ON AUTOMORPHISMS OF $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$.

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Abstract. We investigate the statement "all automorphisms of $\mathcal{P}(\lambda)/[\lambda]^{<\lambda}$ are trivial". We show that MA implies the statement for regular uncountable $\lambda < 2^{\aleph_0}$; that the statement is false for measurable λ if $2^\lambda = \lambda^+$; and that for "densely trivial" it can be forced (together with $2^\lambda = \lambda^{++}$) for inaccessible λ.

1. Introduction

We investigate automorphisms of Boolean algebras of the form

$$P^\lambda_\kappa := \mathcal{P}(\lambda)/[\lambda]^{<\kappa}$$

The instance P^ω_ω, i.e., $\mathcal{P}(\omega)/\mathrm{FIN}$, has been studied extensively for many years.[1] One can study variants for uncountable cardinals λ. Unsurprisingly, the behaviour here tends to be quite different to the countable case. One moderately popular[2] such generalisation is P^λ_ω. Here, we study another obvious generalization of the countable case, P^λ_λ. Some results for general P^λ_κ can be found in [LM16].

The main result of the paper is:

(T1, Thm. 5.2) The following is equiconsistent with an inaccessible: λ is inaccessible, 2^λ is λ^{++} and all automorphisms of P^λ_λ are densely trivial.

Here, $2^\lambda > \lambda^+$ is necessary, at least for measurables:

(T2, Thm. 4.1) If λ is measurable and $2^\lambda = \lambda^+$, then there is a nontrivial automorphism of P^λ_λ.

Remark 1.1. From [SS15, Lem. 3.2] it would follow that T2 holds even when "measurable" is replaced by just "inaccessible". However, the proof there turned out to be incorrect.[3]

For λ below the continuum we get the following result under Martin's Axiom (MA). More explicitly, $\mathrm{MA}_{=\lambda}(\sigma\text{-centered})$ is sufficient, which is the statement that for any σ-centered poset P and $\leq\lambda$ many open dense sets in P there is a filter G meeting all these open sets:

(T3, Thm. 3.1) For $\aleph_0 < \kappa \leq \lambda < 2^{\aleph_0}$ and κ regular, $\mathrm{MA}_{=\lambda}(\sigma\text{-centered})$ implies that every automorphism of P^λ_κ is trivial.

Larson and McKenney [LM16] showed the same under $\mathrm{MA}_{\aleph_1}$ for the case $\lambda = 2^{\aleph_0}$ and $\kappa = \aleph_1$.

Contrast this to the case $\lambda = \kappa = \omega$: Due to results of Veličković, Steprāns and the third author, "Every automorphisms of $\mathcal{P}(\omega)/[\omega]^{<\omega}$ is trivial" is implied by PFA [SS88], in fact even by MA+OCA [Vel93], but not by MA alone [Vel93] (not even for "somewhere trivial" [SS02]).

Date: 2024-03-04.

Supported by Austrian Science Fund (FWF): grants P33420 and P33895 (first author) and T1081 (second author), and Israel Science Foundation (ISF) grant 2320/23 (third author). Variants of Sections 3 and 4 where included in the second author's thesis. This is publication number 1224 of the third author.

[1]Rudin [Rud56a, Rud56b] showed in the 1950s that CH implies that there is a non-trivial automorphism; Shelah [She82] showed that consistently all automorphisms are trivial. Further results can be found, e.g., in [FS14, SS15, Vel93, SS02, Dow19, Far00, FGVV24].

[2]See e.g. [Vel93, SS16, SS18].

[3]A corrected version has been submitted, see `https://shelah.logic.at/papers/990a/`. This version again establishes the result only assuming inaccessibility.

Contents. We start by introducing some notation and basic results in Sec. 2 (p. 2).

The following sections are independent of each other:

In Sec. 3 (p. 3) we show T3, i.e., Thm. 3.1; in Sec. 4 (p. 6), we show T2, i.e., Thm. 4.1; and finally in the main part, Sec. 5 (p. 7) we develop some forcing notions to prove T1, i.e., Thm. 5.2.

Acknowledgments. We thank an anonymous referee for numerous corrections.

2. Definitions

We always assume:

- λ is a cardinal and $\kappa \leq \lambda$ is regular.
- The case $\kappa = \aleph_0$ or $\lambda = \aleph_0$ is included only for completeness sake in the following definitions.
- In Section 3 we will assume that $\aleph_1 \leq \kappa \leq \lambda < 2^{\aleph_0}$.
- In Section 4 we assume that λ is measurable and $\kappa = \lambda$.
- In Section 5 we assume that λ is inaccessible and $\kappa = \lambda$.

Notation:

- We investigate the Boolean algebra (BA) $P^\lambda_\kappa := \mathcal{P}(\lambda)/[\lambda]^{<\kappa}$, i.e., the power set of λ factored by the ideal of sets of size $<\kappa$.
- For $A \subseteq \lambda$, we denote the equivalence class of A with $[A]$. We set $\mathbb{0} := [\emptyset]$.
- $A \subseteq^* B$ means $|B \setminus A| < \kappa$, analogously for $A =^* B$; and "for almost all $\alpha \in A$" means for all but $<\kappa$ many in A. In particular, $A =^* \lambda$ means $A \subseteq \lambda$ and $|\lambda \setminus A| < \kappa$.
- We denote the BA-operations in P^λ_κ with $x \vee y$, $x \wedge y$ and x^c (for the complement).
  So we have $[A] \vee [B] = [A \cup B]$, $[A] \wedge [B] = [A \cap B]$, and $[A]^c = [\lambda \setminus A]$.
- A function $\phi : P^\lambda_\kappa \to P^\lambda_\kappa$ is a BA-automorphism (which we will just call *automorphism*), if it is bijective, compatible with $\wedge$ and the complement, and satisfies $\phi(\mathbb{0}) = \mathbb{0}$.
- Preimages of a function f are denoted by $f^{-1}x$, images by $f''x$.
- We sometimes identify $\eta \in 2^\lambda$ with $\eta^{-1}\{1\} \subseteq \lambda$ without explicitly mentioning it, by referring to η as element of 2^λ or of $P(\lambda)$.

Let us note that P^λ_κ is $<\kappa$-complete[4] and λ^+-cc. Also, any automorphism ϕ is closed under $<\kappa$ unions: $\phi(\bigvee_{i\in I}[A_i]) = \bigvee_{i\in I} \phi([A_i])$.

An automorphism is trivial if it is induced by a function on λ. A standard definition to capture this concept is the following:

Definition 2.1. An automorphism $\phi : P^\lambda_\kappa \to P^\lambda_\kappa$ is trivial, if there is a $g : \lambda \to \lambda$ such that $\phi([A]) = [g^{-1}A]$ for all $A \subseteq \lambda$.

However, we prefer to use forward images instead of inverse images; which can easily be seen to be equivalent:

Definition 2.2.

- For $f : A_0 \to \lambda$ with $A_0 =^* \lambda$, define $\pi_f : P^\lambda_\kappa \to P^\lambda_\kappa$ by $\pi_f([B]) := [f''(B \cap A_0)]$ for all $B \subseteq \lambda$.
- f is an almost permutation, if there are $A_0 =^* \lambda$ and $B_0 =^* \lambda$ with $f : A_0 \to B_0$ bijective.

(Such a π_f is always a well-defined function.)

Lemma 2.3. *Let $\phi : P^\lambda_\kappa \to P^\lambda_\kappa$ be a function. The following are equivalent:*

(1) ϕ is a trivial automorphism.
(2) There is an almost permutation f such that $\phi = \pi_f$.
(3) (Assuming $\kappa > \aleph_0$.) There is a bijection $f : \lambda \to \lambda$ such that $\phi = \pi_f$.

Proof. (1) implies (2): Assume ϕ is a trivial automorphism, witnessed by g.

Then $X := g''\lambda =^* \lambda$ (as $\phi([X]) = [g^{-1}X] = [\lambda]$), and $Y := \{\alpha \in X : |g^{-1}\{\alpha\}| \neq 1\} =^* \emptyset$: Otherwise, pick $y^0_\alpha \neq y^1_\alpha$ for each $\alpha \in Y$ with $g(y^0_\alpha) = g(y^1_\alpha) = \alpha$. So $y^0_\alpha \in g^{-1}C$ iff $y^1_\alpha \in g^{-1}C$ for any $C \subseteq \lambda$. Set $B^i := \{y^i_\alpha : \alpha \in Y\}$ for $i = 0, 1$ and let $[C] = \phi^{-1}([B^0])$. So $\phi([C]) = [g^{-1}C] = [B^0]$,

[4] I.e., if $|I| < \kappa$ then $\bigvee_{i\in I}[A_i] = [\bigcup_{i\in I} A_i]$

i.e., almost all y^0_α are in $g^{-1}C$, but then almost all y^1_α are in $g^{-1}C$ as well, i.e., $[B^0] = \phi([C]) \geq [B^1]$, a contradiction as $B^0 \cap B^1 = \emptyset$.

Set $A_0 := X \setminus Y$, and $B_0 := g^{-1}A_0$. Note that $B_0 =^* \lambda$, as $\mathbb{0} = \phi(\mathbb{0}) = \phi([Y]) = [g^{-1}Y]$. So $g \restriction B_0 \to A_0$ is bijective, and we can set $f : A_0 \to B_0$ the inverse. Then f is an almost permutation, and $\pi = \pi_f$.

(2) implies (1): Let $f : A_0 \to B_0$ be an almost permutation, and $g : B_0 \to A_0$ the inverse (and let g be defined arbitrarily on $\lambda \setminus B_0$). Then $\pi_f([X]) = [f''(X \cap A_0)] = [g^{-1}(X)]$. It remains to be shown that π_f is an automorphism: $\pi_f([\emptyset]) = [f''\emptyset] = [\emptyset]$; $\pi_f([X \cap Y]) = [f''(X \cap Y \cap A_0)] = [f''(X \cap A_0) \cap f''(Y \cap A_0)]$; and $\pi_f([\lambda \setminus X]) = [f''(A_0 \setminus X)] = [B_0 \setminus f''X]$.

(2) implies (3) if $\mathrm{cf}(\kappa) > \aleph_0$: This follows from the follwing lemma. □

Lemma 2.4. *($\kappa > \aleph_0$) Let f be a κ-almost permutation. Then there is an $S =^* \lambda$ such that $f \restriction S : S \to S$ is bijective.*

Proof. Set $X_0 := A_0 = \mathrm{dom}(f)$, and $X_{i+1} := X_i \cap f''X_i \cap f^{-1}X_i$, and $S := \bigcap_{i\in\omega} X_i$.

The X_n are decreasing, and $|\lambda \setminus X_n| < \kappa$ and thus $|\lambda \setminus (f''X_n)| < \kappa$ for $n < \omega$. Accordingly, $|\lambda \setminus S| < \kappa$. We claim that $g := f \restriction S$ is a permutation of S. Clearly it is injective. If $\alpha \in S$ then $\alpha \in X_n$ for all $n \in \omega$, so $f(\alpha) \in X_{n+1}$ for all n. So $g : S \to S$. If $\alpha \in S$, then $\alpha \in X_{n+1}$ for all n, so $f^{-1}(\alpha)$ exists and is in X_n. □

Remark: For $\kappa = \lambda = \omega$, there are trivial automorphisms that are not induced by "proper" bijections $f : \omega \to \omega$, e.g. the automorphism ϕ induced by the almost permutation $n \mapsto n + 1$.[5]

We will investigate somewhere and densely trivial automorphisms. To simplify notation, we assume $\kappa = \lambda > \aleph_0$:

Definition 2.5. ($\lambda > \aleph_0$ regular.) Let $\phi : P^\lambda_\lambda \to P^\lambda_\lambda$ be an automorphism.

- ϕ is trivial on $A \in [\lambda]^\lambda$, if there is an $f : A \to \lambda$ with $\phi([B]) = [f''B]$ for all $B \subseteq A$.
- ϕ is somewhere trivial, if it is trivial on some $A \in [\lambda]^\lambda$.
- ϕ is densely trivial, if for all $A \in [\lambda]^\lambda$ there is a $B \subseteq A$ of size λ such that ϕ is trivial on B.

Just as before it is easy to see that we can assume f to be a full permutation:

Fact 2.6. ($\lambda > \aleph_0$ regular.) An automorphism $\phi : P^\lambda_\lambda \to P^\lambda_\lambda$ is trivial on $A \in [\lambda]^\lambda$ iff there is a bijection $f : \lambda \to \lambda$ such that $\phi([B]) = [f''(B)]$ for all $B \subseteq A$.

Lemma 2.7. *($\lambda > \aleph_0$ regular.) If every automorphism of P^λ_λ is somewhere trivial, then every automorphism of P^λ_λ is densely trivial.*

Proof. Assume π is an automorphism of P^λ_λ, and fix $A \in [\lambda]^\lambda$. If $A =^* \lambda$ and if π is trivial on some B, then π is trivial on $B \cap A \subseteq A$, so we are done. So assume $A \neq^* \lambda$.

Pick some representative $\pi^* : \mathcal{P}(\lambda) \to \mathcal{P}(\lambda)$ of π such that $\pi^*(A)$ and $\pi^*(\lambda \setminus A)$ partition λ, and such that $\pi^*(C) \subseteq \pi^*(A)$ for every $C \subseteq A$. Let $i : \lambda \setminus A \to A$ and $j : \pi^*(\lambda \setminus A) \to \pi^*(A)$ both be bijective. Let π' map $[D]$ to $[\pi^*(D \cap A) \cup j^{-1}\pi^*(i''(D \setminus A))]$. This is an automorphism of P^λ_λ, so it is trivial on some D_0. If $|D_0 \cap A| = \lambda$, we are done, as π' restricted to $D_0 \cap A$ is the same as π and trivial. So assume otherwise. Then π' is trivial on the large set $D_0 \setminus A$. Then π is trivial on $i''(D_0 \setminus A) \subseteq A$. □

3. Under MA, every automorphism is trivial for $\omega_1 \leq \lambda < 2^{\aleph_0}$

Theorem 3.1. *Assume $\aleph_0 < \kappa \leq \lambda < 2^{\aleph_0}$, κ regular, and $\mathrm{MA}_{(=\lambda)}(\sigma\text{-centered})$ holds. Then every automorphism of P^λ_κ is trivial.*

For the proof we will use that we can separate certain sets by closed sets.

A tree T is a subset of $2^{<\omega}$ such that $s \in T \cap 2^n$ and $m \leq n$ implies $s \restriction m \in T$; for such a T we set $\lim(T) = \{\eta \in 2^\omega : (\forall n \in \omega)\, \eta \restriction n \in T\}$. A subset of 2^ω is closed iff it is of the form $\lim(T)$ for some tree T.

[5] A bijection $f : \omega \to \omega$ has infinitely many n such that $f(n) \neq n + 1$, and therefore an infinite set A such that $f''A$ is disjoint to $\{n + 1 : n \in A\}$.

Lemma 3.2. *Assume* $\aleph_0 < \theta \le \lambda < 2^{\aleph_0}$, $\mathrm{cf}(\theta) > \aleph_0$, *and* $\mathrm{MA}_{(=\lambda)}(\sigma\text{-centered})$ *holds. Assume* A_0, A_1 *are disjoint subsets of* 2^ω *of size* $\le \lambda$*;* $|A_0| \ge \theta$*. Then there is a tree* T_0 *in* $2^{<\omega}$ *such that* $|A_0 \cap \lim(T_0)| \ge \theta$ *and* $A_1 \cap \lim(T_0) = \emptyset$*.*

If additionally $|A_1| \ge \theta$*, we get an additional tree* T_1 *such that* $|A_1 \cap \lim(T_1)| \ge \theta$*,* $A_0 \cap \lim(T_1) = \emptyset$*, and* $T_0 \cap T_1 \subseteq 2^n$ *for some* n*.*

Proof of the lemma. In the following we identify an $x \in 2^\omega$ with the according (infinite) branch b in the tree $2^{<\omega}$. So a branch b can be in A_0 or in A_1 (or in neither; but not both, as A_0 and A_1 are disjoint).

We define a poset Q as follows: A condition $q \in Q$ is a triple (n_q, S_q, f_q), where

- $n_q \in \omega$,
- S_q is a tree in $2^{<\omega}$ of the following form: S_q is the union of $2^{\le n_q}$ and finitely many (infinite) branches $\{b_j : j \in m\}$ for some $m \in \omega$, each $b_j \in A_0 \cup A_1$, and $b_j \restriction n_q = b_k \restriction n_q$ implies ($b_j \in A_i$ iff $b_k \in A_i$).

  So every $s \in S_q$ with $|s| > n_q$ is either "in A_0-branches" (i.e., there is one or more $b_j \in A_0$ with $s \in b_j$), or "in A_1-branches" (but not in both).

  Note that an $s \in S_q$ of length n_q is either in A_0-branches, or in A_1-branches, or in neither (but not in both).
- $f_q : S_q \to 2$ such that, for $i = 0, 1$, $f_q(s) = i$ whenever $s \in S_q$, $|s| \ge n_q$ and s is in A_i-branches.

The order on Q is the natural one: $q \le p$ if $n_q \ge n_p$, $S_q \supseteq S_p$ and f_q extends f_p.

Q is σ-centered witnessed by $(n_q, S_q, f_q) \mapsto (n_q, f_q \restriction 2^{\le n_q})$: If p, q are in Q with $n_p = n_q =: n$ and $f_p \restriction 2^{\le n} = f_q \restriction 2^{\le n}$, then $(n, S_p \cup S_q, f_p \cup f_q)$ is a valid condition stronger than both p and q.

For $x \in A_i$, the set D_x of conditions containing x as branch is dense: Given $p \in Q$, let $n_q \ge n_p$ be such that all A_{1-i}-branches in p split off x below n_q; set $S_q := S_p \cup 2^{\le n_q} \cup x$; and set $F_q(s) = i$ for $s \in S_q \setminus S_p$.

Similarly, for all $n \in \omega$, the set D^*_n of conditions q with $n_q \ge n$ is dense as well.

By $\mathrm{MA}_{(=\lambda)}(\sigma\text{-centered})$ and $|A_i| \le \lambda$, we can find a filter G which has nonempty intersection with each D_x for $x \in A_0 \cup A_1$ as well as for each D^*_n. So $F := \bigcup_{p \in G} f_p$ is a total function from $2^{<\omega}$ to 2; and for all $x \in A_i$ there is an $n_x \in \omega$ such that $m \ge n_x$ implies $F(x \restriction m) = i$.

As $|A_0| \ge \theta$ and $\mathrm{cf}(\theta) > \aleph_0$ we can assume that there is an n^*_0 such that $n_x = n^*_0$ for θ many $x \in A_0$. If additionally $|A_1| \ge \theta$, we analogously get an n^*_1 and set $n^* := \max(n^*_0, n^*_1)$; otherwise we set $n^* := n^*_0$. We set $T^*_i := \{s \in 2^{<\omega} : |s| \ge n^*, (\forall n^* \le k \le |s|)\, F(s \restriction k) = i\}$ and generate a tree from it; i.e., we set $T_i := T^*_i \cup \{s \restriction m : m < n^*, s \in T^*_i\}$. As we have seen above, $\lim(T_i) \cap A_i \ge \theta$ for $i = 0$ (and, if $|A_1| \ge \theta$, for $i = 1$ as well). Clearly $T_0 \cap T_1 \subseteq 2^{n^*}$; and $\lim(T_i) \cap A_{i-1}$ is empty, as for any $x \in A_{i-1}$, cofinally many n satisfy $F(x \restriction n) = i - 1$. □

Proof of the theorem. Fix an automorphism π of P^λ_κ represented by some $\pi^* : \mathcal{P}(\lambda) \to \mathcal{P}(\lambda)$, and let π^{-1*} represent π^{-1}. We have to show that π is trivial.

Fix an injective function $\eta : \lambda \to 2^\omega$. Set

$$C_n := \{x \in 2^\omega : x(n) = 0\} \text{ and } \Lambda_n := \eta^{-1} C_n = \{\alpha < \lambda : \eta(\alpha)(n) = 0\}.$$

Define $\nu : \lambda \to 2^\omega$ by

$$\nu(\beta)(n) = 0 \text{ iff } \beta \in \pi^*(\Lambda_n).$$

In the following, "large" means "of cardinality $\ge\kappa$", and "small" means not large. We will show:

- $(*_1)$ $\pi^*(\eta^{-1}C) =^* \nu^{-1}C$ for $C \subseteq 2^\omega$ closed.
- $(*_2)$ $Y \subseteq \lambda$ and $|Y| \ge \kappa$ implies $|\nu'' Y| \ge \kappa$.
- $(*_3)$ If A_0, A_1 are disjoint subsets of 2^ω, $A_0 \subseteq \nu''\lambda$ large, then $\pi^{-1*}(\nu^{-1}A_0) \setminus \eta^{-1}A_1$ is large.
- $(*_4)$ If A_0, A_1 are disjoint subsets of 2^ω, $A_0 \subseteq \eta''\lambda$ large, then $\pi^*(\eta^{-1}A_0) \setminus \nu^{-1}A_1$ is large.

(Note that $(*_2)$ is the only place where we use that κ is regular.)

Proof:

- $(*_1)$ $\pi^*(\eta^{-1}C_n) = \nu^{-1}C_n$ holds by definition of ν. As π honors $<\kappa$-unions and complements, and as the C_n generate the open sets, this equation (with $=^*$) holds whenever C is generated by $<\kappa$-unions and complements from the open sets, in particular, if C is closed.

