# NATURALITY AND DEFINABILITY III

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ABSTRACT. In this paper, we deal with the notions of naturality from category theory and definablity from model theory and study their interactions. In this regard, we present three results. First, we show, under some mild conditions, that naturality implies definablity. Second, by using reverse Easton iteration of Cohen forcing notions, we construct a transitive model of ZFC in which every uniformisable construction is weakly natural. Finally, we show that if $F$ is a natural construction on a class $\mathcal{K}$ of structures which is represented by some formula, then it is uniformly definable without any extra parameters. Our results answer some questions by Hodges and Shelah.

## § 1. INTRODUCTION

We are looking to find some interplay between the notions of naturality from category theory and definability from set theory and model theory. This continues the investigation initiated by Hodges and Shelah in [10] and [11].

In this paper, naturality is in the sense of Eilenberg and Mac Lane [3], i.e., for a class $\mathcal{K}$ of structures, the construction $A \mapsto F(A)$, for $A \in \mathcal{K}$, is natural, if for

*Date*: February 3, 2024.

2010 *Mathematics Subject Classification.* Primary 08A35; 03E35; 18A15.

*Key words and phrases.* Automorphisms of algebraic structures; definable functors; lifting morphisms; forcing techniques; natural constructions; sorted modeles; uniformity.

The second author's research has been supported by a grant from IPM (No. 1402030417). The third author is grateful to an individual who prefers to remain anonymous for providing typing services that were used during the work on the paper. The typists paid by C. F (2022-2025). The third author would like to thank ISF 1838/19: The Israel Science Foundation (ISF) (2019-2023). ISF 2320/23: The Israel Science Foundation (ISF) (2023-2027) NSF-BSF 2021: grant with M. Malliaris, NSF 2051825, BSF 3013005232 (2021-10-2026-09). This is publication 1245 of third author.

every $A \in \mathcal{K}$ and every automorphism $a$ of $A$ there is an automorphism $f(a)$ of $F(A)$ such that the following diagram is commutative

$$\begin{array}{ccc} A & \xrightarrow{\subseteq} & F(A) \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle f(a)} \\ A & \xrightarrow{\subseteq} & F(A) \end{array}$$

and the assignment $A \mapsto F(A)$ has the following two properties:

(1) $f(\mathrm{id}_A) = \mathrm{id}_{F(A)}$, where $\mathrm{id}_A$ is the identity function on $A$,
(2) for automorphisms $a, b$ of $A$, $f(ab) = f(a)f(b)$.

We remark that Hodge and Shelah [11, 10] have other requirements, for example, that there is an embedding of A into $F(A)$ and that $F(A)$ is determined up to isomorphism.

The challenging initial example for naturality was the dual of vector spaces, see [3]. Hodges and Shelah [10, Example 1] observed that the construction of algebraic closure of fields and the construction of divisible hulls are not natural. Another example of a construction which is not natural is that of the divisible hull of an abelian group. Recall that a minimal divisible abelian group containing the abelian group $G$ is called a divisible hull of $G$. By [5, IV, Theorem 2.7], such a thing exists and is unique, and is denoted by $E(G)$. Lambek [6, Page 10] observed that divisible hull $E(G)$ of an abelian group $G$ is not natural.

Given a category $\mathcal{C}$, let $\mathrm{Ob}(\mathcal{C})$ denote the class of its objects, and for $X, Y \in \mathrm{Ob}(\mathcal{C})$, let $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ denote the class of its morphisms from $X$ to $Y$. Now, let $G : \mathcal{C} \to \mathcal{D}$ be a functor and recall that a functor $F : \mathcal{D} \to \mathcal{C}$ is left adjoint to $G$ if there is a canonical isomorphism $\mathrm{Hom}_{\mathcal{C}}(FX, Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, G(Y))$ where $X \in \mathrm{Ob}(\mathcal{D})$ and $Y \in \mathrm{Ob}(\mathcal{C})$. For more details, see Mac Lane's book [13, Page 81]. Hodges and Shelah [10, Example 3] showed that if $G$ is the forgetful functor on structures and $F$ is left adjoint to $G$, then $F$ defines a natural construction. This may be considered as a source to produce several natural constructions. For example, let $\mathcal{T}$ be the category of torsion-free abelian groups. Lambeck [6, Page 11] pointed out the construction $G \mapsto E(G)$ from $\mathcal{T}$ is natural[1]. Indeed, the divisible hull of $G \in Ob(\mathcal{T})$ can also be obtained as $\mathbb{Q} \otimes_{\mathbb{Z}} G$, see [5, IV, Example 2.9]. So, $E : \mathcal{T} \to \mathcal{T}$ is just the tensor product, which is left adjoint to the hom-functor, see [13, Page 80]. Now, the mentioned result [10, Example 3] says that $E : \mathcal{T} \to \mathcal{T}$ is natural. This line of research is contained in [1].

Let $H$ and $G$ be two groups, which are not necessarily abelian. Assume $\varphi : H \twoheadrightarrow G$ is a surjective homomorphism of groups. Following [11, Definition 3.1(i)], we say $\psi \in \mathrm{Hom}(G, H)$ splits $\varphi$, provided $\varphi \circ \psi = \mathrm{id}_G$. In the light of [10, Lemma 1], the naturality condition gives a splitting for $\mathrm{Aut}(F(A)) \to \mathrm{Aut}(A)$. In the absence of $\psi(g_1g_2) = \psi(g_1)\psi(g_2)$, Hodges and Shelah [11, Definition 3.1(ii)] called a map $\psi : G \to H$ is weakly splits $\varphi$, provided $\varphi \circ \psi = \mathrm{id}_G$, $\psi(x^{-1}) = (\psi(x))^{-1}$ and $\psi(e_G) = e_H$. We mention that $\psi$ is not group-homomorphism, instead we assume the induced map is in $\mathrm{Hom}(G, H/\mathcal{Z}(H))$, where $\mathcal{Z}(H)$ is the center of $H$. We say $\varphi$ has lifting (resp. weakly lifting), if some $\psi$ splits (resp. weakly splits) it. From this one may define the notion of weak naturality, see [11, Definition 3.1]. Hodges and Shelah [10] asked:

*Question* 1.1. When does naturality imply definablity?

We use two-sorted models in order to deal with the question. This approach is not new. For example, Harvey Friedman [4] already has considered some constructions in algebra (e.g., direct product construction of pairs of groups) as operations from relational structures of a certain fixed many sorted relational type to structures of an enlarged many sorted relational type. Also, Hodges and Shelah [11] interpreted Question 1.1 in terms of two-sorted models. In sum, Question 1.1 can be discussed naturally in the context of many sorted model theory.

In §2, and for the convenience of the readers, we review the concept of many sorted models. According to its definition, any 2-sorted model $\mathfrak{B}$ gives us two groups $H := \mathrm{Aut}(\mathfrak{B})$ and $G := \mathrm{Aut}(\mathrm{sort}_1(\mathfrak{B}))$, see Definition 2.1. If the restriction map $\varphi : H \to G$ is well-defined and has a weak lifting $\psi$, we say $\mathfrak{B}$ gives us a

[1] While as we mentioned above, this is not the case for the category of abelian groups.

uni-construction problem

$$\mathbf{c} = \langle \mathfrak{B}, \mathrm{sort}_1(\mathfrak{B}), H, G, \varphi, \psi \rangle,$$

(see Definition 2.8). The new concept of uni-construction problem will play the role of weak naturality. Then we introduce the concept of $\chi$-solution (see Definition 2.11), which plays the role of definablity.

Concerning Question 1.1, in §3, we discuss some special kinds of 3-sorted models, and prove the following as our first main result:

**Theorem 1.2.** *Let $\chi$ be a cardinal and let $\mathbf{c}$ be a uni-construction problem. If $\mathbf{c}$ has no lifting, then in a forcing extension, $\mathbf{c}$ has no $\chi$-solution.*

The next auxiliary tool is the uniformity, see Definition 4.1. In particular, a construction is uniformisable if the set-theoretic definition can be given in a form such that for each $A$ the corresponding $B$ is determined uniquely. The precise and the original definition is given by [11, Definition 2.3]. In [11, Theorem 5.1], Hodges and Shelah constructed a transitive model of ZFC in which every uniformisable construction of a prescribed size is weakly natural. This result shows the advantage of weak naturality, as [11, Theorem 4.3] says that there is no transitive model of ZFC in which the natural constructions are exactly the uniformisable ones. Since their result [11, Theorem 5.1] has a cardinality restriction, Hodges and Shelah raised the following natural conjecture.

**Conjecture 1.3.** (See [11, Page 16]) The cardinality restriction can be removed.

In §4, we settle the conjecture, by iterating the main forcing construction of [11, Theorem 5.1] using reverse Easton iteration of suitable Cohen forcing notions. It may be nice to note that the use of forcing techniques in the naturality problems come back to Harvey Friedman [4] where he applied Easton product Cohen-forcing, though the term naturalness in [4] is weaker than here.

Let $M$ be a model of set theory. Suppose some formula represents the construction $F$ on the class $\mathcal{K}$ in $M$. If $F$ is natural and there is only a set of isomorphism types of structures in $\mathcal{K}$, then [10, Theorem 3] states that $F$ is definable in $M$ with parameters. In this regard, Hodges and Shelah raised the following problem:

**Problem 1.4.** (See [10, Problems (A) and (B)])

(i) Can the restriction "$\mathcal{K}$ contains only a set of isomorphism types" be removed?

(ii) If $F$ has a representing formula $\varphi$, is it always possible to define $F$ by a formula whose parameters are those in $\varphi$ and those needed to define $\mathcal{K}$?

In §5, we apply techniques from many sorted models, and reformulate the mentioned result of Hodges-Shelah [10, Theorem 3] about naturality implies definablity.

Suppose $\tau$ is a vocabulary, and $\mathcal{K}$ is a class of $\tau$-models. Suppose $\mathfrak{B}_1, \mathfrak{B}_2 \in \mathcal{K}$ and

$$f_{\mathfrak{A}_1,\mathfrak{A}_2} : \mathfrak{A}_1 := \mathrm{sort}_1(\mathfrak{B}_1) \xrightarrow{\cong} \mathrm{sort}_1(\mathfrak{B}_2) =: \mathfrak{A}_2.$$

Assume we have some construction $F : \mathfrak{A}_i \mapsto \mathfrak{B}_i$. We say $F$ is uniform, if its definition is independent of the choose $\mathfrak{A}_i$, and the extended isomorphism $F(f_{\mathfrak{A}_1,\mathfrak{A}_2})$ in the following commutative diagram:

$$\begin{array}{ccc} \mathfrak{A}_1 & \xrightarrow{\subseteq} & F(\mathfrak{A}_1) \\ {\scriptstyle f_{\mathfrak{A}_1,\mathfrak{A}_2}} \downarrow & & \downarrow {\scriptstyle F(f_{\mathfrak{A}_1,\mathfrak{A}_2})} \\ \mathfrak{A}_2 & \xrightarrow{\subseteq} & F(\mathfrak{A}_2) \end{array}$$

is independent of $f_{\mathfrak{A}_1,\mathfrak{A}_2}$.

