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Torsion-free abelian groups are Borel complete

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Abstract

We prove that the Borel space of torsion-free abelian groups with domain ω is Borel complete, i.e., the isomorphism relation on this Borel space is as complicated as possible, as an isomorphism relation. This solves a long-standing open problem in descriptive set theory, which dates back to the seminal paper on Borel reducibility of Friedman and Stanley from 1989.

1. Introduction

Since the seminal paper of Friedman and Stanley on Borel complexity [4], descriptive set theory has proved itself to be a decisive tool in the analysis of complexity problems for classes of countable structures. A canonical example of this phenomenon is the famous result of Thomas from [17], which shows that the complexity of the isomorphism relation for torsion-free abelian groups of rank $1 \leqslant n < \omega$ (denoted as $\cong_n$) is strictly increasing with n, thus, on one hand, finally providing a satisfactory reason for the difficulties found by many eminent mathematicians in finding systems of invariants for torsion-free abelian groups of rank $2 \leqslant n < \omega$ which were as simple as the one provided by Baer for $n = 1$ (see [2]) and, on the other hand, showing that for no $1 \leqslant n < \omega$ the relation $\cong_n$ is universal among countable Borel equivalence relations. As a matter of fact, abelian group theory has been one of the most important fields

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of mathematics from which inspiration is taken for forging the general theory of Borel complexity as well as for finding some of the most striking applications thereof. The present paper continues this tradition, solving one of the most important problems in the area — a problem open since the above mentioned paper of Friedman and Stanley from 1989. In technical terms, we prove that the space of countable torsion-free abelian groups with domain ω is *Borel complete*.

As we will see in detail below, saying that a class of countable structures is Borel complete means that the isomorphism relation on this class is as complicated as possible, as an isomorphism relation. The Borel completeness of countable abelian group theory is particularly interesting from the perspective of model theory, as this class is model theoretically "low," i.e., stable (in the terminology of [14]). In fact, as already observed in [4], Borel reducibility can be thought of as a weak version of $\mathfrak{L}_{\omega_1,\omega}$-interpretability, and for other classes of countable structures such as groups or fields, much stronger results than Borel completeness exist, as in such cases we can first-order interpret graph theory, but such classes are unstable, while abelian group theory is stable. Reference [9] starts a systematic study of the relations between Borel reducibility and classification theory in the context of $\aleph_0$-stable theories.

Coming back to us, we now introduce the notions from descriptive set theory which are necessary to understand our results, and we try to make a complete historical account of the problems which we tackle in this paper. The starting point of the descriptive set theory of countable structures is the following fact:

Fact 1.1. *The set K^L_ω of structures with domain ω in a given countable language L is endowed with a standard Borel space structure $(\mathrm{K}^L_\omega, \mathcal{B})$. Every Borel subset of this space $(\mathrm{K}^L_\omega, \mathcal{B})$ is naturally endowed with the Borel structure induced by $(\mathrm{K}^L_\omega, \mathcal{B})$.*

For example, if we take $L = \{e, \cdot, ()^{-1}\}$ and we let K' be one of the following,

(a) the set of elements of K^L_ω which are groups;
(b) the set of elements of K^L_ω which are abelian groups;
(c) the set of elements of K^L_ω which are torsion-free abelian groups;
(d) the set of elements of K^L_ω which are n-nilpotent groups for some $n < \omega$,

then we have that K' is a Borel subset of $(\mathrm{K}^L_\omega, \mathcal{B})$, and so Fact 1.1 applies.

Thus, given a class K' as in Fact 1.1, we can consider K' as a standard Borel space, and so we can analyze the complexity of certain subsets of this space or of certain relations on it (i.e., subsets of $K' \times K'$ with the product Borel space structure). Further, this technology allows us to compare pairs of classes of structures or, in another direction, pairs of relations defined on pairs of classes of structures.

Definition 1.2. Let X_1 and X_2 be two standard Borel spaces, and also let $Y_1 \subseteq X_1$ and $Y_2 \subseteq X_2$. We say that Y_1 is reducible to Y_2, denoted as $Y_1 \leqslant_R Y_2$, when there is a Borel map $\mathbf{B} : X_1 \to X_2$ such that for every $x \in X_1$, we have

$$x \in Y_1 \Leftrightarrow \mathbf{B}(x) \in Y_2.$$

Notice that Definition 1.2 covers, in particular, the case $X_1 = K' \times K'$ for K' as in Fact 1.1 and so, for example, Y_1 could be the isomorphism relation on K'. Also, given a Borel space X, we can ask if there are subsets of X which are $\leqslant_R$-maxima with respect to a fixed family of subsets of an arbitrary Borel space (e.g., Borel sets, analytic sets, co-analytic sets, etc). In particular, we have the following definition:

Definition 1.3. Let X_1 be a Borel space, and let $Y_1 \subseteq X_1$. We say that Y_1 is *complete analytic* (resp. complete co-analytic) if for every Borel space X_2 and analytic subset (resp. co-analytic subset) Y_2 of X_2, we have that $Y_2 \leqslant_R Y_1$.

We now introduce the notion of Borel reducibility among equivalence relations.

Definition 1.4. Let X_1 and X_2 be two standard Borel spaces. Also let E_1 be an equivalence relation defined on X_1 and E_2 be an equivalence relation defined on X_2. We say that E_1 is Borel reducible to E_2, denoted as $E_1 \leqslant_B E_2$, when there is a Borel map $\mathbf{B} : X_1 \to X_2$ such that for every $x, y \in X_1$ we have

$$xE_1y \ \Leftrightarrow \mathbf{B}(x)E_2\mathbf{B}(y).$$

Remark 1.5. Note that in the context of Definitions 1.2 and 1.4, $E_1 \leqslant_R E_2$ and $E_1 \leqslant_B E_2$ have two different meanings, as in the first case the witnessing Borel function has domain $X \times X$, while in the second case it has domain X. Furthermore, notice that $E_1 \leqslant_B E_2$ implies $E_1 \leqslant_R E_2$. (However the converse need not hold; see 1.7.)

We now define *Borel completeness*, the notion at the heart of our paper.

Definition 1.6. Let K_1 be a Borel class of structures with domain ω, and let $\cong_1$ be the isomorphism relation on K_1. We say that K_1 is *Borel complete* (or, in more modern terminology, $\cong_1$ is S_∞-complete) if for every Borel class K_2 of structures with domain ω, there is a Borel map $\mathbf{B} : \mathrm{K}_2 \to \mathrm{K}_1$ such that for every $A, B \in \mathrm{K}_2$,

$$A \cong B \ \Leftrightarrow \mathbf{B}(A) \cong_1 \mathbf{B}(B);$$

that is, the isomorphism relation on the space K_2 is Borel reducible (in the sense of Definition 1.4) to the the isomorphism relation on the space K_1.

The following fact will be relevant for our subsequent historical account.

Fact 1.7 ([4]). *Let* K *be a Borel class of structures with domain* ω. *If* K *is Borel complete, then its isomorphism relation is a complete analytic subset of* $K \times K$, *but the converse need not hold as, for example, abelian* p*-groups with domain* ω *have complete analytic isomorphism relation but they are* not *a Borel complete space.*

We now have all the ingredients necessary to be able to understand the problems which we solve in this paper and to introduce the state of the art concerning them. But first a useful piece of notation, which we will use throughout the paper.

Notation 1.8.

(1) We denote by Graph the class of graphs;
(2) we denote by Gp the class of groups;
(3) we denote by AB the class of abelian groups;
(4) we denote by TFAB the class of torsion-free abelian groups;
(5) given a class K, we denote by K_ω the set of structures in K with domain ω.

Convention 1.9. To simplify statements, we use the following convention: when we say that a class K of countable structures is Borel complete, we mean that K_ω is Borel complete. Similarly, when we say that a class K of countable groups is complete co-analytic, we mean that K_ω is a complete co-analytic subset of Gp_ω. Finally, when we say that the isomorphism relation on a class of countable groups is analytic, we mean that restriction of the isomorphism relation on K to $K_\omega \times K_\omega$ is an analytic subset of the Borel space $\mathrm{Gp}_\omega \times \mathrm{Gp}_\omega$ (as a product space).

In [4], together with the general notions just defined, the authors studied some Borel complexity problems for specific classes of countable structures of interest. Among other things they showed (we mention only the results relevant to us) that

(i) countable graphs, linear orders and trees are Borel complete;
(ii) torsion abelian groups have complete analytic $\cong$ but are *not* Borel complete;
(iii) nilpotent groups of class 2 and exponent p (p a prime) are Borel complete;[1]
(iv) the isomorphism relation on finite rank torsion-free abelian groups is Borel.

In [4] Friedman and Stanley explicitly state the following:

[1] As already mentioned in [4], this result is actually a straightforward adaptation of a model theoretic construction due to Mekler [10].

> There is, alas, a missing piece to the puzzle, namely our conjecture that torsion-free abelian groups are complete. [...] We have not even been able to show that the isomorphism relation on torsion-free abelian groups is complete analytic, nor, in another direction, that the class of all abelian groups is Borel complete. We consider these problems to be among the most important in the subject.

The challenge was taken by several mathematicians. The first to work on this problem was Hjorth, who in [6] proved that any Borel isomorphism relation is Borel reducible (in the sense of Definition 1.4) to the isomorphism relation on countable torsion-free abelian groups and that, in particular, the isomorphism relation on TFAB_ω is not Borel (as there is no such Borel equivalence relation), however leaving open the question whether TFAB_ω is a Borel complete class, or even whether the isomorphism relation on TFAB_ω is complete analytic (cf. Definition 1.3 and Fact 1.7).

The problem resisted further attempts of the time, and the interest moved to another very interesting problem on torsion-free abelian groups: for $1 \leqslant n < m < \omega$, is the isomorphism relation $\cong_n$ on torsion-free abelian groups of rank n strictly less complex (in the sense of Definition 1.4) than the isomorphism relation on torsion-free abelian groups of rank m? As mentioned above, the isomorphism relation on torsion-free abelian groups of finite rank is Borel while, as just mentioned, the isomorphim relation on countable torsion-free abelian groups is not, and so the two problems are quite different, but obviously related. Also this problem proved to be very challenging, until Thomas finally gave a positive solution to the problem, in a series of two fundamental papers [16], [17] proving, in particular, that for every $n < \omega$, $\cong_n$ is not universal among countable Borel equivalence relations.

The fundamental work of Thomas thus completely resolved the case of torsion-free abelian groups of finite rank, leaving open the problem for countable torsion-free abelian groups of arbitrary rank, i.e., the problem referred to as "among the most important in the subject" in [4]. The problem remained "dormant" for various years (at the best of our knowledge), until Downey and Montalbán [3] made some important progress showing that the isomorphism relation on countable torsion-free abelian groups is complete analytic, a necessary but not sufficient condition for Borel completeness, as recalled in Fact 1.7. This was of course possible evidence that the isomorphism relation was indeed Borel complete, as conjectured in [4]. Despite this advancement, the problem of Borel completeness of countable torsion-free abelian groups remained unresolved for another 12 years, until this day, when we prove

Main Theorem. *The space* TFAB_ω *is Borel complete*; *in fact there exists a continuous map* $\mathbf{B} : \mathrm{Graph}_\omega \to \mathrm{TFAB}_\omega$ *such that for every* $H_1, H_2, \in$

Graph_ω, *we have*

$$H_1 \cong H_2 \text{ if and only if } \mathbf{B}(H_1) \cong \mathbf{B}(H_2).$$

The techniques employed in the proof of our Main Theorem lead us to (and at the same time were inspired by) classification questions of "rigid" countable abelian groups. One of the most important notions of rigidity in abelian group theory is the notion of endorigidity, where we say that $G \in \mathrm{AB}$ is endorigid if the only endomorphisms of G are multiplication by an integer. The analysis of endorigid abelian groups is an old topic in abelian group theory; famous in this respect is the result of the second author [15] that for every infinite cardinal λ, there is an endorigid torsion-free abelian group of cardinality λ. We prove in Theorem 1.10 below that the classification of the countable endorigid TFAB is an highly untractable problem.

THEOREM 1.10. *The set of endorigid torsion-free abelian groups is a complete co-analytic subset of the Borel space space* TFAB_ω. *In fact, more strongly, there is a Borel function* $\mathbf{F}$ *from the set of tree with domain* ω *into* TFAB_ω *such that*

(i) *if T is well-founded, then $\mathbf{F}(T)$ is endorigid*;

(ii) *if T is not well-founded, then $\mathbf{F}(T)$ has a one-to-one $f \in \mathrm{End}(G)$ which is not multiplication by an integer and such that $G/f[G]$ is not torsion.*

In [12] we extend the ideas behind Theorem 1.10 to a systematic investigation of several classification problems for various rigidity conditions on abelian and nilpotent groups from the perspective of descriptive set theory of countable structures. In another direction, in [13] we study the question of existence of uncountable (co-)Hopfian abelian groups; this work was later continued by the second author et al. in the preprint [1], which settles some questions left open in [13].

We conclude with a few words on the history of this article. At the end of the refereeing process, the referee indicated some points which needed correction in the original version of this paper. Around the same time, Laskowski and Ulrich indicated another point which needed correction in our original submission. The referee also asked us to change the presentation of our Main Theorem and to simply its proof — in particular, separating the algebra from the combinatorics (division which is reflected by the current division in Sections 3 and 4). Here all the points raised there are addressed. We thank the anonymous referee, Laskowski and Ulrich. Meanwhile, Laskowski and Ulrich have found another proof of our Main Theorem; see [8], [7].

2. Notation and preliminaries

For the readers of various backgrounds we try to make the paper self-contained.

2.1. *General notation.*

Definition 2.1.

(1) Given a set X, we write $Y \subseteq_\omega X$ for $\emptyset \neq Y \subseteq X$ and $|Y| < \aleph_0$.
(2) Given a set X and $\bar{x}, \bar{y} \in X^{<\omega}$, we write $\bar{y} \triangleleft \bar{x}$ to mean that $\lg(\bar{y}) < \lg(\bar{x})$ and $\bar{x} \restriction \lg(\bar{y}) = \bar{y}$, where $\bar{x}$ is naturally considered as a function $\lg(\bar{x}) \to X$.
(3) Given a partial function $f : M \to M$, we denote by $\mathrm{dom}(f)$ and $\mathrm{ran}(f)$ the domain and the range of f, respectively.
(4) For $\bar{a} \in B^n$, we write $\bar{a} \subseteq B$ to mean that $\mathrm{ran}(\bar{a}) \subseteq B$, where, as usual, $\bar{a}$ is considered as a function $\{0, \ldots, n-1\} \to B$.
(5) Given a sequence $\bar{f} = (f_i : i \in I)$, we write $f \in \bar{f}$ to mean that there exists $j \in I$ such that $f = f_j$.

2.2. *Groups.*

Notation 2.2. Let G and H be groups.

(1) $H \leqslant G$ means that H is a subgroup of G.
(2) We let $G^+ = G \setminus \{e_G\}$, where e_G is the neutral element of G.
(3) If G is abelian, we might denote the neutral element e_G simply as $0_G = 0$.

Definition 2.3. Let $H \leqslant G$ be groups. We say that H is pure in G, denoted by $H \leqslant_* G$, when if $h \in H$, $0 < n < \omega$, $g \in G$ and (in additive notation) $G \models ng = h$, then there is $h' \in H$ such that $H \models nh' = h$. Given $S \subseteq G$, we denote by $\langle S \rangle_S^*$ the pure subgroup generated by S (the intersection of all the pure subgroups of G containing S).

Observation 2.4. If $H \leqslant_* G \in \mathrm{TFAB}$, $h \in H$, $0 < n < \omega$, $G \models ng = h$, then $g \in H$.

Observation 2.5. Let $G \in \mathrm{TFAB}$, let p be a prime, and let

$$G_p = \{a \in G : a \text{ is divisible by } p^m \text{ for every } 0 < m < \omega\}.$$

Then G_p is a pure subgroup of G.

Proof. This is well known; see, e.g., the discussion in [5, pp. 386–387]. $\square$

Definition 2.6. Let p be a prime. We let

$$\mathbb{Q}_p = \left\{ \frac{m_1}{m_2} : m_1 \in \mathbb{Z}, m_2 \in \mathbb{Z}^+, p \text{ and } m_2 \text{ are coprime} \right\}.$$

2.3. *Trees.*

Definition 2.7. Given an L-structure M, by a partial automorphism of M we mean a partial function $f : M \to M$ such that $f : \langle \mathrm{dom}(f) \rangle_M \cong \langle \mathrm{ran}(f) \rangle_M$.

In Section 5 we shall use the following notions.

Definition 2.8. Let $(T, <_T)$ be a strict partial order.

(1) $(T, <_T)$ is a *tree* when, for all $t \in T$, $\{s \in T : s <_T t\}$ is well-ordered by the relation $<_T$. Notice that according to our definition, a tree $(T, <_T)$ might have more than one root, i.e., more than one $<_T$-minimal element. We say that the tree $(T, <_T)$ is rooted when it has only one $<_T$-minimal element (its root).
(2) A *branch* of the tree $(T, <_T)$ is a maximal chain of the partial order $(T, <_T)$.
(3) A tree $(T, <_T)$ is said to be *well-founded* if it has only finite branches.
(4) Given a tree $(T, <_T)$ and $t \in T$, we let the level of t in $(T, <_T)$, denoted as $\mathrm{lev}(t)$, be the order type of $\{s \in T : s <_T t\}$; recall item (1).

Remark 2.9. Concerning Definition 2.8(4), we will only consider trees $(T, <_T)$ such that, for every $t \in T$, $\{s \in T : s <_T t\}$ is finite, so for us, $\mathrm{lev}(t) \in \omega$.

3. The combinatorial frame

The isomorphism problem for countable models of the theory of two equivalence relations is known to be at least as complex as the isomorphism problem for any other Borel class of countable structures (cf., e.g., [11, p. 295]). In what follows we will reduce this problem to the isomorphism problem for countable TFAB's. Our reduction will consist of an elaborated coding of finite partial automorphisms g's of models with two equivalence relations into partial automorphisms f_g's of TFAB's. For technical reasons, we will actually work with finite sequences $\bar{g}$ of finite partial automorphisms, and to avoid the troublesome case $g = g^{-1}$ we will actually work with objects of the form $(\bar{g}, \iota)$ with $\iota \in \{0, 1\}$. Finally, also for technical reasons, we will consider models of the theory of three equivalence relation, with one of them being equality.

The definition of $\mathfrak{m} \in \mathrm{K}_2^{\mathrm{bo}}(M)$ which we will introduce in Definition 3.4 is phrased just to construct a certain $G_M \in \mathrm{TFAB}$ (see Definition 4.3) for any relevant M and, in fact, as we will see, it will suffice (and it will be very useful to do so) to construct such a G_M just for M the countable homogeneous universal model of the theory of two equivalence relations; cf. 3.2(2). In this regard, below X will serve as set of generators for G_M (in the appropriate sense) and $a \in M$ will be coded as $X'_{\{a\}} \subseteq X$ (in a certain sense). As the $f_{\bar{g}}$'s are finite partial automorphisms related to the partial automorphisms in $\bar{g}$, it is natural to require that if $\bar{g} = (g_0, \ldots, g_n)$, then $f_{\bar{g}}$ maps elements in $\mathrm{dom}(f_{\bar{g}}) \cap X'_{\{a\}}$ into elements in $X'_{\{g_n(a)\}}$. Almost all clauses in Definition 3.4 right below are combinatorial but, not surprisingly, one is more algebraic, namely clause (8). As will be clear from reading Section 4 below, this clause is *crucial* in reconstructing an isomorphism between models of the theory of two equivalence relations from an isomorphism between TFAB_ω's.

Notation 3.1. For Z a set and $0 < n < \omega$, we let $\mathrm{seq}_n(Z) = \{\bar{x} \in Z^n : \bar{x} \text{ injective}\}$.

Hypothesis 3.2.

(1) $\mathbf{K}^{\mathrm{eq}}$ *is the class of models* M *in a vocabulary* $\{\mathfrak{E}_0, \mathfrak{E}_1, \mathfrak{E}_2\}$ *such that each* $\mathfrak{E}_i^M$ *is an equivalence relation and* $\mathfrak{E}_2^M$ *is the equality relation. We use the symbol* $\mathfrak{E}_i$ *to avoid confusion, as the symbol* E_i *appears in* 3.4.
(2) M *is the countable homogeneous universal model in* $\mathbf{K}^{\mathrm{eq}}$.
(3) $\mathcal{G}$ *is essentially the set of finite non-empty partial automorphisms* g *of* M*, but for technical reasons*[2] *it is the set of objects* $g = (\mathbf{h}_g, \iota_g)$*, where*
 (A) (a) $\mathbf{h}_g$ *is a finite non-empty partial automorphism of* M;
  (b) $\iota_g \in \{0, 1\}$;
 (B) *for* $g \in \mathcal{G}$*, we let*
  (a) $g^{-1} = (\mathbf{h}_g^{-1}, 1 - \iota_g)$;
  (b) *for* $a \in M$, $g(a) = \mathbf{h}_g(a)$;
  (c) *for* $\mathcal{U} \subseteq M$, $g[\mathcal{U}] = \{\mathbf{h}_g(a) : a \in \mathcal{U}\}$;
  (d) $g_1 \subseteq g_2$ *means* $\mathbf{h}_{g_1} \subseteq \mathbf{h}_{g_2}$ *and* $\iota_{g_1} = \iota_{g_2}$;
  (e) $g_1 \subsetneq g_2$ *means* $g_1 \subseteq g_2$ *and* $g_1 \neq g_2$;
  (f) $\mathrm{dom}(g) = \mathrm{dom}(\mathbf{h}_g)$ *and* $\mathrm{ran}(g) = \mathrm{ran}(\mathbf{h}_g)$;
  (g) *for* $\mathcal{U} \subseteq M$, $g \restriction \mathcal{U} = (\mathbf{h}_g \restriction \mathcal{U}, \iota_g)$.
(4) *For* $m < \omega$, $\mathcal{G}_*^m = \{(g_0, \ldots, g_{m-1}) \in \mathcal{G}^m : g_0 \subsetneq \cdots \subsetneq g_{m-1}\}$.
(5) $\mathcal{G}_* = \bigcup\{\mathcal{G}_*^m : m < \omega\}$. (*Notice that the empty sequence belongs to* $\mathcal{G}_*$.)

Notation 3.3. (1) We use $s, t, \ldots$ to denote finite non-empty subsets of M and $\mathcal{U}, \mathcal{V}, \ldots$ to denote arbitrary subsets of M. Recall from 2.1 that $\subseteq_\omega$ means finite subset.
(2) For A a set, we let $s \subseteq_1 A$ mean $s \subseteq A$ and $|s| = 1$.
(3) For $\bar{g} = (g_0, \ldots, g_{\lg(\bar{g})-1}) \in \mathcal{G}_*^{\lg(\bar{g})}$ and $s, t \subseteq_\omega M$, we let
 (a) for $a, b \in M$, $\bar{g}(a) = b$ mean that $g_{\lg(\bar{g})-1}(a) = b$;
 (b) $\bar{g}[s] = t$ means that $g_{\lg(\bar{g})-1}[s] = t$;
 (c) $\mathrm{dom}(\bar{g}) = \mathrm{dom}(g_{\lg(\bar{g})-1})$, and $\emptyset$ if $\lg(\bar{g}) = 0$;
 (d) $\mathrm{ran}(\bar{g}) = \mathrm{ran}(g_{\lg(\bar{g})-1})$, and $\emptyset$ if $\lg(\bar{g}) = 0$;
 (e) $\bar{g}^{-1} = (g_i^{-1} : i < \lg(\bar{g}))$;
 (f) $\bar{g}((x_\ell : \ell < n)) = (\bar{g}(x_\ell) : \ell < n)$.

Definition 3.4. In the context of 3.2, let $\mathrm{K}_2^{\mathrm{bo}}(M)$ be the class of objects (called *systems*) $\mathfrak{m}(M) = \mathfrak{m} = (X^{\mathfrak{m}}, \bar{X}^{\mathfrak{m}}, \bar{f}^{\mathfrak{m}}, \bar{E}^{\mathfrak{m}}) = (X, \bar{X}, \bar{f}, \bar{E})$ such that

(1) X is an infinite countable set and $X \subseteq \omega$;
(2) (a) $(X'_s : s \subseteq_1 M)$ is a partition of X into infinite sets;

[2] The reason is that we want to force that $g \neq g^{-1}$.

