# NO UNIVERSAL GROUP IN A CARDINAL SH1029

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ABSTRACT. For many classes of models, there are universal members in any cardinal $\lambda$ which "essentially satisfies GCH, i.e. $\lambda = 2^{<\lambda}$", in particular for the class of a complete first order $T$ (well, if at least if $\lambda > |T|$). But if the class is "complicated enough", e.g. the class of linear orders, we know that if $\lambda$ is "regular and not so close to satisfying GCH" then there is no universal member. Here we find new sufficient conditions (which we call the olive property), not covered by earlier cases (i.e. fail the so-called $\mathrm{SOP}_4$). The advantage of those conditions is witnessed by proving that the class of groups satisfies one of those conditions.

This version has minor changes , compared to the earlier one.

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Annotated Content

§0 Introduction, (labels y,z), pg.3

§1 The olive property, (label d), pg.7

[We give definitions of some versions of the olive property and give an example failing the $\mathrm{SOP}_4$. We phrase relevant set theoretic conditions like $\mathrm{Qr}_1$ (slightly weaker than those used earlier). Then we give complete proof using $\mathrm{Qr}_1(\chi_2, \chi_1, \lambda)$ to deduce $\mathrm{Univ}(\chi_1, \lambda, \mathfrak{k}) \geq \chi_2$ so there is no universal in the class $\mathfrak{k}$ in the cardinal $\lambda$, when $\mathfrak{k}$ has the olive property.]

§2 The class of groups have the olive property, (label s), pg.14

[We prove the stated result. We also deal with the non-existence of universal structures for pairs of classes, e.g. the pair (locally finite groups, groups).]

# § 0. Introduction

## § 0(A). Background and open questions.

A natural and old question is how to characterize the class of cardinals $\lambda$ in which a class $\mathbf{k}$ has a universal member, where $\mathfrak{k}$ is, e.g. the class of models of a first-order theory $T$ with elementary embeddings.

On history see Kojman-Shelah [KS92] and later Dzamonja [Dža05]. Recall that if $\lambda = 2^{<\lambda} > \aleph_0$ then many classes have a universal member in $\lambda$, so assuming GCH, we know when there is a universal model in every $\lambda > |T|$.

For transparency, we consider a first-order countable $T$. Now the class $\mathfrak{k}_T = (\mathrm{Mod}_T, \prec)$ of models of $T$ with elementary embeddings have the amalgamation property, the JEP, and satisfies: $A \subseteq M \in \mathrm{Mod}_T \Rightarrow (\exists N \in \mathrm{Mod}_T)(A \subseteq N \prec M \wedge \|N\| = |A| + \aleph_0)$. From such classes (or just a.e.c. with amalgamation and JEP with $\mathrm{LST}(\mathfrak{k})$ instead of $\aleph_0$) it follows that if $\lambda = \lambda^{<\lambda} > \aleph_0$, <u>then</u> there is a saturated (or universal homogeneous) $M \in \mathrm{Mod}_T$ of cardinality $\lambda$ which implies it is universal for $\mathfrak{k}_T$. If $\lambda = 2^{<\lambda} > \aleph_0$ does not satisfy $\lambda = \lambda^{<\lambda}$, still there is a so-called special model which is universal. So we are interested in the cases where G.C.H. fail.

Recall that on the one hand, Kojman-Shelah [KS92] shows that if $T$ is the theory of dense linear orders or just $T$ has the strict order property, then $T$ fails (in a strong way) to have a universal member in regular cardinals in which cardinal arithmetic is "not close to GCH"; (for regular $\lambda$ this means there is a regular $\mu$ such that $\mu^+ < \lambda < 2^\mu$, while for singular $\lambda$ we need of course $\lambda < 2^{<\lambda}$ and a very weak pcf condition).

By [She96], we can weaken "the strict order property" to the 4-strong order property $\mathrm{SOP}_4$. On consistency see Dzamonja-Shelah [DS04].

Natural questions (we shall address some of them) are the following:

*Question* 0.1. 1) Is there a weaker condition (on $T$) than $\mathrm{SOP}_4$ which suffices?

2) Can we find a best one?

3) Can we find such a condition satisfied for some theory $T$ which is $\mathrm{NSOP}_3$?

*Question* 0.2. 1) Is there $T$ with the class $\mathrm{Univ}(T) \setminus (2^{\aleph_0})^+$ strictly smaller than the one for linear orderings, see Definition 0.12(2); is it better if we restrict ourselves to regular cardinals above $2^{\aleph_0}$?

2) Can we get the above to be $\{\lambda \colon \lambda = 2^{<\lambda}\}$?

3) What about singular cardinals?

*Question* 0.3. 1) Is it consistent that the class of linear orders has a universal member in $\lambda$ such that $2^{<\lambda} > \lambda > 2^{\aleph_0}$. (For $\lambda = \aleph_1 < 2^{\aleph_0}$, the answer is yes, see [She80], a more detailed version is in preparation).

2) The same can be asked for some theory with $\mathrm{SOP}_4$ or the olive property (defined below, e.g. in Definition 0.8.

Recall that by Shelah-Usvyatsov [SU06] the class of groups has $\mathrm{NSOP}_4$ but has $\mathrm{SOP}_3$, so it was not clear where it stands.

*Question* 0.4. 1) Where does the class of groups stand (concerning the existence of a universal member in a cardinal)?

2) Is it consistent that there is a universal locally finite group of cardinality $\aleph_1$? of cardinality $\beth_\omega$? $\lambda = \beth^+_\omega$? of regular cardinals $\lambda \in (\beth^+_\omega, \beth_{\omega+1})$? of other cardinals $\lambda < \lambda^{\aleph_0}$?

Recall (Grossberg-Shelah [GS83]) that if $\mu$ is strong limit of cofinality $\aleph_0$ above a a compact cardinal, then there is a universal locally finite group of cardinality $\mu$ but if $\mu = \mu^{\aleph_0}$ then there is no one.

Concerning singulars

*Question* 0.5. Does $\theta = \mathrm{cf}(\theta)$ and $\theta^{+2} < \mathrm{cf}(\lambda) < \lambda < 2^\theta$ implies $\lambda < \mathrm{univ}(\lambda, T)$?

*Question* 0.6. 0) Characterize the failure of the criterion of [She93b], Džamonja-Shelah [DS04](for consistency).

1) Does $\mathrm{SOP}_3$ (or something weaker) suffice for no universal in $\lambda$ when $\mu = \mu^{<\mu} \ll \lambda < 2^\mu$?

2) Which theories $T$ fail to have a universal in $\lambda$ when $\lambda = \mu^{++} = 2^\mu < 2^{\mu^+}$?

3) We may consider weaker properties of $T$ for no universal in $\lambda, \mu = \mu^{<\mu} \ll \lambda < 2^\mu$.

4) Sort out the variants of the olive property (defined below, e.g. in Definition 0.8).

**Discussion 0.7.** Even for linear orders, the case

- $(*)^1_\lambda$ successor case: $\lambda = \mu^+, \lambda < 2^\mu$ and $2^{<\mu} \le \lambda$ (e.g. for transparency $\mu = \mu^{<\mu}$) is not resolved as we do not necessarily have $\bar{C} = \langle C_\delta : \delta \in S^\lambda_\mu \rangle$ guessing clubs, recall that by [KS92] if $2^\theta > \lambda = \mathrm{cf}(\lambda) > \theta = \mathrm{cf}(\theta), 2^\theta \le \mathrm{univ}(\lambda, T)$, so if $\lambda$ is a successor cardinal $> \lambda_n$, the only open case is $(*)^2_\lambda$. Similarly, if $\lambda$ is a limit cardinal the only open case is
- $(*)^2_\lambda$ limit case: $\lambda$ is singular.

In this case, there are strong pcf restrictions (see [KS92]), so advancement there may eliminate the case.

By some earlier results (see [She93b]) if $\mu = 2^\kappa$, (so $\mu$ is not a strong limit cardinal), and there is no universal in $\lambda$ then there was a sequence $\langle \Lambda_\delta : \delta \in S^\lambda_\mu \rangle, \Lambda_\delta \subseteq {}^{(C_\delta)}\mu$ of cardinality $\lambda$ such that for every sequence $\langle \eta_\delta \subseteq {}^{(C_\delta)}\mu : \delta \in S^\lambda_\mu \rangle$ there is club $E$ of $\lambda$ such that for every $\delta \in E \cap S^\lambda_\mu$ there is a $\nu \in \Lambda_\delta$ such that the functions $\eta_\delta, \nu$ agree on $E \cap \mathrm{nacc}(C_\delta)$.

Using a more complicated $T$ we can replace ${}^{C_\delta}\mu$ by ${}^{(C_\delta \times D_\delta)}\mu$ so the agreement above is on the product $(E \cap \mathrm{nacc}(C_\delta)) \times (E \cap \mathrm{nacc}(C_\delta))$ but of unclear value.

On subsequent works and more on consistency, see [S+a] and [S+b].

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## § 0(B). **What is accomplished.**

What do we achieve? We introduce the "olive property" which is a sufficient condition for a class to have a universal member in $\lambda$ only if $\lambda$ is "close to satisfying G.C.H.", similar to the linear order case. This condition is weaker than $\mathrm{SOP}_4$, hence giving a positive answer to Question 0.1(1). But the condition implies $\mathrm{SOP}_3$ so it does not answer Question 0.1(3), also it is totally unclear whether it is best in any sense and whether its negation has interesting consequences.

However, it answers Question 0.4(1) to a large extent because the class of groups have the olive property and we can also deal with locally finite groups answering some cases of Question 0.4(2); see §2. Also, we try to formalize conditions sufficient for non-existence, see Definition 1.6. As the reader may find the definition of the (variants of the) olive property opaque, we define a simple case used for the class of groups, and the reader may then look first at the class of groups in §2.

