* G_2 *which is induced by* $f_{\bar{g}}$ *(see 4.4(2)), explicitly: if* $k < \omega$ *and for every* $\ell < k$ *we have that* $y^1_\ell \in \mathrm{dom}(f_{\bar{g}})$, $y^2_\ell = f_{\bar{g}}(y^1_\ell)$, $q_\ell \in \mathbb{Q}^+$, *then:*

$$a = \sum_{\ell<k} q_\ell y^1_\ell \in G_2 \;\Rightarrow\; \hat{f}^2_{\bar{g}}(a) = \sum_{\ell<k} q_\ell y^2_\ell.$$

(7) *For* $\ell \in \{0,1\}$ *we let* $\hat{f}^2_{\bar{g}} \restriction G_\ell = \hat{f}^\ell_{\bar{g}}$ *and* $\hat{f}_{\bar{g}} = \hat{f}^1_{\bar{g}}$ *(see 4.4(2)).*

(8) *For* $i \in \{0,1,2\}$, $a = \sum_{\ell<m} q_\ell x_\ell \in G_i$, *with* $(x_\ell : \ell < k) \in \mathrm{seq}_k(X)$ *and* $q_\ell \in \mathbb{Q}^+$, *let* $\mathrm{supp}(a) = \{x_\ell : \ell < m\}$, *i.e., when* $a \in G^+_i$, $\mathrm{supp}(a) \subseteq_\omega X$ *is the smallest subset of* X *such that* $a \in \langle \mathrm{supp}(a)\rangle^*_{G_i}$.

(9) *For* p *a prime and* $a \in G^+_2$ *we define the* p*-support of* a, *denoted as* $\mathrm{supp}_p(a)$, *as: if* $a = \sum\{q_\ell x_\ell : \ell < k\}$ *with* $(x_\ell : \ell < k) \in \mathrm{seq}_k(X)$ *and* $q_\ell \in \mathbb{Q}^+$, *then:*

$$\mathrm{supp}_p(a) = \{x_\ell : \ell < k \text{ and } q_\ell \notin \mathbb{Q}_p\},$$

where we recall that $\mathbb{Q}_p$ *was defined in 2.6.*

Lemma 4.4. *Let* $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_3$ *and* $\ell \in \{0,1,2\}$.

(1) $G_\ell[\mathfrak{m}] \in \mathrm{TFAB}$ *and* $|G_\ell[\mathfrak{m}]| = \aleph_0$.

(2) (a) $\hat{f}^2_{\bar{g}}$ *is a partial automorphisms of* $G_2[\mathfrak{m}]$ *mapping* $G_0[\mathfrak{m}]$ *into itself;*

(b) $\hat{f}_{\bar{g}} = \hat{f}^1_{\bar{g}} = \hat{f}^2_{\bar{g}} \restriction G_{(1,\mathrm{dom}(\bar{g}))}$ *(cf. Def. 4.3(5)(7)), the map* $\hat{f}_{\bar{g}}$ *is a well-defined partial automorphism of* G_1, *and* $\mathrm{dom}(\hat{f}_{\bar{g}})$ *is a pure subgroup of* $G_1[\mathfrak{m}]$, *in fact* $\mathrm{dom}(\hat{f}_{\bar{g}})$ *is the pure closure in* G_1 *of* $\mathrm{dom}(\hat{f}^0_{\bar{g}})$*;*

(c) $\hat{f}_{\bar{g}^{-1}} = \hat{f}^{-1}_{\bar{g}}$*;*

(d) $\bar{g}_1 \subseteq \bar{g}_2 \Rightarrow \hat{f}_{\bar{g}_1} \subseteq \hat{f}_{\bar{g}_2}$*;*

(e) $f_{\bar{g}} \subseteq \hat{f}^0_{\bar{g}} \subseteq \hat{f}^1_{\bar{g}} \subseteq \hat{f}^2_{\bar{g}}$.

(3) *If* $p = p_{(e,\bar{q})}$, $e \in \mathrm{seq}_n(X)/E^{\mathfrak{m}}_n$, $\bar{q} = (q_\ell : \ell < n)$ *is as in 4.1, and* $n \geqslant 1$, *then:*

(a) $\langle \sum_{\ell<n} p^{-m} q_\ell y_\ell : m < \omega, \bar{y} \in e\rangle^*_{G_1} \leqslant G_{(1,p)}$*;*

(b) $G_1 \leqslant \langle \{p^{-m} \sum_{\ell<n} q_\ell y_\ell : m < \omega, \bar{y} \in e\} \cup \mathbb{Q}_p G_0\rangle_{G_2}$*;*

