NOWHERE TRIVIAL AUTOMORPHISMS OF $P(\lambda)/[\lambda]^{<\lambda}$, FOR λ INACCESSIBLE.

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1. Introduction

We investigate the rigidity of the Boolean algebra $P(\lambda)/[\lambda]^{<\lambda}$, for λ inaccessible.

For $\lambda = \omega$ there is extensive literature on this topic (see, e.g., the survey [FGVV24]); some general results on $P(\lambda)/[\lambda]^{<\kappa}$ can be found in [LM16]. In [KLS] it was shown, for λ inaccessible and $2^\lambda = \lambda^{++}$, that conistently every automorphism is densely trivial.

In this paper we show:

(Thm. 5.3) If λ is (strongly) inaccessible and $2^\lambda = \lambda^+$, then there is a nowhere trivial automorphism of the Boolean algebra $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.

Note that the weaker variant "there is a *nontrivial* automorphism" follows from [SS15, Lem. 3.2] (the proof there was faulty, and fixed in [SS]); and for λ measurable, a proof (again only for "nontrivial") was given in [KLS].

We also show:

(Thm. 6.1) It is consistent that λ is inaccessible, 2^λ an arbitrary regular cardinal, and that there is a nowhere trivial automorphism of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.

2. Notation

We will always assume that λ is inaccessible.

For $A \subseteq \lambda$, $[A]^{<\lambda}$ denotes the subsets of A of size less than λ; and $[A]^\lambda$ those of size λ. With $[A]$ we denote the equivalence class of A modulo $[\lambda]^{<\lambda}$. We write $A =^* B$ for $[A] = [B]$, and $A \subseteq^* B$ for $|A \setminus B| < \lambda$.

However, we also use $f[A] := \{f(a) : a \in A\}$. So for example $[f[A]]$ is the equivalence class of the f-image of A. $f \in \mathrm{Sym}(X)$ means that $f : X \to X$ is bijective.

We consider $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ as Boolean algebra. A (Boolean algebra) automorphism π of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ is called trivial on A (for $A \in [\lambda]^\lambda$) if there is an $f \in \mathrm{Sym}(\lambda)$ such that $\pi([B]) = [f[B]]$ for all $B \subseteq A$. π is called nowhere trivial, if there is no such pair (f, A).

For $\delta \le \lambda$, $C \subseteq \delta$ closed and nonempty, and $\alpha \in C$, we set

$$I^*(C \subseteq \delta, \alpha) := \{\beta : \alpha \le \beta < \min\big((C \cup \{\delta\}) \setminus (\alpha + 1)\big)\}.$$

So the $I^*(C \subseteq \delta, \alpha)$, for $\alpha \in C$, form an increasing interval partition of $\delta \setminus \min(C)$.

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3. Approximations

In this section, we define the set AP of "approximations". An approximation $\mathbf{a}$ will induce a "partial monomorphism" $\tilde{\boldsymbol{\pi}}^{\mathbf{a}}$ defined on some $\tilde{\mathbf{B}}^{\mathbf{a}}$ which is trivial, i.e., generated by some $\boldsymbol{\pi}^{\mathbf{a}} \in \mathrm{Sym}(\lambda)$. We will use such approximations to build a nowhere trivial automorphism $\tilde{\boldsymbol{\phi}}$ as limit (i.e., $\tilde{\boldsymbol{\phi}} \restriction \tilde{\mathbf{B}}^{\mathbf{a}} = \tilde{\boldsymbol{\pi}}^{\mathbf{a}}$), cf. Fact 3.6.

Definition 3.1. AP is the set of objects $\mathbf{a}$ consisting of $\mathbf{C}^{\mathbf{a}}$, $\boldsymbol{\pi}^{\mathbf{a}}$ and $\mathbf{B}^{\mathbf{a}}$, such that:

- $\boldsymbol{\pi}^{\mathbf{a}} \in \mathrm{Sym}(\lambda)$.
- $\mathbf{C}^{\mathbf{a}} \subseteq \lambda$ club such that $\boldsymbol{\pi}^{\mathbf{a}} \restriction \varepsilon \in \mathrm{Sym}(\varepsilon)$ for all $\varepsilon \in \mathbf{C}^{\mathbf{a}}$.
- $\mathbf{B}^{\mathbf{a}}$ is a subset of $\mathcal{P}(\lambda)$.

$\boldsymbol{\pi}^{\mathbf{a}}$ induces a (trivial) automorphism of $P(\lambda)/[\lambda]^{<\lambda}$, and $\tilde{\boldsymbol{\pi}}^{\mathbf{a}}$ is the restriction of this automorphism to $\mathbf{B}^{\mathbf{a}}$:

Definition 3.2.
- $\tilde{\mathbf{B}}^{\mathbf{a}} := \mathbf{B}^{\mathbf{a}}/[\lambda]^{<\lambda} = \{[A] : A \in \mathbf{B}^{\mathbf{a}}\}$.
- $\tilde{\boldsymbol{\pi}}^{\mathbf{a}} : \tilde{\mathbf{B}}^{\mathbf{a}} \to P(\lambda)/[\lambda]^{<\lambda}$ is defined by $[A] \mapsto [\boldsymbol{\pi}^{\mathbf{a}}[A]]$.
- For $\mathbf{a} \in \mathrm{AP}$ and $\varepsilon \in \mathbf{C}^{\mathbf{a}}$ we set $I^{\mathbf{a}}_{\varepsilon} := I^*(\mathbf{C}^{\mathbf{a}} \subseteq \lambda, \varepsilon)$.

So the $I^{\mathbf{a}}_{\varepsilon}$ form an increasing interval partition of $\lambda \setminus \min(\mathbf{C}^{\mathbf{a}})$; and $\boldsymbol{\pi}^{\mathbf{a}} \restriction I^{\mathbf{a}}_{\varepsilon} \in \mathrm{Sym}(I^{\mathbf{a}}_{\varepsilon})$.

Definition 3.3. $\mathbf{b} \geq_{\mathrm{AP}} \mathbf{a}$, if $\mathbf{a}, \mathbf{b} \in \mathrm{AP}$ and

(1) $\mathbf{C}^{\mathbf{b}} \subseteq^* \mathbf{C}^{\mathbf{a}}$.
(2) $\boldsymbol{\pi}^{\mathbf{b}} \restriction I^{\mathbf{a}}_{\varepsilon} = \boldsymbol{\pi}^{\mathbf{a}} \restriction I^{\mathbf{a}}_{\varepsilon}$ for all but boundedly many $\varepsilon \in \mathbf{C}^{\mathbf{b}}$.
(3) $\mathbf{B}^{\mathbf{b}} \supseteq \mathbf{B}^{\mathbf{a}}$, and $\tilde{\boldsymbol{\pi}}^{\mathbf{b}}$ extends $\tilde{\boldsymbol{\pi}}^{\mathbf{a}}$.

