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Research Article

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On projections of the tails of a power

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Abstract: Let κ be an inaccessible cardinal, $\mathfrak{U}$ a universal algebra, and $\sim$ the equivalence relation on $\mathfrak{U}^\kappa$ of eventual equality. From mild assumptions on κ, we give general constructions of $\mathcal{E} \in \mathrm{End}(\mathfrak{U}^\kappa/\sim)$ satisfying $\mathcal{E} \circ \mathcal{E} = \mathcal{E}$ which do not descend from $\Delta \in \mathrm{End}(\mathfrak{U}^\kappa)$ having small strong supports. As an application, there exists an $\mathcal{E} \in \mathrm{End}(\mathbb{Z}^\kappa/\sim)$ which does not come from a $\Delta \in \mathrm{End}(\mathbb{Z}^\kappa)$.

Keywords: Endomorphism, idempotence, inaccessible cardinal

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1 Introduction

Questions as to whether a mapping on a quotient can be lifted to the original structure are abundant in mathematics (in algebraic topology, group theory, etc.). When the mapping cannot be (easily) lifted, then it is of particular interest and intrinsic to the quotient structure. Our focus is on the power of a universal algebra and its natural quotient – the set of "tails".

Let κ be an infinite regular cardinal and $\mathfrak{U}$ a universal algebra. We take each ordinal to be the set of ordinals strictly below it. Write $\sigma \sim \tau$ if $\sigma, \tau \in \mathfrak{U}^\kappa$ are eventually equal (i.e. $|\{\alpha \in \kappa : \sigma(\alpha) \neq \tau(\alpha)\}| < \kappa$) and write $p_\sim : \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa/\sim$ for the quotient homomorphism. It is not difficult to produce a $\Delta \in \mathrm{End}(\mathfrak{U}^\kappa)$ for which there is no $\mathcal{E} \in \mathrm{End}(\mathfrak{U}^\kappa/\sim)$ which makes the following diagram commute:

$$\begin{array}{ccc} \mathfrak{U}^\kappa & \xrightarrow{\Delta} & \mathfrak{U}^\kappa \\ \downarrow{\scriptstyle p_\sim} & & \downarrow{\scriptstyle p_\sim} \\ \mathfrak{U}^\kappa/\sim & \xrightarrow{\mathcal{E}} & \mathfrak{U}^\kappa/\sim. \end{array}$$

This happens precisely when Δ makes essential use of early coordinates in defining cofinal coordinates. For example, the homomorphism $(\Delta\sigma)(\alpha) = \sigma(0)$ for all $\alpha \in \kappa$ does not descend to an $\mathcal{E} \in \mathrm{End}(\mathfrak{U}^\kappa/\sim)$ when the underlying set of $\mathfrak{U}$ has at least two elements. By contrast, it is generally less clear how to produce an $\mathcal{E}$ which is not descended from a Δ. In the parlance of calculus, does there exist an epsilon for which there is no delta? We will restrict our attention to the especially challenging situation where $\mathcal{E}$ is required to be a *homomorphic projection* ($\mathcal{E} \circ \mathcal{E} = \mathcal{E}$). The following classical example will motivate us.

Example 1. Letting $\kappa = \aleph_0$, it is well known that the abelian group $\mathbb{Z}^{\aleph_0}/\sim$ has a direct summand which is isomorphic to $\mathbb{Q}$. Taking $\mathcal{E} \in \mathrm{End}(\mathbb{Z}^{\aleph_0}/\sim)$ to be a homomorphic projection whose image is $\mathbb{Q} \leq \mathbb{Z}^{\aleph_0}/\sim$, we claim there is no Δ with $\mathcal{E} \circ p_\sim = p_\sim \circ \Delta$.

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To see this, suppose such a Δ exists. If $\Delta(\mathbb{Z}^{\aleph_0})$ is uncountable, then $p_\sim \circ \Delta(\mathbb{Z}^{\aleph_0}) \simeq \mathbb{Q}$ is uncountable since $p_\sim$ has countable kernel. Thus $\Delta(\mathbb{Z}^{\aleph_0})$ is countable, hence free abelian [1, Theorem 3.8.2], and therefore finitely generated [1, Lemmas 13.2.1, 13.2.3], and so $p_\sim \circ \Delta(\mathbb{Z}^{\aleph_0}) \simeq \mathbb{Q}$ is finitely generated, a contradiction.

It is not obvious how to produce such an $\mathcal{E}$ when $\aleph_0$ is replaced with an uncountable regular cardinal κ since $\mathbb{Z}^\kappa/\sim$ will then be $\aleph_1$-free abelian (hence cotorsion-free) [5], so in particular, we cannot project to a nonzero divisible subgroup. A large cardinal will allow us to determine such a homomorphism, and the map will be of the flavor of the next example.

Example 2. Let κ be a measurable cardinal and $\mathcal{U}$ an ultrafilter on κ witnessing this. Define $\Delta\colon \mathbb{Z}^\kappa \to \mathbb{Z}^\kappa$ by letting $\Delta\sigma$ be the constant sequence $(z_0)_{\alpha\in\kappa}$, where $\{\alpha \in \kappa : \sigma(\alpha) = z_0\} \in \mathcal{U}$. As $\mathcal{U}$ is nonprincipal and κ-complete, Δ descends to a homomorphic projection $\mathcal{E}$ with $\mathcal{E} \circ p_\sim = p_\sim \circ \Delta$.

Of course, the $\mathcal{E}$ in Example 2 descends from a Δ, but the map is defined set-theoretically. Moreover, the construction is quite general as one can replace $\mathbb{Z}$ with any $\mathfrak{U}$ having underlying set smaller than κ.

We will provide two definitions toward stating the main theorem. Letting $\mathfrak{U} = (\chi, \mathcal{S})$, we say a subset $Y \subseteq \kappa$ is a *strong support* for homomorphism $\Theta\colon \mathfrak{U}^\kappa \to \mathfrak{M}$ if $\sigma \restriction Y = \tau \restriction Y$ implies $\Theta\sigma = \Theta\tau$ and, for each $y \in Y$ and $\sigma \in \mathfrak{U}^\kappa$, there exists $x \in \chi$ such that if $\tau \in \mathfrak{U}^\kappa$ is given by

$$\tau(\alpha) = \begin{cases} \sigma(\alpha) & \text{if } \alpha \neq y, \\ x & \text{if } \alpha = y, \end{cases}$$

then $\Theta\sigma \neq \Theta\tau$. A homomorphism may or may not have a strong support. If $\mathbb{F}$ is a field, then we can take $\Theta\colon \mathbb{F}^\kappa \to \mathbb{F}$ to be any homomorphism which extends the homomorphism $\bigoplus_\kappa \mathbb{F} \to \mathbb{F}$ which takes the sum of the entries (such a Θ exists by choosing a basis B_0 of $\bigoplus_\kappa \mathbb{F}$ and extending to a basis $B_1 \supseteq B_0$ of $\mathbb{F}^\kappa$ and defining $\Theta v = 0$ for $v \in B_1 \setminus B_0$). The strong support for this Θ is the entire set κ. On the other hand, if Θ is an extension of the map which takes the constant $(1_\mathbb{F})_{\alpha\in\kappa}$ to $1_\mathbb{F}$ and all elements of $\bigoplus_\kappa \mathbb{F}$ to $0_\mathbb{F}$, then Θ has no strong support.

We say a homomorphism $\mathbf{Av}\colon \mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$ is a *changing average* if, for all $x_0, x_1 \in \chi$, we have

$$\mathbf{Av}(x_0, x_1) = \mathbf{Av}(x_1, x_0) \quad \text{and} \quad \mathbf{Av}(x_0, x_0) = x_0$$

and there exist $x_2, x_3 \in \chi$ for which $x_2 \neq \mathbf{Av}(x_2, x_3) \neq x_3$. In this paper, we give two technical constructions which specialize to the main result (see Remark 2.4 and Theorems 3.8 and 4.10). The map $\pi_\alpha\colon \mathfrak{U}^\kappa \to \mathfrak{U}$ is projection to the α coordinate.

Main Theorem. *Suppose that κ is inaccessible with $\kappa^+ = 2^\kappa$. If $\mathfrak{U} = (\chi, \mathcal{S})$ is a universal algebra with $2 \leq |\chi| < \kappa$, then there exists a homomorphic projection $\mathcal{E} \in \mathrm{End}(\mathfrak{U}^\kappa/\sim)$ such that*

(1) *the image of $\mathcal{E}$ is isomorphic to $\mathfrak{U}^\kappa/\sim$; and*

(2) *for any $\Delta \in \mathrm{End}(\mathfrak{U}^\kappa)$ with $\mathcal{E} \circ p_\sim = p_\sim \circ \Delta$, we have*

$$|\{\alpha \in \kappa : \pi_\alpha \circ \Delta \text{ has no strong support of cardinality} \leq \lambda\}| = \kappa$$

for each $\lambda < \kappa$.

Additionally, if $\mathfrak{U}$ has a changing average,

(3) *there exists $\sigma \in \mathfrak{U}^\kappa$ such that, when $\mathcal{E}[\sigma]_\sim = [\tau]_\sim$, we get*

$$|\{\alpha \in \kappa : \sigma(\alpha) = \tau(\alpha)\}| < \kappa.$$

When there is no changing average, the projection $\mathcal{E}$ essentially spreads some coordinates sideways in a complicated way. When there is a changing average, we obtain a projection which is more algebraically elaborate. Of course, the theorem is most interesting for very specific algebras $\mathfrak{U}$. For example, if $\mathfrak{U}$ is a slender abelian group A (i.e. A is torsion-free and includes no subgroups isomorphic to $\mathbb{Q}$, $\mathbb{Z}^{\aleph_0}$, or the p-adics J_p with p a prime), we know that every homomorphism from A^κ to A has a finite strong support when κ is smaller than the least measurable cardinal [1, Corollary 13.2.10]. The following is immediate.

Corollary 1.1. *Let A be a nontrivial slender abelian group. If κ is an inaccessible which is smaller than any measurable cardinal and $\kappa^+ = 2^\kappa$, then there exists a homomorphism projection $\mathcal{E} \in \mathrm{End}(A^\kappa/\sim)$ for which there is no $\Delta \in \mathrm{End}(A^\kappa)$ such that $\mathcal{E} \circ p_\sim = p_\sim \circ \Delta$. Moreover, if $A = 2A$, then there is $\sigma \in A^\kappa$ for which*

$$|\{\alpha \in \kappa : \sigma(\alpha) = \tau(\alpha)\}| < \kappa \quad \text{for } \mathcal{E}[\sigma]_\sim = [\tau]_\sim.$$

For the last sentence in the corollary, we use the average function $\mathbf{Av}(x_0, x_1) = \frac{x_0+x_1}{2}$. We may let A in Corollary 1.1 be $\mathbb{Z}$ or more generally a free abelian group of rank less than κ, and for an application which involves the second sentence of the corollary, we can take $A = \mathbb{Z}[\frac{1}{2}]$. The hypothesis $\kappa^+ = 2^\kappa$ in the main theorem can be relaxed in a technical way (see Remark 2.4). The assumptions on κ used in our theorem (κ is inaccessible and $\kappa^+ = 2^\kappa$) are quite mild compared to that used in Example 2 (κ is measurable); if μ is the least measurable cardinal, then the set of inaccessible cardinals smaller than μ is of cardinality μ (see [2, Lemma 10.21]).

The proof of our theorem will involve modifications of the main arguments of [3]. We give some preliminaries in Section 2 and then give the first and second constructions in Sections 3 and 4, respectively.

2 Some preliminaries

The reader may reference [2] for most set-theoretic terminology. We review some notational conventions below.

Notation 2.1. We will let

- Card denote the class of cardinals;
- $[X]^\lambda = \{Y : Y \subseteq X, |Y| = \lambda\}$;
- $[X]^{<\lambda} = \{Y : Y \subseteq X, |Y| < \lambda\}$;
- $\mathcal{P}(X)$ be the powerset of X;
- ${}^Y X$ be the set of functions from Y to X;
- $\mathbf{cof}(\alpha)$ denote the cofinality of ordinal α; and
- $X \equiv_\kappa Y$ mean that the symmetric difference $(X \cup Y) \setminus (X \cap Y)$ has cardinality less than κ.

Definition 2.2. For functions $f, g \in {}^\kappa\mathrm{Card}$, we let $\mathfrak{d}_{f,g}$ denote the least cardinality of a family $\mathcal{D} \subseteq \prod_{\nu\in\kappa}[f(\nu)]^{g(\nu)}$ such that, for each $F \in \prod_{\nu\in\kappa} f(\nu)$, there are $G \in \mathcal{D}$ and $\beta \in \kappa$ with $F(\nu) \in G(\nu)$ for $\nu \in \kappa \setminus \beta$.

For the following, see [4, Hypothesis 3.1].

Definition 2.3. For a cardinal κ, we write $\dagger(\kappa, f, g)$ if $f, g \in {}^\kappa(\kappa \cap \mathrm{Card})$ are such that $f(\nu)$ and $g(\nu)$ are infinite regular for all $\nu \in \kappa$ and

(1) $2^{g(\nu)} < f(\nu)$;
(2) if $\nu \in \nu^*$, then $|\nu| \le g(\nu) \le g(\nu^*)$; and
(3) $\mathfrak{d}_{2^f,g} = \kappa^+$.

Remark 2.4. It is clear from (1) and (2) of Definition 2.3 that if $\dagger(\kappa, f, g)$, then $\nu < \kappa$ implies $2^{|\nu|} < \kappa$, and also $\lim_{\nu\to\kappa} g(\nu) = \kappa$. We will also want κ to be a regular cardinal for our arguments, so our attention will be strictly on κ which is inaccessible.

When κ is inaccessible with $\kappa^+ = 2^\kappa$, we can take g and f to be any functions satisfying conditions (1) and (2), take an enumeration $\{H_\xi\}_{\xi\in\kappa^+}$ of $\prod_{\nu\in\kappa} f(\nu)$, $e_\xi : \kappa \to \xi$ to be a bijection for each $\xi \in \kappa^+ \setminus \kappa$, and $G_\xi(\nu) = \{H_{e_{\xi(\eta)}} : \eta \in g(\nu)\}$ to witness $\dagger(\kappa, f, g)$.

It is also possible to have an inaccessible κ and f, g such that $\dagger(\kappa, f, g)$ and $\kappa^+ < 2^\kappa$ (see [4, Corollary 3.2]).

The following appears as [4, Lemma 3.1]; we will provide the proof in extenso since that document has not been refereed and also for the sake of completeness.

Lemma 2.5. *If κ is inaccessible and $\dagger(\kappa, f, g)$, then there is a collection $\{Z_{\eta,\zeta}\}_{\eta\in\kappa^+,\zeta\in\kappa}$ satisfying the following:*

(1) $Z_{\eta,\zeta} \subseteq \eta$;
(2) $|Z_{\eta,\zeta}| \le g(\zeta)$;
(3) *if $\zeta < \zeta^*$, then $Z_{\eta,\zeta} \subseteq Z_{\eta,\zeta^*}$;*

(4) $\bigcup_{\zeta\in\kappa} Z_{\eta,\zeta} = \eta$;
(5) *if* $\eta \in \eta^*$, *then there is* $\beta \in \kappa$ *such that* $Z_{\eta^*,\zeta} \cap \eta = Z_{\eta,\zeta}$ *for* $\zeta \in \kappa \setminus \beta$; *and*
(6) *if* $\eta \in Z_{\eta^*,\zeta}$, *then* $Z_{\eta^*,\zeta} \cap \eta = Z_{\eta,\zeta}$.

Proof. The sets $Z_{\eta,\zeta}$ are defined by induction on the first parameter $\eta \in \kappa^+$. We let $Z_{0,\zeta} = \emptyset$ for all $\zeta \in \kappa$ (as required by condition (1)). That (1)–(6) hold is easy to see. Suppose that $Z_{\eta,\zeta}$ have been defined for all $\eta \in \xi$ and for all $\zeta \in \kappa$. We will treat three cases.

Case one: $\xi = \xi^* + 1$. In this case, we let $Z_{\xi,\zeta} = Z_{\xi^*,\zeta} \cup \{\xi^*\}$. That (1) holds at this stage is quite clear, and (2) holds because $g(\zeta)$ is an infinite cardinal. Condition (3) is clear, and (4) holds at ξ because (4) holds at ξ^*.