(c) *if* $a \in G_1$, *then there are* $k < \omega$, *and, for* $i < k$, $\bar{y}^i \in e$, $s_i \in \mathbb{Q}^+$ *s.t.:*

(i) $a = \sum_{i<k} s_i(\sum_{\ell<n} q_\ell y^i_\ell) \mod(\mathbb{Q}_p G_0 \cap G_1)$*;*

(ii) for all $i < k$, $s_i \sum_{\ell<n} q_\ell y^i_\ell \notin \mathbb{Q}_p G_0$, *and* $\ell < n$ *implies* $s_i q_\ell y^i_\ell \notin \mathbb{Q}_p G_0$*;*

(iii) $s_i \sum\{q^i_\ell y^i_\ell : \ell < n\} \in G_1$.

(4) *In 4.4(3) we may add:* $(\bar{y}^i : i < i_*)$ *is with no repetitions.*

Fact 4.5. *Assume that $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$, $\mathcal{U}, \mathcal{V} \subseteq M$ and $|\mathcal{U}| = |\mathcal{V}| = \aleph_0$. Suppose further that there is $h : M \restriction \mathcal{U} \cong M \restriction \mathcal{V}$. Then there is $\bar{g} = (g_k : k < \omega)$ such that:*
(a) for every $k < \omega$, $g_k \in \mathcal{G}$ (cf. 3.2(3));
(b) for every $k < \omega$, $g_k \subsetneq g_{k+1}$;
(c) $\bigcup_{k<\omega} g_k : M \restriction \mathcal{U} \cong M \restriction \mathcal{V}$.

As mentioned, G_1 will be some sort of universal domain for our construction. This is reflected by the fact that instead of varying $M \in \mathbf{K}^{\mathrm{eq}}$ in Definition 3.4, we fix M to be the countable universal homogeneous model of $\mathbf{K}^{\mathrm{eq}}$, and, for $\mathcal{U} \subseteq M$, we consider the substructure $M \restriction \mathcal{U}$ and the group $G_{(1,\mathcal{U})}$. We intend to show:

$$M \restriction \mathcal{U} \cong M \restriction \mathcal{V} \ \Leftrightarrow\ G_{(1,\mathcal{U})}[\mathfrak{m}] \cong G_{(1,\mathcal{V})}[\mathfrak{m}].$$

The easy direction is course the left-to-right one, which we now establish:

Claim 4.6. *Assume that $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$, $\mathcal{U}, \mathcal{V} \subseteq M$ and $|\mathcal{U}| = |\mathcal{V}| = \aleph_0$. Then:*

$$M \restriction \mathcal{U} \cong M \restriction \mathcal{V} \ \Rightarrow\ G_{(1,\mathcal{U})}[\mathfrak{m}] \cong G_{(1,\mathcal{V})}[\mathfrak{m}].$$

4.2. **Analyzing Isomorphism**

Our aim in this subsection is to prove the converse of Claim 4.6.

Hypothesis 4.7. *Throughout this subsection the following hypothesis holds:*
(1) $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$;
(2) $\mathcal{U}, \mathcal{V} \subseteq M$;
(3) $|\mathcal{U}| = \aleph_0 = |\mathcal{V}|$;
(4) π is an isomorphism from $G_{(1,\mathcal{U})}[\mathfrak{m}]$ onto $G_{(1,\mathcal{V})}[\mathfrak{m}]$.

Our aim in 4.8 and 4.9 below is to show that π essentially comes from a bijection from $X_{\mathcal{U}}$ onto $X_{\mathcal{V}}$, which are respectively the bases of $G_{(1,\mathcal{U})}[\mathfrak{m}]$ and $G_{(1,\mathcal{V})}[\mathfrak{m}]$ (in the appropriate sense). At the bottom of this is the crucial algebraic condition 3.4(8), which puts restrictions on the possible p-supports of certain members of G_1.

The following proof is the only proof we retain in full, the reason for this is that the argument used here is crucial and it is then referred to in the proof of Claim 5.4.

Lemma 4.8. *Let $a \in G_{(1,\mathcal{U})}[\mathfrak{m}]$ and let $b = \pi(a)$.*
(1) For a prime p, $a \in G_{(1,p)} \Leftrightarrow b \in G_{(1,p)}$;
(2) if $a = qx$, for some $q \in \mathbb{Q}^+$ and $x \in X_{\mathcal{U}}$, then for some $y \in X_{\mathcal{V}}$:
  (a) $(x)E_1^{\mathfrak{m}}(y)$;
  (b) $b \in \mathbb{Q}y$, i.e. there exist $m_1, m_2 \in \mathbb{Z}^+$ such that $m_1 b = m_2 y$.