(b) for $s \subseteq_\omega M$, let $X_s = \bigcup_{t \subseteq_1 s} X'_t$;
(c) $\bar{X} = (X_s : s \subseteq_\omega M)$ and so $s \subseteq t \subseteq_\omega M$ implies $X_s \subseteq X_t$;

(3) for $\mathcal{U} \subseteq M$, let $X_{\mathcal{U}} = \bigcup\{X_s : s \subseteq_1 \mathcal{U}\}$ and so $X = X_M = \bigcup\{X_s : s \subseteq_1 M\}$;

(4) $\bar{f} = (f_{\bar{g}} : \bar{g} \in \mathcal{G}_*)$ (recall the definition of $\mathcal{G}_*$ from 3.2(5)) and
(a) $f_{\bar{g}}$ is a finite partial bijection of X and $f_{\bar{g}}$ is the empty function if and only if $\lg(\bar{g}) = 0$;
(b) for $s, t \subseteq_1 M$ and $\bar{g}[s] = t$, we have
$$f_{\bar{g}}(x) = y \text{ implies } (x \in X'_s \text{ if and only if } y \in X'_t);$$
(c) for $s, t \subseteq_1 M$, $(f_{\bar{g}}(x) = y, x \in X'_s, y \in X'_t)$ implies $(\bar{g}[s] = t)$;
(d) $f_{\bar{g}^{-1}} = f_{\bar{g}}^{-1}$ (recall that $\bar{g}^{-1} \neq \bar{g}$, when $\mathrm{dom}(\bar{g}) \neq \emptyset$);

(5) $\bar{g}, \bar{g}' \in \mathcal{G}_*$, $\bar{g} \triangleleft \bar{g}' \Rightarrow f_{\bar{g}} \subsetneq f_{\bar{g}'}$;

(6) we define the graph $(\mathrm{seq}_n(X), R_n^{\mathfrak{m}})$ as $(\bar{x}, \bar{y}) \in R_n^{\mathfrak{m}} = R_n$ when $\bar{x} \neq \bar{y}$ and
$$\text{for some } \bar{g} \in \mathcal{G}_*, \text{ we have } f_{\bar{g}}(\bar{x}) = \bar{y};$$
notice that $f_{\bar{g}}^{-1} = f_{\bar{g}^{-1}} \in \bar{f}$, as $\bar{g} \in \mathcal{G}_*$ implies $\bar{g}^{-1} \in \mathcal{G}_*$;

(7) $\bar{E}^{\mathfrak{m}} = \bar{E} = (E_n : 0 < n < \omega) = (E_n^{\mathfrak{m}} : 0 < n < \omega)$ and, for $0 < n < \omega$, E_n is the equivalence relation corresponding to the partition of $\mathrm{seq}_n(X)$ given by the connected components of the graph $(\mathrm{seq}_n(X), R_n)$;

(8) if p is a prime, $k \geqslant 2$, $\bar{x} \in \mathrm{seq}_k(X)$, $\mathbf{y} = (\bar{y}^i : i < i_*) \in (\bar{x}/E_k^{\mathfrak{m}})^{i_*}$, with the $\bar{y}^i$'s pairwise distinct, $\bar{r} \in \mathbb{Q}^{\mathbf{y}}$, $q_\ell \in \mathbb{Q}_p$ for $\ell < k$, and
$$a_{(\mathbf{y}, \bar{r})}(y) = a_{(\mathbf{y}, \bar{r}, y)} = \sum\{r_{\bar{y}} q_\ell : \ell < k, \bar{y} = \bar{y}^i, i < i_*, y = y_\ell^i\}$$
for $y \in \mathrm{set}(\mathbf{y}) = \bigcup\{\mathrm{ran}(\bar{y}^i) : i < i_*\}$, then we have the following:
$$|\{y \in \mathrm{set}(\mathbf{y}) : a_{(\mathbf{y}, \bar{r})}(y) \notin \mathbb{Q}_p\}| \neq 1,$$
where we recall that $\mathbb{Q}_p$ was defined in Definition 2.6;

(9) if for every $n < \omega$, $g_n \in \mathcal{G}$ and $g_n \subsetneq g_{n+1}$, $\mathcal{U} = \bigcup_{n<\omega} \mathrm{dom}(g_n) \subseteq M$ and $\mathcal{V} = \bigcup_{n<\omega} \mathrm{ran}(g_n) \subseteq M$, then we have the following:
$$\bigcup_{n<\omega} \mathrm{dom}(f_{(g_\ell : \ell < n)}) = X_{\mathcal{U}} \text{ and } \bigcup_{n<\omega} \mathrm{ran}(f_{(g_\ell : \ell < n)}) = X_{\mathcal{V}}.$$

The definition of $\mathfrak{m} \in \mathrm{K}_2^{\mathrm{bo}}(M)$ from 3.4 isolates exactly what is needed for the group theoretic construction from Section 4 to take place. The rest of this section has as its sole purpose to show that an object as in Definition 3.4 exists. To this extent, we introduce an auxiliary class of objects, $\mathrm{K}_1^{\mathrm{bo}}(M)$; cf. Definition 3.5. This definition is devised with a twofold aim in mind: on one hand to put more detailed information on the objects at play in Definition 3.4, and on the other hand to be able to construct the desired $\mathfrak{m} \in \mathrm{K}_2^{\mathrm{bo}}(M)$ as a limit of a sequence of approximations $\mathfrak{m}_\ell \in \mathrm{K}_1^{\mathrm{bo}}(M)$, for $\ell < \omega$, of such an $\mathfrak{m} \in \mathrm{K}_2^{\mathrm{bo}}(M)$. In this process the crucial algebraic condition (8) from Definition 3.4 gets translated in the more technical algebraic condition (11) from

Definition 3.5, showing that this condition is preserved in the limit construction, which will be the most elaborated part of this section.

Definition 3.5. In the context of 3.2, let $\mathrm{K}_1^{\mathrm{bo}}(M)$ be the class of objects $\mathfrak{m}(M) = \mathfrak{m} = (X^{\mathfrak{m}}, \bar{X}^{\mathfrak{m}}, I^{\mathfrak{m}}, \bar{I}^{\mathfrak{m}}, \bar{f}^{\mathfrak{m}}, \bar{E}^{\mathfrak{m}}, Y_{\mathfrak{m}}) = (X, \bar{X}, I, \bar{I}, \bar{f}, \bar{E}, Y)$ such that

(1) X is an infinite countable set and $X \subseteq \omega$;
(2) (a) $(X'_s : s \subseteq_1 M)$ is a partition of X into infinite sets;
  (b) for $s \subseteq_\omega M$, let $X_s = \bigcup_{t \subseteq_1 s} X'_t$;
  (c) $\bar{X} = (X_s : s \subseteq_\omega M)$ and so $s \subseteq t \subseteq_\omega M$ implies $X_s \subseteq X_t$.
(3) For $\mathcal{U} \subseteq M$, let $X_{\mathcal{U}} = \bigcup\{X_s : s \subseteq_1 \mathcal{U}\}$ and so $X = X_M = \bigcup\{X_s : s \subseteq_1 M\}$.
(4) (a) $\bar{I} = (I_n : n < \omega) = (I_n^{\mathfrak{m}} : n < \omega)$ are pairwise disjoint;
  (b) $\bar{g} \in I_n$ implies $\bar{g} \in \mathcal{G}_*^m$ for some $m \leqslant n$;
  (c) I_n is finite.
(5) If $\bar{g}' \triangleleft \bar{g} \in I_n$, then $\bar{g}' \in I_{<n} := \bigcup_{\ell < n} I_\ell$.
(6) $I = I^{\mathfrak{m}} = \bigcup_{n<\omega} I_n$.
(7) $\bar{f} = (f_{\bar{g}} : \bar{g} \in I)$ and
  (a) $f_{\bar{g}}$ is a finite partial bijection of X and $f_{\bar{g}}$ is the empty function if and only if $\lg(\bar{g}) = 0$;
  (b) $\mathrm{dom}(f_{\bar{g}}) \subseteq X_{\mathrm{dom}(\bar{g})}$ and $\mathrm{ran}(f_{\bar{g}}) \subseteq X_{\mathrm{ran}(\bar{g})}$ (cf. Notation 3.3(3c), (3d));
  (c) for $s, t \subseteq_1 M$, $(f_{\bar{g}}(x) = y, x \in X'_s, y \in X'_t)$ implies $(\bar{g}[s] = t)$;
  (d) if $\bar{g} \in I_n$, then $\bar{g}^{-1} \in I_n$ and $f_{\bar{g}^{-1}} = f_{\bar{g}}^{-1}$.
(8) $\bar{g} \triangleleft \bar{g}' \Rightarrow f_{\bar{g}} \subsetneq f_{\bar{g}'}$.
(9) We define the graph $(\mathrm{seq}_n(X), R_n^{\mathfrak{m}})$ as $(\bar{x}, \bar{y}) \in R_n^{\mathfrak{m}} = R_n$ when $\bar{x} \neq \bar{y}$ and
$$\text{for some } \bar{g} \in I, \text{ we have } f_{\bar{g}}(\bar{x}) = \bar{y}.$$
  Notice that $f_{\bar{g}}^{-1} = f_{\bar{g}^{-1}} \in \bar{f}$, as $\bar{g} \in I$ implies $\bar{g}^{-1} \in I$.
(10) (a) $\bar{E}^{\mathfrak{m}} = \bar{E} = (E_n : n < \omega) = (E_n^{\mathfrak{m}} : n < \omega)$, and, for $n < \omega$, E_n is the equivalence relation corresponding to the partition of $\mathrm{seq}_n(X)$ given by the connected components of the graph $(\mathrm{seq}_n(X), R_n)$;
  (b) $Y = Y_{\mathfrak{m}}$ is a non-empty subset of X which <u>includes</u> the following set:
$$\{x \in X : \text{ for some } \bar{g} \in I,\, x \in \mathrm{dom}(f_{\bar{g}})\};$$
  notice that this inclusion may very well be proper;
  (c) $\mathrm{seq}_k(\mathfrak{m}) = \{\bar{x} \in \mathrm{seq}_k(X) : \text{ for some } \bar{g} \in I,\, \bar{x} \subseteq \mathrm{dom}(f_{\bar{g}})\}$; notice that $\mathrm{seq}_k(\mathfrak{m}) \subseteq \mathrm{seq}_k(Y_{\mathfrak{m}})$ but the converse need not hold.
(11) If p is a prime, $k \geqslant 2$, $\bar{x} \in \mathrm{seq}_k(X)$, $\bar{q} \in (\mathbb{Q}_p)^k$, $\mathfrak{s} = (p, k, \bar{x}, \bar{q})$ and $\bar{a} \in \mathcal{A}_{\mathfrak{s}}$, then $\mathrm{supp}_p(\bar{a})$ is not a singleton, <u>where</u> we define $\mathcal{A}_{\mathfrak{s}}$, $\mathcal{A}_{\mathfrak{m}}$ and $\mathrm{supp}_p(\bar{a})$ as follows:
  (a) $\mathcal{A}_{\mathfrak{s}} \subseteq \mathcal{A}_{\mathfrak{m}} = \{(a_y : y \in Z) : Z \subseteq_\omega X \text{ and } a_y \in \mathbb{Q}\}$;
  (b) if $\bar{a} \in \mathcal{A}_{\mathfrak{m}}$, then we let
$$\mathrm{supp}_p(\bar{a}) = \{y \in \mathrm{dom}(\bar{a}) : a_y \notin \mathbb{Q}_p\};$$

(c) if $\mathbf{y} = (\bar{y}^i : i < i_*) \in (\bar{x}/E^{\mathfrak{m}}_k)^{i_*}$ (but abusing notation we may treat $\mathbf{y}$ as a set), with the $\bar{y}^i$'s pairwise distinct and $\bar{r} \in \mathbb{Q}^{\mathbf{y}}$, then $\bar{a} \in \mathcal{A}_{\mathfrak{s}}$, where

$$\bar{a} = \bar{a}_{(\mathbf{y},\bar{r})} = (a_y : y \in \mathrm{set}(\mathbf{y})),$$

and where a_y and $\mathrm{set}(\mathbf{y})$ are defined as follows:

$$a_y = a_{(\mathbf{y},\bar{r})}(y) = a_{(\mathbf{y},\bar{r},y)} = \sum\{r_{\bar{y}} q_\ell : \ell < k, \bar{y} = \bar{y}^i, i < i_*, y = y^i_\ell\},$$

$$\mathrm{set}(\mathbf{y}) = \bigcup\{\mathrm{ran}(\bar{y}^i) : i < i_*\};$$

(d) if $\bar{a} \in \mathcal{A}_{\mathfrak{s}}$ and $\mathrm{supp}_p(\bar{a}) \subseteq Z \subseteq \mathrm{dom}(\bar{a})$, then $\bar{a} \restriction Z \in \mathcal{A}_{\mathfrak{s}}$;

(e) if $\bar{a}, \bar{b} \in \mathcal{A}_{\mathfrak{s}}$, then $\bar{c} = \bar{a} + \bar{b} \in \mathcal{A}_{\mathfrak{s}}$, where $\mathrm{dom}(\bar{c}) = \mathrm{dom}(\bar{a}) \cup \mathrm{dom}(\bar{b})$ and
 (i) $c_y = a_y + b_y$, if $y \in \mathrm{dom}(\bar{a}) \cap \mathrm{dom}(\bar{b})$;
 (ii) $c_y = a_y$, if $y \in \mathrm{dom}(\bar{a}) \setminus \mathrm{dom}(\bar{b})$;
 (iii) $c_y = b_y$, if $y \in \mathrm{dom}(\bar{b}) \setminus \mathrm{dom}(\bar{a})$;

(f) if $\bar{g} \in I^{\mathfrak{m}}$, $Z_1 \subseteq_\omega \mathrm{dom}(f_{\bar{g}})$, $Z_2 = f_{\bar{g}}[Z_1]$ and $\bar{a} = (a_y : y \in Z_2) \in \mathcal{A}_{\mathfrak{s}}$, then

$$\bar{a}^{[f_{\bar{g}}]} = (a_{f_{\bar{g}}(y)} : y \in Z_1) \in \mathcal{A}_{\mathfrak{s}};$$

(g) $\mathcal{A}_{\mathfrak{s}}$ is the minimal subset of $\mathcal{A}_{\mathfrak{m}}$ satisfying clauses (c)–(f).

As mentioned above, members in $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$ are to be thought of as approximations to objects in $\mathrm{K}^{\mathrm{bo}}_2(M)$, but technically an $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$ and an $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_2(M)$ are made of different components, so we give a name to the objects in $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$ which are essentially members of $\mathrm{K}^{\mathrm{bo}}_2(M)$. We call them full; see 3.6.

Definition 3.6. For $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$, we say that $\mathfrak{m}$ is full when in addition to (1)–(11), condition 3.4(9) is satisfied and 3.5(4) is strengthen to 3.4(4) (that is, we ask $I = \mathcal{G}_*$). Explicitly to (1)–(11) from 3.5 we add

(12) if for every $n < \omega$, $g_n \in \mathcal{G}$ and $g_n \subsetneq g_{n+1}$, $\mathcal{U} = \bigcup_{n<\omega} \mathrm{dom}(g_n) \subseteq M$ and $\mathcal{V} = \bigcup_{n<\omega} \mathrm{ran}(g_n) \subseteq M$, then we have the following:

$$\bigcup_{n<\omega} \mathrm{dom}(f_{(g_\ell : \ell < n)}) = X_{\mathcal{U}} \text{ and } \bigcup_{n<\omega} \mathrm{ran}(f_{(g_\ell : \ell < n)}) = X_{\mathcal{V}};$$

(13) $I = \bigcup_{n<\omega} I_n = \mathcal{G}_*$.

We shall concentrate on the $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$ which are, in some sense, with "finite information," i.e., the ones in which both $Y_{\mathfrak{m}}$ and $I^{\mathfrak{m}}$ are finite. Furthermore, we will define a notion of "$\mathfrak{n}$ is a successor of $\mathfrak{m}$." These notions are tailor made for our inductive construction of a full $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$ to take place.

Definition 3.7.

(1) $\mathrm{K}^{\mathrm{bo}}_0(M)$ is the class of $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$ such that $Y_{\mathfrak{m}}$ is finite, and for some $n < \omega$, we have that for every $m \geqslant n$, $I_m = \emptyset$ and $I_0 = \{()\}$. In this case we let $n = n(\mathfrak{m})$ to be the minimal such $n < \omega$ (so $n(\mathfrak{m}) > 0$).

(2) We say that $\mathfrak{n} \in \mathrm{suc}(\mathfrak{m})$ when
   (a) $\mathfrak{n}, \mathfrak{m} \in \mathrm{K}_0^{\mathrm{bo}}(M)$, $X^{\mathfrak{m}} = X^{\mathfrak{n}}$;
   (b) for $s \subseteq_1 M$, $(X'_s)^{\mathfrak{m}} = (X'_s)^{\mathfrak{n}}$;
   (c) for $t \subseteq_\omega M$, $(X_t)^{\mathfrak{m}} = (X_t)^{\mathfrak{n}}$ (follows);
   (d) $n(\mathfrak{n}) = n + 1$, where $n(\mathfrak{m}) = n$;
   (e) if $\ell < n(\mathfrak{m})$, then $I_\ell^{\mathfrak{m}} = I_\ell^{\mathfrak{n}}$ and $\bigwedge_{\bar{g} \in I_\ell^{\mathfrak{m}}} f_{\bar{g}}^{\mathfrak{m}} = f_{\bar{g}}^{\mathfrak{n}}$;
   (f) for some $\bar{g} \in \mathcal{G}_*$, $I_n^{\mathfrak{n}} = \{\bar{g}, \bar{g}^{-1}\}$, $lg(\bar{g}) \leqslant n$ and $\ell < \lg(\bar{g})$ implies

$$\bar{g} \restriction \ell \in \bigcup_{\ell < n} I_\ell^{\mathfrak{m}};$$

   notice that $\bar{g} \notin \bigcup_{\ell<n} I_\ell^{\mathfrak{m}}$ (by Definition 3.5(4a)) and the symmetric condition $\bar{g}^{-1} \restriction \ell \in \bigcup_{\ell<n} I_\ell^{\mathfrak{m}}$ follows from Definition 3.5(7d);
   (g) (α) if $\bar{x} E_k^{\mathfrak{n}} \bar{y}$ and $\neg(\bar{x} E_k^{\mathfrak{m}} \bar{y})$, then $\bar{x} \notin \mathrm{seq}_k(\mathfrak{m})$ or $\bar{y} \notin \mathrm{seq}_k(\mathfrak{m})$;
   (β) $E_k^{\mathfrak{n}} \restriction \mathrm{seq}_k(\mathfrak{m}) = E_k^{\mathfrak{m}} \restriction \mathrm{seq}_k(\mathfrak{m})$.

(3) $<_{\mathrm{suc}}$ on $\mathrm{K}_0^{\mathrm{bo}}(M)$ is the transitive closure of the relation $\mathfrak{n} \in \mathrm{suc}(\mathfrak{m})$.

The heart of this section is the following claim.

Claim 3.8. For M as in 3.2, there exists $\mathfrak{m} \in \mathrm{K}_1^{\mathrm{bo}}(M)$ which is full.

Proof. Our strategy is to construct a full $\mathfrak{m} \in \mathrm{K}_1^{\mathrm{bo}}(M)$ as a limit of members $\mathfrak{m}_\ell \in \mathrm{K}_0^{\mathrm{bo}}(M)$ for $\ell < \omega$. Naturally, $\mathfrak{m}_0$ is not hard to choose; see $(*)_1$ below. Concerning the choice of the $\mathfrak{m}_\ell$'s, in $(*)_3$ below we list our tasks: for every $\bar{g} \in \mathcal{G}^*$, we have a $\bar{g}$-task which is ensuring that $f_{\bar{g}}$ is well defined. Thus, we list $\mathcal{G}^*$ as $(\bar{g}_\ell : \ell < \omega)$ appropriately and in choosing $\mathfrak{m}_{\ell+1}$, a successor of $\mathfrak{m}_\ell$, we take care of the $\bar{g}_\ell$-task. This lead us to the main part of the proof, namely $(*)_2$. Here we are given $\mathfrak{m}$ and appropriate $\bar{g}^\frown(g) \in \mathcal{G}^*$ such that $\bar{g} \in I_{\mathfrak{m}}$, i.e., $f_{\bar{g}}$ is already well defined for $\mathfrak{m}$. Our aim is to define a suitable successor $\mathfrak{n}$ of $\mathfrak{m}$ and, in particular, to define $f_{\bar{g}^\frown(g)}$ for $\mathfrak{n}$. Moreover, to take care of the fullness of the limit we want both $\mathrm{dom}(f_{\bar{g}^\frown(g)})$ and $Y_{\mathfrak{m}}$ to be large enough. This explains the statement of $(*)_2$.

$(*)_1$ $\mathrm{K}_0^{\mathrm{bo}}(M) \neq \emptyset$.

[Why? Let $\mathfrak{m}$ be such that

(a) $|X| = \aleph_0$ and $X \subseteq \omega$;
(b) $(X'_s : s \subseteq_1 M)$ is a partition of X into infinite sets;
(c) for $s \subseteq_\omega M$, $X_s = \bigcup_{t \subseteq_1 s} X'_t$;
(d) $\bar{X} = (X_s : s \subseteq_\omega M)$;
(e) $I_0^{\mathfrak{m}} = \{()\}$, $f_{()}$ is the empty function, $\bar{f} = (f_{()})$ and $I_{1+n} = \emptyset$ for every $n < \omega$;
(f) $Y_{\mathfrak{m}}$ is any finite non-empty subset of X.

Note that $()$ denotes the empty sequence and under this choice of $\mathfrak{m}$, $n(\mathfrak{m}) = 1$, where we recall that the notation $n(\mathfrak{m})$ was introduced in Definition 3.7(1).

Notice also that 3.5(11) is easy to verify for $\mathfrak{m}$ as above, as $\bar{x}/E_k^{\mathfrak{m}}$ is always a singleton.]

$(*)_2$ If $\mathfrak{m} \in \mathrm{K}_0^{\mathrm{bo}}(M)$, $n = n(\mathfrak{m}) > 0$, $\bar{g} = (g_0, \ldots, g_{m-1}) \in I^{\mathfrak{m}}$ (so $n > m$) and
  (i) $g \in \mathcal{G}$;
  (ii) for every $\ell < m$, $g_\ell \subsetneq g$;
  (iii) $\bar{g}^\frown(g) \notin I^{\mathfrak{m}}$;

  then there is $\mathfrak{n} \in \mathrm{K}_0^{\mathrm{bo}}(M)$ such that
  (a) $\mathfrak{n} \in \mathrm{suc}(\mathfrak{m})$;
  (b) $\bar{g}^\frown(g) \in I_n^{\mathfrak{n}}$;
  (c) if $s \subseteq_1 s^+ = \mathrm{dom}(g) \cup \mathrm{ran}(g)$, then $Y_{\mathfrak{n}}$ contains $\min(X'_s \setminus Y_{\mathfrak{m}})$;
  (d) $\mathrm{dom}(f^{\mathfrak{n}}_{\bar{g}^\frown(g)}) = Y_{\mathfrak{m}} \cap X_{\mathrm{dom}(g)}$;
  (e) so $n(\mathfrak{n}) = n(\mathfrak{m}) + 1$.

The proof of $(*)_2$ is clearly the heart of the proof. The choice of $\mathfrak{n}$ in $(*)_{2.3}$ below is natural: we choose $f^{\mathfrak{n}}_{\bar{g}^\frown(g)} = f_*$ "freely," i.e., it extends $f^{\mathfrak{m}}_{\bar{g}}$, it has large enough domain and no "accidental equality" holds. Lastly, $Y_{\mathfrak{n}}$ has to include $Y_{\mathfrak{m}}$, $\mathrm{ran}(f_*)$ and witnesses toward the proof of fullness (cf. $(*)_2$(c)), which will be dealt with in the next successor step, so we are making sure that the induction goes on.

We thus move to the proof of $(*)_2$, where we let $f^{\mathfrak{m}}_{\bar{g}} = f_{\bar{g}}$.

$(*)_{2.1}$ Let $s_* = \mathrm{dom}(g) \subseteq_\omega M$, hence $\mathrm{dom}(\bar{g}) \subsetneq s_*$, and let $u_* = Y_{\mathfrak{m}} \cap X_{s_*}$.

$(*)_{2.2}$ Let f_* be a finite permutation of X satisfying the following:
  (a) f_* obeys 3.5(7a)–(7c) for $\bar{g}^\frown(g)$ and $\mathrm{dom}(f_*) = u_*$;
  (b) f_* extends $f_{\bar{g}}$;
  (c) $\mathrm{dom}(f_*) \cap \mathrm{ran}(f_*) = \mathrm{ran}(f_{\bar{g}})$;
  (d) if $x \in \mathrm{dom}(f_*) \setminus \mathrm{dom}(f_{\bar{g}})$, then $f_*(x) \notin Y_{\mathfrak{m}}$ (so $f_*(x) \notin \mathrm{dom}(f_*)$).

We now define $\mathfrak{n}$, as required in $(*)_2$.

$(*)_{2.3}$ (A) (a) $X^{\mathfrak{n}} = X^{\mathfrak{m}}$ and $\bar{X}^{\mathfrak{n}} = \bar{X}^{\mathfrak{m}}$;
  (b) $I_n^{\mathfrak{n}} = \{\bar{g}^\frown(g), (\bar{g}^{-1})^\frown(g^{-1})\}$;
  (c) $I^{\mathfrak{n}} = I^{\mathfrak{m}} \cup I_n^{\mathfrak{n}}$;
  (d) $I_\ell^{\mathfrak{n}} = I_\ell^{\mathfrak{m}}$ for $\ell \neq n$;
  (e) $f^{\mathfrak{n}}_{\bar{h}} = f^{\mathfrak{m}}_{\bar{h}}$ for $\bar{h} \in I^{\mathfrak{m}}$.

