ON A PROBLEM OF ERDOS AND HAJNAL

SHIMON GARTI, YAIR HAYUT, AND SAHARON SHELAH

ABSTRACT. We address a question of Erdős and Hajnal about the ordinary partition relation $\aleph_{\omega+1} \nrightarrow (\aleph_{\omega+1}, (3)_{\aleph_0})^2$. For $\theta = \mathrm{cf}(\lambda) < \lambda$, assuming $2^\lambda = \lambda^+$ they proved the negative relation $\lambda^+ \nrightarrow (\lambda^+, (3)_\theta)^2$ and asked whether the (local instance of) GCH is indispensable. We show that this negative relation is consistent with λ being strong limit and $2^\lambda > \lambda^+$. The result can be pushed down to $\aleph_{\omega^2}$.

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0. Introduction

Let $G = (V, E)$ be a graph of size λ. One may wonder whether there must be a monochromatic triangle under any edge coloring $c : E \to \theta$. The answer is trivially no, since the graph can be a set of isolated vertices with no edges at all, or a triangle-free graph. Thus in order to make the above question interesting, one should assume that there are many edges (and, in particular, many triangles) in the graph. One possible way to do it uses the following concept. A set of vertices $W \subseteq V$ is called *independent* if $[W]^2 \cap E = \varnothing$. If G is of size λ and there are no independent subsets of size λ in G, then there are many edges in the graph and the question makes more sense.

The above discussion can be formulated in the language of partition calculus, without mentioning graphs at all. The ordinary partition relation $\lambda \to (\kappa, (3)_\theta)^2$ says that for every coloring $c : [\lambda]^2 \to \theta$ there is either $A \in [\lambda]^\kappa$ so that $c''[A]^2 = \{0\}$, or $B \in [\lambda]^3$ and $\gamma \in (0, \theta)$ so that $c''[B]^2 = \{\gamma\}$. A particular interesting case is when $\kappa = \lambda$. In terms of graph theory, one can interpret the coloring as assigning zero to pairs of vertices with no edge, and some color $\gamma \in (0, \theta)$ to edges of a given graph. The positive relation $\lambda \to (\lambda, (3)_\theta)^2$ means that either there is an independent set of size λ, or a monochromatic triangle.

Erdős, Hajnal and Rado investigated this relation in [EHR65]. They established several results, and focused in particular on graphs whose size is a successor of a singular cardinal. A good account appears in the monograph [EHMR84], in which the following is phrased and proved:

Theorem 0.1. *Assume that λ is a singular cardinal and $2^\lambda = \lambda^+$. Then $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$.*

Actually, they proved something a bit stronger, see [EHMR84, Theorem 20.2]. A natural question is whether the assumption $2^\lambda = \lambda^+$ is removable. Let us indicate that if one forces $2^{\mathrm{cf}(\lambda)} \geq \lambda^+$ then a negative result obtains, as mentioned in [EHMR84]. Thus we shall assume from now on that $2^{\mathrm{cf}(\lambda)} < \lambda$, and in fact we shall force the negative relation while λ is a strong limit singular cardinal. The first case, in this context, is $\lambda = \aleph_\omega$. In a collection of unsolved problems [EH71, Problem 5], the pertinent question appears as follows:

Question 0.2. Can one prove without assuming GCH that the relation $\aleph_{\omega+1} \nrightarrow (\aleph_{\omega+1}, (3)_{\aleph_0})^2$ holds?

It appears, again, in [EHMR84, Problem 20.1].[1] Despite the fact that powerful methods for dealing with successors of singular cardinals are available today, the problem is still open. In this paper we show how to obtain $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ even with $2^\lambda > \lambda^+$, where λ is singular and strong

[1]In the monograph [EHMR84], the domain of the coloring is $\aleph_\omega^{\aleph_0}$. Under the assumption $2^{\aleph_\omega} = \aleph_{\omega+1}$, these two entities coincide, i.e. $\aleph_\omega^{\aleph_0} = \aleph_{\omega+1}$.

limit. The main idea is to replace the continuum hypothesis at λ by an appropriate pcf assumption.

The open problems which appear in the lists of Erdős and his colleagues are phrased, frequently, with respect to the first relevant case. In our context, the first case is $\aleph_\omega$. Modern set theory shows that in many cases the first relevant case requires more than the general case. We were able to prove our result at $\aleph_{\omega^2}$, but we still do not know how to obtain the same result at $\aleph_\omega$. Nevertheless, we show that under some appropriate prediction assumption, the main theorem can be forced at $\aleph_\omega$. It remains open, however, whether this prediction assumption is consistent with $2^{\aleph_\omega} > \aleph_{\omega+1}$ where $\aleph_\omega$ is a strong limit cardinal.

The rest of the paper contains two additional sections. In the first one we discuss the stick principle, and we show that this principle, denoted by $\text{stick}(\lambda)$, implies $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ where λ is a strong limit singular cardinal. Although we do not know how to combine the stick at such cardinals with the failure of SCH, it seems that this result is interesting by its own. In the second additional section we prove instances of $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ where $2^\lambda > \lambda^+$, using methods of pcf theory. In that way, we can give an answer to the question of Erdős and Hajnal at $\lambda = \aleph_{\omega^2}$.

Our notation is coherent with [EHMR84]. We shall use the Jerusalem forcing notation, namely we force upwards. A function $f : E \to \mathcal{P}(E)$ is a *set mapping* if $x \notin f(x)$ whenever $x \in E$. A subset $X \subseteq E$ is free for f iff $f(y) \cap X = \varnothing$ whenever $y \in X$. If $\kappa = \mathrm{cf}(\kappa) < \lambda$ then $S^\lambda_\kappa = \{\delta \in \lambda \mid \mathrm{cf}(\delta) = \kappa\}$. If $\aleph_0 < \mathrm{cf}(\lambda)$ then S^λ_κ is a stationary subset of λ. We shall use the idea of indestructibility (at supercompact cardinals) as appeared in the seminal work of Laver, [Lav78]. It is shown there that a supercompact cardinal κ can be made indestructible under any generic extension by κ-directed-closed forcing notions. For basic background concerning Prikry type forcings we refer to [Git10], and to the papers of Magidor [Mag77a] and [Mag77b] in which the basic method of Prikry forcing with interleaved collapses was introduced. We also refer to [Hay23], especially with respect to the Extender-based Prikry forcing with interleaved collapses, to be used later. For background in pcf theory we suggest [AM10] and [She94].