Proof. Item (1) is obvious by Hypothesis 4.7(4). Notice now that:
$(*_0)$ It suffices to prove (2)(b).
Why $(*_0)$? Suppose that $b = \frac{m_2}{m_1} y$ and let $e' = (x)/E_1^{\mathfrak{m}}$ and $p' = p_{(e',(1))}$, then $x \in G_{(1,p')}$, but $a = qx$ and $a \in G_1$, hence $a \in G_{(1,p')}$. Now, applying (1) with (a, b, p') here standing for (a, b, p) there, we get that $b \in G_{(1,p')}$. As $b = \frac{m_2}{m_1} y \in G_1$, we have that $y \in G_{(1,p')}$ and thus:
$(\cdot)$ $G_1 \models (p')^\infty | \, x$ and $G_1 \models (p')^\infty | \, y$.
Now, letting $H_{(p',0)} = \langle x/E_1^{\mathfrak{m}} \rangle_{G_0}$ and $H_{(p',1)} = \langle x/E_1^{\mathfrak{m}} \rangle^*_{G_1}$ we have that:
$(*_{0.1})$ (i) $G_0/H_{(p',0)}$ is canonically $\cong$ to the direct sum of $\langle \mathbb{Z}y : y \in X \setminus x/E_1^{\mathfrak{m}} \rangle$;
(ii) $H_{(p',1)} \cap G_0 = H_{(p',0)}$;
(iii) $G_1/H_{(p',1)}$ naturally extends $G_0/H_{(p',0)}$;

(iv) no non-zero element of $G_1/H_{(p',1)}$ is divisible by $(p')^\infty$.

Why $(*_{0.1})$? Straightforward or see a detailed proof of a more complicated case in [9, 5.15(2)]. This concludes the proof of $(*_0)$.

Coming back to the proof:

$(*_1)$ Let $n < \omega$, $\bar{y} \in \mathrm{seq}_n(X_\mathcal{V})$ and $\bar{q} \in (\mathbb{Q}^+)^n$ be such that $b = \sum\{q_\ell y_\ell : \ell < n\}$.

Trivially, $n > 0$, we shall show that $n = 1$, i.e., that (2)(b) holds. To this extent:

$(*_{1.1})$ Let $q_* \in \omega \setminus \{0\}$ be such that:

- $(\cdot_1)$ $b_1 := q_* b \in G_0[\mathfrak{m}]$;
- $(\cdot_2)$ $q_* q \in \mathbb{Z}$, and $q_* q_\ell \in \mathbb{Z}$, for all $\ell < n$;
- $(\cdot_3)$ for every prime p' we have $p' \mid (q_* q)$ implies $p' \mid (q_* q_\ell)$, for all $\ell < n$.

Let $e = \bar{y}/E_n$, $q'_\ell = q_* q_\ell$ and $\bar{q}' = (q'_\ell : \ell < n)$, so that $q_* q_\ell y_\ell = q'_\ell y_\ell$ and $q'_\ell \in \mathbb{Z}^+$. Let $p = p_{(e,\bar{q}')}$ and let $b_1 = q_* b = \sum\{q'_\ell y_\ell : \ell < n\}$. Notice that we have:

$(*_2)$ $\bigwedge_{\ell<k} p \nmid q'_\ell$ and, for every $\ell < k$, $q'_\ell \in \mathbb{Z}^+ \subseteq \mathbb{Q}_p$.

[Why? Because $p = p_{(e,\bar{q}')}$ has been chosen in 4.1 exactly in this manner.]

Then we have:

$(*_3)$
- (i) $b \in G_{(1,p)}$;
- (ii) $a \in G_{(1,p)}$;
- (iii) if $m < \omega$, then $p^{-m} a \in G_{(1,p)} \leqslant G_1$.

[Why (i)? By the choice of p we have that $b_1 \in G_{(1,p)}$ (cf. Def. 4.3(3)(4)) and so, as $G_{(1,p)}$ is pure in G_1 (cf. Observation 2.5), $b_1 = q_* b$ and $q_* \in \mathbb{Z}$, we have $b \in G_{(1,p)}$ (cf. Observation 2.4). Why (ii)? By (1) and (i), recalling Hyp. 4.7(4). Lastly, (iii) is immediate: by (ii) and the definition of G_1 and of $G_{(1,p)}$ (Definition 4.3(3)(4)).]

$(*_4)$ W.l.o.g. $a = qx \notin \mathbb{Q}_p G_0$ and $pa \in G_0$.