**Definition 0.8.** A (first order) universal theory $T$ has the *olive property* when there are $(\varphi_0, \varphi_1, \psi)$ and a model $\mathfrak{C}$ of $T$ such that:

(a) for some $m, \varphi_0 = \varphi_0(\bar{x}_{[m]}, \bar{y}_{[m]}), \varphi_1 = \varphi_1(\bar{x}_{[m]}, \bar{y}_{[m]}), \psi = \psi(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{z}_{[m]})$ are quantifier free formulas (and $\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{z}_{[m]}$ are $m$-tuples of variables, see Notation 0.10 below)
(b) for every $k$ and $\bar{f} = \langle f_\alpha : \alpha < k\rangle$ where $f_\alpha$ is a function from $\alpha$ to $\{0,1\}$ we can find a model $M$ of $T$ and $\bar{a}_\alpha \in {}^m M$ for $\alpha < k$ such that:
  ($\alpha$) $\varphi_\iota[\bar{a}_\alpha, \bar{a}_\beta]$ for $\alpha < \beta < k$ when $\iota = f_\beta(\alpha)$,
  ($\beta$) $\psi[\bar{a}_\alpha, \bar{a}_\beta, \bar{a}_\gamma]$ when $\alpha < \beta < \lambda$ and $f_\gamma{\restriction}[\alpha, \beta]$ is constantly 0.
(c) there are no $\bar{a}_\ell \in {}^m M$ for $\ell = 0, 1, 2, 3$ such that the following conditions are[1] satisfied in $M$:
  ($\alpha$) $\varphi_0[\bar{a}_0, \bar{a}_\ell]$ for $\ell = 1, 2, 3, \varphi_1[\bar{a}_1, \bar{a}_\ell]$ for $\ell = 1, 2$ and $\varphi_0[\bar{a}_2, \bar{a}_3]$,
  ($\beta$) $\psi[\bar{a}_0, \bar{a}_2, \bar{a}_3]$.

*Concluding Remarks* 0.9. Concerning some things not addressed here we note the following.

1) Concerning the proof here of "there is no universal" we can carry it via defining invariants parallel to Kojman-Shelah [KS92] such that (for transparency $\lambda$ is regular uncountable, see Definition 0.11(5), (7)):

$(*)$ (a) if $M \in \mathrm{Mod}_{T,\lambda}$ then $\mathrm{INV}_\lambda(M)$ is a set of cardinality $\le \lambda$ or just $\le \chi < 2^\lambda$,
  (b) if $M_1, M_2 \in \mathrm{Mod}_{T,\lambda}$ and $M_1$ is elementarily embeddable into $M_2$ then $\mathrm{INV}_\lambda(M_1) \subseteq \mathrm{INV}_\lambda(M_2)$,
  (c) there is a set of $2^\lambda$ objects $\mathbf{x}$ such that $(\exists M \in \mathrm{Mod}_{T,\lambda})(\mathbf{x} \in \mathrm{INV}_\lambda(M))$.

2) We can use more complicated versions of the olive property. In the proof we use one $\delta$ and then one $\alpha \in \mathrm{nacc}(C_\delta) \cap E$ (or less), but we may use several ordinals $\alpha$ resulting in more complicated versions. This will become more pressing if we have a complimentary property, guaranteeing "no universal" or some variant.

## § 0(C). Preliminaries.

*Notation* 0.10. 1) Let $\bar{x}_{[I]} = \langle x_t : t \in I\rangle$ and similarly $\bar{y}_{[I]}, \bar{x}_{[I],\alpha}$, etc. where $\bar{x}_{[I],\ell} = \langle x_{t,\ell} : t \in I\rangle$.

2) For a first-order complete $T, \mathfrak{C}_T$ is the "monster model of $T$" omitting $T$ if clear from the context.

**Definition 0.11.** 1) For a set $A, |A|$ is its cardinality but for a structure $M$ its cardinality is $\|M\|$ while its universe is $|M|$; this applies e.g. to groups.

2) We use $G, H$ for groups, $M, N$ for general models.

3) Let $\mathfrak{k}$ denote a pair $(K_\mathfrak{k}, \le_\mathfrak{k})$, or we may say a class (of models) $\mathfrak{k}$, where:

(a) $K_\mathfrak{k}$ is a class of $\tau_\mathfrak{k}$-structures where $\tau_\mathfrak{k}$ is a vocabulary,
(b) $\le_\mathfrak{k}$ is a partial order on $K_\mathfrak{k}$ such that $M \le_\mathfrak{k} N \Rightarrow M \subseteq N$,

[1]in the class of groups, in clause $(\alpha), \varphi_0[\bar{a}_0, \bar{a}_1], \varphi_1[a_1, a_2], \varphi_1[a_1, \bar{a}_3]$ suffice.

(c) both $K_{\mathfrak{k}}$ and $\le_{\mathfrak{k}}$ are closed under isomorphisms.

4) (a) We may write only $K_{\mathfrak{k}}$ when $\le_{\mathfrak{k}}$ is being a submodel,
(b) We say $f\colon M \to N$ is a $\le_{\mathfrak{k}}$-embedding when $f$ is an isomorphism from $M$ onto some $M_1 \le_{\mathfrak{k}} N$.

5) If $T$ is a first-order theory then $\mathrm{Mod}_T$ is the pair $(\bmod{}_T, \le_T)$ where $\mathrm{mod}_T$ is the class of models of $T$ and $\le_T$ is: $\prec$ if $T$ is complete, $\subseteq$ if $T$ is not complete.

6) We may write $T$ instead of $\mathrm{Mod}_T$, e.g. in Definition 0.12 below.

7) For a class $K$ of structures $K_\lambda = \{M \in K\colon \|M\| = \lambda\}$.

**Definition 0.12.** 1) For a class $\mathfrak{k}$ and a cardinal $\lambda$, a set $\{M_i\colon i < i^*\}$ of models from $K_{\mathfrak{k}}$, is *jointly* $(\lambda, \mathfrak{k})$-*universal* when for every $N \in K_{\mathfrak{k}}$ of size $\lambda$, there is an $i < i^*$ and an $\le_{\mathfrak{k}}$-embedding of $N$ into $M_i$.

2) For $\mathfrak{k}$ and $\lambda$ as above, let (if $\mu = \lambda$ we may omit $\mu$)

$$\begin{aligned}\mathrm{univ}(\lambda, \mu, \mathfrak{k}) := \ & \min\{|\mathscr{M}|\colon \mathscr{M} \text{ is a family of members of } K_{\mathfrak{k}} \text{ each} \\ & \text{of cardinality } \le \mu \text{ which is jointly } \mathfrak{k}\text{-universal for } \lambda\}\end{aligned}$$

Let $\mathrm{Univ}(\mathfrak{k}) := \{\lambda\colon \mathrm{univ}(\lambda, \mathfrak{k}) = 1\}$.

3) For a pair[2] $\bar{\mathfrak{k}} = (\mathfrak{k}_1, \mathfrak{k}_2)$ of classes with $\mathfrak{k}_\iota = (K_{\mathfrak{k}_\iota}, \le_{\mathfrak{k}_\iota})$ as in Definition 0.11(3) for $\iota = 1, 2$ such that $\tau(\mathfrak{k}_1) = \tau(\mathfrak{k}_2)$ and $K_{\mathfrak{k}_1} \subseteq K_{\mathfrak{k}_2}$, let $\mathrm{univ}(\lambda, \mu, \bar{\mathfrak{k}})$ be the minimal $|\mathscr{M}|$ such that $\mathscr{M}$ is a family of members of $K_{\mathfrak{k}_2}$ each of cardinality $\le \mu$ such that every $M \in K_{\mathfrak{k}_1}$ of cardinality $\lambda$ can be $\le_{\mathfrak{k}_2}$-embedded into some member of $\mathscr{M}$.

Dealing with a.e.c.'s (see [She09]) we have the following:

**Definition 0.13.** 1) We say that a formula $\varphi = \varphi(\bar{x}_{[I]})$, in any logic, is $\mathfrak{k}$-upward preserved when $\tau_\varphi \subseteq \tau_{\mathfrak{k}}$ and if $M \le_{\mathfrak{k}} N$ and $\bar{a} \in {}^I M$ then $M \models \varphi[\bar{a}]$ implies $N \models \varphi[\bar{a}]$.

2) For $\bar{\mathfrak{k}}$ as in Definition 0.12(3) we say that a pair $\bar{\varphi}(\bar{x}_{[I]}) = (\varphi_1(\bar{x}_{[I]}), \varphi_2(\bar{x}_{[I]}))$ is $\bar{\mathfrak{k}}$-upward preserving when $\tau_{\varphi_1} \cup \tau_{\varphi_2} \subseteq \tau_{\mathfrak{k}_\iota}$ and if $M_\iota \in K_{\mathfrak{k}_\iota}$ for $\iota = 1, 2, \bar{a} \in {}^I(M_1)$ and $M_1 \le_{\mathfrak{k}_2} M_2$ then $M_1 \models \varphi_1[\bar{a}]$ implies $M_2 \models \varphi_1[\bar{a}]$.

3) In part (2), if $\varphi_0 = \varphi_1$ then we may write $\varphi$ instead of $\bar{\varphi}$. Saying[3] that a sequence $\bar{\psi}$ is $\mathfrak{k}$-upward preserving means that every formula appearing in $\bar{\psi}$ is $\mathfrak{k}$-upward preserving.

**Definition 0.14.** 1) For an ideal $J$ on a set $A$ and a set $B$ let $\mathbf{U}_J(B) = \min\{|\mathscr{P}|\colon \mathscr{P}$ is a family of subsets of $B$, each of cardinality $\le |A|$ such that for every function $f$ from $A$ to $B$ for some $u \in \mathscr{P}$ we have $\{a \in A\colon f(a) \in u\} \in J^+\}$.

2) For an ideal $J$ on a set $A$, a cardinal $\theta$ and a set $B$ let $\mathbf{U}^\theta_J(B) = \min\{|\mathscr{P}|\colon \mathscr{P} \subseteq [B]^{\le |A|}$ and if $f \in {}^A({}^\theta B)$ then for some $u \in \mathscr{P}$ we have $\{a \in A\colon \mathrm{Rang}(f(a)) \subseteq u\} \in J^+\}$. So $\mathbf{U}^\theta_J(B) \le \mathbf{U}_J(|N|^\theta)$.

3) Clearly only $|B|$ matters so we normally write $\mathbf{U}_J(\lambda)$, (see on it [She00a]).

[2]This will be used for $\mathfrak{k}_1$ being the class of locally finite groups and $\mathfrak{k}_2$ being the class of groups.

[3]Pedantically, a pair is a sequence of length 2 so the Definition 0.13(2), 0.13(3) are incompatible, but the intention should be clear from the context.

## § 1. The Olive property

**Definition 1.1.** 1) (Convention)

(a) Let $T$ be a first-order theory and $\mathfrak{C} := \mathfrak{C}_T$ a monster for $T$,

(b) ($\alpha$) $\Delta \subseteq \mathbb{L}(\tau_T)$ a set of formulas,

($\beta$) omitting $\Delta$ means $\Delta = \mathbb{L}(\tau_T)$ if $T$ is complete, $\Delta =$ set of quantifiers free formula otherwise, and we may write qf instead of $\Delta$.