I.e., if $A \in \mathbf{B}^{\mathbf{a}}$, then $\boldsymbol{\pi}^{\mathbf{a}}[A] =^* \boldsymbol{\pi}^{\mathbf{b}}[A]$.

$\leq_{\mathrm{AP}}$ is a nonempty quasi order.

Lemma 3.4. *If $(\mathbf{a}_i)_{i<\delta}$ is an $\leq_{\mathrm{AP}}$ increasing chain such that $\bigcup_{i<\delta} \tilde{\mathbf{B}}^{\mathbf{a}_i} = P(\lambda)/[\lambda]^{<\lambda}$, then $\tilde{\boldsymbol{\phi}} := \bigcup_{i<\delta} \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}$ is an Boolean algebra monomorphism of $P(\lambda)/[\lambda]^{<\lambda}$.*

If additionally $\bigcup_{i<\delta} \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}[\tilde{\mathbf{B}}^{\mathbf{a}_i}] = P(\lambda)/[\lambda]^{<\lambda}$, then $\tilde{\boldsymbol{\phi}}$ is an automorphism.

Proof. We use $\vee$ and ^{c} for the Boolean-algebra-operations, i.e., $[A \cup B] = [A] \vee [B]$, and $[A]^c = [\lambda \setminus A]$. It is enough to show that $\tilde{\boldsymbol{\phi}}$ is injective, honors $\vee$ and ^{c}, and maps $[\emptyset]$ to itself.

For X_1, X_2 in $P(\lambda)/[\lambda]^{<\lambda}$ there is an $i < \delta$ and some $A_1, A_2, A_{\mathrm{union}}$ in $\mathbf{B}^{\mathbf{a}_i}$, such that $[A_j] = X_j$ for $j = 1, 2$ and $[A_{\mathrm{union}}] = [A_1 \cup A_2] = X_1 \vee X_2$. Then

$$\boldsymbol{\pi}^{\mathbf{a}_i}[A_{\mathrm{union}}] =^* \boldsymbol{\pi}^{\mathbf{a}_i}[A_1 \cup A_2] = \boldsymbol{\pi}^{\mathbf{a}_i}[A_1] \cup \boldsymbol{\pi}^{\mathbf{a}_i}[A_2],$$

and

$$\tilde{\boldsymbol{\phi}}(X_1 \vee X_2) = \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}([A_{\mathrm{union}}]) = [\boldsymbol{\pi}^{\mathbf{a}_i}[A_{\mathrm{union}}]] = \\ = \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}([A_1]) \vee \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}([A_2]) = \tilde{\boldsymbol{\phi}}(X_1) \vee \tilde{\boldsymbol{\phi}}(X_2).$$

If $X_1 \neq X_2$, i.e., $A_1 \neq^* A_2$, then $\boldsymbol{\pi}^{\mathbf{a}_i}[A_1] \neq^* \boldsymbol{\pi}^{\mathbf{a}_i}[A_2]$, i.e., $\tilde{\boldsymbol{\phi}}(X_1) \neq \tilde{\boldsymbol{\phi}}(X_2)$.

Similarly we can show $\tilde{\boldsymbol{\phi}}([\lambda \setminus A_1]) = \tilde{\boldsymbol{\phi}}([A_1])^c$ and $\tilde{\boldsymbol{\phi}}([\emptyset]) = [\emptyset]$. □

Definition 3.5. For a pair (f, A) with $A \in [\lambda]^{\lambda}$ and $f \in \mathrm{Sym}(\lambda)$, we say $\mathbf{a} \in \mathrm{AP}$ "spoils (f, A)", if there is an $A' \in [A]^{\lambda} \cap \mathbf{B}^{\mathbf{a}}$ such that $|\boldsymbol{\pi}^{\mathbf{a}}[A'] \cap f[A']| < \lambda$.

If $\tilde{\boldsymbol{\phi}}$ is an automorphism extending such a $\tilde{\boldsymbol{\pi}}^{\mathbf{a}}$, then f cannot witness that $\tilde{\boldsymbol{\phi}}$ is trivial on A. Therefore:

Fact 3.6. If $(\mathbf{a}_i)_{i<\delta}$ is an $\leq_{\mathrm{AP}}$ increasing chain such that

- $\bigcup_{i<\delta} \tilde{\mathbf{B}}^{\mathbf{a}_i} = \bigcup_{i<\delta} \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}[\tilde{\mathbf{B}}^{\mathbf{a}_i}] = P(\lambda)/[\lambda]^{<\lambda}$, and
- for every (f, A) there is an $i < \delta$ such that $\mathbf{a}_i$ spoils (f, A),

then $\tilde{\boldsymbol{\phi}} := \bigcup_{i<\delta} \tilde{\boldsymbol{\pi}}^{\mathbf{a}_i}$ is a nowhere trivial Boolean algebra automorphism of $P(\lambda)/[\lambda]^{<\lambda}$.

We will use this fact both in the case $2^\lambda = \lambda^+$, as well as in the forcing construction to get a nowhere trivial automorphism.

We will often modify an $\mathbf{a} \in \mathrm{AP}$ by replacing $\mathbf{B}^{\mathbf{a}}$ with another $B \subseteq P(\lambda)$. Let the result be $\mathbf{b}$. We call $\mathbf{b}$ "$\mathbf{a}$ with $\mathbf{B}$ replaced by B", or "$\mathbf{a}$ with X added to $\mathbf{B}$" in case $B = \mathbf{B}^{\mathbf{a}} \cup \{X\}$. Obviously $\mathbf{b} \in \mathrm{AP}$, and if $B \supseteq \mathbf{B}^{\mathbf{a}}$ then $\mathbf{b} \geq_{\mathrm{AP}} \mathbf{a}$.

Similarly we can get a stronger approximation by thinning out $\mathbf{C}$. To summarize:

Fact 3.7. If $\mathbf{a} \in \mathrm{AP}$, $D \subseteq \mathbf{C}^{\mathbf{a}}$ club, and $B \subseteq P(\lambda)$ with $B \supseteq \mathbf{B}^{\mathbf{a}}$. Then $\mathbf{b} \geq_{\mathrm{AP}} \mathbf{a}$, for the $\mathbf{b}$ defined by $\boldsymbol{\pi}^{\mathbf{b}} := \boldsymbol{\pi}^{\mathbf{a}}$, $\mathbf{C}^{\mathbf{b}} := D$ and $\mathbf{B}^{\mathbf{b}} := B$.