($*_2$) Fix $x \in 2^\omega$. Then $\eta^{-1}\{x\}$ has at most one element (as η is injective), and $\eta^{-1}\{x\} =^* \pi^{-1*}\nu^{-1}\{x\}$ by ($*_1$). I.e., $\nu^{-1}\{x\}$ is small. And $Y \subseteq \bigcup_{x\in\nu''Y} \nu^{-1}\{x\}$, so as κ is regular we get $|\nu''Y| \geq \kappa$.)

($*_3$) Using the previous lemma (with κ as θ) we get a tree T_0 separating A_0 and A_1. I.e., $\lim(T_0) \cap A_1 = \emptyset$ and $X := \lim(T_0) \cap A_0$ is large. As $X \subseteq A_0 \subseteq \nu''\lambda$, we get that $\nu^{-1}X$ is large. And $\nu^{-1}X = \nu^{-1}\lim(T_0) \cap \nu^{-1}A_0 =^* \pi^*(\eta^{-1}\lim(T_0)) \cap \nu^{-1}A_0$, the last equation by ($*_1$). This implies $\eta^{-1}\lim(T_0) \cap \pi^{-1*}(\nu^{-1}A_0)$ is large, and so $\pi^{-1*}(\nu^{-1}A_0) \setminus \eta^{-1}A_1$ is large.

($*_4$) We get an analogous result when interchanging ν and η and using π^* instead of π^{-1*}.

We claim that the following sets N_i are all small:

(1) $N_1 := \{\alpha \in \lambda : (\neg\exists\beta \in \lambda)\, \eta(\alpha) = \nu(\beta)\}$.
(2) $N_2 := \{\alpha \in \lambda : (\exists^{(\geq 2)}\beta \in \lambda)\, \eta(\alpha) = \nu(\beta)\}$.
(3) $N_3 := \{\beta \in \lambda : (\neg\exists\alpha \in \lambda)\, \eta(\alpha) = \nu(\beta)\}$.

Proof:

(3) Assume N_3 is large. Set $A_0 := \nu''N_3$, which is large by ($*_2$); and $A_1 := \eta''\lambda$. So A_0 and A_1 are disjoint, and by ($*_3$) $\pi^{-1*}\nu^{-1}A_0 \setminus \eta^{-1}A_1$ is large, but $\eta^{-1}A_1 = \lambda$.

(1) Assume N_1 is large. Set $A_0 = \eta''N_1$ (large, as $\underset{\sim}{\eta}$ is injective) and $A_1 := \nu''\lambda$. So A_0 and A_1 are disjoint, and by ($*_4$) $\pi^*(\eta^{-1}A_0) \setminus \nu^{-1}A_1$ is large, but $\nu^{-1}A_1 = \lambda$.

(2) Assume that N_2 is large. For every $\alpha \in N_2$, let $\beta^0_\alpha \neq \beta^1_\alpha$ in λ be such that $\eta(\alpha) = \nu(\beta^0_\alpha) = \nu(\beta^1_\alpha)$. For $i \in \{0,1\}$, set $Y_i := \{\beta^i_\alpha : \alpha \in N_2\}$ and $X_i := \pi^{-1*}(Y_i)$ (without loss of generality disjoint), and $A_i := \eta''X_i$. So the A_i are large and disjoint, and we can find a tree T_0 such that $A_0 \cap \lim(T_0)$ is large, and $A_1 \cap \lim(T_0)$ is empty.

As $A_0 \subseteq \eta''\lambda$, this implies that the inverse η-image of $A_0 \cap \lim(T_0)$ is also large. I.e., $\eta^{-1}(A_0 \cap \lim(T_0)) = \eta^{-1}A_0 \cap \eta^{-1}\lim(T_0) =^* X_0 \cap \pi^{-1*}\nu^{-1}\lim(T_0)$ is large (for the last equation we use ($*_1$)). Therefore also $Y_0 \cap \nu^{-1}\lim(T_0)$ is large, and so, by ($*_2$), $\nu''(Y_0 \cap \nu^{-1}\lim(T_0)) = \lim(T_0) \cap \nu''Y_0$ is large as well.

On the other hand $\lim(T_0)\cap A_1$ is empty, so $0 =^* \pi^*\eta^{-1}(\lim(T_0)\cap A_1) =^* \pi^*\eta^{-1}\lim(T_0) \cap \pi^*\eta^{-1}A_1$. Using ($*_1$) for $\lim(T_0)$, and noting that $\pi^*\eta^{-1}A_1 = Y_1$, this set is (almost) equal to $Y_1 \cap \nu^{-1}\lim(T_0)$ which therefore is also small, and so $\lim(T_0) \cap \nu''Y_1$ is small.

So we know that $\lim(T_0) \cap \nu''Y_0$ is large and $\lim(T_0) \cap \nu''Y_1$ is small, but $\nu''Y_0 = \nu''Y_1$, a contradiction.

Note that this implies:

($*_5$) $X \cap Y$ small implies $\nu''X \cap \nu''Y$ small, for $X, Y \subseteq \lambda$.
($*_6$) $\nu^{-1}\nu''X =^* X$ for $X \subseteq \lambda$.

Proof:

($*_5$) Assume otherwise. Without loss of generality we can assume that X and Y are disjoint, and by (3) that $\nu''X$ and $\nu''Y$ both are subsets of $\eta''\lambda$. Then $\nu''X \cap \nu''Y \subseteq \eta''N_2$ is small.

($*_6$) Set $Y := \nu^{-1}\nu''X \setminus X$. Then $\nu''Y \subseteq N_2 \cup N_3$ is small, and by ($*_2$) Y is small.

Set $D := \lambda \setminus (N_1 \cup N_2)$ and define $e : D \to \lambda$ such that $e(\alpha)$ is the (unique) $\beta \in \lambda$ with $\eta(\alpha) = \nu(\beta)$. Clearly e is injective. We claim that e generates π, i.e., that the following are small (where we can assume $X \subseteq D$):

(4) $N_4 := \pi^*(X) \setminus e''X$.
(5) $N_5 := e''X \setminus \pi^*(X)$.

Proof:

(4) Assume that N_4 is large. Set $Y = \pi^{-1*}(N_4)$, without loss of generality $Y \subseteq X$ and $\pi^*(Y) = N_4$. So $\pi^*(Y)$ is disjoint from $e''Y$ (as it is even disjoint from $e''X$). We set $A_0 := \nu''\pi^*(Y)$ and $A_1 := \nu''e''Y$, by ($*_5$) we can assume they are disjoint, and by ($*_2$) both are large (e is injective).

By ($*_3$), $\pi^{-1*}(\nu^{-1}A_0) \setminus \eta^{-1}A_1$ is large.

$\eta^{-1}(A_1) = Y$, as $\nu(e(\alpha)) = \eta(\alpha)$ for all $\alpha \in D$. And $\pi^{-1*}(\nu^{-1}A_0) =^* Y$ by definition and ($*_6$), a contradiction.

(5) The same proof works: This time we set $Y = e^{-1}N_5$; see that $\pi^*(Y)$ and $e''Y$ are disjoint and large; set $A_0 := \nu''\pi^*(Y)$ and $A_1 := \nu''e''Y$; use $(*_3)$ to see that $Y \setminus \eta^{-1}\nu''e''Y = Y \setminus Y$ is large, a contradiction. □

4. For measureables, GCH implies a nontrivial automorphism

Theorem 4.1. *If λ is measurable and $2^\lambda = \lambda^+$, then there is a nontrivial automorphism of P^λ_λ.*

Proof. Let $\mathcal{D}$ be a normal ultrafilter on λ and denote by $\mathcal{I} := [\lambda]^\lambda \setminus \mathcal{D}$ its dual ideal restricted to sets of size λ.

Since $2^\lambda = \lambda^+$, we can list all permutations of λ as $\{e_\alpha : \alpha < \lambda^+\}$; and analogously all elements of $\mathcal{I}$ as $\{X_\alpha : \alpha < \lambda^+\}$.

We will construct, by induction on $\alpha < \lambda^+$ a set $A_\alpha \in \mathcal{I}$ and a permutation f_α of A_α, such that for $\alpha < \beta$:

(1) $A_\alpha \subseteq^* A_\beta$,
(2) $X_\alpha \subseteq A_{\alpha+1}$,
(3) $f_\alpha(x) = f_\beta(x)$ for almost all $x \in A_\alpha \cap A_\beta$,
(4) there is some $X \subseteq A_{\alpha+1}$ of size λ such that $e''_\alpha X$ and $f''_{\alpha+1} X$ are disjoint.

(Note that by $x \subseteq^* y$ we mean $|y \setminus x| = \lambda$, not $y \setminus x \in \mathcal{I}$; and the same for 'almost all".)
The construction:

- Successor stages $\alpha + 1$: Fix any $B \in \mathcal{I}$ disjoint to A_α such that $A_\alpha \cup B \supseteq X_\alpha$. Set $C := e''_\alpha B \cap A_\alpha$.
  First assume that $|C| = \lambda$. Then set $A_{\alpha+1} = A_\alpha \cup B$ and let $f_{\alpha+1}$ extend f_α by the identity on B. Then (4) is witnessed by $X := e_\alpha^{-1} C$.
  So we assume $|C| < \lambda$. Partition B into large sets B_0, B_1, B_2 such that $e''_\alpha B_i$ is disjoint to A_α for $i = 0, 1$. Set $A_{\alpha+1} := A_\alpha \cup B \cup e''_\alpha B$, and define $f_{\alpha+1}$ on B such that the restriction to B_i is a bijection op $e''_\alpha B_{1-i}$ for $i = 0, 1$, and the restriction to B_2 a bijection to $e''B_2 \setminus A$. Then (4) is witnessed by $X := B_0$.
- Limit stages δ of cofinality $<\lambda$: Let $\xi := \mathrm{cf}(\delta)$ and choose $\langle \alpha_i : i < \xi \rangle$ a cofinal increasing sequence converging to δ. The union $\bigcup_{i<\xi} A_{\alpha_i}$ is, by $<\lambda$ completeness, in $\mathcal{I}$. Remove $< \lambda$ many points to get a subset A_δ such that
  - For all $i < j < \xi$, f_i and f_j agree on $A_{\alpha_i} \cap A_\delta$,
  - For all $i < \xi$, $f_i \restriction (A_{\alpha_i} \cap A_\delta)$ is a full permutation (we can do this as in Lemma 2.4).

  Then f_δ, defined as the union of the f_{α_i}, is a permutation of A_δ and almost extends each f_{α_i}.
- Limit stages δ of cofinality λ: We choose an increasing cofinal sequence $\langle \alpha_i : i < \lambda \rangle$ converging to δ. By induction on $i \in \lambda$ we construct $A'_i =^* A_{\alpha_i}$, such that
  - $A'_i \cap i = \emptyset$,
  - The f_{α_i}'s fully extend each other on the A'_i's, i.e., if $x \in A'_i \cap A'_j$ then $f_{\alpha_i}(x) = f_{\alpha_j}(x)$,
  - $f_{\alpha_i} : A'_i \to A'_i$ is a "full" permutation.

  We can do this by removing from A_{α_i}: the points less than i, the points where f_{α_i} disagrees with some previous f_{α_j} for any $j < i$; and by removing $<\lambda$ many points to get a full permutation.
  Now we can set A_δ and f_δ to be the unions of A'_i and f_{α_i}, respectively, for $i < \delta$. Note that A_δ is in $\mathcal{I}$ (as it is a subset of the diagonal union); and f_δ is a permutation of A_δ satisfying (3).

Note that for all $X \subseteq \lambda$, either $X \in \mathcal{I}$ or $\lambda \setminus X \in \mathcal{I}$ (but not both), i.e., either X or $\lambda \setminus X$ is $\subseteq^* A_\alpha$ for coboundedly many $\alpha < \lambda$.

This allows us to define the automorphism π as follows: For $X \in [\lambda]^\lambda$,

$$\pi([X]) := \begin{cases} [f''_\alpha X] & \text{if } X \in \mathcal{I}, X \subseteq^* A_\alpha \text{ for some } \alpha < \lambda^+ \text{ (Case 1)} \\ [\lambda \setminus f''_\alpha(\lambda \setminus X)] & \text{if } X \notin \mathcal{I}, \lambda \setminus X \subseteq^* A_\alpha \text{ for some } \alpha < \lambda^+ \text{ (Case 2).} \end{cases}$$

Note that in Case 2, $\pi([X]) = [(\lambda \setminus A_\alpha) \cup (A_\alpha \setminus f''_\alpha(A_\alpha \setminus X))] = [(\lambda \setminus A_\alpha) \cup f''_\alpha(X \cap A_\alpha)]$, as $f''_\alpha A_\alpha =^* A_\alpha$.

π is well defined on $[\lambda]^\lambda$, as exactly one of X or $\lambda \setminus X$ will eventually be $\subseteq^* A_\alpha$.

π is an automorphism: $\pi([\emptyset]) = \emptyset$. π honors complements: If X is Case 1, then $\pi([\lambda \setminus X])$ is by definition (Case 2) $[\lambda \setminus f''_\alpha(X)]$. π honors intersections $X \cap Y$: This is clear if both sets are the same Case. Assume that X is Case 1 and Y Case 2. Then $X \cap Y \subseteq X$ is Case 1, and for any α suitable for both X and Y we have

$$\pi([X]) \wedge \pi([Y]) = [f''_\alpha X \cap ((\lambda \setminus A_\alpha) \cup f''_\alpha(Y \cap A_\alpha))] = [f''_\alpha X \cap f''_\alpha(Y \cap A_\alpha)] = [f''_\alpha(X \cap Y))].$$

π is not trivial: Every automorphism e is an e_α for some $\alpha \in \lambda^+$; and according to (4) there is some $X_\alpha \subseteq A_{\alpha+1}$ (and therefore in $\mathcal{I}$) of size λ such that $e''_\alpha X_\alpha$ is disjoint to $f''_{\alpha+1} X_\alpha$, a representative of $\pi([X_\alpha])$. □

5. For inaccessible λ, all automorphisms can be densely trivial

In this section, we always assume the following (in the ground model):

Assumption 5.1. λ is inaccessible and $2^\lambda = \lambda^+$. We set $\mu := \lambda^{++}$.

In the rest of the paper, we will show the following:

Theorem 5.2. *(λ is inaccessible and $2^\lambda = \lambda^+$.) There is a λ-proper, $<\lambda$-closed, λ^{++}-cc poset P (in particular, preserving all cofinalities) that forced: $2^\lambda = \lambda^{++}$, and every automorphism of P^λ_λ is densely trivial.*

By Lemma 2.7, it is enough to show that every automorphism is somewhere trivial.

5.1. The single forcing Q.

Definition 5.3. We fix a strictly increasing sequence $(\theta^*_\zeta)_{\zeta<\lambda}$ with $\theta^*_\zeta < \lambda$ regular and $\theta^*_\zeta > 2^{|\zeta|}$.

- Let $(I^*_\zeta)_{\zeta\in\lambda}$ be an increasing interval partition of λ such that I^*_ζ has size $2^{\theta^*_\zeta}$; and fix a bijection of I^*_ζ and $2^{\theta^*_\zeta}$. Using this (unnamed) bijection, we set $[s] := \{\ell \in I^*_\zeta : \ell > s\}$ for $s \in 2^{<\theta^*_\zeta}$.
  So the $[s]$ are cones, i.e., the set of all branches in I^*_ζ extending s.
  For $\zeta < \lambda$, we set $I^*(<\zeta) := \bigcup_{\ell<\zeta} I^*_\ell$, and analogously $I^*(\leq\zeta) := I^*(<\zeta+1)$, $I^*(\geq\zeta) := \lambda \setminus I^*(<\zeta)$, and $I^*(\geq\zeta, <\xi) := I^*(\geq\zeta) \cap I^*(<\xi)$.
- A condition q of the forcing notion Q is a function with domain λ such that, for all $\zeta \in \lambda$, $q(\zeta)$ is a partial function from I^*_ζ to 2, and such that for a club-set $C^q \subseteq \lambda$
  - if $\zeta \notin C^q$, then $q(\zeta)$ is total,
  - otherwise, the domain of $q(\zeta)$ is $I^*_\zeta \setminus [s^q_\zeta]$ for some $s^q_\zeta \in 2^{<\theta^*_\zeta}$.

  C^q and s^q_ζ are uniquely determined by q; and q is uniquely determined by the partial function $\eta^q : \lambda \to 2$ defined as $\bigcup_{\zeta\in\lambda} q(\zeta)$.
- q is stronger than p if η^q extends η^p.
  (This implies that $C^q \subseteq C^p$, and that s^q_ζ extends s^p_ζ for all $\zeta \in C^q$.)

The following is straightforward:

Lemma 5.4. *Q has size 2^λ, is $<\lambda$-closed and adds a generic real $\underset{\sim}{\eta} := \bigcup_{q\in G} \eta^q$ in 2^λ.*

Proof. $<\lambda$-closure is obvious, but for later reference we would like to point out the "problematic cases":

Let $(p_i)_{i<\delta}$ be decreasing for a limit ordinal $\delta < \lambda$.

As a first approximation, set $\eta^* := \bigcup_{i<\delta} \eta^{p_i}$ (a partial function) and $C^* := \bigcap_{i<\delta} C^{p_i}$ (a club set) and $s^*_\zeta := \bigcup_{i<\delta} s^{p_i}_\zeta \in 2^{\leq\theta^*_\zeta}$ for $s \in C^*$. For $\zeta \notin C^*$, η^* is indeed total on I^*_ζ, and for $\zeta \in C^*$ the domain in I^*_ζ is $I^*_\zeta \setminus [s^*_\zeta]$.

The **problematic case** is when s^*_ζ is unbounded in θ^*_ζ. (This can only happen if $\mathrm{cf}(\delta) = \theta^*_\zeta$, in particular for at most one ζ.) In this case we can just pick any extension η^q of η^* by filling all values in $I^*_{\leq\zeta}$. This gives the desired q, with $C^{q_\delta} = C^* \setminus \zeta + 1$. □

Remarks.

- The limits of $<\lambda$-sequences of conditions are not "canonical" if there are problematic ζ's, as we have to fill in arbitrary values.
- $\underset{\sim}{\eta}$ determines the generic filter, by $G = \{p \in Q : \eta^p \subseteq \underset{\sim}{\eta}\}$. This follows from the following facts:
  - p and q are compatible (as conditions in Q) iff η^p and η^q are compatible as partial functions and $X_{p,q} := \{\zeta \in C^p : s^p_\zeta$ and s^q_ζ are incomparable$\}$ is non-stationary.
  - If p, q are such that $X_{p,q}$ is stationary, then the set of conditions r such that η^r and η^q are incompatible (as partial functions) is dense below p.

5.2. **Properness of Q: Fusion and pure decision.**

Definition 5.5. We say $q \leq_\xi p$, if $q \leq p$, $\xi \in C^q$ and $q \restriction \xi = p \restriction \xi$.
$q \leq^+_\xi p$ means $q \leq_\xi p$ and $q(\xi) = p(\xi)$.

(Note the difference between $q \leq^+_\xi p$ and $q \leq_{\xi+1} p$: The former does not require $\xi + 1 \in C^q$.)

Lemma 5.6. *Let $\delta \leq \lambda$ be a limit ordinal, $\xi \in \lambda$ and $(q_i)_{i<\delta}$ a sequence in Q.*

(1) If $\delta < \lambda$ and $q_j <^+_\xi q_i$ for all $i < j < \delta$, then there is a q_∞ such that $q_\infty <^+_\xi q_i$ for all i.
(2) If $q_j <_{\xi_i} q_i$ for $i < j < \delta$, where $(\xi_i)_{i\in\delta}$ is a strictly increasing[6] sequence in λ, then there is a (canonical) limit q_∞ such that $q_\infty <_{\xi_i} q_i$ for all i.

Proof. (1): We perform the same construction as in the proof of Lemma 5.4. If there is a problematic case ζ, then $\zeta > \xi$ (as for $\zeta' \leq \xi$ the conditions $q_i(\zeta')$ are constant). We can then make η^* total on $I^*(>\xi, \leq\zeta)$. (It may not be enough to make it total on I^*_ζ, as $C^* \setminus \{\zeta\}$ might not be club.)

(2): Define $q_\infty(\zeta) := \bigcup_{i\in\delta} q_i(\zeta)$ for $\zeta \in \lambda$.

This is a non-total function (on I^*_ζ) iff $\zeta \in C^{q_\infty} := \bigcap_{i<\delta} C^{q_i}$, which is closed as intersection of closed sets, and also unbounded: If $\delta < \lambda$ because we have a small intersections of clubs, if $\delta = \lambda$ as it contains each ξ_i.

There are no problematic cases: If ζ is below some ξ_i, then $q_j(\zeta)$ is eventually constant. If ζ is above all ξ_i, which can only happen if $\delta < \lambda$, then $\mathrm{cf}(\delta) \leq \delta \leq \sup(\xi_i) \leq \zeta < \theta^*_\zeta$. □

So Q satisfies fusion; and we will now show that it also satisfies "pure decision"; standard arguments then imply that Q is λ-proper and λ^λ-bounding.

Definition 5.7. Let $\xi \in \lambda$, $q \in Q$.