1. A negative relation from stick

The prediction principle that we need for the combinatorial proof is called *stick*. The idea of stick as a prediction principle is well-articulated in [BBCE, Chapter 4(12)]: "It consults its stick, its rod directs it". Here we need the mathematical incarnation of this idea. We commence with the following definition.

Definition 1.1. Suppose that $\kappa \leq \lambda$.

(ℵ) $⫯(\kappa, \lambda) = \min\{|\mathcal{F}| \mid \mathcal{F} \subseteq [\lambda]^\kappa \wedge \forall y \in [\lambda]^\lambda \exists x \in \mathcal{F}, x \subseteq y\}$.
(ℶ) Denote $⫯(\lambda, \lambda^+)$ by $⫯(\lambda)$.

The stick principle is closely related to the club principle, but no stationary sets are involved in the prediction. This fact makes $⫯(\lambda)$ very useful when λ is a singular cardinal. Let us recall the definition of the club principle (or *tiltan*), which appeared for the first time in [Ost76]. If $\kappa = \mathrm{cf}(\kappa) > \aleph_0$ and $S \subseteq \kappa$ is stationary, then a tiltan sequence $(T_\delta \mid \delta \in S)$ is a sequence of sets, where T_δ is a cofinal subset of δ for each $\delta \in S$,[2] and if $A \in [\kappa^+]^{\kappa^+}$ then $S_A = \{\delta \in S \mid T_\delta \subseteq A\}$ is stationary. One says that $\clubsuit_S$ holds if there exists such a sequence.

In order to force $⫯(\lambda)$ where λ is singular, we will assume that $\clubsuit_S$ holds where $S = S^{\kappa^+}_\kappa$, κ is supercompact and $2^\kappa > \kappa^+$. This assumption, by itself, is forceable. We indicate that this is still insufficient for preserving the stick through Prikry type forcing notions, and this will be elaborated later. The following, however, can be proved.

Lemma 1.2. *It is consistent that κ is supercompact, Laver-indestructible, $2^\kappa > \kappa^+$ and $\clubsuit_S$ holds where $S = S^{\kappa^+}_\kappa$.*

Proof.
Let κ be a Laver-indestructible supercompact cardinal in V. Our strategy is to generalize the method of [She98], in which $\clubsuit_{\aleph_1}$ is forced with $2^{\aleph_0} > \aleph_1$. To this end, we commence with GCH above κ. By [She10], $\diamondsuit_S$ holds at $T = S^{\kappa^{++}}_\kappa$. As shown in [She98], one can define from this assumption a tiltan sequence $(T_\eta \mid \eta \in T)$ that is preserved under any κ^+-complete further extensions. In particular, the usual Cohen forcing to increase 2^{κ^+} will preserve $\clubsuit_T$ and also the supercompactness of κ.

Let W be the generic extension of V after this forcing. So κ is supercompact and Laver-indestructible, $2^{\kappa^+} > \kappa^{++}$ and $\clubsuit_T$ holds at $T = S^{\kappa^{++}}_\kappa$ (moreover, it is preserved by further κ^+-complete extensions). In W let $\mathbb{Q}$ be the collapse of κ^+ to κ, and let $G \subseteq \mathbb{Q}$ be generic over W. Standard arguments show that $2^\kappa > \kappa^+$ in $W[G]$. Let us argue that $\clubsuit_S$ holds in $W[G]$ where $S = S^{\kappa^+}_\kappa$.

[2] We assume, tacitly, that S consists of limit ordinals. There is no loss of generality here since S is stationary.

Firstly, if $\underset{\sim}{B}$ is a $\mathbb{Q}$-name of a set $B \in [\kappa^+]^{\kappa^+} \cap W[G]$ then there are $A \in [\kappa^{++}]^{\kappa^{++}} \cap W$ and $p \in \mathbb{Q}$ such that $p \Vdash A \subseteq \underset{\sim}{B}$. Secondly, given $B \in [\kappa^+]^{\kappa^+} \cap W[G]$ choose a name $\underset{\sim}{B}$ of B and a pair (p, A) as above. In W, pick $\eta \in S$ such that $T_\eta \subseteq A$ and notice that every $q \geq p$ forces $T_\eta \subseteq \underset{\sim}{B}$. Inasmuch as one guess implies stationarily many guesses (see, e.g., [Gar18, Lemma 2.1]) one infers that $\clubsuit_S$ holds in $W[G]$ as required.

$\square_{1.2}$

Our next step is to obtain the above setting with another feature dubbed as *the Galvin property*. There is a substantial connection between this property and density of ground model sets (of a given size) in a Prikry generic extension. This phenomenon was investigated by Gitik in [Git17]. There are several ways to describe the Galvin property, and one can understand it as a saturation property of filters. Recall:

Definition 1.3. Let $\mathscr{F}$ be a normal filter over an uncountable cardinal κ, and assume that $\kappa \leq \mu \leq \lambda$. One says that $\mathrm{Gal}(\mathscr{F}, \mu, \lambda)$ holds iff every family $\mathcal{C} = \{C_\alpha \mid \alpha \in \lambda\} \subseteq \mathscr{F}$ admits a subfamily $\mathcal{D} = \{C_{\alpha_i} \mid i \in \mu\}$ such that $\bigcap \mathcal{D} \in \mathscr{F}$.

Based on the methods of [GS14] one can force κ to be supercompact, $2^\kappa > \kappa^+$ and $\mathrm{Gal}(\mathscr{U}, \kappa^+, \kappa^+)$ for some normal ultrafilter $\mathscr{U}$ over κ, see [BGP23, Corollary 4.6]. Unfortunately, we do not know how to amalgamate the forcing which gives this ultrafilter with the forcing of Lemma 1.2. We show, nevertheless, that if one forces these two properties together, then the stick principle is preserved in Prikry type extensions.

Theorem 1.4. *Assume that:*

- ($\aleph$) *λ is supercompact and $2^\lambda > \lambda^+$.*
- ($\beth$) *The principle $\clubsuit_S$ holds where $S = S^{\lambda^+}_\lambda$.*
- ($\gimel$) *There exists a normal ultrafilter $\mathscr{U}$ over λ with $\mathrm{Gal}(\mathscr{U}, \lambda^+, \lambda^+)$.*

Then one can force a universe in which λ is a strong limit singular cardinal of countable cofinality, $2^\lambda > \lambda^+$ and $\overset{\bullet}{|}(\lambda)$ holds.