In the definition of $\leq_{\mathrm{AP}}$ we require that some things hold "apart from a bounded set", or equivalently, "above some α". We say that α is good for an increasing sequence of $\mathbf{a}_i$, if the requirements for each pair are met above α. We will generally only be able to find such an α for "short sequences":

Definition 3.8.
1. AP_λ is the set of $\mathbf{a} \in \mathrm{AP}$ such that $|\mathbf{B}^{\mathbf{a}}| \leq \lambda$. Analogously for $\mathrm{AP}_{<\lambda}$.
2. $(\mathbf{a}_i)_{i\in J}$ is a "short sequence", if $J < \lambda$ (or more generally, J is a set of ordinals with $|J| < \lambda$), each $\mathbf{a}_i \in \mathrm{AP}_{<\lambda}$, and the sequence is $\leq_{\mathrm{AP}}$-increasing, i.e., $j > i$ in J implies $\mathbf{a}_j \geq_{\mathrm{AP}} \mathbf{a}_i$.
3. Let $\bar{\mathbf{a}} := (\mathbf{a}_i)_{i\in J}$ be short. We say that α is good for $\bar{\mathbf{a}}$, if for all $i \leq k$ in J:
   (a) $\alpha \in \mathbf{C}^{\mathbf{a}_i}$.
   (b) $\mathbf{C}^{\mathbf{a}_k} \subseteq \mathbf{C}^{\mathbf{a}_i}$ above α. (I.e., $\beta \geq \alpha$ and $\beta \in \mathbf{C}^{\mathbf{a}_k}$ implies $\beta \in \mathbf{C}^{\mathbf{a}_i}$.)
   (c) $\boldsymbol{\pi}^{\mathbf{a}_k} \restriction I_\varepsilon^{\mathbf{a}_i} = \boldsymbol{\pi}^{\mathbf{a}_i} \restriction I_\varepsilon^{\mathbf{a}_i}$ for all $\varepsilon \geq \alpha$ in $\mathbf{C}^{\mathbf{a}_k}$.
   (d) $\boldsymbol{\pi}^{\mathbf{a}_i}[A] \setminus \alpha = \boldsymbol{\pi}^{\mathbf{a}_k}[A] \setminus \alpha$, for all $A \in \mathbf{B}^{\mathbf{a}_i}$.
4. For $\mathbf{a}, \mathbf{b}$ in $\mathrm{AP}_{<\lambda}$, we say $\mathbf{b} >_\zeta \mathbf{a}$, if ζ is good for the sequence $\langle \mathbf{a}, \mathbf{b} \rangle$.

So in particular if $\mathbf{b}$ is the result of enlarging $\mathbf{B}$ in $\mathbf{a}$, then $\mathbf{b} >_\zeta \mathbf{a}$ for all $\zeta \in \mathbf{C}^{\mathbf{a}}$.

Fact 3.9.
1. If $\mathbf{a} \in \mathrm{AP}_{<\lambda}$, then $\mathbf{b} \geq_{\mathrm{AP}} \mathbf{a}$ iff $(\exists \zeta \in \lambda)\, \mathbf{b} >_\zeta \mathbf{a}$.
2. If $\bar{\mathbf{a}} = (\mathbf{a}_i)_{i\in J}$ is short, then $\{\alpha \in \lambda : \alpha \text{ good for } \bar{\mathbf{a}}\}$ is club, more concretely it is $\bigcap_{i<\delta} \mathbf{C}^{\mathbf{a}_i} \setminus \alpha^*$ for some $\alpha^* < \lambda$.

Lemma 3.10. *If $\bar{\mathbf{a}}$ is short, then is has an $\leq_{\mathrm{AP}}$-upper-bound $\mathbf{b} \in \mathrm{AP}_{<\lambda}$.*

Proof. Set $D := \bigcap_{i\in J} C^{\mathbf{a}_i}$, and ζ_0 be the smallest $\bar{\mathbf{a}}$-good ordinal. So in particular $\zeta_0 \in D$; and any $\zeta \geq \zeta_0$ is in D iff it is $\bar{\mathbf{a}}$-good.

Fix for now some $\zeta \in D \setminus \zeta_0$. Let ζ^+ be the D-successor of ζ.

For $i \in J$, set $\gamma(\zeta, i)$ to be the ζ-successor of $C^{\mathbf{a}_i}$. Then the sequence $\gamma(\zeta, i)$ is weakly increasing with $i \in J$ and has limit ζ^+. If $\alpha < \gamma(\zeta, i)$ (we also say "α is stable at i"), then $\boldsymbol{\pi}^{\mathbf{a}_i}(\alpha) = \boldsymbol{\pi}^{\mathbf{a}_j}(\alpha)$ for all $j > i$ in J.

We define $\pi^{\lim}(\alpha)$ for all $\alpha \geq \zeta_0$ as $\boldsymbol{\pi}^{\mathbf{a}_i}(\alpha)$ for some i stable for α.

To summarize: Whenever $I := \zeta^+ \setminus \zeta$ for some $\zeta \in D \setminus \zeta_0$ with ζ^+ the D-successor, we get:

1. $(\forall \alpha \in I)\, (\exists i \in J)\, (\forall j > i)\, \pi^{\lim}(\alpha) = \boldsymbol{\pi}^{\mathbf{a}_i}(\alpha)$.
2. $\pi^{\lim} \restriction I \in \mathrm{Sym}(I)$.

(3) If $i \in J$ and $A \in \mathbf{B}^{\mathbf{a}_i}$, then $\pi^{\lim}[A'] = \boldsymbol{\pi}^{\mathbf{a}_i}[A']$ where $A' := A \cap I$.

For (2), note that $\boldsymbol{\pi}^{\mathbf{a}_i} \in \mathrm{Sym}(I)$ for all $i \in J$. If $\alpha_1 \neq \alpha_2 \in I$, then there is an i in J stable for both, and $\boldsymbol{\pi}^{\mathbf{a}_i}(\alpha_1) \neq \boldsymbol{\pi}^{\mathbf{a}_i}(\alpha_2)$. So $\boldsymbol{\pi}^{\lim}$ is injective. And if $\alpha_1 \in I$ and i in J stable for α_1, then there is an $\alpha_2 \in I_\zeta^{\mathbf{a}_i}$ with $\boldsymbol{\pi}^{\lim}(\alpha_2) = \boldsymbol{\pi}^{\mathbf{a}_i}(\alpha_2) = \alpha_1$, so $\pi^{\lim}$ is surjective.