  (B) (a) $n(\mathfrak{n}) = n + 1$;
  (b) $f^{\mathfrak{n}}_{\bar{g}^\frown(g)} = f_*$, $f^{\mathfrak{n}}_{(\bar{g}^{-1})^\frown(g^{-1})} = f_*^{-1}$;
  (c) $Y_{\mathfrak{n}} = Z \cup Z^+$, where (noticing $f_*[Y_{\mathfrak{m}}] = \mathrm{ran}(f_*)$)
    $(\cdot_1)$ $Z = Y_{\mathfrak{m}} \cup f_*[Y_{\mathfrak{m}}]$;
    $(\cdot_2)$ $Z^+ = \{\min(X'_s \setminus Y_{\mathfrak{m}}) : s \subseteq_1 s^+\} \setminus Z$, recalling $(*)_2$(c).

The reason for Z^+ in (B)(c) above it to satisfy condition $(*)_2$(c).

$(*)_{2.3.1}$ $R_k^{\mathfrak{n}}$ and $E_k^{\mathfrak{n}}$ are defined from the information in $(*)_{2.3}$, as in 3.5(9).

Comparing $(\mathrm{seq}_k(X), R_k^{\mathfrak{n}})$ and $(\mathrm{seq}_k(X), R_k^{\mathfrak{m}})$, the set of new edges is

$$\{(\bar{x}, \bar{y}) : (\bar{x}, \bar{y}) \in Z_1^k \cup Z_{-1}^k\},$$

where we let

$(*)_{2.4}$ $Z_1^k = \{(\bar{x}, \bar{y}) : \bar{x} \in \mathrm{seq}_k(\mathrm{dom}(f_*)), f_*(\bar{x}) = \bar{y}, \bar{x} \notin \mathrm{seq}_k(\mathrm{dom}(f_{\bar{g}}))\}$,
$Z_{-1}^k = \{(\bar{x}, \bar{y}) : (\bar{y}, \bar{x}) \in Z_1^k\}$.

Notice that possibly $\bar{x} \subseteq \mathrm{dom}(f_*) \wedge \bar{x} \notin \mathrm{seq}_k(\mathfrak{m})$, and possibly $\bar{x} \subseteq \mathrm{dom}(f_*) \wedge \bar{x} \not\subseteq \mathrm{dom}(f_{\bar{g}}^{\mathfrak{m}}) \wedge \bar{x} \in \mathrm{seq}_k(\mathfrak{m})$ (as witnessed by some $\bar{g}' \in I_{<n}^{\mathfrak{m}}$). Anyhow the union $Z_1^k \cup Z_{-1}^k$ is disjoint, as $\mathrm{dom}(f_*) = u_*$ by $(*)_{2.2}$(a), $u_* \subseteq Y_{\mathfrak{m}}$ by $(*)_{2.1}$, and $x \in \mathrm{dom}(f_*) \setminus \mathrm{dom}(f_{\bar{g}})$ implies $f_*(x) \notin Y_{\mathfrak{m}}$ by $(*)_{2.2}$(d). Notice now that

$(*)_{2.4.1}$ if $\bar{x} \in \mathrm{seq}_k(u_*)$ and $\bar{y} = f_*(\bar{x})$, then

$$\bar{x} \subseteq \mathrm{dom}(f_{\bar{g}}) \Leftrightarrow \bar{y} \subseteq \mathrm{ran}(f_{\bar{g}}) \Rightarrow (\bar{x} \in \mathrm{seq}_k(\mathfrak{m}) \wedge \bar{y} \in \mathrm{seq}_k(\mathfrak{m})).$$

Now, we have

$(*)_{2.5}$ (a) if $(\bar{x}, \bar{y}) \in Z_1^k$, then
 (α) $\bar{x} \in \mathrm{seq}_k(u_*)$ and $\bar{x} \not\subseteq \mathrm{dom}(f_{\bar{g}})$;
 (β) $\bar{y} \subseteq f_*(u_*)$, $\bar{y} \not\subseteq Y_{\mathfrak{m}}$, $\bar{y} \not\subseteq \mathrm{ran}(f_{\bar{g}})$ and $\bar{y} \cap Y_{\mathfrak{m}} \subseteq \mathrm{ran}(f_{\bar{g}})$;
(b) the dual of item (a) for $(\bar{x}, \bar{y}) \in Z_{-1}^k$;
(c) if $\bar{z} \in \mathrm{seq}_k(\mathfrak{n}) \setminus \mathrm{seq}_k(\mathfrak{m})$, then $\bar{z}$ occurs in exactly one edge of $R_k^{\mathfrak{n}}$.

[Why? Item (a)(β) is by $(*)_{2.2}$(d). Item (c) is by $(*)_{2.2}$(c).]

Notice now that

$(*)_{2.6}$ in the graph $(\mathrm{seq}_k(X), R_k^{\mathfrak{n}})$, we have (where $\bar{x} \in \mathrm{seq}_k(X)$ below)
(i) all the new edges have at least one node in $\mathrm{seq}_k(u_*) \setminus \mathrm{seq}_k(\mathrm{dom}(f_{\bar{g}}))$ and one in $\mathrm{seq}_k(f_*[u_*]) \setminus \mathrm{seq}_k(\mathrm{ran}(f_{\bar{g}})) = \mathrm{seq}_k(f_*[u_*]) \setminus \mathrm{seq}_k(Y_{\mathfrak{m}})$;
(ii) every node in $\mathrm{seq}_k(\mathfrak{n}) \setminus \mathrm{seq}_k(Y_{\mathfrak{m}})$ has valency 1;
(iii) if $\bar{x} \not\subseteq Y_{\mathfrak{m}}$ and $\bar{x} \not\subseteq \mathrm{ran}(f_*)$, then $\bar{x}/E_k^{\mathfrak{n}} = \{\bar{x}\}$;
(iv) if $\bar{x} \subseteq Y_{\mathfrak{m}}$ and $\bar{x} \not\subseteq \mathrm{dom}(f_*)$, then $\bar{x}/E_k^{\mathfrak{n}} = \bar{x}/E_k^{\mathfrak{m}}$;
(v) if $\bar{x} \subseteq \mathrm{dom}(f_*)$ (hence $\bar{x} \subseteq Y_{\mathfrak{m}}$), then

$$\bar{x}/E_k^{\mathfrak{n}} = \bar{x}/E_k^{\mathfrak{m}} \cup \{f_*(\bar{y}) : \bar{y} \in \bar{x}/E_k^{\mathfrak{m}}, \bar{y} \subseteq u_*, \bar{y} \not\subseteq \mathrm{dom}(f_{\bar{g}})\};$$

(vi) if $\bar{x} \subseteq \mathrm{dom}(f_{\bar{g}})$ and $\bar{x}/E_k^{\mathfrak{m}} \cap \mathrm{seq}_k(u_*) \subseteq \mathrm{seq}_k(\mathrm{dom}(f_{\bar{g}}))$, then

$$\bar{x}/E_k^{\mathfrak{n}} = \bar{x}/E_k^{\mathfrak{m}} = f_*(\bar{x})/E_k^{\mathfrak{m}};$$

(vii) if $\bar{x} \not\subseteq Y_{\mathfrak{m}}$ but $\bar{x} \subseteq f_*(u_*)$, then $\bar{x}/E_k^{\mathfrak{n}} = f_*^{-1}(\bar{x})/E_k^{\mathfrak{n}}$;
(viii) if $\bar{x} \in \mathrm{seq}_k(Y_{\mathfrak{m}})$, then

$$(\bar{x}/E_k^{\mathfrak{n}}) \cap \mathrm{seq}_k(Y_{\mathfrak{m}}) = (\bar{x}/E_k^{\mathfrak{m}}) \cap \mathrm{seq}_k(Y_{\mathfrak{m}}).$$

Notice also that

$(*)_{2.6.1}$ (a) if $\bar{x}_0, \ldots, \bar{x}_m$ is a path in $(\mathrm{seq}_k(\mathfrak{n}), R_k^{\mathfrak{n}})$ with no repetitions and $0 < \ell < m$, then $\bar{x}_\ell \in \mathrm{seq}_k(\mathfrak{m})$;
(b) $E_k^{\mathfrak{n}} \restriction \mathrm{seq}_k(\mathfrak{m}) = E_k^{\mathfrak{m}} \restriction \mathrm{seq}_k(\mathfrak{m})$ and $E_k^{\mathfrak{n}} \restriction \mathrm{seq}_k(Y_{\mathfrak{m}}) = E_k^{\mathfrak{m}} \restriction \mathrm{seq}_k(Y_{\mathfrak{m}})$.

Now, we claim

$(*)_{2.7}$ $\mathfrak{n} \in \mathrm{K}_0^{\mathrm{bo}}(M)$ and $\mathfrak{n} \in \mathrm{suc}(\mathfrak{m})$.

The only non-trivial thing is to verify that $\mathfrak{n}$ satisfies 3.5(11). In principle, verifying that this holds should be straightforward. As $\mathfrak{n}$ is explicitly defined in an essentially free manner, we should be able to check the algebraic condition 3.5(11). In actuality, though, verifying 3.5(11) would require an explicit description of $\mathcal{A}^{\mathfrak{n}}_{\mathfrak{s}}$. We circumvent this by explicitly defining an $\mathcal{A}'$ such that $\mathcal{A}^{\mathfrak{n}}_{\mathfrak{s}} \subseteq \mathcal{A}'$ (cf. $(*)_{2.7.5}$) and such that $\mathcal{A}'$ satisfies the crucial condition that each $\bar{a} \in \mathcal{A}'$ has non-singleton p-support (cf. $(*)_{2.7.6}$). Notice that in order to show that $\mathcal{A}^{\mathfrak{n}}_{\mathfrak{s}} \subseteq \mathcal{A}'$, it suffices to show that $\mathcal{A}'$ satisfies the minimal set of condition defining $\mathcal{A}^{\mathfrak{n}}_{\mathfrak{s}}$, as defined in 3.5(11), and so it is not hard to achieve, although the proof requires careful checking. Also the proof $(*)_{2.7.6}$ is in principle not hard but it involves a careful checking of many cases.

We thus move to the proof of 3.5(11). To this extent,

$(*)_{2.7.0}$ Let $\mathfrak{s} = (p, k, \bar{x}, \bar{q})$ be as in 3.5(11).

Now, if $\bar{x} \notin \mathrm{seq}_k(Y_{\mathfrak{m}})$ and $\bar{x} \notin \mathrm{seq}_k(\mathrm{ran}(f_*))$, then $\bar{x}/E^{\mathfrak{n}}_k$ is a singleton and so the proof is as in $(*)_1$. Thus, from now on we assume

$(*)_{2.7.1}$ Without loss of generality, $\bar{x} \in \mathrm{seq}_k(Y_{\mathfrak{m}})$ or $\bar{x} \in \mathrm{seq}_k(\mathrm{ran}(f_*))$.

$(*)_{2.7.2}$ (a) Without loss of generality, $\bar{x} \in \mathrm{seq}_k(Y_{\mathfrak{m}})$;
(b) let $\mathfrak{s}$ be is as in 3.5(11) for $\mathfrak{m}$ and $\bar{x}$;
(c) so $\mathcal{A}^{\mathfrak{m}}_{\mathfrak{s}}$ is well defined.

[Why (a)? If $\bar{x} \not\subseteq Y_{\mathfrak{m}}$, then, by $(*)_{2.7.1}$, necessarily $\bar{x} \subseteq \mathrm{ran}(f_*)$, so $f_*^{-1}(\bar{x}) \in \bar{x}/E^{\mathfrak{n}}_k$ and $f_*^{-1}(\bar{x}) \subseteq Y_{\mathfrak{m}}$. By $(*)_{2.6}$(vii), we can replace $\bar{x}$ by $f_*^{-1}(\bar{x})$; (b), (c) are clear.]

$(*)_{2.7.3}$ (a) $\mathcal{A}^{\mathfrak{m}}_{\mathfrak{s}} \subseteq \mathcal{A}^{\mathfrak{n}}_{\mathfrak{s}}$, let $\mathcal{A}^1_{\mathfrak{s}} = \mathcal{A}^{\mathfrak{m}}_{\mathfrak{s}}$, recalling 3.5(11);
(b) let $\mathcal{A}^2_{\mathfrak{s}} = \{\bar{b}^{[f_*^{-1}]} : \bar{b} \in \mathcal{A}^1_{\mathfrak{s}} \text{ and } \mathrm{dom}(\bar{b}) \subseteq \mathrm{dom}(f_*)\}$, where for $\bar{b} = (b_y : y \in Z_1)$ with $Z_1 \subseteq \mathrm{dom}(f_*)$ and $Z_2 = f_*[Z_1]$, we let
$$\bar{b}^{[f_*^{-1}]} = (b_{f_*^{-1}(y)} : y \in Z_2);$$
(c) $\mathcal{A}^2_{\mathfrak{s}} \subseteq \{\bar{b} \in \mathcal{A}^2_{\mathfrak{s}} : \mathrm{dom}(\bar{b}) \subseteq \mathrm{ran}(f_*)\}$;
(d) recalling 3.5(11)(f), notice that for any function h such that $\bar{b}^{[h]}$ is well defined, we have that if $\bar{b}^{[h]} = \bar{c}$, then the following happens:
$$\mathrm{dom}(\bar{b}) \subseteq \mathrm{ran}(h) \text{ and } \mathrm{dom}(\bar{c}) \subseteq \mathrm{dom}(h).$$

$(*)_{2.7.4}$ Let $\mathcal{A}'$ be the set of $\bar{a}$ such that for some $\bar{a}_1 \in \mathcal{A}^1_{\mathfrak{s}}$, $\bar{a}_2 \in \mathcal{A}^2_{\mathfrak{s}}$ and u such that $\mathrm{supp}_p(\bar{a}_1 + \bar{a}_2) \subseteq u \subseteq \mathrm{dom}(\bar{a}_1) \cup \mathrm{dom}(\bar{a}_2)$, we have that $(\bar{a}_1 + \bar{a}_2) \restriction u = \bar{a}$. In this case we call $(\bar{a}_1, \bar{a}_2, u)$ a witness for $\bar{a}$.

Now we crucially claim

$(*)_{2.7.5}$ $\mathcal{A}^{\mathfrak{n}}_{\mathfrak{s}} \subseteq \mathcal{A}'$.

Why $(*)_{2.7.5}$? Obviously $\mathcal{A}'$ satisfies 3.5(11)(a) and 3.5(11)(b) is a definition. By 3.5(11)(g) it suffices to prove that $\mathcal{A}'$ satisfies (c)–(f) from 3.5(11).

$(*)_{2.7.5.1}$ $\mathcal{A}'$ satisfies Clause 3.5(11)(c).

Let $\mathbf{y} = (\bar{y}^i : i < i_*) \in (\bar{x}/E^{\mathfrak{n}}_k)^{i_*}$, $\bar{r} \in \mathbb{Q}^{\mathbf{y}}$ and $\bar{a} = \bar{a}_{(\mathbf{y},\bar{r})}$ be as in the assumptions of Clause 3.5(11)(c). Recall that abusing notation we treat $\mathbf{y}$ as a set. Let

$$\mathbf{y}_1 = \{\bar{y}^i : i < i_*, \bar{y}^i \subseteq Y_{\mathfrak{m}}\},$$
$$\mathbf{y}_2 = \{\bar{y}^i : i < i_*, \bar{y}^i \not\subseteq Y_{\mathfrak{m}} \text{ (so } \bar{y}^i \subseteq \mathrm{ran}(f_*))\}.$$

Easily we have that $\mathbf{y}$ is the disjoint union of $\mathbf{y}_1$ and $\mathbf{y}_2$, and we have

$$\bar{a}_{(\mathbf{y},\bar{r})} = \bar{a}_{(\mathbf{y}_1,\bar{r}\restriction\mathbf{y}_1)} + \bar{a}_{(\mathbf{y}_2,\bar{r}\restriction\mathbf{y}_2)},$$

provided that we show that $\bar{a}_2 = \bar{a}_{(\mathbf{y}_2,\bar{r}\restriction\mathbf{y}_2)} \in \mathcal{A}^2_{\mathfrak{s}}$ (as $\bar{a}_1 = \bar{a}_{(\mathbf{y}_1,\bar{r}\restriction\mathbf{y}_1)} \in \mathcal{A}^1_{\mathfrak{s}}$ is obvious by $(*)_{2.6}$(viii)). We do this. Let $\mathbf{y}'_2 = \{f^{-1}_*(\bar{y}) : \bar{y} \in \mathbf{y}_2\}$. Now, if $\bar{y} \in \mathbf{y}_2$, then $f^{-1}_*(\bar{y}) = f^{-1}_{\bar{g}^\frown(g)}(\bar{y}) \in \bar{x}/E^{\mathfrak{n}}_k \cap \mathrm{seq}_k(Y_{\mathfrak{m}}) \subseteq \bar{x}/E^{\mathfrak{m}}_k$. Why? First of all $f^{-1}_*(\bar{y}) = f^{-1}_{\bar{g}^\frown(g)}(\bar{y})$, by the choice of $\bar{g}^\frown(g)$. Secondly, $f^{-1}_{\bar{g}^\frown(g)}(\bar{y}) \in \bar{x}/E^{\mathfrak{n}}_k$ as $\bar{y} \subseteq f_*[u_*]$, $\bar{y} \notin Y_{\mathfrak{m}}$ and $\bar{y}/E^{\mathfrak{n}}_k = f^{-1}_*(\bar{y})/E^{\mathfrak{n}}_k$, by $(*)_{2.6}$(vii). Thirdly, $f^{-1}_{\bar{g}^\frown(g)}(\bar{y}) \in Y_{\mathfrak{m}}$, by the choice of $f_{\bar{g}^\frown(g)}$. Thus, $f^{-1}_{\bar{g}^\frown(g)}(\bar{y}) \in \bar{x}/E^{\mathfrak{n}}_k \cap Y_{\mathfrak{m}}$, and, by $(*)_{2.6}$(viii) we have that $\bar{x}/E^{\mathfrak{n}}_k \cap Y_{\mathfrak{m}} \subseteq \bar{x}/E^{\mathfrak{m}}_k$. Now let $\bar{r}'_2 = (r'_{(2,\bar{y})} : \bar{y} \in \mathbf{y}'_2)$, where $r'_{(2,\bar{y})} = r_{(2,f_*(\bar{y}))}$. Also, let $\bar{a}'_2 = (a'_{(2,y)} : y \in \mathrm{set}(\mathbf{y}'_2))$, where for $y \in \mathrm{set}(\mathbf{y}'_2)$, we let

$$a'_{(2,y)} = \sum\{r'_{(2,\bar{y})} q_\ell : \bar{y} \in \mathbf{y}'_2 \text{ and } y_\ell = y\}.$$

As $\mathbf{y}'_2 \subseteq \bar{x}/E^{\mathfrak{m}}_k$ and $\mathfrak{m}$ satisfies 3.5(11)(c), easily $\bar{a}'_2 \in \mathcal{A}^{\mathfrak{m}}_{\mathfrak{s}} = \mathcal{A}^1_{\mathfrak{s}}$. Also, easily $y \in \mathrm{set}(\mathbf{y}'_2)$ implies $a'_{(2,y)} = a_{(2,f_*(y))}$ (recall that $r'_{(2,\bar{y})} = r_{(2,f_*(\bar{y}))}$) and so $(\bar{a}'_2)^{[f^{-1}_*]} = \bar{a}_2$. Thus, $\bar{a}_2 \in \mathcal{A}^2_{\mathfrak{s}}$. Now, $\mathbf{y}'_2, \bar{r}'_2$ witness that $\bar{a}'_2 \in \mathcal{A}^1_{\mathfrak{s}}$ and so by the definition of $\mathcal{A}^2_{\mathfrak{s}}$ we are done. This concludes the proof of $(*)_{2.7.5.1}$.

$(*)_{2.7.5.2}$ $\mathcal{A}'$ satisfies Clause 3.5(11)(d).

This is obvious by the definition of $\mathcal{A}'$.

$(*)_{2.7.5.3}$ $\mathcal{A}'$ satisfies Clause 3.5(11)(e).

Let $\bar{a}, \bar{b} \in \mathcal{A}'$, and let $(\bar{a}_1, \bar{a}_2, u)$ be a witness for $\bar{a}$ and $(\bar{b}_1, \bar{b}_2, v)$ be a witness for $\bar{b}$, now $(\bar{a}_1 + \bar{a}_2, \bar{b}_1 + \bar{b}_2, u \cup v)$ is a witness for $\bar{a} + \bar{b}$. Hence, $\bar{c} = \bar{a} + \bar{b} \in \mathcal{A}'$.

$(*)_{2.7.5.4}$ $\mathcal{A}'$ satisfies Clause 3.5(11)(f).

Let $\bar{h} \in I^{\mathfrak{n}}$, $Z_1 \subseteq \mathrm{dom}(f_{\bar{h}})$, $Z_2 = f_{\bar{h}}[Z_1]$ and $\mathrm{dom}(\bar{a}) \subseteq Z_2$. We shall prove that $\bar{a}^{[f_{\bar{h}}]} \in \mathcal{A}'$, where $\bar{a} \in \mathcal{A}'$ and $(\bar{a}_1, \bar{a}_2, u)$ is a witness of this.

Case 1: $u \not\subseteq Y_{\mathfrak{m}}$ *and* $u \not\subseteq \mathrm{ran}(f_*)$. In this case there is no such $\bar{h}$.

Case 2: $u \not\subseteq Y_{\mathfrak{m}}$ *and* $u \subseteq \mathrm{ran}(f_*)$. Notice that $u \not\subseteq Y_{\mathfrak{m}}$, so there is $y \in u \setminus Y_{\mathfrak{m}}$. Now, $y \in u \subseteq \mathrm{dom}(f_*)$. But we have

$$\bar{h} \in I^{\mathfrak{m}} \Rightarrow \ \mathrm{dom}(f_{\bar{h}}) \subseteq Y_{\mathfrak{m}} \ \Rightarrow \ y \notin \mathrm{dom}(f_{\bar{h}}),$$
$$\bar{h} = \bar{g}^\frown(g) \Rightarrow \ \mathrm{dom}(f_{\bar{h}}) = u_* \Rightarrow \ y \notin \mathrm{dom}(f_{\bar{h}}),$$

so necessarily $\bar{h} = (\bar{g}^{-1})^\frown(g^{-1})$ and $f_{\bar{h}} = f_*^{-1}$. Now,
$(\cdot)$ Without loss of generality, $\mathrm{dom}(\bar{a}_1) \subseteq \mathrm{ran}(f_{\bar{g}})$.
[Why? If $z \in \mathrm{dom}(\bar{a}_1) \setminus \mathrm{ran}(f_{\bar{g}})$, then $z \notin u$ so $z \notin \mathrm{supp}_p(\bar{a})$ and $z \notin \mathrm{dom}(\bar{a}_2)$, hence $a_z = a_{(1,z)} \in \mathbb{Q}_p$. Thus, $\bar{a}_1^* = \bar{a}_1 \restriction (\mathrm{dom}(\bar{a}_1) \cap \mathrm{ran}(f_{\bar{g}})) \in \mathcal{A}^1_{\mathfrak{s}}$ and $(\bar{a}_1 + \bar{a}_2) \restriction u = (\bar{a}_1^* + \bar{a}_2) \restriction u$, so we can replace $\bar{a}_1$ by $\bar{a}_1^*$, as $\mathfrak{m}$ satisfies clause (f).]