We prove the following solution to Problem 1.4:

**Theorem 1.5.** *Let $\tau$ be a vocabulary, and let $\mathcal{K}$ be a class of $\tau$-models which is first order definable from a parameter $\mathbf{p}$ such that every $\mathfrak{B} \in \mathcal{K}$ is two sorted and the natural homomorphism $\varphi : \mathrm{Aut}(\mathfrak{B}) \twoheadrightarrow \mathrm{Aut}(\mathrm{sort}_1(\mathfrak{B}))$ splits. Then there exists a class function*

$$F : \{\mathrm{sort}_1(\mathfrak{B}) : \mathfrak{B} \in \mathcal{K}\} \longrightarrow \mathcal{K},$$

*which is uniformly definable from the parameter $\mathbf{p}$ and $\mathrm{sort}_1(F(\mathfrak{A})) = \mathfrak{A}$, where $\mathfrak{A} = \mathrm{sort}_1(\mathfrak{B})$.*

We first show that the proof of Theorem 5.1 can be reduced to the case where $\mathcal{K}$ has only one equivalence class. Let $\mathfrak{A}$ be any model. There may be many $\mathfrak{B}$, and so many $F$, with $F(\mathfrak{A}) = \mathfrak{B}$ and $\mathrm{sort}_1(\mathfrak{B}) = \mathfrak{A}$. In general, the parameters that appear in $\mathfrak{B}$ and $F$ are wild, in the sense that they may be outside of $\mathbf{p}$. The main point of the uniformity theorem is that we can find a distinguished 2-sorted model $\mathfrak{B}$, we call it the universal model, whose definition only uses parameter $\mathbf{p}$. We construct the universal model (see Claim 5.11) with the idea of average from [10, Theorem 3]. By using this, we construct a function $F$ so that $\mathrm{sort}_1(F(\mathfrak{A})) = \mathfrak{A}$, and any parameter that appears in its construction comes from $\mathbf{p}$. One may regard Theorem 5.1 as a generalization of [10, Theorem 3].

For all unexplained definitions from model theory and category theory, see the books of Hodges [8] and Mac Lane [13], respectively. Also, for unexplained definitions from the theory of forcing see the book of Jech [12].

## § 2. The uni-construction problem

In this section we define the concepts of uni-construction problem, lifting, weak lifting and solvability. We use many sorted models, to unify and simplify our treatment.

Let us start by defining many-sorted model structures. They are a suitable vehicle for dealing with statements concerning different types of objects, which are ubiquitous in mathematics. For our purpose, we only consider $n$-sorted model structures, where $n \geq 1$ is a natural number.

**Definition 2.1.** An $n$-sorted model structure $\mathcal{M}$ is of the form

$$\mathcal{M} = \big(\{M_1; \ldots; M_n\}; R_1; \ldots; R_m; f_1; \ldots; f_k; c_1; \ldots; c_l\big),$$

where

(a) the universes $M_1; \ldots; M_n$ are nonempty,

(b) the relations $R_1; \ldots; R_m$ are between elements of the universes. In other words, for each $i$, there is some sequence $s(R_i) = \langle i_1, \ldots, i_s\rangle$ from $\{1, \ldots n\}$ such that

$$R_i \subseteq M_{i_1} \times \ldots \times M_{i_s},$$

(c) the functions $f_1; \ldots; f_k$ are between elements of the universes. In other words, for each $i$, there is some sequence $s(f_i) = \langle i_1, \ldots, i_s, r\rangle$ from $\{1, \ldots n\}$ such that

$$f_i : M_{i_1} \times \ldots \times M_{i_s} \to M_r,$$

(d) the distinguished constants $\{c_1; \ldots; c_l\}$ are in the universes, i.e., for each $i$ there is some $s(c_i) = j \in \{1, \cdots, l\}$ such that $c_i \in M_j$.

*Remark* 2.2. An ordinary single-sorted first order structure

$$\mathcal{M} = \big(M; R_1; \ldots; R_m; f_1; \ldots; f_k; c_1; \ldots; c_l\big)$$

can be identified with the 1-sorted model structure

$$\big(\{M\}; R_1; \ldots; R_m; f_1; \ldots; f_k; c_1; \ldots; c_l\big).$$

*Notation* 2.3. Given an $n$-sorted model structure $\mathcal{M}$ as above, for any non-empty set $I \subseteq \{1, \cdots, n\}$, we can define the $|I|$-sorted model structure $\mathrm{sort}_I(\mathcal{M})$ as

$$\mathrm{sort}_I(\mathcal{M}) = \big(\{M_i : i \in I\}; R_{i_1}; \ldots; R_{i_t}; f_{j_1}; \ldots; f_{j_p}; c_{k_1}; \ldots; c_{k_e}\big)$$

where

(a) $\{R_{i_1}; \ldots; R_{i_t}\}$ is a subset of $\{R_1; \ldots; R_m\}$, consisting of those $R_i$ such that $i_1, \cdots, i_s \in I$, where $s(R_i) = \langle i_1, \ldots, i_s\rangle$,

(b) $\{f_{j_1}; \ldots; f_{j_p}\}$ is a subset of $\{f_1; \ldots; f_k\}$, consisting of those $f_i$ such that $i_1, \cdots, i_s, r \in I$, where $s(f_i) = \langle i_1, \ldots, i_s, r\rangle$,

(c) $\{c_{k_1}; \ldots; c_{k_e}\}$ is a subset of $\{c_1; \ldots; c_l\}$, consisting of those $c_i$ such that $j \in I$, where $s(c_i) = j$.

In the case $I = \{i\}$, we denote the resulting structure by $\mathrm{sort}_i(\mathcal{M})$, and if $I = \{i, j\}$, where $1 \leq i < j \leq n$, then we denote the resulting structure by $\mathrm{sort}_{i,j}(\mathcal{M})$.

The concepts of homomorphism and isomorphism between $n$-sorted models can be defined naturally. We present them for completeness.

**Definition 2.4.** suppose $\mathcal{M} = \big(\{M_1; \ldots; M_n\}; R_1; \ldots; R_m; f_1; \ldots; f_k; c_1; \ldots; c_l\big)$ and $\mathcal{M}' = \big(\{M'_1; \ldots; M'_n\}; R'_1; \ldots; R'_m; f'_1; \ldots; f'_k; c'_1; \ldots; c'_l\big)$ are two $n$-sorted model structures of the same sorts.

(1) A homomorphism $\pi$ from $\mathcal{M}$ to $\mathcal{M}'$, denoted $\pi : \mathcal{M} \to \mathcal{M}'$, is a function $\pi : \bigcup_{i=1}^n M_i \to \bigcup_{i=1}^n M'_i$ such that:
   (a) For each $1 \leq i \leq n, \pi \restriction M_i : M_i \to M'_i$,
   (b) For each $1 \leq i \leq m$, if $s(R_i) = \langle i_1, \ldots, i_s \rangle$ and $(a_1, \cdots, a_s) \in M_{i_1} \times \cdots \times M_{i_s}$, then
$$R_i(a_1, \cdots, a_s) \Leftrightarrow R'_i(\pi(a_1), \cdots, \pi(a_s)),$$
   (c) For each $1 \leq i \leq k$, if $s(f_i) = \langle i_1, \ldots, i_s, r \rangle$ and $(a_1, \cdots, a_s) \in M_{i_1} \times \cdots \times M_{i_s}$, then
$$\pi(f_i(a_1, \cdots, a_s)) = f'_i(\pi(a_1), \cdots, \pi(a_s)),$$
   (d) For each $1 \leq i \leq l$, $\pi(c_i) = c'_i$.

(2) A homomorphism $\pi : \mathcal{M} \to \mathcal{M}'$ is an isomorphism, if for each $1 \leq i \leq n, \pi \restriction M_i : M_i \to M'_i$ is a bijection.

(3) An automorphism of $\mathcal{M}$ is an isomorphism $\pi : \mathcal{M} \to \mathcal{M}$, and denote this by $\pi \in \mathrm{Aut}(\mathcal{M})$.

In what follow, we only work with the concepts of 2-sorted and 3-sorted model structures.

**Discussion 2.5.** For a group $L$, by $e_L$ we mean the unit element. We denote the group operation by $\cdot$, so that for two elements $\ell_1, \ell_2 \in L$, their product is denoted by $\ell_1 \cdot \ell_2$. By $\ell^{-1}$ we mean the inverse of $\ell \in L$. As usual, if $L$ is abelian we use the additive notation $(L, +, -, 0)$.

*Notation* 2.6. Let $H$ be a group which is not necessarily abelian.

i) By $\mathcal{Z}(H)$ we mean the center of $H$, i.e.,
$$\mathcal{Z}(H) := \{h \in H : hx = xh \quad \forall x \in H\}.$$

ii) Since $\mathcal{Z}(H)$ is a normal subgroup of $H$, it induces a group structure on $H/\mathcal{Z}(H)$. By $\pi_H : H \to H/\mathcal{Z}(H)$ we mean the canonical group homomorphism from $H$ onto $H/\mathcal{Z}(H)$ defined by the assignment $h \mapsto h \cdot \mathcal{Z}(H)$.

iii) We denote the set of all group-homomorphisms from a given $G$ to $H$ by $\mathrm{Hom}(G,H)$. In other words, a function $f : G \to H$ belongs to $\mathrm{Hom}(G,H)$ if $f(g_1g_2) = f(g_1)f(g_2)$ for all $g_1, g_2 \in G$.

iv) By $\mathrm{id}_G \in \mathrm{Hom}(G,G)$ we mean the identity map.

**Definition 2.7.** Let $G$ and $H$ be two groups, which are not necessarily abelian. Assume $\varphi : H \twoheadrightarrow G$ is a surjective homomorphism of groups.

1) We say that $\psi \in \mathrm{Hom}(G,H)$ splits $\varphi$, when $\varphi \circ \psi = \mathrm{id}_G$.
2) We say that $\psi$ weakly splits $\varphi$ provided:
   (a) $\psi$ is a function from $G$ into $H$,
   (b) $\varphi \circ \psi = \mathrm{id}_G$,
   (c) the composite mapping $\pi_H \circ \psi$ belongs to $\mathrm{Hom}(G, H/\mathcal{Z}(H))$,
   (d) $\psi(x^{-1}) = (\psi(x))^{-1}$ and $\psi(e_G) = e_H$.
3) We say $\varphi$ has lifting (resp. weakly lifting), if some $\psi$ splits (resp. weakly splits) it.

We now define the concept of uni-construction problem, which plays a key role in this paper.

**Definition 2.8.** 1) We say $\mathbf{c}$ is a uni-construction problem (ucp in short), when

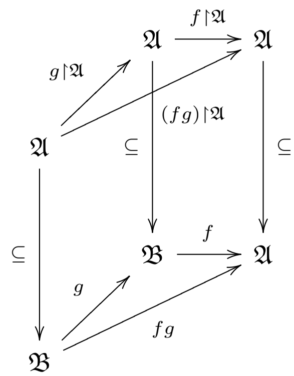
$$\mathbf{c} = \langle \mathfrak{B}_{\mathbf{c}}, \mathfrak{A}_{\mathbf{c}}, H_{\mathbf{c}}, G_{\mathbf{c}}, \varphi_{\mathbf{c}}, \psi_{\mathbf{c}} \rangle = \langle \mathfrak{B}, \mathfrak{A}, H, G, \varphi, \psi \rangle$$

and it satisfies the following conditions:

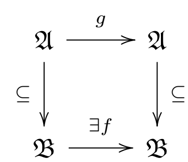
(a) $\mathfrak{B}$ is a two-sorted model,
(b) $\mathfrak{A} := \mathrm{sort}_1(\mathfrak{B})$,
(c) $H = \mathrm{Aut}(\mathfrak{B})$ and $G = \mathrm{Aut}(\mathfrak{A})$,
(d) $\varphi$ is the natural restriction map from $H$ into $G$, i.e. $\varphi(f) = f{\restriction}\mathfrak{A}$:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{f\restriction\mathfrak{A}} & \mathfrak{A} \\ \subseteq\big\downarrow & & \big\downarrow\subseteq \\ \mathfrak{B} & \xrightarrow{f} & \mathfrak{B} \end{array}$$

Furthermore, we assume that $\varphi$ is a group homomorphism. Let us depict the resulting commutative diagram:




(e) $\varphi$ is onto $G$. So, for any $g : \mathfrak{A} \to \mathfrak{A}$ there is an $f$ such that the following diagram commutes




(f) $\psi$ weakly splits $\varphi$.

2) We say $\mathbf{c}$ is a weak uni-constructive problem, if $\mathbf{c}$ is as above and it satisfies items (a)-(e) above.

**Definition 2.9.** Let $\mathbf{c}$ be a (weak) uni-construction problem. Then the classes $\mathcal{K}^1_{\mathbf{c}}$ and $\mathcal{K}^2_{\mathbf{c}}$ are defined as follows:

(a) $\mathcal{K}^1_{\mathbf{c}} := \{\mathfrak{A} : \mathfrak{A} \text{ isomorphic to } \mathfrak{A}_{\mathbf{c}}\}$,

(b) $\mathcal{K}^2_{\mathbf{c}} := \{\mathfrak{B} : \mathfrak{B} \text{ isomorphic to } \mathfrak{B}_{\mathbf{c}}\}$.