For (3): Set $B := \boldsymbol{\pi}^{\mathbf{a}_i}[A']$. As I is above the good ζ_0, we have: $B = \boldsymbol{\pi}^{\mathbf{a}_j}[A']$ for all $j \in J$ with $j > i$. So for $\alpha \in A'$, all $\boldsymbol{\pi}^{\mathbf{a}_j}(\alpha)$ are in B, and also stabilize to $\pi^{\lim}(\alpha)$, which therefore has to be in B. Analogously, we get: If $\alpha \in I \setminus A$, then $\boldsymbol{\pi}^{\mathbf{a}_j}(\alpha) \neq B$ stabilizes to $\pi^{\lim}(\alpha)$, which therefore is not in B. As $\pi^{\lim}[I] = I$, we get $\pi^{\lim}[I \cap A] = B$.

We can now define $\mathbf{b}$ as:

$$\mathbf{C}^{\mathbf{b}} := D \setminus \zeta_0; \quad \boldsymbol{\pi}^{\mathbf{b}}(\alpha) = \begin{cases} \alpha & \text{if } \alpha < \zeta_0 \\ \pi^{\lim}(\alpha) & \text{otherwise;} \end{cases} \quad \mathbf{B}^{\mathbf{b}} := \bigcup_{i \in J} \mathbf{B}^{\mathbf{a}_i}. \qquad \square$$

4. Initial segments

We will work with initial segments of approximations (without the $\mathbf{B}$ part):

Definition 4.1.
- An "initial segment" b consists of a "height" δ^b, a closed $C^b \subseteq \delta^b$ (possibly empty), and a $\pi^b \in \mathrm{Sym}(\delta^b)$ such that $\pi^b \restriction \zeta \in \mathrm{Sym}(\zeta)$ for all $\zeta \in C^b$.
- The set of initial segments is called IS.
- $b >_{\mathrm{IS}} a$, if $\delta^b > \delta^a$, $\delta^a \in C^b$, $C^b \cap \delta^a = C^a$, and $\pi^b \restriction \delta^a = \pi^a$.
- $b \geq_{\mathrm{IS}} a$ if $b >_{\mathrm{IS}} a$ or $b = a$.
- For $\zeta \in C^b$, we set $I_\zeta^b := I^*(C^b \subseteq \delta^b, \zeta)$.

So the I_ζ^b form an increasing interval partition of $\delta^b \setminus \min(C^b)$, and $\pi^b \restriction I_\zeta^b \in \mathrm{Sym}(I_\zeta^b)$.

$\leq_{\mathrm{IS}}$ is a partial order.

Some trivialities:

Fact 4.2. Assume that $\bar{b} = (b_i)_{i<\xi}$, with $\xi \leq \lambda$ limit, is an $<_{\mathrm{IS}}$-increasing sequence.

(1) If $\xi < \lambda$, then the following $b_\xi \in \mathrm{IS}$ is the $\leq_{\mathrm{IS}}$-supremum of $\bar{b}$, and we call it "the limit" of $\bar{b}$: $\delta^{b_\xi} := \bigcup_{i<\xi} \delta^{b_i}$, $C^{b_\xi} := \bigcup_{i<\xi} C^{b_i}$ and $\pi^{b_\xi} := \bigcup_{i<\xi} \pi^{b_i}$.

(2) If $\xi = \lambda$, then to each $B \subseteq P(\lambda)$ there is a $\mathbf{b} \in \mathrm{AP}$ as follows, which we call "a limit" of $\bar{b}$: $\mathbf{C}^{\mathbf{b}} := \bigcup_{i<\lambda} C^{b_i}$ $\boldsymbol{\pi}^{\mathbf{b}} := \bigcup_{i<\lambda} \pi^{b_i}$ and $\mathbf{B}^{\mathbf{b}} := B$.

Let us call an $<_{\mathrm{IS}}$-increasing sequence $\bar{b}$ "continuous" if b_γ is the limit of $(b_\alpha)_{\alpha<\gamma}$ for all limits $\gamma < \delta$. We will only use continuous sequences.

Definition 4.3. Let $\mathbf{a} \in \mathrm{AP}_{<\lambda}$ and $b \in \mathrm{IS}$ with $\delta^b \in \mathbf{C}^{\mathbf{a}}$. We say $c >_{\mathbf{a}} b$, if the following holds:

- $c >_{\mathrm{IS}} b$.
- $(C^c \cup \{\delta^c\}) \setminus \delta^b \subseteq \mathbf{C}^{\mathbf{a}}$.
- For all $\zeta \in C^c \setminus \delta^b$, $\pi^c \restriction I_\zeta^{\mathbf{a}} = \boldsymbol{\pi}^{\mathbf{a}} \restriction I_\zeta^{\mathbf{a}}$.
- For all $A \in \mathbf{B}^{\mathbf{a}}$, $\pi^c[A'] = \boldsymbol{\pi}^{\mathbf{a}}[A']$ where we set $A' := A \cap \delta^c \setminus \delta^b$.

For a short $\bar{\mathbf{a}}$ (with index set J) we say $c >_{\bar{\mathbf{a}}} b$ if $c >_{\mathbf{a}_i} b$ for all $i \in J$.

Lemma 4.4. *Let* $\mathbf{a}, \mathbf{b}$ *in* $\mathrm{AP}_{<\lambda}$ *and* c, d_i $(i < \lambda)$ *in* IS.

(1) $<_{\mathbf{a}}$ *is a partial order*
(2) *If* $\zeta < \lambda$ *and* $(d_i)_{i\in\zeta}$ *is a* $>_{\mathrm{IS}}$*-increasing sequence such that* $d_i >_{\mathbf{a}} c$ *for all* $i < \zeta$*, then also the limit* d_ζ *satisfies* $d_\zeta >_{\mathbf{a}} c$*.*
(3) *If* $\mathbf{b} >_{\delta^c} \mathbf{a}$*, then* $d >_{\mathbf{b}} c$ *implies* $d >_{\mathbf{a}} c$*.*
(4) *Assume* $\bar{c} := (c_i)_{i\in\lambda}$ *is a continuous increasing sequence in* IS *such that for some* $i_0 < \lambda$ *we have* $c_i <_{\mathbf{a}} c_{i+1}$ *for all* $i > i_0$*.*
Then any limit $\mathbf{c} \in \mathrm{AP}$ *of the* $\bar{c}$ *with* $\mathbf{B}^{\mathbf{c}} \supseteq \mathbf{B}^{\mathbf{a}}$ *satisfies* $\mathbf{c} >_{\mathrm{AP}} \mathbf{a}$*.*
(5) *Let* $\bar{\mathbf{a}}$ *be short,* $b \in \mathrm{IS}$*,* δ^b *good for* $\bar{\mathbf{a}}$ *and* $E \subseteq \lambda$ *club.*
Then there is a $c >_{\bar{\mathbf{a}}} b$ *with* $\delta^c \in E$ *and* $C^c = C^b \cup \{\delta^b\}$

Proof. For (5), use (the proof of) Lemma 3.10: Pick any $\delta^c \in D \cap E \setminus (\delta^b + 1)$ and set $C^c = C^b \cup \{\delta^b\}$ and $\pi^c = \pi^{\lim} \restriction \delta^c$.