We prove $(*_4)$. Let $a' = p^{-1} q_* a$, $b' = p^{-1} q_* b$ and $q' = p^{-1} q_*$. So by $(*_3)$ we have that $a', b' \in G_1$ and of course $\pi(a') = b'$. Now, by the choice of b' and q_* (cf. in particular $(*_{1.1})(\cdot_3)$) we have that $pb' \in G_{(0,\mathcal{V})}$, hence $pa' = \pi^{-1}(pb') \in G_{(0,\mathcal{U})}$. Notice that $a' \notin G_{(0,\mathcal{U})}$, as $a' \notin \mathbb{Q}_p G_0$ because $b' \notin \mathbb{Q}_p G_0$, since from $(*_2)$ above, $\bigwedge_{\ell<k} p \nmid q'_\ell$. Noticing that $(a', b', q'_*, b_1, p, \bar{q}')$ satisfies all the demands of $(a, b, q_*, b_1, p, \bar{q}')$ (including $(*_3)$), it follows that:

$(*_{4.1})$
- (a) replacing (a, q, b) with (a', q', b') we can assume that $a = qx \notin \mathbb{Q}_p G_0$;
- (b) if b' belongs to $\mathbb{Q}y$ for some $y \in X_\mathcal{V}$, the the conclusion of (2) is satisfied.

This concludes the proof of $(*_4)$.

Now, by 4.4(3)(c), there are $k < \omega$, and, for $i < k$, $\bar{y}^i \in \bar{y}/E_n$ and $r_i \in \mathbb{Q}^+$ such that:

$(*_5)$
- (a) $qx = a = \sum_{i<k} r_i (\sum_{\ell<n} q'_\ell y^i_\ell) = \sum_{i<k} (\sum_{\ell<n} r_i q'_\ell y^i_\ell) \mod(\mathbb{Q}_p G_0 \cap G_1)$;
- (b) $r_i \sum_{\ell<n} q'_\ell y^i_\ell \in G_1$ and $r_i q'_\ell \notin \mathbb{Q}_p$.

By $(*_4)$, $a = qx \notin \mathbb{Q}_p G_0$, and so clearly $k > 0$. It suffices to prove that $k = 1$, which by $(*_5)$ implies that $n = 1$, i.e., there is $y \in X_\mathcal{V}$ such that $b \in \mathbb{Q}y$. Why does it follow that $n = 1$? As otherwise the LHS of $(*_5)$(a) has p-support a singleton but the RHS of $(*_5)$(a) has p-support of size at least two, a contradiction.

So toward contradiction assume that $k \geqslant 2$. Recalling $(*_4)$ notice that:

$(*_6)$ $qx = a = \sum_{\ell<n} (\sum_{i<k} r_i q'_\ell y^i_\ell) \mod(\mathbb{Q}_p G_0 \cap G_1)$;

Now, let $Z = \{y^i_\ell : i < i_*, \ell < k\}$ and, for $y \in Z$, let:

$$a_y = \sum\{r_i q'_\ell : i < i_*, \ell < k, y^i_\ell = y\}.$$

So, by $(*_6)$ we have:

$(*_7)$ $qx = \sum\{a_y y : y \in Z\} \mod(\mathbb{Q}_p G_0 \cap G_1)$.

Now, as for the sake of contradiction we are assuming that $k \geqslant 2$, recalling that by $(*_2)$ we have that $q'_\ell \in \mathbb{Z}^+ \subseteq \mathbb{Q}_p$, by 3.4(8), we have the following:

$(*_8)$ $\mathrm{supp}_p(\sum_{y \in Y} a_y y) = \{y \in Y : a_y \notin \mathbb{Q}_p\}$ is not a singleton.

Now recall that, by $(*_4)$, $qx = a \notin \mathbb{Q}_p G_0$, hence $\mathrm{supp}_p(qx) = \{x\}$, so it is a singleton. By $(*_8)$, the RHS of $(*_7)$ has a non-singleton p-support whereas the LHS of $(*_7)$ has p-support a singleton, a contradiction. Hence, we are done proving (2). ■

Conclusion 4.9. *(1) There is a sequence $(q^1_x : x \in X_{\mathcal{U}})$ of non-zero rationals and a function $\pi_1 : X_{\mathcal{U}} \to X_{\mathcal{V}}$ such that for every $x \in X_{\mathcal{U}}$ we have that:*

$$\pi(x) = q^1_x(\pi_1(x)) \quad \text{and} \quad \pi_1(x) \in x/E^{\mathfrak{m}}_1.$$