(c) ($\alpha$) $m$ and $n \geq k_\iota \geq 2$ for $\iota = 0, 1, n \geq k_0 + k_1 \geq 3, \eta \in {}^n 2$ be such that $\eta(0) = 0$ and $\eta^{-1}\{0\}$ is not an initial segment and $\eta^{-1}\{\iota\}$ has $\geq k_\iota$ members for $\iota = 0, 1$,

($\beta$) If $\eta(\ell) = \ell \mod 2$ for $\ell < k$ we may write $n$ instead of $\eta$.

(d) ($\alpha$) If $\bar{k} = (k_0, k_1), k_1 \leq k_0 + 1 \leq k_1 + 1$ we may write $k_0 + k_1$ instead of $\bar{k}$ and let $k(\iota) = k_\iota$ for $\iota = 0, 1$,

($\beta$) omitting $m$ means "for some $m$",

($\gamma$) omitting $n, \eta, \bar{k}$ means $n = 3, \eta = \langle 0, 1, 0 \rangle, \bar{k} = (2, 1))$ for some $m$.

(e) ($\alpha$) below we may write $\psi_\iota = \psi_{\iota, k_\iota}$ and $\varphi_\iota = \psi_{\iota, 1}$ for $\iota = 0, 1$,

($\beta$) if $\varphi_0 = \varphi_1 = \varphi$ we may write $\varphi$,

($\gamma$) we may omit $\psi_{3,k}$ when it is a logically true formula.

2) We say $T$ has the $(\Delta, \eta, \bar{k}, m)$-*olive property* <u>when</u> there is a pair $(\bar{\psi}_0, \bar{\psi}_1)$ of sequences of formulas from $\Delta$ witnessing it, see (3).

3) We say $(\bar{\psi}_0, \bar{\psi}_1)$ *witnesses the* $(\Delta, \eta, \bar{k}, m)$-*olive property* (for $T$, with the convention above) <u>when</u> (it is the case for every $\lambda$, but in this definition, by compactness, $\lambda = \aleph_0$ is enough):

(a) $\bar{\psi}_\iota = \langle \psi_{\iota,k}(\bar{x}_{[m],0}, \dots, \bar{x}_{[m],k}) \colon k = 1, \dots, k_\iota \rangle$ for $\iota = 0, 1$ with $\psi_{\iota,k} \in \Delta$,

(b)$_\lambda$ for every $\bar{f} = \langle f_\alpha \colon \alpha < \lambda \rangle$ where $f_\alpha$ is a function from $\alpha$ to $\{0, 1\}$, we can find a model $M$ of $T$ and $\bar{a}_\alpha \in {}^m M$ for $\alpha < \lambda$ such[4] that:

($\alpha$) $\varphi_\iota[\bar{a}_\alpha, \bar{a}_\beta]$ for $\alpha < \beta < \lambda$ when $\iota = f_\beta(\alpha)$, recalling Definition 1.1(1)$(e)(\alpha)$,

($\beta$) $\psi_{\iota,k}(\bar{a}_{\alpha_0}, \dots, \bar{a}_{\alpha_{k-1}}, \bar{a}_\beta)$ when $k \in \{2, \dots, k_\iota\}$ and $\alpha_0 < \dots < \alpha_{k-1} < \beta < \lambda$ and $f_\beta \restriction [\alpha_0, \alpha_{k-1}]$ is constantly $\iota$, so when $k = 1$, it holds trivially,

(c) there are no $\bar{a}_\ell \in {}^m\mathfrak{C}$ for $\ell < n + 1$ such that:

($\alpha$) $\varphi_\iota[\bar{a}_i, \bar{a}_j]$ for $i < j < n + 1$ and $\eta(i) = \iota$,

($\beta$) if $\iota \in \{0, 1\}$, $k \in \{2, \dots, k_\iota\}$, $\ell_0 < \dots < \ell_{k-1}$ are from $\{\ell < n \colon \eta(\ell) = \iota\}$ and $\ell_{k-1} < \ell \leq n$, then $\psi_{\iota,k}[\bar{a}_{\ell_0}, \dots, \bar{a}_{\ell_{k-1}}, \bar{a}_\ell]$.

*Remark* 1.2. This fits the classification of properties of such $T$ in [She00b, 5.15-5.23].

**Definition 1.3.** 1) Let $K$ be a universal class of $\tau$-models, see Definition 0.11(4). We say $K$ has the $\lambda - (\eta, \bar{k}, m)$-*olive property* <u>when</u> some quantifier-free $(\bar{\psi}_0, \bar{\psi}_1)$ witnessing it, that is, $(a) + (b)_\lambda + (c)$ holds (replacing $\mathfrak{C}_T$ by "in some $M \in K_\lambda$").

2) We say that a class (e.g. an a.e.c.) $\mathfrak{k} = (K_\mathfrak{k}, \leq_\mathfrak{k})$ has the $\lambda - (\eta, \bar{k}, m)$-*property* <u>when</u>there are $\bar{\psi}_0, \bar{\psi}_1$ which are $\mathfrak{k}$-upward preserved formulas in any logic (see Definition 0.13) and $(a) + (b)_\lambda + (c)$ of Definition 1.1 holds, replacing $M$ by "some $\mathfrak{C} \in K_\mathfrak{k}$ of cardinality $\lambda$".

[4]Actually clause $(\alpha)$ is a specific case of clause $(\beta)$ provided that in clause $(\beta)$ we allow $k = 1$. Similarly for clauses $(c)(\alpha), (\beta)$.

*Remark* 1.4. 1) Note that for $T$ first order complete, $\mathfrak{k} = \mathrm{Mod}_T = (\mathrm{mod}_T, \prec)$, Definition 1.3(2) gives Definition 1.1 and for $T$ first order universal not complete, $\mathfrak{k} = \mathrm{Mod}_T = (\mathrm{mod}_T, \subseteq)$, Definition 1.3(2) gives Definition 1.1. Similarly for Definition 1.3(1).

2) Of course, for $T$ first order, the $\lambda$ does not matter.

**Claim 1.5.** *Assume $n \geq k_0 + k_1 \geq 3, \eta \in {}^n 2$ and $|\eta^{-1}\{\iota\}| \geq k_\iota \geq 1$ for $\iota = 0, 1$. Then there is a complete first-order countable $T$ having the $(\eta, \bar{k}, 1)$-olive property but $T$ is $\mathrm{NSOP}_4$, is $\mathrm{SOP}_3$ and is categorical in $\aleph_0$.*

*Proof.* Let $\tau = \{P, Q_0, Q_1\}$ where $P$ is a binary predicate and $Q_\iota$ is a $(k_\iota + 1)$-place predicate. Let $T^0_{\eta,\bar{k}}$ be the following universal theory in $\mathbb{L}(\tau)$:

$(*)_1$ a $\tau$-model $M$ is a model of $T^0_{\eta,\bar{k}}$ iff we cannot embed $N^*_{\eta,\bar{k}}$ into $M$ where
- $\oplus$ $N^*_{\eta,\bar{k}}$ is the $\tau$-model with universe $\{a_0, \ldots, a_n\}$ as in $(c)(\alpha), (\beta)$ from Definition 1.1(3) for $\varphi(x_0, x_1) = P(x_0, x_1)$, and $\psi_\iota(x_0, \ldots, x_{k(\iota)}) = Q_\iota(x_0, \ldots, x_{k(\iota)})$, recalling Definition 1.1(1)(e)($\gamma$) and $\ell < k \leq n \Rightarrow a_\ell \neq a_k$.

Now,

$(*)_2$ $T^0_{\eta,\bar{k}}$ has the JEP and the amalgamation property by disjoint union.

[Why? Assume that $M_0 \subseteq M_1, M_0 \subseteq M_2$ are models of $T_0$ (but abusing notation we allow $M_0$ to be empty) and $|M_1| \cap |M_2| = |M_0|$, we define $M = M_1 \cup M_2$, that is,

(a) $|M| = |M_1| \cup |M_2|$,
(b) $P^M = P^{M_1} \cup P^{M_2}$,
(c) $Q^M_\iota = Q^{M_1}_\iota \cup Q^{M_2}_\iota$ for $\iota = 0, 1$.

So $M$ is a $\tau$-model, moreover it is a model of $T$ as in $\oplus$ any pair of distinct elements of $N^*_{\eta,\bar{b}}$ belongs to a relation, i.e. $\ell < k \leq n \Rightarrow (a_\ell, a_k) \in P^M$.]

$(*)_3$ $T_{\eta,\bar{k}}$, the model completion of $T^0_{\eta,\bar{k}}$, is well defined and has elimination of quantifiers.

[Why? As $\tau$ is finite with no function symbols and $(*)_2$.]

$(*)_4$ $T_{\eta,\bar{k}}$ is $\mathrm{NSOP}_4$ (see [She96, 2.5]).

[Why? Because

$(*)_{4.1}$ if (A) then (B), where:
- (A)
  - (a) $A_0, A_1, A_2, A_3$ are disjoint sets,
  - (b) $M_\ell$ is a model of $T^0_{\eta,\bar{k}}$ with universe $A_\ell$ for $\ell = 0, 1, 2, 3$,
  - (c) if $\{\ell(1), \ell(2)\} \in \mathscr{W} := \{\{0,1\}, \{1,2\}, \{2,3\}, \{3,4\}\}$ then $M_{\{\ell(1),\ell(2)\}}$ is a model of $T^0_{k,n}$ with universe $A_{\ell(1)} \cup A_{\ell(2)}$ extending $M_{\ell(1)}$ and $M_{\ell(2)}$.
- (B) $M = \bigcup\{M_{\{\ell(1),\ell(2)\}} : \{\ell(1), \ell(2)\} \in \mathscr{W}\}$ where the union is defined as in the proof of $(*)_2$, is a model of $T^0_{k,n}$ extending all of them.]

[Why? Clearly $M$ is a $\tau$-model and if $f$ embeds $N^*_{\eta,\bar{k}}$ into $M$, as in $(*)_2$ we have $\mathrm{Rang}(f) \subseteq M_{\ell(1),\ell(2)}$ for some $\{\ell(1), \ell(2)\} \in \mathscr{W}$, a contradiction.]

$(*)_5$ $T_{\eta,\bar{k}}$ (and $\mathrm{Mod}_{T^0_{\eta,\bar{k}}}$) has the $(\eta,\bar{k})$-olive property as witnessed by $\varphi(x_0,x_1) = P(x_0,x_1), \psi_\iota(x_0,\dots,x_{k(\iota)}) = Q_\iota(x_0,\dots,x_{k(\iota)})$.