Proof.
Let λ be supercompact and assume that $\mathrm{Gal}(\mathscr{U}, \lambda^+, \lambda^+)$ holds, where $\mathscr{U}$ is a normal ultrafilter $\mathscr{U}$ over λ. We are assuming, further, that $\clubsuit_S$ holds where $S = S^{\lambda^+}_\lambda$ and $2^\lambda > \lambda^+$. Let $(T_\delta \mid \delta \in S)$ be a witness to the tiltan principle at S. Let $\mathbb{P}$ be Prikry forcing through $\mathscr{U}$, and let $G \subseteq \mathbb{P}$ be generic. We claim that $(T_\delta \mid \delta \in S)$ witnesses $\overset{\bullet}{|}(\lambda)$ in the generic extension.

To see this, suppose that A is a subset of λ^+ of size λ^+ in $V[G]$. Let p be a condition which forces this fact. For every $\alpha \in \lambda^+$ let $p \leq p_\alpha = (s_\alpha, B_\alpha) \in \mathbb{P}$ be a condition which forces the minimal ordinal greater than or equals to α in $\underset{\sim}{A}$ to be β_α, namely $p_\alpha \Vdash \check{\beta}_\alpha = \min(\underset{\sim}{A} - \check{\alpha})$. Since the number of possible stems of the form s_α is just λ, there is a fixed $s \in [\lambda]^{<\omega}$ and a set $A' \in [\lambda^+]^{\lambda^+}$ so that $\alpha \in A' \Rightarrow s_\alpha = s$.

Apply $\mathrm{Gal}(\mathscr{U},\lambda^+,\lambda^+)$ to the collection $\{B_\alpha \mid \alpha\in A'\}$ to obtain a set $B\in\mathscr{U}$ and a subcollection $\{B_{\alpha_i}\mid i\in\lambda^+\}$ so that $B\subseteq B_{\alpha_i}$ for every $i\in\lambda^+$. Let $A''=\{\beta_{\alpha_i}\mid i\in\lambda^+\}\in V$, and let $q=(s,B)\in\mathbb{P}$. Since q forces β_{α_i} into A for each $i\in\lambda^+$ one concludes that $q\Vdash A''\subseteq \underset{\sim}{A}$. Since $|A''|=\lambda^+$, there is an ordinal $\delta\in S$ such that $T_\delta\subseteq A''$. It follows that $T_\delta\subseteq A$ in $V[G]$, so the proof is accomplished.

$\square_{1.4}$

Our second statement recasts the story down at $\aleph_\omega$. To this end we need the so-called Prikry forcing with interleaved collapses. As in the previous proof, the result will follow from a density property of the unbounded ground model subsets of λ^+ in the unbounded subsets of $\lambda^+=\aleph_{\omega+1}^{V[G]}$ in the generic extension.

In the simpler version of Theorem 1.4 we had a name for an unbounded set $\underset{\sim}{A}$, and we obtained a sequence $(p_\alpha\mid\alpha\in\lambda^+)$ of conditions, each one of them introduces an ordinal into $\underset{\sim}{A}$. Then, applying the appropriate Galvin property, we constructed a single condition q stronger than λ^+-many of them. Now q forces a ground model set of size λ^+ to be a subset of $\underset{\sim}{A}$. The old set is predicted by a tiltan sequence from the ground model, and hence $\underset{\sim}{A}$ is predicted as well.

The same strategy will be used below in the context of Prikry with interleaved collapses, but here the conditions are more complicated. In particular, every condition contains a component of *promises* for the next collapse. In order to get a single condition that amalgamates λ^+-many such promises, we must pick the measure $\mathscr{U}$ carefully. The following lemma will be instrumental for this process.

Lemma 1.5. *Let λ be a supercompact cardinal. Then there is a generic extension $V[G]$ in which the following statements are true:*

- *(a)* $2^\lambda=\lambda^{++}$.
- *(b)* λ *is supercompact.*
- *(c)* $\mathscr{U}$ *is a normal measure over* λ *and it satisfies* $\mathrm{Gal}(\mathscr{U},\lambda^+,\lambda^+)$.
- *(d)* ${}^{\lambda^+}\jmath_{\mathscr{U}}(\lambda)\subseteq N$, *where* $N=Ult(V[G],\mathscr{U})$ *and* $\jmath_{\mathscr{U}}$ *is the corresponding ultrapower map.*

Proof.
We may assume that λ is Laver-indestructible, and $2^\lambda=\lambda^+$ in the ground model. Let $\ell:\lambda\to V_\lambda$ be the usual Laver diamond. Following [BTFFM17], we define an iteration of length λ^{+3} as a lottery sum in which we have an instance of the generalized Mathias forcing at even steps and $\mathrm{Add}(\lambda,1)$ at odd steps. As shown in [BTFFM17], an initial segment of this iteration generates a normal measure $\mathscr{U}$ over λ which satisfies $\mathrm{Gal}(\mathscr{U},\lambda^+,\lambda^+)$. Indeed, this normal measure has a basis which consists of a $\subseteq^*$-decreasing sequence of length λ^{++}, and this gives the pertinent Galvin property. It also gives $2^\lambda=\lambda^{++}$ in the generic extension, and λ remains supercompact. It remains to prove the closure property ${}^{\lambda^+}\jmath_{\mathscr{U}}(\lambda)\subseteq N$. To this end, we will have to dive into the details of the proof of [BTFFM17].

We commence with a λ^{+3}-supercompact embedding $\jmath : V \to M$ so that $\jmath(\ell)(\lambda)$ is our defined iteration of length λ^{+3} as described in the previous paragraph. We choose the M-generic filter H as done in [BTFFM17], but we add another feature. Let $\tilde{\jmath}$ be lift of $\jmath$. At every odd ordinal, our iteration adds a generic function $F_\alpha : \lambda \to \lambda$. In the construction of H we would like to decide the value of $\tilde{\jmath}(F_\alpha)(\lambda)$ for every such α. Let $h : \lambda^{+3} \to \jmath(\lambda)$ be a bijection in V. By the methods of Gitik and Sharon in [GS08], we can pick a generic filter over M such that $\jmath(F_{2\beta+1})(\lambda) = h(\beta)$ whenever $\beta \in \lambda^{+3}$. We chop the iteration at some α_* such that for every function $g : \lambda \to \lambda$ there exists $\gamma \in \alpha_*$ so that $\tilde{\jmath}(F_\gamma)(\lambda) = \tilde{\jmath}(g)(\lambda)$.