(2) There is a sequence $(q^2_x : x \in X_{\mathcal{V}})$ of non-zero rationals and a function $\pi_2 : X_{\mathcal{V}} \to X_{\mathcal{U}}$ such that:

$$\pi^{-1}(x) = q^2_x(\pi_2(x)).$$

(3) (i) $\pi_2 \circ \pi_1 : X_{\mathcal{U}} \to X_{\mathcal{U}} = id_{\mathcal{U}}$;
(ii) $\pi_1 \circ \pi_2 : X_{\mathcal{V}} \to X_{\mathcal{V}} = id_{\mathcal{V}}$;
(iii) $\pi_1 : X_{\mathcal{U}} \to X_{\mathcal{V}}$ is a bijection.

Our aim in the subsequent claims is to lift the 1-to-1 mapping from $X_{\mathcal{U}}$ onto $X_{\mathcal{U}}$ defined in 4.9 to an isomorphism from $M \restriction \mathcal{U}$ onto $M \restriction \mathcal{V}$. We recall that the equivalence relations $\mathfrak{E}^M_i$ (for $i \in \{0,1,2\}$) were defined in 3.2. We intend to show that our mapping π_1 and $\pi_1^{-1} = \pi_2$ preserve them (and so also their negations). This is done introducing some auxiliary equivalence relations $\mathcal{E}_i$ (for $i \in \{0,1,2\}$) on X which reflect (to some extent) the equivalence relations $\mathfrak{E}^M_i$ on M.

Definition 4.10. *For $i < 3$, let:*

$$\mathcal{E}_i = \{(x,y) : \text{ for some } (a,b) \in \mathfrak{E}^M_i, x \in X'_{\{a\}} \text{ and } y \in X'_{\{b\}}\},$$

where we recall that $\mathfrak{E}^M_i$ was introduced in 3.2.

Claim 4.11. *(1) If $(y_0, y_1) \in (x_0, x_1)/E^{\mathfrak{m}}_2$, $x_0, x_1, y_0, y_1 \in X$ and $i < 3$, then:*

$$x_0 \mathcal{E}_i x_1 \Leftrightarrow y_0 \mathcal{E}_i y_1.$$

(2) The mapping π_1 from 4.9 preserves $\mathcal{E}_i$ and its negation, for all $i < 3$.

Claim 4.12. *There is a bijection $h : \mathcal{U} \to \mathcal{V}$ preserving $\mathfrak{E}^M_i$ and $\neg\mathfrak{E}^M_i$, for all $i < 3$.*

Conclusion 4.13. *$M \restriction \mathcal{U}$ and $M \restriction \mathcal{V}$ are isomorphic members of $\mathbf{K}^{\mathrm{eq}}$.*

Conclusion 4.14. TFAB_ω *is a Borel complete class.*

5. Faithfulness

Notation 5.1. *(1) By $\mathfrak{L}_{\aleph_1,\aleph_0}$-interpretation we mean as in e.g.* [5, Section 5.3].
(2) By $\mathfrak{L}^{\mathrm{pure}}_{\aleph_1,\aleph_0}(\tau_{\mathrm{AB}})$-interpretation we mean an $\mathfrak{L}_{\aleph_1,\aleph_0}$-interpretation in the language of abelian groups $\tau_{\mathrm{AB}} = \{0,+,-\}$ which uses formulas in the closure of the following formulas by negation and countable conjunctions:

$$\{p^m \,|\, x,\ p^m \,|\, (x-y),\ nx = ky,\ x = y : p \in \mathbb{P},\ n,k < \omega\}.$$

(3) Below by "definable" we mean definable by a formula as in (2).

Fact 5.2. *If $\varphi(\bar{x}) \in \mathfrak{L}^{\text{pure}}_{\aleph_1,\aleph_0}(\tau_{\text{AB}})$ and $G \leqslant_* H \in \text{AB}$, for $\bar{a} \in G^{\lg(\bar{y})}$ we have that:*

$$G \models \varphi(\bar{a}) \;\Leftrightarrow\; H \models \varphi(\bar{a}).$$

Definition 5.3. *Let X and G_1 be as in 4.3. For $a \in G_1$ we let:*

$$\mathbb{P}_a = \{p \in \mathbb{P} : p^\infty | \, a\}.$$

Claim 5.4. *Let $\mathbf{B} : \mathbf{K}^{\text{eq}}_\omega \to \text{TFAB}_\omega$ be as in the proof of Proof of Main Theorem of* [9]*. For every $N \in \mathbf{K}^{\text{eq}}_\omega$, we can $\mathfrak{L}^{\text{pure}}_{\aleph_1,\aleph_0}(\tau_{\text{AB}})$-interpret N in $\mathbf{B}(N)$ uniformly.*

Proof. Let X and G_1 be as in 4.3, and $G = G_{\mathcal{U}} = G_{(1,\mathcal{U})}$, so $\mathcal{U} \subseteq M$. Notice that although we fixed $\mathcal{U} \subseteq M$ and G all the formulas below do not depend on this.