- $\mathrm{POSS}^Q(\xi) := 2^{I^*(<\xi)}$. So in the extension $V[G]$, for each ξ there will be exactly one $x \in \mathrm{POSS}^Q(\xi)$ compatible with (or equivalently: an initial segment of) the generic real $\underset{\sim}{\eta}$. We write "$x \subseteq \underset{\sim}{\eta}$" or "$G$ chooses x" for this x.
- $\mathrm{poss}(q, \xi)$ is the set of $x \in \mathrm{POSS}^Q(\xi)$ compatible with η^q (as partial functions), or equivalently: $x \in \mathrm{poss}(q, \xi)$ iff $\neg q \Vdash x \not\subseteq \underset{\sim}{\eta}$. So q forces that exactly one $x \in \mathrm{poss}(q, \xi)$ is chosen by G.
- Let $\underset{\sim}{\tau}$ be a name for an ordinal. We say that q ξ-decides $\underset{\sim}{\tau}$, if there is for all $x \in \mathrm{poss}(q, \xi)$ an ordinal τ^x such that q forces $x \subseteq \underset{\sim}{\eta} \to \underset{\sim}{\tau} = \tau^x$.

Note that for $p \in Q$ and $\zeta \in C^p$, $q \leq^+_\zeta p$ is equivalent to $\mathrm{poss}(q, \zeta+1) = \mathrm{poss}(p, \zeta+1)$, while $q \leq_\zeta p$ is equivalent to $\zeta \in C^q$ and $\mathrm{poss}(q, \zeta) = \mathrm{poss}(p, \zeta)$.

Lemma 5.8. *Assume $p \in Q$, $\zeta \in C^p$, $x \in \mathrm{poss}(p, \zeta+1)$, and $r \leq p$ extends[7] x. Then there is a $q \leq^+_\zeta p$ forcing: $x \subseteq \underset{\sim}{\eta} \to r \in G$. This condition is denoted by $r \vee (p \restriction \zeta + 1)$.*

Proof. We set $q(\ell)$ to be $p(\ell)$ for $\ell \leq \zeta$, and $r(\ell)$ otherwise. If $q' \leq q$ forces $x \subseteq \underset{\sim}{\eta}$ then q' extends x and thus $q' \leq r$. □

Corollary 5.9. *("Pure decision") Let $\underset{\sim}{\tau}$ be a name for an ordinal, $p \in Q$, and $\zeta \in C^p$. Then there is a $q \leq^+_\zeta p$ which $(\zeta+1)$-decides $\underset{\sim}{\tau}$.*

[6] For $\delta = \lambda$, it is enough that the ξ_i converge to λ. For $\delta < \lambda$, we use that the ξ_i are increasing and that $\sup(\xi_i) \geq \mathrm{cf}(\delta)$.

[7] By which we mean $x \subseteq \eta^r$.

Proof. Let $(x_i)_{i\in\delta}$ enumerate $\text{poss}(p,\zeta+1)$, for some $\delta<\lambda$. Set $p_0=p$, and define a $\leq^+_\zeta$-decreasing sequence p_j by induction on $j\leq\delta$: For limits use Lemma 5.6(1), and for successors choose some $r\leq p_i$ deciding $\underset{\sim}{\tau}$ with a stem extending x_i and set p_{i+1} to $r\vee p_i\restriction(\zeta+1)$. □

From fusion and pure decision we get bounding and λ-proper, via "continuous reading of names". This is a standard argument, and we will not give it here; we will anyway prove a more "general" variant (for an iteration of Q's), in Lemmas 5.25 and 5.27.

Fact 5.10.

- Q has continuous reading of names: If $\underset{\sim}{\sigma}$ is a Q-name for a λ-sequence of ordinals, and $p\in Q$, then there is a $q\leq p$ and there are $\xi_i\in\lambda$ such that q ξ_i-decides $\underset{\sim}{\sigma}(i)$ for all $i\in\lambda$.
- Q is λ^λ-bounding. I.e., for every name $\underset{\sim}{\sigma}\in\lambda^\lambda$ and $p\in Q$ there is an $f\in\lambda^\lambda$ and $q\leq p$ such that q forces $f(i)>\underset{\sim}{\sigma}(i)$ for all $i\in\lambda$.
- Q is λ-proper. This means: If N is a $<\lambda$-closed elementary submodel of $H(\chi)$ of size λ containing Q, with χ sufficently large and regular, and if $p\in Q\cap N$, then there is a $q\leq p$ N-generic (i.e., forcing that each name of an ordinal which is in N is evaluated to an ordinal in N).

For completeness, we also mention the following well-known fact (the proof is straightforward):

Fact 5.11. Assume κ is regular, and that the forcing notion R is κ^κ-bounding. Then R preserves the regularity of κ, and every club-subset of κ in the extension contains a ground model club-set.

5.3. The iteration P.

Let us first recall some well-known facts:

Facts 5.12. A $<\lambda$-closed forcing preserves cofinalities $\leq\lambda$ and also the inaccessibilty of λ. The $\leq\lambda$-support iteration of $<\lambda$-closed forcings is $<\lambda$-closed.

We will iterate the forcings Q from the previous section in a $<\lambda$-closed $\leq\lambda$-support iteration of length $\mu:=\lambda^{++}$:

Definition 5.13. Let $(P_\alpha,Q_\alpha)_{\alpha<\mu}$ be the $\leq\lambda$-support iteration such that each Q_α is the forcing Q (evaluated in the P_α-extension). We will write P to denote the limit.

Remark. One way to see that P is proper is to use the framework of [RaS11]. However, we will need an explicit form of continuous reading for P anyway, which in turn gives properness for free.

Definition 5.14. Assume that $w\in[\mu]^{<\lambda}$ and $\xi\in\lambda$.

- $\underset{\sim}{\bar\eta}=(\underset{\sim}{\eta}_\alpha)_{\alpha\in\mu}$ is the sequence of Q_α-generic reals added by P.
- $\text{POSS}(w,\xi):=2^{w\times I^*(<\xi)}$. Exactly one $x\in\text{POSS}(w,\xi)$ is extended by $\underset{\sim}{\bar\eta}$, we write "$x$ is selected by G," or "$x\lhd G$."
- $\text{poss}(p,w,\xi):=\{x\in\text{POSS}(w,\xi):\ \neg p\Vdash\neg x\lhd G\}$.
- Let $\underset{\sim}{\tau}$ be a name of an ordinal. $\underset{\sim}{\tau}$ is (w,ξ)-decided by q, if there are $(\tau^x)_{x\in\text{poss}(q,w,\xi)}$ such that q forces $x\lhd G\rightarrow\underset{\sim}{\tau}=\tau^x$.

Clearly, if $\underset{\sim}{\tau}$ is (w,ξ)-decided by q, and if $q'\leq q$, $w'\supseteq w$ and $\xi'\geq\xi$, then $\underset{\sim}{\tau}$ is (w',ξ')-decided by q'.

Remark. If $q\in P$ (w,ζ)-decides some P_α-name $\underset{\sim}{\tau}$, then the *same* q will generally *not* $(w\cap\alpha,\xi)$-decide $\underset{\sim}{\tau}$ for any ξ.[8]

In the following, whenever we say that q (w,ζ)-decides something, we implicitly assume that $w\in[\mu]^{<\lambda}$ and $\zeta\in\lambda$.

Definition 5.15. Let $\underset{\sim}{\sigma}$ be a P-name for a λ-sequence of ordinals.

[8] For example: For a p-condition Q, let ODD^p be the set of odd elements of C^p (or any other unbounded subset X of C^p such that $C^p\setminus X$ is still club), and set $\text{ODD}^p_?:=\bigcup_{\zeta\in\text{ODD}^p}I^*_\zeta\setminus\text{dom}(\eta^p)$. Note that for any $x:\text{ODD}^p_?\rightarrow 2$, $\eta^p\cup x$ defines a condition in Q (stronger than p). So if we fix any $p(0)\in P_1$, and define the P_1-name $\underset{\sim}{\tau}\in\{0,1\}$ to be 0 iff $\underset{\sim}{\eta}_0\restriction\text{ODD}^{p(0)}_?$ is eventually constant to 0, then $\underset{\sim}{\tau}$ cannot be $(\{0\},\zeta)$-decided by $p(0)$ for any ζ. And if $p(1)$ is any condition with $p(0)\Vdash\eta^{p(1)}(0)=\underset{\sim}{\tau}$, then $\underset{\sim}{\tau}$ is $(\{1\},1)$-decided by $q:=(p(0),p(1))$.

- q continuously reads $\underset{\sim}{\sigma}$, if there are $(w_i, \xi_i)_{i\in\lambda}$ such that q (w_i, ξ_i)-decides $\underset{\sim}{\sigma}(i)$ for each $i \in \lambda$.
- P has continuous reading, if for each such $\underset{\sim}{\sigma}$ and $p \in P$ there is some $q \leq p$ continuously reading $\underset{\sim}{\sigma}$.

The following is a straightforward standard argument:

Fact 5.16. If P has continuous reading, then it is λ^λ-bounding.

As a first step towards pure decision, let us generalize the $\leq_\zeta$-notation we defined for Q:

Definition 5.17. Let $p \in P$, $w \in [\mu]^{<\lambda}$ and $\xi \in \lambda$.

- p fits (w, ξ), if $w \subseteq \mathrm{dom}(p)$ and $p \restriction \alpha \Vdash \xi \in C^{p(\alpha)}$ for all $\alpha \in w$.
- $q \leq_{w,\xi} p$ means: $q \leq p$, and for all $\alpha \in w$, $q \restriction \alpha$ forces $q(\alpha) <_\xi p(\alpha)$.
- $q \leq^+_{w,\xi} p$ is defined analogously using $<^+_\xi$ instead of $<_\xi$.

Obviously $q \leq^+_{w,\xi} p$ implies $q \leq_{w,\xi} p$; and $q \leq_{w,\xi} p$ implies that both p and q fit (w, ξ).

Remark. In contrast to the single forcing (or a product of such forcings), $q \leq_{w,\xi} p$ (or $q \leq^+_{w,\xi} p$) does *not* imply $\mathrm{poss}(q, w, \xi) = \mathrm{poss}(p, w, \xi)$.[9] More explicitly, setting $w = \{0, 1\}$, it is possible that $x \in \mathrm{poss}(p, w, \xi)$ but p does not force that $x(0) \subseteq \underset{\sim}{\eta}_0$ implies $x(1) \in \mathrm{poss}(p(1), \xi)$. (But see Section 5.5.)

5.4. **Continuous reading and properness of P.**

Lemma 5.18. *If q_i is a $\leq^+_{w,\zeta}$-decreasing sequence of length $\delta < \lambda$, then there is an $r \leq^+_{w,\zeta} q_i$ for all $i < \delta$.*

Proof. Set $\mathrm{dom}(r) := \bigcup_{i\in\delta} \mathrm{dom}(q_i)$, without loss of generality closed under limits. By induction on $\alpha \in \mathrm{dom}(r)$ we know that $r \restriction \alpha \leq q_i \restriction \alpha$ for all i, and define $r(\alpha)$ as follows: If $\alpha \in w$, we know that the $q_i(\alpha)$ are $\leq^+_\zeta$-increasing. Using Lemma 5.6(1), we pick some $r(\alpha)$ such that $r(\alpha) \leq^+_\zeta q_i(\alpha)$ for all i. If $\alpha \notin w$, we just pick any $r(\alpha) \leq q_i(\alpha)$ for all i. □

It is easy to see that P satisfies a version of fusion:

Lemma 5.19. *Assume $(p_i)_{i<\delta}$ is a sequence of length $\delta \leq \lambda$, such that $p_j \leq_{w_i,\xi_i} p_i$ for $i \leq j < \delta$, $w_i \in [\mu]^{<\lambda}$ increasing, $\xi_i \in \lambda$ strictly increasing. Set $w_\infty := \bigcup_{i<\delta} w_i$, $\mathrm{dom}_\infty := \bigcup_{i<\delta} \mathrm{dom}(p_i)$ and $\xi_\infty := \sup_{i<\delta} \xi_i$. If $\delta = \lambda$, we additionally assume $w_\infty = \mathrm{dom}_\infty$.*

Then there is a limit q_∞ with $\mathrm{dom}(q_\infty) = \mathrm{dom}_\infty$ such that $q_\infty \leq_{w_i,\xi_i} p_i$ for all $i < \delta$.

If $\delta < \lambda$, then q_∞ fits (w_∞, ξ_∞).

(If $w_\infty = \mathrm{dom}_\infty$, then the limit q_∞ is "canonical".)

Proof. We define $q_\infty(\alpha)$ by induction on dom_∞. We assume that we already have $q' := q_\infty \restriction \alpha$ which satisfies $q' \leq_{w_i\cap\alpha,\xi_i} p_i$ for all $i < \delta$.

Case 1: $\alpha \notin w_\infty$ (this can only happen if $\delta < \lambda$): We know that q' forces that $(p_i(\alpha))_{i<\delta}$ is a decreasing sequence, and we just pick some $q_\infty(\alpha)$ stronger then all of them.

Case 2: $\alpha \in w_\infty$: Let i^* be minimal such that $\alpha \in w_{i^*}$. We know that q' forces for all $i^* \leq i < j < \delta$ that $p_j(\alpha) <_{\xi_i} p_i(\alpha)$, so according to Lemma 5.6(2) there is a limit $q_\infty(\alpha) <_{\zeta_i} p_i(\alpha)$ (so in particular $q' \Vdash \zeta_i \in C^{q_\infty(\alpha)}$ for all $i \geq i^*$).

Now assume $\delta < \lambda$. If $\alpha \in w_\infty$, then it is in w_i for coboundedly many $i < \delta$. In other words, $p_j \restriction \alpha \Vdash \zeta_i \in C^{p_j(\alpha)}$ for coboundedly many $i \in \delta$ and all $j > i$, which implies $q_\infty \restriction \alpha \Vdash \xi_\infty \in C^{q_\infty(\alpha)}$. □

[9] An example: $\mathrm{dom}(p) = \mathrm{dom}(q) = w = \{0, 1\}$, $\min(C^{p(0)}) = \min(C^{q(0)}) = \xi$, and both $p(0)$ and $q(0)$ have trunk $a \in \mathrm{POSS}^Q(\xi)$. $p(0)$ forces that $p(1) = q(1)$, that $\min(C^{p(1)}) = \xi$ and that the trunk of $p(1)$ is either b or c (elements of $\mathrm{POSS}^Q(\xi)$); both are possible with $p(0)$. Now $q(0) \leq^+_\xi p(0)$ decides that the trunk of $p(1)$ is b. Then $q \leq^+_{w,\xi} p$, and (a, c) is in $\mathrm{poss}(p, w, \xi) \setminus \mathrm{poss}(q, w, \xi)$. In particular $(a, c) \in \mathrm{poss}(p, w, \xi)$ but p does not force that $a \subseteq \underset{\sim}{\eta}_0$ implies $c \in \mathrm{poss}(p(1), \xi)$.

Preliminary Lemma 5.20. *Let p fit (w,ζ), $x \in \text{poss}(p,w,\zeta+1)$, and let $r \le p$ extend x, i.e., $r \Vdash x \lhd G$. Then there is a $q \le^+_{w,\zeta} p$ forcing that $x \lhd G$ implies $r \in G$.*

Proof. Set $\text{dom}(q) := \text{dom}(r)$. We define $q(\alpha)$ by induction on $\alpha \in \text{dom}(q)$ and show inductively:

- $q \restriction \alpha \le^+_{w\cap\alpha,\zeta} p \restriction \alpha$.
- $q \restriction \alpha \Vdash (x \restriction \alpha \lhd G_\alpha \to r \restriction \alpha \in G_\alpha)$.

For notational convenience, we assume $\text{dom}(p) = \text{dom}(r)$ (by setting $p(\alpha) = \mathbb{1}_Q$ for any α outside the original domain of p).

Assume we already have constructed $q_0 = q \restriction \alpha$. Work in the P_α-extension $V[G_\alpha]$ with $q_0 \in G$.
Case 1: $r \restriction \alpha \notin G_\alpha$. Set $q(\alpha) := p(\alpha)$.
Case 2: $r \restriction \alpha \in G_\alpha$. Then $r(\alpha) \le p(\alpha)$. If $\alpha \notin w$, we set $q(\alpha) := r(\alpha)$; otherwise we set $q(\alpha)$ to be $r(\alpha) \vee (p(\alpha) \restriction \zeta+1)$ as in Lemma 5.8.

If $\alpha \in w$, then in both cases we get $q \restriction \alpha \vDash q(\alpha) \le^+_\zeta p(\alpha)$. Also, if $G_{\alpha+1}$ selects $x \restriction (\alpha+1)$, then at stage α we used, by induction, Case 2; so then $r(\alpha) \in G(\alpha)$ as $x(\alpha) \subseteq \underset{\sim}{\eta}_\alpha$. □

We can iterate the construction for all elements of $\text{poss}(w,\zeta+1)$, which gives us:

Lemma 5.21. *If p fits (w,ζ) and $\underset{\sim}{\tau}$ is a name for an ordinal, then there is a $q \le^+_{w,\zeta} p$ which $(w,\zeta+1)$-decides $\underset{\sim}{\tau}$.*

Proof. We enumerate $\text{poss}(p,w,\zeta+1)$ as $(x_i)_{i\in\delta}$. We start with $p_0 := p$. Inductively we construct p_ℓ: If at step ℓ, if x_ℓ is not in $\text{poss}(p_\ell,w,\zeta+1)$ any more, then we set $p_{\ell+1} := p_\ell$. Otherwise, pick an $r \le p_\ell$ that decides $\underset{\sim}{\tau}$ to be some τ^{x_ℓ} and extends x_ℓ. Then apply 5.20 to get $p_{\ell+1} \le^+_{w,\zeta} p_\ell$ which forces that $x_\ell \lhd G$ implies $\underset{\sim}{\tau} = \tau^{x_\ell}$. At limits use Lemma 5.18. □

For the proof of Lemma 5.23 we will need a variant where the "height" ζ is not the same for all elements of w, more specifically:

Preliminary Lemma 5.22. *Assume that p fits (w,ζ) and $p \restriction \alpha^* \Vdash \zeta^* \in C^{p(\alpha^*)}$, and that $\underset{\sim}{\tau}$ is a name for an ordinal. Then there is a $q \le^+_{w,\zeta} p$ such that $q \restriction \alpha^* \Vdash q(\alpha^*) \le^+_{\zeta^*} p(\alpha^*)$ and there is a (ground model) set A of size $<\lambda$ such that $q \Vdash \underset{\sim}{\tau} \in A$.*

Proof. This is just a notational variation of the previous proof. For notational simplicity we assume $\alpha^* \notin w$.

First we have to modify 5.20: A candidate is a pair (x,a) where $x \in \text{POSS}(w,\zeta)$ and $a^* \in \text{POSS}^Q(\zeta^*)$. Assume that (x,a) is a candidate, that $p \in P$ fits (w,ζ) and that $p \restriction \alpha^* \Vdash \zeta^* \in C^{p(\alpha^*)}$, and assume that $r \le p$ extends (x,a), i.e., $r \Vdash (x \lhd G \,\&\, a^* \subseteq \underset{\sim}{\eta}_{\alpha^*})$. Then there is a q such that

$$(*) \qquad q \le^+_{w,\zeta} p, \quad q \restriction \alpha^* \Vdash q(\alpha^*) \le^+_{\zeta^*} p(\alpha^*), \text{ and } \quad q \Vdash \big((x \lhd G \,\&\, a^* \subseteq \underset{\sim}{\eta}_{\alpha^*}) \to r \in G\big).$$

The same proof works, with the obvious modifications:

When defining $q(\alpha)$, we inductively show:

- $q \restriction \alpha \le^+_{w\cap\alpha,\zeta} p \restriction \alpha$ and if $\alpha > \alpha^*$ then $q \restriction \alpha^* \Vdash q(\alpha^*) \le^+_{\zeta^*} p(\alpha^*)$,
- $q \restriction \alpha \Vdash \big((x \restriction \alpha \lhd G_\alpha \,\&\, a^* \subseteq \underset{\sim}{\eta}_{\alpha^*}) \to r \restriction \alpha \in G_\alpha\big)$, unless $\alpha \le \alpha^*$ in which case we omit the clause about α^*.

Again, in the P_α-extension we have:
Case 1: $r \restriction \alpha \notin G_\alpha$. Set $q(\alpha) := p(\alpha)$.
Case 2: $r \restriction \alpha \in G_\alpha$. Then $r(\alpha) \le p(\alpha)$. If $\alpha \notin w \cup \{\alpha^*\}$, we set $q(\alpha) := r(\alpha)$; otherwise we set $q(\alpha)$ to be $r(\alpha) \vee (p(\alpha) \restriction \zeta+1)$ as in Lemma 5.8.

Then we can show $(*)$ as before.

We then enumerate all candidates (there are $<\lambda$ many) as (x_ℓ, a_ℓ), and at step ℓ, if (x_ℓ,a_ℓ) is compatible with p_ℓ, use $(*)$ to decide $\underset{\sim}{\tau}$ to be some $\underset{\sim}{\tau}^\ell$. □

We will now show that P is λ^λ-bounding and proper. We first give two preliminary lemmas that assume this is already the case for all $P_{\beta'}$ with $\beta' < \beta$.

Preliminary Lemma 5.23. *Let $\beta \le \mu$, and assume that $P_{\beta'}$ is λ^λ-bounding for all $\beta' < \beta$.*

Assume $p \in P_\beta$ fits (w, ζ), $\tilde{C} \subseteq \lambda$ is club, and $\alpha^* < \beta$.

Then there is a $q \le^+_{w,\zeta} p$ and a $\xi \in \tilde{C}$ such that q fits $(w \cup \{\alpha^*\}, \xi)$.