Let $\mathscr{U}$ be the measure derived from the initial segment of the iteration up to α_*, and let $\jmath_{\mathscr{U}}$ be associated ultrapower embedding. Set $N = Ult(V[G], \mathscr{U})$. We must show that every sequence of ordinals from $\jmath_{\mathscr{U}}(\lambda)$ of length λ^+ belongs to N. Define $\imath : N \to M[H]$ by letting $\imath([r]) = \jmath(r)(\lambda)$. It is routine to check that $\imath$ is an elementary embedding. Observe that $\mathrm{crit}(\imath) \geq \jmath(\lambda)$. Indeed, if $\alpha \in \jmath(\lambda)$ then there is an ordinal γ such that $\jmath(F_\gamma)(\lambda) = \alpha$ and then $\imath([F_\gamma]) = \alpha$. Since $\mathrm{crit}(\imath)$ is the first ordinal which does not belong to the image of $\imath$, one concludes that $\mathrm{crit}(\imath) \geq \jmath(\lambda)$. But actually $\mathrm{crit}(\imath) > \jmath(\lambda)$, since $\imath(\jmath_{\mathscr{U}}(\lambda)) = \jmath(\lambda)$.

To sum up, $\mathrm{crit}(\imath) > \jmath(\lambda)$ and from the fact that $\imath(\jmath_{\mathscr{U}}(\lambda)) = \jmath(\lambda)$ we conclude that $\jmath_{\mathscr{U}}(\lambda) = \jmath(\lambda)$. By elementarity, $\jmath(\lambda)$ is an inaccessible cardinal in N. Observe that $M[H]$ is closed under sequences of length λ^+, and $\mathrm{cf}(\jmath(\lambda)) > \lambda^+$. Consequently, $V^N_{\jmath(\lambda)} = V^M_{\jmath(\lambda)} \supseteq {}^{\lambda^+}\jmath(\lambda)$, so the proof is accomplished.

$\square_{1.5}$

Equipped with the above lemma, we can prove the following.

Theorem 1.6. *If there exists a supercompact cardinal λ in the ground model such that $2^\lambda > \lambda^+$, $\mathscr{U}$ is a normal measure over λ and it satisfies $\mathrm{Gal}(\mathscr{U}, \lambda^+, \lambda^+)$ and $\clubsuit_S$ holds at $S = S^{\lambda^+}_\lambda$ then one can force a universe in which $\aleph_\omega$ is a strong limit cardinal, $2^{\aleph_\omega} > \aleph_{\omega+1}$, and $\overset{\bullet}{|}(\aleph_\omega)$ holds.*

Proof.
Let λ and $\mathscr{U}$ be as in the assumptions of the theorem. Let $\jmath_{\mathscr{U}}$ be the associated ultrapower embedding. We define a forcing notion $\mathbb{P}$ as follows. A condition $p \in \mathbb{P}$ is a quadruple $(s, \vec{f}, A, F) = (s^p, \vec{f}^p, A^p, F^p)$, where:

- (a) $s = (\rho_0, \ldots, \rho_{n-1}) \in [\lambda]^{<\omega}$ is an increasing sequence of inaccessible cardinals.
- (b) $\ell g(\vec{f}) = n+1$, where $f_k \in \mathrm{Col}(\rho^{++}_{k-1}, <\rho_k)$ for every $k \leq n$, stipulating $\rho_{-1} = \aleph_0$ and $\rho_n = \lambda$.
- (c) $A \in \mathscr{U}$ and $\min(A) > \max(s), \mathrm{rang}(f_n)$.
- (d) $\mathrm{dom}(F) = A$ and $F(\alpha) \in \mathrm{Col}(\alpha^{++}, <\lambda)$ for every $\alpha \in \mathrm{dom}(F)$.

For the order, suppose that $p, q \in \mathbb{P}$. We shall say that $p \leq_{\mathbb{P}} q$ iff the following requirements are met:

- (a) $s^p \trianglelefteq s^q$ and $s^q - s^p \subseteq A^q$.

(b) For every $\ell \leq n, f^p_\ell \subseteq f^q_\ell$.
(c) Denoting $n = |s^p|, m = |s^q|$ let $\rho_n, \ldots, \rho_{m-1}$ be the increasing enumeration of the elements of $s^q - s^p$, and require that for every $n < \ell \leq m, F^p(\rho_\ell) \subseteq f^q_\ell$.
(d) $A^q \subseteq A^p$.
(e) For every $\beta \in A^q, F^p(\beta) \subseteq F^q(\beta)$.

The forcing notion $\mathbb{P}$ is traditionally described as Prikry forcing for singularizing λ with interleaved collapses, without a guiding generic. It is known that $\mathbb{P}$ enjoys the Prikry property, see e.g. [Git10]. Likewise, if $G \subseteq \mathbb{P}$ is V-generic then $\lambda^+ = \aleph^{V[G]}_{\omega+1}$.

Suffice it to show that every unbounded subset of $\lambda^+ = \aleph^{V[G]}_{\omega+1}$ contains an old unbounded subset of λ^+ from the ground model. Indeed, an appropriate tiltan sequence in the ground model will predict the old unbounded set and hence also the new unbounded set of $\lambda^+ = \aleph^{V[G]}_{\omega+1}$ in the generic extension, exactly as in the proof of Theorem 1.4. Let us verify that the desired density property is forced by $\mathbb{P}$. Essentially, the reason for this property to hold true is the Galvin property of $\mathscr{U}$, but let us elaborate.

Let $\underset{\sim}{B}$ be a name for an unbounded subset of $\lambda^+ = \aleph^{V[G]}_{\omega+1}$. Fix an arbitrary condition $p \in \mathbb{P}$. We choose a sequence $(p_\alpha \mid \alpha \in \lambda^+)$ of conditions above p with the following two properties:

($\aleph$) For each $\alpha \in \lambda^+$ the condition $p_\alpha \geq p$ decides an ordinal β_α to be the least element of $\underset{\sim}{B}$ above α.
($\beth$) The conditions $[F_\alpha]_{\mathscr{U}} \in \mathrm{Col}(\lambda^{++}, < \jmath(\lambda))^{Ult(V,\mathscr{U})}$ at each p_α form an increasing sequence.