The rest is straightforward. □

We now turn to spoiling (f, A):

Definition 4.5. Given $f \in \mathrm{Sym}(\lambda)$ and $A \in [\lambda]^\lambda$, we define $c >^{f,A} b$ by: $c >_{\mathrm{IS}} b$, $f \restriction \delta^c \in \mathrm{Sym}(\delta^c)$, and there is a $\xi^* \in A \cap \delta^c \setminus \delta^b$ with $f(\xi^*) \neq \pi^c(\xi^*)$.

We write $c >^{f,A}_{\bar{\mathbf{a}}} b$ for: $c >_{\bar{\mathbf{a}}} b$ & $c >^{f,A} b$

Lemma 4.6. *Assume* $(b_i)_{i\in\lambda}$ *is* $<_I S$*-increasing such that unboundedly often* $b_{i+1} >^{f,A} b_i$*. Then for some* $A' \in [A]^\lambda$*, every limit* $\mathbf{b}$ *of* $(b_i)_{i\in\lambda}$ *with* $A' \in \mathbf{B}^{\mathbf{b}}$ *spoils* (f, A)*.*

Proof. By taking a subsequence, we can assume that for all odd i (i.e., $i = \delta+2n+1$ with δ limit or 0 and $n \in \omega$) $b_{i+1} >^{f,A} b_i$.

For i odd, set $I_i := \delta^{b_{i+1}} \setminus \delta^{b_i}$ and let $\xi_i \in I_i$ satisfy $f(\xi_i) \neq \pi^{b_{i+1}}(\xi_i) = \boldsymbol{\pi}^{\mathbf{b}}(\xi_i)$.

If i is odd, then $\boldsymbol{\pi}^{\mathbf{b}} \restriction I_i \in \mathrm{Sym}(I_i)$ and $f \restriction \delta^{b_{i+1}} \in \mathrm{Sym}(\delta^{b_{i+1}})$.

So if $i < j$ are both odd, then $f(\zeta_j) > \delta^{b_{i+1}} > \boldsymbol{\pi}^{\mathbf{b}}(\zeta_i)$; and if $j < k$ are both odd then $f(\zeta_j) < \delta^{b_j} \leq \boldsymbol{\pi}^{\mathbf{b}}(\zeta_k)$. This means that $f(\zeta_j)$ is different to all $\boldsymbol{\pi}^{\mathbf{b}}(\zeta_i)$ for i odd.

So we can set $A' = \{\zeta_j : j \text{ odd}\}$ and get that $f[A']$ is disjoint to $\boldsymbol{\pi}^{\mathbf{b}}[A']$. So $\mathbf{b}$ with A' added to $\mathbf{B}$ spoils (f, A). □

Lemma 4.7. *If* $\bar{\mathbf{a}}$ *is short,* $b \in \mathrm{IS}$*,* δ^b *good for* $\bar{\mathbf{a}}$*,* $f \in \mathrm{Sym}(\lambda)$ *and* $A \in [\lambda]^\lambda$*, then there is some* $d >^{f,A}_{\bar{\mathbf{a}}} b$*.*

Proof. Let $\mathbf{B} := \bigcup_{i\in J} \mathbf{B}^{\mathbf{a}_i}$. Let $\zeta_0 < \lambda$ be the supremum of all $\mathbf{C}^{\mathbf{a}_i}$-successors of δ^b.

Set $E := \{\zeta \in \lambda : f \restriction \zeta \in \mathrm{Sym}(\zeta)\}$ (a club-set). Pick $\zeta_1 \in E$ such that $|A \cap (\zeta_1 \setminus \zeta_0)| > |2^{\mathbf{B}}|$. Pick $c >_{\bar{\mathbf{a}}} b$ with $\delta^c \in E \setminus \zeta_1$ and such that $C^c = C^b \cup \{\delta^b\}$.

Set $I := \delta^c \setminus \zeta_0$. For α, β in $I \cap A$ set $\alpha \sim \beta$ iff $(\forall A \in \mathbf{B})\,(\alpha \in A \leftrightarrow \beta \in A)$. As there are at most $|2^{\mathbf{B}}|$ many equivalence classes, there have to be $\beta_0 \neq \beta_1$ in $I \cap A$ with $\beta_0 \sim \beta_1$.

If $\pi^c(\beta_i) \neq f(\beta_i)$ for $i = 0$ or $i = 1$, set $d := c$. Otherwise, defines d as follows:

$$\delta^d = \delta^c,\ C^d = C^c, \text{ and } \pi^d(\alpha) := \begin{cases} \pi^c(\beta_1) & \text{if } \alpha = \beta_0, \\ \pi^c(\beta_0) & \text{if } \alpha = \beta_1, \\ \pi^c(\alpha) & \text{otherwise.} \end{cases}$$

Set $I := \delta^d \setminus \delta^b$. As $\beta_0 \sim \beta_1$ we have $\pi^d[A \cap I] = \pi^c[A \cap I] = \boldsymbol{\pi}^{\mathbf{a}_i}[A \cap I]$ for all $i \in J$ and $A \in \mathbf{B}^{\mathbf{a}_i}$ (as $c >_{\bar{\mathbf{a}}} b$).

And as the β_0, β_1 are above ζ_0, and $I^{\mathbf{a}_i}_{\delta^b}$ is below ζ_0 for all $i \in J$, we have $\pi^d \restriction I^{\mathbf{a}_i}_{\delta^b} = \pi^c \restriction I^{\mathbf{a}_i}_{\delta^b} = \boldsymbol{\pi}^{\mathbf{a}_i} \restriction I^{\mathbf{a}_i}_{\delta^b}$.