[Why? In Definition 1.1(3), clause (a) holds trivially, and clause (c) is obvious from the choice of $T^0_{\eta,\bar{k}}$. For clause $(\mathrm{b})_\lambda$ we are given $\langle f_\alpha \colon \alpha < \lambda\rangle$ where $f_\alpha$ is a function from $\alpha$ to $\{0,1\}$ and we have to find $M$ as there. We define a $\tau$-model $M$ with:

- universe $\{a^*_\alpha \colon \alpha < \lambda\}$ such that $\alpha < \beta \Rightarrow a^*_\alpha \neq a^*_\beta$,
- $P^M = \{(a^*_\alpha, a^*_\beta) \colon \alpha < \beta < \lambda\}$,
- $Q^M_\iota = \{(a^*_{\alpha_0},\dots,a^*_{\alpha_{k(\iota)-1}}, a_\beta) \colon \alpha_0 < \dots < \alpha_{k(\iota)-1} < \beta$ and $f_\beta{\restriction}[\alpha_0,\alpha_{k(\iota)-1}]$ is constantly $\iota\}$.

It suffices to prove that $M$ is a model of $T^0_{\eta,\bar{k}}$. So toward a contradiction assume $h$ embeds $N^*_{\eta,\bar{k}}$ into $M$ and consider $h(a^*_\ell) = a_{g(\ell)}$ where $g \colon \{0,\dots,n\} \to \lambda$; recalling $\ell_1 < \ell_2 \leq n \Rightarrow (a^*_{\ell_1} \neq a^*_{\ell_2})$ and $h$ is an embedding, necessarily $g$ is a one-to-one function. For $\ell < n$, recall that $N^*_{\eta,\bar{k}} \models$ "$P(a^*_\ell, a^*_{\ell+1})$" but $h$ is an embedding so $M \models$ "$P[a^*_{g(\ell)}, a^*_{g(\ell+1)}]$", but if $g(\ell) \geq g(\ell+1)$ this fails by the choice of $P^M$, hence $g(\ell) < g(\ell+1)$. Now let $i_* := \min\{i \colon \eta(i) = 1\}$. Let $i_0 < \dots < i_{k(0)-1}$ be from $\eta^{-1}\{0\}$ such that $i_0 = 0$ (recall Definition 1.1(1)(c)($\alpha$) and $i_{k(\iota)-1}$ is maximal in $\eta^{-1}\{1\}$ hence $i_* \in [i_0, i_{k(0)-1})$. Now $N^*_{\eta,\bar{k}} \models Q_0[a^*_{i_0},\dots,a^*_{i_{k(0)-1}}, a^*_n]$ hence $M \models Q_0[a^*_{g(i_0)},\dots,a^*_{g(i_{k(0)-1})}, a^*_{g(n)}]$ and this implies that $f_{g(n)}{\restriction}[g(i_0), g(i_{k(0)-1}))]$ is constantly 0 hence in particular $f_{g(n)}(g(i_*)) = 0$. Similarly let $j_0 < \dots < j_{k(1)-1}$ be from $\eta^{-1}\{1\}$ such that $j_0 = i_*$; now $N^*_{\eta,\bar{k}} \models Q_1[a^*_{j_0},\dots,a^*_{j_{k(1)-1}}, a^*_n]$ hence $M \models Q_1[a_{g(0)},\dots,a_{g(j_{k(1)-1})}, a_n]$ hence, $f_{g(n)}{\restriction}[g(j_0), g(j_{k(1)-1})]$ is constantly 1, hence in particular $f_{g(n)}(g(i_*)) = 1$, a contradiction, so $(*)_5$ holds indeed.]

$(*)_6$ $T_{\eta,\bar{k}}$ has the $\mathrm{SOP}_3$.

Why? Again let $i_* = \min\{i \colon \eta(i) = 1\}, u_0 = \{0,\dots,i_*-1\}, u_1 = \{i_*\}$, and $u_2 = \{i_*+1,\dots,n\}$.

Note that,

$(*)_{6.1}$ $(u_0,u_1,u_2)$ is a partition of $\{0,\dots,n\}$.

$(*)_{6.2}$ if $\iota < 2$ and $(a_{\ell_0},\dots,a_{\ell_{k(\iota)}}) \in Q^{N^*_{\eta,\bar{k}}}_\iota$ then $\{\ell_0,\dots,\ell_{k(\iota)}\} \cap u_j = \emptyset$ for some $j \leq 2$.

[Why? Otherwise as $\ell_0 < \ell_1 < \dots$ necessarily $\ell_0 < i_*, \ell_{k(\iota)} > \iota_*$ and $i_* = \ell_k$ for some $k \in (0,k(\iota))$. But then $\eta(\ell_0) = 0 \neq 1 = \eta(k)$, in contradiction to Definition 1.1(3)(c).]

The rest should be clear by considering the proof of the model completion of the theory of triangle-free graphs having $\mathrm{SOP}_3$, see [She96, §2]. $\square_{1.5}$

As in earlier cases, we apply a kind of guessing of clubs (almost suitable also for them i.e. for the proof with strict order and $\mathrm{SOP}_4$). An unexpected gain is that here we use a weaker version: there is no requirement

$$\alpha < \lambda \Rightarrow \lambda > |\{C_\delta \cap \alpha \colon \delta \in S \text{ satisfies } \alpha \in \mathrm{nacc}(C_\delta)\}|,$$

but it is unclear how this helps. Also here the use of the pair $(\bar{\mathscr{A}}, \bar{\mathbf{g}})$ may be helpful.

**Definition 1.6.** 1) For $\lambda$ regular uncountable and $\chi_2 > \chi_1 \geq \lambda$ let $\mathrm{Qr}_1(\chi_2,\chi_1,\lambda)$ mean that there are $S, \bar{C}, I, \bar{\mathscr{A}}, \bar{\mathbf{g}}$ witnessing it, which means that:

$\boxplus$ (a) $S \subseteq \lambda$ and $I$ is an ideal on $S$,
(b) $\bar{C} = \langle C_\delta \colon \delta \in S\rangle$,
(c) $C_\delta \subseteq \delta$, note that possibly $\sup(C_\delta) < \delta$,
(d) ($\alpha$) $\bar{\mathbf{g}} = \langle \bar{g}_j \colon j < \chi_2\rangle$,
($\beta$) $\bar{g}_j = \langle g_{j,\delta} \colon \delta \in S\rangle$,
($\gamma$) $g_{j,\delta} \colon C_\delta \to \{0,1\}$.
(e) ($\alpha$) $\bar{\mathscr{A}} = \langle \bar{\mathscr{A}}_j \colon j < \chi_2\rangle$,
($\beta$) $\bar{\mathscr{A}}_j = \langle \mathscr{A}_{j,\delta} \colon \delta \in S\rangle$,
($\gamma$) $\mathscr{A}_{j,\delta} \subseteq \mathscr{P}(\mathrm{nacc}(C_\delta)$.
(f) $\mathbf{U}_I(\chi_1) < \chi_2$; see Definition 0.14, if $\chi_1 = \lambda$ then we stipulate $\mathbf{U}_I(\chi_1) = \chi_1$ hence this means $\chi_1 < \chi_2$,
(g) if $j_1 \neq j_2$ are $< \chi_2, \delta \in S, A_1 \in \mathscr{A}_{j_1,\delta}$ and $A_2 \in \mathscr{A}_{j_2,\delta}$, then there is $\gamma \in A_1 \cap A_2$ such that $g_{j_1,\delta}(\gamma) \neq g_{j_2,\delta}(\gamma)$,
(h) if $j < \chi_2$ and $E$ is a club of $\lambda$, then for some $Y \in I^+$ hence $Y \subseteq S$ for every $\delta \in Y$ we have $\mathrm{nacc}(C_\delta) \cap E \in \mathscr{A}_{j,\delta}$.

2) For $\ell = 1,2,3$ let $\mathrm{Qr}_\ell(\chi_2,\chi_1,\lambda)$ be defined by:

- if $\ell = 1$ as above,
- if $\ell = 2$ as above but there is a sequence $\langle J_\delta \colon \delta \in S\rangle$ of ideals on $\mathrm{nacc}(C_\delta)$ such that $\mathscr{A}_{j,\delta} = \{\mathrm{nacc}(C_\delta)\backslash X \colon X \in J_\delta\}$,
- if $\ell = 3$ we use clauses (a)-(g) from part (1) and,

(h)$^-$ $(h)^-$ if $E_j$ is a club of $\lambda$ for $j < \chi_2$ and $\langle \xi_j \colon j < \chi_2\rangle$ is a sequence of ordinals with $\sup(\{\xi_j \colon j < \chi_2\}) < \chi_2$ then we can find $j_1 < j_2 < \chi_2, \delta \in S$ and $\gamma \in \mathrm{nacc}(C_\delta)$ such that $\xi_{j_1} = \xi_{j_2}, \gamma \in E_{j_1} \cap E_{j_2}$ and $g_{j_1,\delta}(\gamma) \neq g_{j_2,\delta}(\gamma)$.

3) $\mathrm{Qr}_{\ell,\iota}(\chi_2,\chi_1,\lambda)$ is defined as in $\mathrm{Qr}_\ell(\chi_2,\chi_1,\lambda)$ but $g_{j,\delta} \colon \mathrm{nacc}(C_\delta) \to \iota$, etc.

*Remark* 1.7. Can we weaken the conclusion of clause (h) of 1.6(1), etc. to:

- $\{\alpha \in \mathrm{nacc}(C_\delta) \colon \sup(\alpha \cap E) > \max(C_\delta \cap \alpha)\} \in \mathscr{A}_{j,\delta}$.

That is, this suffices in Theorem 1.9 but there is no clear gain so we have not looked into it.

**Fact 1.8.** 1) $\mathrm{Qr}_2(\chi_2,\chi_1,\lambda) \Rightarrow \mathrm{Qr}_1(\chi_2,\chi_1,\lambda) \Rightarrow \mathrm{Qr}_3(\chi_2,\chi_1,\lambda)$.
2) We have $\mathrm{Qr}_1(\chi_2,\chi_1,\lambda)$ and even $\mathrm{Qr}_2(\chi_2,\chi_1,\lambda)$ when:

(a) $\kappa^+ \leq \lambda \leq \chi_1 < \chi_2 < 2^\kappa$,
(b) $\kappa = \mathrm{cf}(\kappa), \lambda = \mathrm{cf}(\lambda)$,
(c) $\mathbf{U}_I(\chi_1) < \chi_2$ when $\lambda < \chi_1$ and $I$ an ideal on $S$ so $S \notin I$,
(d) $S \subseteq S^\lambda_\kappa$ is stationary, $\bar{C} = \langle C_\delta \colon \delta \in S\rangle$ guess clubs, $C_\delta \subseteq \delta, \mathrm{otp}(C_\delta) = \kappa$,
(e) $I = \{A \subseteq S$: for some club $E$ of $\lambda$ for no $\delta \in S$ do we have $\mathrm{nacc}(C_\delta) \cap E \in J^{\mathrm{bd}}_{C_\delta}\}$.