Condition (5) is true by induction. Let $\eta < \xi$ be given. If $\eta < \xi^*$ then by induction we select $\beta \in \kappa$ such that $Z_{\xi^*,\zeta} \cap \eta = Z_{\eta,\zeta}$ for all $\zeta \in \kappa \setminus \beta$, and clearly $Z_{\xi,\zeta} \cap \eta = Z_{\eta,\zeta}$ for all $\zeta \in \kappa \setminus \beta$. If $\eta = \xi^*$ then we can simply let $\beta = 0$. Condition (6) holds by induction, looking at cases $\eta < \xi^*$ and $\eta = \xi$.

Case two: ξ is a limit ordinal and $\mathbf{cof}(\xi) < \kappa$. Let $\{\gamma_\nu\}_{\nu<\mathbf{cof}(\xi)}$ be a strictly increasing sequence with $\sup_{\nu<\mathbf{cof}(\xi)} \gamma_\nu = \xi$. As (5) holds below ξ, for each pair ν, ν^* with $\nu < \nu^* < \mathbf{cof}(\xi)$, we select $\beta_{\nu,\nu^*} \in \kappa$ such that $Z_{\gamma_{\nu^*},\zeta} \cap \gamma_\nu = Z_{\gamma_\nu,\zeta}$ for all $\zeta \in \kappa \setminus \beta_{\nu,\nu^*}$. As (4) holds below ξ, for each pair ν, ν^* with $\nu < \nu^* < \mathbf{cof}(\xi)$, select $\epsilon_{\nu,\nu^*} \in \kappa$ such that $\gamma_\nu \in Z_{\gamma_{\nu^*},\epsilon_{\nu,\nu^*}}$. Let $\delta = \sup\{\beta_{\nu,\nu^*}\}_{\nu<\nu^*<\mathbf{cof}(\xi)} \cup \{\epsilon_{\nu,\nu^*}\}_{\nu<\nu^*<\mathbf{cof}(\xi)}$, and as $\mathbf{cof}(\xi) < \kappa$, we have $\delta \in \kappa$. If $\alpha > \delta$, we have for all $\nu < \nu^* < \mathbf{cof}(\xi)$ that

$$\gamma_\nu \in Z_{\gamma_{\nu^*+1},\epsilon_{\nu,\nu^*+1}} \cap \gamma_{\nu^*} \subseteq Z_{\gamma_{nu^*+1},\max(\epsilon_{\nu,\nu^*+1},\beta_{\nu^*,\nu^*+1})} \cap \gamma_{\nu^*} \subseteq Z_{\gamma_{\nu^*+1},\alpha} \cap \gamma_{\nu^*} = Z_{\gamma_{\nu^*}},$$

where the first two inclusions hold because condition (3) holds below ξ, and the last equality holds by choice of β_{ν^*,ν^*+1}. Picking $\alpha \in \kappa \setminus (\max(\mathbf{cof}(\xi), \delta))$, we therefore know that, for all $\zeta > \alpha$ and all $\nu < \nu^* < \mathbf{cof}(\xi)$, the equality $Z_{\gamma_{\nu^*},\zeta} \cap \gamma_\nu = Z_{\gamma_\nu,\zeta}$ holds. Let

$$Z_{\xi,\zeta} = \begin{cases} \emptyset & \text{if } \zeta \le \alpha, \\ \bigcup_{\nu<\mathbf{cof}(\xi)} Z_{\gamma_\nu,\zeta} & \text{if } \zeta \ge \alpha. \end{cases}$$

That (1) holds is clear. To see that condition (2) holds, we point out that

$$|Z_{\xi,\zeta}| \le \mathbf{cof}(\xi) g(\zeta) \le |\alpha| g(\zeta) \le |\zeta| g(\zeta) \le g(\zeta),$$

where the last inequality follows from condition (2) of $\dagger(\kappa, f, g)$. For condition (3), we notice that, given $\zeta < \zeta^* < \kappa$, we either have $\zeta \le \alpha$, in which case $Z_{\xi,\zeta} = \emptyset \subseteq Z_{\xi,\zeta^*}$, or $\alpha < \zeta$, and since $Z_{\gamma_\nu,\zeta} \subseteq Z_{\gamma_\nu,\zeta^*}$ holds for each $\nu < \mathbf{cof}(\xi)$, we get $Z_{\xi,\zeta} \subseteq Z_{\xi,\zeta^*}$. Condition (4) is clear by induction. To check condition (5), we let $\eta \in \xi$ be given and pick $\nu < \mathbf{cof}(\xi)$ such that $\eta < \gamma_\nu$. As condition (5) holds at γ_ν, we select $\beta \in \kappa \setminus \alpha$ such that $\zeta \in \kappa \setminus \beta$ implies $Z_{\gamma_\nu,\zeta} \cap \eta = Z_{\eta,\zeta}$. Recall that, by how α was selected, we have $Z_{\gamma_{\nu^*},\zeta} \cap \gamma_{\nu^{**}} = Z_{\gamma_{\nu^{**}},\zeta}$ for all $\nu^{**} < \nu^* < \mathbf{cof}(\xi)$ and $\zeta \in \kappa \setminus \beta$. Then, for $\zeta \in \kappa \setminus \beta$, we have

$$Z_{\xi,\zeta} \cap \eta = \Big(\bigcup_{\nu^*<\mathbf{cof}(\xi)} Z_{\gamma_{\nu^*},\zeta}\Big) \cap \eta = \Big(\bigcup_{\nu^*<\mathbf{cof}(\xi)} Z_{\gamma_{\nu^*},\zeta}\Big) \cap \gamma_\nu \cap \eta = Z_{\gamma_\nu,\zeta} \cap \eta = Z_{\eta,\zeta}.$$

For condition (6), we suppose that $\eta \in Z_{\xi,\zeta}$. As $Z_{\xi,\eta} \neq \emptyset$, we have $\zeta > \alpha$, so $Z_{\xi,\zeta} = \bigcup_{\nu<\mathbf{cof}(\xi)} Z_{\gamma_\nu,\zeta}$. Select $\nu < \mathbf{cof}(\xi)$ for which $\eta \in Z_{\gamma_\nu,\zeta}$. Then $Z_{\gamma_\nu,\zeta} \cap \eta = Z_{\eta,\zeta}$ since condition (6) holds at γ_ν. As $\zeta \in \kappa \setminus \beta$, we know that

$$Z_{\gamma_{\nu^*},\zeta} \cap \gamma_{\nu^{**}} = Z_{\gamma_{\nu^{**}},\zeta} \quad \text{for all } \nu^{**} < \nu^* < \mathbf{cof}(\xi),$$

and so

$$Z_{\xi,\zeta} \cap \eta = \Big(\bigcup_{\nu^*<\mathbf{cof}(\xi)} Z_{\gamma_{\nu^*},\zeta}\Big) \cap \eta = \Big(\bigcup_{\nu^*<\mathbf{cof}(\xi)} Z_{\gamma_{\nu^*},\zeta}\Big) \cap \gamma_\nu \cap \eta = Z_{\gamma_\nu,\zeta} \cap \eta = Z_{\eta,\zeta},$$

and all conditions are satisfied.

Case three: $\mathbf{cof}(\xi) = \kappa$. We take $\{\gamma_\nu\}_{\nu\in\kappa}$ to be a strictly increasing sequence such that $\sup_{\nu\in\kappa} \gamma_\nu = \xi$. As κ is regular and condition (5) holds below ξ, we can take $\{\beta_\theta\}_{\theta\in\kappa}$ to be a strictly increasing sequence of ordinals in κ such that $\beta_0 = 0$, and when $\nu < \theta$, we have for all $\zeta \ge \beta_\theta$ that $Z_{\gamma_\theta,\zeta} \cap \gamma_\nu = Z_{\gamma_\nu,\zeta}$. Let $Z_{\xi,\zeta} = Z_{\gamma_\theta,\zeta}$ for $\beta_\theta \le \zeta < \beta_{\theta+1}$. We check conditions (1)–(6).

To see why (1) and (2) hold, let $\zeta \in \kappa$ be given and take $\theta \in \kappa$ to be the unique element such that $\beta_\theta \leq \zeta < \beta_{\theta+1}$, and notice that

$$Z_{\xi,\zeta} = Z_{\gamma_\theta,\zeta} \subseteq \gamma_\theta \subseteq \xi \quad \text{and} \quad |Z_{\xi,\zeta}| = |Z_{\beta_\theta,\zeta}| \leq g(\zeta).$$

For (3), we let $\zeta < \zeta^* < \kappa$ and take $\beta_\theta \leq \zeta < \beta_{\theta+1}$ and $\beta_{\theta^*} \leq \zeta^* < \beta_{\theta^*+1}$. If $\theta = \theta^*$, then

$$Z_{\xi,\zeta} = Z_{\gamma_\theta,\zeta} \subseteq Z_{\gamma_{\theta^*},\zeta^*} = Z_{\xi,\zeta^*}$$

since condition (3) holds below ξ, and if $\theta < \theta^*$, then

$$Z_{\xi,\zeta} = Z_{\gamma_\theta,\zeta} \subseteq Z_{\gamma_\theta,\zeta^*} = Z_{\gamma_{\theta^*},\zeta^*} \cap \gamma_\theta \subseteq Z_{\gamma_{\theta^*},\zeta^*} = Z_{\xi,\zeta^*}.$$

For condition (4), we notice first that $\bigcup_{\zeta\in\kappa} Z_{\xi,\kappa} \subseteq \xi$ as condition (1) holds. For the reverse inclusion, we let $\alpha \in \xi$ be given. Select $\nu \in \kappa$ such that $\alpha \in \gamma_\nu$, and as condition (4) holds at γ_ν, we select $\zeta^* \in \kappa$ such that $\alpha \in Z_{\gamma_\nu,\zeta^*}$. Select $\nu < \theta \in \kappa$ so that there exists $\zeta^{**} \in \kappa$ with $\zeta^{**} > \zeta^*$ and $\beta_\theta \leq \zeta^{**} < \beta_{\theta+1}$. Then

$$\alpha \in Z_{\gamma_\nu,\zeta^*} \subseteq Z_{\gamma_\nu,\zeta^{**}} = Z_{\gamma_\theta,\zeta^{**}} \cap \gamma_\nu \subseteq Z_{\gamma_\theta,\zeta^{**}} = Z_{\xi,\zeta^{**}} \subseteq \bigcup_{\zeta\in\kappa} Z_{\xi,\zeta}.$$

For condition (5), we let $\eta \in \xi$. Select $\theta \in \kappa$ large enough that $\eta \in \gamma_\theta$. As condition (5) holds at γ_θ, we select $\beta \in \kappa$ such that $\zeta \in \kappa \setminus \beta$ implies $Z_{\gamma_\theta,\zeta} \cap \eta = Z_{\eta,\zeta}$. Fixing $\zeta \in \kappa \setminus \max(\beta, \beta_\theta)$, we select $\theta^* \in \kappa$ with $\beta_{\theta^*} \leq \zeta < \beta_{\theta^*+1}$. Then $\theta \leq \theta^*$ and $Z_{\xi,\zeta} = Z_{\gamma_{\theta^*},\zeta}$, so in particular,

$$Z_{\eta,\zeta} = Z_{\gamma_\theta,\zeta} \cap \eta = Z_{\gamma_{\theta^*},\zeta} \cap \gamma_\theta \cap \eta = Z_{\xi,\zeta} \cap \eta.$$

To see condition (6), if $\eta \in Z_{\xi,\zeta} = Z_{\gamma_\theta,\zeta}$, then

$$Z_{\xi,\zeta} \cap \eta = Z_{\gamma_\theta,\zeta} \cap \eta = Z_{\eta,\zeta}$$

since condition (6) holds below ξ, and all conditions hold. □

Definition 2.6. If $\mathfrak{B}$ is an atomic Boolean algebra, we say that a function $\Phi\colon \mathfrak{B} \to \mathfrak{B}$ is a *ȷ-mapping* if

(a) $\Phi \circ \Phi = \Phi$;
(b) $\Phi(\mathbf{Atoms}(\mathfrak{B})) \subseteq \mathbf{Atoms}(\mathfrak{B})$;
(c) $\Phi \restriction (\mathbf{Atoms}(\mathfrak{B}) \setminus \Phi(\mathbf{Atoms}(\mathfrak{B})))$ is a bijection with $\Phi(\mathbf{Atoms}(\mathfrak{B}))$; and
(d) $\Phi(\bigcup_{A\in X} A) = \bigcup_{A\in X} \Phi(A)$ for any $X \subseteq \mathbf{Atoms}(\mathfrak{B})$,

where $\mathbf{Atoms}(\mathfrak{B})$ denotes the set of atoms of $\mathfrak{B}$.

Now we give the crucial lemma from which the constructions will follow, which is a modification of the ideas in [3].

Lemma 2.7. *Assume κ is inaccessible and $\dagger(\kappa, f, g)$ and let $\{I_\nu\}_{\nu\in\kappa}$ be a partition of κ such that $|I_\nu| = f(\nu)$. Let $p\colon \kappa \to \kappa$ be defined by $\alpha \in I_{p(\alpha)}$. For each $\nu \in \kappa$, write $I_\nu = I_{0,\nu} \sqcup I_{1,\nu}$, where $|I_{0,\nu}| = |I_{1,\nu}| = |I_\nu|$, and let $\psi_\nu\colon I_\nu \to I_{1,\nu}$ be a function such that $\psi_\nu \restriction I_{1,\nu}$ is identity and $\psi_\nu \restriction I_{0,\nu}$ is a bijection with $I_{1,\nu}$.*

Then there exist $\{\mathfrak{B}_{\xi,\nu}\}_{\xi\in\kappa^+,\nu\in\kappa}$ and $\{\Phi_{\xi,\nu}\}_{\xi\in\kappa^+,\nu\in\kappa}$ such that

(1) *$\mathfrak{B}_{\xi,\nu}$ is an atomic Boolean subalgebra of $\mathcal{P}(I_\nu)$ and $\Phi_{\xi,\nu}\colon \mathfrak{B}_{\xi,\nu} \to \mathfrak{B}_{\xi,\nu}$ is a ȷ-mapping for each $\xi \in \kappa^+$ and $\nu \in \kappa$;*
(2) *if $\xi \in \eta \in \kappa^+$, then there is a $\beta \in \kappa$ such that $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\eta,\nu}$ and $\Phi_{\xi,\nu} \subseteq \Phi_{\eta,\nu}$ for all $\nu \in \kappa \setminus \beta$;*
(3) *for $A \in \mathcal{P}(\kappa)$, there are $\xi \in \kappa^+$ and $\beta \in \kappa$ with $A \cap I_\nu \in \mathfrak{B}_{\xi,\nu}$ for all $\nu \in \kappa \setminus \beta$;*
(4) *for any collection $\{F_\alpha\}_{\alpha\in\kappa}$ of sets such that $|F_\alpha| < f(p(\alpha))$ and $F_\alpha \subseteq I_{0,p(\alpha)}$, there are $\xi \in \kappa^+$, $\beta \in \kappa$ and a sequence $\{\alpha_\nu\}_{\nu\in\kappa\setminus\beta}$ for which $F_{\alpha_\nu} \in \mathfrak{B}_{\xi,\nu}$ and $\alpha_\nu \in I_{1,\nu} \setminus \Phi_{\xi,\nu}(F_{\alpha_\nu})$.*

Proof. Let $\{G_\xi\}_{\xi\in\kappa^+}$ witness $\mathfrak{d}_{2^f,g} = \kappa^+$. For each $\nu \in \kappa$, let $\{E_{\theta,\nu}\}_{\theta\in 2^{f(\nu)}}$ enumerate $\mathcal{P}(I_\nu)$ and let $\{L_{\theta,\nu}\}_{\theta\in 2^{f(\nu)}}$ enumerate all functions from $I_{1,\nu}$ to $[I_{0,\nu}]^{<f(\nu)}$.