*Remark* 2.10. The assignment $\mathrm{sort}_1(\mathfrak{B}) \overset{F}{\mapsto} \mathfrak{B}$ on the domain of $\mathcal{K}^1_{\mathbf{c}}$ is not in general single-valued; but by clause (e) from Definition 2.8 it is single-valued up to isomorphism over $\mathfrak{B}$. This means $\mathfrak{B}_1 \cong \mathfrak{B}_2$ provided $\mathrm{sort1}(\mathfrak{B}_1) \cong \mathrm{sort}_1(\mathfrak{B}_2)$. In particular, suppose $\mathfrak{A} \cong \mathrm{sort}_1(\mathfrak{B})$. Then

$$F(\mathrm{sort}_1(\mathfrak{B})) = \mathfrak{B} \cong F(\mathfrak{A}).$$

**Definition 2.11.** Let $\mathbf{c}$ be a (weak) uni-construction problem.

(a) We say $\mathbf{c}$ is solvable, when there is a class function $F : \mathcal{K}^1_{\mathbf{c}} \to \mathcal{K}^2_{\mathbf{c}}$, such that for each $\mathfrak{B} \in \mathcal{K}^2_{\mathbf{c}}$, $F(\mathrm{sort}_1(\mathfrak{B})) = \mathfrak{B}$.

(b) Let $F$ be as in clause (a). We say $F$ is definable, if there is a formula $\theta$ such that

$$F(\mathfrak{A}) = \mathfrak{B} \Leftrightarrow \theta(\mathfrak{A}, \mathfrak{B}),$$

for all two sorted models $\mathfrak{B}$ with $\mathfrak{A} := \mathrm{sort}_1(\mathfrak{B})$.

(c) We say $\mathbf{c}$ is purely solvable, when there is an $F$ as above which is definable using only $(\mathfrak{B}_{\mathbf{c}}, \mathfrak{A}_{\mathbf{c}})$ as a parameter.

(d) We say $\mathbf{c}$ is $\chi$-solvable, when there is $F$ as above, which is definable by some parameter $a \in \mathscr{H}(\chi)$.

The notion of expansion or reduction between many sorted model structures is defined in the natural way. We give its definition for completeness.

**Definition 2.12.** Let $n_2, n_1 \geq 1$, and suppose

$$\mathfrak{B}_1 = \big(\{M^1_1; M^1_2; \cdots; M^1_{n_1}\}; R^1_1; \ldots; R^1_{m_1}; f^1_1; \ldots; f^1_{k_1}; c^1_1; \ldots; c^1_{l_1}\big)$$

and

$$\mathfrak{B}_2 = \big(\{M^2_1; M^2_2; \cdots; M^2_{n_2}\}; R^2_1; \ldots; R^2_{m_2}; f^2_1; \ldots; f^2_{k_2}; c^2_1; \ldots; c^2_{l_2}\big)$$

are many-sorted models. We say $\mathfrak{B}_2$ expands $\mathfrak{B}_1$, or $\mathfrak{B}_1$ is a reduction of $\mathfrak{B}_2$, if $\mathfrak{B}_1$ is obtained from $\mathfrak{B}_2$ by leaving out some sorts, relations, functions, and constants. In other words,

(a) $n_2 \geq n_1$ and $\{M^1_1; M^1_2; \cdots; M^1_{n_1}\} \subseteq \{M^2_1; M^2_2; \cdots; M^2_{n_2}\}$,

(b) For each $1 \leq i \leq m_1$, there is some $1 \leq j \leq m_2$ such that $R^1_i = R^2_j$,

(c) For each $1 \leq i \leq k_1$, there is some $1 \leq j \leq k_2$ such that $f^1_i = f^2_j$,

(d) For each $1 \leq i \leq l_1$, there is some $1 \leq j \leq l_2$ such that $c^1_i = c^2_j$.

Note that relations, functions, and constants which are not meaningful are at the same time dropped.

*Remark* 2.13. If $\mathcal{M}$ is an $n$-sorted model structure as in Definition 2.1 and $I \subseteq \{1, \cdots, n\}$ is non-empty, then $\mathrm{sort}_I(\mathcal{M})$, as defined in 2.3, is a reduction of $\mathcal{M}$.

Let us start with the following easy observation.

**Observation 2.14.** *Let* $\mathbf{c}$ *and* $\mathbf{d}$ *be two uni-construction problems, satisfying the following conditions:*

(a) $\mathfrak{A}_{\mathbf{c}} = \mathfrak{A}_{\mathbf{d}}$,

(b) $\mathfrak{B}_{\mathbf{d}}$ *expands* $\mathfrak{B}_{\mathbf{c}}$.

*If* $\mathbf{d}$ *is solvable, then* $\mathbf{c}$ *is solvable.*

*Proof.* Assume $\mathbf{d}$ is solvable, and let $F_{\mathbf{d}} : \mathcal{K}^1_{\mathbf{d}} \to \mathcal{K}^2_{\mathbf{d}}$ witness it. Let $\mathfrak{A} \in \mathcal{K}^1_{\mathbf{c}}$. Since $\mathcal{K}^1_{\mathbf{d}} = \mathcal{K}^1_{\mathbf{c}}$, we can define $F_{\mathbf{d}}(\mathfrak{A})$. Let $F_{\mathbf{c}}(\mathfrak{A})$ be the reduction of $F_{\mathbf{d}}(\mathfrak{A})$ into the language of $\mathfrak{B}_{\mathbf{c}}$. Then $F_{\mathbf{c}} : \mathcal{K}^1_{\mathbf{c}} \to \mathcal{K}^2_{\mathbf{c}}$ is well-defined, and it witnesses that $\mathbf{c}$ is solvable. □

## § 3. From uni-construction to naturality

The main result of this section is Theorem 3.14, which shows that for a given cardinal $\chi$, if a uni-construction problem has no lifting, then it has no $\chi$-solution in some forcing extension of the universe.

**Hypothesis 3.1.** Let $\mathfrak{C}$ be a 3-sorted model.

(i) Let us fix the following 2-sorted models $\mathbf{c}_{1,2}$, $\mathbf{c}_{2,3}$ and $\mathbf{c}_{1,3}$ via defining their first and second sorts:

(1) $\mathfrak{A}_{\mathbf{c}_{1,2}} = \mathrm{sort}_1(\mathfrak{C}), \mathfrak{B}_{\mathbf{c}_{1,2}} = \mathrm{sort}_{1,2}(\mathfrak{C})$,

(2) $\mathfrak{A}_{\mathbf{c}_{2,3}} = \mathrm{sort}_{1,2}(\mathfrak{C}), \mathfrak{B}_{\mathbf{c}_{2,3}} = \mathfrak{C}$,

(3) $\mathfrak{A}_{\mathbf{c}_{1,3}} = \mathrm{sort}_1(\mathfrak{C}), \mathfrak{B}_{\mathbf{c}_{1,3}} = \mathfrak{C}$.

(ii) We assume in addition to $(i)$ that $\mathbf{c}_{1,2}, \mathbf{c}_{2,3}$ and $\mathbf{c}_{1,3}$ are weak uni-construction problems. In particular, there are the following surjective group homomorphisms induced by the canonical restriction maps:

(1) $\varphi_{\mathbf{c}_{1,2}} : \mathrm{Aut}(\mathrm{sort}_{1,2}(\mathfrak{C})) \to \mathrm{Aut}(\mathrm{sort}_1(\mathfrak{C}))$,

(2) $\varphi_{\mathbf{c}_{2,3}} : \mathrm{Aut}(\mathfrak{C}) \to \mathrm{Aut}(\mathrm{sort}_{1,2}(\mathfrak{C}))$,

(3) $\varphi_{\mathbf{c}_{1,3}} : \mathrm{Aut}(\mathfrak{C}) \to \mathrm{Aut}(\mathrm{sort}_1(\mathfrak{C}))$.

**Lemma 3.2.** *(Transitivity) Let $\mathfrak{C}$ be as in Hypothesis 3.1. If $\mathbf{c}_{1,2}$ and $\mathbf{c}_{2,3}$ are solvable, then $\mathbf{c}_{1,3}$ is solvable.*

*Proof.* According to Definition 2.11 there are definable class functions

$F_{\mathbf{c}_{1,2}} : \mathcal{K}^1_{\mathbf{c}_{1,2}} \to \mathcal{K}^2_{\mathbf{c}_{1,2}}$, and

$F_{\mathbf{c}_{2,3}} : \mathcal{K}^1_{\mathbf{c}_{2,3}} \to \mathcal{K}^2_{\mathbf{c}_{2,3}}$.

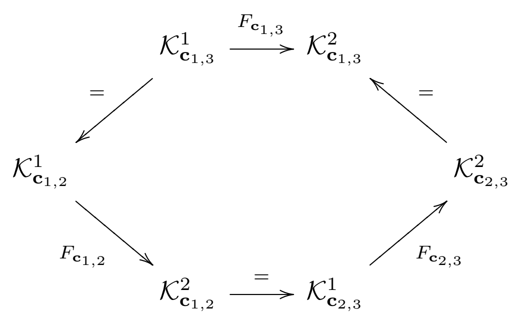
Recall from Hypothesis 3.1 that the following three equalities are satisfied

(i) $\mathcal{K}^1_{\mathbf{c}_{1,2}} = \mathrm{sort}_1(\mathfrak{C}) = \mathcal{K}^1_{\mathbf{c}_{1,3}}$,

(ii) $\mathcal{K}^1_{\mathbf{c}_{2,3}} = \mathrm{sort}_{1,2}(\mathfrak{C}) = \mathcal{K}^2_{\mathbf{c}_{1,2}}$,

(iii) $\mathcal{K}^2_{\mathbf{c}_{2,3}} = \mathfrak{C} = \mathcal{K}^2_{\mathbf{c}_{1,3}}$.

This enables us to get the composition $F_{\mathbf{c}_{1,3}} = F_{\mathbf{c}_{2,3}} \circ F_{\mathbf{c}_{1,2}}$. Let us summarize things in the following commutative diagram:




Then the function $F_{\mathbf{c}_{1,3}}$ witnesses that $\mathbf{c}_{1,3}$ is solvable. □

Let us reformulate the transitivity in terms of $\chi$-solvablity:

**Lemma 3.3.** *Let $\mathfrak{C}$ be as in Hypothesis 3.1. If $\mathbf{c}_{1,2}$ and $\mathbf{c}_{2,3}$ are $\chi$-solvable, then $\mathbf{c}_{1,3}$ is $\chi$-solvable.*

**Conclusion 3.4.** *Let $\mathfrak{C}$ be as in Hypothesis 3.1 and let $\chi$ be a cardinal. If $\mathbf{c}_{1,3}$ is not $\chi$-solvable and $\mathbf{c}_{2,3}$ is solvable. Then $\mathbf{c}_{1,2}$ is not $\chi$-solvable.*

*Proof.* Assume by the way of contradiction that $\mathbf{c}_{1,2}$ is $\chi$-solvable. It follows from Lemma 3.3 that $\mathbf{c}_{1,3}$ is $\chi$-solvable, a contradiction. □

**Corollary 3.5.** *Let $\mathfrak{C}$ be as in Hypothesis 3.1, and let $\chi$ be an infinite cardinal. Assume in addition that*

(a) $\varphi_{\mathbf{c}_{1,3}}$ *has no weak lifting, and*

(b) $\varphi_{\mathbf{c}_{2,3}}$ *has a lifting.*

*Then there is a forcing extension of the universe in which $\mathbf{c}_{1,2}$ is not $\chi$-solvable.*

*Proof.* Fix an infinite cardinal $\chi$. As $\varphi_{\mathbf{c}_{2,3}}$ has a lifting, in the light of [10, Theorem 3] we observe that it is solvable. Hence, for some cardinal $\chi' \geq \chi$, it is $\chi'$-solvable. As $\varphi_{\mathbf{c}_{1,3}}$ has no weak lifting, thus by [10, Theorem 4], we can find a generic extension $V[G]$ of the universe, in which $\varphi_{\mathbf{c}_{1,3}}$ is not $\chi'$-solvable. In view of Conclusion 3.4 we know that $\mathbf{c}_{1,2}$ is not $\chi'$-solvable. Hence not $\chi$-solvable in $V[G]$, as well. □

*Notation* 3.6. For a group $G$ and an automorphism $\psi$ of $G$, let:

- $\psi^0 := \mathrm{id}_G$ denote the identity automorphism,
- For $n > 0, \psi^n$ is the composition of $\psi, n$-times,
- For $n > 0$, $\psi^{-n}$ is defined as $\psi^{-n}(x) = (\psi^n(x))^{-1}$.