If additionally $\alpha^* \in \mathrm{dom}(p)$ and $p \restriction \alpha^* \Vdash \zeta^* \in C^{p(\alpha^*)}$ for some $\zeta^* \in \lambda$, then we can additionally get $q \restriction \alpha^* \Vdash q(\alpha^*) \le^+_{\zeta^*} p(\alpha^*)$.

Proof. For notational simplicity assume $\alpha^* \notin w$ and $\min(\tilde{C}) > \max(\zeta, \zeta^*)$. By induction on $\alpha \le \beta$ we show that the result holds for all w, α^* with $w \cup \{\alpha^*\} \subseteq \alpha$.

Successor case $\alpha + 1$: Set $w_0 := w \cap \alpha$.

By our assumption P_α is λ^λ-bounding, so every club-set in the P_α-extension contains a ground-model club (see Fact 5.11). In particular, $C^{p(\alpha)}$ contains some ground-model C^*. By Lemma 5.21 (or 5.22, if $\alpha^* < \alpha$) there is a $p' \le^+_{w_0,\zeta} p \restriction \alpha$ (also dealing with α^*, if $\alpha^* < \alpha$) leaving only $<\!\lambda$ many possibilities for C^*. So we can intersect them all, resulting in C'. Set $C'' := C' \cap \tilde{C}$. Apply the induction hypothesis in P_α to get $q' \le^+_{w_0,\zeta} p'$ and ξ in C'' such that q' fits (w_0, ξ) (and also $(\{\alpha^*\}, \xi)$, if $\alpha^* < \alpha$). Set $q := q' \cup \{(\alpha, p(\alpha))\}$, so trivially $q \le^+_{w,\zeta} p$ (and, if $\alpha = \alpha^*$, then $q \restriction \alpha \Vdash q(\alpha) \le^+_{\zeta^*} p(\alpha)$), and q fits $(w \cup \{\alpha\}, \xi)$.

Limit case: If w is bounded in α there is nothing to do. So assume w is cofinal.

Set $\alpha_0 := \min(w \setminus \alpha^*)$ and $w_0 := (w \cap \alpha_0) \cup \{\alpha^*\}$. Use the induction hypothesis in P_{α_0} using $(p \restriction \alpha_0, w_0, \zeta, \alpha^*, \zeta^*)$ as $(p, w, \zeta, \alpha^*, \zeta^*)$. This gives us some $p'_0 \le^+_{w \cap \alpha_0, \zeta} p \restriction \alpha_0$ fitting (w_0, ζ_0) and dealing with α^*, for some $\zeta_0 \in \tilde{C}$. Set $p_0 := p' \wedge p$.

Enumerate $w \setminus w_0$ increasingly as $(\alpha_i)_{i<\delta}$, and set $w_j := w_0 \cup \{\alpha_i : \ i < j\}$ for $j \le \delta$.

We will construct p'_i in P_{α_i} and $(\zeta_i)_{i<\delta}$ a strictly increasing sequence in $\tilde{C}$, and we set $p_j := p'_j \wedge p$ and will get: p_ℓ fits (w_ℓ, ζ_ℓ),and $p_\ell \le^+_{w_i,\zeta_i} p_i$ for all $i < \ell \le j$.

For successors $\ell = i + 1$, we use the induction hypothesis in $P_{\alpha_{i+1}}$, using $(p_i \restriction \alpha_{i+1}, w_i, \zeta_i, \alpha_i, \zeta)$ as $(p, w, \zeta, \alpha^*, \zeta^*)$. This gives us $p'_{i+1} \le^+_{w_i,\zeta_i} p_i \restriction \alpha_{i+1}$ and some $\zeta_{i+1} > \zeta_i$ in $\tilde{C}$ such that p_{i+1} fits (w_{i+1}, ζ_{i+1}) and $p_{i+1} \restriction \alpha_i \Vdash p_{i+1}(\alpha_i) \le^+_\zeta p_i(\alpha_i)$.

For j limit, we set $\zeta_j := \sup_{i<j} \zeta_i$ (which is in $\tilde{C}$), and let p_j be a limit of the $(p_i)_{i<j}$. I.e., $\mathrm{dom}(p_j) = \bigcup_{i<j} \mathrm{dom}(p_i)$, and for $\beta \in \mathrm{dom}(p_j)$ let $p_j(\beta)$ be as follows: If $\beta \notin w$, fix some condition $p_j(\beta)$ stronger than all $p_i(\beta)$. Otherwise, there is a minimal $i_0 < j$ such that $\beta \in w_{i_0}$, and $p_\ell(\beta) <^+_{\zeta_i} p_i(\beta)$ for all $i_0 \le i < \ell < j$. In that case let $p_j(\beta)$ be the (canonical) limit of the $(p_i(\beta))_{i_0 \le i < j}$, and note that $\zeta_j \in C^{p_j(\beta)}$. ☐

Preliminary Lemma 5.24. *Let $\beta \le \mu$, and assume that $P_{\beta'}$ is λ^λ-bounding for all $\beta' < \beta$.*

Assume that $p \in P_\beta$ fits (w, ζ), and $\underset{\sim}{\sigma}$ is a P_β-name for a λ-sequence of ordinals. Then there is a $q \le^+_{w,\zeta} p$ continuously reading $\underset{\sim}{\sigma}$.

Proof. Set $p_0 := p$, $\zeta_0 := \zeta$, $w_0 := w$. We construct by induction on $i < \lambda$ p'_i, p_i, ζ_i, α_i and w_i as follows:

- Given p_j, w_j, and ζ_j, pick $\alpha_j \in \mathrm{dom}(p_j) \setminus w_j$ by bookkeeping (so that in the end the domains of all conditions will be covered).
- Successor $j = i + 1$: Set $w_{i+1} := w_i \cup \{\alpha_i\}$. Find $p'_{i+1} \le^+_{w_i,\zeta_i} p_i$ and $\zeta_{i+1} > \zeta_i$ such that p'_{i+1} fits (w_{i+1}, ζ_{i+1}) (using the previous preliminary lemma).
- Limit j: Let p'_j be the canonical limit of the $(p_i)_{i<j}$, $\zeta_j := \sup_{i<j}(\zeta_i)$, and $w_j := \bigcup_{i<j} w_i$. Note that p'_j fits (w_j, ζ_j).
- In any case, given p'_j we pick some $p_j \le^+_{w_j,\zeta_j} p'_j$ which $(w_j, \zeta_j + 1)$-decides $\underset{\sim}{\sigma}(\zeta_j)$.

Then the limit q of the p_i continuously reads $\underset{\sim}{\sigma}$. ☐

Lemma 5.25. *P has continuous reading (and in particular is λ^λ-bounding).*

Proof. Assume by induction that $P_{\beta'}$ is λ^λ-bounding for all $\beta < \beta'$. Then the previous lemma gives us that P_β has continuous reading of names, and thus is λ^λ-bounding. ☐

The same construction shows λ-properness:

Definition 5.26. Let $\chi \gg \mu$ be sufficiently large and regular. An "elementary model" is an $M \preceq H(\chi)$ of size λ which is $<\lambda$-closed and contains λ and μ (and thus P).

Lemma 5.27. *If M is an elementary model containing $p \in P$, then there is a $q \leq p$ which is strongly M-generic in the following sense: For each P-name $\underset{\sim}{\tau}$ in M for an ordinal, q (w, ζ)-decides $\underset{\sim}{\tau}$ via a decision function in M (so in particular $q \Vdash \underset{\sim}{\tau} \in M$).*

(The decision function being in M is equivalent to $w \subseteq M$, as M is $<\lambda$ closed.)

Proof. Let $\underset{\sim}{\sigma}$ be a sequence of all P-names for ordinals that are in M. Starting with $p_0 \in M$, perform the successor step of the previous construction within M; as M is closed the limits at steps $<\lambda$ are in M as well. Then the λ-limit is M-generic. □

5.5. **Canonical conditions.** We will use conditions that "continuously read themselves."

Definition 5.28. $p \in P$ is (w, ζ)-canonical if p fits (w, ζ) and $p(\alpha) \restriction (\zeta+1)$ is $(w \cap \alpha, \zeta+1)$-decided by $p \restriction \alpha$ for all $\alpha \in w$.

Facts 5.29. Let p be canonical for (w, ζ).

(1) If $q \leq^+_{w,\zeta} p$, then q is canonical for (w, ζ) and $\mathrm{poss}(p, w, \zeta+1) = \mathrm{poss}(q, w, \zeta+1)$
(2) Let $x \in \mathrm{poss}(p, w, \zeta+1)$. There is a naturally defined $p \wedge x \leq p$ such that $p \Vdash (p \wedge x \in G \leftrightarrow x \triangleleft G)$. $\{p \wedge x : x \in \mathrm{poss}(p, w, \zeta+1)\}$ is a maximal antichain below p.
(3) Let $x \in \mathrm{poss}(p, w, \zeta+1)$. In an intermediate P_α-extension $V[G_\alpha]$ with $x \restriction \alpha \triangleleft G_\alpha$ the rest of x, i.e., $x \restriction [\alpha, \mu]$, is compatible with p/G_α in the quotient forcing.
Or equivalently: If $r_0 \leq p \restriction \alpha$ in P_α extends $x \restriction \alpha$, then there is an $r \leq r_0$ extending x.

Definition 5.30. Assume $p \in P$, and $\underset{\sim}{\sigma}$ is a P-name for a λ-sequence of ordinals. Let $E \subseteq \lambda$ be a club-set and $\bar{w} = (w_\zeta)_{\zeta \in E}$ an increasing sequence in $[\mu]^{<\lambda}$.

p canonically reads $\underset{\sim}{\sigma}$ as witnessed by $\bar{w}$ if the following holds:

- $\mathrm{dom}(p) = \bigcup_{\zeta \in E} w_\zeta$.
- p is (w_ζ, ζ)-canonical for all $\zeta \in E$.
- $p \restriction \alpha \Vdash C^{p(\alpha)} = E \setminus (\zeta'_\alpha)$ for some (ground model) ζ'_α.
- $\underset{\sim}{\sigma} \restriction I^*(\leq\zeta+1)$ is $(w_\zeta, \zeta+1)$-decided by p for all $\zeta \in E$.

If σ is the constant 0 sequence (or any sequence in V), we just say "p is canonical" (as witnessed by $\bar{w}$).

Lemma 5.31. *For p, $\underset{\sim}{\sigma}$ as above, there is a $q \leq p$ canonically reading $\underset{\sim}{\sigma}$.*

If $p \in P_\alpha$ and $\underset{\sim}{\sigma}$ is a P_α-name for some $\alpha < \mu$, then $q \in P_\alpha$.

Proof. We just have to slightly modify the proof of Lemma 5.24.

We will construct p_j, ξ_j and α_j by induction on $j \in \lambda$, setting $w_j := \{\alpha_i : i < j\}$, such that for $0 < j < k$ the following holds:

- $p_k \leq^+_{w_j, \xi_j} p_j$.
- p_j is (w_j, ξ_j)-canonical.
- p_j (w_j, ξ_j+1)-decides $\underset{\sim}{\sigma} \restriction I^*(\leq\xi_j+1)$.
- In p_k, for $\alpha_j \in w_k$, $\{\zeta_i : j < i < k\}$ is (forced to be) an initial segment of $C^{p_k(\alpha_j)}$.
- The α_j are chosen (by some book-keeping) so that $\{\alpha_i : i \in \lambda\} = \bigcup_{i \in \lambda} \mathrm{dom}(p_i)$.

Then the limit of the p_j is as required, with $E = \{\xi_i : i \in \lambda\}$ and, for $\zeta = \xi_j$ in E, we use w_j as w_ζ.

Set $p_0 \leq p$ such that $|\mathrm{dom}(p_0)| = \lambda$, and set $\xi_0 := 0$. Assume we already have p_i, α_i for $i < j$ (so we also have w_j).

- For j limit, let s be a limit of $(p_i)_{i<j}$, and set $\xi_j := \sup_{i<j} \xi_i$. Note that s fits (w_j, ξ).
- Successor case $j = i+1$: Find $s_0 \leq^+_{w_i, \xi_i} p_i$ and $\xi_j > \xi_i$ such that s fits (w_j, ξ_j). (As in Lemma 5.23. Recall that $w_j = w_i \cup \{\alpha_i\}$.)
  Strengthen s_0 to $s \leq^+_{w_i, \xi_i}$ so that:
  - s still fits (w_j, ξ_j),
  - the trunk at α_i has length ξ_j, i.e., $s \restriction \alpha_i \Vdash \min(C^{s(\alpha_i)}) = \xi_j$),

 - for $\alpha_{i'}$, $i' < i$, there are no elements in $C^{s(\alpha_{i'})}$ between ξ_i and ξ_j.
- Construct $s^* \restriction \alpha$ by recursion on $\alpha \in w_j$, such that $s^* \restriction \alpha \leq^+_{w_j\cap\alpha,\xi_j} s \restriction \alpha$ and $s^* \restriction \alpha$ $(w_j \cap \alpha, \xi_j + 1)$-decides $s(\alpha) \restriction (\xi_j + 1)$ (which is the same as $s^*(\alpha) \restriction (\xi_j + 1)$). This gives $s^* \leq^+_{w_j,\xi_j} s$.
- Find $p_j \leq^+_{w_j,\xi_j} s^*$ which $(w_j, \xi_j + 1)$ decides $\underset{\sim}{\sigma} \restriction I^*(\leq\xi + 1)$.
- Choose $\alpha_j \in \mathrm{dom}(p_j) \setminus w_j$ by bookkeeping. □

Facts 5.32. (1) If a P_β-name $\underset{\sim}{x} \subseteq \lambda$ is continuously read (by some P_β-condition p), and $\mathrm{cf}(\beta) > \lambda$, then there is an $\alpha < \beta$ such that: $p \in P_\alpha$, and $\underset{\sim}{x}$ is already a P_α-name (formally: there is a P_α-name $\underset{\sim}{y}$ such that $p \Vdash \underset{\sim}{x} = \underset{\sim}{y}$).
(2) There are at most $|\alpha|^\lambda \leq \lambda^+$ many pairs[10] $(p, \underset{\sim}{x})$ such that p canonically reads $\underset{\sim}{x}$ in P_α.

5.6. **Δ systems.** In this section we define Δ-systems and show that such systems exist, which we will in the indirect proofs of Lemmas 5.39 and 5.54.

In Section 5.10 we will then fix a specific Δ-system for the rest of the paper.

From now on, we assume that p_* forces

$$\underset{\sim}{\pi} : \mathcal{P}(\lambda) \to \mathcal{P}(\lambda) \text{ represents the automorphism } \underset{\sim}{\phi} : P^\lambda_\lambda \to P^\lambda_\lambda, \tag{5.33}$$

and we set, for $\beta \in \mu$,

$$\underset{\sim}{a}_\beta := \underset{\sim}{\pi}(\underset{\sim}{\eta}_\beta),$$

where, as usual, we identify $\underset{\sim}{\eta}_\beta \in 2^\lambda$ with $\underset{\sim}{\eta}_\beta^{-1}\{1\} \subseteq \lambda$.

Note that, other than $\underset{\sim}{\eta}_\beta$, $\underset{\sim}{a}_\beta$ is a priori not a $P_{\beta+1}$-name (but see Section 5.9).

We also fix a P-name for a representation of the inverse automorphism $\underset{\sim}{\phi}^{-1}$. Abusing notation, we call it $\underset{\sim}{\pi}^{-1}$.

With $S^\mu_{\lambda^+}$ we denote the stationary subset of μ consisting of ordinals with cofinality λ^+.

Definition 5.34. Let $S \subseteq S^\mu_{\lambda^+}$ be stationary, $\chi \gg \mu$ sufficiently large and regular, and $z \in H(\chi)$. "An elementary S-system" (using parameter z) is a sequence $(M_\beta, p_\beta)_{\beta\in S}$ such that, for each $\beta \in S$, M_β is an elementary model (as in Definition 5.26) and contains z, β, p_*, $\underset{\sim}{\phi}$, $\underset{\sim}{\pi}$ and $\underset{\sim}{\pi}^{-1}$, and $p_\beta \in P \cap M_\beta$ canonically reads $\underset{\sim}{a}_\beta$ witnessed by some $(w^{p_\beta}_\zeta)_{\zeta\in E^{p_\beta}}$, which $E^{p_\beta} \subseteq \lambda$ club (cf. Def. 5.30).

By a simple Δ-system argument we can make an S-system homogeneous:

Definition 5.35. $(M_\beta, p_\beta)_{\beta\in S}$ forms a "Δ-system", if $\bar{M}, \bar{p}$ is an elementary S-system with parameter z, and is homogeneous in the following sense: For β and $\beta_1 < \beta_2$ in S, we get:
(1) $M_{\beta_1} \cap M_{\beta_2} \cap \mu$ is constant. We call this set the "heart" and, abusing notation, denote it with Δ. Obviously $\Delta \supseteq \lambda$, $\Delta \supseteq \mathrm{dom}(p_*)$, $\lambda^+ \in \Delta$, etc.
(2) $M_\beta \cap \beta = \Delta$. So in particular β is the minimal element of M_β above Δ. All the non-heart elements of M_{β_2} are above all elements of M_{β_1}. I.e., $\sup(M_{\beta_1} \cap \mu) < \beta_2$.
(3) There is an $\in$-isomorphism $h^*_{\beta_1,\beta_2} : M_{\beta_1} \to M_{\beta_2}$, mapping β_1 to β_2, p_{β_1} to p_{β_2}, $\underset{\sim}{a}_{\beta_1}$ to $\underset{\sim}{a}_{\beta_2}$ and fixing $\lambda, \mu, \underset{\sim}{\phi}, \underset{\sim}{\pi}$ as well as each α in Δ.

Note that this implies that the continuous reading of $\underset{\sim}{a}_\beta$ works the same way for all β. In particular the E^{p_β} are that same E for all β; and if F^β_ζ is the function mapping $\mathrm{POSS}(w^{p_\beta}_\zeta, \zeta+1)$ to the value of $\underset{\sim}{a}_\beta \restriction I^*(\leq\zeta + 1)$ (for $\zeta \in E$), then $h^*_{\beta_1,\beta_2}(F^{\beta_1}_\zeta) = F^{\beta_2}_\zeta$ and in particular $h^*_{\beta_1,\beta_2}(w^{p_{\beta_1}}_\zeta) = w^{p_{\beta_2}}_\zeta$; i.e., they are the same apart from shifting coordinates above Δ.

Lemma 5.36. *Assume $S \subseteq S^\mu_{\lambda^+}$ is stationary.*
- *For every $z \in H(\chi)$ and $(p'_\beta)_{\beta\in S}$ there are M_β and $p_\beta \leq p'_\beta$ such that $\bar{M}, \bar{p}$ is an S-system with parameter z.*

[10] Depending on the formal definition, we could/should add "modulo equivalence", i.e., there is a $\leq|\alpha|^\lambda$-sized set Z of such pairs such that whenever p canonically reads $\underset{\sim}{y}$ in P_α then there is a $\underset{\sim}{x}$ such that $(p, \underset{\sim}{x}) \in Z$ and $p \Vdash \underset{\sim}{x} = \underset{\sim}{y}$.

- *If $\bar{M},\bar{p}$ is an S-system then there is an $S'\subseteq S$ stationary such that $(M_\beta,p_\beta)_{\beta\in S'}$ is a Δ-system on S'.*

Proof. The first item is trivial, using the fact that everything can be read canonically.

Using $2^\lambda=\lambda^+$, a standard Δ-system argument (or: Fodor's Lemma argument) lets us thin out S to some S^2 so that $(M_\beta\cap\mu)_{\beta\in S^2}$ satisfies (1–3). For $\beta\in S^2$ let $\iota_\beta: M_\beta\cup\{M_\beta\}\to H(\lambda^+)$ be the transitive collapse, and assign to β the tuple of the ι_β-images of the following objects:

- M_β, p_β, $\underset{\sim}{a}_\beta$, μ, $\underset{\sim}{\phi}$, $\underset{\sim}{\pi}$, and E^{p_β}.
- For $\zeta\in E^{p_\beta}$, the object $w^{p_\beta}_\zeta$,
- For $\zeta\in E^{p_\beta}$ and $\gamma\in w^{p_\beta}_\zeta$, the object $F^{p_\beta}_\gamma$.

Again, there are $|H(\lambda^+)|^\lambda<\mu$ many possibilities, so the objects are constant on a stationary $S'\subseteq S^2$.

For $\alpha<\beta$ in S', we define $h^*_{\beta_1,\beta_2}:=\iota^{-1}_{\beta_2}\circ\iota_{\beta_1}$. (Note that $\iota_{\beta_1}(\alpha)=\iota_{\beta_2}(\alpha)$ for $\alpha\in\Delta$.) □

So in particular if we have a Δ-system on S, then $p_\beta\restriction\sup(\Delta)=p_\beta\restriction\beta\in M_\beta$ is the same for all $\beta\in S$, and outside of Δ the domains of the p_β are disjoint for $\beta\in S$. In particular we get:

Fact 5.37. For a Δ-system with domain S, and $A\subseteq S$ of size $\leq\lambda$, the union of the $(p_\beta)_{\beta\in A}$ is a condition in P (and stronger than each p_β).

Whenever $r\in P_\beta\cap M_\beta$ (as is the case for $r=p_\beta\restriction\beta$), we know that $r\in P_\alpha$ for $\alpha\in\Delta$ (as M_β knows that β has cofinality λ^+).