We inductively define the $\{\mathfrak{B}_{\xi,\nu}\}_{\xi\in\kappa^+,\nu\in\kappa}$ and $\{\Phi_{\xi,\nu}\}_{\xi\in\kappa^+,\nu\in\kappa}$. Let $\{Z_{\eta,\zeta}\}_{\eta\in\kappa^+,\zeta\in\kappa}$ be the collection given by Lemma 2.5. For $\nu \in \kappa$, we let $\mathfrak{B}_{\Psi,\nu} = \{\emptyset, I_\nu, I_{0,\nu}, I_{1,\nu}\}$ and let $\Psi_\nu\colon \mathfrak{B}_{\Psi,\nu} \to \mathfrak{B}_{\Psi,\nu}$ be the ȷ-mapping induced by ψ_ν.

For $\nu \in \kappa$, let $\mathfrak{B}_{0,\nu} = \mathfrak{B}_{\Psi,\nu}$ and let $\phi_{0,\nu}$ be the empty mapping. Suppose that $\mathfrak{B}_{\xi,\nu}$ and $\phi_{\xi,\nu}$ have been defined for all $\zeta \in \eta$ and $\nu \in \kappa$ satisfying the following conditions:

(1) for $\xi \in \eta$ and $\nu \in \kappa$,
  (a) $\phi_{\xi,\nu}\colon D_{\xi,\nu} \to D_{\xi,\nu}$ is such that
    (i) $\phi_{\xi,\nu} \restriction (D_{\xi,\nu} \cap I_{0,\nu})$ is a bijection with $D_{\xi,\nu} \cap I_{1,\nu}$,
    (ii) $\phi_{\xi,\nu} \restriction (D_{\xi,\nu} \cap I_{1,\nu})$ is the identity, and
    (iii) for $x \in D_{\xi,\nu} \cap I_{0,\nu}$, we have $\phi_{\xi,\nu} \circ (\psi_\nu \restriction I_{0,\nu})^{-1} \circ \phi_{\xi,\nu}(x) = \psi_\nu(x)$;
  (b) $|D_{\xi,\nu}| < f(\nu)$;
  (c) $\mathfrak{B}_{\xi,\nu}$ is an atomic Boolean subalgebra of $\mathcal{P}(I_\nu)$ such that
    (i) $\mathfrak{B}_{\psi,\nu} \subseteq \mathfrak{B}_{\xi,\nu}$, and
    (ii) for all $X \in \mathfrak{B}_{\xi,\nu}$, we have $\psi_\nu(X), \psi_\nu^{-1}(X) \in \mathfrak{B}_{\xi,\nu}$;
  (d) $D_{\xi,\nu} \in \mathfrak{B}_{\xi,\nu}$;
  (e) $|\mathbf{Atoms}(\mathfrak{B}_{\xi,\nu})| \leq 2^{g(\nu)}$;

(2) if $\overline{\eta} \in \eta$ and $\xi \in Z_{\overline{\eta},\nu}$ then
  (a) $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\overline{\eta},\nu}$;
  (b) $\phi_{\xi,\nu} \subseteq \phi_{\overline{\eta},\nu}$;
  (c) $\Psi_\nu \subseteq \Phi_{\xi,\nu} \subseteq \Phi_{\overline{\eta},\nu}$, where $\Phi_{\xi,\nu}\colon \mathfrak{B}_{\xi,\nu} \to \mathfrak{B}_{\xi,\nu}$ is a J-mapping given by

$$\Phi_{\xi,\nu}(A) = \phi_{\xi,\nu}(A \cap D_{\xi,\nu}) \cup \psi_\nu(A \setminus D_{\xi,\nu}).$$

By condition (6) of Lemma 2.5 and induction hypothesis (2)(a), we see that if $\xi \in \xi^*$ and $\xi, \xi^* \in Z_{\eta,\nu}$, then $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\xi^*,\nu}$. Moreover, because $|Z_{\eta,\nu}| \leq g(\nu)$, it follows from induction hypothesis (1)(e) that, letting

$$\mathcal{J}_{\eta,\nu} = \Big\{ \bigcap_{\xi \in Z_{\eta,\nu}} j(\xi) \;\Big|\; j \in \prod_{\xi \in Z_{\eta,nu}} \mathbf{Atoms}(\mathfrak{B}_{\xi,\nu}) \Big\},$$

we have

$$|\mathcal{J}_{\eta,\nu}| \leq (2^{g(\nu)})^{g(\nu)} = 2^{g(\nu)} \quad \text{for each } \nu.$$

Furthermore, letting $\mathcal{K}_{\eta,\nu}$ be the partition of I_ν generated by $\{E_{\theta,\nu}\}_{\theta \in G_\eta(\nu)}$, we see also $|\mathcal{K}_{\eta,\nu}| \leq 2^{|G_\eta(\nu)|} = 2^{g(\nu)}$.

We point out that the union $\bigcup_{\xi \in Z_{\eta,\nu}} \phi_{\xi,\nu}$ is a function, for if $\xi, \xi^* \in Z_{\eta,\nu}$ and $\xi \in \xi^*$, then $Z_{\eta,\nu} \cap \xi^* = Z_{\xi^*,\nu}$ (by condition (6) of Lemma 2.5), and by induction hypothesis (2)(b), we have $\phi_{\xi,\nu} \subseteq \phi_{\xi^*,\nu}$. By induction hypothesis (1)(a), we see that $\bigcup_{\xi \in Z_{\eta,\nu}} \phi_{\xi,\nu}$ has domain $\bigcup_{\xi \in Z_{\eta,\nu}} D_{\xi,\nu}$ and moreover

(i) $(\bigcup_{\xi \in Z_{\eta,\nu}} \phi_{\xi,\nu}) \restriction (\bigcup_{\xi \in Z_{\eta,\nu}} D_{\xi,\nu} \cap I_{0,\nu})$ is a bijection with $\bigcup_{\xi \in Z_{\eta,\nu}} D_{\xi,\nu} \cap I_{1,\nu}$;
(ii) $(\bigcup_{\xi \in Z_{\eta,\nu}} \phi_{\xi,\nu}) \restriction (\bigcup_{\xi \in Z_{\eta,\nu}} D_{\xi,\nu} \cap I_{1,\nu})$ is the identity; and
(iii) for $x \in \bigcup_{\xi \in Z_{\eta,\nu}} D_{\xi,\nu} \cap I_{0,\nu}$, we have $(\bigcup_{\xi \in Z_{\eta,\nu}} \phi_{\xi,\nu}) \circ (\psi_\nu \restriction I_{0,\nu})^{-1} \circ (\bigcup_{\xi \in Z_{\eta,\nu}} \phi_{\xi,\nu})(x)$ equal to $\psi_\nu(x)$.

We let

$$\underline{D_{\eta,\nu}} = \bigcup_{\xi \in Z_{\eta,\nu}} D_{\xi,\nu} \quad \text{and} \quad \underline{\phi_{\eta,\nu}} = (\psi_\nu \restriction I_\nu \setminus \underline{D_{\eta,\nu}}) \cup \bigcup_{\xi \in Z_{\eta,\nu}} \phi_{\xi,\nu}.$$

Also, since $f(\nu)$ is regular and $|D_{\xi,\nu}| < f(\nu)$ for all ξ, we have $|\underline{D_{\eta,\nu}}| < f(\nu)$. Clearly, for $x \in I_{0,\nu}$, we have

$$\underline{\phi_{\eta,\nu}} \circ (\psi_\nu \restriction I_{0,\nu})^{-1} \circ \underline{\phi_{\eta,\nu}}(x) = \psi_\nu(x).$$

We let $\underline{\mathcal{A}_{\eta,\nu}}$ be the partition of I_ν generated by $\mathcal{J}_{\eta,\nu} \cup \mathcal{K}_{\eta,\nu} \cup \{I_{0,\nu}, I_{1,\nu}\}$, so $|\underline{\mathcal{A}_{\eta,\nu}}| \leq 2^{g(\nu)}$. As $\underline{\mathcal{A}_{\eta,\nu}}$ is a partition of I_ν, for each $x \in I_{0,\nu}$, there are unique $W_{0,x}, W_{1,x}, W_{2,x}, W_{3,x} \in \underline{\mathcal{A}_{\eta,\nu}}$ such that

- $x \in W_{0,x}$;
- $\psi_\nu(x) \in W_{1,x}$;
- $(\psi_\nu \restriction I_{0,\nu})^{-1} \circ \underline{\phi_{\eta,\nu}}(x) \in W_{2,x}$; and
- $\underline{\phi_{\eta,\nu}}(x) \in W_{3,x}$.

Thus, for $x \in I_{0,\nu}$, we let

$$A_x = W_{0,x} \cap \psi_\nu^{-1}(W_{1,x}) \cap \underline{\phi_{\eta,\nu}}^{-1}(\psi_\nu(W_{2,x})) \cap \underline{\phi_{\eta,\nu}}^{-1}(W_{3,x}) \cap I_{0,\nu}$$

and define $\mathcal{H}_0 \subseteq \mathcal{P}(I_{0,\nu})$ by $\mathcal{H}_0 = \{A_x\}_{x \in I_{0,\nu}}$. It is clear that $\mathcal{H}_0$ is a partition of $I_{0,\nu}$. Let $\mathcal{H}_1 = \{\psi_\nu(A_x)\}_{x \in I_{0,\nu}}$ and $\mathcal{A}_{\eta,\nu} = \mathcal{H}_0 \cup \mathcal{H}_1$, and it is clear that $\mathcal{A}_{\eta,\nu}$ is a partition of I_ν and $|\mathcal{A}_{\eta,\nu}| \leq 2^{g(\nu)}$.

We note that, for any $x \in I_{0,\nu}$, we have $\psi_\nu(A_x) \in \mathcal{H}_1$ (by definition of $\mathcal{H}_1$) but also $\underline{\phi_{\eta,\nu}}(A_x) \in \mathcal{H}_1$. To see why this latter claim is true, we let $x \in I_{0,\nu}$ be given and let $y = (\psi_\nu \restriction I_{0,\nu})^{-1} \circ \underline{\phi_{\eta,\nu}}(x)$. We have

$$(\psi_\nu \restriction I_{0,\nu})^{-1} \circ \underline{\phi_{\eta,\nu}}(y) = x.$$

Then $W_{0,x} = W_{2,y}$, $W_{1,x} = W_{3,y}$, $W_{2,x} = W_{0,y}$, $W_{3,x} = W_{1,y}$. It is readily seen that

$$\begin{aligned}
\underline{\phi_{\eta,\nu}}(A_x) &= \underline{\phi_{\eta,\nu}}(W_{0,x} \cap \psi_\nu^{-1}(W_{1,x}) \cap \underline{\phi_{\eta,\nu}}^{-1}(\psi_\nu(W_{2,x})) \cap \underline{\phi_{\eta,\nu}}^{-1}(W_{3,x}) \cap I_{0,\nu}) \\
&= \underline{\phi_{\eta,\nu}}(W_{0,x}) \cap \underline{\phi_{\eta,\nu}} \circ (\psi_\nu \restriction I_{0,\nu})^{-1}(W_{1,x}) \\
&\qquad \cap \underline{\phi_{\eta,\nu}} \circ (\underline{\phi_{\eta,\nu}} \restriction I_{0,\nu})^{-1}(\psi_\nu(W_{2,x})) \cap \underline{\phi_{\eta,\nu}} \circ (\underline{\phi_{\eta,\nu}} \restriction I_{0,\nu})^{-1}(W_{3,x}) \\
&= \underline{\phi_{\eta,\nu}}(W_{0,x}) \cap \underline{\phi_{\eta,\nu}} \circ (\psi_\nu \restriction I_{0,\nu})^{-1}(W_{1,x}) \cap \psi_\nu(W_{2,x}) \cap W_{3,x} \\
&= \underline{\phi_{\eta,\nu}}(W_{2,y}) \cap \underline{\phi_{\eta,\nu}} \circ (\psi_\nu \restriction I_{0,\nu})^{-1}(W_{3,y}) \cap \psi_\nu(W_{0,y}) \cap W_{1,y} \\
&= \psi_\nu \circ (\psi_\nu \restriction I_{0,\nu})^{-1}(\underline{\phi_{\eta,\nu}}(W_{2,y}) \cap \underline{\phi_{\eta,\nu}} \circ (\psi_\nu \restriction I_{0,\nu})^{-1}(W_{3,y}) \cap \psi_\nu(W_{0,y}) \cap W_{1,y}) \\
&= \psi_\nu((\psi_\nu \restriction I_{0,\nu})^{-1} \circ \underline{\phi_{\eta,\nu}}(W_{2,y}) \cap (\psi_\nu \restriction I_{0,\nu})^{-1} \circ \underline{\phi_{\eta,\nu}} \circ (\psi_\nu \restriction I_{0,\nu})^{-1}(W_{3,y}) \\
&\qquad \cap (\psi_\nu \restriction I_{0,\nu})^{-1} \circ \psi_\nu(W_{0,y}) \cap (\psi_\nu \restriction I_{0,\nu})^{-1}(W_{1,y})) \\
&= \psi_\nu((\psi_\nu \restriction I_{0,\nu})^{-1} \circ \underline{\phi_{\eta,\nu}}(W_{2,y}) \cap (\underline{\phi_{\eta,\nu}} \restriction I_{0,\nu})^{-1}(W_{3,y}) \cap W_{0,y} \cap (\psi_\nu \restriction I_{0,\nu})^{-1}(W_{1,y})) \\
&= \psi_\nu(\psi_\nu^{-1} \circ \underline{\phi_{\eta,\nu}}(W_{2,y}) \cap \underline{\phi_{\eta,\nu}}^{-1}(W_{3,y}) \cap W_{0,y} \cap \psi_\nu^{-1}(W_{1,y}) \cap I_{0,\nu}) \\
&= \psi_\nu(\underline{\phi_{\eta,\nu}}^{-1}(\psi_\nu(W_{2,y})) \cap \underline{\phi_{\eta,\nu}}^{-1}(W_{3,y}) \cap W_{0,y} \cap \psi_\nu^{-1}(W_{1,y}) \cap I_{0,\nu}) \\
&= \psi_\nu(A_y) \in \mathcal{H}_1.
\end{aligned}$$

It is similarly seen that if $A \subseteq I_{1,\nu}$ and $A \in \mathcal{A}_{\eta,\nu}$, then both $(\psi_\nu \restriction I_{0,\nu})^{-1}(A) \in \mathcal{A}_{\eta,\nu}$ and $(\underline{\phi_{\eta,\nu}})^{-1}(A) \in \mathcal{A}_{\eta,\nu}$. As $f(\nu)$ is regular and $|\mathcal{A}_{\eta,\nu}| \le 2^{g(\nu)} < f(\nu)$, we have some $A_{\eta,\nu} \in \mathcal{A}_{\eta,\nu}$ with $|A_{\eta,\nu}| = f(\nu)$, and moreover, we can take $A_{\eta,\nu} \subseteq I_{0,\nu}$ by replacing $A_{\eta,\nu}$ with $(\psi_\nu \restriction I_{0,\nu})^{-1}(A_{\eta,\nu})$ if necessary.

We let $B^1_\nu \in [\psi_\nu(A_{\eta,\nu})]^{|G_\eta(\nu)|}$. Select $B^0_\nu \in [A_{\eta,\nu}]^{|G_\eta(\nu)|}$ such that the intersection $(\bigcup_{x \in B^1_\nu} L_{\theta,\nu}(x)) \cap B^0_\nu = \emptyset$ for all $\theta \in G_\eta(\nu)$ and $\psi_\nu(B^0_\nu) \cap B^1_\nu = \emptyset$. Let $B^2_\nu = (\psi_\nu \restriction I_{0,\nu})^{-1}(B^1_\nu))$ and $B^3_\nu = \psi_\nu(B^0_\nu)$. Let

$$\phi \colon B^0_\nu \cup B^1_\nu \cup B^2_\nu \cup B^3_\nu \to B^1_\nu \cup B^3_\nu$$

be a function such that

- $\phi \restriction (B^1_\nu \cup B^3_\nu)$ is the identity map;
- $\phi \restriction B^0_\nu$ is a bijection with B^1_ν;
- $\phi \restriction B^2_\nu$ is the bijection with B^3_ν such that, for $x \in B^2_\nu$, we have $\phi \circ (\psi_\nu \restriction X_{0,\nu})^{-1} \circ \phi(x) = \psi_\nu(x)$.