3) If $\kappa^+ < \lambda$ and clauses (a), (b) of part (2) hold and $S \subseteq S^\lambda_\kappa$ is stationary then there is $\bar{C}$ as required in clause (d).

*Proof.* 1) Easy.
2) The proof is straightforward.
3) Clause (d) follows by [She93a, §2]. $\square_{1.8}$

**Theorem 1.9.** *1) If $T$ is complete, with the 3-olive property and $\lambda > \kappa^+$ and $\lambda, \kappa$ are regular, $2^\kappa > \lambda \geq \kappa^{++} + |T|$, <u>then</u> $T$ has no universal model in $\lambda$ (for $\prec$).*

*2) If $T$ is complete, with the $(\eta, \bar{k}, m)$-olive property and $\lambda = \mathrm{cf}(\lambda) \geq |T|$ and $\mathrm{Qr}_1(\chi_2, \chi_1, \lambda)$, <u>then</u> $\mathrm{univ}(\chi_1, \lambda, T) \geq \chi_2$.*

*3) Similarly for class $\mathfrak{k}$ of models with the $\lambda-(\mathrm{qf}, \eta, \bar{k})$-olive property, see Definition 1.3(2), so e.g. for universal $K$ with the JEP and the $\lambda-(\mathrm{qf}, \eta, \bar{k})$-olive property.*

*4) Like part (3) for a pair $\bar{\mathfrak{k}}$.*

*5) We can weaken $\mathrm{Qr}_1(\chi_2, \chi_1, \lambda)$ to $\mathrm{Qr}_{1,\theta}(\chi_2, \chi_1, \lambda)$ <u>when</u> $\theta = 2^\partial, \chi_1 = \chi_1^\partial, \partial < \lambda$.*

*Remark* 1.10. 1) We can use $\mathrm{Qr}_3$ instead of $\mathrm{Qr}_1$ by the same proof but the gain is not clear.

2) Assume $T$ is as in 1.9(1), $\lambda \in \mathrm{Univ}(T)$. If e.g. $\lambda = \mu^+, \mu = \mu^{<\mu} = 2^\partial, \chi_1 = \lambda = \chi_2$ (so $T$ have a universal in $\lambda$), failure of $\mathrm{Qr}_1(\lambda, \lambda, \lambda)$ implies: there is $\mathscr{F} \subseteq {}^\mu\mu$ such that $(\forall \eta \in {}^\mu\mu)(\exists \nu \in \mathscr{F})(\exists^\mu i < \mu)(\eta(i) = \nu(i))$.

*Proof.* 1) It follows from (2) by Fact 1.8(2).

2) Let $(\bar{\psi}_0, \bar{\psi}_1)$, i.e. $\bar{\psi}_\iota = \langle \psi_{\iota,k}(\bar{x}_0, \dots, \bar{x}_k) \colon k = 1, \dots, k_\iota \rangle$ for $\iota = 0, 1$ witness the $(\Delta, \eta, \bar{k}, m)$-olive property. For simplicity we can, without loss of generality assume that $m = 1$ and the formulas $\psi_\iota$ are quantifier-free and $T$ has only predicates and its vocabulary is finite. To make this proof also be a proof of Theorem 1.9(3) let $\leq_{\mathfrak{k}}$ be $\prec$ on $\mathrm{mod}_T$. Let $S, \bar{C}, \bar{\mathscr{A}}, \bar{\mathbf{g}}$ witness $\mathrm{Qr}_1(\chi_2, \chi_1, \lambda)$. For each $j < \chi_2$ we define $\bar{f}_j$ by:

$(*)_1$ (a) $\bar{f}_j = \langle f_{j,\alpha} \colon \alpha < \lambda \rangle$,
(b) $f_{j,\alpha} \colon \alpha \to \{0, 1\}$ is defined by:
($\alpha$) if $\beta < \alpha \in S$ then $f_{j,\alpha}(\beta) = g_{j,\alpha}(\min(C_\alpha \setminus \beta))$,
($\beta$) if $\beta < \alpha \in \lambda \backslash S$ then $f_{j,\alpha}(\beta) = 0$.

For each $j < \chi_2$ we can find $M_j \models T$ of cardinality $\lambda$ and pairwise distinct elements $\langle a_{j,\alpha} \colon \alpha < \lambda \rangle$ of $M_j$ satisfying $(\mathrm{b})_\lambda$ of Definition 1.1(3) for $\bar{f}_j$. Let $M_{j,\alpha} = M_j \restriction \cup \{a_{j,\beta} \colon \beta < \alpha\}$. Let the function $h^0_j \colon \lambda \to M_j$ be defined by $h^0_j(\alpha) = a_{j,\alpha}$.

Let $\mathscr{P} \subseteq [\chi_1]^\lambda$ witness $\mathbf{U}_J(\chi_1) < \chi_2$, so if $\lambda = \chi_1$ we use $\mathscr{P} = \{\lambda\}$; without loss of generality $u \in \mathscr{P} \wedge \alpha < \chi_1 \wedge |u \cap \alpha| = \lambda \Rightarrow u \cap \alpha \in \mathscr{P}$. For $u \in \mathscr{P}$ or just $u \in [\chi_1]^\lambda$ let $h^1_u$ be a one-to-one function from $u$ onto $\lambda$.

Towards a contradiction assume that there are $\xi_* < \chi_2$ and a sequence $\langle \mathfrak{A}_\xi \colon \xi < \xi_* \rangle$ of models of $T$, each of cardinality $\leq \chi_1$ witnessing $\mathrm{univ}(\chi_1, \lambda, T) < \chi_2$, even equal to $|\xi_*|$. Without loss of generality the universe of $\mathfrak{A}_\xi$ is $\alpha_\xi \leq \chi_1$ for $\xi < \xi_*$. So for every $j < \chi_2$ there are $\xi = \xi_j < \xi_*$ and[5] an $\leq_{\mathfrak{k}}$-embedding $h^2_j$ of $M_j$ into $\mathfrak{A}_\xi$, hence there is $u_j \in \mathscr{P}$ such that $W_j := \{\alpha \in S \colon h^2_j(a_{j,\alpha}) \in u_j\} \in I^+$ and let $v_j \supseteq u_j \cup \mathrm{Rang}(h^2_j)$ be such that $v_j \in [\alpha_{\xi_j}]^\lambda \subseteq [\chi_1]^\lambda$ and $\mathfrak{A}_j \restriction v_j \prec \mathfrak{A}_j$ and let $\langle \gamma_{j,\alpha} \colon \alpha < \lambda \rangle$ list the members of $v_j$.

Let $h'_j = h^2_{v_j} \circ (h^0_j \restriction w_j)$ and let $h_j = h^1_{v_j} \circ h^2_j \circ (h^0_j \restriction W_j)$; they are functions from $W_j$ into $\mathfrak{A}_{\xi_j}, \lambda$ respectively. Let $N_j := (\mathfrak{A}_{\xi_j} \restriction v_j, P^{N_j}_*)$ be the expansion of $\mathfrak{A}_{\xi_j} \restriction v_j$ by the relation $P^{N_j}_* = \mathrm{Rang}(h'_j)$ and let,

[5] Recall that now $\leq_{\mathfrak{k}} = \prec \restriction \mathrm{mod}_T$.

$$\begin{aligned} E_j = \{\delta < \lambda \colon \quad & \delta \text{ is a limit ordinal}, (\forall \alpha < \lambda)((h_{v_j})^{-1}(\alpha) \in \{\gamma_{j,\beta} \colon \beta < \delta\} \\ & \equiv \alpha < \delta) \text{ and } N_j \restriction \{\gamma_{j,\alpha} \colon \alpha < \delta\} \prec N_j\}, \end{aligned}$$

which clearly is a club in $\lambda$. Hence by clause (h) of Definition 1.6(1) there is an ordinal $\delta_j \in E_j \cap S$ such that $A_j := \mathrm{nacc}(C_\delta) \cap E_j$ belongs to $\mathscr{A}_{j,\delta}$.

As $\xi_* < \chi_2, |\mathscr{P}| < \chi_2$ and $|\{h_j(a_{j,\delta}) \colon j < \chi_2, \delta \in S\}| < \sup\{\|\mathfrak{A}_\xi\| \colon \xi < \xi_*\} \leq \chi_1 < \chi_2$ by the pigeon-hole-principle there are $j_1, j_2$ such that:

$(*)_2$
- (a) $j_1 = j(1) < j_2 = j(2)$,
- (b) $\xi_{j(1)} = \xi_{j(2)}$,
- (c) $\delta_{j_1} = \delta_{j_2}$ call it $\delta$ (so $\delta \in S$),
- (d) $u_{j_1} = u_{j_2}$ call it $u$, so $u = u_{j_\iota} \subseteq |N_{j_\iota}|$ for $\iota = 1, 2$,
- (e) $h^2_{j_1}(a_{j_1,\delta}) = h^2_{j_2}(a_{j_2,\delta})$ call it $b$, so $b \in \mathrm{Rang}(h^2_{j_1}) \cap \mathrm{Rang}(h^2_{j_2})$.

By clause (g) of Definition 1.6(1), there is $\gamma \in A_{j_1} \cap A_{j_2}$ such that $g_{j_1,\delta}(\gamma) \neq g_{j_2,\delta}(\gamma)$.

Now we shall choose $\alpha_\ell$ by induction on $\ell < n$ such that:

$(*)_3$
- (a) $\alpha_\ell \in W_{j_{\eta(\ell)}}$,
- (b) $\alpha_\ell < \gamma$ but $\alpha_\ell > \sup(C_\delta \cap \gamma)$,
- (c) $\langle \alpha_0, \ldots, \alpha_\ell \rangle$ is increasing,
- (d) in the model $N_{j_{\eta(\ell)+1}}$ the elements $h^2_{j_{\eta(\ell)+1}}(a_{j_1,\delta}) = b, h^1_{j_{\eta(\ell)+1}}(a_{j_1,\alpha_\ell})$ realize the same quantifier type over $\{h_{j_{\eta(\ell(1))+1}}(a_{j_\eta,\alpha_{\ell(1)}}) \colon \ell(1) < \ell\}$ or at least for all relevant (finitely many) formulas.