**Definition 3.7.** Let $G'$ be a group, and let $\psi$ be an automorphism of $G'$. We are going to define a group-structure over the following set:

$$G := \{y^n x : n \in \mathbb{Z} \textit{ and } x \in G'\},$$

via the following rules:

(a) The identity element is $y^0 e_{G'}$,

(b) The multiplication $(y^n x_1) \times (y^m x_2)$ is defined by $y^{n+m}(\psi^m(x_1)\psi^n(x_2))$, and recall that $\psi^0$ is the identity map.

(c) The inverse of $(y^n x)$ is $y^{-n}x^{-1}$, i.e.,

$$(y^n x) \times (y^{-n}x^{-1}) = y^{n-n}\psi^{-n}(x)\psi^n(x^{-1}) = y^0\psi^0(e_{G'}) = y^0 e_{G'} = e_G.$$

We denote the resulting group by $G := \mathbb{Z} \ltimes_\psi G'$.

*Remark* 3.8. Adopt the notation of Definition 3.7.

(1) The set $\{y^n e_{G'} \mid n \in \mathbb{Z}\}$ is a subgroup of $G$, and there is an isomorphism $\pi : (\mathbb{Z}, +) \to \{y^n e_{G'} \mid n \in \mathbb{Z}\}$ of groups, where $\pi(n) := y^n e_{G'}$.

(2) The assignment $x \mapsto y^0 x$ defines a morphism $G' \to G$.

(3) For simplicity, let us denote $y^0 x = x$ and $ye_{G'} = y$. This shows that

$$x \times y^n := y^0 x \times y^n e_{G'} = y^{0+n}(\psi^n(x)\psi^0(e_{G'})) = y^n \psi^n(x).$$

Similarly, by using the above identification, we have $y^{-n}xy^n = \psi^n(x)$.

The above construction is a special case of HNN extensions, where HNN is an abbreviation for Higman-Neumann-Neumann. Here, is an algebraic definition equipped with a topological motivation:

**Discussion 3.9.** (See [7, §11: HNN extensions])

(i) Suppose $G$ is a group, $A$ and $B$ are two isomorphic subgroups of $G$ and $f : A \xrightarrow{\cong} B$. Then the group having the presentation

$$(G; p \mid p^{-1}ap = f(a) \quad \forall a \in A)$$

is called an HNN extension of $G$;

(ii) Consider a connected topological space X with homeomorphic disjoint subspaces A and B, and let $f : A \to B$ be a homeomorphism. Let $\hat{X}$ be $X$ that we add a handle from $A$ to $B$. Then fundamental group of $\hat{X}$ is the HNN extension over the fundamental group of $X$ (see [7, Theorem 11.75]).

*Notation* 3.10. Suppose $\{H_i : i \in I\}$ is a family of groups. Let $H$ be the family of all sequences $(h_i)_{i\in I}$ so that $h_i \in H_i$ with the property that $\{i : h_i \neq e_{H_i}\}$ is finite, and we define

$$\text{supp}((h_i)_{i\in I}) := \{i : h_i \neq e_{H_i}\}.$$

Now, take $(h_i), (g_i) \in H$ and define $(h_i)_{i\in I} \times (g_i)_{i\in I} := (h_i g_i)_{i\in I} \in H$. By this multiplicative operation, $H$ equips with a structure of group. A common notation for $H$, is $\bigoplus_{i\in I} H_i$.

**Proposition 3.11.** *Let $G_1$ and $G_2$ be two groups and suppose there is an onto homomorphism $\varphi_{1,2} : G_2 \twoheadrightarrow G_1$ that has no lifting but has a weak lifting. Then, there are $G_3$ and $\varphi_{2,3}$, such that:*

*(a) $G_3$ is a group;*

*(b)* $\varphi_{2,3} : G_3 \twoheadrightarrow G_2$ *is a surjective homomorphism which has a lifting;*

*(c)* $\varphi_{1,3} = \varphi_{1,2} \circ \varphi_{2,3} \in \mathrm{Hom}(G_3, G_1)$ *is a surjective homomorphism with no weak lifting.*

*Proof.* (a): First, we introduce the auxiliary group $G_3' := \bigoplus\{G_{2,n} : n \in \mathbb{Z}\}$, where $G_{2,n} \cong G_2$ for each $n \in \mathbb{Z}$. Let also $\psi_n$ denote the corresponding isomorphism $\psi_n : G_2 \to G_{2,n}$. Note that:

(1) $\bigcup\limits_{n\in\mathbb{Z}} G_{2,n}$ generates $G_3'$,

(2) $G_{2,n} \cap \Sigma\{G_{2,k} : k \in \mathbb{Z}\backslash\{n\}\} = \{e_{G_3'}\}$, and

(3) the family of groups $\{G_{2,n} : n \in \mathbb{Z}\}$ pairwise commutes.

Let $\psi_*$ be the automorphism of $G_3'$ such that, for each $m$,

$$\psi_* {\restriction} G_{2,m} = \psi_{m+1} \circ \psi_m^{-1}.$$

Set $G_3 := \mathbb{Z} \ltimes_{\psi_*} G_3'$. Recall from Definition 3.7 that $G_3$ is generated by $G_3' \cup \{y\}$, subject to the following relations

$$y^{-1}xy = \psi_*(x) \quad \forall x \in G_3'.$$

(b): Let $\varphi_{2,3} : G_3 \to G_2$ be the unique homomorphism from $G_3$ onto $G_2$ such that:

- $\bullet_1$ $\varphi_{2.3}{\restriction}G_{2,n} = \psi_n^{-1}$ for $n \in \mathbb{Z}$,
- $\bullet_2$ $\varphi_{2.3}(y) = e_{G_2}$.

Now, $\psi_0$ is clearly a lifting of $\varphi_{2,3}$, i.e. clause (b) holds.

($c$): In order to prove clause (c), we first prove the following claim.

**Claim 3.12.** *$G_3$ has trivial center.*

*Proof.* Suppose $x \in G_3'\backslash\{e_{G_3'}\}$. Clearly $y^{-1}xy \neq x$, for example using $\mathrm{supp}(x)$ (see Notation 3.10). Suppose $\alpha \in G_3\backslash G_3'$. According to Definition 3.7, $\alpha$ has the form $y^n x_1$ with $n \in \mathbb{Z}, x_1 \in G_3'$ and $n \neq 0 \vee x_1 \neq e_{G_3'}$. Let $x_2 \in G_2'\backslash\{e_{G_2'}\}$ be such that

$$\ell := \min(\mathrm{supp}(x_2)) > \max(\mathrm{supp}(x_1)) + n.$$

Suppose on the way of contradiction that $y^n x_1$ and $x_2$ commute with each other. This yields that

$$\begin{aligned} y^n(x_1x_2) &= (y^nx_1)x_2 \\ &= x_2(y^nx_1) \\ &= x_2(yy^{n-1}x_1) \\ &= (x_2y)y^{n-1}x_1 \\ &= (y\psi_*(x_2))y^{n-1}x_1 \\ &= y^n(\psi_*^n(x_2)x_1). \end{aligned}$$

By definition, this implies

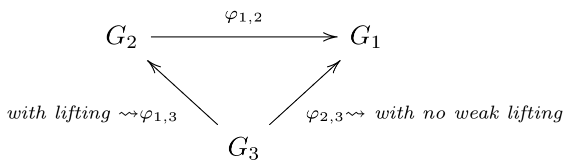
$$\psi_*^n(x_2)x_1 = x_1x_2 \quad (+)$$

Recall that $\psi_*\restriction G_{2,m} = \psi_{m+1}\circ\psi_m^{-1}$. We combine this equality along with (+), to see some coordinate of its left hand side is nonzero, but the corresponding coordinate in the right hand side is nonzero. This contradiction shows that $G_3$ has trivial center, as claimed. □

Now, we proceed the proof of the proposition. By definition $\varphi_{1,3} \in \mathrm{Hom}(G_3, G_1)$ is onto. We show that it has no weak lifting. Suppose towards contradiction that there is a function $\psi : G_1 \to G_3$ such that $\varphi_{1,3}\circ\psi = \mathrm{id}_{G_1}$ and the composite mapping $\pi_3\circ\psi$ belongs to $\mathrm{Hom}(G_1, G_3/\mathcal{Z}(G_3))$, where $\pi_3 : G_3 \to G_3/\mathcal{Z}(G_3)$ is the canonical homomorphism. But, according to Claim 3.12, $G_3$ has trivial center. So $\psi$ is a homomorphism from $G_1$ into $G_3$, hence $\varphi_{2,3}\circ\psi$ is a homomorphism from $G_1$ into $G_2$; in fact, is one to one as $\varphi_{1,3}\circ\psi$ is. This contradicts our assumption that $\varphi_{1,2}$ has no lifting. □

**Corollary 3.13.** *Let $\chi$ be a cardinal. Suppose $\varphi \in \mathrm{Hom}(G_2, G_1)$ is onto, and does not split. Then there exists a weak uni-construction problem* $\mathbf{c}$*, such that in some forcing extension,* $\mathbf{c}$ *is not $\chi$-solvable.*

*Proof.* Let $G_1$ and $G_2$ be as above and set $\varphi_{1,2} := \varphi$. By Proposition 3.11, we can find a group $G_3$ and two homomorphisms $\varphi_{2,3}$ and $\varphi_{1,3}$, which fit in the following diagram




Without loss of generality the groups $G_1, G_2$ and $G_3$ are pairwise disjoint. We define the three sorted model $\mathfrak{C}$ as follows:

- (∗) (a) the set of elements of $\mathrm{sort}_\ell(\mathfrak{C})$ is the set of elements of $G_\ell$ for $\ell = 1, 2, 3$;
  - (b) $F^{\mathfrak{C}}_\ell = \varphi_{\ell,\ell+1}$ for $\ell = 1, 2$;
  - (c) for $\ell = 1, 2, 3$ and $a \in G_\ell$, the homomorphism $F^{\mathfrak{C}}_{\ell,a} : G_\ell \to G_\ell$ is defined as $F^{\mathfrak{C}}_{\ell,a}(b) = a \cdot b$ for $b \in G_\ell$.

Now, we show that the assumptions of Corollary 3.5 hold. To this end, note that we consider the sorts only as a set of elements and forget about the multiplication in $G_i$. The entire structure is given by the functions. We bring the following two claims:

- (i) $G_1 \cong \mathrm{Aut}(\mathrm{sort}_1(\mathfrak{C})), G_2 \cong \mathrm{Aut}(\mathrm{sort}_{1,2}(\mathfrak{C}))$ and $G_3 \cong \mathrm{Aut}(\mathfrak{C})$ and
- (ii) $\varphi_{i,j} : \mathrm{Aut}(\mathrm{sort}_{i,j}(\mathfrak{C})) \to \mathrm{Aut}(\mathrm{sort}_i(\mathfrak{C}))$ is the natural restriction map.

For clause (i), we only show that $G_3 \cong \mathrm{Aut}(\mathfrak{C})$, as the other cases can be proved in a similar way.