One can construct this sequence by induction on λ^+ by virtue of the closure of the forcing in the ultrapower.

Consider the sequence $([F_\alpha]_{\mathscr{U}} \mid \alpha \in \lambda^+)$. By the closure of $Ult(V, \mathscr{U})$, the sequence belongs to the ultrapower and, moreover, there is a single condition $[F_{\lambda^+}]_{\mathscr{U}}$ so that $[F_\alpha]_{\mathscr{U}} \leq [F_{\lambda^+}]_{\mathscr{U}}$ for every $\alpha \in \lambda^+$.

For every $\alpha \in \lambda^+$ let $B_\alpha = A_\alpha \cap \{\gamma \in \lambda \mid F_\alpha(\gamma) \leq F_{\lambda^+}(\gamma)\}$, so $B_\alpha \in \mathscr{U}$ for every $\alpha \in \lambda^+$. Recall that $\mathrm{Gal}(\mathscr{U}, \lambda^+, \lambda^+)$ holds and hence there are $B \in \mathscr{U}$ and $I \in [\lambda^+]^{\lambda^+}$ such that $\alpha \in I \Rightarrow B \subseteq B_\alpha$. Since the number of pairs of the form $(s_\alpha, \vec{f_\alpha})$ is λ, there is a single pair $(s, \vec{f})$ and a set $J \in [I]^{\lambda^+}$ such that $\alpha \in J \Rightarrow (s_\alpha, \vec{f_\alpha}) = (s, \vec{f})$.

Let $q = (s, \vec{f}, B, [F_{\lambda^+}]_{\mathscr{U}})$ and let $A'' = \{\beta_\alpha \mid \alpha \in J\}$. The set A'' belongs to the ground model. Notice that $q \in \mathbb{P}$ and $p_\alpha \leq q$ for every $\alpha \in J$. Consequently, $q \Vdash A'' \subseteq \underset{\sim}{A}$, so we are done.

$\square_{1.6}$

In the above theorems we force Prikry through a normal ultrafilter which satisfies an additional property. We still do not know how to force our tiltan setting along with this instance of the Galvin property. Likewise, we do not know whether Prikry forcing through any normal ultrafilter preserves the desired prediction principle. The following problem summarizes our ignorance.

Question 1.7. Let κ be a measurable cardinal and denote $S^{\kappa^+}_\kappa$ by S. Let $(T_\delta \mid \delta \in S)$ be a tiltan sequence and let $\mathscr{U}$ be a normal ultrafilter over κ.

- (α) Let $\mathbb{P}$ be Prikry forcing through $\mathscr{U}$ and let $G \subseteq \mathbb{P}$ be V-generic. Does $(T_\delta \mid \delta \in S)$ witness $\stick(\kappa)$ in $V[G]$?
- (β) Is it consistent that $\mathrm{Gal}(\mathscr{U}, \kappa^+, \kappa^+)$ holds, $2^\kappa > \kappa^+$ and $\clubsuit_S$ holds?

We proceed to the combinatorial result. Our goal is to prove the relation $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ from the stick principle $\stick(\lambda)$. Negative partition relations at successors of a regular cardinal κ follow from $\stick(\kappa)$ as shown in [CGW20]. Here we apply a similar idea to successors of singular cardinals. We need the following lemma about free sets from [HM75]. The lemma and its proof also appear in [EHMR84, Lemma 20.3].

Lemma 1.8. *Let κ be a regular cardinal. Suppose that $E = \bigcup_{\alpha\in\kappa} E_\alpha$, and $|E_\alpha| > \kappa$ for every $\alpha \in \kappa$. Assume further that $f : E \to \mathcal{P}(E)$ is a set mapping, and $|f(x) \cap E_\alpha| < \kappa$ for every $x \in E, \alpha \in \kappa$. Then there exists a free set X for f so that $X \cap E_\alpha \neq \varnothing$ for every $\alpha \in \kappa$.*

We can state now the following:

Theorem 1.9. *suppose that $\theta = \mathrm{cf}(\lambda) < \lambda$ and assume that $\stick(\lambda)$ holds. Then $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$.*

Proof.
Let $(\kappa_i \mid 1 \leq i \in \theta)$ be an increasing sequence of infinite cardinals such that $\lambda = \bigcup_{i\in\theta} \kappa_i$. Notice that the enumeration of these cardinals begins with κ_1 since we wish to save the first color to the full-sized independent subsets of the graph. We shall define a partition $\mathcal{P} = (\mathcal{P}_i \mid i \in \theta)$ of $[\lambda^+]^2$.[3] Then, essentially, for $\alpha < \beta < \lambda^+$ we will set $c(\alpha, \beta) = i$ iff $\{\alpha, \beta\} \in \mathcal{P}_i$.[4] The partition $\mathcal{P}$ will be based on a sequence of set-mappings in the following way. For every $i \in (0, \theta)$ we shall define $f_i : [\lambda^+ \to [\lambda^+]^{\leq\kappa_i}$ such that $f_i(\alpha) \subseteq \alpha$ for every $i \in \theta, \alpha \in \lambda^+$. We let $\mathcal{P}_i = \{\{\alpha, \beta\} \mid \alpha < \beta < \lambda^+, \alpha \in f_i(\beta)\}$. This procedure defines $\mathcal{P}_i$ for $i > 0$, and we let $\mathcal{P}_0 = [\lambda^+]^2 - \bigcup\{\mathcal{P}_i \mid 1 \leq i < \theta\}$.

The construction of each f_i is by induction on $\alpha \in \lambda^+$, where at the αth stage, $f_i(\alpha)$ is defined simultaneously for each $i \in (0, \theta)$. Fix $\alpha \in \lambda^+$ and suppose that $f_i(\gamma)$ is already defined for every $\gamma \in \alpha$ and every $i \in \theta$. Let $(T_\eta \mid \eta \in \lambda^+)$ be a $\stick(\lambda)$ sequence, so $T_\eta \in [\lambda^+]^\lambda$ for every $\eta \in \lambda^+$. Let $\mathcal{S}_\alpha = \{T_\eta \mid \eta \in \alpha, T_\eta \subseteq \alpha\}$. Notice that $|\mathcal{S}_\alpha| \leq |\alpha| \leq \lambda$ and hence there exists a decomposition of the form $\mathcal{S}_\alpha = \bigcup\{\mathcal{S}^\alpha_i \mid 1 \leq i \in \theta\}$, where $i < j \Rightarrow \mathcal{S}^\alpha_i \cap \mathcal{S}^\alpha_j = \varnothing$ and $|\mathcal{S}^\alpha_i| \leq \kappa_i$ for every $i \in (0, \theta)$.