If we succeed, then in the model $\mathfrak{A}_{\xi_*}$ which extends $N_{j_1}$ and $N_{j_2}$ the sequence $\langle h^2_{j_{\eta(\ell)}}(a_{j_{\eta(\ell)},\alpha_\ell}) \colon \ell < n \rangle {}^\frown \langle b \rangle$ realizes the "forbidden" type, that is, the one from clause (c) of Definition 1.1, which is a contradiction.

As $\delta \in W_j \cap E_{j_{\eta(\ell)}}$ by the choice of $E_{j_{\eta(\ell)}}$ we can carry the induction.

3) Similarly.

4) As in [She93b] and the above, just use $\partial$-tuples of $\bar{a}$'s. $\square_{1.9}$

A sufficient condition for cases of $\mathrm{Qr}_i$ is the following.

**Definition 1.11.** Let $\mathrm{Qr}_4(\lambda)$ mean: $\lambda = \mu^+$ and $\langle C_\delta, D_\delta \colon \delta \in S \rangle$ satisfies $C_\delta \subseteq \delta, D_\delta$ a filter on $\mathrm{nacc}(C_\delta)$ such that $\mathscr{P}(\mathrm{nacc}(C_\delta)) \,/\, D_\delta$ satisfies the $2^\mu$-c.c. and, for every club $E$ of $\lambda$ for some $\delta \in S, E \cap \mathrm{nacc}(C_\delta) \in D^+_\delta$.

Note the extreme case:

**Conclusion 1.12.** *1) If (A) then (B), where:*

(A)
- $\bullet_1$ $\lambda = \mu^+$ *and* $2^\mu > \lambda \vee \mathfrak{d}_\mu = \lambda$,
- $\bullet_2$ $\bar{C} = \langle C_\delta \colon \delta \in S \rangle$ *guess club and* $\mathrm{otp}(C_\delta) \geq \mu$,
- $\bullet_3$ $T$ *is a first-order complete theory with the olive property of cardinality* $\leq \lambda$.

(B) $T$ *has no universal model in* $\lambda$.

*2) We can replace (A)($\bullet_1$) by:*

- $\mathfrak{d}_\mu > \lambda \wedge \mathrm{cf}(\mu) = \mu$ *or just there is* $\mathscr{A} \subseteq \{C \subseteq \lambda \colon \mathrm{otp}(C) = \mu\}$ *has cardinality* $\lambda$ *and it guess club of* $\lambda$.

*Proof.* Easy. $\square_{1.12}$

## § 2. The class of Groups has the olive property

### § 2(A). General Groups.

We shall try to prove that the class of groups has a universal member almost only when cardinal arithmetic is close to G.C.H. The following theorem does this.

**Theorem 2.1.** *The class of groups has the olive property (see Definition 0.8 or 1.1(1)(d)($\gamma$)), in fact, the $(\eta, \bar{k}, m)$-olive property, where $\eta = \langle 0, 1, 0\rangle, \bar{k} = (2, 1)$, and $m = 6$.*

Why does Theorem 2.1 suffice? Because then we can use Theorem 1.9(3); or see Conclusions 2.16, 2.17, we break the proof into a series of definitions and claims; we may replace the use of HNN extensions (in Claim 2.13) and free amalgamation (in Claim 2.12) by the proof of Claim 2.14.

**Definition 2.2.** Let $\bar{\psi} \coloneqq \bar{\psi}^{\rm grp}_{\rm olive}$ be $(\psi_{0,1}, \psi_{0,2}, \psi_{1,1})$ defined as follows (letting $m = 6$):

(a) $\psi_{0,1} = \varphi_0 = \varphi_0(\bar{x}_{[m]}, \bar{y}_{[m]}) = y_5^{-1} x_0 y_5 = x_2$,
(b) $\psi_{1,1} = \varphi_1 = \varphi_1(\bar{x}_{[m]}, \bar{y}_{[m]}) = x_5^{-1} y_1 x_5 = y_3 \wedge x_5^{-1} y_4 x_5 = y_4$,
(c) $\psi_{0,2} = \psi(\bar{x}_{[m]}, \bar{y}_{[m]}, \bar{z}_{[m]}) = (\sigma_*(x_0, y_1, z_4) = e \wedge \sigma_*(x_2, y_3, z_4) \neq e)$, on $\sigma_*$, see below.

**Definition/Claim 2.3.** There is a $\sigma_* = \sigma_*(x, y, z)$ such that:

(a) $\sigma_*$ is a group word,
(b) for some group $G$ and $a, b, c \in G$ we have "$\sigma_*(a, b, c) \neq e_G$",
(c) for any group $G$ and $a, b, c \in G$ we have $e \in \{a, b, c\} \Rightarrow \sigma^G(a, b, c) = e_G$.

*Proof.* Straightforward, e.g. $(x^{-1}y^{-1}xy)^{-1}z^{-1}(x^{-1}y^{-1}xy)z$. $\square_{2.3}$

**Claim 2.4.** *The $\bar{\psi}$ from Definition 2.2 satisfies clause (c) of Definition 0.8, i.e., for no group $G$ and $\bar{a}_\ell \in {}^m G$ for $\ell < 4$ do the formulas there hold.*

*Remark* 2.5. We prove more: there are no group $G$ and $\bar{a}_\ell \in {}^m G$ for $\ell = 0, 1, 2, 3$ such that $\varphi_0[\bar{a}_0, \bar{a}_1]$, $\varphi_1[\bar{a}_1, \bar{a}_2]$, $\varphi_1[\bar{a}_1, \bar{a}_3]$, and $\psi[\bar{a}_0, \bar{a}_2, \bar{a}_3]$.

*Proof.* Assume towards a contradiction that $G$ and $\langle \bar{a}_\ell \colon \ell < 4\rangle$ form a counterexample. Notice that conjugation by $a_{1,5}$ is an automorphism of $G$, which we call $g$.

Now,

- $g(a_{0,0}) = a_{0,2}$ by Definition 2.2(a) as $G \models \varphi_0[\bar{a}_0, \bar{a}_1]$,
- $g(a_{2,1}) = a_{2,3}$ by first conjunct of Definition 2.2(b) as $G \models \varphi_1[\bar{a}_1, \bar{a}_2]$,
- $g(a_{3,4}) = a_{3,4}$ by the second conjunct of Definition 2.2(b) as $G \models \varphi_1[\bar{a}_1, \bar{a}_3]$.

Together we have:

- $g(\sigma_*(a_{0,0}, a_{2,1}, a_{3,4})) = \sigma_*(a_{0,2}, a_{2,3}, a_{3,4})$.

But this contradicts $G \models \psi[\bar{a}_0, \bar{a}_2, \bar{a}_3]$ (see clause Definition 2.2(c)). $\square_{2.4}$

**Definition 2.6.** Let $\bar{f} \in \mathbf{F}_\lambda$, i.e. $\bar{f} = \langle f_\alpha \colon \alpha < \lambda\rangle$ where, for each $\alpha < \lambda$, $f_\alpha \colon \alpha \to \{0, 1\}$.

1) Let $X_{\bar f} := X_{\bar f,m}$, where we let $X_{\bar f,k} := \{x_{\alpha,\ell} \colon \alpha < \lambda, \ell < k\}$ for $k \le m$; recall that here $m = 6$.

2) Let $\bar x_{\alpha,k} := \langle x_{\alpha,\ell} \colon \ell < k \rangle$ for $k \le m$ and let $\bar x_\alpha := \bar x_{\alpha,m}$.

3) For $\ell = 0, 1$ we define the set $\Gamma^\ell_{\bar f}$ of equations (pedantically, for $\ell = 1$ conjunctions of two equations) as follows:

$$\{\varphi_\ell(\bar x_\alpha, \bar x_\beta) \colon \alpha < \beta < \lambda \text{ and } f_\beta(\alpha) = \ell\}.$$

4) We define the set $\Gamma^2_{\bar f}$ of equations as follows:

$$\{\sigma_*(x_{\alpha,0}, x_{\beta,1}, x_{\gamma,4}) = e \colon \alpha < \beta < \gamma < \lambda \text{ and } f_\gamma{\restriction}[\alpha,\beta] \text{ is constantly } 0\}.$$

5) Let $G^5_{\bar f}$ be the group generated by $X_{\bar f,5}$ freely except for the equations in $\Gamma^2_{\bar f}$. Note that the $x_{\alpha,5}$'s are not mentioned in $\Gamma^2_{\bar f}$.

6) Let $G^6_{\bar f}$ be the group generated by $X_{\bar f,6}$ freely except for the equations in $\Gamma^0_{\bar f} \cup \Gamma^1_{\bar f} \cup \Gamma^2_{\bar f}$.

**Discussion 2.7.** 1) For our purpose we have to show that for $\alpha < \beta < \gamma$ (and $\bar f \in \mathbf{F}_\lambda$) we have:

$$G_{\bar f,6} \models \text{“}\psi[\bar x_\alpha, \bar x_\beta, \bar x_\gamma]\text{” iff } f_\gamma \restriction [\alpha,\beta] = 0_{[\alpha,\beta]}.$$

For proving the “if” implication, assume $f_\gamma \restriction [\alpha,\beta] = 0_{[\alpha,\beta]}$. Now the satisfaction of “$\sigma_*(x_{\alpha,0}, x_{\beta,1}, x_{\gamma,4}) = e$” is obvious by the role of $\Gamma^2_{\bar f}$, the analysis below is intended to prove the other half, “$\sigma_*(x_{\alpha,2}, x_{\beta,3}, x_{\gamma,4}) \ne e$”. For proving the “only if” implication it suffices to prove that “$\sigma_*(x_{\alpha,0}, x_{\beta,1}, x_{\gamma,4}) \ne e$” when $f_\gamma{\restriction}[\alpha,\beta] \ne 0_{[\alpha,\beta]}$. In both cases, we prove that this holds in $G^5_{\bar f}$ and then prove that $G^5_{\bar f} \subseteq G^6_{\bar f}$ in a natural way.

2) Of course, we also have to prove $G^6_{\bar f} \models \varphi_i(\bar x_\alpha, \bar x_\beta)$ when $\alpha < \beta < \lambda$ and $f_\beta(\alpha) = \iota$.