Now define

$$D_{\eta,\nu} = \bigcup_{\xi \in Z_{\eta,\nu}} D_{\xi,\nu} \cup B^0_\nu \cup B^1_\nu \cup B^2_\nu \cup B^3_\nu$$

We point out that the sets $B^0_\nu, B^1_\nu, B^2_\nu, B^3_\nu$ are pairwise disjoint by construction. Because

$$D_{\eta,\nu} = \underline{D_{\eta,\nu}} \cup B^0_\nu \cup B^1_\nu \cup B^2_\nu \cup B^3_\nu,$$

we have $|D_{\eta,\nu}| < f(\nu)$.

Note that $A_{\eta,\nu} \cup \psi_\nu(A_{\eta,\nu})$ is disjoint from $\bigcup_{\xi \in Z_{\eta,\nu}} D_{\xi,\nu}$. To see this, given $\xi \in Z_{\eta,\nu}$, we have by induction hypothesis (1) (d) that $D_{\xi,\nu} \in \mathfrak{B}_{\xi,\nu}$, and we have already seen that if $\xi \in \xi^* \in Z_{\eta,\nu}$, we have $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\xi^*,\nu}$, so in particular, $D_{\xi,\nu}$ is a union of elements in $\mathcal{J}_{\eta,\nu}$. As $|A_{\eta,\nu}| = f(\nu) > |D_{\xi,\nu}|$ and $A_{\eta,\nu}$ is in the partition $\mathcal{A}_{\eta,\nu}$, we see that $A_{\eta,\nu} \cap D_{\xi,\nu} = \emptyset$. But also $\psi_\nu(A_{\eta,\nu}) \in \mathcal{A}_{\eta,\nu}$, and as $|\psi_\nu(A_{\eta,\nu})| = f(\nu)$, we argue as before that $\psi_\nu(A_{\eta,\nu}) \cap D_{\xi,\nu} = \emptyset$.

Thus it is also the case that each of the sets $B^0_\nu, B^1_\nu, B^2_\nu, B^3_\nu$ is disjoint from $\bigcup_{\xi \in Z_{\eta,\nu}} D_{\xi,\nu}$. Let

$$\phi_{\eta,\nu} = \bigcup_{\xi \in Z_{\eta,\nu}} \phi_{\xi,\nu} \cup \phi.$$

By construction, $\phi_{\eta,\nu} \restriction (D_{\eta,\nu} \cap I_{0,\nu})$ is a bijection with $D_{\eta,\nu} \cap I_{1,\nu}$ and $\phi_{\eta,\nu} \restriction (D_{\eta,\nu} \cap I_{1,\nu})$ is the identity, and for $x \in D_{\eta,\nu} \cap I_{0,\nu}$, we have $\phi_{\eta,\nu} \circ (\psi_\nu \restriction I_{0,\nu})^{-1} \circ \phi_{\eta,\nu}(x) = \psi_\nu(x)$. We have also seen that $|D_{\eta,\nu}| < f(\nu)$. Thus (1) (a) and (1) (b) hold.

Let $\mathfrak{B}_{\eta,\nu}$ be the Boolean subalgebra of $\mathcal{P}(I_\nu)$ generated by $\mathcal{A}_{\eta,\nu} \cup \{B^0_\nu, B^1_\nu, B^2_\nu, B^3_\nu\}$. It is easy to see by our construction that $\mathfrak{B}_{\eta,\nu}$ is atomic with

$$\mathbf{Atoms}(\mathfrak{B}_{\eta,\nu}) = (\mathcal{A}_{\eta,\nu} \setminus \{A_{\eta,\nu}, \psi_\nu(A_{\eta,\nu})\}) \cup \{B^0_\nu, B^2_\nu, A_{\eta,\nu} \setminus (B^0_\nu \cup B^2_\nu), B^1_\nu, B^3_\nu, \psi_\nu(A_{\eta,\nu}) \setminus (B^1_\nu \cup B^3_\nu)\}.$$

For $A \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$, we know if $A \subseteq I_{0,\nu}$, then $\psi_\nu(A) \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$, and if $A \subseteq I_{1,\nu}$, then

$$(\psi_\nu \restriction I_{0,\nu})^{-1}(A) \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu}) \quad \text{and} \quad \psi_\nu(A) = A.$$

Also,

$$\bigcup_{A \subseteq I_{0,\nu}, A \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})} A = I_{0,\nu},$$

so $\mathfrak{B}_{\psi,\nu} \subseteq \mathfrak{B}_{\eta,\nu}$, and $X \in \mathfrak{B}_{\eta,\nu}$ implies $\psi_\nu(X), \psi_\nu^{-1}(X) \in \mathfrak{B}_{\eta,\nu}$ (since this is true when X is an atom), so inductive hypothesis (1) (c) holds. It is also clear that if $A \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$ with $A \subseteq D_{\eta,\nu} \cap I_{0,\nu}$, then $\phi_{\eta,\nu}(A) \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$, and for $A \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$ with $A \subseteq D_{\eta,\nu} \cap I_{1,\nu}$, we have $(\phi_{\eta,\nu} \restriction (D_{\eta,\nu} \cap I_{0,\nu}))^{-1}(A) \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$.

We note that, for $\xi \in Z_{\eta,\nu}$, we have $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\eta,\nu}$, i.e. (2) (a) holds as well, since by construction $\mathcal{J}_{\eta,\nu} \subseteq \mathfrak{B}_{\eta,\nu}$.

For (1) (d), we recall that if $\xi, \xi^* \in Z_{\eta,\nu}$ with $\xi \in \xi^*$, we have $\xi \in Z_{\eta,\nu} \cap \xi^* = Z_{\xi^*,\nu}$, and by induction hypotheses (1) (c) and (2) (a), we have $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\xi,\nu}$ and both are atomic. Therefore, when $\xi^* \in Z_{\eta,\nu}$, we have $D_{\xi^*,\nu} \in \mathfrak{B}_{\xi^*,\nu}$ (by hypothesis (1) (d)) and also $\bigcup_{\xi \in Z_{\xi^*,\nu}} D_{\xi,\nu} \in \mathfrak{B}_{\xi^*,\nu}$, so $D_{\xi^*,\nu} \setminus (\bigcup_{\xi \in Z_{\xi^*,\nu}} D_{\xi,\nu}) \in B_{\xi^*,\nu}$. Therefore,

$$D_{\eta,\nu} = B^0_\nu \cup B^1_\nu \cup B^2_\nu \cup B^3_\nu \cup \Big(\bigcup_{\xi^* \in Z_{\eta,\nu}} \Big(D_{\xi^*,\nu} \setminus \bigcup_{\xi \in Z_{\xi^*,\nu}} D_{\xi,\nu} \Big) \Big) \in \mathfrak{B}_{\eta,\nu}.$$

For (1) (e), we point out that $|\mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})| = |\mathcal{A}_{\eta,\nu}| + 4 \leq 2^{g(\nu)}$, and it is quite clear that $\phi_{\xi,\nu} \subseteq \phi_{\eta,\nu}$ for $\xi \in Z_{\eta,\nu}$ so that (2) (b) holds as well.

We must check (2) (c). Let $\xi \in Z_{\eta,\nu}$ be given, along with $A \in \mathfrak{B}_{\xi,\nu}$. Then $A \cap A_{\eta,\nu} = A_{\eta,\nu}$ or $A \cap A_{\eta,\nu} = \emptyset$, and similarly $A \cap \psi_\nu(A_{\eta,\nu}) = \psi_\nu(A_{\eta,\nu})$. Therefore, $A \cap (B^0_\nu \cup B^2_\nu) = (B^0_\nu \cup B^2_\nu)$ or $A \cap (B^0_\nu \cup B^2_\nu) = \emptyset$, and similarly $A \cap (B^1_\nu \cup B^3_\nu) = B^1_\nu \cup B^3_\nu$ or $A \cap (B^1_\nu \cup B^3_\nu) = \emptyset$. Therefore,

$$\phi(A \cap (B^0_\nu \cup B^1_\nu \cup B^2_\nu \cup B^3_\nu)) = \psi_\nu(A \cap (B^0_\nu \cup B^1_\nu \cup B^2_\nu \cup B^3_\nu)).$$

We also have for $\xi^* \in Z_{\eta,\nu} \setminus \xi$ that $D_{\xi^*,\nu} \setminus D_{\xi,\nu} \in \mathfrak{B}_{\xi^*,\nu}$, and applying induction hypothesis (2) (c), we obtain

$$\phi_{\xi^*,\nu}(A \cap (D_{\xi^*,\nu} \setminus D_{\xi,\nu})) = \psi_\nu(A \cap (D_{\xi^*,\nu} \setminus D_{\xi,\nu})).$$

Thus

$$\begin{aligned}
\phi_{\eta,\nu}(A \cap D_{\eta,\nu}) &= \phi_{\xi,\nu}(A \cap D_{\xi,\nu}) \cup \bigcup_{\xi^* \in Z_{\eta,\nu} \setminus \xi} \phi_{\xi^*,\nu}(A \cap (D_{\xi^*,\nu} \setminus D_{\xi,\nu})) \cup \phi(A \cap (B^0_\nu \cup B^1_\nu \cup B^2_\nu \cup B^3_\nu)) \\
&= \phi_{\xi,\nu}(A \cap D_{\xi,\nu}) \cup \bigcup_{\xi^* \in Z_{\eta,\nu} \setminus \xi} \psi_\nu(A \cap (D_{\xi^*,\nu} \setminus D_{\xi,\nu})) \cup \psi_\nu(A \cap (B^0_\nu \cup B^1_\nu \cup B^2_\nu \cup B^3_\nu)) \\
&= \phi_{\xi,\nu}(A \cap D_{\xi,\nu}) \cup \psi_\nu(A \cap (D_{\eta,\nu} \setminus D_{\xi,\nu})),
\end{aligned}$$

and so

$$\begin{aligned}
\Phi_{\eta,\nu}(A) &= \phi_{\eta,\nu}(A \cap D_{\eta,\nu}) \cup \psi_\nu(A \setminus D_{\eta,\nu}) \\
&= \phi_{\xi,\nu}(A \cap D_{\xi,\nu}) \cup \psi_\nu(A \cap (D_{\eta,\nu} \setminus D_{\xi,\nu})) \cup \psi_\nu(A \setminus D_{\eta,\nu}) \\
&= \phi_{\xi,\nu}(A \cap D_{\xi,\nu}) \cup \psi_\nu(A \setminus D_{\xi,\nu}) \\
&= \Phi_{\xi,\nu}(A),
\end{aligned}$$

and so $\Phi_{\xi,\nu} \subseteq \Phi_{\eta,\nu}$. The fact that $\Phi_{\eta,\nu}$ is a J-mapping is immediate from our observations on how $\phi_{\eta,\nu}$ and ψ_ν behave on $\mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$, and so (2) (c) holds, and all induction hypotheses hold.

Now we check that properties (1)–(4) in the statement of the lemma hold. Property (1) is immediate from the construction. For property (2), we let $\xi \in \eta \in \kappa^+$ be given. By condition (4) of Lemma 2.5, we select $\beta \in \kappa$ so that $\xi \in Z_{\eta,\beta}$, and by condition (3) of Lemma 2.5, we see that, for $\nu \in \kappa \setminus \beta$, the membership $\xi \in Z_{\eta,\nu}$ holds. By induction hypotheses (2) (a) and (2) (b), we get that $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\eta,\nu}$ and $\Phi_{\xi,\nu} \subseteq \Phi_{\eta,\nu}$ hold for $\nu \in \kappa \setminus \beta$.

For property (3) of our lemma, we let $A \in \mathcal{P}(\kappa)$ be given. Let $A \cap I_\nu = E_{\theta_\nu,\nu}$ for each $\nu \in \kappa$. By the selection of G, we pick $\xi \in \kappa^+$ and $\beta \in \kappa$ such that $\theta_\nu \in G_\xi(\nu)$ for all $\nu \in \kappa \setminus \beta$. Then, by construction, we have $E_{\theta_\nu,\nu} \in \mathfrak{B}_{\xi,\nu}$ for all $\nu \in \kappa \setminus \beta$.

For property (4) of our lemma, we let $\{F_\alpha\}_{\alpha\in\kappa}$ be a collection of sets such that $|F_\alpha| < f(p(\alpha))$ and $F_\alpha \subseteq I_{0,p(\alpha)}$. Then, for $\nu \in \kappa$, we have the function $L_\nu \colon I_{1,\nu} \to [I_{0,\nu}]^{<f(\nu)}$ given by $L_\nu(\alpha) = F_\alpha$. Let $\{\theta_\nu\}_{\nu\in\kappa}$, where $\theta_\nu \in 2^{f(\nu)}$, be the sequence such that $L_{\theta_\nu,\nu} = L_\nu$. Then, by the selection of G, there is $\xi^* \in \kappa^+$ and $\beta^* \in \kappa$ such that $\theta_\nu \in G_{\xi^*}(\nu)$ for all $\nu \in \kappa \setminus \beta^*$. Then, by construction, we see that, for $\nu \in \kappa \setminus \beta^*$, we have nonempty pairwise disjoint atoms $B^0_\nu, B^1_\nu \in \mathfrak{B}_{\xi,\nu}$ with $B^0_\nu \subseteq I_{0,\nu}$, $B^1_\nu \subseteq I_{1,\nu}$, $\Phi_{\xi,\nu}(B^0_\nu) = B^1_\nu$, and $(\bigcup_{\alpha\in B^1_\nu} L_{\theta_\nu,\nu}(x)) \cap B^0_\nu = \emptyset$. Select $\alpha_\nu \in B^1_\nu$. By statement (4), we select $\xi^{**} \in \kappa^+$ and $\beta^{**} \in \kappa \setminus \beta^*$ such that $F_{\alpha_\nu} \in \mathfrak{B}_{\xi^{**},\nu}$ for each $\nu \in \kappa \setminus \beta^{**}$. Let $\xi = \max\{\xi^*, \xi^{**}\} + 1$, and by conditions (3) and (4) of Lemma 2.5, select $\beta \in \kappa \setminus \beta^{**}$ such that $\xi^*, \xi^{**} \in Z_{\xi,\beta}$. Then, by induction hypotheses (2) (a) and (2) (b), we have for $\nu \in \kappa \setminus \beta$ that $\mathfrak{B}_{\xi^*,\nu}, \mathfrak{B}_{\xi^{**},\nu} \subseteq \mathfrak{B}_{\xi,\nu}$ and $\Phi_{\xi^*,\nu} \subseteq \Phi_{\xi,\nu}$. Then, in particular, for $\nu \in \kappa \setminus \beta$, we have $\alpha_\nu \notin \Phi_{\xi,\nu}(F_\alpha)$ since $F_{\alpha_\nu} \subseteq I_{0,\nu} \setminus B^0_\nu$, $\Phi_{\xi^*,\nu}$ is a J-mapping, $B^0_\nu \in \mathbf{Atoms}(\mathfrak{B}_{\xi^*,\nu})$, and $\Phi_{\xi,\nu} \supseteq \Phi_{\xi^*,\nu}$. □

We remind the reader of some concepts introduced in the introduction and give some observations.

Definition 2.8. If κ is a regular cardinal, we say functions $\sigma_0, \sigma_1 \in {}^\kappa\chi$ are *eventually equal* if

$$\{\alpha \in \kappa : \sigma_0(\alpha) \neq \sigma_1(\alpha)\} \equiv_\kappa \emptyset.$$

Eventual equality determines an equivalence relation which we will write as $\sigma_0 \sim \sigma_1$ and let ${}^\kappa\chi/\!\sim$ denote the set of equivalence classes. If $|\chi| < \kappa$, then $\sigma_0 \sim \sigma_1$ if and only if, for all $x \in \chi$, we have $\sigma_0^{-1}(\{x\}) \equiv_\kappa \sigma_1^{-1}(\{x\})$. We let $p_\sim$ denote the function $\sigma \mapsto [\sigma]_\sim$. When χ is the underlying set for a universal algebra $\mathfrak{U}$, then ${}^\kappa\chi$ is the underlying set for the algebra $\mathfrak{U}^\kappa$ and the function $p_\sim$ is a homomorphism from $\mathfrak{U}^\kappa$ to $\mathfrak{U}^\kappa/\!\sim$.