[3] The elements of this partition are not required to be disjoint, so we use here the term *partition* in an unusual way.

[4] We say *essentially* since the elements of the partition here are not necessarily disjoint, so the formal definition of the coloring will take the first i for which $\{\alpha, \beta\} \in \mathcal{P}_i$.

In order to define $f_i(\alpha)$ for each $i \in (0,\theta)$, fix an ordinal i and apply Lemma 1.8, where κ_i^+ here stands for κ there, and $f_i \upharpoonright \alpha$ here stands for f there. Notice that $|f_i(\gamma)| \leq \kappa_i$ for each $\gamma \in \alpha$ by the induction hypothesis, so the assumptions of the lemma hold. By the conclusion of the lemma, there exists a free set $X = X_{\alpha i}$ for $f_i \upharpoonright \alpha$ which satisfies $X \cap T \neq \varnothing$ for every $T \in \mathcal{S}_i^\alpha$. By removing elements from X if needed, we may assume that $|X| \leq |\mathcal{S}_i^\alpha| \leq \kappa_i$, so we can define $f_i(\alpha) = X = X_{\alpha i}$. This completes the definition of our set mappings, and consequently the definition of $\mathcal{P}$, the partition of $[\lambda^+]^2$.

We define, at this stage, the coloring $c : [\lambda^+]^2 \to \theta$ by letting $c(\alpha,\beta) = i$ iff $i \in \theta$ is the first ordinal so that $\{\alpha,\beta\} \in \mathcal{P}_i$. We claim that c witnesses the negative relation to be proved. To see this, let us show firstly that there are no $\alpha < \beta < \delta < \lambda^+$ and $i \in \theta$ such that $c(\alpha,\beta) = c(\alpha,\delta) = c(\beta,\delta) = i$. Indeed, if $\alpha < \beta < \delta < \lambda^+$ and $c(\alpha,\delta) = c(\beta,\delta) = i$ then $\{\alpha,\delta\},\{\beta,\delta\} \in \mathcal{P}_i$. This means that $\alpha,\beta \in f_i(\delta) = X$. But X is a free set with respect to $f_i \upharpoonright \delta$, and $\beta \in X$, hence $f_i(\beta) \cap X = \varnothing$. Since $\alpha \in X$ one concludes that $\alpha \notin f_i(\beta)$. Therefore, $c(\alpha,\beta) \neq i$.

Secondly, we argue that there is no 0-monochromatic subset of λ^+ of size λ^+. To see this, fix $A \in [\lambda^+]^{\lambda^+}$. Choose an ordinal $\eta \in \lambda^+$ such that $T_\eta \subseteq A$. If $\xi > \eta$ and $\xi > \sup(T_\eta)$ then, by definition, $T_\eta \in \mathcal{S}_\xi$. Since A is unbounded in λ^+, one can choose $\xi > \eta, \sup(T_\eta)$ such that $\xi \in A$. Recall that we had a partition $\mathcal{S}_\xi = \bigcup\{\mathcal{S}_i^\xi \mid 1 \leq i \in \theta\}$, hence $T_\eta \in \mathcal{S}_i^\xi$ for some $i \in (0,\theta)$.

By the choice of $f_i(\xi)$ we know that $T_\eta \cap f_i(\xi) \neq \varnothing$, so one can choose $\alpha \in T_\eta \cap f_i(\xi)$. The fact that $\alpha \in f_i(\xi)$ implies that $\{\alpha,\xi\} \in \mathcal{P}_i$. Hence $c(\alpha,\xi) \neq 0$. Since $\alpha \in T_\eta \subseteq A$ and $\xi \in A$, one concludes that $c''[A]^2 \neq \{0\}$, so we are done.

$\square_{1.9}$

2. A negative relation from pcf arguments

In this section we take a different path for proving the negative relation $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ where λ is a strong limit singular cardinal and $2^\lambda > \lambda^+$. The idea is to assume the negative relation at an unbounded sequence of cardinals below λ (by assuming GCH at these cardinals) and to obtain the negative relation at λ^+ by means of pcf theory. Recall that $\mathrm{tcf}(\prod_{i\in\theta}\lambda_i, J) = \kappa$ iff there is a J-cofinal and increasing sequence in the product $\prod_{i\in\theta}\lambda_i$, and κ is the minimal length of such a sequence. We commence with the combinatorial theorem, followed by a description of the ways to force the assumptions in this theorem.

Theorem 2.1. *Assume that:*

(a) $\mu > \mathrm{cf}(\mu) = \theta$.
(b) μ *is a strong limit cardinal.*
(c) $2^\mu > \mu^+$.
(d) $(\mu_i \mid i \in \theta)$ *is increasing and* $\mu = \bigcup_{i\in\theta}\mu_i$.
(e) $\mu_i > \mathrm{cf}(\mu_i) = \theta$ *for every* $i \in \theta$.
(f) μ_i *is a strong limit cardinal for every* $i \in \theta$.
(g) $2^{\mu_i} = \mu_i^+$ *for every* $i \in \theta$.
(h) $\mathrm{tcf}(\prod_{i\in\theta}\mu_i^+, J_\theta^{\mathrm{bd}}) = \mu^+$.

Then $\mu^+ \nrightarrow (\mu^+, (3)_{\mathrm{cf}(\mu)})^2$.

Proof.
For every $i \in \theta$ let $c_i : [\mu_i^+]^2 \to \theta$ be a witness to the negative relation $\mu_i^+ \nrightarrow (\mu_i^+, (3)_\theta)^2$. This negative relation follows from assumption (g). Our goal is to define a coloring $c : [\mu^+]^2 \to \theta$ by combining the c_is together in such a way that the corresponding negative relation at μ^+ will follow.