**Claim 2.8.** *1) If $\alpha < \beta < \gamma < \lambda$ and $f_\gamma{\restriction}[\alpha,\beta] \ne 0_{[\alpha,\beta]}$, <u>then</u>*

$$G^5_{\bar f} \models \text{“}\sigma_*[x_{\alpha,0}, x_{\beta,1}, x_{\gamma,4}] \ne e\text{”}.$$

*2) If $\alpha < \beta < \gamma < \lambda$, <u>then</u> $G^5_{\bar f} \models$ “$\sigma_*(x_{\alpha,2}, x_{\beta,3}, x_{\gamma,4}) \ne e$”.*

*Proof.* 1) Use 2.9 below with $X = \{x_{\xi,\ell} \colon \xi \in \{\alpha,\beta,\gamma\} \text{ and } \ell < 5\}$.

2) Use 2.9(2) below with $X = \{x_{\xi,\ell} \colon \xi < \lambda, \ell < 5 \text{ and } \ell > 0\}$. $\square_{2.8}$

**Observation 2.9.** *1) If $x_{\alpha,\ell}, x_{\beta,k} \in X^5_{\bar f}$ and $(\alpha,\ell) \ne (\beta,k)$, <u>then</u> $G^5_{\bar f} \models$ “$x_{\alpha,\ell} \ne x_{\beta,k}$”.*

*2) If $X \subseteq X^5_{\bar f}$ and $(\sigma_*(x_{\alpha,0}, x_{\beta,1}, x_{\gamma,4}) = e) \in \Gamma^2_{\bar f} \Rightarrow \{x_{\alpha,0}, x_{\beta,1}, x_{\gamma,4}\} \nsubseteq X$, <u>then</u> $X$ generates freely a subgroup of $G^5_{\bar f}$.*

*Proof.* 1) Let $G' := \bigoplus\{\mathbb{Z}x \colon x \in X^5_{\bar f}\}$, it is an abelian group, and let $G'' := \bigoplus\{\mathbb{Z}x_{\alpha,i} \colon \alpha < \lambda, i \notin \{\ell,k\}\}$, it is a subgroup. So $G'/G''$ by clause (c) of Definition 2.3 because $\{0,1,4\} \nsubseteq \{\ell,k\}$, satisfies all the equations in $\Gamma^2_{\bar f}$ and it satisfies the desired inequality. As $G^5_{\bar f}$ is generated by $X^5_{\bar f}$ freely except for the equations in $\Gamma^2_{\bar f}$, the desired result follows. Alternatively, use part (2).

2) Let $H := H_X$ be the group generated by $X$ freely. We define a function $F$ from $X^5_{\bar f}$ into $H$ by:

$$F(x) := \begin{cases} x, \text{ if } x \in X, \\ e_H, \text{ if } x \in X_{\bar f^5} \setminus X. \end{cases}$$

Now $F$ respects every equation form $\Gamma^2_{\bar f}$ by Claim 2.3(c), hence $f$ induces a homomorphism from $G^5_{\bar f}$ into $H$, really onto. Thus, the desired conclusion follows. $\square_{2.9}$

**Definition 2.10.** For $\beta < \lambda$ we define a partial function $F_\beta \colon X^5_{\bar f} \to X^5_{\bar f}$ as follows:

- if $\alpha < \beta$ and $f_\beta(\alpha) = 0$, <u>then</u> $F_\beta(x_{\alpha,0}) := x_{\alpha,2}$,
- if $\gamma > \beta$ and $f_\gamma(\beta) = 1$ <u>then</u> $F_\beta(x_{\gamma,1}) := x_{\gamma,3}$ and $F_\beta(x_{\gamma,4}) := x_{\gamma,4}$.

**Claim 2.11.** *1) $F_\beta$ is a well-defined partial one-to-one function from $X^5_{\bar f}$ to $X^5_{\bar f}$.*
*2) The domain and the range of $F_\beta$ satisfy the criterion of Observation 2.9(2).*

*Proof.* 1) It is a function as no $x_{\alpha,\ell}$ appears in two cases. Also if $F_\beta(x_{\alpha_1,\ell}) = x_{\alpha_2,k}$ then $\alpha_1 = \alpha_2 \wedge (\ell, k) \in \{(0,2),(1,3),(4,4)\}$, so $F_\beta$ is one-to-one.
2) Assume $[\sigma_(x_{\alpha_1,0}, x_{\alpha_2,1}, x_{\alpha_3,4}) = e] \in \Gamma^2_{\bar f}$ so,

$(*)_1$ $\alpha_1 < \alpha_2 < \alpha_3$,

and,

$(*)_2$ $f_{\alpha,3} \restriction [\alpha_1, \alpha_2] = 0_{[\alpha_1,\alpha_2]}$.

First, toward contradiction assume $\{x_{\alpha_1,0}, x_{\alpha_2,1}, x_{\alpha_3,4}\} \subseteq \mathrm{dom}(F_\beta)$.
Now if $\alpha_1 \geq \beta$ then $x_{\alpha_1,0} \notin \mathrm{dom}(F_\beta)$, just inspect Definition 2.10 so necessarily $\alpha_1 < \beta$ and similarly $f_\beta(\alpha_1) = 0$ (but not used).
If $\alpha_2 \leq \beta$ then $x_{\alpha_2,1} \notin \mathrm{dom}(F_\beta)$, so $\beta < \alpha_2$ and similarly $f_{\alpha_2}(\beta) = 1$ (again not used) so together we get $\alpha_1 < \beta < \alpha_2$. Also as $x_{\alpha_3,4} \in \mathrm{dom}(F_\beta)$, it follows that ($\beta < \alpha_3$ which follows by earlier inequalities and) $f_{\alpha_3}(\beta) = 1$, therefore $\beta$ witnesses that $f_{\alpha_3} \restriction [\alpha_1, \alpha_2]$ is not constantly zero; but this is a contradiction to $[\sigma(x_{\alpha_1,0}, x_{\alpha_2,0}, x_{\alpha_3,0}) = e] \in \Gamma^2_{\bar f}$.
Second, assume towards contradiction that $\{x_{\alpha_2,0}, x_{\alpha_1,2}, x_{\alpha_3,4}\} \subseteq \mathrm{Rang}(F_\beta)$, but "$x_{\alpha_2,0} \in \mathrm{Rang}(F_\beta)$" is impossible by Definition 2.10. $\square_{2.11}$

**Claim 2.12.** *To prove $G^5_{\bar f} \subseteq G^6_{\bar f}$ any of the following conditions suffice:*

(a) *there are a group $H$ extending $G^5_{\bar f}$ and $y_\zeta \in G$ for $\zeta < \lambda$ such that:*

$$\zeta < \lambda \wedge F_\zeta(x_{\varepsilon_1,\ell_1}) = x_{\varepsilon_2,\ell_2} \Rightarrow H \models \text{“}y_\zeta^{-1} x_{\varepsilon_1,\ell_1} y_\zeta = x_{\varepsilon_2,\ell_2}\text{”}.$$

(b) *for each $\zeta < \lambda$ there is a group $H$ extending $G^5_{\bar f}$ and $y \in G$ such that:*

$$F_\zeta(x_{\varepsilon_1,\ell_1}) = x_{\varepsilon_2,\ell_2} \Rightarrow H \models \text{“}y^{-1} x_{\varepsilon_1,\ell_1} y = x_{\varepsilon_2,\ell_2}\text{”}.$$

*Proof.* <u>Clause (a) suffice:</u>
We define a function $F$ from $X^6_{\bar f}$ into $H$ by:

$$F(x_{\varepsilon,\ell}) := \begin{cases} x_{\varepsilon,\ell}, \text{ if } \ell < 5 \wedge \varepsilon < \lambda, \\ y_\varepsilon, \text{ if } \ell = 5 \wedge \varepsilon < \lambda, \end{cases}$$

where in the first case, $x_{\varepsilon,\ell} \in G^5_{\bar{f}} \subseteq H$.

Check that the mapping $F$ respects the equations in $\Gamma^0_{\bar{f}} \cup \Gamma^1_{\bar{f}} \cup \Gamma^2_{\bar{f}}$ hence it induces a homomorphism $F^1$ from $G^6_{\bar{f}}$ into $H$, and for every group word

$$\sigma = \sigma(\ldots, x_{\varepsilon_i,\ell_i}, \ldots)_{i<n}, x_{\varepsilon_i,\ell_i} \in X^5_{\bar{f}},$$

we have $G^6_{\bar{f}} \models$ "$\sigma = e$" $\Rightarrow G^5_{\bar{f}} \models$ "$\sigma = e$", so we are done.

Clause (b) suffice:

Let $(H_\zeta, y_\zeta)$ for $\zeta < \lambda$ be as guaranteed by the assumption, i.e. clause (b). Without loss of generality, $\zeta \neq \xi < \lambda = G_\zeta \cap G_\xi = G^5_{\bar{f}}$. Now clause (a) follows by using free amalgamation of $\langle H_\zeta \colon \zeta < \lambda \rangle$ over $G^5_{\bar{f}}$, we know it is as required in clause (a), see e.g. [LS77]. $\square_{2.12}$

**Claim 2.13.** *1) Clause (b) of Claim 2.12 holds.*

*2) The conclusion of Claim 2.12 holds also for $G^6_{\bar{f}}$.*

*3) The conclusions of Claim 2.8 hold also for $G^6_{\bar{f}}$.*

*Proof.* 1) By the theorems on HNN extensions (see [LS77]) applied with the group being $G^5_{\bar{f}}$ and the partial automorphism $\pi_\zeta$ being the one $F_\zeta$ induced, i.e.,

- $\mathrm{dom}(\pi_\zeta)$ is the subgroup of $G^5_{\bar{f}}$ generated by $\mathrm{dom}(F_\zeta)$,
- $\pi_\zeta(x_{\varepsilon,\ell}) = F_\zeta(x_{\varepsilon,\ell})$ for $x_{\varepsilon,\ell} \in \mathrm{dom}(F_\zeta)$.

By Claim 2.11(2) and Observation 2.9(2) we know that $\pi_\zeta$ is indeed an isomorphism.

2) Follows by Claims 2.12 and 2.13(1).