In order to see $G_3 \cong \mathrm{Aut}(\mathfrak{C})$, define the map

$$\theta : G_3 \to \mathrm{Aut}(\mathfrak{C})$$

which sends some element $c \in G_3$ to the map $\sigma_c : \mathfrak{C} \to \mathfrak{C}$, which is defined via

$$\sigma_c(g) = \begin{cases} c \cdot g & \text{if } g \in G_3 \\ \varphi_{2,3}(c) & \text{if } g \in G_2 \\ \varphi_{1,3}(c) & \text{if } g \in G_1 \end{cases}$$

We have to show that $\theta$ is well-defined. To see this, let $c \in G_3$. First, we show that $\theta(c)$ is a homomorphism of $\mathfrak{C}$. We consider to $a \in G_3$, and show $\sigma_c$ behaves well with respect to $F^{\mathfrak{C}}_{a,3}$, in the following sense:

$$\sigma_c(F^{\mathfrak{C}}_{a,3}(b)) = c \cdot b \cdot a = \sigma_c(b) \cdot a = F^{\mathfrak{C}}_{a,3}(\sigma_c(b)),$$

where, $b \in G_3$. Furthermore, it behaves well with respect to $F^{\mathfrak{C}}_{2,3}$:

$$\begin{aligned}
\sigma_c(F^{\mathfrak{C}}_{2,3}(a)) &= \sigma_c(\varphi_{2,3}(a)) \\
&= \varphi_{2,3}(c) \cdot \varphi_{2,3}(a) \\
&= \varphi_{2,3}(c \cdot a) \\
&= \varphi_{2,3}(\sigma_c(a)) \\
&= F^{\mathfrak{C}}_{2,3}(\sigma_c(a)).
\end{aligned}$$

We apply similar arguments for $a, b \in G_i$ with $i = 1, 2$. Whence $\sigma_c$ is indeed a homomorphism of $\mathfrak{C}$. It is easy to see that each $\theta(c)$ is a bijection, as it corresponds to left translation in the groups. Thus $\theta(c) \in \mathrm{Aut}(\mathfrak{C})$, and $\theta$ is well-defined.

Now, it remains to show that $\theta$ is an isomorphism of groups. In order to be an homomorphism, it has to respect the function symbols. Let $c_1, c_2 \in G_3$ be arbitrary. Then

$$\theta(c_1 \cdot c_2)(a) = c_1 \cdot c_2 \cdot a = \theta(c_1)(c_2 \cdot a) = (\theta(c_1)\theta(c_2))(a),$$

for any $a \in G_3$ and similarly for $a \in G_i$ with $i = 1, 2$.

Clearly, $\theta(c^{-1}) = \theta(c)^{-1}$. Finally, the map $\theta$ is injective, as for $c \neq c'$ and $a \in G_3$, we have that

$$\sigma_c(a) = c \cdot a \neq c' \cdot a = \sigma_{c'}(a).$$

For surjectivity, consider an arbitrary automorphism $\sigma \in \mathrm{Aut}(\mathfrak{C})$. It is easy to check that for $c = \sigma(e_{G_3})$, we have $\sigma = \sigma_c$, by using the identity

$$\sigma(F^{\mathfrak{C}}_a(e_{G_3})) = F^{\mathfrak{C}}_a(\sigma(e_{G_3})).$$

This concludes that $\theta : G_3 \to \mathrm{Aut}(\mathfrak{C})$ is an isomorphism of groups.

To prove clause (ii), we have to show that the maps $\varphi_{i,j}$ correspond to the natural restriction maps between the automorphism groups. This is immediate, as we saw that any automorphism of $\mathfrak{C}$ corresponds to the right translation on each of the sorts and if $\sigma$ restricted to the $j^{th}$ sort is the translation by $b \in G_j$, then it is translation by $\varphi_{i,j}(b)$ on the sort $i$.

In the light of Proposition 3.11, and in view of the way we defined the 3-sorted model $\mathfrak{C}$, we observe that $\varphi_{\mathbf{c}_{1,3}}$ has no weak lifting, and $\varphi_{\mathbf{c}_{2,3}}$ has a lifting. This allows us to apply Corollary 3.5 and deduce that $\mathbf{c}_{1,2}$ is not $\chi$-solvable in some forcing extension. So, it is enough to take $\mathbf{c} := \mathbf{c}_{1,2}$, and the corollary follows. □

Now, we are ready to prove the main result of this section.

**Theorem 3.14.** *Let $\chi$ be a cardinal and $\mathbf{c}$ be a uni-construction problem. If $\mathbf{c}$ has no lifting, then in some forcing extension, $\mathbf{c}$ has no $\chi$-solution.*

*Proof.* Recall that $G_{\mathbf{c}} := \mathrm{Aut}(\mathfrak{A}_{\mathbf{c}})$ and $H_{\mathbf{c}} := \mathrm{Aut}(\mathfrak{B}_{\mathbf{c}})$. We set $G_1 := G_{\mathbf{c}}, G_2 := H_{\mathbf{c}}$ and $\varphi_{1,2} := \varphi_{\mathbf{c}}$. In the light of Proposition 3.11 we can find a group $G_3$, an a surjective homomorphism $\varphi_{2,3} : G_3 \twoheadrightarrow G_2$ which has a lifting such that the homomorphism $\varphi_{1,3} = \varphi_{1,2} \circ \varphi_{2,3} \in \mathrm{Hom}(G_3, G_1)$ has no weak lifting. Without loss of generality we may assume that

$$a \in \mathfrak{B}_{\mathbf{c}} \wedge b \in G_3 \Rightarrow a \neq b.$$

Let $\langle a_\alpha : \alpha < \alpha_* \rangle$ list the elements of $\mathfrak{B}_{\mathbf{c}}$. We define a 3-sorted model $\mathfrak{C}$ as follows:

(∗) (a) $\mathrm{sort}_{1,2}(\mathfrak{C}) = \mathfrak{B}_{\mathbf{c}}$ so $\mathrm{sort}_1(\mathfrak{C}) = \mathfrak{A}_{\mathbf{c}}$;

(b) the set of elements of $\mathrm{sort}_3(\mathfrak{C})$ is the set of elements of $G_3$;

(c) $F^{\mathfrak{C}}_{1,\alpha} : G_3 \to \mathrm{sort}_1(\mathfrak{C})$ is defined by the help of $\varphi_{1,3}$. More precisely, for any $b \in G_3$, we set $F^{\mathfrak{C}}_{1,\alpha}(b) = (\varphi_{1,3}(b))(a_\alpha)$;

(d) $F^{\mathfrak{C}}_{2,\alpha} : G_3 \to \mathfrak{B}_{\mathbf{c}}$ is defined by:

$$b \in G_3 \Rightarrow F^{\mathfrak{C}}_{2,\alpha}(b) = (\varphi_{2,3}(b))(a_\alpha);$$

(e) $F^{\mathfrak{C}}_{3,c} : G_3 \to G_3$, where $c \in G_3$, is defined by:

$$b \in G_3 \Rightarrow F^{\mathfrak{C}}_{3,c}(b) = cb.$$

By the same vein as in the proof of Corollary 3.13, we observe that

(∗∗) (a) $\mathfrak{A}_{\mathbf{c}_{1,2}} = \mathrm{sort}_1(\mathfrak{C}), \mathfrak{B}_{\mathbf{c}_{1,2}} = \mathrm{sort}_{1,2}(\mathfrak{C})$,

(b) $\mathfrak{A}_{\mathbf{c}_{2,3}} = \mathrm{sort}_{1,2}(\mathfrak{C}), \mathfrak{B}_{\mathbf{c}_{2,3}} = \mathfrak{C}$,

(c) $\mathfrak{A}_{\mathbf{c}_{1,3}} = \mathrm{sort}_1(\mathfrak{C}), \mathfrak{B}_{\mathbf{c}_{1,3}} = \mathfrak{C}$.

By our construction, it is easily seen tha $\varphi_{\mathbf{c}_{1,3}}$ has no weak lifting, and $\varphi_{\mathbf{c}_{2,3}}$ has a lifting.

In view of Corollary 3.5 we are able to find a forcing extension $V[G]$ of the universe in which $\mathbf{c}_{1,2}$ has no $\chi$-solution. But $\mathbf{c}_{1,2} = \mathbf{c}$, and hence $\mathbf{c}$ has no $\chi$-solution in $V[G]$. The theorem follows. $\square$

## § 4. A global consistency result

Let $\mathbf{c}$ be a definable uni-construction problem via a formula $\theta$. Recall that there is a class function $F : \mathcal{K}^1_{\mathbf{c}} \to \mathcal{K}^2_{\mathbf{c}}$, such that for each $\mathfrak{B} \in \mathcal{K}^2_{\mathbf{c}}$, $F(\mathrm{sort}_1(\mathfrak{B})) = \mathfrak{B}$ and that

$$F(\mathfrak{A}) = \mathfrak{B} \Leftrightarrow \theta(\mathfrak{A}, \mathfrak{B}),$$

for all two sorted models $\mathfrak{B}$ with $\mathfrak{A} := \mathrm{sort}_1(\mathfrak{B})$.

**Definition 4.1.** Let $c$ be as above. We say $\mathbf{c}$ is uniformly definable if there is a formula $\Theta$ so that

$$\exists y \theta(y, \mathfrak{B}) \to \exists y \Theta(y, \mathfrak{B}) \wedge \forall y (\Theta(y, \mathfrak{B}) \to \theta(y, \mathfrak{B})).$$

**Definition 4.2.** For an infinite cardinal $\lambda$, let $\mathbb{S}_\lambda$ be the forcing notion

$$\mathbb{S}_\lambda = \{p : \lambda^{++} \times \lambda^{++} \times \lambda^{++} \to 2 : |p| \leq \lambda\},$$

ordered by reverse inclusion.

Thus $\mathbb{S}_\lambda$ is forcing equivalent to $\mathrm{Add}(\lambda^+, \lambda^{++})$, the Cohen forcing for adding $\lambda^{++}$-many Cohen subsets of $\lambda^+$. Furthermore, in view of [11, Lemma 5.2], we observe that it is $\lambda^+$-closed and satisfies the $\lambda^{++}$-c.c.

**Fact 4.3.** (Hodges-Shelah) Let $M$ be an inner model of $\mathrm{ZFC} + \mathrm{GCH}$ and let $\lambda$ be an infinite cardinal of $M$. Let $\mathbb{Q}$ be the forcing notion $\mathbb{S}_\lambda$ as computed in $M$ and let $\mathbf{G}$ be $\mathbb{Q}$-generic over $M$. Then the following holds in $M[\mathbf{G}]$:

$(*)_\lambda$ : suppose $\mathbf{c}$ is a uniformisable uni-construction problem, such that $\mathbf{c}$ is defined using parameters from $V$, $\mathcal{B}_{\mathbf{c}} \in V$ and $\mathcal{B}_{\mathbf{c}}$ and $\mathrm{Aut}(\mathcal{B}_{\mathbf{c}})$ have size $\leq \lambda$. Then $\mathbf{c}$ is weakly natural.

*Proof.* This is in [11, Theorem 5.1]. $\square$

In this section we are going to prove a global version of this theorem, which removes both the cardinality assumption and the parameter assumption from the above result. The proof uses the reverse Easton iteration of forcing notions, where we refer to [2] and [12, Chapter 21] for more details on this subject.