$(\star_1)$ Let $\mathcal{E}_\star = \{(a,b) \in G_1 : a \neq 0 \neq b \wedge ma = nb, \text{ for some } m,n \in \mathbb{Z}^+\}$.

$(\star_2)$ $\mathcal{E}_\star$ is a definable equivalence relation.

$(\star_3)$ From here until $(\star_7)$, fix $\hat{x} \in X$.

$(\star_4)$ We define a formula $\psi^{\hat{x}}(a)$ (so a is a free variable) saying the following:
- (a) a is p^∞-divisible for every prime $p \in \mathbb{P}_{\hat{x}}$.
- (b) a is not p^∞-divisible for every $p \in \mathbb{P}_{\sum_{\ell<k} q_\ell x_\ell}$, where:
  - (i) $k \geqslant 2$;
  - (ii) $(x_\ell : \ell < k) \in \text{seq}_k(X)$;
  - (iii) $\bar{q} \in (\mathbb{Z}^+)^k$.
- (c) $a \neq 0$.

$(\star_5)$ If $a \in G$, $y \in \hat{x}/E^{\mathfrak{m}}_1 \cap \mathcal{U}$ and $a \in y/\mathcal{E}_\star$, then $G \models \psi^{\hat{x}}(a)$.

Why? Easy, recalling that $G \leqslant_* G_1$.

$(\star_6)$ (a) If $a \in G$ and $|\text{supp}(a)| \geqslant 2$, then $G \models \neg\psi^{\hat{x}}(a)$;
(b) If $y \in \mathcal{U}$, $a = qy \in G$ and $y \notin \hat{x}/E^{\mathfrak{m}}_1$, then $G \models \neg\psi^{\hat{x}}(a)$.

Why clause (a)? As in the proof of 4.8. Why clause (b)? Easy.

$(\star_7)$ If $a \in G$, then $G \models \psi^{\hat{x}}(a)$ iff $a \in \bigcup\{y/\mathcal{E}_\star : y \in \hat{x}/E^{\mathfrak{m}}_1\}$.

[Why? By $(\star_5)$ and $(\star_6)$.]

$(\star_8)$ If $\hat{x} \in X$, then for every $\mathcal{U} \subseteq M$ we have:

$$\hat{x} \in X_{\mathcal{U}} \;\Leftrightarrow\; \hat{x}/\mathcal{E}_\star \subseteq G_{(1,\mathcal{U})} = G.$$

Why? Easy (natural but not necessary for what follows).

$(\star_9)$ For $\hat{x}, \hat{y} \in X$, we define a formula $\psi^{\hat{x}-\hat{y}}(a)$ (so also here a is a free variable) saying the following:
- (a) a is p^∞-divisible for every prime $p \in \mathbb{P}_{\hat{x}-\hat{y}}$;
- (b) a is not p^∞-divisible when for some $x \neq y \in X$ we have $(x,y) \notin (\hat{x},\hat{y})/E^{\mathfrak{m}}_2$ and $p \in \mathbb{P}_{x-y}$.

$(\star_{10})$ If $G \models \psi^{\hat{x}}(a) \wedge \psi^{\hat{y}}(b) \wedge \psi^{\hat{x}-\hat{y}}(a-b)$, then for some x_1, y_1 and $q \in \mathbb{Q}^+$ we have:
- (a) $x_1, y_1 \in X_{\mathcal{U}}$;
- (b) $a = qx_1 \in G$ and $b = qy_1 \in G$;
- (c) $(x_1,y_1)E^{\mathfrak{m}}_2(\hat{x},\hat{y})$, so $x_1 \in \hat{x}/E^{\mathfrak{m}}_1 \cap X_{\mathcal{U}}$, $y_1 \in \hat{y}/E^{\mathfrak{m}}_1 \cap X_{\mathcal{U}}$.