Let $\bar{a}_1' = \bar{a}_1^{[f_{\bar{g}}]} = \bar{a}_1^{[f_*]}$; this is well defined. I belongs to $\mathcal{A}^1_{\mathfrak{s}}$ and has domain $\subseteq \mathrm{dom}(f_*)$. Also, $\mathrm{dom}(\bar{a}_2) \subseteq \mathrm{ran}(f_*)$ and $\bar{a}_2 \in \mathcal{A}^2_{\mathfrak{s}}$, hence $\bar{a}_2' = a_2^{[f_*]} \in \mathcal{A}^1_{\mathfrak{s}}$ and it has domain $\subseteq \mathrm{dom}(f_*)$. By 3.5(11)(e) and the above we have that $\bar{a}' = \bar{a}_1' + \bar{a}_2' \in \mathcal{A}^1_{\mathfrak{s}}$. Also, $\mathrm{supp}_p(\bar{a}') \subseteq f_*^{-1}[u] \subseteq \mathrm{dom}(\bar{a}_1' + \bar{a}_2')$, hence $\bar{a}' \restriction f_*^{-1}[u] \in \mathcal{A}^1_{\mathfrak{s}}$. Thus,

$$\begin{aligned}
\bar{a}^{[f_{\bar{h}}]} &= \bar{a}^{[f_*]} \\
&= ((\bar{a}_1 + \bar{a}_2) \restriction u)^{[f_*^{-1}]} \\
&= (\bar{a}_1 + \bar{a}_2)^{[f_*]} \restriction f_*^{-1}[u] \\
&= (\bar{a}_1^{[f_*]} + \bar{a}_2^{[f_*]}) \restriction f_*^{-1}[u] \\
&= (\bar{a}_1' + \bar{a}_2') \restriction f_*^{-1}[u] \\
&= \bar{a}' \restriction f_*^{-1}[u] \in \mathcal{A}^1_{\mathfrak{s}}.
\end{aligned}$$

Case 3: $u \subseteq Y_{\mathfrak{m}}$ *and* $\bar{h} = f^{\mathfrak{n}}_{\bar{g}^\frown(g)} = f_*$. In this case we have
$(\cdot)$ Without loss of generality, $\mathrm{dom}(\bar{a}_2) \subseteq \mathrm{ran}(f_{\bar{g}})$.
[Why? If $y \in \mathrm{dom}(\bar{a}_2) \setminus \mathrm{ran}(f_{\bar{g}})$, then (recalling $\mathrm{dom}(\bar{a}_2) \setminus \mathrm{ran}(f_{\bar{g}}) \subseteq f_*[u_*] \setminus \mathrm{ran}(f_{\bar{g}}) \subseteq f_*[u_*] \setminus u$) we have that $y \notin u$ so $y \notin \mathrm{supp}_p(\bar{a})$ and $y \notin \mathrm{dom}(\bar{a}_1)$, hence $a_y = a_{(2,y)} \in \mathbb{Q}_p$. Thus, by 3.5(11)(d), $\bar{a}_2^* = \bar{a}_2 \restriction (\mathrm{dom}(\bar{a}_2) \cap \mathrm{ran}(f_{\bar{g}})) \in \mathcal{A}^2_{\mathfrak{s}}$ and $(\bar{a}_1 + \bar{a}_2) \restriction u = (\bar{a}_1 + \bar{a}_2^*) \restriction u$, so we can replace $\bar{a}_2$ by $\bar{a}_2^*$, as $\mathfrak{m}$ satisfies clause (f).]

Let $\bar{a}_2' = \bar{a}_2^{[f_{\bar{g}}]} = \bar{a}_2^{[f_*]}$. This is well defined, and it belongs to $\mathcal{A}^1_{\mathfrak{s}}$ (by the definition of $\mathcal{A}^2_{\mathfrak{s}}$, recalling $\bar{a}_2 \in \mathcal{A}^2_{\mathfrak{s}}$). Also, $\bar{a}_2'$ has domain $\subseteq \mathrm{dom}(f_*)$. Now, as $\mathfrak{m} \in \mathrm{K}_0^{\mathrm{bo}}(M)$, $f_{\bar{g}} \in I^{\mathfrak{m}}$ and $\bar{a}_2' \in \mathcal{A}^1_{\mathfrak{s}} = \mathcal{A}^{\mathfrak{m}}_{\mathfrak{s}}$, recalling 3.5(11)(e), we have $\bar{a}_2 = (\bar{a}_2')^{[f_{\bar{g}}^{-1}]} \in \mathcal{A}^1_{\mathfrak{s}}$, so as $\mathfrak{m} \in \mathrm{K}_0^{\mathrm{bo}}(M)$, we have $\bar{a}_1 + \bar{a}_2 \in \mathcal{A}^1_{\mathfrak{s}}$. Thus, as $\mathcal{A}^1_{\mathfrak{s}} \subseteq \mathcal{A}'$, we are done.

Case 4: $u \subseteq Y_{\mathfrak{m}}$ *and* $\bar{h} \in I_{\mathfrak{m}}$. This is similar to Case 3.

Case 5: $u \subseteq Y_{\mathfrak{m}}$ *and* $\bar{h} = f^{\mathfrak{n}}_{(\bar{g}^{-1})^\frown(g^{-1})} = f_*^{-1}$. As $u \subseteq Y_{\mathfrak{m}}$ and $u \subseteq \mathrm{dom}(f^{\mathfrak{n}}_{\bar{h}}) = \mathrm{dom}(f_*^{-1}) = \mathrm{ran}(f_*)$, necessarily we have

$$u \subseteq Y_{\mathfrak{m}} \cap \mathrm{ran}(f_*) = \mathrm{ran}(f_{\bar{g}}) \ \ (\text{cf. } (*)_{2.2}(\text{c})).$$

As in earlier cases,

$(\cdot)$ Without loss of generality, $\mathrm{dom}(\bar{a}_2) \subseteq \mathrm{ran}(f_{\bar{g}})$.

So we can finish as in Case 4.

Thus, indeed $\mathcal{A}^{\mathfrak{n}}_{\mathfrak{s}} \subseteq \mathcal{A}'$, by (g) of the definition of $\mathcal{A}^{\mathfrak{n}}_{\mathfrak{s}}$ in 3.5(11) and $(*)_{2.7.5.1}$-$(*)_{2.7.5.4}$. Hence, we finished proving $(*)_{2.7.5}$.

$(*)_{2.7.6}$ If $\bar{a} \in \mathcal{A}'$, then $\mathrm{supp}_p(\bar{a})$ is not a singleton.

We prove $(*)_{2.7.6}$. Let $\bar{a} \in \mathcal{A}'$, and let $(\bar{a}_1, \bar{a}_2, u)$ be a witness of this. For $\ell = 1, 2$, let $\bar{a}'_\ell = \bar{a}_\ell \restriction \mathrm{supp}_p(\bar{a}_\ell)$. Then, by 3.5(11)(d), we have that $\bar{a}'_1 \in \mathcal{A}^1_{\mathfrak{s}} = \mathcal{A}^{\mathfrak{m}}_{\mathfrak{s}}$. Also, $\bar{a}^*_2 = \bar{a}_2^{[f_*]} \in \mathcal{A}^1_{\mathfrak{s}}$, by the definition of $\mathcal{A}^2_{\mathfrak{s}}$, and so, as $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_0(M)$, $\bar{a}^*_2 \restriction \{y \in X : f_*(y) \in \mathrm{dom}(\bar{a}'_2)\} \in \mathcal{A}^1_{\mathfrak{s}}$. Clearly we have the following:

$$\begin{aligned}\mathrm{dom}_p(a^*_2) &:= \{y \in \mathrm{dom}(\bar{a}^*_2) : \bar{a}^*_{(2,y)} \notin \mathbb{Q}_p\} \\ &= \{y \in \mathrm{dom}(a^*_2) : a_{(2,f_*(y))} = a_{(2,y)} \notin \mathbb{Q}_p\} \\ &= \{f_*(y) : y \in \mathrm{dom}_p(\bar{a}_2) = \{y \in \mathrm{dom}(\bar{a}^*_2) : \bar{a}_{(2,y)} \notin \mathbb{Q}_p\}\}.\end{aligned}$$

Thus, $\bar{a}^*_2 \restriction \mathrm{dom}_p(\bar{a}^*_2) = (\bar{a}'_2)^{[f_*]}$. As $\bar{a}^*_2 = \bar{a}_2^{[f_*]} \in \mathcal{A}^1_{\mathfrak{s}}$, recalling that $\mathcal{A}^1_{\mathfrak{s}} = \mathcal{A}^{\mathfrak{m}}_{\mathfrak{s}}$, by 3.5(11)(d), we have that $(\bar{a}'_2)^{[f_*]} = \bar{a}^*_2 \restriction \mathrm{dom}_p(\bar{a}^*_2) \in \mathcal{A}^{\mathfrak{m}}_{\mathfrak{s}}$. Hence, by the definition of $\mathcal{A}^2_{\mathfrak{s}}$ (as $\bar{a} = \bar{b}^{[f_*]}$ if and only if $\bar{b} = \bar{a}^{[f_*^{-1}]}$), $\bar{a}'_2 \in \mathcal{A}^2_{\mathfrak{s}}$. So we have

(a) if $y \in \mathrm{dom}(\bar{a}_1) \cap \mathrm{dom}(\bar{a}_2)$, then
  - $(\cdot)$ $y \notin \mathrm{supp}_p(\bar{a}_1)$ implies $y \in \mathrm{supp}_p(\bar{a}_1 + \bar{a}_2)$ if and only if $y \in \mathrm{supp}_p(\bar{a}_2)$;
  - $(\cdot)$ $y \notin \mathrm{supp}_p(\bar{a}_2)$ implies $y \in \mathrm{supp}_p(\bar{a}_1 + \bar{a}_2)$ if and only if $y \in \mathrm{supp}_p(\bar{a}_1)$;

(b) if $y \in \mathrm{dom}(\bar{a}_1) \setminus \mathrm{dom}(\bar{a}_2)$, then $y \in \mathrm{supp}_p(\bar{a}_1)$ if and only if $y \in \mathrm{supp}_p(\bar{a}_1 + \bar{a}_2)$;

(c) if $y \in \mathrm{dom}(\bar{a}_2) \setminus \mathrm{dom}(\bar{a}_1)$, then $y \in \mathrm{supp}_p(\bar{a}_2)$ if and only if $y \in \mathrm{supp}_p(\bar{a}_1 + \bar{a}_2)$.

Hence,

$(*)_{2.7.6.1}$ Without loss of generality, $\bar{a} = \bar{a}_1 + \bar{a}_2$ and $\bar{a}_\ell = \bar{a}_\ell \restriction \mathrm{supp}_p(\bar{a}_\ell)$ for $\ell = 1, 2$.

[Why? Letting $u' = \mathrm{dom}(\bar{a}'_1) \cup \mathrm{dom}(\bar{a}'_2)$, we have

(a) $u' \subseteq u$;

(b) $\mathrm{dom}(\bar{a}'_1), \mathrm{dom}(\bar{a}'_2) \subseteq u$;

(c) $\bar{a}'_1 + \bar{a}'_2 \restriction \mathrm{supp}_p(\bar{a}'_1 + \bar{a}'_2) = \bar{a}_1 + \bar{a}_2 \restriction \mathrm{supp}_p(\bar{a}_1 + \bar{a}_2)$.

So $(*)_{2.7.6.1}$ holds indeed.]

With $(*)_{2.7.6.1}$ in mind, we now get back to the proof of $(*)_{2.7.6}$.

Case A: $\mathrm{supp}_p(\bar{a}_1) \not\subseteq \mathrm{ran}(f_{\bar{g}})$ *and* $\mathrm{supp}_p(\bar{a}_2) \not\subseteq \mathrm{ran}(f_{\bar{g}})$. As $\mathrm{supp}_p(\bar{a}_1) \not\subseteq \mathrm{ran}(f_{\bar{g}})$, we can choose $y_1 \in \mathrm{supp}_p(\bar{a}_1) \setminus \mathrm{ran}(f_{\bar{g}})$, and similarly we can choose $y_2 \in \mathrm{supp}_p(\bar{a}_2) \setminus \mathrm{ran}(f_{\bar{g}})$. Now $\mathrm{dom}(\bar{a}_1) \subseteq Y_{\mathfrak{m}}$ and $\mathrm{dom}(\bar{a}_2) \subseteq f_*[Y_{\mathfrak{m}}]$, hence $\mathrm{dom}(\bar{a}_1) \cap \mathrm{dom}(\bar{a}_2) \subseteq Y_{\mathfrak{m}} \cap f_*[Y_{\mathfrak{m}}] = \mathrm{ran}(f_{\bar{g}})$ (recall $(*)_{2.2}$(c)), so necessarily $y_1 \notin \mathrm{dom}(\bar{a}_2)$ and $y_2 \notin \mathrm{dom}(\bar{a}_1)$ (by the choice of y_1 and y_2). Hence, letting $\bar{a} = (a_y : y \in u)$ and recalling the definition of $\bar{a} = \bar{a}_1 + \bar{a}_2$ from 3.5(11e), we have

$(\cdot)$ $y_1 \in \mathrm{dom}(\bar{a}_1) \setminus \mathrm{dom}(\bar{a}_2)$, so $a_{y_1} = a_{(1,y_1)}$;
$(\cdot)$ $y_2 \in \mathrm{dom}(\bar{a}_2) \setminus \mathrm{dom}(\bar{a}_1)$, so $a_{y_2} = a_{(2,y_2)}$.
But $a_{(1,y_1)}, a_{(2,y_2)} \notin \mathbb{Q}_p$ (as $y_\ell \in \mathrm{supp}_p(\bar{a}_\ell)$, for $\ell = 1, 2$) and so $a_{y_1}, a_{y_2} \notin \mathbb{Q}_p$, and, as obviously $y_1 \neq y_2$, we are done. This concludes the proof of Case A.

Case B: $\mathrm{supp}_p(\bar{a}_2) \subseteq \mathrm{ran}(f_{\bar{g}})$, *equivalently, by* $(*)_{2.7.6.1}$, $\mathrm{dom}(\bar{a}_2) \subseteq \mathrm{ran}(f_{\bar{g}})$. Define $\mathbf{y}'_2 = \{f_*^{-1}(\bar{y}) : \bar{y} \in \mathbf{y}_2\}$, where, recalling $\bar{x}$ is from $\mathfrak{s}$ (cf. $(*)_{2.7.0}$), we let

$$\mathbf{y}_2 = \{\bar{y} \in \bar{x}/E^{\mathfrak{n}}_k : \bar{y} \not\subseteq Y_{\mathfrak{m}} \text{ (so } \bar{y} \subseteq \mathrm{ran}(f_*))\}.$$

Now let

$$\bar{a}'_2 = (a'_{(2,y)} : y \in \mathrm{set}(\mathbf{y}'_2)),$$

where

$$y \in \mathrm{set}(\mathbf{y}'_2) \;\Rightarrow\; a'_{(2,y)} = a_{(2,f_*(y))}.$$

Now, we have

$(\cdot_1)$ $\mathbf{y}'_2 \subseteq \bar{x}/E^{\mathfrak{m}}_k$;
$(\cdot_2)$ $\bar{a}'_2 \in \mathcal{A}^1_{\mathfrak{s}}$;
$(\cdot_3)$ $(\bar{a}'_2)^{[f_*]} = \bar{a}_2$;
$(\cdot_4)$ $\mathrm{dom}(\bar{a}'_2) \subseteq \mathrm{dom}(f_{\bar{g}})$;
$(\cdot_5)$ $\bar{a}_2 \in \mathcal{A}^1_{\mathfrak{s}}$.

[Why? Concerning $(\cdot_1)$, if $\bar{y}' \in \mathbf{y}'_2$, then by the choice of $\mathbf{y}'_2$, there is $\bar{y} \in \mathbf{y}_2$ such that $f_*^{-1}(\bar{y}) = \bar{y}'$. Furthermore, by the choice of $\mathbf{y}_2$, we have $\bar{y} \in \bar{x}/E^{\mathfrak{n}}_k$ and $\bar{y} \not\subseteq Y_{\mathfrak{m}}$ (so $\bar{y} \subseteq \mathrm{ran}(f_*)$). By the definition of $E^{\mathfrak{n}}_k$ we have $\bar{y}' \in \bar{x}/E^{\mathfrak{n}}_k$. Thus, by $(*)_{2.6}$(viii), we have $\bar{y}' \in \bar{x}/E^{\mathfrak{m}}_k$, so $(\cdot_1)$ indeed holds. Also, $(\cdot_2)$ is by $(\cdot_1)$ and $(\cdot_3)$ is because we defined $\bar{a}'_2 = (a'_{(2,y)} : y \in \mathrm{set}(\mathbf{y}'_2))$. Moving to the remaining clauses, we have that $(\cdot_4)$ holds as $\mathrm{supp}(\bar{a}_2) \subseteq \mathrm{ran}(f_{\bar{g}})$. Finally, concerning $(\cdot_5)$, recalling that $f_{\bar{g}} \subseteq f_*$, by $(\cdot_3)$+$(\cdot_4)$ we have that $(\bar{a}'_2)^{[f_{\bar{g}}]} = \bar{a}_2$, and as $\bar{a}'_2 \in \mathcal{A}^1_{\mathfrak{s}} = \mathcal{A}^{\mathfrak{m}}_{\mathfrak{s}}$, by 3.5(11)(e) we have $\bar{a}_2 \in \mathcal{A}^1_{\mathfrak{s}} = \mathcal{A}^{\mathfrak{m}}_{\mathfrak{s}}$.]

Now let $\bar{a}_* = \bar{a}_1 + \bar{a}_2$, as each summand is in $\mathcal{A}^1_{\mathfrak{s}}$ (notice that the second summand is in $\mathcal{A}^1_{\mathfrak{s}}$ by $(\cdot_6)$). Then also $\bar{a}_* \in \mathcal{A}^1_{\mathfrak{s}}$, recalling that $\mathcal{A}^1_{\mathfrak{s}} = \mathcal{A}^{\mathfrak{m}}_{\mathfrak{s}}$ and $\mathfrak{m}$ satisfies condition 3.5(8)(e). Also, clearly $\bar{a} = \bar{a}_*$, but the latter belonging to $\mathcal{A}^1_{\mathfrak{s}}$, we have that $\mathrm{supp}_p(\bar{a})$ is not a singleton, recalling that $\mathcal{A}^1_{\mathfrak{s}} = \mathcal{A}^{\mathfrak{m}}_{\mathfrak{s}}$ and $\mathfrak{m}$ satisfies 3.5(11).

Case C: $\mathrm{supp}_p(\bar{a}_1) \subseteq \mathrm{ran}(f_{\bar{g}})$, *equivalently, by* $(*)_{2.7.6.1}$, $\mathrm{dom}(\bar{a}_1) \subseteq \mathrm{ran}(f_{\bar{g}})$. This case is similar to Case B. Recalling $\bar{x}$ is from $\mathfrak{s}$ (cf. $(*)_{2.7.0}$), let

$$\mathbf{y}_2 = \{\bar{y} \in \bar{x}/E^{\mathfrak{n}}_k : \bar{y} \not\subseteq Y_{\mathfrak{m}} \text{ (so } \bar{y} \subseteq \mathrm{ran}(f_*))\},$$
$$\mathbf{y}'_2 = \{f_*^{-1}(\bar{y}) : \bar{y} \in \mathbf{y}_2\},$$
$$\bar{a}'_2 = (a'_{(2,y)} : y \in \mathrm{set}(\mathbf{y}'_2)),$$

where

$$y \in \mathrm{set}(\mathbf{y}'_2) \;\Rightarrow\; a'_{(2,y)} = a_{(2,f_*(y))}.$$

Now let $Y_1 = \mathrm{supp}_p(\bar a_1) \subseteq \mathrm{ran}(f_{\bar g})$ and $Y_1' = f_{\bar g}^{-1}(Y_1) \subseteq \mathrm{dom}(f_{\bar g})$. Then we let

$(\cdot_a)$ $\bar a_1' = (a'_{(1,y)} : y \in Y_1')$, where

$(\cdot_b)$ $a'_{(1,y)} = a_{(1,f_{\bar g}(y))}$;

$(\cdot_c)$ $Y_2 = \mathrm{dom}(\bar a_2)$, $Y_2' = f_*^{-1}[Y_2] = f_{\bar g}^{-1}[Y_2]$;

$(\cdot_d)$ $\bar a_2' = (a'_{(2,y)} : y \in Y_2')$, where

$(\cdot_e)$ $a'_{(2,y)} = a'_{(2,f_*(y))}$.

Then

$(\cdot_1)$ $\mathbf{y}_2' \subseteq \bar x / E_k^{\mathfrak m}$ (recall that $\mathfrak s = (p, k, \bar x, \bar q)$ and 3.5(11)(e));

$(\cdot_2)$ $\bar a_2' \in \mathcal A^1_{\mathfrak s}$;

$(\cdot_3)$ $(\bar a_1')^{[f_*^{-1}]} = \bar a_1$;

$(\cdot_4)$ $\mathrm{dom}(\bar a_1') \subseteq \mathrm{dom}(f_{\bar g})$;

$(\cdot_5)$ $\bar a_1' \in \mathcal A^1_{\mathfrak s}$;

$(\cdot_6)$ $(\bar a_2')^{[f_*^{-}]} = \bar a_2$.

Now let $\bar a_* = \bar a_1 + \bar a_2$, and let $\bar a_*' = \bar a_1' + \bar a_2'$, so that $(\bar a_*')^{[f_*^{-1}]} = \bar a_*$. As $\bar a_1' \in \mathcal A^1_{\mathfrak s}$ by $(\cdot_5)$ and $\bar a_2' \in \mathcal A^1_{\mathfrak s}$ by $(\cdot_2)$, then by 3.5(8)(e), we also have $\bar a_*' = \bar a_1' + \bar a_2' \in \mathcal A^1_{\mathfrak s}$. Hence $\mathrm{supp}_p(\bar a_*')$ is not a singleton (as $\mathfrak m \in \mathrm{K}_1^{\mathrm{bo}}(M)$) and so also $\mathrm{supp}_p(\bar a_*)$ is not a singleton.

So we have finished proving $(*_{2.7.6})$; i.e., $\bar a \in \mathcal A'$ implies that $\mathrm{supp}_p(\bar a)$ is not a singleton. Thus, we also finished proving $(*)_2$, as by $(*_{2.7.5})$ we have $\bar a \in \mathcal A^{\mathfrak n}_{\mathfrak s} \Rightarrow \bar a \in \mathcal A'$, and so by $(*_{2.7.6})$ we are done; i.e., we have verified that $\mathfrak n$ satisfies 3.5(11).

$(*)_3$ We can choose an $<_{\mathrm{suc}}$-increasing sequence $(\mathfrak m_\ell : \ell < \omega)$ in $\mathrm{K}_0^{\mathrm{bo}}(M)$ whose limit $\mathfrak m$ is as wanted, i.e., $\mathfrak m \in \mathrm{K}_2^{\mathrm{bo}}(M)$.

We show this. We can find a list $(\bar g^\ell : \ell < \omega)$ of $\bigcup_{m<\omega} \mathcal G^m_*$ such that

$(*)_{3.1}$ (i) $\lg(\bar g^\ell) \leqslant \ell$;

(ii) if $\bar g^\ell \lhd \bar g^k$, then $\ell < k$;

(iii) $\lg(\bar g^\ell) = 0$ if and only if $\ell = 0$;

(iv) note that for $\ell < \lg(\bar g)$, $g^k_\ell \neq (g^k_\ell)^{-1}$

(v) $\bar g^{2\ell+2} = (\bar g^{2\ell+1})^{-1}$;

(vi) if $\lg(\bar g^{2\ell+1}) > 1$, then there is a unique $i < \ell$ such that

$(\cdot_1)$ $\bar g^{2i+1} \lhd \bar g^{2\ell+2}$;

$(\cdot_2)$ $\bar g^{2i+2} \lhd \bar g^{2\ell+1}$;

$(\cdot_3)$ $\lg(\bar g^{2\ell+1}) = \lg(\bar g^{2\ell+2}) = \lg(\bar g^{2i+1}) + 1 = \lg(\bar g^{2i+2}) + 1$.

Why do we ask what we ask in $(*)_{3.1}$? Clause (i) is just for clarity. Clause (ii) is needed because defining $\mathfrak m_{k+1}$ we would like to ensure $\bar g_k \in I^{\mathfrak m_{k+1}}$, in the interesting case $\bar g_k \notin I^{\mathfrak m_k}$. But $\bar g' \lhd \bar g_k$ implies $f_{\bar g'} \subseteq f_{\bar g_k}$, so it makes sense to take care of $\bar g_k$ only after all the $\bar g' \lhd \bar g_k$ have been taken care of, but this means $\bar g' \lhd \bar g_k$ implies $\bar g' \in I^{\mathfrak m_k}$. Concerning clause (vi), the point is that in $(*)_2$ we only took care of having $\mathrm{dom}(\bar f_{\bar g^\frown(g)})$ be large enough, but not of $\mathrm{ran}(\bar f_{\bar g^\frown(g)})$. But, by our bookkeeping, if $\bar{g}_\ell$ is not empty sequence, $\bar{g}_\ell \triangleleft \bar{g}_k$ and $\lg(\bar{g}_\ell)+1 = \lg(\bar{g}_k)$, then k is odd if and only if ℓ is even. Hence if k is odd, then choosing $\bar{g}_k$ will increase the domain of $f^{\mathfrak{m}_{k+1}}_{\bar{g}_k}$ to include $Y_{\mathfrak{m}_k} \cap X_{\mathrm{dom}(\bar{g}_k)}$ (cf. $(*_2)$(d)), and if ℓ is odd, then choosing $\bar{g}_\ell$ will increase the domain of $f^{\mathfrak{m}_{\ell+1}}_{\bar{g}_\ell}$ to include $Y_{\mathfrak{m}_\ell} \cap X_{\mathrm{dom}(\bar{g}_\ell)}$. But $Y_{\mathfrak{m}_\ell} \subseteq Y_{\mathfrak{m}_k}$, so always $Y_{\mathfrak{m}_\ell} \cap X_{\mathrm{dom}(\bar{g}_\ell)} \subseteq \mathrm{dom}(f^{\mathfrak{m}_{k+1}}_{\bar{g}_k})$. Mutatis mutandi we have that $Y_{\mathfrak{m}_\ell} \cap X_{\mathrm{ran}(\bar{g}_\ell)} \subseteq \mathrm{ran}(f^{\mathfrak{m}_{k+1}}_{\bar{g}_k})$. Clearly this suffices.