Instead of "$r\in P_\alpha$ for some $\alpha\in\Delta$" we will sometimes just state the weaker but shorter $r\in P_{\sup(\Delta)}$.

Remark. This is an important effect also for some names. Generally, a P_β-name in M_β is of course not a P_α-name for any $\alpha<\beta$ (just take the P_β-generic filter G_β). However, as we will explicitly state in Lemma 5.42, such names for subsets of λ are, modulo some condition, P_α-names for some $\alpha\in\Delta$ and independent of β. In the specific case of the P_β-name $p_\beta(\beta)$ we do not have to increase the condition:

Definition and Lemma 5.38. *$\tilde{p}:=p_\beta(\beta)$ is a $P_{\sup(\Delta)}$-name independent of $\beta\in S$.*

Proof. $p_\beta(\beta)\restriction\zeta+1$ is $(w^{p_\beta}_\zeta,\zeta+1)$-determined for cofinally many $\zeta\in E$, where $w^{p_\beta}_\zeta\in[\beta]^{<\lambda}$ is a subset of M_β. So $w^{p_\beta}_\zeta\subseteq\Delta$, and the isomorphisms between the M_β guarantee that each $w^{p_\beta}_\zeta$ is the same, and that $p_\beta(\beta)\restriction\zeta+1$ is decided the same way. So $\tilde{p}$ is a P_γ-name for $\gamma=\sup(w^{p_\beta}_\zeta)_{\zeta\in E}$. This γ is independent of $\beta\in S$, and is in Δ. So $\tilde{p}$ is actually a P_α-name for some $\alpha\in\Delta$; and certainly a $P_{\sup(\Delta)}$-name. □

For later reference we note:

Lemma 5.39. *For all but non-stationary many β, p_* forces $\underset{\sim}{a}_\beta\notin V_\beta$.*

(Here, V_β denotes the P_β-extension of the ground model.)

Proof. Assume that $p_\beta\leq p_*$ forces that $\underset{\sim}{a}_\beta=\underset{\sim}{x}_\beta$ for a P_β-name $\underset{\sim}{x}_\beta$ for all $\beta\in S^*$ stationary. We can also assume that p_β canonically reads $\underset{\sim}{a}_\alpha$. Pick M_β containing p_β and $S\subseteq S^*$ such that $(M_\beta,p_\beta)_{\beta\in S}$ is a Δ-system, where we can assume (or get from homogeneity) that $h^*_{\beta_0,\beta_1}(\underset{\sim}{x}_{\beta_0})=\underset{\sim}{x}_{\beta_1}$. So the $\underset{\sim}{x}_\beta$ are P_β-names in M_β and therefore $P_{\sup(\Delta)}$-names, and are the same for all β. Choose $\beta_1>\beta_0$ in S. So $p_{\beta_0}\wedge p_{\beta_1}$ force that $\underset{\sim}{a}_{\beta_0}=\underset{\sim}{x}=\underset{\sim}{a}_{\beta_1}$, which contradicts the injectivity of $\underset{\sim}{\phi}$ and the fact that $\underset{\sim}{\eta}_{\beta_0}\neq\underset{\sim}{\eta}_{\beta_1}$. □

5.7. **Preservation of cofinalities, catching canonical names.**

Corollary 5.40. *P is λ^{++}-cc and preserves all cofinalities.*

Proof. Cofinalities $\leq\lambda$ are preserved as P is $<\lambda$-closed.

Cofinality λ^+ is preserved by properness: Assume that it is forced by p that κ has a cofinal λ-sequence $\underset{\sim}{\bar{\alpha}}:=(\underset{\sim}{\alpha}_i)_{i\in\lambda}$. Then there is an elementary model M containing p and $\underset{\sim}{\bar{\alpha}}$. If $q\leq p$ is

M-generic, and G a P-generic filter containing q, then $\underset{\sim}{\alpha}_i[G] \in M$ for all $i < \lambda$, so $M \cap \kappa$ is a cofinal subset of κ of size λ in the ground model.

Cofinality $\geq \lambda^{++}$ is preserved as P has the λ^{++}-cc, which we have shown in a very roundabout way with the fact about Δ-systems: If $(p'_\alpha)_{\alpha\in\mu}$ are arbitrary conditions, then (M_β, p_β) form a Δ-system from some $p_\beta < p'_\beta$ and stationary S, and any two (in fact, $\leq\lambda$ many) p_β are compatible for $\beta \in S$. □

Remark 5.41. This shows that P is (μ, λ)-Knaster, i.e., for every $A \in [P]^\mu$ there is a $B \in [A]^\mu$ which is λ-linked.

The λ^{++}-cc also implies: For every name $\underset{\sim}{x}$ for a subset of λ (or of λ^+) there is a $\beta < \mu$ and a P_β-name $\underset{\sim}{y}$ such that the empty condition forces that $\underset{\sim}{x} = \underset{\sim}{y}$.

Given $\alpha < \mu$, there are $<\!\mu$ many pairs $(p, \underset{\sim}{x})$ where p canonically reads $\underset{\sim}{x} \subseteq \lambda$ in P_α, see Fact 5.32(2). So there is a $g(\alpha) < \mu$ such that for each such $p, \underset{\sim}{x}$, both $\underset{\sim}{\pi}(\underset{\sim}{x})$ and $\underset{\sim}{\pi}^{-1}(\underset{\sim}{x})$ are equivalent (modulo the empty condition) to some $P_{g(\alpha)}$-name. Let $C^* \subseteq \mu$ be the club set with $(\zeta \in C^* \,\&\, \alpha < \zeta) \;\rightarrow\; g(\alpha) < \zeta$.

Given a Δ-system on S we can restrict it to a Δ-system on $S \cap C^*$; so we will assume from now on that each Δ-system we consider satisfies $S \subseteq C^*$.

To summarize:

Lemma 5.42. *(1) If $\beta \in S$, $p \in P_\beta$ and $\underset{\sim}{x}$ a P_β-name for a subset of λ, then there is an $\alpha < \beta$ and a $q \leq p$ canonically reading $\underset{\sim}{x}$, $\underset{\sim}{\pi}(\underset{\sim}{x})$, $\underset{\sim}{\pi}^{-1}(\underset{\sim}{x})$ as P_α-names.*

More explicitly: There is a P_α-name $\underset{\sim}{y}$ which is canonically read by q such that $q \Vdash \underset{\sim}{y} = \underset{\sim}{x}$. (And analogously for $\underset{\sim}{\pi}(\underset{\sim}{x})$ and $\underset{\sim}{\pi}^{-1}(\underset{\sim}{x})$ instead of $\underset{\sim}{x}$.)

(2) If additionally $p \leq p_\beta \restriction \beta$ in P_β and $(p, \underset{\sim}{x}) \in M_\beta$, then we can additionally get: $\underset{\sim}{x}$, $\underset{\sim}{\pi}(\underset{\sim}{x})$ and $\underset{\sim}{\pi}^{-1}(\underset{\sim}{x})$ are P_α-names in M_β independent of $\beta \in S$.

More explicitly: Let $\underset{\sim}{y}$ be as above (for $\underset{\sim}{x}$). Then $\alpha \in \Delta$, q and $\underset{\sim}{y}$ are in M_β, and if $\beta' \in S$ and $h := h^*_{\beta,\beta'}$, then h acts as identity on α, q, and $\underset{\sim}{y}$, and ($M_{\beta'}$ knows that) $q \Vdash \underset{\sim}{y} = h(\underset{\sim}{x})$. (And analogously for $\underset{\sim}{\pi}(\underset{\sim}{x})$ and $\underset{\sim}{\pi}^{-1}(\underset{\sim}{x})$ instead of $\underset{\sim}{x}$.)

Proof. (1): Use Lemma 5.31 to get a $q_1 \in P_\beta$ canonically reading $\underset{\sim}{x}$. And if $\beta \in S$ then $\mathrm{cf}(\beta) = \lambda^+$, so $\mathrm{dom}(p)$ is bounded by some $\alpha' < \beta$ and, by Fact 5.32(1), $q_1 \in P_{\alpha_1}$ for some $\alpha' \leq \alpha_1 < \beta$. As $\beta \in C^*$, $\underset{\sim}{\pi}(\underset{\sim}{x})$ and $\underset{\sim}{\pi}^{-1}(\underset{\sim}{x})$ are P_β-names. So repeat the same argument to get $q \leq q_1$ in P_α canonically reading all three subsets of λ.

(2): Apply (1) inside M_β. As $\alpha \in \beta \cap M_\beta$, we get $\alpha \in \Delta$. As q canonically reads itself as well as $\underset{\sim}{y}$, we know that h does not change q and $\underset{\sim}{y}$. As h is an isomorphism, we know that $h(q) = q$ forces that $h(\underset{\sim}{x}) = h(\underset{\sim}{y}) = \underset{\sim}{y}$. □

5.8. **Majority decisions.** For any (a_1, a_2, a_3) with $a_i \in \{0,1\}$ there is a $b \in \{0,1\}$ such that $b = a_i$ for at least two $i \in \{1,2,3\}$. We write $b = \mathrm{major}_{i=1,2,3}(a_i)$.

Similarly, if f_1, f_2, f_3 are functions $A \to 2$ we write $\mathrm{major}_{i=1,2,3}(f_i)$ for the function $A \to 2$ that maps ℓ to $\mathrm{major}_{i=1,2,3}(f_i(\ell))$.

The following is a central point of the whole construction:

Lemma 5.43. *Let $(M_\alpha, p_\alpha)_{\alpha\in S}$ be a Δ-system. Pick $\beta_0 < \beta_1 < \beta_2 < \beta_3$ in S.*

(1) p_* forces: If $\underset{\sim}{\eta}_{\beta_0} =^* \mathrm{major}_{i=1,2,3}(\underset{\sim}{\eta}_{\beta_i})$, then $\underset{\sim}{a}_{\beta_0} =^* \mathrm{major}_{i=1,2,3}(\underset{\sim}{a}_{\beta_i})$.

(2) Let $s = \bigwedge_{i<4} p_{\beta_i}$. Recall that $s(\beta_i)$ is the same $P_{\sup(\Delta)}$-name called $\tilde{p}$ for all i. We can strengthen s by strengthening, for $i = 1,2,3$, the condition $s(\beta_i) = \tilde{p}$ to some P_{β_0+1}-names $r_i \leq \tilde{p}$ (without changing $C^{\tilde{p}}$) such that the resulting condition forces $\underset{\sim}{\eta}_{\beta_0} = \mathrm{major}_{i=1,2,3}(\underset{\sim}{\eta}_{\beta_i})$.

(We do not have to strengthen $s(\beta_0)$ for this, i.e., we can use $r_0 := \tilde{p}$.)

We describe this by "$(r_i)_{i<4}$ honors majority".

Recall that $\nu_1 =^* \nu_2$ denotes that $\nu_1(\ell) = \nu_2(\ell)$ for all but $<\!\lambda$ many $\ell \in \lambda$.

Proof. (1) Identifying 2^λ with $P(\lambda)$, we have $\text{major}_{i=1,2,3} f_i = (f_1 \cap f_2) \cup (f_2 \cap f_3) \cup (f_1 \cap f_3)$ for any tuple $(f_i)_{i=1,2,3}$. As $\underset{\sim}{\pi}$ represents an automorphism, we get $\underset{\sim}{\pi}(\text{major}_{i=1,2,3}(f_i)) =^* \text{major}_{i=1,2,3}(\underset{\sim}{\pi}(f_i))$. Apply this to $f_i := \underset{\sim}{\eta}_{\beta_i}$.

(2) Work in the P_{β_0+1}-extension. Recall $\tilde{p} := p_{\beta_0}(\beta_0)$. So both $\tilde{p}$ and $\underset{\sim}{\eta}_{\beta_0}$ are already determined, and $\underset{\sim}{\eta}_{\beta_0}$ extends $\eta^{\tilde{p}}$. Set $r_0 := \tilde{p}$.

Set $s_1 := (0,0)$, $s_2 := (0,1)$, $s_3 := (1,0)$. For $\zeta \in C^{\tilde{p}}$ and $i = 1,2,3$, we define $r_i(\zeta) \supseteq \tilde{p}(\zeta)$ as follows:

$$\text{Extend } s^{\tilde{p}}_\zeta \text{ by } s_i, \text{i.e., } s^{r_i}_\zeta := (s^{\tilde{p}}_\zeta)^\frown s_i; \text{ and set } r_i(\zeta)(\ell) := \underset{\sim}{\eta}_\beta(\ell) \text{ for } \ell \in [s^{\tilde{p}}_\zeta] \setminus [s^{r_i}_\zeta]. \tag{5.44}$$

So η^{r_i} agrees on its domain with $\underset{\sim}{\eta}_{\beta_0}$, and each $\ell \in \lambda$ is in $\text{dom}(\eta^{r_i})$ for at least two $i \in \{1,2,3\}$. Accordingly, an extension by a generic filter G with $r_i \in G(\beta_i)$ for all $i < 4$ will satisfy $\underset{\sim}{\eta}_{\beta_0} = \text{major}_{i=1,2,3}(\underset{\sim}{\eta}_{\beta_i})$. (We do not even have to assume that any $p_\beta \in G$.) □

Remark 5.45. Let p'_{β_1} be the condition where we strengthen $p_{\beta_1}(\beta_1)$ to r_1. Note that p'_{β_1} is not in M_{β_1}, as $\beta_0 \notin M_{\beta_1}$ and r_1 is defined using $\underset{\sim}{\eta}_{\beta_0}$. Similarly (basically the same): $r_1[G_{\beta_1}] \notin M_{\beta_1}[G_{\beta_1}]$, even if we assume that G_{β_1} is M_{β_1}-generic. But generally we will not be interested in M_β-generic conditions or extensions (we needed generic conditions only in Lemma 5.27, which in turn is needed for Corollary 5.40). And while usually most conditions we consider can be constructed within (and therefore will be elements of) some M_β, this is generally not required (an example are the s_i's in the following Lemma).

The same proof works if we do not start with the p_β but with any stronger conditions, as long as they still "cohere" in the way that the p_{β_i} cohere:

Lemma 5.46. *Let $(M_\alpha, p_\alpha)_{\alpha \in S}$ be a Δ-system, $\beta_0 < \beta_1 < \beta_2 < \beta_3$ in S, and $s_i \leq p_{\beta_i}$ for $i = 0,1,2,3$ such that:*

- $\text{dom}(s_i) \subseteq M_{\beta_i}$
- *$s^* := s_i \restriction \beta_i$ is the same for all i,*
- *s^* forces that the $s_i(\beta_i)$ are the same for all i.*
  (In the usual sense: The $s_i(\beta_i)$ are continuously read from generics below β_0 in the same way for each $i < 4$.)

Then there is condition stronger than all s_i forcing that $\underset{\sim}{\eta}_{\beta_0} = \text{major}_{i=1,2,3}(\underset{\sim}{\eta}_{\beta_i})$ and thus $\underset{\sim}{a}_{\beta_0} =^* \text{major}_{i=1,2,3}(\underset{\sim}{a}_{\beta_i})$.

5.9. **$\underset{\sim}{a}_\beta$ is in the $\beta+1$-extension.** We now show that $\underset{\sim}{a}_\beta$ can be assumed to be a P_β-name.

The following definitions, in particular everything concerning the notion of coherence, is used only in this section. In the rest of the paper, we will use from this section only Lemma 5.54, i.e., the fact that $\underset{\sim}{a}_\beta \in V_{\beta+1}$.

Remark. Why do we introduce this (rather annoying) notion of coherence? Well, we would like to simultaneously construct something like $s_i \leq p_{\beta_i}$ where each s_i ends up in M_{β_i}. We cannot directly do this in M_{β_0}, as M_{β_0} does not know about, e.g., β_1. So instead, we construct four different $s'_i \leq p_{\beta_0}$ in M_{β_0} in such a way (a "coherent" way) and use $s_i := h^*_{\beta_0,\beta_i}(s'_i)$.

Let us for now (until Lemma 5.54) fix an arbitrary Δ-system $(M_\beta, p_\beta)_{\beta \in S}$ as well as $\beta_0 < \beta_1 < \beta_2 < \beta_3$ in S. For notational convenience, set

$$\beta := \beta_0.$$

Definition 5.47.
- $\bar{q} = (q_i)_{i<4}$ in M_β is called coherent, if each q_i is stronger than p_β and $q_i \restriction (\beta+1)$ is the same for all $i < 4$.
- If $\bar{q}$ is coherent, then $\bigwedge_{i<4} h^*_{\beta,\beta_i}(q_i)$ is a valid condition in P, and we call it q^*.
  I.e., q^* is the union of the copies of q_i in M_{β_i}; and the copy for q_0 is just q_0.
  $r \in P$ is called coherent, if $r = q^*$ for some coherent $\bar{q} \in M_\beta$.

Facts.
- The p_{β_i} are coherent, more correctly:
  The condition $\bigwedge_{i\in 4} p_{\beta_i}$ is coherent; equivalently: The tuple $\big(h^{*-1}_{\beta,\beta_i}(p_{\beta_i})\big)_{i<4}$ is coherent.
- Any coherent r is stronger than $\bigwedge_{i<4} p_{\beta_i}$.
- If $\bar q$ is coherent, $r_i \leq q_i$ in M_β for $i<4$, and $r_i \restriction \beta_i$ is the same for all $i<4$, then $\bigwedge_{i<4} h^*_{\beta,\beta_i}(r_i)$ is (a valid condition and) compatible with q^*.
- $r\in P$ is coherent iff: $\mathrm{dom}(r) \subseteq \bigcup_{i<4} M_{\beta_i}$, $r\restriction(\mu\cap M_{\beta_i}) \in M_{\beta_i}$ is stronger than p_{β_i}, and each $r(\beta_i)$ is forced to be the same condition.
  In that case, $r = q^*$ for $q_i := h^{*-1}_{\beta,\beta_i}(r_i)$ and $r_i := r\restriction(\mu\cap M_{\beta_i})$

Lemma 5.48. *If r is coherent, then it can be strengthened*[11] *to force*[12] $\underset{\sim}{a}_{\beta_0} = \mathrm{major}_{i=1,2,3}\, \underset{\sim}{a}_{\beta_i}$.

Proof. This follows from Lemma 5.46, using $s_i := r\restriction(\mu\cap M_{\beta_i})$. □

Definition 5.49.
- $\bar w = (w_i)_{i<4}$ is coherent, if $w_i \in [\mu]^{<\lambda}$ is in M_β and $w_i \cap (\beta+1)$ is independent of i.
  In the following we always assume that $\bar q$ and $\bar w$ are coherent.
- $\bar q$ fits $(\bar w,\zeta)$, if each q_i fits (w_i,ζ).
- $\bar q$ is $(\bar w,\zeta)$-canonical, if each q_i is (w_i,ζ)-canonical.
- $\bar r \leq^+_{\bar w,\zeta} \bar q$ means: $\bar r$ is coherent, and $r_i \leq^+_{w_i,\zeta} q_i$ for all $i<4$.
- $\bar x = (x_i)_{i<4}$ is defined to be in $\mathrm{poss}(\bar q,\bar w,\zeta)$ if $x_i \in \mathrm{poss}(q_i,w_i,\zeta)$ and $x_i\restriction\beta$ is independent of i. Such a $\bar x$ will be called coherent possibility.
  (Note that the $x_i(\beta)$ in a coherent possibility can be different for different $i<4$. Also note that such a $\bar x$ is automatically in M_β, which is $<\lambda$-closed.)

Note that if $\bar r \leq^+_{\bar w,\zeta} \bar q$ and $\bar q$ is $(\bar w,\zeta)$-canonical, then $\bar r$ and $\bar q$ have the same coherent $(\bar w,\zeta+1)$-possibilities, see Fact 5.29(1).

Several of the previous constructions result in coherent 4-tuples when applied to coherent 4-tuples. In particular:

Lemma 5.50.
(1) *Assume $(\bar q^j)_{j\in\delta}$ is a sequence of coherent 4-tuples such that, for each $i<4$, the i-part $(q^j_i)_{j\in\delta}$ satisfies the assumptions of Lemma 5.18.*
Then for each i, the lemma (in M_β) gives us a limit r, which we call q^δ_i.
We can choose the q^δ_i so that they form a coherent 4-tuple.
(2) *The same applies to Lemma 5.19. I.e., we can get a coherent fusion limit from a λ-sequence of coherent tuples.*
(3) *Assume $\bar p$ fits $(\bar w,\zeta)$, and $\alpha_i\in\mu$ such that $w'_i := w_i\cup\{\alpha_i\}$ is coherent. Then there is a $\xi>\zeta$ and a $\bar q \leq^+_{\bar w,\zeta} \bar p$ which fits $(\bar w',\xi)$ and is $(\bar w',\xi)$-canonical.*
(4) *Assume $\bar q$ is coherent and (for simplicity) $(\bar w,\zeta)$-canonical with $\beta\in w_i$ (which is independent of $i<4$), and $\underset{\sim}{\tau}_i$ are names of ordinals. Then there is an $\bar r \leq^+_{\bar w,\zeta} \bar q$ such that $\underset{\sim}{\bar\tau}$ is $(\bar w,\zeta+1)$-decided by $\bar r$.*
By this we mean that $\underset{\sim}{\tau}_i$ is $(w_i,\zeta+1)$-decided by r_i for all $i<4$.

Proof. For the first items, we just have to look at the proofs of the according lemmas (For (3) this is 5.23 and 5.24) and note that coherent input gives us coherent output. In the following we will prove (4). We work in M_β.