Why? The existence of $x_1, y_1 \in X_{\mathcal{U}}$ such that $a \in x_1/\mathcal{E}_\star \cap X_{\mathcal{U}}$ and $b \in y_1/\mathcal{E}^G_\star \cap X_{\mathcal{U}}$ holds by $(\star_7)$ and the assumption. Furthermore, as $G \models \psi^{\hat{x}}(a) \wedge \psi^{\hat{y}}(b) \wedge \psi^{\hat{x}-\hat{y}}(a+b)$, then necessarily $x_1, y_1 \in X_{\mathcal{U}}$. Let now $a = q_1 x_1$ and $b = q_2 y_1$, for $q_1, q_2 \in \mathbb{Q}^+$. For the sake of contradiction suppose that $q_1 \neq q_2$. As $G \models \psi^{\hat{x}-\hat{y}}(a-b)$ we know that for every $p \in \mathbb{P}_{\hat{x}-\hat{y}}$ we have that $G \models p^\infty \,|\, (q_1 x_1 - q_2 y_1)$. Let $q \in \mathbb{Z}^+$ be such that $qq_1, qq_2 \in \mathbb{Z}$ and let $p \in \mathbb{P}_{\hat{x}-\hat{y}}$ be $> |qq_1| + |qq_2|$. Now, we can find n and

$(q^\ell, x^\ell, y^\ell : \ell < n)$ such that $x^\ell, y^\ell \in X_{\mathcal{U}}$, $q_\ell \in \mathbb{Z}^+$, $q^\ell(x^\ell - y^\ell) \in G$, $q \in \mathbb{Z}^+$ and $(x^\ell, y^\ell) \in (\hat{x}, \hat{y})/E_2^{\mathfrak{m}}$ and we have the following:

$$q(q_1 x_1 - q_2 y_1) = \sum_{\ell<n} q^\ell(x^\ell - y^\ell) \ \mathrm{mod}(\mathbb{Q}_p G_0 \cap G_1).$$

But analyzing the equation above we have that the sum of the coefficients on the LHS is $q(q_1 - q_2) \neq 0$ (recall that by assumption $q_1 \neq q_2$), whereas on the RHS it is zero, a contradiction. Finally, the fact that $(x_1, y_1)E_2^{\mathfrak{m}}(\hat{x}, \hat{y})$ is by (ii) of $(\star_9)$.

$(\star_{11})$ Recalling 3.2(1), for $i = 0, 1, 2$, let $\chi_i'(a, b)$ be the formula:

$$\bigvee\{\psi^{\hat{x}}(a) \wedge \psi^{\hat{y}}(b) \wedge \psi^{\hat{x}-\hat{y}}(a-b) : \hat{x}, \hat{y} \in X \text{ and } G_1 \models \hat{x}\mathcal{E}_i\hat{y}\}.$$

$(\star_{12})$ For $i = 0, 1, 2$, let $\chi_i(a, b)$ be the formula:

$$\exists a_1, b_1(a\mathcal{E}_\star a_1 \wedge b\mathcal{E}_\star^G b_1 \wedge \chi'(a_1, b_1)).$$

$(\star_{13})$ For $\mathcal{U} \subseteq M$, $a, b \in G = G_{(1,\mathcal{U})}$ and $i < 3$, we have that TFAE:
- (a) $G \models \chi_i(a, b)$;
- (b) for some $\mathcal{E}_i$-equivalence class $Y \subseteq X_{\mathcal{U}}$ we have $a, b \in \bigcup\{x/\mathcal{E}_\star : x \in Y\}$.

Why? The interesting direction is "(a) implies (b)". So assume that $G \models \chi_i(a, b)$, then there are $a_1 \in a/\mathcal{E}_\star^G$ and $b_1 \in b/\mathcal{E}_\star^G$ such that $G \models \chi_i'(a_1, b_1))$. Hence, for some $\hat{x}, \hat{y} \in X$ we have that:
- (i) $G \models \psi^{\hat{x}}(a_1) \wedge \psi^{\hat{y}}(b_1) \wedge \psi^{\hat{x}-\hat{y}}(a_1 - b_1)$;
- (ii) $\hat{x}\mathcal{E}_i\hat{y}$.

Now, by $(\star_{10})$, for some $x \in \hat{x}/E_1^{\mathfrak{m}} \cap X_{\mathcal{U}}$, $y \in \hat{y}/E_1^{\mathfrak{m}} \cap X_{\mathcal{U}}$ and $q \in \mathbb{Q}^+$ we have that $a' = qx, b' = qy \in G_1$ and $(x, y)E_2^{\mathfrak{m}}(\hat{x}, \hat{y})$. But by 4.11(1) we have that $(x\mathcal{E}_i y)$ iff $(\hat{x}\mathcal{E}_i\hat{y})$, and so by (ii) above we are done. This is enough for our purposes as we can now interpret a model isomorphic to $M \restriction \mathcal{U}$ in $G_{(1,\mathcal{U})} = G$ in the following manner:

$(\star_{14})$
- (a) the domain of the interpretation is $\{a \in G : G \models \chi_2(a, a)\}$;
- (b) equality is interpreted as $\chi_2(a, b)$ (recall that $\mathfrak{E}_2$ is $=$ on M, cf. 3.2(1));
- (c) we interpret $\mathfrak{E}_i$ as $\chi_i(a, b)$.