Now, by induction on $\ell < \omega$, we choose $\mathfrak{m}_\ell \in \mathrm{K}^{\mathrm{bo}}_0$ such that $n(\mathfrak{m}_\ell) \leqslant \ell + 1$ and $\mathfrak{m}_{\ell+1} \in \mathrm{suc}(\mathfrak{m}_\ell)$ or $\mathfrak{m}_{\ell+1} = \mathfrak{m}_\ell$. We proceed as follows:

$(*)_{3.2}$ $(\ell = 0)$ use $(*)_1$;
 $(\ell = k+1)$ $(\cdot_1)$ if $\bar{g}^{k+1} \in I^{\mathfrak{m}_k}$, then $\mathfrak{m}_\ell = \mathfrak{m}_k$ (if this occurs, then k is odd);
 $(\cdot_2)$ if $\bar{g}^{k+1} \notin I^{\mathfrak{m}_k}$, let $m_k = \lg(\bar{g}^{k+1}) - 1$, so $\bar{g}^{k+1} \restriction m_k \in I^{\mathfrak{m}_k}$, and use $(*)_2$ with the pair $n(\mathfrak{m}_k)$, $\bar{g}^{2k+1}$ here standing for $n, \bar{g}^\frown(g)$ there.

Clearly $\mathfrak{m} = \lim_{\ell<\omega}(\mathfrak{m}_\ell) \in \mathrm{K}^{\mathrm{bo}}_1(M)$. Notice that by $(*)_{3.1}$, we have

$(*)_{3.3}$ if $\bar{g}^k \triangleleft \bar{g}^\ell \triangleleft \bar{g}^m$, then
 (i) $f_{\bar{g}^k} \subseteq f_{\bar{g}^\ell} \subseteq f_{\bar{g}^m}$;
 (ii) $Y_{\mathfrak{m}_k} \cap X_{\mathrm{dom}(f_{\bar{g}^k})} \subseteq \mathrm{dom}(f_{\bar{g}^m})$;
 (iii) $Y_{\mathfrak{m}_k} \cap X_{\mathrm{ran}(f_{\bar{g}^k})} \subseteq \mathrm{ran}(f_{\bar{g}^m})$;
 (iv) if $s \subseteq_1 \mathrm{dom}(f_{\bar{g}^k})$, then $\min(X'_s \setminus Y_{\mathfrak{m}_k}) \in \mathrm{dom}(f_{\bar{g}^m})$ (see $(*)_{2.3}$(B)(c)$(\cdot_1)$);
 (v) if $s \subseteq_1 \mathrm{ran}(f_{\bar{g}^k})$, then $\min(X'_s \setminus Y_{\mathfrak{m}_k}) \in \mathrm{ran}(f_{\bar{g}^m})$ (see $(*)_{2.3}$(B)(c)$(\cdot_2)$).

Thus we are only left to show that $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_1(M)$ is full, that this, that $\mathfrak{m}$ satisfies conditions (12) and (13) from 3.6. For this, notice that

(i) Definition 3.6(12) holds by the definition of $\mathfrak{m}_{k+1} \in \mathrm{suc}_{\mathfrak{m}_k}$, recalling $(*)_{3.3}$(iv)(v);
(ii) Definition 3.6(13) holds as the $\bar{g}^\ell$'s list $\mathcal{G}_*$. □

Corollary 3.9. $\mathrm{K}^{\mathrm{bo}}_2(M) \neq \emptyset$.

Proof. This is obvious by Claim 3.8, simply comparing Definitions 3.4, 3.5 and 3.6. □

4. Borel completeness of torsion-tree abelian groups

4.1. *The definition of the groups $G_{(1,\mathcal{U})}$.*

Definition 4.1. Let $\mathrm{K}^{\mathrm{bo}}_3(M)$ be the class of $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_2(M)$ expanded with a sequence $\bar{p} = \bar{p}^{\mathfrak{m}}$ of prime numbers without repetitions such that we have the following:

(1) $\bar{p} = (p_{(e,\bar{q})} : e \in \mathrm{seq}_n(X)/E^{\mathfrak{m}}_n$ for some $0 < n < \omega$ and $\bar{q} \in (\mathbb{Z}^+)^n)$;
(2) for every $\ell < n$, $p \nmid q_\ell$.

FACT 4.2. *Clearly every element of* $\mathfrak{m} \in \mathrm{K}_2^{\mathrm{bo}}(M)$ *can be expanded to an element of* $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$ *and, as we showed in 3.9 that* $\mathrm{K}_2^{\mathrm{bo}}(M) \neq \emptyset$*, we have* $\mathrm{K}_3^{\mathrm{bo}}(M) \neq \emptyset$.

We try to give some intuition on the group $G_1 = G_1[\mathfrak{m}]$ which we are about to introduce in Definition 4.3. This group will be some sort of universal domain for our construction, and in fact all the TFAB_ω's which will be in the range of our Borel reduction from $\mathbf{K}_2^{\mathrm{eq}}$ (cf. Hypothesis 3.2) to TFAB_ω will be pure subgroups of this group G_1. The group G_1 naturally interpolates between $G_0 = \bigoplus\{\mathbb{Z}x : x \in X\}$ and $G_2 = \bigoplus\{\mathbb{Q}x : x \in X\}$, which have respectively the minimal and the maximal amount of divisibility possible. Clearly, the groups G_0 and G_2 do not code anything of the universal countable model $M \in \mathbf{K}_2^{\mathrm{eq}}$ (cf. Hypothesis 3.2). Thus, we want to find a subgroup $G_0 \leqslant G_1 \leqslant G_2$ which does encode M. We do this adding divisibility conditions to G_0 which depend on the relation $E_n^{\mathfrak{m}}$ from 3.4. So the first step is that for every $a \in G_0^+$, we choose a prime p_a and require the following condition:

$$G_0 \models a = \sum_{\ell<k} q_\ell x_\ell \neq 0 \ \Rightarrow\ G_1 \models p_a^\infty \,|\, a.$$

However, we want the partial permutations $f_{\bar{g}}$ of X from 3.4 to induce partial automorphisms $\hat{f}^1_{\bar{g}}$ of our desired group G_1, and so we naturally demand

$$\iota \in \{1,2\},\ a_\iota = \sum_{\ell<k} q_\ell x^\iota_\ell,\ \bigwedge_{\ell<k} f_{\bar{g}}(x^1_\ell) = x^2_\ell \ \Rightarrow\ p_{a_1} = p_{a_2}.$$

Formally, this translates into a choice of $p_{(e,\bar{q})}$ as in Section 4.1, where condition 4.1(2) is simply a useful technical requirement. We finally define our "universal" group G_1.

Definition 4.3. Let $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$.

(1) Let $G_2 = G_2[\mathfrak{m}]$ be $\bigoplus\{\mathbb{Q}x : x \in X\}$.
(2) Let $G_0 = G_0[\mathfrak{m}]$ be the subgroup of G_2 generated by X, i.e., $\bigoplus\{\mathbb{Z}x : x \in X\}$.
(3) Let $G_1 = G_1[\mathfrak{m}]$ be the subgroup of G_2 generated by
   (a) G_0;
   (b) $p^{-m}(\sum_{\ell<n} q_\ell x_\ell)$, where
      (i) $0 < m < \omega$;
      (ii) $\bar{x} = (x_\ell : \ell < n) \in \mathrm{seq}_n(X)$, $e = \bar{x}/E_n^{\mathfrak{m}}$, $n > 0$;
      (iii) $\bar{q}$ is as in 4.1;
      (iv) $p = p_{(e,\bar{q})}$ (so a prime, recalling Definition 4.1);
   (c) [follows] for every $a \in G_1$, there are $i_* < \omega$ and, for $i < i_*$, k_i, $\bar{x}_i \in \mathrm{seq}_{k_i}(X)$, $\bar{q}_i \in (\mathbb{Z}^+)^{k(i)}$, $e_i = \bar{x}_i/E^{\mathfrak{m}}_{k_i}$, $p_i = p_{(e_i,\bar{q}_i)}$ (hence $\bar{q}_i$ is as in 4.1), $m(i) \geqslant 0$ and $r^i \in \mathbb{Z}^+$ such that the following condition holds:

$$a = \sum\{p_i^{-m(i)} r^i q_{(i,\ell)} x_{(i,\ell)} : i < i_*, \ell < k_i\}.$$

(4) For a prime p, we have

$$G_{(1,p)} = \{a \in G_1 : a \text{ is divisible by } p^m, \text{ for every } 0 < m < \omega\}.$$

(Notice that by Observation 2.5, $G_{(1,p)}$ is always a pure subgroup of G_1.)

(5) For $\mathcal{U} \subseteq M$, we let

$$G_{(1,\mathcal{U})}[\mathfrak{m}] = G_{(1,\mathcal{U})}[\mathfrak{m}(M)] = G_{(1,\mathcal{U})} = \langle y : y \in X_u, u \subseteq_1 \mathcal{U}\rangle^*_{G_1} = \langle X_{\mathcal{U}}\rangle^*_{G_1}.$$

The notation $\mathfrak{m}(M)$ is from the second line of Definition 3.4 and $X_{\mathcal{U}}$ is from 3.4(3).

(6) For $f_{\bar{g}} \in \bar{f}^{\mathfrak{m}}$ (cf. Definition 3.4(4)), let $\hat{f}^2_{\bar{g}}$ be the unique partial automorphism of G_2 which is induced by $f_{\bar{g}}$ (see 4.4(2)), explicitly: if $k < \omega$ and for every $\ell < k$ we have that $y^1_\ell \in \mathrm{dom}(f_{\bar{g}})$, $y^2_\ell = f_{\bar{g}}(y^1_\ell)$, $q_\ell \in \mathbb{Q}^+$, then

$$a = \sum_{\ell<k} q_\ell y^1_\ell \in G_2 \;\Rightarrow\; \hat{f}^2_{\bar{g}}(a) = \sum_{\ell<k} q_\ell y^2_\ell.$$

(7) For $\ell \in \{0,1\}$, we let $\hat{f}^2_{\bar{g}} \restriction G_\ell = \hat{f}^\ell_{\bar{g}}$ and $\hat{f}_{\bar{g}} = \hat{f}^1_{\bar{g}}$ (see 4.4(2)).

(8) For $i \in \{0,1,2\}$, $a = \sum_{\ell<m} q_\ell x_\ell \in G_i$, with $(x_\ell : \ell < k) \in \mathrm{seq}_k(X)$ and $q_\ell \in \mathbb{Q}^+$, let $\mathrm{supp}(a) = \{x_\ell : \ell < m\}$, i.e., when $a \in G^+_i$, $\mathrm{supp}(a) \subseteq_\omega X$ is the smallest subset of X such that $a \in \langle \mathrm{supp}(a)\rangle^*_{G_i}$.

(9) For p a prime and $a \in G^+_2$, we define the p-support of a, denoted as $\mathrm{supp}_p(a)$, as follows: if $a = \sum\{q_\ell x_\ell : \ell < k\}$ with $(x_\ell : \ell < k) \in \mathrm{seq}_k(X)$ and $q_\ell \in \mathbb{Q}^+$, then

$$\mathrm{supp}_p(a) = \{x_\ell : \ell < k \text{ and } q_\ell \notin \mathbb{Q}_p\},$$

where we recall that $\mathbb{Q}_p$ was defined in 2.6.

Lemma 4.4. *Let $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_3$ and $\ell \in \{0,1,2\}$.*

(1) *$G_\ell[\mathfrak{m}] \in \mathrm{TFAB}$ and $|G_\ell[\mathfrak{m}]| = \aleph_0$.*

(2) (a) *$\hat{f}^2_{\bar{g}}$ is a partial automorphisms of $G_2[\mathfrak{m}]$ mapping $G_0[\mathfrak{m}]$ into itself;*

(b) *$\hat{f}_{\bar{g}} = \hat{f}^1_{\bar{g}} = \hat{f}^2_{\bar{g}} \restriction G_{(1,\mathrm{dom}(\bar{g}))}$ (cf. Definition 4.3(5), (7)), the map $\hat{f}_{\bar{g}}$ is a well-defined partial automorphism of G_1, and $\mathrm{dom}(\hat{f}_{\bar{g}})$ is a pure subgroup of $G_1[\mathfrak{m}]$; in fact $\mathrm{dom}(\hat{f}_{\bar{g}})$ is the pure closure in G_1 of $\mathrm{dom}(\hat{f}^0_{\bar{g}})$;*

(c) *$\hat{f}_{\bar{g}^{-1}} = \hat{f}^{-1}_{\bar{g}}$;*

(d) *$\bar{g}_1 \subseteq \bar{g}_2 \Rightarrow \hat{f}_{\bar{g}_1} \subseteq \hat{f}_{\bar{g}_2}$;*

(e) *$f_{\bar{g}} \subseteq \hat{f}^0_{\bar{g}} \subseteq \hat{f}^1_{\bar{g}} \subseteq \hat{f}^2_{\bar{g}}$.*

(3) *If $p = p_{(e,\bar{q})}$, $e \in \mathrm{seq}_n(X)/E^{\mathfrak{m}}_n$, $\bar{q} = (q_\ell : \ell < n)$ is as in 4.1, and $n \geqslant 1$, then*

(a) *$\langle \sum_{\ell<n} p^{-m} q_\ell y_\ell : m < \omega, \bar{y} \in e\rangle^*_{G_1} \leqslant G_{(1,p)}$;*

(b) *$G_1 \leqslant \langle \{p^{-m} \sum_{\ell<n} q_\ell y_\ell : m < \omega,\ \bar{y} \in e\} \cup \mathbb{Q}_p G_0\rangle_{G_2}$;*

(c) *if $a \in G_1$, then there are $k < \omega$, and, for $i < k$, $\bar{y}^i \in e$, $s_i \in \mathbb{Q}^+$ such that*

(i) *$a = \sum_{i<k} s_i(\sum_{\ell<n} q_\ell y^i_\ell) \mod(\mathbb{Q}_p G_0 \cap G_1)$;*

(ii) *for all* $i < k$, $s_i \sum_{\ell<n} q_\ell y^i_\ell \notin \mathbb{Q}_p G_0$, *and* $\ell < n$ *implies* $s_i q_\ell y^i_\ell \notin \mathbb{Q}_p G_0$;

(iii) $s_i \sum\{q^i_\ell y^i_\ell : \ell < n\} \in G_1$.

(4) *In Lemma 4.4(3) we may add that* $(\bar{y}^i : i < i_*)$ *is with no repetitions.*

Proof. Item (1) is clear. Concerning item (2), clause (a) holds as $f_{\bar{g}}$ is a partial one-to-one function from X to X; while for clause (b) it suffices to prove that given $\sum_{\ell<k} q_\ell y^1_\ell$ and $\sum_{\ell<k} q_\ell y^2_\ell$ as in Definition 4.3(6), we have that

$$\sum_{\ell<k} q_\ell y^1_\ell \in G_1 \Rightarrow \sum_{\ell<k} q_\ell y^2_\ell \in G_1.$$

In order to verify this it suffices to consider the case in which $a := \sum_{\ell<k} q_\ell y^1_\ell$ is one of the generators of G_1 from 4.3(3). Thus, to conclude, it suffices to notice that $f_{\bar{g}}$ maps $\bar{y}^1 = (y^1_\ell : \ell < k)$ to $\bar{y}^2 = (y^2_\ell : \ell < k)$; hence $\bar{y}^2 \in \bar{y}^1/E^{\mathfrak{m}}_k$ and recall 4.3(3b). This shows (2)(b). Finally, items (2)(c)–(e) are easy, and so we omit details.

Concerning item (3), if $\bar{y} \in e$ and $0 < m < \omega$, then $p^{-m} \sum_{\ell<k} q_\ell y_\ell$ is one of the generators of G_1. As this holds for every $0 < m < \omega$, it follows that $\sum_{\ell<k} p^{-m} q_\ell y_\ell \in G_{(1,p)}$, by the definition of $G_{(1,p)}$. As $G_{(1,p)}$ is a subgroup of G_1, for every $\bar{y} \in e$, we have that $\sum_{\ell<n} q_\ell y_\ell \in G_{(1,p)} \leqslant G_1$. Let $Z_{(e,\bar{q})} = \{\sum_{\ell<n} q_\ell y_\ell : \bar{y} \in e\} \subseteq G_{(1,p)}$. Then $\langle Z_{(e,\bar{q})}\rangle^*_{G_1} \leqslant G_{(1,p)}$, because by Definition 4.3(4) we have that $G_{(1,p)}$ is a pure subgroup of G_1 (cf. Observation 2.5). This proves (3)(a).

Concerning (3)(b)(c), assume

$(*_1)$ $a \in G^+_1$.

By 4.3(3)(c), we have

$(*_2)$ As $a \in G_1$, we can find
- (a) $i_* < \omega$;
- (b) for $i < i_*$, $e_i = \bar{x}_i/E_{k_i}$, $\bar{x}_i \in \mathrm{seq}_{k_i}(X)$, $\bar{q}^i = (q^i_\ell : \ell < k_i) \in (\mathbb{Z}^+)^{k_i}$;
- (c) $r^i \in \mathbb{Z}^+$, $\bar{y}^i \in e_i$, $b_i = \sum_{\ell<k_i} q^i_\ell y^i_\ell \in G_0$;
- (d) $p_i = p_{(e_i,\bar{q}^i)}$;
- (e) $a = \sum_{i<i_*} p_i^{-m(i)} r^i b_i$, where $m(i) < \omega$;
- (f) $(b_i : i < i_*)$ is with no repetitions;
- (g) $p_i^{-m(i)} r^i b_i \in G_1$.

Now let

$(*_3)$ $V = \{i < i_* : p_i = p = p_{(e,\bar{q})}$ and $p_i^{-m(i)} r^i b_i \notin \mathbb{Q}_p G_0\}$,

where we recall that the object $p_{(e,\bar{q})}$ is from the statement of lemma and, in particular, it is fixed. Notice also that if $i \in V$, then $(e, \bar{q}) = (e_i, \bar{q}^i)$. Hence, we have

$(*_4)$ (a) if $i \in i_* \setminus V$, then $p_i^{-m(i)} r^i b_i \in \mathbb{Q}_p G_0$;

(b) $i \in V$ implies $\bar{y}^i \in e$ and $\bar{q}^i = \bar{q}$;

(c) if $i \in V$ and $\ell(1), \ell(2) < k$, then

$$p_i^{-m(i)} r^i q^i_{\ell(1)} \in \mathbb{Q}_p \;\Leftrightarrow\; p_i^{-m(i)} r^i q^i_{\ell(2)} \in \mathbb{Q}_p \;\Leftrightarrow\; p_i^{-m(i)} r^i b_i \in \mathbb{Q}_p G_0;$$

(d) if $i \in V$, then $p_i^{-m(i)} r^i b_i \notin \mathbb{Q}_p G_0$;

(e) if $i \in V$ and $\ell < k$, then $p_i^{-m(i)} r^i q^i_\ell y^i_\ell \notin \mathbb{Q}_p G_0$.

[Notice that in the first equivalence of $(*_4)$(c) we use $\ell < k \Rightarrow q_\ell \in \mathbb{Z}^+, p \not| q_\ell$.]

By $(*_4)$, we have

$(*_5)$ (a) $a = \sum\{p_i^{-m(i)} r^i b_i : i \in V\} \bmod (\mathbb{Q}_p G_0 \cap G_1)$;

(b) $i \in V$ implies $p_i^{-m(i)} r^i b_i \notin \mathbb{Q}_p G_0$.

So, defining s_i as $p^{-m(i)} r^i$, we are done proving (3)(b)(c). Finally, (4) is easy. $\square$

Fact 4.5. *Assume that* $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$, $\mathcal{U}, \mathcal{V} \subseteq M$ *and* $|\mathcal{U}| = |\mathcal{V}| = \aleph_0$. *Suppose further that there is* $h : M \restriction \mathcal{U} \cong M \restriction \mathcal{V}$. *Then there is* $\bar{g} = (g_k : k < \omega)$ *such that*

(a) *for every* $k < \omega$, $g_k \in \mathcal{G}$ (*cf.* Hypothesis 3.2(3));

(b) *for every* $k < \omega$, $g_k \subsetneq g_{k+1}$;

(c) $\bigcup_{k<\omega} g_k : M \restriction \mathcal{U} \cong M \restriction \mathcal{V}$.

Proof. Let $h : M \restriction \mathcal{U} \cong M \restriction \mathcal{V}$. We can choose an increasing sequence $(n_k : k < \omega)$ such that $g_k = h \cap (n_k \times n_k)$ (pedantically $g = (h \cap (n_k \times n_k), 1)$ recalling 3.2(3)) is strictly increasing and $\bigcup_{k<\omega} g_k = h$. $\square$

As mentioned, G_1 will be some sort of universal domain for our construction. This is reflected by the fact that instead of varying $M \in \mathbf{K}^{\mathrm{eq}}$ in Definition 3.4, we fix M to be the countable universal homogeneous model of $\mathbf{K}^{\mathrm{eq}}$ and, for $\mathcal{U} \subseteq M$, we consider the substructure $M \restriction \mathcal{U}$ and the group $G_{(1,\mathcal{U})}$. We intend to show

$$M \restriction \mathcal{U} \cong M \restriction \mathcal{V} \;\Leftrightarrow\; G_{(1,\mathcal{U})}[\mathfrak{m}] \cong G_{(1,\mathcal{V})}[\mathfrak{m}].$$

The easy direction is course the left-to-right one, which we now establish:

Claim 4.6. Assume that $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$, $\mathcal{U}, \mathcal{V} \subseteq M$ and $|\mathcal{U}| = |\mathcal{V}| = \aleph_0$. Then

$$M \restriction \mathcal{U} \cong M \restriction \mathcal{V} \;\Rightarrow\; G_{(1,\mathcal{U})}[\mathfrak{m}] \cong G_{(1,\mathcal{V})}[\mathfrak{m}].$$

Proof. Let $(g_k : k < \omega)$ be as in Fact 4.5, $s_k = \mathrm{dom}(g_k)$ and $t_k = \mathrm{ran}(g_k)$. Then

(i) for $k < \omega$, $\bar{g}_k = (g_\ell : \ell \leqslant k)$, so $\bar{g}_k \in \mathcal{G}_*^{k+1}$ (cf. Hypothesis 3.2(4) and 4.5(a), (b));

(ii) $\bigcup_{k<\omega} g_k : M \restriction \mathcal{U} \cong M \restriction \mathcal{V}$ (cf. 4.5(c));

(iii) for every $k < \omega$, we have that $\bar{g}_k \in \mathcal{G}_*$ and so, by 3.4(4), $f_{\bar{g}_k} \in \bar{f}^{\mathfrak{m}}$.

Notice also that by 3.4(9) we have

($\star_1$) (d) $\bigcup_{k<\omega} \mathrm{dom}(f_{\bar{g}_k}) = \bigcup_{k<\omega} X_{s_k} = X_{\mathcal{U}}$;
(e) $\bigcup_{k<\omega} \mathrm{ran}(f_{\bar{g}_k}) = \bigcup_{k<\omega} X_{h[s_k]} = X_{\mathcal{V}}$.

Hence, we have

($\star_2$) $\bigcup_{k<\omega} \hat{f}_{\bar{g}_k}$ is an isomorphism from $G_{(1,\mathcal{U})}$ onto $G_{(1,\mathcal{V})}$ (cf. 4.3(5), (7)).

[Why? By 4.3(5), (6), (7), 4.4(2b) and 3.4(9).] □

4.2. *Analyzing isomorphism.* Our aim in this subsection is to prove the converse of Claim 4.6.

Throughout this subsection the following hypothesis holds:

Hypothesis 4.7.

(1) $\mathfrak{m} \in \mathrm{K}_3^{\mathrm{bo}}(M)$;
(2) $\mathcal{U}, \mathcal{V} \subseteq M$;
(3) $|\mathcal{U}| = \aleph_0 = |\mathcal{V}|$;
(4) π *is an isomorphism from* $G_{(1,\mathcal{U})}[\mathfrak{m}]$ *onto* $G_{(1,\mathcal{V})}[\mathfrak{m}]$.

Our aim in Lemma 4.8 and Conclusion 4.9 below is to show that π essentially comes from a bijection from $X_{\mathcal{U}}$ onto $X_{\mathcal{V}}$, which are respectively the bases of $G_{(1,\mathcal{U})}[\mathfrak{m}]$ and $G_{(1,\mathcal{V})}[\mathfrak{m}]$ (in the appropriate sense). At the bottom of this is the crucial algebraic condition 3.4(8), which puts restrictions on the possible p-supports of certain members of G_1.

Lemma 4.8. *Let* $a \in G_{(1,\mathcal{U})}[\mathfrak{m}]$, *and let* $b = \pi(a)$.

(1) *For a prime* p, $a \in G_{(1,p)} \Leftrightarrow b \in G_{(1,p)}$;
(2) *if* $a = qx$ *for some* $q \in \mathbb{Q}^+$ *and* $x \in X_{\mathcal{U}}$, *then for some* $y \in X_{\mathcal{V}}$,
(a) $(x)E_1^{\mathfrak{m}}(y)$;
(b) $b \in \mathbb{Q}y$, *i.e., there exist* $m_1, m_2 \in \mathbb{Z}^+$ *such that* $m_1 b = m_2 y$.