3) By Claims 2.8 and 2.13(2). $\square_{2.13}$

Now, we prove Theorem 2.1:

*Proof.* Should be clear by now. $\square_{2.1}$

## § 2(B). **Locally Finite Groups.**

**Claim 2.14.** *The pair $(K_{\mathrm{lfgr}}, K_{\mathrm{gr}})$ of classes, i.e. (locally finite groups, groups), has the olive property, as witnessed by $\bar{\varphi}$ from Definition 2.2.*

*Proof.* We rely on Observation 2.15 below and use its notation. Let

$$J = \{(\alpha, \beta, \gamma) \colon \alpha < \beta < \gamma < \lambda \text{ and } f_\gamma \restriction [\alpha, \beta] = 0_{[\alpha,\beta]}\}.$$

Let $G^5_{\bar{f}}, G^6_{\bar{f}}$ be as in the proof of Theorem 2.1, that in Definition 2.6. Now for $\bar{\alpha} = (\alpha_0, \alpha_1, \alpha_2) \in J$, let $\pi^5_{\bar{\alpha}}$ be the function from $X_{\bar{f},5}$ (see Definition 2.6(5)) into $K$, ($K$ is from 2.15) defined as follows:

$(*)_1$ $\pi^5_{\bar{\alpha}}(x_{\beta,k})$ is
- $e_K$, if $\beta \notin \{\alpha_0, \alpha_1, \alpha_2\}$,
- $z_{\ell,k}$, if $\beta = \alpha_\ell, \ell \leq 2$.

Now,

$(*)_2$ $\pi^5_{\bar{\alpha}}$ respects the equations from $\Gamma^2_{\bar{f}}$.

[Why? The equation $\sigma_*(x_{\alpha_0,0}, x_{\alpha_1,1}, x_{\alpha_2,4}) = e$ holds as $K$ satisfies $\sigma_(z_{0,0}, z_{1,1}, z_{2,4}) = e$. For the other equations see Definition 2.3(c); recall that the equations for the cases of $\varphi_0, \varphi_1$ do not appear, see Definition 2.6(5).]

Let $\pi^6_{\bar\alpha}$ be the following function from $X^6_{\bar f}$ into $K$:

$(*)_3$ $\pi^6_{\bar\alpha}(x)$ is
- $\pi^5_{\bar\alpha}(x)$ when $x \in X^5_{\bar f}$,
- $z_s$ when $x = x_{\beta,5}, \beta < \lambda$ and $s = s_{\bar\alpha,\beta} := (\{\ell \le 2 \colon \alpha_\ell < \beta$ and $f_\beta(\alpha_\ell) = 0\}, \{\ell \le 2 \colon \beta < \alpha_\ell$ and $f_{\alpha_\ell}(\beta) = 1\})$.

[Why is $\pi^6_{\bar\alpha}$ as required? The least obvious point is: why $s \in S_*$? Let $s = (u_1, u_2)$, now $\ell_1 \in u_1 \wedge \ell_2 \in u_2 \Rightarrow \alpha_{\ell_1} < \beta < \alpha_{\ell_2} \Rightarrow \ell_1 < \ell_2$ and $(\{0\}, \{1,2\}) \neq s$ because $f_{\alpha_2} \restriction [\alpha_0, \alpha_1]$ is constantly zero.]

$(*)_4$ $\pi^6_{\bar\alpha}$ respects the equations in $\Gamma^0_{\bar f} \cup \Gamma^1_{\bar f}$.

[Why? Check the definitions.]

By $(*)_2, (*)_4$ there is a homomorphism $\pi_{\bar\alpha}$ from $G_{\bar f}$ into $K$ extending $\Pi^6_{\bar\alpha}$. Let $G_*$ be the product of $J$-copies of $K$, i.e.,

$(*)_5$ (a) the set of elements of $G_*$ is the set of functions $g$ from $J$ into $K$,
(b) $G_* \models$ "$g_1 g_2 = g_3$" <u>iff</u> $\bar\alpha \in J \Rightarrow K \models$ "$g_1(\bar\alpha) g_2(\bar\alpha) = g_3(\bar\alpha)$".

Now,

$(*)_6$ $G_*$ is a locally finite group.
$(*)_7$ for $\alpha < \lambda, k < m$ let $\bar g_\beta = \langle g_{\beta,k} \colon k < m\rangle$ where $g^*_{\beta,k} \in G_*$ be defined by $(g_{\beta,k}(\bar\alpha))(x) = \pi^6_{\bar\alpha}(x_{\beta,k})$.
$(*)_8$ $G_*, \langle \bar g_\beta \colon \beta < \lambda\rangle$ witnesses the olive property.

[Why? Check.]

So we are done. $\square_{2.14}$

**Observation 2.15.** *There are $K, z_{i,k}$ for $i < 3$, $k < m$ and $\langle \pi_s \colon s \in S_*\rangle$ such that:*

(a) *$K$ is a finite group,*
(b) *$z_{i,k} \in K$,*
(c) *$\sigma_*(z_{0,0}, z_{1,2}, z_{2,4}) = e$ but $\sigma_*(z_{0,2}, z_{1,3}, z_{2,4}) \neq e$,*
(d) *$S_* = \{(u_1, u_2) \colon u_1, u_2 \subseteq \{0,1,2\}$ and $(\forall \ell_1 \in u_1)(\forall \ell_2 \in u_2)(\ell_1 < \ell_2)$ but $(u_1, u_2) \neq (\{0\}, \{1,2\})$,*
(e) *for $s = (u_1, u_2) \in S_*$ we have: $\pi_s$ is a partial automorphism of $K$ such that:*
 ($\alpha$) *if $\ell \in u_1$ then $\pi_s(x_{\ell,0}) = x_{\ell,2}$,*
 ($\beta$) *if $\ell \in u_2$ then $\pi_s(x_{\ell,1}) = x_{\ell,2}, \pi_s(z_{\ell,4}) = z_{\ell,4}$,*
(f) *moreover, there are $z_s \in K$ for $s \in S_*$ such that $(\forall x \in \mathrm{dom}(\pi_s))(\pi_s(x) = z_s^{-1} x z_s)$.*

*Proof.* First, we ignore clause (f). We use finite nilpotent groups. Let $n_2 := 6m$, $n_1 := \binom{n_2}{2}$, $n_0 := \binom{n_1}{2}$ and let $f_\ell \colon [n_{\ell+1}]^2 \to n_\ell$ be one-to-one for $\ell = 0, 1$.

Let $K_1$ be the group generated by $\{y_{j,\ell} \colon j \le 2, \ell < n_j\}$ freely except for the following equations:

$(*)_1$ (a) $y_{j,\ell} \cdot y_{j,\ell} = e$,

(b) $[y_{j+1,\ell_1}, y_{j+1,\ell_2}] = y_{j,f\{\ell_1,\ell_2\}}$, i.e. $y^{-1}_{j+1,\ell_1} y^{-1}_{j+1,\ell_2} y_{j+1,\ell_1} y_{j+1,\ell_2} = y_{j,f\{\ell_1,\ell_2\}}$, when $j < 2, \ell_1 < \ell_2 < n_{j+1}$,

(c) $[y_{j_1,\ell_1}, y_{j_2,\ell_2}] = e$ when $(j_1 = 0 = j_2) \vee (j_1 \neq j_2 \leq 2)$ and $\ell_1 < n_{j_1}, \ell_2 < n_{j_2}$.

Clearly, $K_1$ is finite.

Let $z'_{i,\ell} = y_{2,6i+\ell}$ for $i < 3, \ell < m$, let $\ell_*$ be such that $[[z'_{0,0}, z'_{1,1}], z'_{2,4}] = y_{0,\ell_*}$. Let $K_0$ be the subgroup $\{e, y_{0,\ell_*}\}$ of $K$, it is a normal subgroup as it is included in the center of $K_1$ and let $K_2 := K_1/K_0$ and we define $z_{i,\ell}$ as $z'_{i,\ell}/K_0$.

Now,

$(*)_2$ $K_2, \langle z_{i,\ell} \colon i \leq 2, \ell < m\rangle$ are as required in (a)-(e) of the claim.

[Why? We should just check that for $s \in S_*$ there is $\pi_s$ as required, i.e. that some subgroups of $K_2$ generated by subsets of $\langle z_{i,\ell} \colon i \leq 2, \ell < m\rangle$ are isomorphic, but as none of them included $\{z_{0,0}, z_{1,1}, z_{2,4}\}$ and the way $K_2$ was defined this is straightforward.]

Lastly, there is a finite group $K$ extending $K_2$ and $z_s \in K$ for $s \in S$ such that:

$(*)_3$ $x \in \mathrm{dom}(\pi_s) \Rightarrow z_s^{-1} x z_s = \pi_s(x)$.

Why? Simply because $K_2$ can be considered as a group of permutations of the set $K_2$ (e.g. multiplying from the right), and it is easy to find $z_s \in \mathrm{Sym}(K_2)$ as required. $\square_{2.15}$

**Conclusion 2.16.** *Assume* $\mathrm{Qr}_1(\chi_1, \chi_2, \lambda)$.

*Then there is no sequence $\langle G_\alpha \colon \alpha < \alpha_*\rangle$ of length $< \chi_2$ of groups of cardinality $\leq \chi_1$ such that any locally finite group $H$ of cardinality $\lambda$ can be embedded into at least one of them.*

The following is an example.

**Conclusion 2.17.** *1) If $\mu = \mathrm{cf}(\mu), \mu^+ < \lambda = \mathrm{cf}(\lambda) < 2^\mu$, then there is no group of cardinality $\lambda$ universal for the class of locally finite groups.*

*2) For example, if $\aleph_2 \leq \lambda = \mathrm{cf}(\lambda) < 2^{\aleph_0}$ this applies.*

## § 2(C). The Class of Groups is not Amenable.

We have claimed (in earlier versions of [She17]) that the class of groups is amenable (see Dzamonja-Shelah [DS04]) but this is not true.

An easy way to prove it is the following claim.

**Claim 2.18.** *For some group $G_0$, if $G \supseteq G_0$ and $T = \mathrm{Th}(G)$ then $T$ is not amenable.*

*Proof.* In any forcing extension $\mathbf{V}_1 = \mathbf{V}^{\mathbb{P}}$ of $\mathbf{V}$ we have:

$(*)$ If $\mathbf{V}_1 \models$ "$\lambda := \mu^+ = 2^\mu + \diamondsuit_{S^\lambda_\mu}$, $\mathbb{Q}$ is $\mu^+$-c.c., $(< \mu)$-complete forcing notion", then in $\mathbf{V}_1^{\mathbb{Q}}, T$ has no universal member in $\lambda$, moreover $\mathrm{univ}(\lambda, \lambda, T) \leq \lambda^+$.

[Why? Because in $\mathbf{V}_1$ there is $\bar{C} = \langle C_\delta \colon \delta \in S^\lambda_\mu\rangle$ guessing clubs hence this holds also in $\mathbf{V}_1^{\mathbb{Q}}$, so we can apply Theorems 1.9 and 2.1 or directly 2.16.]

From $(*)$, by [DS04] we get a contradiction to amenability (using a suitable $\mathbb{Q}$). $\square_{2.18}$

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