We need two mathematical objects to define our coloring. The first is a scale $(f_\alpha \mid \alpha \in \mu^+)$ in the product $(\prod_{i\in\theta}\mu_i^+, J_\theta^{\mathrm{bd}})$. The second is a system of functions $(h_i \mid i \in \theta)$ where $h_i \in {}^\theta\theta$ is injective and $h_i(0) = 0$ for each $i \in \theta$. Also, if $i < j < \theta$ then $rang(h_i) \cap rang(h_j) = \{0\}$. Suppose that $\alpha < \beta < \mu^+$. Let $i(\alpha,\beta)$ be the minimal $j \in \theta$ so that $f_\alpha(j) \neq f_\beta(j)$. Such an ordinal always exists since $f_\alpha <_{J_\theta^{\mathrm{bd}}} f_\beta$. We define the coloring $c : [\mu^+]^2 \to \theta$ as follows. Given $\alpha < \beta < \mu^+$ let $i = i(\alpha,\beta)$ and let $c(\alpha,\beta) = h_i(c_i(f_\alpha(i), f_\beta(i)))$. Let us show that c exemplifies the negative relation $\mu^+ \nrightarrow (\mu^+, (3)_{\mathrm{cf}(\mu)})^2$.

(ℵ) Assume that $A \in [\mu^+]^{\mu^+}$. For every $i \in \theta$ let $A_i = \{f_\alpha(i) \mid \alpha \in A\}$. Set $X = \{i \in \theta \mid \mu_i^+ = \bigcup A_i\}$, and notice that $X = \theta \bmod J_\theta^{\mathrm{bd}}$. Fix $i \in X$. For every $\varepsilon \in \mu_i^+$ we choose $\alpha_\varepsilon \in A$ such that $f_{\alpha_\varepsilon}(i) \geq \varepsilon$. Since $|\prod_{j\in i}\mu_j^+| < \mu_i^+$ there are a set $B_i \in [\mu_i^+]^{\mu_i^+}$ and a fixed element $g \in \prod_{j\in i}\mu_j^+$ such that if $\varepsilon < \zeta$ are taken from B_i then $\alpha_\varepsilon < \alpha_\zeta$ and $f_{\alpha_\varepsilon} \restriction i = g$ for every $\varepsilon \in B_i$. Since c_i witnesses the negative relation $\mu_i^+ \nrightarrow (\mu_i^+, (3)_\theta)^2$, one can choose $\varepsilon, \zeta \in B_i$ such that $\varepsilon < \zeta$ and

$c_i(\varepsilon,\zeta) \neq 0$. But then $c(\alpha_\varepsilon, \alpha_\zeta \neq 0$, so the proof of the first case is accomplished.

($\beth$) Assume that $\alpha < \beta < \gamma < \mu^+$. If $i(\alpha,\beta) \neq i(\alpha,\gamma)$ or $i(\alpha,\beta) \neq i(\beta,\gamma)$ or $i(\alpha,\gamma) \neq i(\beta,\gamma)$ then $\{\alpha,\beta,\gamma\}$ cannot be c-monochromatic with any color $\xi > 0$ since for $i \neq j$ one has $rang(h_i) \cap rang(h_j) = \{0\}$ and by the definition of c. If $i(\alpha,\beta) = i(\alpha,\gamma) = i(\beta,\gamma) = i$ then $c \restriction [\{\alpha,\beta,\gamma\}]^2 = \{\xi\}$ with $\xi > 0$ implies $c \restriction [\{f_\alpha(i), f_\beta(i), f_\gamma(i)\}]^2 = \{\xi\}$, since h_i is injective. But this is impossible by the choice of c_i, so we are done.

$\square_{2.1}$

A corollary to the above theorem gives an answer to the question of Erdős and Hajnal. Within the proof of this corollary we force with $\mathbb{Q}_{\bar\mu}$ from [GS12, Definition 2.3]. For being self-contained, we unfold the definition of this forcing notion.

Let μ be supercompact, and let $\bar\mu = (\mu_i \mid i \in \mu)$ be an increasing sequence of regular cardinals so that $2^{|i|} < \mu_i$ for every $i \in \mu$. A condition $p \in \mathbb{Q}_{\bar\mu}$ is a pair $(\eta, f) = (\eta^p, f^p)$ such that $\ell g(\eta) \in \mu$ and $\eta \in \prod_{i\in\ell g(\eta)} \mu_i$. We refer to η as the stem of p. Also, $f \in \prod_{i\in\mu} \mu_i$ and $\eta \triangleleft f$. If $p, q \in \mathbb{Q}_{\bar\mu}$ then $p \leq q$ iff $\eta^p \trianglelefteq \eta^q$ and $f^p(j) \leq f^q(j)$ for every $j \in \mu$.

Intuitively, $\mathbb{Q}_{\bar\mu}$ adds a function $h \in \prod_{i\in\mu} \mu_i$ which dominates every old function in this product. If $2^\mu = \mu^+$ in the ground model then $\mathbb{Q}_{\bar\mu}$ is μ^+-cc. Also, $\mathbb{Q}_{\bar\mu}$ is $(< \mu)$-strategically-closed. Hence one can iterate $\mathbb{Q}_{\bar\mu}$. If $\theta = \mathrm{cf}(\theta) > \mu$ is the length of the iteration then the generic functions added at each step form a scale. Moreover, upon singularizing μ either by Prikry or by Magidor forcing one preserves the properties of this scale, thus forcing $\mathrm{tcf}(\prod_{i\in\mathrm{cf}(\mu)} \mu_i, J^{\mathrm{bd}}_{\mathrm{cf}(\mu)}) = \theta$ in the generic extension.

Corollary 2.2. *Assuming the existence of large cardinals in the ground model, one can force $\mu^+ \nrightarrow (\mu^+, (3)_{\mathrm{cf}(\mu)})^2$ with $2^\mu > \mu^+$ and μ is a strong limit cardinal.*

Proof.
Our goal is to force the assumptions of Theorem 2.1. Let μ be a supercompact cardinal and fix a regular cardinal $\aleph_0 \leq \theta \in \mu$. We may assume that μ is Laver-indestructible, and GCH holds above μ. Let $(\mu_i \mid i \in \mu)$ be an increasing sequence of singular cardinals so that $\mathrm{cf}(\mu_i) = \theta$ for every $i \in \mu$ and $\mu = \bigcup_{i\in\mu} \mu_i$. We may assume that $2^{\mu_i} = \mu_i^+ < \mu_{i+1}$ for every $i \in \mu$.