Definition 2.9. Let $\mathfrak{U} = (\chi, \mathcal{S})$ and $\mathfrak{M} = (\Omega, \mathcal{S})$ be universal algebras and $\Theta \colon \mathfrak{U}^Z \to \mathfrak{M}$ a homomorphism from the power $\mathfrak{U}^Z$. A subset $Y \subseteq Z$ is a *strong support* for Θ if $\sigma \restriction Y = \tau \restriction Y$ implies $\Theta\sigma = \Theta\tau$ and, for any $y \in Y$ and $\sigma \in \mathfrak{U}^Z$, there exists some $x \in \chi$ such that $\tau \in \mathfrak{U}^Z$ given by

$$\tau(z) = \begin{cases} \sigma(z) & \text{if } z \neq y, \\ x & \text{if } z = y \end{cases}$$

has $\Theta\sigma \neq \Theta\tau$.

Observation 2.10. We point out that a strong support, if it exists, is unique provided $\chi \neq \emptyset$. To see this, suppose that Y_0 and Y_1 are strong supports for Θ. If $y \in Y_0 \setminus Y_1$, then in particular, $Z \neq \emptyset$, and as $\chi \neq \emptyset$, we can select $\sigma \in \mathfrak{U}^Z$. As Y_0 is a strong support, pick $x \in \chi$ such that

$$\tau(z) = \begin{cases} \sigma(z) & \text{if } z \neq y, \\ x & \text{if } z = y \end{cases}$$

has $\Theta\sigma \neq \Theta\tau$, but as Y_1 is a strong support and $\tau \restriction Y_0 = \sigma \restriction Y_1$, we have $\Theta\sigma = \Theta\tau$, a contradiction.

Observation 2.11. A homomorphic function $\Theta \colon \mathfrak{U}^Z \to \mathfrak{M}$ with nonempty domain and strong support Y is constant if and only if $Y = \emptyset$. If $y \in Y$, then we select σ in the domain of Δ and select $x \in \chi$ such that letting $\tau(y) = x$ and $\tau \restriction (Z \setminus \{y\}) = \sigma \restriction (Z \setminus \{y\})$ gives $\Theta\tau \neq \Theta\sigma$, so Δ is not constant. On the other hand, if $Y = \emptyset$, then for σ and τ in the domain of Θ, we get $\sigma \restriction \emptyset = \tau \restriction \emptyset$, so $\Theta\sigma = \Theta\tau$.

Lemma 2.12. *Suppose that κ is an infinite regular cardinal, $\mathfrak{U} = (\chi, \mathcal{S})$ with $1 \leq |\chi| < \kappa$, and*

$$\begin{array}{ccc} \mathfrak{U}^\kappa & \xrightarrow{\Delta} & \mathfrak{U}^\kappa \\ \downarrow{\scriptstyle p_\sim} & & \downarrow{\scriptstyle p_\sim} \\ \mathfrak{U}^\kappa/\!\sim & \xrightarrow{\mathcal{E}} & \mathfrak{U}^\kappa/\!\sim \end{array}$$

is a commuting diagram. If $J \subseteq \kappa$ is such that $|J| = \kappa$ and $Y_\alpha \subseteq \kappa$ is a strong support for $\pi_\alpha \circ \Delta$ for each $\alpha \in J$, then $\bigcap_{\alpha\in J} Y_\alpha = \emptyset$.

Proof. Assume the hypotheses and suppose for contradiction that $\zeta \in \bigcap_{\alpha\in J} Y_\alpha$. Select $x_0 \in \chi$ and let $\sigma \in \mathfrak{U}^\kappa$ be constantly x_0. For each $\alpha \in J$, there exists some $x_\alpha \in \chi$ such that if $\tau \restriction (\kappa \setminus \{\zeta\}) = \sigma \restriction (\kappa \setminus \{\zeta\})$ and $\tau(\zeta) = x_\alpha$, then $(\pi_\alpha \circ \Delta)\sigma \neq (\pi_\alpha \circ \Delta)\tau$. As κ is regular, there exist some $J' \subseteq J$ and $x \in \chi$ with $|J'| = |J| = \kappa$, and for each $\alpha \in J'$, we have $(\pi_\alpha \circ \Delta)\sigma \neq (\pi_\alpha \circ \Delta)\tau$, where $\tau \restriction (\kappa \setminus \{\zeta\}) = \sigma \restriction (\kappa \setminus \{\zeta\})$ and $\tau(\zeta) = x$. Then, as $|J'| = \kappa$, we get

$$\mathcal{E}[\sigma]_\sim = [\Delta\sigma]_\sim \neq [\Delta\tau]_\sim = \mathcal{E}[\tau]_\sim,$$

but of course, $[\tau]_\sim = [\sigma]_\sim$, a contradiction. □

3 First construction

We are now prepared to define the first homomorphic projection.

Construction 3.1. Suppose that κ is inaccessible and $\dagger(\kappa, f, g)$, and that $\mathfrak{U} = (\chi, \mathcal{S})$ is a universal algebra, with $2 \leq |\chi| < \kappa$. Take $\{I_\nu\}_{\nu\in\kappa}$, $\{\mathfrak{B}_{\xi,\nu}\}_{\xi\in\kappa^+,\nu\in\kappa}$, $\{\Phi_{\xi,\nu}\}_{\xi\in\kappa^+,\nu\in\kappa}$, $\{I_{0,\nu}\}_{\nu\in\kappa}$, and $\{I_{1,\nu}\}_{\nu\in\kappa}$ as in Lemma 2.7.

Given $\sigma \in {}^\kappa\chi$, for each $x \in \chi$, we select $\xi_x \in \kappa^+$ and $\beta_x \in \kappa$ such that $\sigma^{-1}(\{x\}) \cap I_\nu \in \mathfrak{B}_{\xi_x,\nu}$ for all $\nu \in \kappa \setminus \beta_x$. Let $\eta \in \kappa^+$ be greater than all elements in $\{\xi_x\}_{x\in\chi}$. For each $x \in \chi$, select $\beta'_x \in \kappa$ such that $\mathfrak{B}_{\xi_x,\nu} \subseteq \mathfrak{B}_{\eta,\nu}$ and $\Phi_{\xi_x,\nu} \subseteq \Phi_{\eta,\nu}$ for all $\nu \in \kappa \setminus \beta'_x$. Select $\beta \in \kappa$ which is greater than all elements in $\{\beta_x\}_{x\in\chi} \cup \{\beta'_x\}_{x\in\chi}$. Then we have for all $\nu \in \kappa \setminus \beta$ and $x \in \chi$ that $\sigma^{-1}(\{x\}) \in \mathfrak{B}_{\xi_x,\nu} \subseteq \mathfrak{B}_{\eta,\nu}$ and $\Phi_{\xi_x,\nu} \subseteq \Phi_{\eta,\nu}$. Let $\tau \in {}^\kappa\chi$ be such that

$$\tau^{-1}(\{x\}) \cap I_\nu = (\sigma^{-1}(\{x\}) \cap I_{0,\nu}) \cup \Phi_{\eta,\nu}(\sigma^{-1}(\{x\}) \cap I_{0,\nu})$$

for all $\nu \in \kappa \setminus \beta$ (define $\tau \restriction (\bigcup_{\nu\in\beta} I_\nu)$ arbitrarily). Let this function which assigns σ to τ, defined via various arbitrary choices for each σ, be denoted $G\colon \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$.

Lemma 3.2. *The function G in Construction 3.1 gives a well-defined $\mathcal{E}_0\colon \mathfrak{U}^\kappa/\!\sim \to \mathfrak{U}^\kappa/\!\sim$ which is a homomorphic projection.*

Proof. To see that the element τ is well-defined up to $\sim$, suppose that instead we select $\overline{\xi_x} \in \kappa^+$ and $\overline{\beta_x} \in \kappa$ such that $f^{-1}(\{x\}) \cap I_\nu \in \mathfrak{B}_{\overline{\xi_x},\nu}$ for all $\nu \in \kappa \setminus \overline{\beta_x}$ and select $\overline{\eta} \in \kappa^+$ greater than all elements in $\{\overline{\xi_x}\}_{x\in\chi}$, $\overline{\beta'_x}$ such that $\mathfrak{B}_{\overline{\xi_x},\nu} \subseteq \mathfrak{B}_{\overline{\eta},\nu}$ and $\Phi_{\overline{\xi_x},\nu} \subseteq \Phi_{\overline{\eta},\nu}$ for all $\nu \in \kappa \setminus \overline{\beta'_x}$, $\overline{\beta} \in \kappa$ greater than all elements in $\{\overline{\beta_x}\}_{x\in\chi} \cup \{\overline{\beta'_x}\}_{x\in\chi}$. Define $\overline{\tau} \in {}^\kappa\chi$ by

$$\overline{\tau}^{-1}(\{x\}) \cap I_\nu = (\sigma^{-1}(\{x\}) \cap I_{0,\nu}) \cup \Phi_{\overline{\eta},\nu}(\sigma^{-1}(\{x\}) \cap I_{0,\nu})$$

for $\nu \in \kappa \setminus \beta$ and define $\overline{\tau} \restriction (\bigcup_{\nu\in\overline{\beta}} I_\nu)$ arbitrarily.

Now select $\underline{\eta} \in \kappa^+$ which is greater than both η and $\overline{\eta}$. Select $\underline{\beta} \in \kappa$ large enough that $\mathfrak{B}_{\eta,\nu} \subseteq \mathfrak{B}_{\underline{\eta}\nu}$, $\Phi_{\eta,\nu} \subseteq \Phi_{\underline{\eta},\nu}$, $\mathfrak{B}_{\overline{\eta},\nu} \subseteq \mathfrak{B}_{\underline{\eta}\nu}$, and $\Phi_{\overline{\eta},\nu} \subseteq \Phi_{\underline{\eta},\nu}$ for all $\nu \in \kappa \setminus \underline{\beta}$. Then, for $\nu \in \kappa \setminus \underline{\beta}$, we have

$$\begin{aligned}\overline{\tau}^{-1}(\{x\}) \cap I_\nu &= (\sigma^{-1}(\{x\}) \cap I_{0,\nu}) \cup \Phi_{\overline{\eta},\nu}(\sigma^{-1}(\{x\}) \cap I_{0,\nu}) \\ &= (\sigma^{-1}(\{x\}) \cap I_{0,\nu}) \cup \Phi_{\eta,\nu}(\sigma^{-1}(\{x\}) \cap I_{0,\nu}) \\ &= \tau^{-1}(\{x\}),\end{aligned}$$

and so $\tau \sim \overline{\tau}$.

Notice also that if $\sigma \sim \sigma_1$, then we select $\overline{\beta} \in \kappa \setminus \beta$ such that $\nu \in \kappa \setminus \overline{\beta}$ implies $\sigma^{-1}(\{x\}) \cap I_\nu = \sigma_1^{-1}(\{x\}) \cap I_\nu$. Then $\sigma_1^{-1}(\{x\}) \cap I_\nu \in \mathfrak{B}_{\xi_x,\nu}$ for all $\nu \in \kappa \setminus \overline{\beta}$ and $x \in \chi$, and for all $\nu \in \kappa \setminus \alpha$ and $x \in \chi$, we have

$$\sigma_1^{-1}(\{x\}) \in \mathfrak{B}_{\xi_x,\nu} \subseteq \mathfrak{B}_{\eta,\nu} \quad \text{and} \quad \Phi_{\xi_x,\nu} \subseteq \Phi_{\eta,\nu}.$$

Then we define $\tau_1 \in {}^\kappa\chi$ to be such that

$$\tau_1^{-1}(\{x\}) \cap I_\nu = (\sigma_1^{-1}(\{x\}) \cap I_{0,\nu}) \cup \Phi_{\eta,\nu}(\sigma_1^{-1}(\{x\}) \cap I_{0,\nu}) \quad \text{for all } \nu \in \kappa \setminus \overline{\beta}$$

and let $\tau_1 \restriction (\bigcup_{\nu\in\overline{\beta}} I_\nu)$ be arbitrary. Then it is clear that $\tau_1 \restriction (\bigcup_{\nu\in\kappa\setminus\overline{\beta}} I_\nu) = \tau \restriction (\bigcup_{\nu\in\kappa\setminus\overline{\beta}} I_\nu)$, so $\tau \sim \tau_1$.

We let $\mathcal{E}_0[\sigma]_\sim = [\tau]_\sim$. We have seen that $\mathcal{E}_0$ is well-defined. To see that $\mathcal{E}_0 \circ \mathcal{E}_0 = \mathcal{E}_0$, we point out that, for σ and τ as above, we have $\tau \restriction \bigcup_{\nu\in\kappa\setminus\beta} I_{0,\nu} = \sigma \restriction \bigcup_{\nu\in\kappa\setminus\beta} I_{0,\nu}$, and by how $\mathcal{E}_0$ was defined, it is immediate that $\mathcal{E}_0[\tau]_\sim = [\tau]_\sim$.

We now check that $\mathcal{E}_0$ is a homomorphism. Let $s(w_0, \dots, w_{n-1}) \in \mathcal{S}$ be given, together with $[\sigma_0]_\sim, \dots, [\sigma_{n-1}]_\sim$. For each $x \in \chi$, select $\xi_x \in \kappa^+$ and $\beta_x \in \kappa$ such that $\sigma_i{-}1(\{x\}) \cap I_\nu \in \mathfrak{B}_{\xi_x,\nu}$ for all $\nu \in \kappa \setminus \beta_x$ and $0 \leq i < n$. Take $\eta \in \kappa^+$ greater than $\sup\{\xi_x\}_{x\in\chi}$. For each $x \in \chi$, select $\beta'_x \in \kappa$ such that $\mathfrak{B}_{\xi_x,\nu} \subseteq \mathfrak{B}_{\eta,\nu}$ for all $\nu \in \kappa \setminus \beta'_x$. Select $\beta \in \kappa$ greater than $\sup\{\beta_x\}_{x\in\chi} \cup \{\beta'_x\}_{x\in\chi}$.

For each $0 \leq i < n$, we let τ_i be given by

$$\{\alpha \in I_\nu : \tau_i(\alpha) = x\} = (\{\alpha' \in I_{0,\nu} : \sigma_i(\alpha') = x\}) \cup \Phi_{\eta,\nu}(\{\alpha' \in I_{0,\nu} : \sigma_i(\alpha') = x\})$$

for $x \in \chi$ and $\nu \in \kappa \setminus \beta$, and $\tau_i \restriction \bigcup_{\nu\in\beta} I_\nu$ is defined arbitrarily. We have $\mathcal{E}_0[\sigma_i]_\sim = [\tau_i]_\sim$ for all i. Let σ be given by $\{\alpha \in I_\nu : \sigma(\alpha) = x\} = \{\alpha \in I_\nu : s(\sigma_0(\alpha), \dots, \sigma_{n-1}(\alpha)) = x\}$ for $x \in \chi$ and $\nu \in \kappa \setminus \beta$, and $\sigma \restriction \bigcup_{\nu\in\beta} I_\nu$ is arbitrary. Thus $[\sigma]_\sim = s([\sigma_0]_\sim, \dots, [\sigma_{n-1}]_\sim)$.

Let τ be given by

$$\{\alpha \in I_\nu : \tau(\alpha) = x\} = \{\alpha \in I_\nu : s(\tau_0(\alpha), \dots, \tau_{n-1}(\alpha)) = x\}$$

for $x \in \chi$ and $\nu \in \kappa \setminus \beta$ and $\tau \restriction \bigcup_{\nu\in\beta} I_\nu$ arbitrary, so $s([\tau_0]_\sim, \dots, [\tau_{n-1}]_\sim) = [\tau]_\sim$. Let $\overline{\tau}$ be given by

$$\{\alpha \in I_\nu : \overline{\tau}(\alpha) = x\} = (\{\alpha' \in I_{0,\nu} : \sigma(\alpha') = x\} \cup \Phi_{\eta,\nu}(\{\alpha' \in I_{0,\nu} : \sigma(\alpha') = x\})$$

for $x \in \chi$ and $\nu \in \kappa \setminus \beta$ and $\overline{\tau} \restriction \bigcup_{\nu\in\beta} I_\nu$ arbitrary, so that $\mathcal{E}_0[\sigma]_\sim = [\overline{\tau}]_\sim$.