Enumerate all coherent possibilities as $(\bar x_k)_{k\in K}$. Set $\bar r^0 := \bar q$. We now construct $\bar r^{k+1}$ from $\bar r := \bar r^k$ where we assume $\bar r^k \leq^+_{\bar w,\zeta} \bar q$.
- Find s_0 stronger than r_0 and extending x_0, deciding $\underset{\sim}{\tau}_0$.
- $s^* := (s_0\restriction\beta)\wedge r_1$ is stronger than r_1, as $\bar r$ is coherent. Strengthen $s^*(\beta) = r_1(\beta) = r_0(\beta)$ to $s_0(\beta)$, but replace the trunk with $x_1(\beta)$. Then $s^*\restriction\beta$ forces that $s^*(\beta)\leq r_1(\beta)$, as $x_1\restriction\beta = x_0\restriction\beta$ and as $x_1(\beta)$ is guaranteed to be possible, because r_1 is canonical. Further strengthen s^* (above β) to extend (the rest of) x_1; and then strengthen the whole condition once more to decide $\underset{\sim}{\tau}_1$. Call the result s_1.

[11] to a condition that will generally not be coherent
[12] Here we write β_0 instead of β to stress the interaction with $\beta_1,\dots,\beta_3$, but recall that $\beta := \beta_0$.

- Do the same for $i = 2$, starting with s_1, resulting in s_2, and then for $i = 3$, starting with s_2, resulting in some s_3.

  So $s_i \leq r_i$ extends x_i and decides $\underaccent{\sim}{\tau}_i$, and $s_3 \restriction \beta \leq s_i \restriction \beta$ and $s_3(\beta)$ is stronger than $s_i(\beta)$ "above $\zeta + 1$".
- We define $r'_i \leq r_i$ as follows: $\mathrm{dom}(r'_i) = (\mathrm{dom}(s_3) \cap \beta) \cup \mathrm{dom}(s_i)$. We define $r'_i(\alpha)$ inductively such that $r'_i \restriction \alpha \leq^+_{w_i \cap \alpha, \zeta} r_i$ forces that $x_i \restriction \alpha \triangleleft G$ implies $s_i \restriction \alpha \in G$.
  - For $\alpha \leq \beta$:
    If $s_3 \restriction \alpha \notin G_\alpha$, set $r'_i(\alpha) = r_i(\alpha)$. Assume otherwise. So $s_3(\alpha)$ is defined and stronger than $r_i(\alpha) = r_3(\alpha)$. If $\alpha \notin w_i$ (which implies $\alpha < \beta$), set $r'_i(\alpha) = s_3(\alpha)$. Otherwise, use $s_3(\alpha) \vee (r_3(\alpha) \restriction \zeta + 1)$, as in Lemma 5.8.
  - For $\alpha > \beta$, we do the same but we use s_i instead of s_3. In more detail:
    If $s_i \restriction \alpha \notin G_\alpha$, set $r'_i(\alpha) = r_i(\alpha)$. Assume otherwise. If $\alpha \notin w_i$, set $r'_i(\alpha) = s_i(\alpha)$. Otherwise, use $s_i(\alpha) \vee (r_i(\alpha) \restriction \zeta + 1)$.

  We can use this $\bar{r}'$ as $\bar{r}^{k+1}$: It is coherent, $\bar{r}' \leq^+_{\bar{w},\zeta} \bar{r}^k$, and r'_i decides $\underaccent{\sim}{\tau}_i$ assuming $x_i \triangleleft G$. $\square$

Coherent tuples $\bar{q}$ naturally define a P-condition q^*. However, we have to assume that $\bar{q}$ is canonical to guarantee that coherent $\bar{q}$ possibilities correspond to q^*-possibilities:

Lemma 5.51. *Assume $\bar{q}$ and $\bar{w}$ coherent. We set $w^* := \bigcup_{i<4} h^*_{\beta,\beta_i}(w_i)$. Let $\bar{x}$ be in $\mathrm{poss}(\bar{q}, \bar{w}, \zeta+1)$.*

(1) $\bar{q}$ fits $(\bar{w}, \zeta)$ iff q^* fits (w^*, ζ).
(2) $\bar{r} \leq^+_{\bar{w},\zeta} \bar{q}$ iff $r^* \leq^+_{w^*,\zeta} q^*$.
(3) Assume $\bar{q}$ fits $(\bar{w}, \zeta)$. Then $\bar{q}$ is $(\bar{w}, \zeta)$-canonical iff q^* is (w^*, ζ)-canonical.
(4) Assume that $\bar{q}$ is $(\bar{w}, \zeta)$-canonical. Let x^* be the union of the $h^*_{\beta,\beta_i}(x_i)$. Then $x^* \in \mathrm{poss}(q^*, w^*, \zeta + 1)$; and every element of $\mathrm{poss}(q^*, w^*, \zeta + 1)$ is such an x^* for some $\bar{x} \in \mathrm{poss}(\bar{q}, \bar{w}, \zeta + 1)$.
(5) Assume that $\bar{q}$ is $(\bar{w}, \zeta)$-canonical. Then $\bar{q}$ $(\bar{w}, \zeta+1)$-decides $(\underaccent{\sim}{\tau}_i)_{i<4}$ iff q^* $(w^*, \zeta+1)$-decides all $h^*_{\beta,\beta_i}(\underaccent{\sim}{\tau}_i)$.

Proof. Assume $\alpha \in w_i$. Set $\alpha' := h^*_{\beta,\beta_i}(\alpha) \in w^*$ and $q' := h^*_{\beta,\beta_i}(q_i)$.

(1) Assume q_i, α satisfy $q_i \restriction \alpha \Vdash \zeta \in C^{q_i(\alpha)}$. By absoluteness they satisfy it in M_β, so the h^*_{β,β_i}-images q', α' satisfy it in M_{β_i}, which again is absolute; and $q^* \restriction \alpha' \leq q' \restriction \alpha'$ forces that $q^*(\alpha') = q'(\alpha')$. For the other direction, assume (in M_β) some $s \leq q_i \restriction \alpha$ forces $\zeta \notin C^{q_i(\alpha)}$. Then $h^*_{\beta,\beta_i}(s)$ is compatible with q^* and forces $\zeta \notin C^{q'_i(\alpha')} = C^{q^*(\alpha')}$.

In the same way we can show (2), as well as (5) and the trivial directions of (3), (4). E.g., if $\bar{q}$ is $(\bar{w}, \zeta)$-canonical, then q^* is (w^*, ζ)-canonical. For this, use the fact that every element $y^* \in \mathrm{poss}(q^*, w^*, \zeta + 1)$ "induces" a coherent possibility $\bar{y}$ (which is true whether $\bar{q}$ is canonical or not). And if additionally $\bar{x} \in \mathrm{poss}(\bar{q}, \bar{w}, \zeta + 1)$, then $x^* \in \mathrm{poss}(q^*, w^*, \zeta + 1)$; and if each q_i forces that $x_i \triangleleft G$ implies $\underaccent{\sim}{\tau}_i = x^i$, then q^* forces that $x^* \triangleleft G$ implies $h^*_{\beta,\beta_1}(\underaccent{\sim}{\tau}_i) = h^*_{\beta,\beta_1}(x^i)$.

We omit the (also straightforward) proofs of the other directions of (3) and (4) (which we do not need in this paper). $\square$

In the following, whenever we mention q^* or w^*, we assume $\bar{w}$, $\bar{q}$ to be coherent and in M_β. We will (and can) use x^* only if $\bar{q}$ additionally is canonical (otherwise x^* will generally not be a possibility for q^*). In this case, every P-generic filter containing q^* will select an x^* for some coherent possibility $\bar{x}$.

Lemma 5.52. *Assume $\bar{q}$ is coherent, $\underaccent{\sim}{\sigma}_i$ are P-names in M_β for elements of 2^λ, and[13] $q_0 \Vdash \underaccent{\sim}{\sigma}_0 \notin V_{\beta+1}$. Then there is a coherent $\bar{r} \leq \bar{q}$, and sequences $(\zeta^j)_{j\in\lambda}$ and $(\bar{w}^j)_{j\in\lambda}$ such that $\bar{r}$ is $(\bar{w}^j, \zeta^j)$-canonical for all j, and for all $\bar{x} \in \mathrm{poss}(\bar{r}, \bar{w}^j, \zeta^j + 1)$ there is some $\ell \in I^*(>\zeta^j, <\zeta^{j+1})$ and $\bar{b} = (b_i)_{i<4}$, with $b_i \in 2$, violating majority[14] such that for all $i < 4$*

$$r_i \Vdash x_i \triangleleft G \rightarrow \underaccent{\sim}{\sigma}_i(\ell) = b_i.$$

As the p_{β_i} are coherent, we can apply the lemma to $\underaccent{\sim}{\sigma}_i := \underaccent{\sim}{a}_\beta$ (for all i) and get:

[13] As usual, $V_{\beta+1}$ denoted the $P_{\beta+1}$-extension.
[14] I.e., $b_0 = 1 - \mathrm{major}_{i=1,2,3}(b_i)$.

Corollary 5.53. *If $p_\beta \Vdash \underset{\sim}{a}_\beta \notin V_{\beta+1}$, then there is a coherent $r^* \le \bigwedge_{i<4} p_{\beta_i}$ forcing that*

$$\neg \left(\underset{\sim}{a}_{\beta_0} =^* \text{major}_{i=1,2,3}(\underset{\sim}{a}_{\beta_i})\right).$$

Proof of the lemma. We will construct (in M_β), by induction on $j \in \lambda$, ζ^j, $\bar{w}^j$ and $\bar{r}^j$ with $r_i^0 = q_i$, such that the following holds:

(1) $\bar{r}^j$ is coherent.
(2) $\bar{w}^j$ is coherent, for each $i < 4$ the w_i^j are increasing with j, and their union covers $\bigcup_{j\in\lambda} \text{dom}(r_i^j)$.
(3) $\bar{r}^j$ is $(\bar{w}^j, \zeta^j)$-canonical.
(4) $\bar{r}^k \le^+_{\bar{w}^j,\zeta^j} \bar{r}^j$ for $j < k$.
(5) If $\bar{x} \in \text{poss}(\bar{r}^j, \bar{w}^j, \zeta^j + 1)$, then there is an $\ell \in I^*(>\zeta^j, <\zeta^{j+1})$ and a $b \in 2$ such that for at least two i_1, i_2 in $\{1,2,3\}$, r_i^{j+1} forces that $x_i \lhd G$ implies

$$(*) \qquad \underset{\sim}{\sigma}_0(\ell) = 1 - b, \quad \underset{\sim}{\sigma}_{i_1}(\ell) = b, \quad \underset{\sim}{\sigma}_{i_2}(\ell) = b.$$

Then we take the usual fusion limits, as in Lemma 5.50(2), and are done.

For limits j, let $\bar{r}'$ be a (coherent) limit of $(\bar{r}^{j'})_{j'<j}$, and set $\zeta^* := \sup_{j'<j}(\zeta^{j'})$ and $w_i^* := \bigcup_{j'<j} w_i^{j'}$ for each $i < 4$. Note that $\bar{r}'$ fits $(\bar{w}^*, \zeta^*)$. Then we can find coherent $\bar{r}^* \le^+_{\bar{w}^*,\zeta^*} \bar{r}'$ which is $(\bar{w}^*, \zeta^*)$-canonical, as in Lemma 5.50(3).

In successor cases $j = j' + 1$ set $(\bar{r}^*, \bar{w}^*, \zeta^*) := (\bar{r}^{j'}, \bar{w}^{j'}, \zeta^{j'})$.

In any case we want to construct $\bar{r}^j$, $\bar{w}^j$, and ζ^j.

Enumerate $\text{poss}(\bar{r}^*, \bar{w}^*, \zeta^* + 1)$ as $(\bar{x}^k)_{k\in K}$.

We define $\bar{s}^k$ for $k \le K$, with $\bar{s}^0 := \bar{r}^*$ and, as usual, taking (coherent) limits at limits, such that:

- $\bar{s}^k$ is coherent.
- $\bar{s}^\ell \le^+_{\bar{w}^*,\zeta^*} \bar{s}^k$ for $k < \ell < K$. (This implies that $\bar{s}^k$ is $(\bar{w}^*, \zeta^*)$-canonical.)
- There is a ξ^k and an $\ell \in I^*(>\zeta^*, <\xi^k)$ and a $b \in 2$ such that

$$(**) \qquad s_0^{k+1} \Vdash x_0^k \lhd G \to \underset{\sim}{\tau}_0(\ell) = 1 - b \quad \text{and} \quad (\exists^{\ge 2} i \in \{1,2,3\})\ s_i^{k+1} \Vdash x_i^k \lhd G \to \underset{\sim}{\tau}_i(\ell) = b.$$

Assume we can construct these $\bar{s}^k$, ξ^k for all $k \in K$, then let $\bar{s}^K$ be again a (coherent) limit. We set $w_i^j := w_i^* \cup \{\alpha_j\}$ such that $\bar{w}^j$ is coherent (and such that, by bookkeeping, all elements of $\text{dom}(p_i^j)$ will be eventually covered), and find some $\zeta^j > \sup_{k\in K}(\xi^k)$ and $\bar{r}^j \le^+_{\bar{w}^*,\zeta^*} r^*$ which is $(\bar{w}^j, \zeta^j)$-canonical, again as in Lemma 5.50(3). Then $\bar{r}^j$, $\bar{w}^j$ and ζ^j are as required.

So it remains to construct, for $k \in K$, $\bar{s}^{k+1}$ and ξ^k, which we will do in the rest of the proof. Set $\bar{s} := \bar{s}^k$, $\bar{x} := \bar{x}^k$, $\bar{w} := \bar{w}^*$ and $\zeta := \zeta^*$. Recall that $\bar{s}$ is $(\bar{w}, \zeta)$-canonical, $\bar{x} \in \text{poss}(\bar{s}, \bar{w}, \zeta)$, and we are looking for $\bar{s}^{k+1} \le^+_{\bar{w},\zeta} \bar{s}$ which satisfies $(**)$ for $\bar{x}$.

Set $s_i' := s_i \wedge x_i$. It is enough to construct $t_i \le s_i'$ such that:

- Both $t_i \restriction \beta$ and $t_i(\beta) \restriction (\lambda \setminus \zeta + 1)$ are independent of i.
- $t_0 \Vdash \underset{\sim}{\tau}_0(\ell) = 1 - b$ and $(\exists^{\ge 2} i \in \{1,2,3\})\ t_i \Vdash \underset{\sim}{\tau}_i(\ell) = b$.

Then we can define $\bar{s}^{k+1}$ in the usual way: $\text{dom}(s_i^{k+1}) = \text{dom}(t_i)$ (and we can assume $\text{dom}(s_i) = \text{dom}(t_i)$, by using trivial conditions). For $\alpha \in \text{dom}(t_i)$, if $t_i \restriction \alpha \notin G_\alpha$ then set $s_i^{k+1}(\alpha)$ to be $s_i(\alpha)$, otherwise $t_i(\alpha) \vee (s_i(\alpha) \restriction \zeta + 1)$ if $\alpha \in w_i$ and $t_i(\alpha)$ otherwise. The resulting $\bar{s}^{k+1} \le^+_{\bar{w},\zeta} \bar{s}$ is coherent and s_i^{k+1} forces that $x_i \lhd G$ implies $t_i \in G$.

We have to introduce more notation: Fix $j \ne i$, and $a \le s_j'$ and $b \le s_i' \restriction \beta + 1$ (in $P_{\beta+1}$) such that $b \restriction \beta \le a$ and $b \restriction \beta$ forces that $b(\beta)$ is stronger than $a(\beta)$ above ζ (i.e., $b \restriction \beta \Vdash (\forall \xi > \zeta)\, b(\beta)(\xi) \supseteq a(\beta)(\xi)$). Then we define $b^{[j]} \wedge a$ by

$$(b^{[j]} \wedge a)(\alpha)(\xi) = \begin{cases} b(\alpha)(\xi) & \text{if } \alpha < \beta, \\ x_j(\beta)(\xi) & \text{if } \alpha = \beta \text{ and } \xi \le \zeta, \\ b(\beta)(\xi) & \text{if } \alpha = \beta \text{ and } \xi > \zeta, \\ a(\alpha)(\xi) & \text{otherwise.} \end{cases}$$

Note that $b^{[j]} \wedge a$ is stronger than a, but generally not stronger than b.

By our assumption, q_0 and therefore s'_0 forces $\underset{\sim}{a}_0 \notin V_{\beta+1}$. So in an intermediate model $V[G_{\beta+1}]$, there is some $\ell \in I^*(>\zeta)$ such that $s'_0/G_{\beta+1}$ does not decide $\underset{\sim}{a}_0(\ell)$. Back in V, fix some $b_0 \le s'_0 \restriction (\beta+1)$ in $P_{\beta+1}$ which determines this ℓ.

Find $r'_1 \le b_0^{[1]} \wedge s'_1$ which determines $\underset{\sim}{a}_1(\ell)$ to be j_1 for some $j_1 \in 2$. Find $r'_2 \le (r'_1 \restriction \beta+1)^{[2]} \wedge s'_2$ which determines $\underset{\sim}{a}_2(\ell)$ to be j_2; analogously find $r'_3 \le (r'_2 \restriction \beta+1)^{[3]} \wedge s'_3$ which determines $\underset{\sim}{a}(\ell)$ to be some j_3. Let $j \in 2$ be equal to at least two of j_1, j_2, j_3.

Set $p := (r'_3 \restriction \beta+1)^{[0]} \wedge s'_0$. In any $P_{\beta+1}$-extension honoring $p \restriction \beta+1$, $\underset{\sim}{a}_0(\ell)$ is not determined by $p/G_{\beta+1}$, i.e., there is an $t_0 \le p$ forcing that $\underset{\sim}{a}(\ell) = 1-j$.

We now set and $t_i := (t_0 \restriction \beta+1)^{[i]} \wedge r'_i$ for $i = 1,2,3$. Note that $t_i \le r'_i \le s'_i$ extends x_i and forces $\underset{\sim}{a}_i(\ell)$ to be $1-j$ if $i=0$ and to be j for at least two i in $\{1,2,3\}$. □

We can now easily show:

Lemma 5.54. *For all but non-stationary many $\beta \in S^{\mu}_{\lambda^+}$*

$$p_* \Vdash \underset{\sim}{a}_\beta \in V_{\beta+1}$$

Proof. We started in this section with an arbitrary Δ-system and showed that Corollary 5.53 and Lemma 5.48 holds for this system.

We now use a specific Δ-system:

Assume towards a a contradiction that on a non-stationary set S' there are $p_\beta \le p_*$ forcing $\underset{\sim}{a}_\beta \notin V_{\beta+1}$. By strengthening we can assume that p_β canonically reads $\underset{\sim}{a}_\beta$. Let M_β contain p_β and let $S \subseteq S'$ be such that $(M_\beta, p_\beta)_{\beta\in S}$ is a Δ-system. Fix $\beta_0 < \beta_1 < \beta_2 < \beta_3$ in S. By Corollary 5.53 we get a coherent $\bar{r}$ stronger than $\bar{p}$ such that $r^* \Vdash \neg\big(\underset{\sim}{a}_{\beta_0} =^* \text{major}_{i=1,2,3}(\underset{\sim}{a}_{\beta_i})\big)$. This contradicts Lemma 5.48. □

5.10. **Fixing the Δ-system.** We now know that there is a stationary set $S^0 \subseteq S^{\mu}_{\lambda^+}$ such that for all $\beta \in S^0$, $\underset{\sim}{a}_\beta$ is forced (by p_*) to be in $V_{\beta+1}$ but not in V_β (see Lemmas 5.39 and 5.54).

For each $\beta \in S^0$ there is a $p'_\beta \le p_*$ in P forcing that $\underset{\sim}{a}_\beta$ is equal to some $P_{\beta+1}$-name, call it $\underset{\sim}{a}^*_\beta$, and we choose $p_\beta \le p'_\beta$ (we only have to strengthen the part below $\beta+1$) which canonically reads $\underset{\sim}{a}^*_\beta$.[15]

We now fix, as usual, for each $\beta \in S^0$, some elementary model M_β containing p_β, and fix $S \subseteq S^0$ such that $(M_\beta, p_\beta)_{\beta\in S}$ is a Δ-system.

So $p_{**} := p_\beta \restriction \beta \le p_*$ is independent of $\beta \in S$ (it is a P_α-condition for some $\alpha \in \Delta$, independent of $\beta \in S$); and $\underset{\sim}{a}^*_\beta$ is read continuously by $p_\beta \restriction \beta+1$ via $(w'_\zeta)_{\zeta\in E'}$ for some $E' \subseteq \lambda$ club, with $w'_\zeta \subseteq \beta+1$. As usual, due to homogeneity E' is independent of $\beta \in S$, and the w'_ζ are independent of β apart from the shifting of the final coordinate β via the mapping $h^*_{\beta_0,\beta_1}$; the same holds for the decision functions that map $\text{poss}(p_\zeta, w'_\zeta, \zeta+1)$ to $\underset{\sim}{a}_\beta \restriction I^*(<\zeta+1)$

Let E be the limit points of E', and set $w_\zeta := \bigcup_{\nu<\zeta} w'_\nu$. Then $\underset{\sim}{a}_\beta \restriction I^*(<\xi)$ is (w_ξ, ξ)-determined by p_β for all $\xi \in E$.