So $d >_{\bar{\mathbf{a}}} b$. □

5. $2^\lambda = \lambda^+$ for λ inaccessible implies a nowhere trivial automorphism

Lemma 5.1. *Every increasing sequence in* AP_λ *of length* $<\lambda^+$ *has an upper bound.*

Proof. We can assume without loss of generality that the increasing sequence is $\bar{a} := (\mathbf{a}_i)_{i\in\xi}$ with $\xi \le \lambda$.

For $i<\xi$, enumerate[1] $\mathbf{B}^{\mathbf{a}_i}$ as $\{x_i^j : j \le \lambda\}$, and set $B_i^j := \{x_i^k : k \le j\}$ for $j<\lambda$. We enumerate in a way so that the B_i^j are increasing with $i<\xi$. Let $\mathbf{a}_i^j$ be $\mathbf{a}_i$ with $\mathbf{B}$ replaced by B_i^j, and for $\ell<\lambda$ set $\bar{\mathbf{a}}^\ell := (\mathbf{a}_k^\ell)_{k<\min(\ell,\xi)}$. Note that $\bar{\mathbf{a}}^\ell$ is short.

$\mathbf{c} \in \mathrm{AP}$ is an upper bound of $\bar{\mathbf{a}}$ iff it is an upper bound of all $\mathbf{a}_k^\ell$ for $\ell<\lambda$ and $k<\min(\ell,\xi)$.

We now construct by induction on $\ell<\lambda$ a $<_{\mathrm{IS}}$-increasing continuous sequence $(c^\ell)_{\ell\in\lambda}$, such that δ^{c^ℓ} is $\bar{\mathbf{a}}^\ell$-good:

- At limits γ we let c^γ be the limit of the $(c^k)_{k<\gamma}$, and note that (by induction) its height it is $\bar{\mathbf{a}}^\gamma$-good.
- For $j=\ell+1$, let E be the club set of $\bar{\mathbf{a}}^{\ell+1}$-good ordinals, and choose, as in Lemma 4.4(5) $c^{\ell+1} >_{\bar{\mathbf{a}}^\ell} c^\ell$ with $\delta^{c^{\ell+1}} \in E$.

Let $\mathbf{c}$ be the limit of the c^ℓ with $\mathbf{B}^{\mathbf{c}} := \bigcup_{i<\xi} \mathbf{B}^{\mathbf{a}_i}$.

We claim that $\mathbf{c} \ge_{\mathrm{AP}} \mathbf{a}_j^\ell$ for all $\ell<\lambda$ and $j<\min(\ell,\xi)$. Assume that $k>\max(i,j)$.

- By Lemma 4.4(3):
  δ^{c^k} (which is $\bar{\mathbf{a}}^k$-good and so, by definition, $\mathbf{a}_j^k$-good) is $\mathbf{a}_j^\ell$-good, as $\mathbf{a}_j^k >_{\delta^{c^k}} \mathbf{a}_j^\ell$.
  Also, $c^{k+1} >_{\bar{\mathbf{a}}^k} c^k$, so (by definition) $c^{k+1} >_{\mathbf{a}_j^k} c^k$, and so $c^{k+1} >_{\mathbf{a}_j^\ell} c^k$.
- By Lemma 4.4(4) we get $\mathbf{c} >_{\mathrm{AP}} \mathbf{a}_j^\ell$, as required. □

Lemma 5.2. *Given* $\mathbf{a} \in \mathrm{AP}_\lambda$, $f \in \mathrm{Sym}(\lambda)$ *and* $A \in [\lambda]^\lambda$, *there is a* $\mathbf{b} \ge_{\mathrm{AP}} \mathbf{a}$ *which is in* AP_λ *and spoils* (f,A).

Proof. Enumerate $\mathbf{B}^{\mathbf{a}}$ as $\{x^j : j\in\lambda\}$ and let $\mathbf{a}^j$ be $\mathbf{a}$ with $\mathbf{B}$ replaced by $\{x^i : i<j\}$. So $\mathbf{a}^j \in \mathrm{AP}_{<\lambda}$. We construct a continuous increasing sequence b^i $(i<\lambda)$ in IS such that δ^{b^i} is $\mathbf{a}^i$-good: Given b^i, we find $b^{i+1} >_{\mathbf{a}^i}^{f,A} b^i$ as in Lemma 4.7. Let $\mathbf{b}$ be the limit of the b^i with $\mathbf{B}^{\mathbf{b}} = \mathbf{B}^{\mathbf{a}} \cup \{A'\}$ as in Lemma 4.6.

And $\mathbf{b} >_A P\mathbf{a}^j$ for all $j<\lambda$ and therefore $\mathbf{b} >_{\mathrm{AP}} \mathbf{a}$. □

We can now easily show:

Theorem 5.3. *If* λ *is (strongly) inaccessible and* $2^\lambda = \lambda^+$, *then there is a nowhere trivial automorphism of the Boolean algebra* $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.

Proof. We construct, by induction on $i \in \lambda^+$, an increasing chain of $\mathbf{a}_i$ in AP_λ, such that:

- For limit i, we take limits according to Lemma 5.1.
- For odd successors $i=j+1=\delta+2n+1$ (δ limit, $n\in\omega$), pick by bookkeeping some X_j and let $\mathbf{a}_{j+1}$ be the same as $\mathbf{a}_j$ but with X_j and $(\boldsymbol{\pi}^{\mathbf{a}_j})^{-1}[X_j]$ added to $\mathbf{B}$.
- For even successors $i=j+1=\delta+2n+2$, we pick by book-keeping an $f_j \in \mathrm{Sym}(\lambda)$ and an $A_j \in [\lambda]^\lambda$. Then we choose $\mathbf{a}_{j+1} \ge_{\mathrm{AP}} \mathbf{a}_j$ spoiling (f_j, A_j), using Lemma 5.2.

[1] with lots of repetitions

Then $\tilde{\phi} := \bigcup_{i<\lambda} \tilde{\pi}^{\mathbf{a}_i}$ is a nowhere trivial automorphism according to Fact 3.6. $\square$

6. Forcing a nowhere trivial automorphism with $2^\lambda > \lambda^+$, λ inaccessible

Theorem 6.1. *Assume λ is inaccessible, $2^\lambda = \lambda^+$ and $\mu > \lambda^+$ is regular. Then there is a cofinality preserving ($<\lambda$-closed and λ^+-cc) poset which forces: $2^\lambda = \mu$, and there is a nowhere trivial automorphism of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.*

For the rest of this section we fix a μ as in the lemma.