**Theorem 4.4.** *Let* $\mathbf{c}$ *be a uniformisable uni-construction problem. There exists a GCH and cofinality preserving class generic extension* $V[\mathbf{G}]$ *of the universe in which* $\mathbf{c}$ *is weakly natural.*

*Proof.* For a given an infinite cardinal $\lambda$, let $\mathbb{S}_\lambda$ be as Definition 4.2. Let

$$\mathbb{P} = \big\langle \langle \mathbb{P}_\lambda : \lambda \in \mathrm{Ord} \rangle, \langle \underset{\sim}{\mathbb{Q}}_\lambda : \lambda \in \mathrm{Ord} \rangle \big\rangle$$

be the reverse Easton iteration of forcing notions, such that for each ordinal $\lambda$, $\underset{\sim}{\mathbb{Q}}_\lambda$ is forced to be the trivial forcing notion except $\lambda$ is an infinite cardinal, in which case we let

$$\Vdash_{\mathbb{P}_\lambda} \text{“}\underset{\sim}{\mathbb{Q}}_\lambda = \underset{\sim}{\mathbb{S}}_\lambda\text{”}.$$

Thus a condition in $\mathbb{P}$ is a partial function $p$ such that:

(1) $\mathrm{dom}(p)$ is a set of ordinals,
(2) if $\lambda \in \mathrm{dom}(p)$, then $p \restriction \lambda \Vdash_{\mathbb{P}_\lambda}$ “$p(\lambda) \in \underset{\sim}{\mathbb{Q}}_\lambda$”,
(3) for any regular cardinal $\kappa$, $|\mathrm{supp}(p) \cap \kappa| < \kappa$, where

$$\mathrm{supp}(p) := \big\{\lambda \in \mathrm{dom}(p) : p \restriction \lambda \Vdash_{\mathbb{P}_\lambda} \text{“}p(\lambda) \neq 1_{\underset{\sim}{\mathbb{Q}}_\lambda}\text{”}\big\}.$$

Let also

$$\mathbf{G} = \big\langle \langle \mathbf{G}_\lambda : \lambda \in \mathrm{Ord} \rangle, \langle \mathbf{H}_\lambda : \lambda \in \mathrm{Ord} \rangle \big\rangle$$

be $\mathbb{P}$-generic over $V$. Thus for each infinite cardinal $\lambda$, $\mathbf{G}_\lambda = \mathbb{P}_\lambda \cap \mathbf{G}$ is $\mathbb{P}_\lambda$-generic over $V$ and $H_\lambda$ is $\underset{\sim}{\mathbb{S}}_\lambda[\mathbf{G}_\lambda]$-generic over $V[\mathbf{G}_\lambda]$. We are going to show that $V[\mathbf{G}]$ is as required. To this end, let us recall the following well-known result from [2] (also see [12, Chapter 21]).

**Fact 4.5.** Adopt the above notation. Then the following assertions are valid:

(1) $V[\mathbf{G}]$ is a GCH and cofinality preserving class generic extension of $V$,

(2) $V[\mathbf{G}]$ and $V[\mathbf{G}_\lambda]$ contain the same $\lambda$-sequences of ordinals.

Recall that $\mathbf{c}$ is a uniformisable uni-construction problem. Let $\lambda$ be a large enough cardinal such that $\mathcal{B}_\mathbf{c}$, $\mathrm{Aut}(\mathcal{B}_\mathbf{c})$ and the parameters occurring in the definition of $\mathbf{c}$ and the formula uniformizing it are all in $V[\mathbf{G}_\lambda]$, and $|\mathcal{B}_\mathbf{c}|, |\mathrm{Aut}(\mathcal{B}_\mathbf{c})| \leq \lambda$. By Fact 4.3, applied to the model $V[\mathbf{G}_\lambda]$ and the uni-construction problem $\mathbf{c}$, we conclude that

$$V[\mathbf{G}_\lambda][H_\lambda] \models \text{"}\mathbf{c} \text{ is weakly natural"}.$$

On the other hand, $V[\mathbf{G}]$ is a generic extension of $V[\mathbf{G}_\lambda][H_\lambda]$ by a class forcing notion which adds no new subsets to $\lambda^+$. Thus

$$V[\mathbf{G}] \models \text{"}\mathbf{c} \text{ is weakly natural"}.$$

The theorem follows. $\square$

## § 5. Uniformity

In this section we deal with Problem 1.4. Our main result is Theorem 5.1, which can be considered as a generalization of [10, Theorem 3]. We state our result in terms of two sorted models.

**Theorem 5.1.** *Let $\tau$ be a vocabulary, and let $\mathcal{K}$ be a class of $\tau$-models such that*

- *(i) $\mathcal{K}$ is first order definable from a parameter $\mathbf{p}$,*
- *(ii) every $\mathfrak{B} \in \mathcal{K}$ is two sorted,*
- *(iii) the natural homomorphism $\varphi_\mathfrak{B} : \mathrm{Aut}(\mathfrak{B}) \twoheadrightarrow \mathrm{Aut}(\mathrm{sort}_1(\mathfrak{B}))$ splits.*

*Then there exists a class function $F$, which is uniformly definable from the parameter $\mathbf{p}$ and furnished the following assertions:*

- *(a) the domain of $F$ is $\mathcal{K}_1 = \{\mathrm{sort}_1(\mathfrak{B}) : \mathfrak{B} \in \mathcal{K}\}$,*
- *(b) if $\mathfrak{A} \in \mathcal{K}_1$, then $F(\mathfrak{A}) \in \mathcal{K}$ and $\mathrm{sort}_1(F(\mathfrak{A})) = \mathfrak{A}$.*

*Proof.* We prove the theorem in a sequence of claims. We first show that we can be reduced to the case where $\mathcal{K}$ has only one equivalence class. To this end, define the class function $\mathbf{H}$, with $\mathrm{dom}(\mathbf{H}) = \mathcal{K}$, by

$$\mathbf{H}(\mathfrak{B}) := \Big\{\mathfrak{B}' : \mathfrak{B}' \text{ is a } \tau\text{-model isomorphic to } \mathfrak{B} \text{ with universe the cardinal } \|\mathfrak{B}\|\Big\}.$$

Clearly, $\mathbf{H}$ is definable from the parameter $\mathbf{p}$. For every $\mathbf{x} \in \mathrm{Range}(\mathbf{H})$, set

(1) $\mathcal{K}_{\mathbf{x}} := \{\mathfrak{B} \in \mathcal{K} : \mathbf{H}(\mathfrak{B}) = \mathbf{x}\}$,

(2) $\mathcal{K}_{\mathbf{x},1} := \{\mathrm{sort}_1(\mathfrak{B}) : \mathfrak{B} \in \mathcal{K}_{\mathbf{x}}\}$.

We remark that $\langle K_{\mathbf{x}} : \mathbf{x} \in \mathrm{Rang}(\mathbf{H})\rangle$ is a partition of $\mathcal{K}$, which is uniformly definable using the parameter $\mathbf{p}$, so it suffices to uniformly deal with $\mathcal{K}_{\mathbf{x}}$ for each $\mathbf{x} \in \mathrm{Range}(\mathbf{H})$. Thus, let us fix some $\mathbf{x} \in \mathrm{Range}(\mathbf{H})$.

Now, we consider to $\Upsilon$. By definition, it is the family of all pairs $(\mathfrak{B}, \psi)$ equipped with the following properties:

$\Upsilon_1$) $\mathfrak{B} \in \mathcal{K}$ is such that for some $\mathfrak{B}' \in \mathbf{x}$ the function $b \mapsto (\mathfrak{B}', b)$, for $b \in \mathfrak{B}'$, is an isomorphism from $\mathfrak{B}'$ onto $\mathfrak{B}$.

$\Upsilon_2$) $\psi$ is a weak lifting corresponds to $\mathfrak{B}$, more precisely a lifting for $\varphi_{\mathfrak{B}}$.

Let us collect all of them with a new name:

(3) Let $\mathbf{y}_{\mathbf{x}} := \big\{(\mathfrak{B}, \psi) : (\mathfrak{B}, \psi) \in \Upsilon\big\}$.

To make things be easier, we bring a series of claims (see Claim 5.2-5.11):

**Claim 5.2.** $\mathbf{y}_{\mathbf{x}}$ *is non-empty.*

*Proof.* By our hypothesis, for each $\mathfrak{B} \in \mathcal{K}$, there is a weak lifting for $\varphi_{\mathfrak{B}}$, and hence $\mathbf{y}_{\mathbf{x}}$ is non-empty, as requested. □

Let $S = \mathbf{y}_{\mathbf{x}}$, and for each $s \in S$ set $(\mathfrak{B}_s, \psi_s) = s$. It then follows that $\mathbf{y}_{\mathbf{x}} = \{(\mathfrak{B}_s, \psi_s) : s \in S\}$. For any $s \in S$, we set

(4) $\mathfrak{A}_s := \mathrm{sort}_1(\mathfrak{B}_s)$.

By replacing each $\mathfrak{B}_s$ by $\mathfrak{B}_s \times \{s\}$ if necessary, we may assume that the $\mathfrak{B}_s$'s are pairwise disjoint. For each $b \in \bigcup_{s \in S} \mathfrak{B}_s$, we let

$$r(b) := \text{the unique } s \in S \text{ such that } b \in \mathfrak{B}_s.$$

**Claim 5.3.** $\mathbf{x}$ *is definable from* $\mathbf{y}_{\mathbf{x}}$.

*Proof.* Recall that $\mathbf{x}$ is equal to

$$\Big\{\mathfrak{B}' : \exists(\mathfrak{B},\psi) \in \mathbf{y}_{\mathbf{x}}\big(\textit{the function } b \mapsto (\mathfrak{B}', b) \textit{ induces } \mathfrak{B}' \xrightarrow{\cong} \mathfrak{B}\big)\Big\},$$

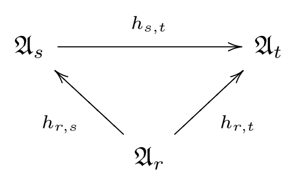
which concludes the result. □

Let $\langle h_{s,t} : \mathfrak{A}_s \xrightarrow{\cong} \mathfrak{A}_t\rangle_{s,t\in S}$ be a family of isomorphisms. Let us introduce the following terminology:

**Definition 5.4.** We say the $\langle h_{s,t} : \mathfrak{A}_s \xrightarrow{\cong} \mathfrak{A}_t\rangle_{s,t\in S}$ is a commutative $\mathfrak{A}_\bullet$-family provided:

- $h_{r,s} \circ h_{s,t} = h_{r,t}$ and
- $h_{s,s} = \mathrm{id}_{\mathfrak{A}_s}$,

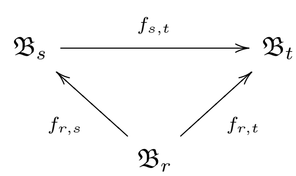
where $r, s, t \in S$. In other terms, we have the following commutative diagram:




(5) By the first automorphism class of $\mathbf{y}_{\mathbf{x}}$ we mean

$$\mathrm{Aut}_1(\mathbf{y}_{\mathbf{x}}) := \big\{\bar{h} = \langle h_{s,t} : s,t \in S\rangle : \bar{h} \textit{ is a commutaive } \mathfrak{A}_\bullet - \textit{family}\big\}.$$

In the same vein, we say the family $\langle f_{s,t} : \mathfrak{B}_s \xrightarrow{\cong} \mathfrak{B}_t\rangle_{s,t\in S}$ of isomorphisms is a commutative $\mathfrak{B}_\bullet$-family provided $f_{r,s}\circ f_{s,t} = f_{r,t}$ and $f_{s,s} = \mathrm{id}_{\mathfrak{B}_s}$ for all $r, s, t \in S$. So, the diagram




is commutative. The second automorphism class of $\mathbf{y}_{\mathbf{x}}$ is:

(6) $\mathrm{Aut}_2(\mathbf{y}_{\mathbf{x}}) := \big\{\bar{f} = \langle f_{s,t} : s,t \in S\rangle : \bar{f} \textit{ is a commutaive } \mathfrak{B}_\bullet - \textit{family}\big\}.$

For $\mathfrak{A} \in \mathcal{K}_{\mathbf{x},1}$, the isomorphism class of $\mathfrak{A}$ with respect to $\mathbf{y}_{\mathbf{x}}$, is defined by:

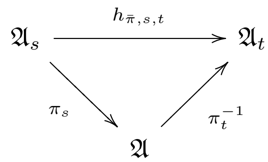
(7) $\mathrm{iso}_{\mathbf{y}_{\mathbf{x}}}(\mathfrak{A}) := \{\bar{\pi} : \bar{\pi} = \langle \pi_s : s \in S\rangle \textit{ such that } \pi_s : \mathfrak{A}_s \xrightarrow{\cong} \mathfrak{A}\}.$

Suppose $\bar{\pi} \in \mathrm{iso}_{\mathbf{y}_{\mathbf{x}}}(\mathfrak{A})$ and $s, t \in S$. We set

(8) $h_{\bar{\pi},s,t} = \pi_t^{-1} \circ \pi_s \in \mathrm{iso}(\mathfrak{A}_s, \mathfrak{A}_t)$,

(9) $\bar{h}_{\bar{\pi}} = \langle h_{\bar{\pi},s,t} : s, t \in S \rangle$.