Proof. Item (1) is obvious by 4.7(4). Notice now that

($*_0$) It suffices to prove (2)(b).

Why ($*_0$)? Suppose that $b = \frac{m_2}{m_1} y$, and let $e' = (x)/E_1^{\mathfrak{m}}$ and $p' = p_{(e',(1))}$. Then $x \in G_{(1,p')}$, but $a = qx$ and $a \in G_1$, hence $a \in G_{(1,p')}$. Now, applying (1) with (a, b, p') here standing for (a, b, p) there, we get that $b \in G_{(1,p')}$. As $b = \frac{m_2}{m_1} y \in G_1$, we have that $y \in G_{(1,p')}$ and thus

($\cdot$) $G_1 \models (p')^\infty |\, x$ and $G_1 \models (p')^\infty |\, y$.

Now, letting $H_{(p',0)} = \langle x/E_1^{\mathfrak{m}} \rangle_{G_0}$ and $H_{(p',1)} = \langle x/E_1^{\mathfrak{m}} \rangle^*_{G_1}$ we have that

($*_{0.1}$) (i) $G_0/H_{(p',0)}$ is canonically $\cong$ to the direct sum of $\langle \mathbb{Z}y : y \in X \backslash x/E_1^{\mathfrak{m}} \rangle$;
(ii) $H_{(p',1)} \cap G_0 = H_{(p',0)}$;
(iii) $G_1/H_{(p',1)}$ naturally extends $G_0/H_{(p',0)}$;
(iv) no non-zero element of $G_1/H_{(p',1)}$ is divisible by $(p')^\infty$.

Why $(*_{0.1})$? This is straightforward; see a detailed proof of a more complicated case in 5.15(2). This concludes the proof of $(*_0)$.

Coming back to the proof,

$(*_1)$ Let $n < \omega$, $\bar{y} \in \mathrm{seq}_n(X_{\mathcal{V}})$ and $\bar{q} \in (\mathbb{Q}^+)^n$ be such that $b = \sum\{q_\ell y_\ell : \ell < n\}$.

Trivially, $n>0$. We shall show that $n=1$, i.e., that (2)(b) holds. To this extent,

$(*_{1.1})$ Let $q_* \in \omega \setminus \{0\}$ be such that
- $(\cdot_1)$ $b_1 := q_* b \in G_0[\mathfrak{m}]$;
- $(\cdot_2)$ $q_* q \in \mathbb{Z}$, and $q_* q_\ell \in \mathbb{Z}$ for all $\ell < n$;
- $(\cdot_3)$ for every prime p', we have that $p' \mid (q_* q)$ implies $p' \mid (q_* q_\ell)$ for all $\ell < n$.

Let $e = \bar{y}/E_n$, $q'_\ell = q_* q_\ell$ and $\bar{q}' = (q'_\ell : \ell < n)$, so that $q_* q_\ell y_\ell = q'_\ell y_\ell$ and $q'_\ell \in \mathbb{Z}^+$. Let $p = p_{(e,\bar{q}')}$, and let $b_1 = q_* b = \sum\{q'_\ell y_\ell : \ell < n\}$. Notice that we have

$(*_2)$ $\bigwedge_{\ell<k} p \nmid q'_\ell$ and, for every $\ell < k$, $q'_\ell \in \mathbb{Z}^+ \subseteq \mathbb{Q}_p$.

[Why? Because $p = p_{(e,\bar{q}')}$ has been chosen in 4.1 exactly in this manner.]

Then we have

$(*_3)$ (i) $b \in G_{(1,p)}$;
(ii) $a \in G_{(1,p)}$;
(iii) if $m < \omega$, then $p^{-m} a \in G_{(1,p)} \leqslant G_1$.

[Why (i)? By the choice of p we have that $b_1 \in G_{(1,p)}$ (cf. Definition 4.3(3), (4)) and so, as $G_{(1,p)}$ is pure in G_1 (cf. Observation 2.5), $b_1 = q_* b$ and $q_* \in \mathbb{Z}$, we have $b \in G_{(1,p)}$ (cf. Observation 2.4). Why (ii)? By (1) and (i), recalling 4.7(4). Lastly, (iii) is immediate: by the definition of G_1 and of $G_{(1,p)}$ (Definition 4.3(3), (4)).]

$(*_4)$ Without loss of generality, $a = qx \notin \mathbb{Q}_p G_0$ and $pa \in G_0$.

We prove $(*_4)$. Let $a' = p^{-1} q_* a$, $b' = p^{-1} q_* b$ and $q' = p^{-1} q_*$. So by $(*_3)$ we have that $a', b' \in G_1$ and of course $\pi(a') = b'$. Now, by the choice of b' and q_* (in particular, cf. $(*_{1.1})(\cdot_3)$) we have that $pb' \in G_{(0,\mathcal{V})}$, hence $pa' = \pi^{-1}(pb') \in G_{(0,\mathcal{U})}$. Notice that $a' \notin G_{(0,\mathcal{U})}$, as $a' \notin \mathbb{Q}_p G_0$ because $b' \notin \mathbb{Q}_p G_0$, since from $(*_2)$ above, $\bigwedge_{\ell<k} p \nmid q'_\ell$. Noticing that $(a', b', q'_*, b_1, p, \bar{q}')$ satisfies all the demands of $(a, b, q_*, b_1, p, \bar{q}')$ (including $(*_3)$), it follows that

$(*_{4.1})$ (a) replacing (a, q, b) with (a', q', b') we can assume that $a = qx \notin \mathbb{Q}_p G_0$;
(b) if b' belongs to $\mathbb{Q}y$ for some $y \in X_{\mathcal{V}}$, the the conclusion of (2) is satisfied.

This concludes the proof of $(*_4)$.

Now, by 4.4(3), there are $k < \omega$ and, for $i < k$, $\bar{y}^i \in \bar{y}/E_n$ and $r_i \in \mathbb{Q}^+$ such that

$(*_5)$ (a) $qx = a = \sum_{i<k} r_i (\sum_{\ell<n} q'_\ell y^i_\ell) = \sum_{i<k} (\sum_{\ell<n} r_i q'_\ell y^i_\ell) \mod (\mathbb{Q}_p G_0 \cap G_1)$;
(b) $r_i \sum_{\ell<n} q'_\ell y^i_\ell \in G_1$ and $r_i q'_\ell \notin \mathbb{Q}_p$.

By $(*_4)$, $a = qx \notin \mathbb{Q}_p G_0$, and so clearly $k > 0$. It suffices to prove that $k = 1$, which by $(*_5)$ implies that $n = 1$; i.e., there is $y \in X_{\mathcal{V}}$ such that $b \in \mathbb{Q}y$. Why does it follow that $n = 1$? Because otherwise, the left-hand side of $(*_5)$(a) has p-support a singleton but the right-hand side of $(*_5)$(a) has p-support of size at least two, a contradiction.

So toward contradiction assume that $k \geqslant 2$. Recalling $(*_4)$, notice that

$(*_6)$ $qx = a = \sum_{\ell<n}(\sum_{i<k} r_i q'_\ell y^i_\ell) \mod(\mathbb{Q}_p G_0 \cap G_1)$.

Now, let $Z = \{y^i_\ell : i < i_*, \ell < k\}$ and, for $y \in Z$, let

$$a_y = \sum\{r_i q'_\ell : i < i_*, \ell < k, y^i_\ell = y\}.$$

So, by $(*_6)$ we have

$(*_7)$ $qx = \sum\{a_y y : y \in Z\} \mod(\mathbb{Q}_p G_0 \cap G_1)$.

Now, since for the sake of contradiction we are assuming that $k \geqslant 2$, recalling that by $(*_2)$ we have that $q'_\ell \in \mathbb{Z}^+ \subseteq \mathbb{Q}_p$, by 3.4(8), we have the following:

$(*_8)$ $\mathrm{supp}_p(\sum_{y \in Y} a_y y) = \{y \in Y : a_y \notin \mathbb{Q}_p\}$ is not a singleton.

Now recall that, by $(*_4)$, $qx = a \notin \mathbb{Q}_p G_0$, hence $\mathrm{supp}_p(qx) = \{x\}$, so it is a singleton. By $(*_8)$, the right-hand side of $(*_7)$ has a non-singleton p-support whereas the left-hand side of $(*_7)$ has p-support a singleton, a contradiction. Hence, we are done proving (2). □

Conclusion 4.9.

(1) There is a sequence $(q^1_x : x \in X_{\mathcal{U}})$ of non-zero rationals and a function $\pi_1 : X_{\mathcal{U}} \to X_{\mathcal{V}}$ such that for every $x \in X_{\mathcal{U}}$, we have that

$$\pi(x) = q^1_x(\pi_1(x)) \quad \text{and} \quad \pi_1(x) \in x/E^{\mathfrak{m}}_1.$$

(2) There is a sequence $(q^2_x : x \in X_{\mathcal{V}})$ of non-zero rationals and a function $\pi_2 : X_{\mathcal{V}} \to X_{\mathcal{U}}$ such that

$$\pi^{-1}(x) = q^2_x(\pi_2(x)).$$

(3) (i) $\pi_2 \circ \pi_1 : X_{\mathcal{U}} \to X_{\mathcal{U}} = id_{\mathcal{U}}$;
(ii) $\pi_1 \circ \pi_2 : X_{\mathcal{V}} \to X_{\mathcal{V}} = id_{\mathcal{V}}$;
(iii) $\pi_1 : X_{\mathcal{U}} \to X_{\mathcal{V}}$ is a bijection.

Proof. (1) is by 4.8; we elaborate. To this extent, let $R = \{(x, y) : x, y \in X \text{ and } \pi(x) \in \mathbb{Q}^+ y\}$. Now, we have

$(*_1)$ For all $x \in X_{\mathcal{U}}$, there is $y \in X_{\mathcal{V}}$ such that $R(x, y)$.

[Why? By 4.8(2b) there is $y \in X_{\mathcal{V}}$ such that $\pi(x) \in \mathbb{Q}y$, as π is an automorphism, necessarily $\pi(x) \neq 0$ and so $\pi(x) \in \mathbb{Q}^+ y$.]

$(*_2)$ If $x \in X_{\mathcal{U}}$ and $(x, y_1), (x, y_2) \in R$, then $y_1 = y_2$.

[Why? By the definition of R, there are $q_1, q_2 \in \mathbb{Q}^+$ such that $q_1 y_1 = \pi(x) = q_2 y_2$. As $q_1, q_2 \neq 0$, necessarily $q_1 = q_2$ and $y_1 = y_2$.]

Together, R is the graph of a function that we call π_1. Lastly, $\pi_1(x) \in x/E_1^{\mathfrak{m}}$ by 4.8(2a). Thus we proved (1).

(2) is by part (1) applied to π^{-1} (and $\mathcal{V}, \mathcal{U}$).

(3) is by (1) and (2). Why? For example, for (i), we have that

$$\pi^{-1} \circ \pi(x) = \pi^{-1}(q^1_x(\pi_1(x))) = q^2_{\pi_1(x)} q^1_x(\pi_2 \circ \pi_1(x)) = x,$$

which implies that $\pi_2 \circ \pi_1(x) = x$; (ii) is similar, and (iii) follows from (i) and (ii). □

Our aim in the subsequent claims is to lift the one-to-one mapping from $X_{\mathcal{U}}$ onto $X_{\mathcal{U}}$ defined in 4.9 to an isomorphism from $M \restriction \mathcal{U}$ onto $M \restriction \mathcal{V}$. We recall that the equivalence relations $\mathfrak{E}_i^M$ (for $i \in \{0, 1, 2\}$) were defined in 3.2. We intend to show that our mappings π_1 and $\pi_1^{-1} = \pi_2$ preserve them (and so also their negations). This is done introducing some auxiliary equivalence relations $\mathcal{E}_i$ (for $i \in \{0, 1, 2\}$) on X which reflect (to some extent) the equivalence relations $\mathfrak{E}_i^M$ on M.

Definition 4.10. For $i < 3$, let

$$\mathcal{E}_i = \{(x, y) : \text{ for some } (a, b) \in \mathfrak{E}_i^M, x \in X'_{\{a\}} \text{ and } y \in X'_{\{b\}}\},$$

where we recall that $\mathfrak{E}_i^M$ was introduced in 3.2.

Claim 4.11.

(1) If $(y_0, y_1) \in (x_0, x_1)/E_2^{\mathfrak{m}}$, $x_0, x_1, y_0, y_1 \in X$ and $i < 3$, then

$$x_0 \mathcal{E}_i x_1 \Leftrightarrow y_0 \mathcal{E}_i y_1.$$

(2) The mapping π_1 from 4.9 preserves $\mathcal{E}_i$ and its negation for all $i < 3$.

Proof. (1) Suppose that $(y_0, y_1) \in (x_0, x_1)/E_2^{\mathfrak{m}}$. Then it is enough to prove

($\star_1$) If $\bar{g} \in \mathcal{G}_*$, $f_{\bar{g}}(x_\ell) = y_\ell$ for $\ell = 0, 1$, then $x_0 \mathcal{E}_i x_1 \Leftrightarrow y_0 \mathcal{E}_i y_1$.

For $\ell = 0, 1$, let $x_\ell \in X'_{s_\ell}$ for $s_\ell \subseteq_1 M$, and $y_\ell \in X'_{t_\ell}$ for $t_\ell \subseteq_1 M$. Now, as $f_{\bar{g}}(x_\ell) = y_\ell$, by 3.4(4)(d) we have that $\bar{g}[s_\ell] = t_\ell$. So $\bar{g}(s_0, s_1) = (t_0, t_1)$, and so, as $\bar{g} \in \mathcal{G}_*$ we have that $s_0 \mathfrak{E}_i s_1 \Leftrightarrow t_0 \mathfrak{E}_i^M t_1$. This implies $x_0 \mathcal{E}_i x_1 \Leftrightarrow y_0 \mathcal{E}_i y_1$.

Concerning (2), also using $\pi_2, \mathcal{V}, \mathcal{U}$ it suffices to prove that for $x, y \in X_{\mathcal{U}}$, we have

$$x \mathcal{E}_i y \Rightarrow \pi_1(x) \mathcal{E}_i \pi_1(y).$$

To this extent, suppose that $x \mathcal{E}_i y$ and let $s \subseteq_1 \mathcal{U}$ be such that $x, y \in X_{s/\mathfrak{E}_i^M}$. (As $s/\mathfrak{E}_i^M \subseteq M$, we are using 3.4(3) to give meaning to the expression $X_{s/\mathfrak{E}_i^M}$.) If $x = y$, then the conclusion is trivial, so we assume that $x \neq y$.

($\star_{1.1}$) Let $e = (x, y)/E_2^{\mathfrak{m}}$, $\bar{q} = (1, 1)$ and $p = p_{(e, \bar{q})}$.

Now, we claim

$(\star_{1.2})$ There is $0 < m < \omega$ such that $p^{-m}q_x\pi_1(x) + p^{-m}q_y\pi_1(y) \notin \mathbb{Q}_pG_0 \cap G_1$.

[Why? First of all, as $q_x, q_y \in \mathbb{Q}^+$ and $x \neq y \Rightarrow \pi_1(x) \neq \pi_1(y)$ and $\pi(x+y) \in G_1$, we have that $0 \neq a = q_x\pi_1(x) + q_y\pi_1(y) \in G_1$ and so we are done, recalling that by the definition of $\mathbb{Q}_p$ we have that for every $b \in G_1^+$, there is $m < \omega$ such that $p^{-m}b \notin \mathbb{Q}_pG_0$.]

So fix an $m < \omega$ as in $(\star_{1.2})$. Now, by the choice of p, we have that $p^{-m}(x+y) \in G_{(1,p)} \leqslant G_1$ and so we have that the following is satisfied:

$$p^{-m}q_x\pi_1(x) + p^{-m}q_y\pi_1(y) = p^{-m}\pi(x) + p^{-m}\pi(y) = \pi(p^{-m}(x+y)) \in G_{(1,p)}.$$

Therefore, by Lemma 4.4(3) applied with $((x,y)/E_2^{\mathfrak{m}}, (1,1), p, p^{-m}q_x\pi_1(x) + p^{-m}q_y\pi_1(y))$ standing for $(e, \bar{q}, p_{(e,\bar{q})}, a)$ there, there are $(x_j, y_j) \in (x,y)/E_2^{\mathfrak{m}}$ and $r_j \in \mathbb{Q}^+$ for $j < j_*$ such that

$(\star_2)$ (a) $((x_j, y_j) : j < j_*)$ is with no repetitions;

(b) $p^{-m}q_x\pi_1(x) + p^{-m}q_y\pi_1(y) = \sum_{j<j_*} r_j(x_j + y_j) \mod(\mathbb{Q}_pG_0 \cap G_1)$.

Now, by (1), recalling $x\mathcal{E}_iy$, for $j < j_*$ there are $s_j \subseteq_1 M$ such that $x_j, y_j \in X_{s_j/\mathfrak{E}_i^M}$. Next, by $(\star_{1.2})$, the left-hand side of $(\star_2)$(b) is not in $\mathbb{Q}_pG_0 \cap G_1$, so the same happens for the right-hand side of $(\star_2)$(b), hence, necessarily, $\{s_j/\mathfrak{E}_i^M : j < j_*\} \neq \emptyset$ (i.e., $j_* \geqslant 1$), let $(t_\ell/\mathfrak{E}_i^M : \ell < \ell_*)$ list it without repetitions, with $t_\ell \in \{s_j : j < j_*\}$ for each $\ell < \ell_*$. Then let

$$u_\ell = \{j < j_* : s_j/\mathfrak{E}_i^M = t_\ell/\mathfrak{E}_i^M\}.$$

So we have

$(\star_3)$ $p^{-m}q_x\pi_1(x) + p^{-m}q_y\pi_1(y) = \sum_{\ell<\ell_*}\sum_{j\in u_\ell} r_j(x_j + y_j) \mod(\mathbb{Q}_pG_0 \cap G_1)$.

Now, for $\ell < \ell_*$, let $c_\ell = \sum_{j\in u_\ell} r_j(x_j + y_j)$. Then

$(\star_4)$ $p^{-m}q_x\pi_1(x) + p^{-m}q_y\pi_1(y) = \sum_{\ell<\ell_*} c_\ell \mod(\mathbb{Q}_pG_0 \cap G_1)$.

$(\star_5)$ $(\mathrm{supp}_p(c_\ell) : \ell < \ell_*)$ is a sequence of pairwise disjoint sets.

[Why? As $\mathrm{supp}(c_\ell) \subseteq X_{t_\ell/\mathfrak{E}_i^M}$, recall the $t_\ell/\mathfrak{E}_i^M$'s are with no repetitions.]

$(\star_6)$ If $c_\ell \notin \mathbb{Q}_pG_0$, then $|\mathrm{supp}_p(c_\ell)| \geqslant 2$.

[Why? Recall that $c_\ell = \sum_{j\in u_\ell} r_j(x_j + y_j)$, and let $Y_\ell = \bigcup\{\{x_j, y_j\} : j \in u_\ell\}$ and, for $z \in Y_\ell$, let $a_z = \sum\{r_j : j \in u_\ell,\ x_j = z\} + \sum\{r_j : j \in u_\ell,\ y_j = z\}$. Now we can apply 3.4(8) with

$$(p, 2, (x,y), ((x_j,y_j) : j \in u_\ell), (1,1), (r_j : j \in u_\ell), (a_z : z \in Y_\ell))$$

here standing for $(p, k, \bar{x}, \mathbf{y}, \bar{r}, \bar{a}_{(\mathbf{y},r)})$ there, and get $|\{z \in Y_\ell : a_z \notin \mathbb{Q}_p\}| \neq 1$. But this means that $|\mathrm{supp}_p(c_\ell)| \neq 1$, but $|\mathrm{supp}_p(c_\ell)| \neq 0$ as $c_\ell \notin \mathbb{Q}_pG_0$, hence $|\mathrm{supp}_p(c_\ell)| \geqslant 2$, as promised. This concludes the proof of $(\star_6)$.]

$(\star_7)$ $V = \{\ell < \ell_* : c_\ell \notin \mathbb{Q}_pG_0\}$ has exactly one member.

[Why? If $V = \emptyset$, then the right-hand side of $(\star_4)$ is in $\mathbb{Q}_pG_0$ but not the left-hand side, recalling $(\star_5)$ and the choice of $m < \omega$ in $(\star_{1.2})$, a contradiction. On the other hand, if $|V| \geqslant 2$, then the right-hand side of $(\star_3)$ has p-support of size $\sum_{\ell<\ell_*} |\mathrm{supp}_p(c_\ell)| \geqslant 2|V| > 2$, but the p-support of the left-hand side of $(\star_3)$ has cardinality 2, a contradiction.]

Let k be the unique member of V. Then we have the following:

$$\begin{aligned}\{\pi_1(x), \pi_1(y)\} &= \mathrm{supp}_p(p^{-m} q_x \pi_1(x) + p^{-m} q_y \pi_1(y)) \\ &= \mathrm{supp}_p(\textstyle\sum_{\ell<\ell_*} c_\ell) \\ &= \mathrm{supp}_p(c_k) \subseteq X_{t_k/\mathfrak{E}_i^M}.\end{aligned}$$

So $\pi_1(x), \pi_1(y) \in X_{t_k/\mathfrak{E}_i^M}$ and as $X_{t_k/\mathfrak{E}_i^M}$ is an $\mathcal{E}_i$-equivalence class (by the definition of $\mathcal{E}_i$), then $\pi_1(x)\mathcal{E}_i\pi_1(y)$, as desired. This concludes the proof of the claim. □

Claim 4.12. There is a bijection $h : \mathcal{U} \to \mathcal{V}$ preserving $\mathfrak{E}_i^M$ and $\neg\mathfrak{E}_i^M$ for all $i < 3$.

Proof. By 4.11(2), we have

$(*_1)$ If $x, y \in X_{\mathcal{U}}$ and $i < 3$, then $x\mathcal{E}_i y \Leftrightarrow \pi_1(x)\mathcal{E}_i\pi_1(y)$.

Now apply $(*_1)$ for $i = 2$ and recall that by 3.2(1) $\mathfrak{E}_2^M$ is equality on M. Then

$(*_2)$ $\exists s \subseteq_1 \mathcal{U}(x, y \in X'_s) \Leftrightarrow \exists t \subseteq_1 \mathcal{V}(\pi_1(x), \pi_1(y) \in X'_t)$.

Now, as $X_{\mathcal{U}} = \bigcup_{s\subseteq_1 \mathcal{U}} X'_s$ and $X_{\mathcal{V}} = \bigcup_{s\subseteq_1 \mathcal{V}} X'_s$, there is a function $\mathbf{h}_1$ from $\mathcal{U}$ into $\mathcal{V}$ such that (not distinguishing $a \in \mathcal{U}$ with $\{a\} \subseteq_1 \mathcal{U}$)

$(*_3)$ If $x \in X'_s$, $s \subseteq_1 \mathcal{U}$, then $\pi_1(x) \in X'_{\mathbf{h}_1(s)}$.

As $\pi_2 = \pi_1^{-1}$ and π_2 is a function from $X_{\mathcal{V}}$ onto $X_{\mathcal{U}}$ (cf. 4.9), we have that

$(*_4)$ $\mathbf{h}_1 : \mathcal{U} \to \mathcal{V}$ is one-to-one and onto.

Finally, applying 4.11(2) to i and recalling the definition of $\mathcal{E}_i$, we get

$(*_5)$ For $i = 0, 1$, $a \neq b \in \mathcal{U}$ implies $a\mathfrak{E}_i^M b \Leftrightarrow \pi_1(a)\mathfrak{E}_i^M\pi_1(b)$. □

Conclusion 4.13. $M \restriction \mathcal{U}$ and $M \restriction \mathcal{V}$ are isomorphic members of $\mathbf{K}^{\mathrm{eq}}$.

In a work in preparation, (among other things) we intend to give a characterization of the automorphism groups of the groups $G_{(1,\mathcal{U})}$ that we construct above.

4.3. *The proof of the Main Theorem.* Notice that in this subsection 4.7 is no longer assumed.

Conclusion 4.14. Let $\mathfrak{m}[M] \in \mathrm{K}_3^{\mathrm{bo}}$, $\mathcal{U}, \mathcal{V} \subseteq M$ and $|\mathcal{U}| = |\mathcal{V}| = \aleph_0$. Then

$$(\star) \qquad M \restriction \mathcal{U} \cong M \restriction \mathcal{V} \Leftrightarrow G_{(1,\mathcal{U})}[\mathfrak{m}] \cong G_{(1,\mathcal{V})}[\mathfrak{m}].$$

Proof. If the left-hand side of $(\star)$ holds, then by 4.6 also the right-hand side of $(\star)$ holds. If the right-hand side of $(\star)$ holds, then the assumptions in 4.7 are fulfilled and thus 4.13 holds, and so the left-hand side of $(\star)$ holds. □

Convention 4.15. In Fact 1.1 and Notation 1.8(5), instead of considering structures with domain ω we could have considered structures with domain an infinite subset of ω. We take the liberty of not distinguishing between these two variants. This happens most notably in the proof of the Main Theorem right below.