We will construct a $<\lambda$-support iteration $(P_\alpha, Q_\alpha)_{\alpha<\mu}$. We call the final limit P. We denote the P_α-extension $V[G_\alpha]$ by V_α.

Each Q_α and therefore also each P_α will be $<\lambda$-closed.

So $x \in \mathrm{AP}$, $x <_{\mathrm{AP}} y$, as well as IS (as set) are absolute between P_α-extensions (and $|\mathrm{IS}| = \lambda$).

Each Q_α will add a $\mathbf{a}^*_\alpha \in \mathrm{AP}$, such that the $\mathbf{a}^*_\alpha$ are $<_{\mathrm{AP}}$-increasing in α.

By induction we assume we live in the P_α-extension V_α where we already have the increasing sequence $(\mathbf{a}^*_i)_{i<\alpha}$. (We do not claim that this sequence has an upper bound in V_α.)

We now define Q_α, which we will just call Q to improve readability.

Definition 6.2. $q \in Q$ consists of:

1. A $b^q \in \mathrm{IS}$, also called "trunk of q".
   We also write δ^q, π^q C^q and I^q_β instead of δ^{b^q} etc.
2. A set $X^q \in [\alpha]^{<\lambda}$, and for $\beta \in X^q$, a set $\mathbf{B}^q_\beta \in [\mathbf{B}^{\mathbf{a}^*_\beta}]^{<\lambda}$, such that the $\mathbf{B}^q_\beta$ are increasing in β.
3. For $\beta \in X^q$ set $\mathbf{a}^q_\beta$ to be $\mathbf{a}^*_\beta$ with $\mathbf{B}$ replaced by $\mathbf{B}^q_\beta$. Set $\bar{\mathbf{a}}^q := (\mathbf{a}^q_\beta)_{\beta\in X^q}$ (which is short).
4. We require δ^{b^q} to be good for $\bar{\mathbf{a}}^q$.

("Short" and "good" are defined in Definition 3.8.) As we use Q as forcing poset, we follow the notation that $r \leq_Q q$ means that r is stronger than q (whereas in $<_{\mathrm{AP}}$ and $<_{\mathrm{IS}}$ the stronger object is the larger one).

Definition 6.3. $r \leq_Q q$ if:

1. $b^r \geq_{\bar{\mathbf{a}}^q} b^q$ (see Definition 4.3).
2. $X^r \supseteq X^q$, and $\mathbf{B}^r_\beta \supseteq \mathbf{B}^q_\beta$ for $\beta \in X^q$.

The following follows immediately from the definitions:

Fact 6.4. Assume that $r \leq_Q q$, $b \in \mathrm{IS}$ and that δ^b is good for $\bar{\mathbf{a}}^r$. Then $c \geq_{\bar{\mathbf{a}}^r} b$ implies $c \geq_{\bar{\mathbf{a}}^q} b$.

This implies that $\leq_Q$ is transitive. (It even is a partial order.)

Lemma 6.5. *For $q \in Q$, the following holds (in V_α): Let $E \subseteq \lambda$ be club.*

1. *For $\beta < \alpha$ and $A \in \mathbf{B}^{\mathbf{a}^*_\beta}$ there is an $r <_Q q$ with $\delta^r \in E$, $\beta \in X^r$ and $A \in \mathbf{B}^r_\beta$.*
2. *For any $A \in [\lambda]^\lambda$ and $f \in \mathrm{Sym}(\lambda)$ (both in V_α) there is an $r \leq_Q q$ with $b^r <^{f,A} b^q$.*

(3) Q is λ-centered, witnessed by the function that maps q to its trunk, b^q.
(Actually, even $<\lambda$ many conditions with the same trunk have lower bound.)

(4) Q_α is $<\lambda$-closed.
Moreover, a sequence $(q_i)_{i\in\xi}$ ($\xi < \lambda$) has a canonical limit r, and the trunk of r is the union of the trunks of the q_i.

Proof. (1): Extend $\bar{\mathbf{a}}^q$ in the obvious way to $\bar{\mathbf{a}}^r$: Add β to the index set, set $\mathbf{B}^r_\beta := \{A\} \cup \bigcup_{\zeta\in X^q\cap(\beta+1)} \mathbf{B}^q_\zeta$, and add A to all $\mathbf{B}^q_\zeta$ for $\zeta \in X^q \setminus \beta$. Let $E' := \{\zeta \in \lambda : \zeta \text{ good for } \bar{\mathbf{a}}^r\}$. Then E' is club according to Fact 3.9, so we can use Lemma 4.4(5) to find $b^r >_{\bar{\mathbf{a}}^q} b^q$ with $\delta^r \in E \cap E'$.

(2) This is Lemma 4.7.

(3) Let $(q_i)_{i\in\mu}$, $\mu < \lambda$ all have the same trunk b. Then the following r is a condition in Q: $b^r = b$, $X^r = \bigcup_{i<\mu} X^{q_i}$ and $B^r_\zeta = \bigcup_{i<\mu\ \&\ \zeta\in X^{q_i}} B^{q_i}_\zeta$.

(4) Let $(q_i)_{i<\zeta}$ with $\zeta < \lambda$ be $<_Q$-decreasing. Then the obvious union r is an element of Q and stronger than each q_i:

b^r is the union of the b^{q_i}, as in Fact 4.2, and $X^r := \bigcup_{i<\zeta} X^{q_i}$ and $\mathbf{B}^r_\beta := \bigcup_{i<\zeta,\beta\in X^{q_i}} \mathbf{B}^{q_i}_\beta$ for each $\beta \in X^r$.

Then δ^r is good for $\mathbf{a}^r_\beta$ for $\beta \in X^r$: It is enough to show that δ^r is good for all $\mathbf{a}^{q_i}_\beta$ (for sufficiently large i). Fix such an i. If $j > i$, then δ^{q_j} is good for $\bar{\mathbf{a}}^{q_j}$ and therefore for $\mathbf{a}^{q_j}_\beta$ and therefore for $\mathbf{a}^{q_i}_\beta$. So the limit δ^r is good as well.

Similarly one can argue that $b^r >_{\bar{\mathbf{a}}^{q_i}} b^{q_i}$ for all $i < \zeta$. □

Definition 6.6. Let $G(\alpha)$ be Q_α-generic. We define $\mathbf{a}^*_\alpha$ (in $V_{\alpha+1}$) as follows:
$\mathbf{C}^{\mathbf{a}^*_\alpha} := \bigcup_{q\in G(\alpha)} C^q$, $\boldsymbol{\pi}^{\mathbf{a}^*_\alpha} := \bigcup_{q\in G(\alpha)} \pi^q$, and $\mathbf{B}^{\mathbf{a}^*_\alpha} := P(\lambda)$.