In the P_β-extension, only $\underset{\sim}{\eta}_\beta$ remains undetermined, i.e., there are f_ξ for $\xi \in E$ such that p_β/G_β forces $\underset{\sim}{a}_\beta \restriction I^*(<\xi) = f_\xi(\underset{\sim}{\eta}_\beta \restriction I^*(<\xi))$. The f_ξ are canonically read from $p_\beta \restriction \beta$ in a way independent of β (due to homegeneity).

Recall that $x \in \text{poss}(\tilde{p}, \xi)$ is equivalent to: $x \in 2^{I^*(<\xi)}$ and x extends $\eta^{\tilde{p}} \restriction I^*(<\xi)$. So the domain of f_ξ is $\text{poss}(\tilde{p}, \xi)$.

To summarize:

Fact 5.55. $(M_\beta, p_\beta)_{\beta\in S}$ satisfies:

- $p_\beta \restriction \beta =: p_{**} \le p_*$ is a $P_{\sup(\Delta)}$-condition independent of $\beta \in S$.
- $p_\beta(\beta) =: \tilde{p}$ is a $P_{\sup(\Delta)}$-name independent of $\beta \in S$,
- There is a club-set $E \subseteq \lambda$ and, for $\xi \in E$, $P_{\sup(\Delta)}$-names $\underset{\sim}{f}_\xi : \text{poss}(\tilde{p}, \xi) \to 2^{I^*(<\xi)}$ such that for all $\beta \in S$ and $\xi \in E$

$$p_\beta \Vdash \underset{\sim}{a}_\beta \restriction I^*(<\xi) = \underset{\sim}{f}_\xi(\underset{\sim}{\eta}_\beta \restriction I^*(<\xi)).$$

[15]So $p_\beta \restriction \beta+1$ reads $\underset{\sim}{a}^*_\beta$, but generally the whole p_β may be required to force $\underset{\sim}{a}_\beta = \underset{\sim}{a}^*_\beta$.

- If $\beta \in S$, $\underset{\sim}{x} \subseteq \lambda$ is a P_β-name, $q \leq p_{**}$ in P_β and $q, \underset{\sim}{x}$ are in M_β, then we can find $\alpha \in \Delta$ and $p'_{**} \leq q$ in P_α which continuously reads $\underset{\sim}{x}$, $\underset{\sim}{\tau}(\underset{\sim}{x})$ and $\underset{\sim}{\tau}^{-1}(\underset{\sim}{x})$ independently[16] of β.

The last item follows from Lemma 5.42; and we will use it several times: Before Corollary 5.59 we find $p^2_{**} \leq p_{**}$ to get names for U, F_ξ etc. that are independent of β; before Lemma 5.63 we get $p^3_{**} \leq p^2_{**}$ to get independent names for some unions, intersections and $\underset{\sim}{\pi}$-images; and finally after Corollary 5.70 we choose $q \leq p^3_{**}$ to get an independent name for the generator f_{gen}.

5.11. **Local reading.** So we know that we can determine initial segments of $\underset{\sim}{a}_\beta$ from initial segments of $\underset{\sim}{\eta}_\beta$, more specifically, we can determine $\underset{\sim}{\eta}_\beta \restriction I$ from $\underset{\sim}{a}_\beta \restriction I$ for $I := I^*(<\xi)$.

In this section we show that on unboundedly many disjoint intervals of the form $A := I^*(\geq \xi, < \nu)$, we can read $\underset{\sim}{a}_\beta \restriction A$ from just $\underset{\sim}{\eta}_\beta \restriction A$ (without having to use the $\underset{\sim}{\eta}_\beta$-values below A).

The following definition (the notion of candidate) is only used in this section. In the rest of the paper we only need Corollary 5.59.

In the following, we work in V_β, the P_β-extension $V[G_\beta]$ where we assume $\beta \in S$ and $p_{**} \in G_\beta$.

Definition 5.56. (In V_β)

- For $A \subseteq \lambda$ and $\bar{x} = (x_i)_{i<4}$, $x_i : A \to 2$, we say $\bar{x}$ honors majority above ζ, if
$$x_0(\ell) = \text{major}_{i=1,2,3}\, x_i(\ell) \text{ for all } \ell \in A \cap I^*(\geq\zeta).$$
We say $\bar{x}$ honors $\tilde{p}$, if each x_i is compatible with $\eta^{\tilde{p}}$ (as partial functions).
- $\bar{x} = (x_i)_{i<4}$ is a (ζ_0, ζ_1)-candidate, (for $\zeta_0 \leq \zeta_1$ both in E) if the $x_i \in \text{poss}(\tilde{p}, \zeta_1)$ honor majority above ζ_0.
 (As elements of $\text{poss}(\tilde{p}, \zeta_1)$ they automatically honor $\tilde{p}$.)
- If $\bar{x}$ is a (ζ_0, ζ_1)-candidate, we say "$\bar{y}$ extends $\bar{x}$" if $\bar{y}$ is a (ζ_1, ζ_2)-candidate[17] for some $\zeta_2 \geq \zeta_1$ and each y_i extends x_i.
 Equivalently, $\bar{y} = \bar{x}^\frown \bar{b}$ for some $\bar{b}$, with $b_i : I^*(\geq\zeta_1, <\zeta_2) \to 2$, which honors both majority and $\tilde{p}$.
- A (ζ_0, ζ_1)-candidate $\bar{y}$ is "good", if for every candidate $\bar{z}$ of height $\xi > \zeta_1$ that extends $\bar{y}$ we have:

$$(*_1) \qquad f_\xi(z_0)(\ell) = \text{major}_{i=1,2,3}\, f_\xi(z_i)(\ell) \text{ for all } \ell \in I^*(\geq\zeta_1, <\xi).$$

Preliminary Lemma 5.57. *(In V_β.) Every candidate can be extended to a good candidate.*

Proof. Assume otherwise, i.e., there is a (ζ', ζ_0)-candidate $\bar{x}$ which is a counterexample, which means:

$(*_2)$ Whenever $\bar{y}$ is a (ζ_0, ζ_1)-candidate extending $\bar{x}$ then there is a $\xi > \zeta_1$ and a (ζ_1, ξ)-candidate $\bar{z}$ extending $\bar{y}$ which violates $(*_1)$.

We now construct $r_0 \leq \tilde{p}$ and, for $i = 1, 2, 3$, Q_β-names $r_i \leq \tilde{p}$. All these conditions live on the same $C^* \subseteq E$ with $\min(C^*) = \zeta_0$. The trunk of r_i is x_i.

We now construct inductively $C^* \restriction \zeta$ and $r_i \restriction \zeta$.

Assume we have determined that $\zeta \in C^*$ and we have constructed each r_i below ζ. Set $r_0(\zeta) := \tilde{p}(\zeta)$ and pick $r_i(\zeta)$ as in (5.44), i.e., they have majority $\underset{\sim}{\eta}_\beta$ and leave enough freedom to form a valid condition.

We will now construct the C^*-successor ξ of ζ, together with r_i on $I^*(>\zeta, <\xi)$.

Enumerate all $(\zeta_0, \zeta + 1)$-candidates extending $\bar{x}$ as $(\bar{y}^k)_{k\in K}$.

Let $\bar{a}^0$ be the empty 4-tuple and set $\xi_0 := \zeta + 1$. We will construct, for $k \in K$, ξ_k and some $\bar{a}^k$ that honors majority and $\tilde{p}$, where a^k_i has domain $I^*(\geq\zeta + 1, <\xi_k)$ and extends a^j_i if $j < k$.

If k is a limit, let $\bar{a}^x$ be the (pointwise) union of $\bar{a}^j$ with $j < k$, and set $\xi_k := \sup_{j<k}(\xi_j)$.

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[16] This means: $p'_{**} \in M_\gamma$ for all $\gamma \in S$, and there is a way (independent of $\gamma \in S$) to continuously read $\underset{\sim}{y}_1, \underset{\sim}{y}_2, \underset{\sim}{y}_3$ modulo p'_{**} from the generics below α, and for all $\gamma \in S$ we have that $p'_{**} \wedge p_\gamma$ forces $\underset{\sim}{y}_1 = \underset{\sim}{x}'$, $\underset{\sim}{y}_2 = \underset{\sim}{\tau}(\underset{\sim}{x}')$ and $\underset{\sim}{y}_3 = \underset{\sim}{\tau}^{-1}(\underset{\sim}{x}')$, where $\underset{\sim}{x}' := h^*_{\beta,\gamma}(\underset{\sim}{x})$.

[17] or equivalently, a (ζ_0, ζ_2)-candidate

Assume we already have $\bar{a}^j$. Extend $\bar{y}^j{}^\frown\bar{a}^j$ to some candidate $\bar{y}^j{}^\frown\bar{a}^{j+1}$ of some height ξ_{j+1} in E such that

$$(*_3) \qquad \bar{y}^j{}^\frown\bar{a}^{j+1} \text{ violates } (*_1) \text{ for some } \ell \in I^*(\geq\xi_j, <\xi_{j+1}).$$

We can due that due to $(*_2)$.

So in the end we get some $\xi > \zeta$ in E and $\bar{b}^\zeta$ with domain $I^*(>\zeta, <\xi)$ honoring majority and $\tilde{p}$ such that

$(*_4)$ for every $(\zeta_0, \zeta+1)$-candidate $\bar{y}$ extending $\bar{x}$, $\bar{y}^\frown\bar{b}^\zeta$ is a (ζ_0, ξ)-candidate violating $(*_1)$ for some $\ell \in I^*(>\zeta, <\xi)$.

We then define C^* below $\xi + 1$ by adding only ξ, i.e., ξ is the C^*-successor of ζ. We extend the conditions r_i by b_i^ζ for $i < 4$. I.e., we have $\eta^{r_i}(\ell) = b_i^\zeta(\ell)$. This ends the construction of $r_i \leq \tilde{p}$.

Back in V, assume that $(*_2)$ is forced by some $q' \leq p_\beta \restriction \beta$. Pick an increasing sequence β_i ($i < 4$) in S. We take the union of q' and the p_{β_i}, call it s, and strengthen $s(\beta_i) = \tilde{p}$ to r_i. The resulting condition s' forces the following:

- $\underset{\sim}{a}_{\beta_i} \restriction I^*(<\xi) = f_\xi(\underset{\sim}{\eta}_{\beta_i} \restriction I^*(<\xi))$ for all $\xi \in C^*$. This is because $s' \leq p_{\beta_i}$, cf. Fact 5.55.
- The $\underset{\sim}{\eta}_{\beta_i}$ honor majority above ζ_0. This is because for all $\zeta \in C^*$, the $r_i(\zeta)$ are chosen as in (5.44) and therefore honor majority; and for $\zeta \in \lambda \setminus (C^* \cup \zeta_0)$ we use values $\bar{b}$ which honor majority.
- Accordingly, the $\underset{\sim}{a}_{\beta_i}$ honor majority above some $\gamma < \lambda$, cf. Lemma 5.43(1). Pick ζ_1 such that $\sup(I^*(<\zeta_1)) > \gamma$.
- So for all $\xi > \zeta_1$ the $f_\xi(\underset{\sim}{\eta}_{\beta_i} \restriction I^*(<\xi))$ honor majority above ζ_1.
- Pick some $\zeta > \zeta_0, \zeta_1$ in C^* with C^*-successor ξ. By construction of the r_i, $\underset{\sim}{\eta}_{\beta_i} \restriction I^*(\geq\zeta+1, <\xi)$ is b_i^ζ. As r_i extends x_i, $\bar{y} := \underset{\sim}{\eta}_{\beta_i} \restriction I^*(<\zeta+1)$ is a $(\zeta_0, \zeta+1)$-candidate extending $\bar{x}$. So by $(*_4)$, the $\underset{\sim}{\eta}_{\beta_i} \restriction I^*(<\xi)$ violate $(*_1)$ at some $\ell \in I^*(>\zeta, <\xi)$, a contradiction. □

Let $U \subseteq \lambda$ be club. Set U^{ODD} to be the odd elements[18] of U. For $\xi \in U^{\text{ODD}}$ with U-successor ν, set

$$A_\xi^U := I^*(\geq\xi, <\nu)$$

Lemma 5.58. *(In V_β.) There is an $r_0 \leq \tilde{p}$, a club $U \subseteq C^{r_0} \subseteq E$ and, for $\xi \in U^{\text{ODD}}$, an $F_\xi : 2^{A_\xi^U} \to 2^{A_\xi^U}$ such that*

- *$r_0 \wedge p_\beta/G_\beta$ forces that $F_\xi(\underset{\sim}{\eta}_\beta \restriction A_\xi^U) = \underset{\sim}{a}_\beta \restriction A_\xi^U$.*
- *F_ξ is not constant: There are, for $k = 0, 1$, z_ξ^k in $\text{poss}(r_0, I^*(<\nu))$ and $\ell_\xi \in A_\xi^U$ such that $F_\xi(z_\xi^k \restriction A_\xi^U)(\ell_\xi) = k$. (Again, ν is the U-successor of ξ.)*

(Note: Only those elements of $2^{A_\xi^U}$ that are compatible with r_0 are relevant as arguments for F_ξ.)

Proof. We construct r_i for $i < 4$ and U iteratively; C^{r_i} will be independent of i, call it C.

All r_i have the same trunk as $\tilde{p}$; i.e., $\min(C) = \min(C^{\tilde{p}}) =: \zeta_0$ and $r_i \restriction \zeta_0 := \tilde{p} \restriction \zeta_0$. We also set $\min(U) = \zeta_0$.

For all $\zeta \in C$, we choose some $r_i^*(\zeta)$ as in (5.44), i.e., $r_0^*(\zeta) = \tilde{p}(\zeta)$, and the $r_i^*(\zeta)$ for $i = 1, 2, 3$ are such that the majority of their generics would be the $r_0^*(\zeta)$-generic.

Assume that we already know that some ζ is in U (which is a subset of C), and that we know $r_i \restriction \zeta$ for $i < 4$.

We now construct the U-successor ξ of ζ, $C \restriction [\zeta, \xi]$, and $r_i(\nu)$ for $i < 4$ and $\nu \in [\zeta, \xi)$.

- Even case: If ζ is an even element of U, we start with $r_i(\zeta) := r_i^*(\zeta)$, but then add a "shield", or "isolator" above ζ: As in the previous proof, we iterate over all $\zeta + 1$-candidates $\bar{y}^j$, but but in $(*_3)$, instead of violating $(*_1)$ for some ℓ, we demand that $\bar{y}^j{}^\frown\bar{z}^{j+1}$ is good.

[18] I.e., if $(u_\alpha)_{\alpha<\lambda}$ is the canonical enumeration of U, then $\zeta \in U$ is in U^{ODD} if $\zeta = u_{\delta+2n+1}$ for δ a limit (or 0) and $n \in \omega$.

(We already know that every candidate can be extended to a good one.) Accordingly, we get some $\xi > \zeta$ and $\bar b^\zeta$ with domain $I^*(>\zeta, <\xi)$ (and honoring majority and $\tilde p$) such that $\bar y{}^\frown \bar b^\zeta$ is good for every candidate $\bar y$ of height $\zeta+1$; i.e.:

$(*'_4)$ If $\bar z$ is a $(\zeta+1,\nu)$-candidate whose restriction to $I^*(>\zeta,<\xi)$ is $\bar b^\zeta$, then the $f_\nu(z_i)$ honor majority above ξ.

We now let this ξ be the successor of ζ in both C and U (and extend each $p_i(\zeta)$ by b_i).

- Odd case: Now assume ζ is odd in U. Then we choose some $\xi > \zeta$ in $C^{\tilde p}$ large enough such that there are, for $k = 0, 1$, z^k_ξ in $\mathrm{poss}(\tilde p, \xi)$ compatible with all the r_0 constructed so far, such that the $f_\xi(z^k_\xi)(\ell) = k$ for some $\ell > I^*(<\zeta)$. (Such ξ and ℓ have to exist as $\underset{\sim}{a}_\beta$ is not in V_β.)

  We let C restricted to $[\zeta, \xi]$ be the same as $C^{\tilde p}$, and set $r_i(\nu) := r^*_i(\nu)$ for $\nu \in C \cap [\zeta, \xi)$. (For $\zeta \in [\zeta, \xi) \setminus C$ there is no freedom left, i.e., $\tilde p(\zeta)$ is already completely determined, so the only choice for any $r \leq \tilde p$ is $r(\zeta) = \tilde p(\zeta)$.)

This ends the construction of U and of r_i (for $i < 4$).

Pick $\xi \in U^{\text{ODD}}$, let ζ be the U-predecessor and ν the U-successor. We have to show that we can determine (modulo p_β) $\underset{\sim}{a}_\beta \restriction I^*(\geq \xi, < \nu)$ from $\underset{\sim}{\eta}_\beta \restriction I^*(\geq \xi, < \nu)$ alone. (We already know that we can determine it from $\underset{\sim}{\eta}_\beta \restriction I^*(< \nu)$.)

Fix any $z^\zeta_* \in \mathrm{poss}(r_0, \zeta+1)$. Let $x_0 \in \mathrm{poss}(r_0, \nu)$. In particular x_0 extends b^ζ_0. For $i = 1, 2, 3$, let x_i be the copy of x_0 with the initial segment $x_0 \restriction \xi$ replaced by $z^\zeta_*{}^\frown b^\zeta_i$. Note that $\bar x$ is a candidate extending $\bar b^\zeta$. Accordingly the $f_\nu(x_i)$ honor majority above ξ. So we can define

$$F_\xi(x_0 \restriction A^U_\xi) := \mathrm{major}_{i=1,2,3}\, f_\nu(x_i) \restriction A^U_\xi = f_\nu(x_0) \restriction A^U_\xi .$$

This is well-defined,[19] and $r_0 \wedge p_\beta / G_\beta$ forces that $F_\xi(x_0 \restriction A^U_\xi) = \underset{\sim}{a}_\beta \restriction A^U_\xi$. □

We now summarize this lemma, which was shown in V_β for some $\beta \in S$, from the point of view of the ground model. The lemma only uses the parameters $\underset{\sim}{\eta}_\beta$ and $\underset{\sim}{a}_\beta$ (and $\tilde p$, which is just $\underset{\sim}{\eta}_\beta(\beta)$), so by absoluteness M_β knows that the Lemma is forced by p_{**}. Accordingly, we can find P_β-names for U, F_ξ etc in M_β. Using the last item of Fact 5.55, we can strengthen p_{**} to p^2_{**} to canonically read these names:

Corollary 5.59. *There is an $\alpha \in \Delta$, a $p^2_{**} \leq p_{**}$ in P_α and P_α-names for: A condition $r_0 \leq \tilde p$, a set U and a sequence $(F_\xi, z^0_\xi, z^1_\xi, \ell^0_\xi, \ell^1_\xi)_{\xi \in U}$, such that the following holds for all $\beta \in S$, where we set*

p^+_β to be the condition $p^2_{**} \wedge p_\beta$ where we strengthen $p_\beta(\beta)$ to r_0.

(1) α, the condition p^2_{**} and all the names are in M_β.
(2) $p^2_{**} \Vdash U \subseteq C^{r_0} \subseteq \lambda$ club.
(3) for $k = 0, 1$: $p^2_{**} \Vdash\ \forall \xi \in U^{\text{ODD}} \left(z^k_\xi \in \mathrm{poss}(r_0, I^*(<\nu)) \,\&\, \ell_\xi \in A^U_\xi \,\&\, F_\xi(z^k_\xi \restriction A^U_\xi)(\ell_\xi) = k \right)$.
(4) $p^+_\beta \Vdash (\forall \xi \in U^{\text{ODD}})\, F_\xi(\underset{\sim}{\eta}_\beta \restriction A^U_\xi) = \underset{\sim}{a}_\beta \restriction A_\xi$, where we define

A_ξ to be $I^*(\geq\xi, <\nu)$ with ν the U-successor of ξ.

5.12. **Finding the generator.** In this section we use these p^2_{**}, r_0, $(F_\xi, z^0_\xi, z^1_\xi, \ell^0_\xi, \ell^1_\xi)_{\xi \in U}$.

We start working in $V_\beta = V[G_\beta]$, where we assume $p^2_{**} \in G_\beta$.

Let $\xi \in U^{\text{ODD}}$ and ν its U-successor. Set

$$A_\xi := I^*(\geq\xi, <\nu), \qquad A^?_\xi := A_\xi \setminus \mathrm{dom}(\eta^{r_0}),$$

$$\text{ODD} := \bigcup_{\xi \in U^{\text{ODD}}} A_\xi, \qquad \text{ODD}^? := \bigcup_{\xi \in U^{\text{ODD}}} A^?_\xi = \text{ODD} \setminus \mathrm{dom}(\eta^{r_0}). \tag{5.60}$$

[19] Assume y and x in $\mathrm{poss}(r_0, \nu)$ are the identical restricted to A^U_ξ. Then y defines the same $(x_i)_{i=1,2,3}$ and thus the same F_ξ.

For F_ξ it is enough to use $\underset{\sim}{\eta}_\beta \restriction A^?_\xi$ as input (the part in $A_\xi \setminus A^?_\xi$ is determined anyway by r_0), and *every* element of $2^{A^?_\xi}$ is compatible with r_0 (and thus a possible input for F_ξ). Identifying 2^B and $\mathcal{P}(B)$ as usual, we get:

$$F_\xi : \mathcal{P}(A^?_\xi) \to \mathcal{P}(A_\xi)$$

is such that p^+_β / G_β forces

$$F_\xi(\underset{\sim}{\eta}_\beta \cap A^?_\xi) = \underset{\sim}{a}_\beta \cap A_\xi,$$

We now define

$$F : \mathcal{P}(\text{ODD}^?) \to \mathcal{P}(\text{ODD}) \quad \text{by} \quad x \mapsto \bigcup_{\xi \in U^{\text{ODD}}} F_\xi(x \cap A^?_\xi).$$

So in particular p^+_β / G_β forces that

$$F(\underset{\sim}{\eta}_\beta \cap \text{ODD}^?) = \underset{\sim}{a}_\beta \cap \text{ODD}. \tag{5.61}$$

Note that for every $z \subseteq \text{ODD}^?$ (in V_β that is) there is an $r' \leq r_0$ forcing that $\underset{\sim}{\eta}_\beta \cap \text{ODD}^? = z$. ($C' := U \setminus U^{\text{ODD}}$ is club, so it is enough to leave freedom at C' and we may assign arbitrary values everywhere else.)