Recall the following:

FACT 4.16. *The class* $\mathbf{K}^{\mathrm{eq}}_\omega$ *is Borel complete. In fact, there is a continuous map from* Graph_ω *into* $\mathbf{K}^{\mathrm{eq}}_\omega$ *which preserves isomorphism and its negation.*

Proof. See, e.g., [11, pg. 295]. □

Proof of the Main Theorem. Let M be as in 3.2. Fix $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_3(M)$ (cf. Fact 4.2) and assume without loss of generality, that $G_1[\mathfrak{m}]$ has the set of elements ω. For every $H \in \mathbf{K}^{\mathrm{eq}}_\omega$, we define $F[H] : H \to M$ by defining $F[H](n)$ by induction on $n < \omega$ as the minimal $k < \omega$ such that $\{(\ell, F[H](\ell)) : \ell < n\} \cup \{(n, k)\}$ is an isomorphism from $H \restriction (n + 1)$ onto $M \restriction (\{F[H](\ell) : \ell < n\} \cup \{k\})$. The map $H \mapsto M \restriction \{F[H](n) : n < \omega\}$ is clearly continuous. We will show that the map $F' : M \restriction \mathcal{U} \mapsto G_{(1,\mathcal{U})}[\mathfrak{m}]$, for $\mathcal{U} \subseteq M$ infinite, is also continuous (cf. 4.15), thus concluding that the map $\mathbf{B} := F' \circ F : H \mapsto G_{(1,\{F[H](n)\,:\,n<\omega\})}[\mathfrak{m}]$ is a continuous map from $\mathbf{K}^{\mathrm{eq}}_\omega$ into TFAB_ω (cf. 4.15) so, by 4.14 and 4.16, we are done.

In order to show that F' is continuous, first recall that $\mathfrak{m} \in \mathrm{K}^{\mathrm{bo}}_3$ is fixed (cf. 4.1) and so, in particular, $\bar{p}$ is fixed. Now, given $a \in G_1[\mathfrak{m}]$, we have to compute from $\mathcal{U}$ whether $a \in G_{(1,\mathcal{U})}[\mathfrak{m}]$ or not. To this extent, let $a = \sum\{q^a_\ell x^a_\ell : \ell < n\}$ with the x_ℓ's pairwise distinct and $q_\ell \in \mathbb{Q}^+$. Now, as by 3.4(3), $X_\mathcal{U} = \bigcup\{X_s : s \subseteq_\omega \mathcal{U}\} = \bigcup\{X'_s : s \subseteq_1 \mathcal{U}\}$ and the latter is a partition of X, for every $\ell < n$, there is a unique finite $s^a_\ell \subseteq M$ such that the following conditions holds:

$$a \in G_{(1,\mathcal{U})}[\mathfrak{m}] \Leftrightarrow \bigwedge_{\ell<n} s^a_\ell \subseteq \mathcal{U}.$$

This suffices to show continuity of F', thus concluding the proof of the theorem. □

Remark 4.17. We observe that in the context of the proof of the Main Theorem, we can choose both M and $\mathfrak{m}$ to be computable stuctures, in the sense of computable model theory; i.e., all the relations and functions of the structure are computable.

5. Completeness of endorigid torsion-free abelian groups

The aim of this section is to show that deciding whether a group $G \in \mathrm{TFAB}_\omega$ is endorigid is a complete co-analytic problem. We do this by reducing a well-known problem to the endorigidity problem, namely the problem of deciding whether a tree with domain ω has an infinite branch, which is well known to be complete co-analytic. So the idea here is to code a tree T into a TFAB_ω $G[T]$. The way we code the tree T is reminiscent of the coding used for the proof of the Main Theorem. Also in this case X will be a basis of the group $G[T]$, and we will code an element $t \in T$ via a partial permutation f_t of X. As in Section 4, the group $G[T]$ that we wish to construct will interpolate between between $G_0 = \bigoplus\{\mathbb{Z}x : x \in X\}$ and $G_2 = \bigoplus\{\mathbb{Q}x : x \in X\}$, via a set of tailored divisibility conditions which code the behavior of the partial maps f_t's, which in turn code the elements of the tree T.

In 5.1–5.9 we deal with the combinatorial part; then we will define the groups.

Hypothesis 5.1. *Throughout this section the following hypothesis stands*:

(1) $T = (T, <_T)$ *is a rooted tree with* ω *levels, and we denote by* $\mathrm{lev}(t)$ *the level of* t.
(2) $T = \bigcup_{n<\omega} T_n$, $T_n \subseteq T_{n+1}$, *and* $t \in T_n$ *implies that* $\mathrm{lev}(t) \leqslant n$.
(3) $T_0 = \emptyset$, T_n *is finite, and we let* $T_{<n} = \bigcup_{\ell<n} T_\ell$ (*so* $T_{<(n+1)} = T_n$).
(4) *If* $s <_T t \in T_{n+1}$, *then* $s \in T_n$.
(5) T *is countable.*

Definition 5.2. Let $\mathrm{K}_1^{\mathrm{ri}}(T)$ be the following class of objects:

$$\mathfrak{m}(T) = \mathfrak{m} = (X, X_n^T, \bar{f}^T : n < \omega) = (X, X_n, \bar{f} : n < \omega)$$

satisfying the following requirements:

(a) $X_0 \neq \emptyset$ and, for $n < \omega$, X_n is finite and $X_n \subsetneq X_{n+1}$, and $X_{<n} = \bigcup_{\ell<n} X_\ell$;
(b) $\bar{f} = (f_t : t \in T)$;
(c) if $n>0$ and $t \in T_n \setminus T_{<n}$, then f_t is a one-to-one function from X_{n-1} into X_n;
(d) for every $t \in T$, $X_0 \cap \mathrm{ran}(f_t) = \emptyset$;
(e) if $s \leqslant_T t \in T_n$, then $f_s \subseteq f_t$;
(f) if $t \in T_{n+1} \setminus T_n$, $f_t(x) = y$ and $y \in X_n$, then for some $s <_T t$, $x \in \mathrm{dom}(f_s)$;
(g) if $s, t \in T_n$ and $y \in \mathrm{ran}(f_s) \cap \mathrm{ran}(f_t)$, then for some $r \in T_n$ such that $r \leqslant_T s, t$, we have that $y \in \mathrm{ran}(f_r)$, equivalently, $\mathrm{ran}(f_s) \cap \mathrm{ran}(f_t) = \mathrm{ran}(f_r)$, for $r = s \wedge t$, where $\wedge$ is the natural semi-lattice operation taken in the tree $(T, <_T)$;
(h) $X_{n+1} \supsetneq \bigcup\{\mathrm{ran}(f_t) : t \in T_{n+1} \setminus T_n\} \cup X_n$;
(i) we let $X = X^{\mathfrak{m}} = \bigcup_{n<\omega} X_n$;
(j) if $f_s(x) = f_t(x)$ and $x \in X_n \setminus X_{<n}$, then we have the following:

$$f_s \restriction X_n = f_t \restriction X_n \text{ and } X_n \subseteq \mathrm{dom}(f_s) \cap \mathrm{dom}(f_t).$$

Notation 5.3. For $x \in X$, we let $\mathbf{n}(x)$ be the unique $n < \omega$ such that $x \in X_n \setminus X_{<n}$ (so, e.g., $x \in X_0$ implies $\mathbf{n}(x) = 0$).

Convention 5.4. Fix $\mathfrak{m} = (X, X_n, \bar{f} : n < \omega) \in \mathrm{K}_1^{\mathrm{ri}}(T)$ (cf. Definition 5.2 and Claim 5.6 below).

Observation 5.5. In the context of Definition 5.2, we have

(1) If $m < n < \omega$, $t \in T_n \setminus T_{<n}$ and for every $s <_T t$ we have $s \in T_m$, then
$$(X_{n-1} \setminus X_m) \cap \mathrm{ran}(f_t) = \emptyset.$$
(2) If $t \in T$, then for every $x \in \mathrm{dom}(f_t)$, we have that $x \neq f_t(x)$. Moreover there is a unique $0 < n < \omega$ such that $x \in X_{n-1}$ and $f_t(x) \in X_n \setminus X_{n-1}$, and for some $s \in T_n \setminus T_{<n}$, we have $s \leqslant_T t$ and $f_s(x) = f_t(x)$.

Proof. First we prove (1). By Definition 5.2(c) we know that f_t is one-to-one from X_{n-1} into X_n. If $n = 1$, then $m = 0$ and so $X_{n-1} = X_0 = X_m$, thus the conclusion is trivial. Suppose then that $n > 1$, let $y \in (X_{n-1} \setminus X_m) \cap \mathrm{ran}(f_t)$, and let $x \in \mathrm{dom}(f_t)$ be such that $f_t(x) = y$. Then, by Definition 5.2(f) there exists $s <_T t$ such that $x \in \mathrm{dom}(f_s)$. But then, using the assumption in (1), we have that $s \in T_m$ (so $m = 0$ is impossible by Definition 5.1(3)). Hence, by Definition 5.2(c), $\mathrm{ran}(f_s) \subseteq X_m$, so $y = f(x) \in X_m$, contradicting the fact that $y \in (X_{n-1} \setminus X_m)$.

Now we prove (2). Assume that x, t, and thus also f_t, are fixed and $x \in \mathrm{dom}(f_t)$. Let $s \leqslant_T t$ be $\leqslant_T$-minimal such that $f_s(x)$ is well defined, and let $n < \omega$ be such that $s \in T_n \setminus T_{<n}$. (Notice that $n \geqslant 1$ since $T_0 = \emptyset$.) Clearly, there is unique $m < \omega$ such that $x \in X_m \setminus X_{<m}$. As $x \in \mathrm{dom}(f_s)$ and $s \in T_n \setminus T_{<n}$ necessarily $m < n$, so $x \in X_{<n}$. But by the choice of s we have that $r <_T s$ implies $x \notin \mathrm{dom}(f_r)$. By the last two sentences and Definition 5.2(f) we have $f_s(x) \in X_n \setminus X_{<n}$, but $f_t(x) = f_s(x)$. □

Claim 5.6. For T as in 5.1, $\mathrm{K}^{\mathrm{ri}}_1(T) \neq \emptyset$ (cf. Definition 5.2).

Proof. The proof is straightforward. □

Definition 5.7. On X (cf. Convention 5.4) we define the following:

(1) For $x \in X$, $\mathrm{suc}(x) = \{f_t(x) : t \in T, x \in \mathrm{dom}(f_t)\}$.
(2) For $x, y \in X$, we let $x <_X y$ if and only if for some $0 < n < \omega$ and $x_0, \ldots, x_n \in X$, we have that $\bigwedge_{\ell<n} x_{\ell+1} \in \mathrm{suc}(x_\ell)$, $x = x_0$ and $y = x_n$.
(3) $\mathrm{seq}_k(X) = \{\bar{x} \in \mathrm{seq}_k(X) : \bar{x} \text{ is injective}\}$.
(4) We say that $\bar{x} \in \mathrm{seq}_k(X)$ is reasonable when the following happens:
$$n(1) < n(2),\ x_{i(1)} \in X_{n(1)} \setminus X_{<n(1)},\ x_{i(2)} \in X_{n(2)} \setminus X_{<n(2)} \Rightarrow i(1) < i(2).$$
(5) $<^k_X$ is the partial order on $\mathrm{seq}_k(X)$ defined as $\bar{x}^1 <^k_X \bar{x}^2$ if and only if $\bar{x}^1, \bar{x}^2 \in \mathrm{seq}_k(X)$ and there are $0 < n < \omega$, $\bar{y}^0, \ldots, \bar{y}^n \in \mathrm{seq}_k(X)$ and $t_0, \ldots, t_{n-1} \in T$ such that for every $\ell < n$, we have that $f_{t_\ell}(\bar{y}^\ell) = \bar{y}^{\ell+1}$, and $(\bar{x}^1, \bar{x}^2) = (\bar{y}^0, \bar{y}^n)$.
(6) Notice that for $k = 1$, we have that $<^k_X = <_X$, where $<_X$ is as in (2) (ignoring the difference between x and (x), for $x \in X$).
(7) For $k \geqslant 1$, let E_k be the closure of $\{(\bar{x}, \bar{y}) : \bar{x} <^k_X \bar{y}\}$ to an equivalence relation.

Observation 5.8. If $\bar{x}^1 \leqslant^k_X \bar{x}^2$ (cf. 5.7(5)), then there is a unique $\bar{t}$ such that

(a) $\bar{t} \in T^n$ for some $n < \omega$;
(b) $f_{\bar{t}}(\bar{x}^1) = \bar{x}^2$, where $f_{\bar{t}} = (f_{t_{n-1}} \circ \cdots \circ f_{t_0})$ and $t_0, \ldots, t_{n-1}$ are as in Definition 5.7(5);
(c) for every $\ell < n$, there is no $s <_T t_\ell$ such that $f_s(\bar{y}_\ell)$ is well defined, where $(\bar{y}_0, \ldots, \bar{y}_{n-1})$ are as in Definition 5.7(5);
(d) if $\bar{t}' \in T^n$ is as in clauses (a)–(b) above and $\ell < n$, then $t_\ell \leqslant_T t'_\ell$.

Observation 5.9.

(1) $(X, <_X)$ is a tree with ω levels.
(2) Every $z \in X_0$ is a root of the tree $(X, <_X)$; further, for every $n < \omega$, some $z \in X_{n+1} \setminus X_n$ is a root of the tree $(X, <_X)$, and so $z/E_1 \cap X_n = \emptyset$.
(3) If $y \in X_{n+1} \setminus X_n$, then for at most one $x \in X_n$, we have $y \in \mathrm{suc}(x)$.
(4) If $y \in \mathrm{suc}(x)$, then $\{t \in T : f_t(x) = y\}$ is a cone of T.
(5) If $\bar{x} \in \mathrm{seq}_k(X)$, then some permutation of $\bar{x}$ is reasonable (cf. Definition 5.7(4)).
(6) If $f_t(\bar{x}) = \bar{y}$ and $\bar{x}$ is reasonable, then so is $\bar{y}$.
(7) For every $1 \leqslant k < \omega$, $(\mathrm{seq}_k(X), <^k_X)$ is a tree with ω levels.
(8) If $\bar{x} <^k_X \bar{y}$ and $\bar{x}$ is reasonable, then $\bar{y}$ is also reasonable.
(9) If $\bar{x} \in \mathrm{seq}_k(X)$ is reasonable, $\bar{x} \leqslant^k_X \bar{y}^1 = (y^1_0, \ldots, y^1_{k-1})$, $\bar{x} \leqslant^k_X \bar{y}^2 = (y^2_0, \ldots, y^2_{k-1})$ and $y^1_{k-1} = y^2_{k-1}$, then $\bar{y}^1 = \bar{y}^2$.
(10) For every $t \in T$, $\mathrm{dom}(f_t)$ is X_n for some $n < \omega$, and we have
$$x, y \in \mathrm{dom}(f_t) \wedge \mathbf{n}(x) = \mathbf{n}(y) \;\Rightarrow\; \mathbf{n}(f_t(x)) = \mathbf{n}(f_t(y)).$$
(11) Like (10) with $\bar{t} \in T^n$, $n \geqslant 1$, where we let
$$f_{\bar{t}} = (f_{t_{n-1}} \circ \cdots \circ f_{t_0}).$$
(12) Recalling the notation from (11), if $f_{\bar{t}_1}(x) = f_{\bar{t}_2}(x)$, then $\lg(\bar{t}_1) = \lg(\bar{t}_2) = k$. Moreover, letting $\bar{t}_1 = (t_{(1,\ell)} : \ell < k)$ and $\bar{t}_2 = (t_{(2,\ell)} : \ell < k)$, if $\bar{t}_1$ is as in 5.8, then we have that $\ell < \lg(\bar{t}_1)$ implies that $t_{(1,\ell)} \leqslant_T t_{(2,\ell)}$.

Proof. Items (1)–(2) are clear, where (2) is by 5.2(h). Item (3) is by Definition 5.2(f)–(g). Items (4) and (5) are also easy (and (4) is not used (except in 5.11(1)) but we retain it to give the picture). Item (6) can be proved for $t \in T_n \setminus T_{<n}$ by induction on $n < \omega$. Finally, (7) and (8) are easy, and (9) is easy to see using 5.8 and 5.2(j). Also clauses (10), (11) are easy, and clause (12) holds by 5.8. □

Definition 5.10. Let $\mathfrak{m} \in \mathrm{K}^{\mathrm{ri}}_1(T)$ (i.e., as in Convention 5.4).

(1) Let $G_2 = G_2[\mathfrak{m}]$ be $\bigoplus\{\mathbb{Q}x : x \in X\}$.
(2) Let $G_0 = G_0[\mathfrak{m}]$ be the subgroup of G_2 generated by X, i.e., $\bigoplus\{\mathbb{Z}x : x \in X\}$.
(3) For $t \in T$, let

(a) $H_{(2,t)} = \bigoplus\{\mathbb{Q}x : x \in \mathrm{dom}(f_t)\}$;
(b) $I_{(2,t)} = \bigoplus\{\mathbb{Q}x : x \in \mathrm{ran}(f_t)\}$;
(c) $\hat{f}^2_t$ is the (unique) isomorphism from $H_{(2,t)}$ onto $I_{(2,t)}$ such that $x \in \mathrm{dom}(f_t)$ implies that $\hat{f}^2_t(x) = f_t(x)$ (cf. Definition 5.2(c)).

(4) For $t \in T$, we define $H_{(0,t)} := H_{(2,t)} \cap G_0$ and $I_{(0,t)} := I_{(2,t)} \cap G_0$.
(5) For $\hat{f}^2_t$ as above, we have that $\hat{f}^2_t[H_{(0,t)}] = I_{(0,t)}$. We define $\hat{f}^0_t$ as $\hat{f}^2_t \restriction H_{(0,t)}$.
(6) We define the partial order $<_*$ on G_0^+ by letting $a <_* b$ if and only if $a \neq b \in G_0^+$ and, for some $0 < n < \omega$, $a_0, \ldots, a_n \in G_0, a_0 = a, a_n = b$ and
$$\ell < n \Rightarrow \exists t \in T(\hat{f}^0_t(a_\ell) = a_{\ell+1}).$$
(7) For $a = \sum_{\ell<m} q_\ell x_\ell$, with $x_\ell \in X$ and $q_\ell \in \mathbb{Q}^+$, let $\mathrm{supp}(a) = \{x_\ell : \ell < m\}$.
(8) For $a \in G_2^+$, let $\mathbf{n}(a)$ be the minimal $n < \omega$ such that $a \in \langle X_n \rangle^*_{G_2}$.

While the aim of Definition 5.10 should be clear from the explanations given at the beginning of this section, the reader may wonder what is the aim of Lemma 5.11 and Claim 5.12. In the crucial proof of this section we will show that given an endomorphism π of G_1 and $x \in X$ we have that $\pi(x)$ has the form qy for some $y \in Y$ and $q \in \mathbb{Q}$. This requires a detailed analysis of supports, hence 5.11 and 5.12.

Lemma 5.11.

(1) *If $\{t \in T : \hat{f}^2_t(a) = b\} \neq \emptyset$, then it is a cone of T.*
(2) *$<_* \restriction X = <_X$ (where $<_X$ is as in Definition 5.7(2)).*
(3) *$(G_0^+, <_*)$ is a countable tree with ω levels (recall 5.1(1)).*
(4) *If $s \leqslant_T t$, then $\hat{f}^\ell_s \subseteq \hat{f}^\ell_t$ for $\ell \in \{0, 2\}$.*
(5) *If $t \in T$, $\hat{f}^2_t(a) = b$ and $a \in G_0^+$, then $\mathbf{n}(a) < \mathbf{n}(b)$ (cf. Definition 5.10(8)).*
(6) *If $a <_* b$ (so $a, b \in G_0^+$), then the sequence $(a_\ell : \ell \leqslant n)$ from 5.10(6) is unique.*
(7) *If $a_1 <_* a_2$ and, for $\ell \in \{1, 2\}$, $a_\ell = \sum_{i<k} q^\ell_i x^\ell_i$, $q^\ell_i \in \mathbb{Q}^+$, $\bar{x}^\ell = (x^\ell_i : i < k) \in \mathrm{seq}_k(X)$, then maybe after replacing $\bar{x}^1$ with a permutation of it we have*
$$\bar{x}^0 \leqslant^k_X \bar{x}^1 \text{ and } q^1_i = q^2_i \text{ (for } i < k).$$

Proof. Unraveling definitions, we elaborate only on item (5). As $a \neq 0$, let $a = \sum_{i \leqslant n} q_i x_i$, $x_i \in X$ with no repetitions, $q_i \in \mathbb{Q}^+$. Let $x_i \in X_{k(i)} \setminus X_{<k(i)}$ and without loss of generality, $k(i) \leqslant k(i+1)$ for $i < n$ (cf. Observation 5.9(5)). Clearly $a \in \langle X_{k(n)} \rangle^*_{G_2}$ but $a \notin \langle X_{<k(n)} \rangle^*_{G_2}$. As $\hat{f}^2_t(a)$ is well defined, clearly $\{x_i : i \leqslant n\} \subseteq \mathrm{dom}(f_t)$ and $b = \hat{f}^2_t(a) = \sum_{i \leqslant n} q_i f_t(x_i)$ and, as f_t is one-to-one, the sequence $(f_t(x_i) : i \leqslant n)$ is with no repetitions. By Observation 5.5(2) applied with n there as $k(n)$ here, $f_t(x_n) \notin \langle X_{k(n)} \rangle^*_{G_2}$, hence we have that $\mathbf{n}(b) \geqslant n(f_t(x_n)) > k(n) = \mathbf{n}(a)$. □

Claim 5.12. If (A), then (B), where

(A) (a) $a, b_\ell \in G_2^+$ for $\ell < \ell_*$;

(b) $a \leqslant_* b_\ell$ and the b_ℓ's are with no repetitions;
(c) $a = \sum\{q_i x_i : i < j\}$, $q_i \in \mathbb{Q}^+$;
(d) $\bar{x} = (x_i : i < j) \in \mathrm{seq}_j(X)$ and it is reasonable;

(B) for $\ell < \ell_*$, there are $\bar{y}^\ell = (y_{(\ell,i)} : i < j)$ such that
(a) $y_{(\ell,i)} =: y^\ell_i \in X$ and $\bar{x} \leqslant^j_X \bar{y}^\ell$ (cf. Definition 5.7(5));
(b) $b_\ell = \sum\{q_i y_{(\ell,i)} : i < j\}$, (so the $\bar{y}^\ell$ are pairwise distinct, as the b_ℓ's are);
(c) $(y_{(\ell,i)} : i < j) \in \mathrm{seq}_j(X)$ and it is reasonable;
(d) if $j > 1$ and $\ell_* > 1$, then there are $\ell_1 \neq \ell_2 < \ell_*$ and $i_1, i_2 < j$ such that
(i) if $\ell < \ell_*$, $i < j$ and $y_{(\ell,i)} = y_{(\ell_1,i_1)}$, then $(\ell, i) = (\ell_1, i_1)$;
(ii) if $\ell < \ell_*$, $i < j$ and $y_{(\ell,i)} = y_{(\ell_2,i_2)}$, then $(\ell, i) = (\ell_2, i_2)$.

Proof. By the definition of $\leqslant_*$ there are $(y_{(\ell,i)} : i < j, \ell < \ell_*)$, and by 5.9(6) and 5.11(7) they satisfying clauses (a)–(c) of (B). Recall that $(\{\bar{y} : \bar{x} \leqslant^j_X \bar{y}\}, \leqslant^j_X)$ is a tree (as $(\mathrm{seq}_j(X), \leqslant^j_X)$ is a tree). We now show (B)(d). There are two cases.

Case 1: *$\{\bar{y}^\ell : \ell < \ell_*\}$ is not linearly ordered by $\leqslant^j_X$.* Then there are $\ell(1) \neq \ell(2) < \ell_*$ such that $\bar{y}^{\ell(1)}, \bar{y}^{\ell(2)}$ are locally $\leqslant^j_X$-maximal. So we can choose $i_1, i_2 < j$ such that we have the following:

$$x^{\ell_1}_{i_1} \in X_{\mathbf{n}(b_{\ell_1})} \setminus X_{<\mathbf{n}(b_{\ell_1})} \text{ and } x^{\ell_2}_{i_2} \in X_{\mathbf{n}(b_{\ell_2})} \setminus X_{<\mathbf{n}(b_{\ell_2})}.$$

Notice that using the assumption that the sequences are reasonable we can choose $i_1 = j - 1 = i_2$; see 5.11(5) and 5.9(9). Hence, ℓ_1, ℓ_2, i_1, i_2 are as required for (d).

Case 2: *Not Case* 1. So without loss of generality, we have that for every $\ell < \ell_* - 1$, $\bar{y}^\ell <^j_X \bar{y}^{\ell+1}$. Now, for $\ell < \ell_*$ and $i < j$, let $n(\ell, i) = \mathbf{n}(y^\ell_i)$. Then let

$(\cdot_1)$ $i(1) < j$ be such that $i < j$ implies $n(0, i) \geqslant n(0, i(1))$;
$(\cdot_2)$ $i(2) < j$ be such that $i < j$ implies $n(\ell_* - 1, i) \leqslant n(\ell_* - 1, i(2))$.