For $x \in \chi$, we let $N_x = \{(x_0, \dots, x_{n-1}) \in \chi^n : s(x_0, \dots, x_{n-1}) = x\}$ and note that, for a fixed $x \in \chi$ and $\nu \in \kappa \setminus \beta$, we have

$$\begin{aligned}
\{\alpha \in I_\nu : \tau(\alpha) = x\} &= \{\alpha \in I_\nu : s(\tau_0(\alpha), \dots, \tau_{n-1}(\alpha)) = x\} \\
&= \{\alpha \in I_\nu : (\tau_0(\alpha), \dots, \tau_{n-1}(\alpha)) \in N_x\} \\
&= \bigcup_{(x_0,\dots,x_{n-1})\in N_x} \bigcap_{i=0}^{n-1} (\{\alpha' \in I_{0,\nu} : \sigma_i(\alpha') = x_i\}) \cup \Phi_{\eta,\nu}(\{\alpha' \in I_{0,\nu} : \sigma_i(\alpha') = x_i\})) \\
&= \bigcup_{(x_0,\dots,x_{n-1})\in N_x} \left(\left\{\alpha' \in I_{0,\nu} : \bigwedge_{i=0}^{n-1} \sigma_i(\alpha') = x_i\right\}\right) \cup \Phi_{\eta,\nu}\left(\left\{\alpha' \in I_{0,\nu} : \bigwedge_{i=0}^{n-1} \sigma_i(\alpha') = x_i\right\}\right) \\
&= (\{\alpha' \in I_{0,\nu} : \sigma(\alpha') = x\}) \cup \Phi_{\eta,\nu}(\{\alpha' \in I_{0,\nu} : \sigma(\alpha') = x\}) \\
&= \{\alpha \in I_\nu : \overline{\tau}(\alpha) = x\},
\end{aligned}$$

and so $[\tau]_\sim = [\overline{\tau}]_\sim$ and $\mathcal{E}_0$ is a homomorphism. □

Lemma 3.3. *Suppose that $\Delta : \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$ such that $p_\sim \circ \Delta = \mathcal{E}_0 \circ p_\sim$. If Y_α is a strong support for $\pi_\alpha \circ \Delta$ for each $\alpha \in J \subseteq \kappa$, then $|\{\alpha \in J : Y_\alpha = \emptyset\}| < \kappa$.*

Proof. We suppose on the contrary that $|\{\alpha \in J : Y_\alpha = \emptyset\}| = \kappa$, and without loss of generality, we replace J with $\{\alpha \in J : Y_\alpha = \emptyset\}$. Without loss of generality, we further replace J with a subset such that $p \restriction J$ is injective while maintaining $|J| = \kappa$. By Observation 2.11, for each $\alpha \in J$, the function $\pi_\alpha \circ \Delta$ is constant, say $\pi_\alpha \circ \Delta$ is constantly x_α. Using the fact that $|\chi| \geq 2$, we select $x'_\alpha \in \chi \setminus \{x_\alpha\}$. Take $\sigma \in \mathfrak{U}^\kappa$ to have $\sigma \restriction I_{p(\alpha)}$ be constantly x'_α and σ is defined arbitrarily elsewhere. Then, letting $\tau \in \mathfrak{U}^\kappa$ be such that $\mathcal{E}_0[\sigma]_\sim = [\tau]_\sim$, it is easily seen by the definition of $\mathcal{E}_0$ that

$$\{\alpha \in J : \tau \restriction I_{p(\alpha)} \text{ is constantly } x'_\alpha\} \equiv_\kappa J,$$

but on the other hand, we have $(\pi_\alpha \circ \Delta)\sigma = x_\alpha$ for every $\alpha \in J$ so that $\mathcal{E}_0[\sigma]_\sim = [\tau]_\sim \neq [\Delta\sigma]_\sim$, contradicting $p_\sim \circ \Delta = \mathcal{E}_0 \circ p_\sim$. □

Lemma 3.4. *Suppose that we have a homomorphism $\Delta : \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$ such that $p_\sim \circ \Delta = \mathcal{E}_0 \circ p_\sim$. Suppose also that $J \subseteq \kappa$ with $|J| = \kappa$, and for each $\nu \in J$ and each $\alpha \in I_\nu$, the homomorphism $\pi_\alpha \circ \Delta$ has strong support Y_α with $|Y_\alpha| < \kappa$. Then there exists $J' \subseteq J$ with $|J'| = \kappa$, and for all $\nu \in J'$ and $\alpha \in I_\nu$, we have $Y_\alpha \subseteq I_{0,\nu}$.*

Proof. Assume the hypotheses. Since κ is regular, we know for each $\nu \in J$ that $|I_\nu \cup \bigcup_{\alpha\in I_\nu} Y_\alpha| < \kappa$. Then, by Lemma 2.12, it is straightforward to select by induction a $J'' \subseteq J$ such that the collection $\{I_\nu \cup \bigcup_{\alpha\in I_\nu} Y_\alpha\}_{\nu\in J''}$ is pairwise disjoint and $|J''| = \kappa$. Let $J' = \{\nu \in J'' : Y_\alpha \subseteq I_{0,\nu} \text{ for all } \alpha \in I_\nu\}$.

Suppose for contradiction that $|J'' \setminus J'| = \kappa$. For each $\nu \in J'' \setminus J'$, we select $\alpha_\nu \in I_\nu$ such that $Y_{\alpha_\nu} \not\subseteq I_{0,\nu}$, so let $t_{\alpha_\nu} \in Y_\alpha \setminus I_{0,\nu}$. Let $x_0 \in \chi$ and define $\sigma \in \mathfrak{U}^\kappa$ to be the function which is constantly x_0. For each t_{α_ν}, select $x_{t_{\alpha_\nu}} \in \chi$ such that if τ agrees with σ, except $\tau(t_{\alpha_\nu}) = x_{t_{\alpha_\nu}}$, then $(\pi_{\alpha_\nu} \circ \Delta)\tau \neq (\pi_{\alpha_\nu} \circ \Delta)\sigma$. Then, more generally, if $\tau \in \mathfrak{U}^\kappa$ is such that $\tau \restriction Y_\alpha \setminus \{t_{\alpha_\nu}\}$ is constantly x_0 and $\tau(t_{\alpha_\nu}) = x_{t_{\alpha_\nu}}$, then $(\pi_{\alpha_\nu} \circ \Delta)\tau \neq (\pi_{\alpha_\nu} \circ \Delta)\sigma$. Define $\tau_0 \in \mathfrak{U}^\kappa$ by

$$\tau_0(\gamma) = \begin{cases} x_{t_{\alpha_\nu}} & \text{if } \gamma = t_{\alpha_\nu}, \\ x_0 & \text{otherwise.} \end{cases}$$

If we take $\tau_1 \in \mathfrak{U}^\kappa$ to be such that $[\tau_1]_\sim = \mathcal{E}_0[\tau_0]_\sim$, then by the construction of $\mathcal{E}_0$, it is the case that

$$\{\nu \in J'' \setminus J' : \tau_1(\alpha) = x_0 \text{ for all } \alpha \in I_\nu\} \equiv_\kappa J'' \setminus J',$$

but on the other hand, we know that $(\pi_{\alpha_\nu} \circ \Delta)\tau_0 \neq x_0$ for all $\nu \in J'' \setminus J'$ so that $[\Delta\tau_0]_\sim \neq [\tau_1]_\sim = \mathcal{E}_0[\tau_0]_\sim$ contradicting the assumption that $p_\sim \circ \Delta = \mathcal{E}_0 \circ p_\sim$. Therefore, we have $|J'' \setminus J'| < \kappa$, so $|J'| = \kappa$, and the lemma is proved. □

Lemma 3.5. *Suppose $\Delta : \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$ is a homomorphism such that $p_\sim \circ \Delta = \mathcal{E}_0 \circ p_\sim$. For each $J \subseteq \kappa$ with $|J| = \kappa$, there exists $J' \subseteq J$ with $|J'| = \kappa$ and such that, for every $\nu \in J'$, there exists $\alpha_\nu \in I_\nu$ such that the homomorphism $\pi_{\alpha_\nu} \circ \Delta$ does not have a strong support Y_{α_ν} with $|Y_{\alpha_\nu}| < f(\nu)$.*

Proof. Suppose that the conclusion fails. Then there is a $J \subseteq \kappa$ with $|J| = \kappa$, and for every $\nu \in J$ and $\alpha \in I_\nu$, the homomorphism $\pi_\alpha \circ \Delta$ has a strong support Y_α with $|Y_\alpha| < f(\nu)$. By removing fewer than κ elements in J, without loss of generality, we replace J with a subset such that, for every $\nu \in J$ and $\alpha \in I_\nu$, we have $Y_\alpha \neq \emptyset$ by Lemma 3.3. As $f(\nu) < \kappa$ for all $\nu \in \kappa$, we have by Lemma 3.4 a $J' \subseteq J$ such that $|J'| = \kappa$, and for all $\nu \in J'$ and $\alpha \in I_\nu$, the inclusion $Y_\alpha \subseteq I_{0,\nu}$ holds.

Now we define a collection $\{F_\alpha\}_{\alpha \in \kappa}$ by letting $F_\alpha = Y_\alpha$ for $\nu \in J'$ and $\alpha \in I_\nu$, and for $\nu \in \kappa \setminus J'$ and $\alpha \in I_\nu$, we let $F_\alpha = \emptyset$. For this collection $\{F_\alpha\}_{\alpha \in \kappa}$, we have $|F_\alpha| < f(p(\alpha))$ and $F_\alpha \subseteq I_{0,p(\alpha)}$. By Lemma 2.7, there are $\xi \in \kappa^+$ and $\beta \in \kappa$ and sequence $\{\alpha_\nu\}_{\nu \in \kappa \setminus \beta}$ for which $F_{\alpha_\nu} \in \mathfrak{B}_{\xi,\nu}$ and $\alpha_\nu \in I_{1,\nu} \setminus \Phi_{\xi,\nu}(F_{\alpha_\nu})$. For each $\nu \in J'$, we select $t_{\alpha_\nu} \in F_{\alpha_\nu}$. Let $x_0 \in \chi$ and $\sigma \in \mathfrak{U}^\kappa$ be the constant function with output x_0. As F_{α_ν} is a strong support for $\pi_{\alpha_\nu} \circ \Delta$, we pick $x_{t_{\alpha_\nu}} \in \chi$ such that, for any $\tau \in \mathfrak{U}^\kappa$ such that $\tau \restriction (F_{\alpha_\nu} \setminus \{t_{\alpha_\nu}\}) = \sigma \restriction (F_{\alpha_\nu} \setminus \{t_{\alpha_\nu}\})$ and $\tau(t_{\alpha_\nu}) = x_{t_{\alpha_\nu}}$, we have $(\pi_{\alpha_\nu} \circ \Delta)\tau \neq (\pi_{\alpha_\nu} \circ \Delta)\sigma$. Define $\tau_0 \in \mathfrak{U}^\kappa$ by

$$\tau_0(\gamma) = \begin{cases} x_{t_{\alpha_\nu}} & \text{if } \gamma = t_{\alpha_\nu}, \\ x_0 & \text{otherwise.} \end{cases}$$

Now, by Lemma 2.7 parts (2) and (3), select a $\xi^* \in \kappa^+ \setminus \xi$ and $\beta^* \in \kappa \setminus \beta$ such that, for $\nu \in \kappa \setminus \beta^*$, we have

- $\{t_{\alpha_\nu}\} \in \mathfrak{B}_{\xi^*,\nu}$; and
- $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\xi^*,\nu}$ and $\Phi_{\xi,\nu} \subseteq \Phi_{\xi^*,\nu}$.

Let $\tau_1 \in \mathfrak{U}^\kappa$ be given by

$$\tau_1(\gamma) = \begin{cases} x_{t_{\alpha_\nu}} & \text{if } \gamma = t_{\alpha_\nu} \text{ or } \Phi_{\xi^*,\nu}(\{t_{\alpha_\nu}\}) = \{\gamma\} \text{ with } \nu \in J' \setminus \beta^*, \\ x_0 & \text{otherwise.} \end{cases}$$

By how $\mathcal{E}_0$ is defined, we have $\mathcal{E}_0[\tau_0]_\sim = [\tau_1]_\sim$, but on the other hand, we have

$$(\pi_{\alpha_\nu} \circ \Delta\tau)(\tau_0) = x_{t_{\alpha_\nu}} \neq x_0 = \pi_{\alpha_\nu}\tau_1$$

for all $\nu \in J' \setminus \beta^*$, and as $|J' \setminus \beta^*| = \kappa$, we see that $[\Delta\tau_0]_\sim \neq [\tau_1]_\sim = \mathcal{E}_0[\tau_0]_\sim$, contradicting $p_\sim \circ \Delta = \mathcal{E}_0 \circ p_\sim$. □

Lemma 3.6. *Suppose that $\Delta : \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$ is a homomorphism such that $p_\sim \circ \Delta = \mathcal{E}_0 \circ p_\sim$ and that $\lambda < \kappa$. The set*

$$\{\alpha \in \kappa : \pi_\alpha \circ \Delta \text{ has no strong support of cardinality} \leq \lambda\}$$

has cardinality κ.

Proof. Assume the hypotheses. Recall that $\dagger(\kappa, f, g)$ implies that $\lim_{\nu\to\kappa} f(\nu) = \kappa$. Take $J = \{\nu \in \kappa : \lambda < f(\nu)\}$, so $|J| = \kappa$ as κ is regular. By Lemma 3.5, we obtain $J' \subseteq J$ with $|J'| = \kappa$, and for every $\nu \in J'$, we have $\alpha_\nu \in I_\nu$ such that $\pi_{\alpha_\nu} \circ \Delta$ does not have a strong support of cardinality $< f(\nu)$; in particular, $\pi_{\alpha_\nu} \circ \Delta$ does not have support of cardinality at most λ. □

Lemma 3.7. *The image of $\mathcal{E}_0$ is isomorphic to $\mathfrak{U}^\kappa/{\sim}$.*

Proof. For each $\nu \in \kappa$, we let $P_\nu : I_{0,\nu} \to I_\nu$ be a bijection and define bijection $P : \bigcup_{\nu\in\kappa} I_{0,\nu} \to \kappa$ by $\bigcup_{\nu\in\kappa} P_\nu$. Let $H : \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$ be given by $(H(\sigma))(\alpha) = \sigma(P^{-1}(\alpha))$. It is clear that H is a homomorphism. Letting $\overline{H} : \mathfrak{U}^\kappa/{\sim} \to \mathfrak{U}^\kappa/{\sim}$ be given by $\overline{H}([\sigma]_\sim) = [H(\sigma)]_\sim$, it is clear that $\overline{H}$ is also a homomorphism.

We claim that $\overline{H} \restriction \mathrm{im}(\mathcal{E}_0)$ is an isomorphism from $\mathrm{im}(\mathcal{E}_0)$ to $\mathfrak{U}^\kappa/{\sim}$. Letting $\tau \in \mathfrak{U}^\kappa$ be given, we take $\sigma \in \mathfrak{U}^\kappa$ to be such that $\sigma \restriction I_{0,\nu}$ is given by $\sigma(\alpha) = \tau(P_\nu(\alpha))$ for each $\nu \in \kappa$ and let $\sigma \restriction (\bigcup_{\nu\in\kappa} I_{1,\nu})$ be defined arbitrarily. Then it is easy to see that $\overline{H}(\mathcal{E}_0[\sigma]_\sim) = [\tau]_\sim$. Thus $\overline{H} \restriction \mathrm{im}(\mathcal{E}_0)$ is onto $\mathfrak{U}^\kappa/{\sim}$.

To see that $\overline{H} \restriction \mathrm{im}(\mathcal{E}_0)$ is injective, suppose that $[\sigma_0]_\sim, [\sigma_1]_\sim \in \mathrm{im}(\mathcal{E}_0)$ are such that $\overline{H}([\sigma_0]_\sim) = \overline{H}([\sigma_1]_\sim)$. Then $H(\sigma_0) \sim H(\sigma_1)$, so in particular, there exists some $\overline{\beta} \in \kappa$ such that $\nu \in \kappa \setminus \overline{\beta}$ implies $H(\sigma_0) \restriction I_\nu = H(\sigma_1) \restriction I_\nu$. Then $\sigma_0 \restriction I_{0,\nu} = \sigma_1 \restriction I_{0,\nu}$ for all $\nu \in \kappa \setminus \overline{\beta}$. Taking $\sigma \in \mathfrak{U}^\kappa$ to be such that $\sigma \restriction I_{0,\nu} = \sigma_0 \restriction I_{0,\nu}$ for all $\nu \in \kappa \setminus \overline{\beta}$, it is easy to see that $[\sigma_0]_\sim = \mathcal{E}_0[\sigma]_\sim = [\sigma_1]_\sim$. □

We anthologize the relevant facts.