Lemma 6.7. *$P_{\alpha+1}$ forces:*

(1) $\mathbf{a}^*_\alpha >_{\mathrm{AP}} \mathbf{a}^*_\beta$ for all $\beta < \alpha$.

(2) $\mathbf{a}^*_\alpha$ spoils (f, A) for all $(f, A) \in V_\alpha$.

The proof consists of straightforward density arguments:

Proof. For (1) we know that by there is some $q \in G(\alpha)$ with $\beta \in X^q$. This implies that $\mathbf{C}^{\mathbf{a}^*_\alpha} \subseteq \mathbf{C}^{\mathbf{a}^*_\beta}$ above δ^q and that $\boldsymbol{\pi}^{\mathbf{a}^*_\alpha} \restriction I^{\mathbf{a}^*_\beta}_\zeta = \boldsymbol{\pi}^{\mathbf{a}^*_\beta} \restriction I^{\mathbf{a}^*_\beta}_\zeta$ for all $\zeta \in \mathbf{C}^{\mathbf{a}^*_\beta} \setminus \delta^q$. We can also assume that a given $A \in \mathbf{B}^{\mathbf{a}^*_\beta}$ is in $\mathbf{B}^q_\beta$, which implies that $\boldsymbol{\pi}^{\mathbf{a}^*_\alpha}[A] = \boldsymbol{\pi}^{\mathbf{a}^*_\beta}[A]$ above δ^q.

For (2) and $(f, A) \in V_\alpha$ we know by Lemma 6.5(2) that for $q \in G(\alpha)$ of unbounded heights there are $r(q)$ in $G(\alpha)$ such that $b^{r(q)} >^{f,A} b^q$. I.e, in $V_{\alpha+1}$, $\mathbf{a}^*_\alpha$ is a limit of an $<_{\mathrm{IS}}$-increasing sequence as in Lemma 4.6, therefore $\mathbf{a}^*_\alpha$ spoils (f, A) (as A' certainly is in $\mathbf{B}^{\mathbf{a}^*_\alpha} = P(\lambda)$). □

So P adds a sequence $(\mathbf{a}^*_\alpha)_{\alpha<\mu}$ that we can use in Fact 3.6 to get a nowhere trivial automorphism. We will now show that P is λ^+-cc, which finishes the proof of Theorem 6.1.

Lemma 6.8. *Set $t(p) := (b^{p(\alpha)})_{\alpha\in\mathrm{dom}(p)}$ (i.e., the sequence of trunks). Then the following set D is dense: p in D if there is an $x \in V$ such that the empty condition forces $t(p) = x$.*

Proof. We claim that the lemma holds for P_α, by induction on P_α. Successors and limits of cofinality $\geq\lambda$ are clear.

Let α be a limit with cofinality $\kappa < \lambda$, and $(\alpha_i)_{i\in\kappa}$ cofinal in α, $\alpha_0 = 0$. Set $D_j := D \cap P_{\alpha_j}$ (by induction dense in P_{α_j}). We construct by induction on $j \in \kappa$ a decreasing sequence $p_j \in P_\alpha$ such that $p_0 = p$ and $p_j \restriction \alpha_j \in D$:

Successors: Given p_j, we find $r \leq p_j \restriction \alpha_{j+1}$ in D_{j+1} and set $p_{j+1} := r \wedge p_j$ (which is the same as $r \wedge p$).

Limits: Given $(p_i)_{i<\xi}$ with $\xi \leq \kappa$, let p_ξ be the pointwise canonical limit. Note that we can calculate (in V) each $p_\xi(\beta)$ from the sequence $(p_i(\beta))_{i<\xi}$ (it is just the union). □

Lemma 6.9. *(Assuming $2^\lambda = \lambda^+$ in the ground model.) P is λ^+-cc.*

Proof. Assume $(a_i)_{i\in\lambda^+}$ is a sequence in P. For every a_i find an $a'_i \leq a$ in D. By Fodor (or the Delta-system lemma) there is an $X \subseteq \lambda^+$ of size λ^+ such that $\{\mathrm{dom}(a'_i) : i \in X\}$ form a Delta system with heart Δ, and furthermore we can assume that $t(a'_i) \restriction \Delta$ (the sequence of trunks restricted to Δ) is the same for all $i \in X$. (There are $\lambda^{|\Delta|} = \lambda < \lambda^+$ many such restrictions.) Then for i, j in X, the conditions a'_i and a'_j (and therefore also a_i and a_j) are compatible. □

Remark 6.10. Generally, preserving λ^+-cc for $\lambda > \omega_1$ is much more cumbersome than for $\lambda = \omega$, as there is no obvious universal theorem analogous to "the finite support iteration of ccc forcings is ccc". In our case, it was very easy to show λ^+-cc manually. However, we could have used existing iteration theorems. We give two examples (but there surely are many more). Note that the following theorems do not require λ to be inaccessible.

(1) From [Shi99] (generalising the $\lambda = \aleph_1$ case from [Bau83, Lem. 4.1]):
  - Definition [Shi99, p. 237]: Q is λ-centered closed, if a centered subset D of Q of size $<\lambda$ has a lower bound.
  - Lemma [Shi99, p. 237]: Assume $2^{<\lambda} = \lambda$. Let P be a $<\lambda$-support iteration such that each iterand is (forced to be) λ-linked and λ-centered closed. Then P is λ^+-cc.

  It is easy to see that our Q satisfies the requirements (Q is even λ-centered and "λ-linked closed").

(2) From [BGS21] (generalizing the $\lambda = \aleph_1$ case from [She78, 3.1]):
  - [BGS21, Def. 2.2.2]: Q is "stationary λ^+-Knaster", if for every sequence $(p_i)_{i<\lambda^+}$ in Q there exists a club $E \subseteq \lambda^+$ and a regressive function f on $E \cap S^{\lambda^+}_\lambda$ such that p_i and p_j are compatible whenever $f(i) = f(j)$.
  - [BGS21, Lem. 2.2.5]: Assume that P is a $<\lambda$-support iteration of iterands that all are: stationary λ^+-Knaster, strategically $<\lambda$-closed, and any two compatible conditions have a greatest lower bound, as do decreasing ω-sequences. Then P is stationary λ^+-Knaster.

  Note that our Q satisfies the requirements, and that our proof of λ^+-cc actually shows stationary λ^+-Knaster.

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