This property can be summarized by the subjoined diagram:



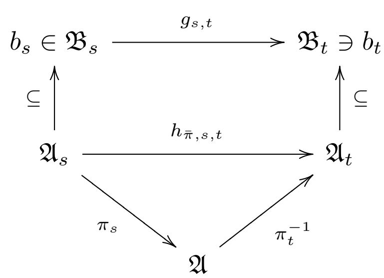

The proof of next claim is evident.

**Claim 5.5.** *Suppose* $\mathfrak{A} \in \mathcal{K}_{\mathbf{x},1}$ *and* $\bar{\pi} \in \mathrm{iso}_{\mathbf{y}_\mathbf{x}}(\mathfrak{A})$. *Then* $\bar{h}_{\bar{\pi}} \in \mathrm{Aut}_1(\mathbf{y}_\mathbf{x})$.

Recall that $\psi_s$ is a weak lifting corresponds to $\mathfrak{B}_s$. For $\mathfrak{A} \in \mathcal{K}_{\mathbf{x},1}$, suppose $\bar{\pi} \in \mathrm{iso}_{\mathbf{y}_\mathbf{x}}(\mathfrak{A})$ and $\bar{g} \in \mathrm{Aut}_2(\mathbf{y}_\mathbf{x})$ are such that $g_{s,s} = \psi_s(h_{\bar{\pi},s,s})$ and $h_{\bar{\pi},s,t} \subseteq g_{s,t}$ for $s, t \in S$. Let also $\bar{b} = \langle b_s : s \in S \rangle$.

**Definition 5.6.** We say the triple $(\bar{\pi}, \bar{g}, \bar{b})$ is *matched*, if $\bar{\pi}$, $\bar{g}$ and $\bar{b}$ are as above, and $b_s \in \mathfrak{B}_s$ we have $g_{s,t}(b_t) = b_s$ for $s, t \in S$.

Let $\Sigma(\mathbf{x}, \mathfrak{A})$ denote the family of all matched triples $(\bar{\pi}, \bar{g}, \bar{b})$. The following diagram summarizes the above situation:




In order to define the "universal model", we first consider to:

(10) $X_{\mathbf{y}_\mathbf{x}}(\mathfrak{A}) := \big\{(\bar{\pi}, \bar{g}, \bar{b}) : (\bar{\pi}, \bar{g}, \bar{b}) \in \Sigma(\mathbf{x}, \mathfrak{A})\big\}$.

For $\mathfrak{A} \in \mathcal{K}_{\mathbf{x},1}$, let $E_\mathfrak{A}$ be the following two place relation on $X_{\mathbf{y}_\mathbf{x}}(\mathfrak{A})$:

$$(\bar{\pi}, \bar{g}_1, \bar{b}_1) E_\mathfrak{A} (\pi_2, \bar{g}_2, \bar{b}_2)$$

if and only if

- $(\bar{\pi}_1, \bar{g}_1, \bar{b}_1)$ and $(\bar{\pi}_2, \bar{g}_2, \bar{b}_2)$ belong to $X_{\mathbf{y}_\mathbf{x}}(\mathfrak{A})$,

- letting $\widetilde{\pi}_s = \pi_{1,s}^{-1}\pi_{2,s}$ and $h_s = \psi_s(\widetilde{\pi}_s^{-1})$, we have $h_s(b_{1,s}) = b_{2,s}$.

Note that $\widetilde{\pi}_s$ is an automorphism of $\mathfrak{A}_s$, hence so is $\widetilde{\pi}_s^{-1}$. Consequently, $h_s = \psi_s(\widetilde{\pi}_s^{-1})$ is a member of $H_s = \mathrm{Aut}(\mathfrak{B}_s)$ which extends $\widetilde{\pi}_s$. Since $b_{1,2}, b_{2,s} \in \mathfrak{B}_s$ so $h_s(b_{1,s}) = b_{2,s} \in \mathfrak{B}_s$ is meaningful.

**Claim 5.7.** *$E_{\mathfrak{A}}$ is an equivalence relation.*

*Proof.* It is clearly reflexive and symmetric. To show that $E_{\mathfrak{A}}$ is transitive, assume that

(a) $(\bar{\pi}_1, \bar{g}_1, \bar{b}_1) E_{\mathfrak{A}} (\bar{\pi}_2, \bar{g}_2, \bar{b}_2)$,
(b) $(\bar{\pi}_2, \bar{g}_2, \bar{b}_2) E_{\mathfrak{A}} (\bar{\pi}_3, \bar{g}_3, \bar{b}_3)$.

According to the definition of $E_{\mathfrak{A}}$, we have:

(c) $(\psi_s(\pi_{1,s}^{-1}\pi_{2,s})^{-1})(b_{1,s}) = b_{2,s}$ for $s \in S$
(d) $(\psi_s(\pi_{2,s}^{-1}\pi_{3,s})^{-1})(b_{2,s}) = b_{3,s}$ for $s \in S$.

Hence

$$\begin{aligned} b_{3,s} &= (\psi_s(\pi_{2,s}^{-1}\pi_{3,s})^{-1})(b_{2,s}) \\ &= (\psi_s(\pi_{2,s}^{-1}\pi_{3,s})^{-1})((\psi_s(\pi_{1,s}^{-1}\pi_{2,s})^{-1})(b_{1,s})) \\ &= (\psi_s\big((\pi_{2,s}^{-1}\pi_{3,s})^{-1} \circ (\pi_{1,s}^{-1}\pi_{2,s})^{-1}\big)(b_{1,s})) \\ &= \psi_s((\pi_{1,s}^{-1}\pi_{3,s})^{-1})(b_{1,s}), \end{aligned}$$

as requested. □

We define a model $\mathfrak{B}'_{\mathfrak{A}}$ as follows:[2]

(11.1) the universe of $\mathfrak{B}'_{\mathfrak{A}}$ is $X_{\mathbf{y_x}}(\mathfrak{A})$,
(11.2) For any $R \in \tau$ which is an $n$-place relation, we set

$$R^{\mathfrak{B}'_{\mathfrak{A}}} := \Big\{ \langle (\bar{\pi}_\ell, \bar{g}_\ell, \bar{b}_\ell) : \ell < n \rangle \in X_{\mathbf{y_x}}(\mathfrak{A})^n : \exists s \in S \textit{ such that } \big(\langle b_{\ell,s} : \ell < n \rangle \in R^{\mathfrak{B}_s}\big) \Big\}.$$

**Claim 5.8.** *In the definition of $R^{\mathfrak{B}'_{\mathfrak{A}}}$, we can replace "$\exists s \in S$" by "$\forall s \in S$".*

*Proof.* Suppose $s, t \in S$ and $\langle (\bar{\pi}_\ell, \bar{g}_\ell, \bar{b}_\ell) : \ell < n \rangle \in X_{\mathbf{y_x}}(\mathfrak{A})^n$. We have to show that

$$\langle b_{\ell,s} : \ell < n \rangle \in R^{\mathfrak{B}_s} \Leftrightarrow \langle b_{\ell,t} : \ell < n \rangle \in R^{\mathfrak{B}_t}.$$

[2]For simplicity, we assume that the structures $\mathfrak{B}$ are relational, so they only contain relations.

This follows from the fact that $g_{s,t} : \mathfrak{B}_s \to \mathfrak{B}_t$ is an isomorphism and $g_{s,t}(b_{\ell,s}) = b_{\ell,t}$. □

**Claim 5.9.** *Let $\iota = 1, 2$ and $n$ be any integer. Assume $(\bar\pi^\iota_\ell, \bar g^\iota_\ell, \bar b^\iota_\ell) \in X_{\mathbf{y}_\mathbf{x}}(\mathfrak{A})$ and $(\bar\pi^1_\ell, \bar g^1_\ell, \bar b^1_\ell) E_{\mathfrak{A}} (\bar\pi^2_\ell, \bar g^2_\ell, \bar b^2_\ell)$ for all $\ell < n$. Then*

$$\langle(\bar\pi^1_\ell, \bar g^1_\ell, \bar b^1_\ell) : \ell < n\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}} \Leftrightarrow \langle(\bar\pi^2_\ell, \bar g^2_\ell, \bar b^2_\ell) : \ell < n\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}}.$$

*Proof.* Assume $\langle(\bar\pi^1_\ell, \bar g^1_\ell, \bar b^1_\ell) : \ell < n\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}}$. For each $\ell < n$, we have

$$(\bar\pi^1_\ell, \bar g^1_\ell, \bar b^1_\ell) E_{\mathfrak{A}} (\bar\pi^2_\ell, \bar g^2_\ell, \bar b^2_\ell).$$

This in turns imply that

$$\psi_s((\pi^2_{\ell,s})^{-1}\pi^1_{\ell,s})(b^1_{\ell,s}) = b^2_{\ell,s},$$

where $s \in S$. As $\psi_s((\pi^2_{\ell,s})^{-1}\pi^1_{\ell,s})$ is an automorphism of $\mathfrak{B}_s$, we have

$$\langle b^1_{\ell,s} : \ell < n\rangle \in R^{\mathfrak{B}_s} \Leftrightarrow \langle b^2_{\ell,s} : \ell < n\rangle \in R^{\mathfrak{B}_s}.$$

This implies that:

$$\begin{aligned}\Big\langle(\bar\pi^1_\ell, \bar g^1_\ell, \bar b^1_\ell) : \ell < n\Big\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}} \quad &\Rightarrow \exists s \in S \text{ such that } \langle b^1_{\ell,s} : \ell < n\rangle \in R^{\mathfrak{B}_s} \\ &\Rightarrow \langle b^2_{\ell,s} : \ell < n\rangle \in R^{\mathfrak{B}_s} \\ &\Rightarrow \Big\langle(\bar\pi^2_\ell, \bar g^2_\ell, \bar b^2_\ell) : \ell < n\Big\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}}.\end{aligned}$$

By symmetry, we have

$$\Big\langle(\bar\pi^2_\ell, \bar g^2_\ell, \bar b^2_\ell) : \ell < n\Big\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}} \Rightarrow \Big\langle(\bar\pi^1_\ell, \bar g^1_\ell, \bar b^1_\ell) : \ell < n\Big\rangle \in R^{\mathfrak{B}'_{\mathfrak{A}}}.$$

The claim follows. □

In the light of Claim 5.9 we observe that $E_{\mathfrak{A}}$ is a congruence relation on $\mathfrak{B}'_{\mathfrak{A}}$. Here, we define a function $\mathbf{k}$ with domain $\mathfrak{A}$ as follows:

$$\mathbf{k}(a) = \Big\{(\bar\pi, \bar g, \bar b) \in X_{\mathbf{y}_\mathbf{x}}(\mathfrak{A}) : b_s = \pi_s^{-1}(a) \text{ for all } s \in S\Big\}.$$

**Claim 5.10.** *If $a \in \mathfrak{A}$, then $\mathbf{k}(a)$ is an $E_{\mathfrak{A}}$-equivalence class.*

*Proof.* Let us first show that $\mathbf{k}$ is closed under the relation $E_{\mathfrak{A}}$, in the sense that $(\bar\pi_2, \bar g_2, \bar b_2) \in \mathbf{k}(a)$ provided $(\bar\pi_1, \bar g_1, \bar b_1) \in \mathbf{k}(a)$ and $(\bar\pi_1, \bar g_1, \bar b_1) E_{\mathfrak{A}} (\bar\pi_2, \bar g_2, \bar b_2)$. To this end, let $s \in S$ and recall that

- $b_{1,s} = \pi_{1,s}^{-1}(a)$,
- $\psi_s(\pi_{2,s}^{-1}\pi_{1,s})(b_{1,s}) = b_{2,s}$.