Theorem 3.8. *Suppose that κ is inaccessible, $\dagger(\kappa, f, g)$, and that $\mathfrak{U} = (\chi, \mathcal{S})$ is a universal algebra with $2 \le |\chi| < \kappa$. Then there is a homomorphic projection $\mathcal{E}_0 : \mathfrak{U}^\kappa/{\sim} \to \mathfrak{U}^\kappa/{\sim}$, with the image of $\mathcal{E}_0$ isomorphic to $\mathfrak{U}^\kappa/{\sim}$, such that, for any homomorphism $\Delta : \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$ making*

$$\begin{array}{ccc} \mathfrak{U}^\kappa & \xrightarrow{\Delta} & \mathfrak{U}^\kappa \\ \downarrow p_\sim & & \downarrow p_\sim \\ \mathfrak{U}^\kappa/{\sim} & \xrightarrow{\mathcal{E}_0} & \mathfrak{U}^\kappa/{\sim} \end{array}$$

commute, the set

$$\{\alpha \in \kappa : \pi_\alpha \circ \Delta \text{ has no strong support of cardinality} \le \lambda\}$$

is of cardinality κ for every $\lambda < \kappa$.

4 A homomorphic projection using averages

We begin with a definition.

Definition 4.1. Let $\mathfrak{U} = (\chi, \mathcal{S})$ be a universal algebra. We say that a homomorphism $\mathbf{Av} : \mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$ is an *average* if, for all $x_0, x_1 \in \chi$, we have $\mathbf{Av}(x_0, x_1) = \mathbf{Av}(x_1, x_0)$ and $\mathbf{Av}(x_0, x_0) = x_0$.

Construction 4.2. Suppose that κ is inaccessible and $\dagger(\kappa, f, g)$ and that $\mathfrak{U} = (\chi, \mathcal{S})$ is a universal algebra with $2 \le |\chi| < \kappa$ and that $\mathbf{Av}$ is an average for $\mathfrak{U}$. Take $\{I_\nu\}_{\nu\in\kappa}$, $\{\mathfrak{B}_{\xi,\nu}\}_{\xi\in\kappa^+,\nu\in\kappa}$, $\{\Phi_{\xi,\nu}\}_{\xi\in\kappa^+,\nu\in\kappa}$, and $\{I_{0,\nu}\}_{\nu\in\kappa}$ and $\{I_{1,\nu}\}_{\nu\in\kappa}$ as in Lemma 2.7.

Recall that, for $\xi \in \kappa^+$ and $\nu \in \kappa$, the mapping $\Phi_{\xi,\nu}$ is a J-mapping on $\mathfrak{B}_{\xi,\nu}$, and particularly, the restriction of $\Phi_{\xi,\nu}$ to $\{A \in \mathbf{Atoms}(\mathfrak{B}_{\xi,\nu}) : A \subseteq I_{0,\nu}\}$ gives a bijection with $\{A \in \mathbf{Atoms}(\mathfrak{B}_{\xi,\nu}) : A \subseteq I_{1,\nu}\}$. Let

$$\jmath_{\xi,\nu} : \mathbf{Atoms}(\mathfrak{B}_{\xi,\nu}) \to \mathbf{Atoms}(\mathfrak{B}_{\xi,\nu})$$

be the involution of atoms defined by

$$\jmath_{\xi,\nu} = \Phi_{\xi,\nu} \restriction \{A \in \mathbf{Atoms}(\mathfrak{B}_{\xi,\nu}) : A \subseteq I_{0,\nu}\} \cup (\Phi_{\xi,\nu} \restriction \{A \in \mathbf{Atoms}(\mathfrak{B}_{\xi,\nu}) : A \subseteq I_{0,\nu}\})^{-1}.$$

Given $\sigma \in {}^\kappa\chi$, for each $x \in \chi$, we select $\xi_x \in \kappa^+$ and $\beta_x \in \kappa$ such that $\sigma^{-1}(\{x\}) \cap I_\nu \in \mathfrak{B}_{\xi_x,\nu}$ for all $\nu \in \kappa \setminus \beta_x$. Let $\eta \in \kappa^+$ be greater than all elements in $\{\xi_x\}_{x\in\chi}$. For each $x \in \chi$, select $\beta'_x \in \kappa$ such that $\mathfrak{B}_{\xi_x,\nu} \subseteq \mathfrak{B}_{\eta,\nu}$ and $\Phi_{\xi_x,\nu} \subseteq \Phi_{\eta,\nu}$ for all $\nu \in \kappa \setminus \beta'_x$. Select $\beta \in \kappa$ which is greater than all elements in $\{\beta_x\}_{x\in\chi} \cup \{\beta'_x\}_{x\in\chi}$. Then we

have for all $\nu \in \kappa \setminus \beta$ and $x \in \chi$ that $\sigma^{-1}(\{x\}) \in \mathfrak{B}_{\xi_x,\nu} \subseteq \mathfrak{B}_{\eta,\nu}$ and $\Phi_{\xi_x,\nu} \subseteq \Phi_{\eta,\nu}$. Let $\tau \in {}^{\kappa}\chi$ be defined by letting $\tau \restriction A = \mathbf{Av}(x_0, x_1)$ for $A \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$, where $\sigma \restriction A$ is constantly x_0 and $\sigma \restriction \mathfrak{I}_{\eta,\nu}(A)$ is x_1 (and define τ arbitrarily on $\bigcup_{\nu\in\eta} I_\nu$). Call this function $G_1 : \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$.

Lemma 4.3. *The function G given in Construction 4.2 induces a well-defined homomorphic projection*

$$\mathcal{E}_1 : \mathfrak{U}^\kappa/{\sim} \to \mathfrak{U}^\kappa/{\sim}.$$

Proof. We check well-definedness first. Imagine that we select $\overline{\xi_x} \in \kappa^+$ and $\overline{\beta_x}$ such that $f^{-1}(\{x\}) \cap I_\nu \in \mathfrak{B}_{\overline{\xi_x},\nu}$ for all $\nu \in \overline{\beta_x}$, and select $\overline{\eta} \in \kappa^+$ with $\eta > \sup\{\overline{\xi_x}\}_{x\in\chi}$, set $\{\overline{\beta'_x}\}_{x\in\chi}$ with $\mathfrak{B}_{\overline{\xi_x},\nu} \subseteq \mathfrak{B}_{\overline{\eta},\nu}$ and $\Phi_{\overline{\xi_x},\nu} \subseteq \Phi_{\overline{\eta},\nu}$ for all $\nu \in \kappa \setminus \overline{\beta'_x}$ and $\overline{\beta} > \sup\{\overline{\beta_x}\}_{x\in\chi} \cup \{\overline{\beta'_x}\}_{x\in\chi}$. Let $\overline{\tau}$ be given by $\overline{\tau} \restriction A' = \mathbf{Av}(x_0, x_1)$, where $A' \in \mathbf{Atoms}(\mathfrak{B}_{\overline{\eta},})$ and $\sigma \restriction A'$ is constantly x_0 and $\sigma \restriction \mathfrak{I}_{\overline{\eta},\nu}(A')$ is x_1, and define $\overline{\tau} \restriction \bigcup_{\nu\in\kappa\setminus\overline{\beta}}$ arbitrarily.

Select $\underline{\eta} \in \kappa^+$ with $\underline{\eta} > \eta, \overline{\eta}$ and $\underline{\beta} \in \kappa$ which is large enough that $\mathfrak{B}_{\eta,\nu} \cup \mathfrak{B}_{\overline{\eta},\nu} \subseteq \mathfrak{B}_{\underline{\eta},\nu}$ and $\Phi_{\eta,\nu} \cup \Phi_{\overline{\eta},\nu} \subseteq \Phi_{\underline{\eta},\nu}$ for all $\nu \in \kappa \setminus \underline{\beta}$. Now we fix $\nu \in \kappa \setminus \underline{\beta}$. Take $A \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$ and $A' \in \mathbf{Atoms}(\mathfrak{B}_{\overline{\eta},\nu})$ such that $A \cap A' \neq \emptyset$. We know $\sigma \restriction A$ and $\sigma \restriction A'$ are both constant, so $\sigma \restriction A \cup A'$ is constant, say constantly x_0. If $A \subseteq I_{0,\nu}$, and therefore $A' \subseteq I_{0,\nu}$, then $\Phi_{\underline{\eta},\nu}(A) = \Phi_{\eta,\nu}(A) = \mathfrak{I}_{\eta,\nu}(A)$ and $\Phi_{\underline{\eta},\nu}(A') = \Phi_{\overline{\eta},\nu}(A') = \mathfrak{I}_{\overline{\eta},\nu}(A')$. Then $\sigma \restriction \mathfrak{I}_{\eta,\nu}(A)$ and $\sigma \restriction \mathfrak{I}_{\overline{\eta},\nu}(A')$ are each constant functions, and $\mathfrak{I}_{\eta,\nu}(A) \cap \mathfrak{I}_{\overline{\eta},\nu}(A') \neq \emptyset$, so $\sigma \restriction \mathfrak{I}_{\eta,\nu}(A) \cup \mathfrak{I}_{\overline{\eta},\nu}(A')$ is constant, say constantly x_1. Therefore, $\tau \restriction (A \cap A')$ is constantly $\mathbf{Av}(x_0, x_1)$ and so is $\overline{\tau} \restriction (A \cap A')$. Then $\tau \restriction (\bigcup_{\nu\in\kappa\setminus\overline{\beta}} I_\nu) = \overline{\tau} \restriction (\bigcup_{\nu\in\kappa\setminus\overline{\beta}} I_\nu)$ and $\tau \sim \overline{\tau}$.

When $\sigma \sim \sigma_1$, the element $\tau_1 \in \mathfrak{U}^\kappa$ with $\sigma_1 \mapsto \tau_1$ under Construction 4.2 has $\tau \sim \tau_1$. The check is straightforward and follows appropriate changes to the comparable claim in Lemma 3.2. Thus we have a well-defined function $\mathcal{E}_1$.

To see that $\mathcal{E}_1 \circ \mathcal{E}_1 = \mathcal{E}_1$, we take σ, η and β and τ as before. It is clear that, for every $x \in \chi$ and $\nu \in \kappa \setminus \beta$, we have $\tau^{-1}(\{x\}) \cap I_\nu \in \mathfrak{B}_{\eta,\nu}$. For each atom $A \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$, we have $\tau \restriction A$ and $\tau \restriction \mathfrak{I}_{\eta,\nu}(A)$ are each constantly x, by how τ is constructed. Then, since $\mathbf{Av}(x, x) = x$, we see that again $\mathcal{E}_1[\tau]_\sim = [\tau]_\sim$, so $\mathcal{E}_1 \circ \mathcal{E}_1 = \mathcal{E}_1$.

Now we check that $\mathcal{E}_1$ is a homomorphism. Let $s(w_0, \dots, w_{n-1}) \in \mathcal{S}$ and also $[\sigma_0]_\sim, \dots, [\sigma_{n-1}]_\sim$. We take $\eta \in \kappa^+$ large enough and $\beta \in \kappa$ large enough that, for every $x \in \chi$ and $\nu \in \kappa \setminus \beta$, we have $\sigma_i^{-1}(x) \cap I_\nu \in \mathfrak{B}_{\eta,\nu}$. For $A \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$ and $0 \leq i < n$, take $\tau_i \restriction A$ to be constantly $\mathbf{Av}(x_0, x_1)$, where $\sigma_i \restriction A$ and $\sigma_i \restriction \mathfrak{I}_{\eta,\nu}(A)$ are constantly x_0 and x_1, respectively, and τ_i is defined arbitrarily elsewhere. Then $\mathcal{E}_1([\sigma_i]_\sim) = [\tau_i]_\sim$.

Let σ be defined by having $\sigma \restriction A$ be constantly x when $A \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$, where $\sigma_i \restriction A$ is constantly x_i and $s(x_0, \dots, x_{n-1}) = x$, and define σ arbitrarily elsewhere. Thus $[\sigma]_\sim = s([\sigma_0]_\sim, \dots, [\sigma_{n-1}]_\sim)$. Let τ be defined by having $\tau \restriction A$ be constantly x when $A \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$, where $\tau_i \restriction A$ is constantly x_i and $s(x_0, \dots, x_{n-1}) = x$, and define τ arbitrarily elsewhere. Thus $\mathcal{E}_1[\sigma]_\sim = [\tau]_\sim$. Let $\overline{\tau}$ be defined by $\overline{\tau} \restriction A$ to be constantly $\mathbf{Av}(x_0, x_1)$, where $\sigma \restriction A$ and $\sigma \restriction \mathfrak{I}_{\eta,\nu}(A)$ are constantly x_0 and x_1, respectively, and $\overline{\tau}$ is defined arbitrarily elsewhere. Then $\mathcal{E}_1[\sigma]_\sim = [\overline{\tau}]_\sim$.

Fixing $\nu \in \kappa \setminus \beta$ and $A \in \mathbf{Atoms}(\mathfrak{B}_{\eta,\nu})$, we let $\sigma_i \restriction A$ be constantly $x_{0,i}$ and $\sigma_i \restriction \mathfrak{I}_{\eta,\nu}(A)$ be constantly $x_{1,i}$. Then $\tau_i \restriction A$ is constantly $\mathbf{Av}(x_{0,i}, x_{1,i})$, and $\tau \restriction A$ is constantly $s(\mathbf{Av}(x_{0,0}, x_{1,0}), \dots, \mathbf{Av}(x_{0,n-1}, x_{1,n-1}))$, and $\overline{\tau} \restriction A$ is constantly $\mathbf{Av}(s(x_{0,0}, \dots, x_{0,n-1}), s(x_{1,0}, \dots, x_{1,n-1}))$. Then $\tau \restriction A = \overline{\tau} \restriction A$ since $\mathbf{Av}$ is a homomorphism, and so $\tau \sim \overline{\tau}$. □

Lemma 4.4. *Suppose that $\Delta : \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$ is such that $p_\sim \circ \Delta = \mathcal{E}_1 \circ p_\sim$. If Y_α is a strong support for $\pi_\alpha \circ \Delta$ for each $\alpha \in J \subseteq \kappa$, then $|\{\alpha \in J : Y_\alpha = \emptyset\}| < \kappa$.*

Proof. Assume the hypotheses, and suppose for contradiction that $|\{\alpha \in J : Y_\alpha = \emptyset\}| = \kappa$. Without loss of generality, we can assume that $Y_\alpha \neq \emptyset$ for all $\alpha \in J$ and that $p \restriction J$ is injective. Define function $r : J \to \{0, 1\}$ by $\alpha \in I_{r(\alpha),p(\alpha)}$. As $|\chi| \geq 2$, there exist $x_0, x_1 \in \chi$ such that $\mathbf{Av}(x_0, x_1) \neq x_0$. Define $\sigma_0 \in \mathfrak{U}^\kappa$ to be constantly x_0 and define $\sigma_1 \in \mathfrak{U}^\kappa$ by

$$\sigma_1(\gamma) = \begin{cases} x_1 & \text{if } \gamma \in I_{1-r(\alpha),p(\alpha)} \text{ for some } \alpha \in J, \\ x_0 & \text{otherwise.} \end{cases}$$

It is clear that $\mathcal{E}_1[\sigma_0]_\sim = [\sigma_0]_\sim$ and $\mathcal{E}_1[\sigma_1]_\sim = [\tau_1]_\sim$, where

$$\tau_1(\gamma) = \begin{cases} \mathbf{Av}(x_0, x_1) & \text{if } \gamma \in I_{p(\alpha)} \text{ for some } \alpha \in J, \\ x_0 & \text{otherwise,} \end{cases}$$

and in particular,

$$|\{\alpha \in J : (\pi_\alpha \circ \Delta)\sigma_0 \neq (\pi_\alpha \circ \Delta)\sigma_1\}| = \kappa,$$

whereas, by Observation 2.11,

$$\{\alpha \in J : \pi_\alpha \circ \Delta \text{ is constant}\} \equiv_\kappa J,$$

which gives a contradiction. □

Lemma 4.5. *Suppose that we have a homomorphism $\Delta : \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$ such that $p_\sim \circ \Delta = \mathcal{E}_1 \circ p_\sim$. Suppose also that $J \subseteq \kappa$ with $|J| = \kappa$ and, for each $\nu \in J$ and each $\alpha \in I_\nu$, the homomorphism $\pi_\alpha \circ \Delta$ has strong support Y_α with $|Y_\alpha| < \kappa$. Then there exists $J' \subseteq J$ with $|J'| = \kappa$, and for all $\nu \in J'$ and $\alpha \in I_\nu$, we have*

(a) *$Y_\alpha \subseteq I_\nu$; and*

(b) *$Y_\alpha \cap I_{0,\nu} \neq \emptyset$.*

Proof. By the same proof as that of Lemma 3.4 but with every mention of $Y_{0,\nu}$ being replaced with Y_ν, we obtain a $J'' \subseteq J$ with $|J''| = \kappa$, and for all $\nu \in J''$ and $\alpha \in I_\nu$, we have $Y_\alpha \subseteq I_\nu$.