We force with $\mathbb{Q}_{\bar\mu}$ followed by Magidor forcing to make $\theta = \mathrm{cf}(\mu)$ to obtain the assumption $\mathrm{tcf}(\prod_{i\in\theta} \mu_i^+, J^{\mathrm{bd}}_\theta) = \mu^+$. If $\theta = \aleph_0$ then one can simply use Prikry forcing. Thus, the length of the iteration should be an ordinal of cofinality μ^+. We increase 2^μ to any desired point (this can be done by choosing the length of the iteration to be in the desired cardinality). Notice that $2^{\mu_i} = \mu_i^+$ remains true, as $\mathbb{Q}_{\bar\mu}$ is χ-strategically-closed for every $\chi \in \mu$ and the component of Prikry or Magidor forcing also preserve this fact.

Thus the assumptions of Theorem 2.1 hold in the generic extension, and the corollary follows.

$\square_{2.2}$

It seems that the above method cannot be applied to $\aleph_\omega$. The main point is that our singular cardinal μ of cofinality θ should be a limit of singular cardinals with the same cofinality. Thus, the negative colorings along the way are always with the same number of colors (namely, θ) and hence one can produce a coloring over the cardinal μ^+ with θ-many colors. Since there are no singular cardinals below $\aleph_\omega$ at all, the above proof is inapplicable as it is to this case. However, $\aleph_{\omega^2}$ seems suitable for this pattern of proof. Indeed, the cofinality of $\aleph_{\omega^2}$ is ω and it is a limit of singular cardinals of countable cofinality.

Theorem 2.3. *Assuming the existence of large cardinals in the ground model, one can force $\mu^+ \nrightarrow (\mu^+, (3)_{\mathrm{cf}(\mu)})^2$ with $2^\mu > \mu^+$ and μ is a strong limit cardinal, where $\mu = \aleph_{\omega^2}$.*

Proof.
Let μ be a strong cardinal and let $\lambda \geq \mu^{++}$ be a regular cardinal. Let E be a (μ, λ)-extender and let $\jmath : V \to M \cong \mathrm{Ult}(V, E)$ be the canonical embedding, where $M \supseteq V_{\mu^{++}}$. We assume GCH in the ground model. In order to force the above statement at μ we use the Extender-based Prikry forcing, and in order to obtain the negative relation at $\mu = \aleph_{\omega^2}$ we use the same forcing with interleaved collapses.

Let G be V-generic for this forcing notion. Notice that μ is a strong limit cardinal in $V[G]$, and $2^\mu = \mu^{++}$. Likewise, μ is a singular cardinal of countable cofinality in the generic extension, and GCH still holds below μ in $V[G]$. We can add the collapses to make $\mu = \aleph_{\omega^2}$ in $V[G]$.

Let $(\rho_n \mid n \in \omega)$ be the Prikry forcing added through the (unique) normal ultrafilter of E. It is known that $\mathrm{tcf}(\prod_{n\in\omega} \mu_n^+, J_\omega^{\mathrm{bd}}) = \mu^{++}$ in the generic extension. Moreover, up to a modification of a proper initial segment, this is the only sequence with true cofinality μ^{++} in this product. Hence, if $(\mu_n \mid n \in \omega)$ is an increasing sequence of singular cardinals with countable cofinality such that $\mu = \bigcup_{n\in\omega} \mu_n$ then $\mathrm{tcf}(\prod_{n\in\omega} \mu_n^+, J_\omega^{\mathrm{bd}}) = \mu^+$ in $V[G]$. For these facts we refer to [Hay23]. We see that all the assumptions of Theorem 2.1 hold, and therefore $\mu^+ \nrightarrow (\mu^+, (3)_{\mathrm{cf}(\mu)})^2$. In the setting of the Extender-based Prikry forcing with interleaved collapses we can make $\mu = \aleph_{\omega^2}$ in $V[G]$. This is the first infinite cardinal which can be represented as a limit of a sequence $(\mu_n \mid n \in \omega)$ as above, so the proof is accomplished.

$\square_{2.3}$

We conclude with three open problems. The first problem is related to $\aleph_\omega$. Using the methods of this paper, this problem boils down to a more general problem about the stick principle. We believe that this principle at a successor of a strong limit singular cardinal μ is strictly weaker than the set-theoretical assumption $2^\mu = \mu^+$, but we do not know how to force it.

Question 2.4. Is it consistent that $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ where $\lambda = \aleph_\omega$ is a strong limit cardinal and $2^\lambda > \lambda^+$? More generally, is it consistent that λ is a strong limit singular cardinal, $\text{⫯}(\lambda)$ holds but $2^\lambda > \lambda^+$?

Maybe the most interesting problem which issues from our study is whether the negative relation holds in ZFC. We believe that the positive relation $\lambda^+ \rightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ is consistent, but we do not know how to prove this:

Question 2.5. Is it consistent that λ is a strong limit singular cardinal and $\lambda^+ \rightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$? Is it forceable at $\lambda = \aleph_\omega$?

Another interesting problem is the consistency strength of the main result of the current paper. In order to force the failure of SCH at λ, as done in our results, one has to assume the existence of a measurable cardinal κ with $o(\kappa) = \kappa^{++}$ in the ground model. This fundamental result was proved by Gitik in [Git89] and in [Git91]. In our constructions we started from a supercompact cardinal in the ground model. The gap between these large cardinals invites the following:

Question 2.6. Let λ be a strong limit singular cardinal.

(א) What is the consistency strength of the negative relation $\lambda^+ \nrightarrow (\lambda^+, (3)_{\mathrm{cf}(\lambda)})^2$ with $2^\lambda > \lambda^+$?

(ב) What is the consistency strength of the same negative relation with $2^\lambda > \lambda^+$ where $\lambda = \aleph_\omega$?

(ג) What is the consistency strength of the negative relation at every strong limit singular cardinal λ, in a universe in which $2^\lambda > \lambda^+$ at every such λ?

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Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel

Email address: shimon.garty@mail.huji.ac.il

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Jerusalem 91904, Israel

Email address: yair.hayut@mail.huji.ac.il

Institute of Mathematics The Hebrew University of Jerusalem, Jerusalem 91904, Israel and Department of Mathematics Rutgers University New Brunswick, NJ 08854, USA

Email address: shelah@math.huji.ac.il

URL: http://www.math.rutgers.edu/~shelah