It then follows that

$$\begin{aligned} b_{2,s} &= \psi_s(\pi_{2,s}^{-1}\pi_{1,s})(b_{1,s}) \\ &= \psi_s(\pi_{2,s}^{-1}\pi_{1,s})(\pi_{1,s}^{-1}(a)) \\ &= \pi_{2,s}^{-1}\pi_{1,s}\pi_{1,s}^{-1}(a) \\ &= \pi_{2,s}^{-1}(a). \end{aligned}$$

In sum, $(\bar{\pi}_2, \bar{g}_2, \bar{b}_2) \in \mathbf{k}(a)$, as requested. Next, we show $(\bar{\pi}_1, \bar{g}_1, \bar{b}_1) E_{\mathfrak{A}} (\bar{\pi}_2, \bar{g}_2, \bar{b}_2)$ provided $(\bar{\pi}_1, \bar{g}_1, \bar{b}_1), (\bar{\pi}_2, \bar{g}_2, \bar{b}_2) \in \mathbf{k}(a)$. Let $s \in S$ and set $h_s = \psi_s(\pi_{2,s}^{-1}\pi_{1,s})$. We have to show that $h_s(b_{1,s}) = b_{2,s}$:

$$\begin{aligned} h_s(b_{1,s}) &= \psi_s(\pi_{2,s}^{-1}\pi_{1,s})(b_{1,s}) \\ &= \psi_s(\pi_{2,s}^{-1}\pi_{1,s})(\pi_{1,s}^{-1}(a)) \\ &= \pi_{2,s}^{-1}\pi_{1,s}\pi_{1,s}^{-1}(a) \\ &= \pi_{2,s}^{-1}(a) \\ &= b_{2,s}. \end{aligned}$$

We are done. The claim follows immediately. □

Now, we are ready to introduce the universal model:

**Claim 5.11.** *$\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}}$ is isomorphic to $\mathfrak{B}_s$, for $s \in S$.*

*Proof.* Fix $s \in S$, and let $\phi : \mathfrak{A}_s \simeq \mathfrak{A}$ be an isomorphism. Set

$$Y_{s,\phi} := \{(\bar{\pi}, \bar{g}, \bar{b}) \in X_{\mathbf{y}_\mathbf{x}}(\mathfrak{A}) : \pi_s = \phi\},$$

and define a function

$$\rho_{s,\phi} : Y_{s,\phi} \to \mathfrak{B}_s$$

as $\rho_{s,\phi}((\bar{\pi}, \bar{g}, \bar{b})) = b_s$. Clearly, $\rho_{s,\phi}$ is well-defined. Next, we bring the following two auxiliary observations:

$(*)_{5.11.1}$ Suppose $x_1 = (\bar{\pi}_1, \bar{g}_1, \bar{b}_1)$ and $x_2 = (\bar{\pi}_2, \bar{g}_2, \bar{b}_2)$ are in $Y_{s,\phi}$. Then

$$x_1 E_{\mathfrak{A}} x_2 \Rightarrow \rho_{s,\phi}(x_1) = \rho_{s,\phi}(x_2).$$

Indeed, suppose that $x_1 E_{\mathfrak{A}} x_2$. It then follows that $\psi_s(\pi_{2,s}^{-1}\pi_{1,s})(b_{1,s}) = b_{2,s}$. But as $\pi_{1,s} = \phi = \pi_{2s}$, we have

$$\psi_s(\pi_{2,s}^{-1}\pi_{1,s})(b_{1,s}) = \psi_s(\mathrm{id}_{\mathfrak{A}_s})(b_{1,s}) = \mathrm{id}_{\mathfrak{B}_s}(b_{1,s}) = b_{1,s},$$

and thus $\rho_{s,\phi}(x_1) = b_{1,s} = b_{2,s} = \rho_{s,\phi}(x_2)$. This completes the proof of $(*)_{5.11.1}$.

Suppose $\bar{\pi}_2 = \langle \pi_s^2 : s \in S \rangle \in \mathrm{iso}_{\mathbf{y_x}}(\mathfrak{A})$ and let $x_1 = (\bar{\pi}_1, \bar{g}_1, \bar{b}_1) \in X_{\mathbf{y_x}}(\mathfrak{A})$. Recall that $\widetilde{\pi}_s = \pi_{1,s}^{-1}\pi_{2,s}$. For simplicity, we set:

- $\widetilde{\Pi}_s := \psi_s(\widetilde{\pi}_s) \in \mathrm{Aut}(\mathfrak{B}_s)$.

The second auxiliary observation is:

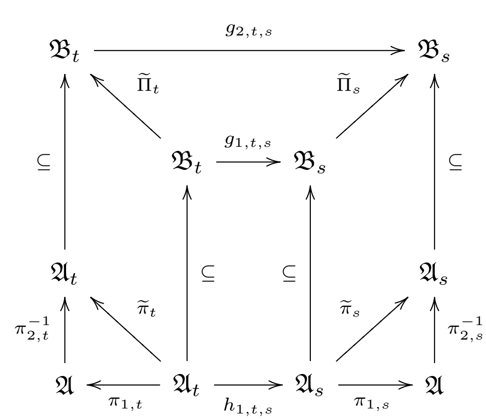
$(*)_{5.11.2}$ Suppose $\bar{\pi}_* = \langle \pi_s^* : s \in S \rangle \in \mathrm{iso}_{\mathbf{y_x}}(\mathfrak{A})$ and $s(*) \in S$. Let $x_1 = (\bar{\pi}_1, \bar{g}_1, \bar{b}_1) \in X_{\mathbf{y_x}}(\mathfrak{A})$. Then there is $x_2 = (\bar{\pi}_2, \bar{g}_2, \bar{b}_2) \in [x_1]_{E_{\mathfrak{A}}}$ such that
- $\bar{\pi}_2 = \bar{\pi}_*$ and
- $b_{2,s(*)} = \widetilde{\Pi}_{s(*)}^{-1} b_{1,s(*)}$.

[Why? In order to define $x_2$, we set

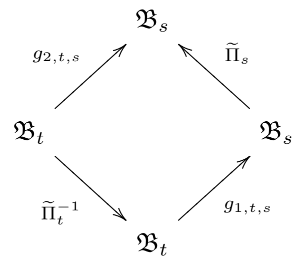
(i) $\bar{\pi}_2 = \bar{\pi}_*$,

(ii) for $s, t \in S$, $g_{2,t,s} = \psi_t((\pi_{1,t}^{-1}\pi_{2,t})^{-1}) \circ g_{1,t,s} \circ \psi_s(\pi_{1,s}^{-1}\pi_{2,s})$.

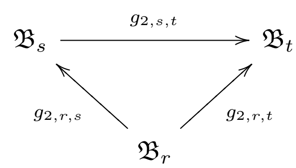
The following commutative diagram summarizes the above situation:



Let us depict the resulting commutative diagram:




It is easily seen that $x_2$ as defined above is as required. Indeed, by (i), $\bar{\pi}_2 = \bar{\pi}_*$. Also, it is clear that for each $s \in S$, $g_{2,s,s} = \mathrm{id}_{\mathfrak{B}_s}$. Next, we show the following diagram




is commutative where $r, s, t \in S$. To check this, we first recall that $\widetilde{\pi}_s = \pi_{1,s}^{-1}\pi_{2,s}$, then we have

$$\begin{aligned} g_{2,r,s} \circ g_{2,s,t} &= \big(\psi_r(\widetilde{\pi}_r)^{-1} \circ g_{1,r,s} \circ \psi_s(\widetilde{\pi}_s)\big) \circ \big(\psi_s(\widetilde{\pi}_s)^{-1} \circ g_{1,s,t} \circ \psi_t(\widetilde{\pi}_t)\big) \\ &= \psi_r(\pi_{1,r}^{-1}\pi_{2,r})^{-1} \circ g_{1,r,s} \circ g_{1,s,t} \circ \psi_t(\pi_{1,t}^{-1}\pi_{2,t}) \\ &= \psi_r(\pi_{1,r}^{-1}\pi_{2,r})^{-1} \circ g_{1,r,t} \circ \psi_t(\pi_{1,t}^{-1}\pi_{2,t}) \\ &= g_{2,r,t}. \end{aligned}$$

Also,

$$\begin{aligned} g_{2,s(*),s}(b_{2,s(*)}) &= g_{2,s(*),s}(\widetilde{\Pi}_{s(*)}^{-1} b_{1,s(*)}) \\ &= \widetilde{\Pi}_s^{-1} g_{1,s(*),s}(b_{1,s(*)}) \\ &= \widetilde{\Pi}_s^{-1}(b_1 s) \\ &= b_2 s. \end{aligned}$$

In sum, we proved that

$$x_2 = (\bar{\pi}_2, \bar{g}_2, \bar{b}_2) \in X_{\mathbf{y}_\mathbf{x}}(\mathfrak{A}).$$

Next, we are going to show that $x_2 E_{\mathfrak{A}} x_1$. To this end, let $s \in S$, and set

- $\pi_s = \pi_{1,s}^{-1}\pi_{2,s}$,
- $h_s = \psi_s(\pi_s^{-1})$.

Then:

$$\begin{aligned} h_s(b_{1,s}) &= \psi_s(\pi_{2,s}^{-1}\pi_{1,s})(b_{1,s}) \\ &= \widetilde{\Pi}_s(g_{1,s(*),s}(b_{1,s(*)})) \\ &= g_{2,s,s(*)}(\widetilde{\Pi}_{s(*)}b_{1,s(*)}) \\ &= g_{2,s,s(*)}(b_{2,s(*)}) \\ &= b_{2,s}, \end{aligned}$$

i.e., $h_s(b_{1,s}) = b_{2,s}$, from which the equivalence $x_2 E_{\mathfrak{A}} x_1$ follows. This completes the proof of $(*)_{5.11.2}$. ]

It follows from $(*)_{5.11.1}$ and $(*)_{5.11.2}$ that

$$\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}} = Y_{s,\phi}/E_{\mathfrak{A}} \simeq \mathfrak{B}_s.$$

Claim 5.11 follows. □

Given any $\mathfrak{A} \in \mathcal{K}_{\mathbf{x},1}$, we are going to define an isomorphism

$$F_{\mathfrak{A}} : \mathfrak{A} \to \mathrm{sort}_1(\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}}),$$

uniformly definable from $\mathbf{p}$. To this end, we set

$$F_{\mathfrak{A}}(a) := \left\{ \left[(\bar{\pi}, \bar{g}, \bar{b})\right]_{E_{\mathfrak{A}}} : (\bar{\pi}, \bar{g}, \bar{b}) \in X_{\mathbf{y_x}}(\mathfrak{A}) \text{ and } \bigwedge_{s\in S} \pi_s(b_s) = a \right\}.$$

In other words,

$$F_{\mathfrak{A}}(a) = \left\{ \left[(\bar{\pi}, \bar{g}, \bar{b})\right]_{E_{\mathfrak{A}}} : (\bar{\pi}, \bar{g}, \bar{b}) \in \mathbf{k}(a) \right\}.$$

Due to Claim 5.10, we know $\mathbf{k}(a)$ is an $E_{\mathfrak{A}}$-equivalence class. In particular, $F_{\mathfrak{A}}(a)$ is singleton. Recall from Claim 5.11 that we have constructed the 2-sorted model $\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}}$. Now, for $\mathfrak{A} \in \mathcal{K}_{\mathbf{x},1}$, let $\mathfrak{B}_{\mathfrak{A}}$ be defined as follows:

- $\mathfrak{B}_{\mathfrak{A}}$ has as universe $|\mathfrak{A}| \cup \mathrm{sort}_2(\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}})$,
- the mapping $F_{\mathfrak{A}} \cup \mathrm{id}_{\mathrm{sort}_2(\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}})}$ is an isomorphism from $\mathfrak{B}_{\mathfrak{A}}$ onto $\mathfrak{B}'_{\mathfrak{A}}/E_{\mathfrak{A}}$.

In sum, $F(\mathfrak{A}) = \mathfrak{B}_{\mathfrak{A}}$ is the required function we were looking for, and the theorem follows. □

**Acknowledgement.** The authors thank the referee for reading the paper and providing valuable comments.

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