Now suppose for contradiction that $J_1 = \{\nu \in J'' : \text{there exists } \alpha \in I_\nu \text{ such that } Y_\alpha \cap I_{0,\nu} = \emptyset\}$ is of cardinality κ. Select for each $\nu \in J_1$ an $\alpha_\nu \in I_\nu$ with $Y_\alpha \cap I_{0,\nu} = \emptyset$. As $|\chi| \geq 2$, select $x_0, x_1 \in \chi$ such that $\mathbf{Av}(x_0, x_1) \neq x_0$. Let $\sigma_0 \in \mathfrak{U}^\kappa$ be constantly x_0. Let $\sigma_1 \in \mathfrak{U}^\kappa$ be given by

$$\sigma_1(\gamma) = \begin{cases} x_1 & \text{if } \gamma \in I_{0,\nu} \text{ for some } \nu \in J_0, \\ x_0 & \text{otherwise.} \end{cases}$$

Clearly, $\mathcal{E}_1[\sigma_0]_\sim = [\sigma_0]_\sim$ and $\mathcal{E}_1[\sigma_1]_\sim = [\tau_1]_\sim$, where

$$\tau_1(\gamma) = \begin{cases} \mathbf{Av}(x_0, x_1) & \text{if } \gamma \in I_\nu \text{ for some } \nu \in J_1, \\ x_0 & \text{otherwise.} \end{cases}$$

As $Y_{\alpha_\nu} \cap I_{0,\nu} = \emptyset$ for each $\nu \in J_1$, we see that $(\pi_{\alpha_\nu} \circ \Delta)\sigma_0 = (\pi_{\alpha_\nu} \circ \Delta)\sigma_1$, but on the other hand, it is also the case that $\pi_{\alpha_\nu}\sigma_0 = x_0 \neq \mathbf{Av}(x_0, x_1) = \pi_{\alpha_\nu}\tau_1$ for all $\nu \in J_1$, contradicting $p_\sim \circ \Delta = \mathcal{E}_1 \circ p_\sim$. Thus we let $J' = J'' \setminus J_1$, and the claim is true. □

Lemma 4.6. *Suppose that $\Delta : \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$ is a homomorphism such that $p_\sim \circ \Delta = \mathcal{E}_1 \circ p_\sim$. For each $J \subseteq \kappa$ with $|J| = \kappa$, there exists $J' \subseteq J$ with $|J'| = \kappa$ and such that, for every $\nu \in J'$, there exists $\alpha_\nu \in J'$ such that the homomorphism $\pi_{\alpha_\nu} \circ \Delta$ does not have a strong support Y_{α_ν} with $|Y_{\alpha_\nu}| < f(\nu)$.*

Proof. Suppose for contradiction that there is some $J \subseteq \kappa$ with $|J| = \kappa$ such that, for every $\nu \in J$ and $\alpha \in I_\nu$, the homomorphism $\pi_\alpha \circ \Delta$ has strong support Y_α with $|Y_\alpha| < f(\nu)$. By Lemma 4.4, we can assume without loss of generality that $Y_\alpha \neq \emptyset$ for each $\alpha \in I_\nu$ with $\nu \in J$. By Lemma 4.5, we have a $J' \subseteq J$ with $|J'| = \kappa$, and for all $\nu \in J'$ and $\alpha \in I_\nu$, we get $Y_\alpha \subseteq I_\nu$ and $Y_\alpha \cap I_{0,\nu} \neq \emptyset$. Let collection $\{F_\alpha\}_{\alpha \in \kappa}$ be defined by $F_\alpha = Y_\alpha \cap I_{0,\nu}$ when $\alpha \in I_\nu$ and $\nu \in J'$ and let $F_\alpha = \emptyset$ when $\alpha \in I_\nu$ for $\nu \in \kappa \setminus J'$. Now we have $|F_\alpha| < f(p(\alpha))$ and $F_\alpha \subseteq I_{0,p(\alpha)}$ for each $\alpha \in \kappa$. Now, by Lemma 2.7 part (4), we have $\xi \in \kappa^+$ and $\beta \in \kappa$ and sequence $\{\alpha_\nu\}_{\nu \in \kappa \setminus \beta}$ such that, for each $\nu \in \kappa \setminus \beta$, we have both $F_{\alpha_\nu} \in \mathfrak{B}_{\xi,\nu}$ and $\alpha_\nu \in I_{1,\nu} \setminus \Phi_{\xi,\nu}(F_{\alpha_\nu})$.

For each $\nu \in J'$, pick $t_{\alpha_\nu} \in F_{\alpha_\nu} \cap I_{0,\nu}$. Let $x_0 \in \chi$ and $\sigma \in \mathfrak{U}^\kappa$. Since F_{α_ν} is a strong support for $\pi_{\alpha_\nu} \circ \Delta$, it is possible to select $x_{t_{\alpha_\nu}} \in \chi$ such that, when $\tau \in \mathfrak{U}^\kappa$ satisfies $\tau \restriction (F_{\alpha_\nu} \setminus \{t_{\alpha_\nu}\}) = \sigma \restriction (F_{\alpha_\nu} \setminus \{t_{\alpha_\nu}\})$ and $\tau(t_{\alpha_\nu}) = x_{t_{\alpha_\nu}}$, we have $(\pi_{\alpha_\nu} \circ \Delta)\tau \neq (\pi_{\alpha_\nu} \circ \Delta)\sigma$. Let $\tau_0 \in \mathfrak{U}^\kappa$ be given by

$$\tau_0(\gamma) = \begin{cases} x_{t_{\alpha_\nu}} & \text{if } \gamma = t_{\alpha_\nu}, \\ x_0 & \text{otherwise.} \end{cases}$$

By Lemma 2.7 (2) and (3), we select $\xi^* \in \kappa^+ \setminus \xi$ and $\beta^* \in \kappa \setminus \beta$ such that, for any $\nu \in \kappa \setminus \beta^*$, we have $\{t_{\alpha_\nu}\} \in \mathfrak{B}_{\xi^*,\nu}$ and $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\xi^*,\nu}$ and $\Phi_{\xi,\nu} \subseteq \Phi_{\xi^*,\nu}$. Take $\tau_1 \in \mathfrak{U}^\kappa$ to be

$$\tau_0(\gamma) = \begin{cases} \mathbf{Av}(x_{t_{\alpha_\nu}}, x_0) & \text{if } \gamma = t_{\alpha_\nu} \text{ or } \Phi_{\xi^*,\nu}(\{t_{\alpha_\nu}\}) = \{\gamma\}, \\ x_0 & \text{otherwise.} \end{cases}$$

It is clear that $\mathcal{E}_1[\tau_0]_\sim = [\tau_1]_\sim$, but we also know $(\pi_{\alpha_\nu} \circ \Delta)\tau_0 \neq x_0 = \pi_{\alpha_\nu}\tau_1$ for all $\nu \in J' \setminus \beta^*$, and since $|J' \setminus \beta^*| = \kappa$, we get $[\Delta\tau_0]_\sim \neq [\tau_1]_\sim = \mathcal{E}_1[\tau_0]_\sim$, which contradicts $p_\sim \circ \Delta = \mathcal{E}_1 \circ p_\sim$. □

Lemma 4.7. *Suppose that $\Delta\colon \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$ is a homomorphism such that $p_\sim \circ \Delta = \mathcal{E}_1 \circ p_\sim$. The set*

$$\{\alpha \in \kappa : \pi_\alpha \circ \Delta \text{ has no strong support of cardinality of } \leq \lambda\}$$

has cardinality κ for every $\lambda < \kappa$.

Proof. The proof is that of Lemma 3.6 with obvious modifications. □

Lemma 4.8. *The image of $\mathcal{E}_1$ is isomorphic to $\mathfrak{U}^\kappa/\sim$.*

Proof. We begin as in Lemma 3.7. For $\nu \in \kappa$, let $P_\nu\colon I_{0,\nu} \to I_{1,\nu}$ be a bijection and let $P\colon \bigcup_{\nu\in\kappa} I_{0,\nu} \to \kappa$ be the bijection $\bigcup_{\nu\in\kappa} P_\nu$. Then $H\colon \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$ is given by $(H(\sigma))(\alpha) = \sigma(P^{-1}(\alpha))$. It is clear that H is a homomorphism. Let $\overline{H}\colon \mathfrak{U}^\kappa/\sim \to \mathfrak{U}^\kappa/\sim$ be given by $\overline{H}([\sigma]_\sim) = [H(\sigma)]_\sim$, and $\overline{H}$ is also a homomorphism.

We will show that $\overline{H} \restriction \mathrm{im}(\mathcal{E}_1)$ is an isomorphism from $\mathrm{im}(\mathcal{E}_1)$ to $\mathfrak{U}^\kappa/\sim$. Let $\tau \in \mathfrak{U}^\kappa$ be given. We will define an element $\sigma \in \mathfrak{U}^\kappa$ in stages. First we let σ be such that $\sigma \restriction I_{0,\nu}$ is given by $\sigma(\alpha) = \tau(P_\nu(\alpha))$. By Lemma 2.7 part (3), $|\chi| < \kappa$, and the fact that κ is regular, we can select $\xi \in \kappa^+$ and $\beta \in \kappa$ large enough that $\nu \in \kappa \setminus \beta$ and $x \in \chi$ imply that $K_{x,\nu} := \{\alpha \in I_{0,\nu} : \sigma(\alpha) = x\} \in \mathfrak{B}_{\xi,\nu}$. Then $\Phi_{\xi,\nu}(K_{x,\nu}) \in \mathfrak{B}_{\xi,\nu}$ for all $\nu \in \kappa \setminus \beta$. For each $A \in \mathbf{Atoms}(\mathfrak{B}_{\xi,\nu})$ with $A \subseteq I_1, \nu$, we let $\sigma \restriction A$ be the constant x_0 where $\sigma \restriction \lrcorner_{\xi,\nu}(A)$ is x_0. At all the (negligible) places where σ is not yet defined, define it arbitrarily. Clearly, $\mathcal{E}_1[\sigma]_\sim = [\sigma]_\sim$ and $\overline{H}([\sigma]_\sim) = [\tau]_\sim$, so $\overline{H} \restriction \mathrm{im}(\mathcal{E}_1)$ is onto $\mathfrak{U}^\kappa/\sim$.

To see that $\overline{H} \restriction \mathrm{im}(\mathcal{E}_1)$ is injective, suppose that $[\sigma_0]_\sim, [\sigma_1]_\sim \in \mathrm{im}(\mathcal{E}_1)$ are such that $\overline{H}([\sigma_0]_\sim) = \overline{H}([\sigma_1]_\sim)$. Then $H(\sigma_0) \sim H(\sigma_1)$. Then, in particular, there exists $\beta \in \kappa$ such that $\sigma_0 \restriction I_{0,\nu} = \sigma_1 \restriction I_{0,\nu}$ for each $\nu \in \kappa \setminus \beta$. Then, again using Lemma 2.7 part (3), we can select $\beta' \in \kappa$, with $\beta' > \beta$, and $\xi \in \kappa^+$ such that $\nu \in \kappa \setminus \beta'$ and $x \in \chi$ imply that $K_{x,\nu} := \{\alpha \in I_{0,\nu} : \sigma_0(\alpha) = x\} \in \mathfrak{B}_{\xi,\nu}$. Now, for any atom A of $\mathfrak{B}_{\xi,\nu}$, ν appropriately large, with $A \subseteq I_{0,\nu}$, we have that $\sigma_0 \restriction A$ has the same constant output as $\sigma_0 \restriction \Phi_{\xi,\nu}(A)$ (since $[\sigma_0]_\sim \in \mathrm{im}(\mathcal{E}_1)$). But, similarly, $\sigma_1 \restriction A$ has the same constant output as $\sigma_1 \restriction \Phi_{\xi,\nu}(A)$ when ν is large enough. Since we have already seen that $\sigma_0 \restriction I_{0,\nu} = \sigma_1 \restriction I_{0,\nu}$ for large enough $\nu \in \kappa$, we have in fact that $\sigma_0 \restriction I_\nu = \sigma_1 \restriction I_\nu$ for all such ν. Thus $\sigma_0 \sim \sigma_1$. □

Definition 4.9. We say an average $\mathbf{Av}\colon \mathfrak{U} \times \mathfrak{U} \to \mathfrak{U}$ is *changing* provided there exist $x_0, x_1 \in \mathfrak{U}$ such that

$$x_0 \neq \mathbf{Av}(x_0, x_1) \neq x_1.$$

Theorem 4.10. *Suppose that κ is a inaccessible, $\dagger(\kappa, f, g)$, and that $\mathfrak{U} = (\chi, \mathcal{S})$ is a universal algebra with $2 \leq |\chi| < \kappa$ and that $\mathbf{Av}$ is an average for $\mathfrak{U}$. Then there is a homomorphic projection $\mathcal{E}_1\colon \mathfrak{U}^\kappa/\sim \to \mathfrak{U}^\kappa/\sim$, with the image of $\mathcal{E}_1$ isomorphic to $\mathfrak{U}^\kappa/\sim$, such that, for any homomorphism $\Delta\colon \mathfrak{U}^\kappa \to \mathfrak{U}^\kappa$ making*

$$\begin{array}{ccc} \mathfrak{U}^\kappa & \xrightarrow{\Delta} & \mathfrak{U}^\kappa \\ \downarrow{\scriptstyle p_\sim} & & \downarrow{\scriptstyle p_\sim} \\ \mathfrak{U}^\kappa/\sim & \xrightarrow{\mathcal{E}_1} & \mathfrak{U}^\kappa/\sim \end{array}$$

commute, the set

$$\{\alpha \in \kappa : \pi_\alpha \circ \Delta \text{ has no strong support of cardinality } \leq \lambda\}$$

is of cardinality κ for every $\lambda < \kappa$. Moreover, if $\mathbf{Av}$ is changing, then there exists $\sigma \in \mathfrak{U}^\kappa$ such that, when $\mathcal{E}_1[\sigma]_\sim = [\tau]_\sim$, we get $\{\alpha \in \kappa : \sigma(\alpha) = \tau(\alpha)\} \equiv_\kappa \emptyset$.

Proof. Everything has been checked already except the claim in the last sentence. Selecting $x_0, x_1 \in \chi$ witnessing that $\mathbf{Av}$ is changing, we let $\sigma \in \mathfrak{U}$ be given by

$$\sigma(\alpha) = \begin{cases} x_0 & \text{if } \alpha \in \bigcup_{\nu\in\kappa} I_{0,\nu}, \\ x_1 & \text{if } \alpha \in \bigcup_{\nu\in\kappa} I_{1,\nu}, \end{cases}$$

and it is clear that, letting $\tau \in \mathfrak{U}^\kappa$ be constantly $\mathbf{Av}(x_0, x_1)$, we have $\mathcal{E}_1[\sigma]_\sim = [\tau]_\sim$. □

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