■

Proof of Theorem 1.1. This follows from 5.4. ■

6. Pure embeddability is a complete analytic quasi-order

Fact 6.1. *There is a Borel map* $\mathbf{B}$ *from* Graph_ω *into* $\mathbf{K}_\omega^{\mathrm{eq}}$ *such that we have:*

$$H_1 \text{ embeds into } H_2 \ \Leftrightarrow\ \mathbf{B}(H_1) \text{ embeds into } \mathbf{B}(H_2).$$

Proof. This is folklore but we add details for the benefit of the reader. For a graph $H = (H, R^H)$ with domain $\subseteq \omega$ we define a model $M = \mathbf{B}(H)$ of the theory of two equivalence relations with set of elements $\omega \cup \omega \times \omega$ defining E_1^M, E_2^M as follows:

(1) E_1^M partitions M into the sets $X_n = \{n\} \cup \{(n, m) : m < \omega\}$, for $n < \omega$;
(2) $E_2^M = \{(n, m) : n, m < \omega\} \cup \{((n, m), (m, n)) : nR^H m\} \cup =_M$.

Notice that H is first-order interpretable in $\mathbf{B}(H)$ as follows:

(A) the domain of the interpretation is the set of elements $\varphi_0(x)$ such that x/E_2 has at least three elements and equality is interpreted as equality;
(B) the edge relation on $\varphi_0(M)$ is defined as $\varphi_R(x, y)$ iff there are x_1 and y_1 s.t.:

$$xE_1x_1 \wedge x_1E_2y_1 \wedge y_1E_1y.$$

It is then easy to see that the Borel map $H \mapsto \mathbf{B}(H)$ is as wanted. ■

Proof of Theorem 1.2. First of all, notice that with a slight abuse of notation (but not a problematic one) we consider models with domain $\subseteq \omega$ instead of simply ω. Notice now that the Borel map $\mathbf{B}$ from the proof of 4.14 is such that for $H_1, H_2 \in \mathbf{K}^{\mathrm{eq}}_{\omega}$ we have that:

$$H_1 \text{ embeds into } H_2 \;\Leftrightarrow\; \mathbf{B}(H_1) \text{ embeds purely into } \mathbf{B}(H_2),$$

and so by Fact 6.1 we are done as it was proved in [6] that embeddability between countable graphs is a complete analytic quasi-order. Finally, it is easy to see that all the torsion-free abelian groups in our construction are elementary equivalent to $\mathbb{Z}^{(\omega)}$ and it is well-known that elementary embeddability among models of a complete theory of TFAB corresponds to pure embeddability, see e.g. [5, Appendix 6.2]. ■

References

[1] F. Calderoni and S. Thomas. *The bi-embeddability relation for countable abelian groups.* Trans. Amer. Math. Soc. **371** (2019), no. 03, 2237-2254.

[2] R. Camerlo and S. Gao. *The completeness of the isomorphism relation for countable Boolean algebras.* Trans. Amer. Math. Soc. **353** (2001), No. 02, 491-518.

[3] H. Friedman and L. Stanley. *A Borel reducibility theory for classes of countable structures.* J. Symb. Log. **54** (1989), no. 03, 894-914.

[4] S. Gao. *Invariant descriptive set theory.* Taylor & Francis Inc, 2008.

[5] W. Hodges. *Model theory.* Cambridge University Press, 1993.

[6] Alain Louveau and Christian Rosendal. *Complete analytic equivalence relations.* Trans. Amer. Math. Soc. **357** (1943), no. 12, 4839-4866.

[7] A. H. Mekler. *Stability of nilpotent groups of class 2 and prime exponent.* J. Symb. Log. **46** (1981), no. 04, 781-788.

[8] G. Paolini and S. Shelah. *Anti-classification results for rigidity conditions in abelian and nilpotent groups.* Submitted, available on the ArXiv at this link.

[9] G. Paolini and S. Shelah. *Torsion-free abelian groups are Borel complete.* Ann. of Math. (2), accepted for publication (04.12.2023), link here.

[10] D. Rossenger. *Degree spectra of analytic complete equivalence relations.* J. Symb. Log. **87** (2022), no. 04, 1663-1676.

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