Back in the ground model V, using the last item of Fact 5.55 again, we can strengthen p^2_{**} to p^3_{**} so that

(5.62) $\quad p^3_{**}$ canonically reads each of the following (countably many) sets:[20]

- $(A_\xi)_{\xi \in U^{\text{ODD}}}$, ODD, r_0, $(A^?_\xi)_{\xi \in U^{\text{ODD}}}$, $\text{ODD}^?$ (actually, these are already read by r^2_{**}).
- The closure of these sets under $\underset{\sim}{\pi}$, $\underset{\sim}{\pi}^{-1}$, finite unions, and finite intersections.

In particular, the (names for) all these sets are independent of $\beta \in S$, modulo p^3_{**}.[21]

Lemma 5.63. *(In V) $p^3_{**} \Vdash |\underset{\sim}{\pi}(\text{ODD}^?) \cap \text{ODD}| = \lambda$.*

Proof. Let $q \leq p^3_{**}$ in P_β be arbitrary. We have to show that q does not force (in P_β) $|\underset{\sim}{\pi}(\text{ODD}^?) \cap \text{ODD}| < \lambda$.

For $\xi \in U^{\text{ODD}}$ and $k = 0, 1$, use r_0, p^+_β, z^k_ξ and ℓ_ξ as in Corollary 5.59 and set $b^k_\xi := z^k_\xi \cap A^?_\xi$.

For $k = 0, 1$, set $B^k := \bigcup_{\xi \in U^{\text{ODD}}} (b^k_\xi)$. Note that $F(B^1) \setminus F(B^0)$ contains $\{\ell_\xi : \xi \in U^{\text{ODD}}\}$, a set of size λ.

Pick increasing $(\beta_i)_{i<4}$ in S with $\beta_0 = \beta$. Set $s := q \wedge \bigwedge_{i<4} p^+_{\beta_i} \in P$.

Now for each $i < 4$, strengthen $s(\beta_i)$ (i.e., r_0) as follows: At the even intervals in some way that together they honor majority; and at the odd intervals (where we do not have to leave freedom) to the value $B^{\text{sgn}(i)}$ (where $\text{sgn}(k) = 0$ for $k = 0$ and 1 for $k = 1, 2, 3$).

Accordingly, we have

$$\underset{\sim}{\pi}(\underset{\sim}{\eta}_{\beta_i}) \cap \text{ODD} = F(\underset{\sim}{\eta}_{\beta_i} \cap \text{ODD}^?) = F(B^{\text{sgn}(i)}),$$

or, when we split $\underset{\sim}{\pi}(\underset{\sim}{\eta}_{\beta_i})$ into the parts in and out of $\underset{\sim}{\pi}(\text{ODD}^?)$:

$$\Big(\big(\underset{\sim}{\pi}(\underset{\sim}{\eta}_{\beta_i}) \setminus \underset{\sim}{\pi}(\text{ODD}^?)\big) \cap \text{ODD} \Big) \cup \Big(\underset{\sim}{\pi}(\underset{\sim}{\eta}_{\beta_i}) \cap \underset{\sim}{\pi}(\text{ODD}^?) \cap \text{ODD} \Big) =^* F(B^{\text{sgn}(i)})$$

Now assume towards a contradiction that $\underset{\sim}{\pi}(\text{ODD}^?) \cap \text{ODD} =^* \emptyset$. Then we get:

$$\big(\underset{\sim}{\pi}(\underset{\sim}{\eta}_{\beta_i}) \setminus \underset{\sim}{\pi}(\text{ODD}^?)\big) \cap \text{ODD} =^* F(B^{\text{sgn}(i)}). \tag{5.64}$$

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[20] We can do this for λ many sets, of course; but we cannot assume e.g. that $\underset{\sim}{\pi}(z) \in V_\beta$ for *all* $z \in V_\beta$, let alone that each such $\underset{\sim}{\pi}(z)$ is canonically read by p^3_{**}.

[21] But we need p^+_β to force that these names have anything to do with $\underset{\sim}{a}_\beta$.

But on the other hand we have:

$$\underset{\sim}{\eta}_{\beta_0} \setminus \text{ODD}^? = \text{major}_{i=1,2,3}(\underset{\sim}{\eta}_{\beta_i} \setminus \text{ODD}^?), \text{ so}$$
$$\underset{\sim}{\pi}(\underset{\sim}{\eta}_{\beta_0}) \setminus \underset{\sim}{\pi}(\text{ODD}^?) =^* \underset{\sim}{\pi}\big(\underset{\sim}{\eta}_{\beta_0} \setminus \text{ODD}^?\big) = \underset{\sim}{\pi}\big(\text{major}_{i=1,2,3}(\underset{\sim}{\eta}_{\beta_i} \setminus \text{ODD}^?)\big) =^*$$
$$=^* \text{major}_{i=1,2,3}\big(\underset{\sim}{\pi}(\underset{\sim}{\eta}_{\beta_i} \setminus \text{ODD}^?)\big) =^* \text{major}_{i=1,2,3}\big(\underset{\sim}{\pi}(\underset{\sim}{\eta}_{\beta_i}) \setminus \underset{\sim}{\pi}(\text{ODD}^?), \text{ and}$$
$$\big(\underset{\sim}{\pi}(\underset{\sim}{\eta}_{\beta_0}) \setminus \underset{\sim}{\pi}(\text{ODD}^?)\big) \cap \text{ODD} =^* \text{major}_{i=1,2,3}\Big(\big(\underset{\sim}{\pi}(\underset{\sim}{\eta}_{\beta_i}) \setminus \underset{\sim}{\pi}(\text{ODD}^?)\big) \cap \text{ODD}\Big).$$

Applying (5.64) to both sides of the last line, we get $F(B^0) =^* \text{major}_{i=1,2,3} F(B^{\text{sgn}(i)}) = F(B^1)$, a contradiction. $\square$

Set

(5.65) $$\underset{\sim}{X} := \text{ODD}^? \cap \underset{\sim}{\pi}^{-1}(\text{ODD}).$$

By choice of p^3_{**}, $\underset{\sim}{X}$ and $\underset{\sim}{\pi}(\underset{\sim}{X})$ are canonically read by p^3_{**} (and independent of β).

We now show that $F(z) \cap \underset{\sim}{\pi}(\underset{\sim}{X}) = \underset{\sim}{\pi}(z)$ for $z \subseteq \underset{\sim}{X}$. Again, here we are talking about $z \in V_\beta$. To make that more explicit, let us formulate in the ground model V:

Lemma 5.66. *For $\beta \in S$,*
$$p^3_{**} \Vdash_{P_\beta} \Big(|\underset{\sim}{X}| = \lambda, \text{ and for all } z \subseteq \underset{\sim}{X},\ p^+_\beta/G_\beta \Vdash \underset{\sim}{\pi}(z) =^* F(z) \cap \underset{\sim}{\pi}(\underset{\sim}{X})\Big).$$

(Note that, other than $F(z)$, $\underset{\sim}{\pi}(z)$ will generally not be in V_β, and we have to force with p^+_β/G_β.)

Proof. Work in V_β. $|\underset{\sim}{X}| = \lambda$ follows from Lemma 5.63, as $\underset{\sim}{\pi}(\underset{\sim}{X}) =^* \underset{\sim}{\pi}(\text{ODD}^?) \cap \text{ODD}$.

Set $y := \underset{\sim}{\eta}_\beta \cap \text{ODD}^?$. So by (5.61), $p^+_\beta/G_\beta \le r_0$ forces: $F(y) = \underset{\sim}{\pi}(\underset{\sim}{\eta}_\beta) \cap \text{ODD}$. As $\underset{\sim}{\pi}(\underset{\sim}{X}) \subseteq^* \text{ODD}$, we get $F(y) \cap \underset{\sim}{\pi}(\underset{\sim}{X}) =^* \underset{\sim}{\pi}(\underset{\sim}{\eta}_\beta) \cap \underset{\sim}{\pi}(\underset{\sim}{X})$. Then $y \subseteq^* \underset{\sim}{\pi}^{-1}(\text{ODD})$ (or equivalently, $y \subseteq^* \underset{\sim}{X}$) implies $y =^* y \cap \underset{\sim}{\pi}^{-1}(\text{ODD}) = \underset{\sim}{\eta}_\beta \cap \underset{\sim}{X}$ and thus $\underset{\sim}{\pi}(y) =^* \underset{\sim}{\pi}(\underset{\sim}{\eta}_\beta) \cap \underset{\sim}{\pi}(\underset{\sim}{X})$. To summarize:

$(*)$ $$p^+_\beta/G_\beta \Vdash \Big(y \subseteq^* \underset{\sim}{X} \ \to\ \underset{\sim}{\pi}(y) =^* F(y) \cap \underset{\sim}{\pi}(\underset{\sim}{X}),\ \text{for } y := \underset{\sim}{\eta}_\beta \cap \text{ODD}^?\Big)$$

Now back in V assume towards a contradiction that some $q \le p^+_\beta$ forces that the lemma fails, i.e., that $\underset{\sim}{z} \subseteq \underset{\sim}{X}$ in V_β is a counterexample (in the final extension). By absoluteness, we can assume that q and $\underset{\sim}{z}$ are in M_β, in particular $\underset{\sim}{z}$ is a P_β-name in M_β. Strengthen $q \restriction \beta$ to canonically read $\underset{\sim}{z}$. So for every $\beta' \in S$, $h^*_{\beta,\beta'}(\underset{\sim}{z})$ will be evaluated in $V_{\beta'}$ to the same $z \subseteq \lambda$ as $\underset{\sim}{z}$ in V_β.

Chose a β' above $\text{supp}(q)$. Then we can strengthen $q \wedge p_{\beta'}$ at index β', i.e., r_0, to some r_1 that forces $\underset{\sim}{\eta}_\beta \cap \text{ODD}^? = h^*_{\beta,\beta'}(\underset{\sim}{z})$. (Recall that we can fix the values in the odd intervals, as the even intervals still form a club). Let G be P-generic containing $q \wedge p^+_{\beta'} \wedge r_1$. Then we have:

- The evaluation of $h^*_{\beta,\beta'}(\underset{\sim}{z})$ in $V_{\beta'}$, is the same as the evaluation of $\underset{\sim}{z}$ in V_β, call it z.
- Also the evaluation of $\underset{\sim}{X}$ and F are the same β and β', cf. (5.62).
- $z \subseteq \underset{\sim}{X}$ is a counterexample (as this is forced by q).
  In particular, $z \subseteq \underset{\sim}{X}$ and $\underset{\sim}{\pi}(z) \ne^* F(z) \cap \underset{\sim}{\pi}(X)$ in the final extension.
- $p_{\beta'} \wedge r_1$ forces in $V_{\beta'+1}$ that $\underset{\sim}{\eta}_\beta \cap \text{ODD}^? = z$; also we have just seen that $z \subseteq \underset{\sim}{X}$; and so $\underset{\sim}{\pi}(z) =^* F(z) \cap \underset{\sim}{\pi}(X)$ by $(*)$, a contradiction. $\square$

For $\xi \in U^{\text{ODD}}$, we define the following P_β-names (independent of β):[22]

$$\underset{\sim}{x}_\xi := A^?_\xi \cap \underset{\sim}{X} \qquad\qquad \underset{\sim}{y}_\xi := A_\xi \cap \underset{\sim}{\pi}(\underset{\sim}{X})$$
$$\text{so} \quad \bigcup_{\xi \in U^{\text{ODD}}} \underset{\sim}{x}_\xi = \underset{\sim}{X} \qquad\qquad \bigcup_{\xi \in U^{\text{ODD}}} \underset{\sim}{y}_\xi = \text{ODD} \cap \underset{\sim}{\pi}(\underset{\sim}{X}) =^* \underset{\sim}{\pi}(\underset{\sim}{X}),$$

as well as

[22] More concretely, canonically read by p^3_{**}, see (5.62).

$$F'_\xi : \mathcal{P}(\underset{\sim}{x}_\xi) \to \mathcal{P}(\underset{\sim}{y}_\xi) \qquad \text{by} \quad a \mapsto F_\xi(a) \cap \underset{\sim}{\pi}(\underset{\sim}{X}),$$

$$\text{and} \quad F' : P(\underset{\sim}{X}) \to P(\underset{\sim}{\pi}(\underset{\sim}{X})) \quad \text{by} \quad z \mapsto \bigcup_{\xi \in U^{\text{ODD}}} F'_\xi(z \restriction \underset{\sim}{x}_\xi) = F(z) \cap \underset{\sim}{\pi}(\underset{\sim}{X}).$$

So the p^3_{**} forces that for all $z \in V_\beta$ the following is forced by p^+_β/G_β:

(5.67) $\quad z \subseteq \underset{\sim}{X} \ \to F'(z) =^* \underset{\sim}{\pi}(z), \quad$ in particular $F'(\underset{\sim}{X}) =^* \underset{\sim}{\pi}(\underset{\sim}{X}), \quad$ also $F'(z) \subseteq \underset{\sim}{\pi}(\underset{\sim}{X})$ for all z

Lemma 5.68. *p^3_{**} forces: For almost all $\xi \in U^{\text{ODD}}$, F'_ξ is a Boolean algebra isomorphism from $P(\underset{\sim}{x}_\xi)$ to $P(\underset{\sim}{y}_\xi)$.*

Proof. All and nothing: We claim that for almost all ζ, $F'_\zeta(\underset{\sim}{x}_\zeta) = \underset{\sim}{y}_\zeta$. Assume that $\ell \in \underset{\sim}{y}_\zeta \setminus F'_\zeta(\underset{\sim}{x}_\zeta) \subseteq \underset{\sim}{\pi}(\underset{\sim}{X})$. Then $\ell \in \underset{\sim}{\pi}(\underset{\sim}{X})$, and ℓ is not in $F'(\underset{\sim}{X}) =^* \underset{\sim}{\pi}(\underset{\sim}{X})$, so there cannot be many such ℓ. Similarly $F'_\zeta(\emptyset) = \emptyset$ for almost all ζ .

Unions: We claim that for almost all ζ, $F'_\zeta(a) \cup F'_\zeta(b) = F'_\zeta(a \cup b)$ for all subsets a, b of $\underset{\sim}{x}_\zeta$. Let $A \subseteq \lambda$ be the set of counterexamples, i.e., for $\xi \in A$ there are $\ell_\xi \in \underset{\sim}{y}_\xi$, and a_ξ, b_ξ subsets of $\underset{\sim}{x}_\zeta$ such that $\ell_\xi \in \big(F'_\xi(a_\xi) \cup F'_\xi(b_\xi)\big) \Delta F'_\xi(a_\xi \cup b_\xi)$. Set $x := \bigcup_{\xi \in A} a_\xi$ and $y := \bigcup_{\xi \in A} b_\xi$. Then ℓ_ξ is in $\big(F'(x) \cup F'(y)\big) \Delta F'(x \cup y) =^* \emptyset$, so A cannot be large.

Complements: We claim that for almost all ξ, $F'_\xi(a) \cap F'_\xi(\underset{\sim}{x}_\xi \setminus a) = \emptyset$. Let A be the set of counterexamples, i.e., for $\xi \in A$ there is an $a_\xi \subseteq \underset{\sim}{x}_\xi$ and $\ell \in \underset{\sim}{y}_\xi$ such that $\ell_\xi \in F'_\xi(a_\xi) \cap F'_\xi(\underset{\sim}{x}_\xi \setminus a_\xi)$. Then ℓ_ξ is in $F'(\bigcup_{\zeta \in A} a_\zeta) \cap F'(\bigcup_{\zeta \in A} \underset{\sim}{x}_\xi \setminus a_\zeta) =^* \emptyset$, so A cannot be large.

Injectivity: We already know that union and complements (and thus disjointness) are preserved, so it is enough to show that a nonempty set is mapped to a nonempty set.

Assume this fails often, then we get an $x \subseteq \underset{\sim}{X}$ of size λ such that $\emptyset = F'(x) =^* \underset{\sim}{\pi}(x)$, a contradiction.

Surjectivity: Assume surjectivity fails often; i.e., there are many $b_\zeta \subseteq \underset{\sim}{\pi}(\underset{\sim}{X}) \cap \text{ODD}$ not in the range of F'_ζ. Let y be the union of those b_ζ. Pick $x \subseteq \lambda$ such that $\underset{\sim}{\pi}(x) =^* y \subseteq \underset{\sim}{\pi}(\underset{\sim}{X})$. So we can assume $x \subseteq \underset{\sim}{X}$ and so $F'(x) =^* y$, which implies that $F_\zeta(x \cap \underset{\sim}{x}_\zeta) = y \cap A_\zeta = b_\zeta$ for almost all ζ, a contradiction. □

Lemma 5.69. *For each $\beta \in S$: p^3_{**} forces (in P_β): There is a $f_{gen} : \underset{\sim}{X} \to \underset{\sim}{\pi}(\underset{\sim}{X})$ bijective such that for all $z \subseteq \underset{\sim}{X}$ (in V_β), p^+_β/G_β forces $\underset{\sim}{\pi}(z) =^* f''_{gen} z$.*

Proof. Every Boolean algebra isomorphism from $P(A)$ to $P(B)$ is generated by a bijection from A to B (the restriction to the atoms). So there is an $U' \subseteq U^{\text{ODD}}$ with $|U^{\text{ODD}} \setminus U'| < \lambda$ such that $\zeta \in U'$ implies that F'_ζ is generated by some bijection $g_\zeta : \underset{\sim}{x}_\zeta \to \underset{\sim}{y}_\zeta$. So F' is generated by $g := \bigcup_{\zeta \in U'} g_\zeta$; and we can change g into a bijection from $\underset{\sim}{X}$ to $\underset{\sim}{\pi}(\underset{\sim}{X})$ by changing less than λ many values. □

We now strengthen p^3_{**} to some q to continuously read f_{gen} (independently of β), again using Fact 5.55.

So to summarize, we have the following (where we start with the Δ-system $(M_\beta, p_\beta)_{\beta \in S}$ of Section 5.10):

Corollary 5.70. *There is $\alpha \in \Delta$, $q \in P_\alpha$ stronger than all $p_\beta \restriction \beta$ and canonically reading $r_0 \leq \tilde{p}$, $\underset{\sim}{X}$, f_{gen} and $\underset{\sim}{\pi}(\underset{\sim}{X})$, such that the following holds for all $\beta \in S$:*

- *$q \wedge p_\beta$ with the condition[23] at index β strengthened to r_0 is a valid condition, called p^{++}_β.*
- *α, p^{++}_β and the names are in M_β.*
- *q forces in P_β: $|\underset{\sim}{X}| = \lambda$, $f_{gen} : \underset{\sim}{X} \to \underset{\sim}{\pi}(\underset{\sim}{X})$ is a bijection, and if $z \subseteq \underset{\sim}{X}$ is in V_β, then $p^{++}_\beta/G_\beta \Vdash \underset{\sim}{\pi}(z) =^* f''_{gen} z$.*

5.13. Putting everything together.

Corollary 5.71. *(Assuming λ is inaccessible and $2^\lambda = \lambda^+$.) P forces that every automorphism of P^λ_λ is somewhere trivial.*

[23] which is $p_\beta(\beta) = \tilde{p}$

Proof. Assume towards a contradiction that some p_* forces that $\underset{\sim}{\phi}$ is a nowhere trivial automorphism represented by $\underset{\sim}{\pi}$.

As described in Section 5.10 we find a Δ-system $(M_\beta, p_\beta)_{\beta\in S}$ with $p_\beta \restriction \beta \leq p_*$ for all $\beta \in S$, and we find q, $\underset{\sim}{X}$, f_{gen} as in Corollary 5.70, so in particular: $q \leq p_\beta \restriction \beta$ for all S; and q forces that $|\underset{\sim}{X}| = \lambda$ and that $f_{\text{gen}} : \underset{\sim}{X} \to \underset{\sim}{\pi}(\underset{\sim}{X})$ is a bijection.

As $\underset{\sim}{\pi}$ is nowhere trivial, f_{gen} cannot be a generator, i.e., there is some $z \subseteq \underset{\sim}{X}$ with $\underset{\sim}{\pi}(z) \neq^* f''_{\text{gen}} z$. Fix a name for this z and let $q^* \leq q$ canonically read z.

Pick $\beta \in S$ above $\text{dom}(q^*)$. So $q^* \wedge p^{++}_\beta$ is a valid condition, which forces that in the final extension $V[G]$ the following holds:

- $z \subseteq \underset{\sim}{X}$ with $\underset{\sim}{\pi}(z) \neq^* f''_{\text{gen}} z$, as this is forced by q^*.
- $z \in V_\beta$, as q^* canonically reads z.
- So by Corollary 5.70 and as $p^{++}_\beta \in G$, we get $\underset{\sim}{\pi}(z) =^* f''_{\text{gen}} z$, a contradiction. □

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