Then $(0, i(1))$, $(\ell_* - 1, i(2))$ are as required. Since $\bar{y}^\ell$ is reasonable for $\ell < \ell_*$, we can actually choose $i(1), i(2)$ such that $i(1) = 0$ and $i(2) = j_* - 1$. □

Now we turn to the groups which we shall actually use, i.e., the groups $G_1 = G_1[T]$ defined in 5.13(2) below. Our aim is to include among the partial automorphisms of G_1 all the maps of the form $\hat{f}_t$, i.e., the maps induced by the f_t's, but we want in addition that G_1 is minimal modulo this. So to each $a \in G^+_0$ we assign a prime number p_a and add $p^{-m}a$ to G_1 for all $m < \omega$. But in order to respect the $\hat{f}_t$'s, when $a \in \mathrm{dom}(\hat{f}_t)$ we have to also add $p^{-m}\hat{f}_t(a)$ to G_1, for all $m < \omega$. Of course all the $\hat{f}_s$'s have to respect this, so we also add $p^{-m}\hat{f}_{\bar{t}}(a)$ to G_1 for all $m < \omega$, where $\bar{t} = (t_0, \dots, t_n)$ and $\hat{f}_{\bar{t}} = \hat{f}_{t_n} \circ \cdots \circ \hat{f}_{t_0}$

(and $\hat{f}_{\bar{t}}(a)$ is well defined). This is done in 5.13. In 5.14–5.15 we analyze the groups $G_{(1,p)} = \{a \in G_1 : G_1 \models p^\infty | \, a\}$.

Definition 5.13. Let $(p_a : a \in G_0^+)$ be a sequence of pairwise distinct primes such that

$$a = \sum_{\ell<k} q_\ell x_\ell,\ q_\ell \in \mathbb{Z}^+,\ (x_\ell : \ell < k) \in \mathrm{seq}_k(X) \Rightarrow p_a \not| \, q_\ell.$$

(1) For $a \in G_0^+$, let
$$\mathbb{P}_a^{\leqslant_*} = \{p_b : b \in G_0^+, b \leqslant_* a\}.$$
(2) Let $G_1 = G_1[\mathfrak{m}] = G_1[\mathfrak{m}(T)] = G_1[T]$ be the subgroup of G_2 generated by
$$\{m^{-1}a : a \in G_0^+,\ m \in \omega \setminus \{0\} \text{ a power of a prime from } \mathbb{P}_a^{\leqslant_*}\}.$$
(3) For a prime p, let
$$G_{(1,p)} = \{a \in G_1 : a \text{ is divisible by } p^m \text{ for every } 0 < m < \omega\}.$$
(Notice that by Observation 2.5, $G_{(1,p)}$ is always a pure subgroup of G_1.)
(4) For $b \in G_1^+$, let $\mathbb{P}_b = \{p_a : a \in G_0^+, G_1 \models \bigwedge_{m<\omega} p_a^m | \, b\}$.
(5) For $t \in T$ and $\ell \in \{0,1,2\}$, let
$$H_{(\ell,t)} = \langle x : x \in \mathrm{dom}(f_t)\rangle^*_{G_\ell} \quad \text{and} \quad I_{(\ell,t)} = \langle x : x \in \mathrm{ran}(f_t)\rangle^*_{G_\ell}$$

Remark 5.14.
(1) If $a, b \in G_1^+$ and $\mathbb{Q}a = \mathbb{Q}b \subseteq G_2$, then $\mathbb{P}_a = \mathbb{P}_b$.
(2) If $a \leqslant_* b$, then $\mathbb{P}_a \subseteq \mathbb{P}_b$.

Proof. The proof is essentially due to Observation 2.5. □

Here we look more deeply at G_1. The crucial point is that any endomorphism of G_1 maps $G_{(1,p)} = \{a \in G_1 :$ for all $m < \omega,\ p^m | \, a\}$ into itself, and so the following characterization of $G_{(1,p)}$ will allow us to reconstruct information on the action of the f_t's on X, and thus eventually to reconstruct the tree T, to some extent.

Lemma 5.15.
(1) *For $b \in G_0^+$, we have that $\mathbb{P}_b^{\leqslant_*} = \mathbb{P}_b$.*
(2) *If $p = p_a$, $a \in G_0^+$, then*
$$G_{(1,p)} = \langle b \in G_0^+ : a \leqslant_* b\rangle^*_{G_1}.$$
(3) *For $t \in T$, $H_{(1,t)} := H_{(2,t)} \cap G_1$ and $I_{(1,t)} := I_{(2,t)} \cap G_1$ are pure in G_1.*
(4) *For $\hat{f}_t^2$ as in Definition 5.10(3c), $\hat{f}_t^2[H_{(1,t)}] \subseteq I_{(1,t)}$. We define*
$$\hat{f}_t^1 = \hat{f}_t^2 \restriction H_{(1,t)},$$
and for $\bar{t}$ a finite sequence of members of T, we let
$$\hat{f}_{\bar{t}}^1 = (\cdots \circ \hat{f}_{t_\ell}^1 \circ \cdots).$$

(5) *$\hat{f}^1_t \restriction H_{(1,t)} = \hat{f}^2_t \restriction H_{(1,t)}$ is into $I_{(1,t)} \leqslant G_1$ but it is not onto $I_{(1,t)}$.*

(6) *Assume $a = \sum_{\ell<k} q_\ell x_\ell \in G_0$, $k > 0$, $x_\ell \in X$, $q_\ell \in \mathbb{Q}^+$, $\bar{x} = (x_\ell : \ell < k) \in \mathrm{seq}_k(X)$ and $p = p_a$. If $b \in G_{(1,p)}$, then there are $j > 0$, $m > 0$ and, for $i < j$, $\bar{y}^i$, b_i and $q'_i \in \mathbb{Q}^+$ such that the following conditions are verified:*
   (a) *for $i < j$, $\bar{x} \leqslant^k_X \bar{y}^i$;*
   (b) *$(b_i = \sum_{\ell<k} q_\ell y^i_\ell : i < j)$ is linearly independent;*
   (c) *$b = \sum\{q'_i b_i : i < j\}$;*
   (d) *for $i < j$, $ma \leqslant_* mb_i$.*

Proof. Item (1) is easy. We prove item (2). The RTL inclusion is clear by 5.13(2). We prove the other implication. To this extent,

($*_1$) $a \in G^+_0$, $p = p_a$, and we let $W_p := \{b \in G^+_0 : a \leqslant_* b\}$.

We claim

($*_2$) W_p is a linearly independent subset of G_2, as a $\mathbb{Q}$-vector space.

[Why ($*_2$)? Let $k \geqslant 1$, $\bar{x} \in \mathrm{seq}_k(X)$, $\bar{q} \in (\mathbb{Z}^+)^k$ and $a = \sum\{q_\ell x_\ell : \ell < k\}$. (Recall that $a \in G^+_0$.) Without loss of generality, $\bar{x}$ is reasonable. Now, toward contradiction, assume that $n \geqslant 1$, $b_i \in W_p$ for $i < n$, $(b_i : i < n)$ is without repetitions and there are $q^i \in \mathbb{Q}^+$ for $i < n$, such that

($*_{2.1}$) $\sum\{q^i b_i : i < n\} = 0$.

For each $i < n$, let $b_i = \sum\{q_\ell x_{(i,\ell)} : \ell < k\}$, where $\bar{x} \leqslant^k_X \bar{x}_i = (x_{(i,\ell)} : \ell < k)$. As $a \in G^+_1$, clearly $n > 1$, and by 5.9(7) there is $i_* < n$ such that $\bar{x}_{i_*}$ is $<^k_X$-maximal in $\{\bar{x}_i : i < n\}$. As $\bar{x}$ is reasonable, so is $\bar{x}_{i_*}$ and so $x_{(i_*,n-1)}$ appears exactly once in ($*_{2.1}$), so recalling $q^{n-1} \in \mathbb{Q}^+$ we get a contradiction, and so ($*_2$) holds indeed.]

($*_3$) Let $\mathcal{U} \subseteq X$ be such that
   (a) if $y \in \mathcal{U}$, then $y \notin \langle W_p \rangle^*_{G_2}$;
   (b) $\mathcal{U} \cup W_p$ is linearly independent;
   (c) under conditions (a), (b), $\mathcal{U}$ is maximal.

Clearly $\mathcal{U}$ is well defined, and we have

($*_4$) (a) the disjoint union $\mathcal{U} \cup W_p$ is a basis of G_2, as a $\mathbb{Q}$-vector space;
   (b) let $h \in \mathrm{End}(G_2)$ be such that $h \restriction \mathcal{U}$ is the identity and $h(a) = 0$ for all $a \in W_p$.

Now we define

($*_5$) (a) $G'_1 := (\sum\{\mathbb{Q}y : y \in \mathcal{U}\}) + G_1$;
   (b) $G''_1 = \sum\{\mathbb{Q}_p y : y \in \mathcal{U}\} + G_1$.

Also, we have

($*_6$) (a) $h \restriction G''_1 \in \mathrm{End}(G''_1)$;
   (b) if $d \in G''_1$ and $G''_1 \models p^\infty \mid d$, then $d = 0$;
   (c) $G'_1 = G''_2$.

[Why $(*_6)$)? Concerning clause (a), we just have to prove that if $b \in G_0^+$, $p' \in \mathbb{P}_b$, so $p' = p_d$ for some $d \leqslant_* b$, then, for every $m < \omega$, $h(p_d^{-m} b) = p_d^{-m} h(b) \in G_1''$. Now, if $d = a$, then $p_d = p$ and $a \in W_p$, hence $h(b) = 0$, so in this case we are fine. If on the other hand $d \neq a$, then $p_d \neq p$. Notice that the support of $h(b)$ is a subset of $\mathcal{U}$. Now, $\{b' \in G_1'' : \mathrm{supp}(b') \subseteq \mathcal{U}\} = \bigoplus \{\mathbb{Q}_p x : x \in \mathcal{U}\}$, which is p_d-divisible, so clause (a) holds indeed. Finally clauses (b) and (c) hold by the definitions of $\mathbb{Q}_p$ and G_1'.]

Now, let c be any member of $G_{(1,p)}$. As $h \restriction G_1'' \in \mathrm{End}(G_1'')$, clearly $h(c) \in G_1''$, and as $m < \omega$ implies $p^{-m} c \in G_{(1,p)}$, clearly $G_1'' \models p^\infty | \, h(c)$. By $(*_6)$(b), it follows that $h(c) = 0$, but this implies that c belongs to the kernel of $h \restriction G_1$, which is $\langle W_p \rangle^*_{G_1}$. As c was any member of $G_{(1,p)}$, we are done. This concludes the proof of item (2).

Concerning item (3), notice

$$H_{(1,t)} = \langle \mathbb{Z} x : x \in \mathrm{dom}(f_t) \rangle^*_{G_1},$$
$$I_{(1,t)} = \langle \mathbb{Z} x : x \in \mathrm{ran}(f_t) \rangle^*_{G_1}.$$

Item (4) is by item (2) and the following observation, if $f_t(x) = y$, then we have $x \leqslant_* y$ (recall 5.7(2)), and so $\mathbb{P}_x \subseteq \mathbb{P}_y$ (cf. 5.14(2)). Concerning item (5), assume that $0 < n < \omega$, $t \in T_n \setminus T_{<n}$, $x \in X_{n-1} \setminus X_{<n-1}$ and let $y = f_t(x) \in X_n \setminus X_{<n}$ (cf. Observation 5.5). Notice that in particular, $x <_* y$. So p_y is well defined, since $y \in G_0^+$, and we have the following:

(a) $G_1 \models p_y \nmid x$, and so $H_{(1,t)} \models p_y \nmid x$ (as $H_{(1,t)}$ is pure in G_1; cf. item (3)).
(b) $G_1 \models \bigwedge_{m<\omega} p_y^m \, | \, y$.

[Why (b)? By the definition of G_1. Why (b)? Recalling that $x <_* y$.]

Do you mean to repeat "Why (b)?

But then, since by item (4) $\hat{f}_t \restriction H_{(1,t)}$ is an embedding of $H_{(1,t)}$ into $I_{(1,t)}$, we have that $\hat{f}_t[H_{(1,t)}] \models p_y \nmid f(x) \wedge f(x) = y$. On the other hand, since $I_{(1,t)}$ is pure in G_1 (cf. (3) of this lemma), we have that for every $m < \omega$, $p_y^{-m} y \in I_{(t,1)}$ (cf. 2.4). Finally, item (6) is by clause (2) and unraveling definitions. □

We now prove the main theorem of this section, namely Theorem 5.16. Notice that in 5.16(2) below, we prove more than needed in order to show that the set of endorigid groups in TFAB_ω is complete co-analytic, as, in combination with 5.16(1) and 5.16(3), it would suffice to show that if T is well-founded, then there is an endomorphism of G_1 which is not multiplication by an integer. We show that in addition, such an endomorphism can be taken to be one-to-one and such that $G_1 / f[G_1]$ is not torsion.

THEOREM 5.16. *Let* $\mathfrak{m}(T) \in \mathrm{K}_1^{\mathrm{ri}}(T)$.

(1) *We can modify the construction so that* $G_1[\mathfrak{m}(T)] = G_1[T]$ *has domain* ω *and the function* $T \mapsto G_1[T]$ *is Borel (for* T *a tree with domain* ω*).*

(2) *If T is not well-founded, then $G_1[T] = G_1$ has a one-to-one $f \in \mathrm{End}(G_1)$ which is not multiplication by an integer and such that $G_1/f[G_1]$ is not torsion.*

(3) *If T is well-founded, then $G_1[T]$ is endorigid.*

Proof. Item (1) is easy. We prove item (2). Let $(t_n : n < \omega)$ be a strictly increasing infinite branch of T. By Lemma 5.11(4), $(\hat{f}^2_{t_n} : n < \omega)$ is increasing, by Definition 5.10(3c), $\hat{f}^2_{t_n}$ embeds $H_{(2,t_n)}$ into $I_{(2,t_n)}$, thus $\hat{f}^2 = \bigcup_{n<\omega} \hat{f}^2_{t_n}$ is an embedding of G_2 into $\bigcup_{n<\omega} H_{(2,t_n)}$. Now, $(H_{(2,t_n)} : n < \omega)$ is a chain of pure subgroups of G_2 with limit G_2, because, recalling 5.2(e), we have that

$$H_{(2,t_n)} = \mathrm{dom}(f_{t_n}) \subseteq \mathrm{dom}(f_{t_{n+1}}) \subseteq H_{(2,t_{n+1})},$$

and by 5.2(c) we have that $\bigcup_{n<\omega} H_{(2,t_n)} = G_2$. Thus $\hat{f}^1 := \hat{f} \restriction G_1 = \bigcup_{n<\omega} \hat{f}^1_{t_n} = \bigcup_{n<\omega} \hat{f}^2_{t_n} \restriction H_{(1,t_n)}$ is an embedding of G_1 into G_1 (cf. Lemma 5.15(3), (5)). In fact we have that $\mathrm{dom}(\hat{f}^1_{t_n}) = H_{(1,t_n)}$ (cf. Lemma 5.15(3), (5)) and $G_1 = \bigcup_{n<\omega} H_{(1,t_n)}$, where $(H_{(1,t_n)} : n < \omega)$ is chain of pure subgroups of G_1 with limit G_1. Clearly $\hat{f}^1$ is not of the form $g \mapsto mg$ for some $m \in \mathbb{Z} \setminus \{0\}$, since for every $x \in \mathrm{dom}(f_t)$, we have $x \neq f_t(x)$ (cf. Observation 5.5(2)). We claim that $G_1/\hat{f}^1[G_1]$ is not torsion. To this extent, first of all notice that $X_0 \neq \emptyset$ (by Definition 5.2(a)) and $X_0 \cap \mathrm{ran}(f_{t_n}) = \emptyset$ (by Definition 5.2(d)). Thus, we have the following:

$$\mathrm{ran}(\hat{f}^1) \subseteq G^2_{X \setminus X_0} := \sum\{\mathbb{Q}x : x \in X \setminus X_0\} = \langle X \setminus X_0 \rangle^*_{G_2}.$$

Now, let $x \in X_0$. Then $x \in G_1 \setminus \mathrm{ran}(\hat{f}^1)$, and moreover for $q \in \mathbb{Q} \setminus \{0\}$,

$$qx \notin G^2_{X \setminus X_0} \text{ and so } qx \notin \mathrm{ran}(\hat{f}^1).$$

So, in particular, for every $0 < n < \omega$, we have that $nx \notin \mathrm{ran}(\hat{f}^1)$, hence $n(x/(\mathrm{ran}(\hat{f}^1)) \neq 0$. This concludes the proof of item (2).

We now prove item (3). To this extent, suppose that $(T, <_T)$ is well-founded and, letting $G_1 = G_1[T]$, suppose that $\pi \in \mathrm{End}(G_1)$. We shall show that there is $m \in \mathbb{Z}$ such that for every $a \in G_1$, $\pi(a) = ma$; i.e., G_1 is endorigid. We recall that the equivalence relation E_1 (used below) was defined in Definition 5.7(7).

Case 1: The set $Y = \{x/E_1 :$ for some $y \in x/E_1, \pi(y) \notin \mathbb{Q}y\}$ is infinite.

($*_1$) Choose x_i, n_i, for $i < \omega$, such that
- (a) n_i is increasing with i;
- (b) $x_i \in X_{n_{i+1}} \setminus X_{n_i}$;
- (c) $\pi(x_i) \notin \mathbb{Q}x_i$, $\mathrm{supp}(\pi(x_i)) \subseteq X_{n_{i+1}}$;
- (d) $X_{n_i} \cap x_i/E_1 = \emptyset$;
- (e) $(x_i/E_1 : i < \omega)$ are pairwise distinct (this actually follows).

Note that for $i < \omega$, we have

($*_2$) $\mathrm{supp}(\pi(x_i)) \subseteq x_i/E_1$, hence $\mathrm{supp}(\pi(x_i)) \subseteq X_{n_{i+1}} \setminus X_{n_i}$.

[Why? We apply 5.15(6) with $(x_i, \pi(x_i), 1, (1))$ here standing for $(a, b, k, (q_\ell : \ell < k))$ thereso, in particular, $p = p_{x_i}$. In order to be able to apply 5.15(6) we need that $b = \pi(a) \in G_{(1,p)}$, but this is clear in our case as $\pi \in \mathrm{End}(G_1)$ and $p = p_{x_i}$. But then applying 5.15(6) and writing $b = \pi(x_i)$ as there, we get what we need.]

For $r < \omega$, $(\mathrm{supp}(x_\ell) : \ell \leqslant r)$ is a sequence of non-empty sets and $\mathrm{supp}(x_\ell) \subseteq X_{n_{\ell+1}} \setminus X_{n_\ell}$, so it is a sequence of pairwise disjoint non-empty sets. Now, for $r < \omega$, let

$$x_r^+ = \sum_{\ell \leqslant r} x_\ell,\ p_r = p_{x_r^+} \text{ and } \bar{x}_r = (x_\ell : \ell \leqslant r).$$

As $\pi \in \mathrm{End}(G_1)$, clearly $\pi(x_r^+) \in G_{(1,p_r)}$, hence by 5.15(6) applied to x_r^+, $\pi(x_r^+)$ here standing for a, b there, we can find $j_r, m_r > 0$, and for $j < j_r$, $\bar{y}^{(r,j)}$, b_j^r, $q_j^r \in \mathbb{Q}^+$ such that the following hold:

$(*_3)$ (a) for $j < j_r$, $\bar{x}_r \leqslant_X^{r+1} \bar{y}^{(r,j)}$;
  (b) $(b_j^r = \sum_{\ell \leqslant r} y_\ell^{(r,j)} : j < j_r)$ is linearly independent;
  (c) $\pi(x_r^+) = \sum\{q_j^r b_j^r : j < j_r\}$;
  (d) for $j < j_r$, $m_r x_r^+ \leqslant_* m_r b_j^r$ (and $m_r b_j^r \in G_0^+$).

$(*_{3.1})$ We define $f_{()}^1$ as the identity on X, hence, for $j < j_r$, the following are equivalent:
  $(\cdot_1)$ $\bar{y}^{(r,j)} = \bar{x}_r$;
  $(\cdot_2)$ $f_{()}^1(\bar{x}_r) = \bar{y}^{(r,j)}$;
  $(\cdot_3)$ for all $0 < n < \omega$ and $\bar{t} \in T^n$, $f_{\bar{t}}^1(\bar{x}_r) \neq \bar{y}^{(r,j)}$.

As $\bar{x}_r \leqslant_X^{r+1} \bar{y}^{(r,j)}$ we can apply 5.8 and find a finite sequence $\bar{t}_j^r \in T^{<\omega}$ such that

$(*_4)$ if $\bar{x}_r <_X^{r+1} \bar{y}^{(r,j)}$, then
  (a) $f_{\bar{t}_j^r}^1(\bar{x}_r) = \bar{y}^{(r,j)}$;
  (b) $\hat{f}_{\bar{t}_j^r}^1(x_r^+) = b_j^r$;
  (c) for $\ell \leqslant r$ and $j < j_r$, we have $f_{\bar{t}_j^r}^1(x_\ell) \neq x_\ell$ and $\lg(\bar{t}_j^r) > 0$;
  (d) $\bar{t}_j^r = (t_{(j,\ell)}^r : \ell < \lg(\bar{t}_j^r))$;
  (e) if $j < j_r$ and $\ell < \lg(\bar{t}_j^r)$, then

$$t <_T t_{(j,\ell)}^r \ \Rightarrow \ \bigvee_{m \leqslant r} f_{\bar{t}_j^r \restriction \ell}(x_m) \notin \mathrm{dom}(f_t);$$

  (f) in (e) this is equivalent to the following condition:

$$t <_T t_{(j,\ell)}^r \ \Rightarrow \ f_{\bar{t}_j^r \restriction \ell}(x_r) \notin \mathrm{dom}(f_t);$$

  (g) $(\bar{t}_j^r : j < j_r)$ is without repetitions;
  (h) $(f_{\bar{t}_j^r}(x_r) : j < j_r)$ is without repetitions.

Why $(*_4)$? Concerning (e), recall 5.8. Concerning (f)–(h), recalling 5.9(10), note that if $\lg(\bar{t}^r_j) > 0$, then $\mathrm{dom}(f_{\bar{t}^r_j})$ is X_n for some $n < \omega$; so $x_r \in \mathrm{dom}(f_{\bar{t}^r_j})$ and

$(*_{4.1})$ $\ell \leqslant r \Rightarrow x_\ell \in \mathrm{dom}(f_{\bar{t}^r_j})$.

This concludes the proof of $(*_4)$.

$(*_5)$ For $r < \omega$ and $\ell \leqslant r$, we have $\pi(x_\ell) = \sum\{y^{(r,j)}_\ell : j < j_r\}$.

[Why? Because $\{y^{(r,j)}_\ell : j < j_r\} \subseteq x_\ell/E_1$ and $(x_\ell/E_1 : \ell \leqslant r)$ is a sequence of pairwise disjoint sets.]

Now, by induction on $r < \omega$, we choose i_r and y_r such that

$(*_6)$ (a) $i_r < j_r$, $y_r = f_{\bar{t}^r_{i_r}}(x_r) \neq x_r$, hence $\lg(\bar{t}^r_{i_r}) > 0$;
(b) if $r > 0$, then $f_{\bar{t}^r_{i_r}}(x_{r-1}) = y_{r-1}$.

Why $(*_6)$ is possible? For $r = 0$, recall that $\pi(x_r) \notin \mathbb{Q}x_r$. For $r \geqslant 1$, by $(*_5)$,

$$\sum\{f_{\bar{t}^r_j}(x_{r-1}) : j < j_r\} = \pi(x_{r-1}) = \sum\{f_{\bar{t}^{r-1}_j}(x_{r-1}) : j < j_{r-1}\}.$$

Now, by $(*_4)$(h) the sum in the right-hand side is without repetitions and of course $f_{\bar{t}^{r-1}_{i_{r-1}}}(x_{r-1})$ appears in it, hence it belongs to the support on the left-hand side, so for some $i_r < j_r$,

$$f_{\bar{t}^r_{i_r}}(x_{r-1}) = f_{\bar{t}^{r-1}_{i_{r-1}}}(x_{r-1}) = y_{r-1}.$$

As $f_{\bar{t}^r_{i_r}}(x_{r-1}) \neq x_{r-1}$, clearly $\lg(\bar{t}^r_{i_r}) > 0$ and so $x_r \neq y_r$. This proves $(*_6)$.

Now, $f_{\bar{t}^{r-1}_{i_{r-1}}}(x_{r-1}) = f_{\bar{t}^r_{i_r}}(x_{r-1})$ and $\bar{t}^{r-1}_{i_{r-1}}$ satisfies $(*_4)$(e), hence by 5.9(12) we have $\lg(\bar{t}^{r-1}_{i_{r-1}}) = \lg(\bar{t}^r_{i_r})$ and $\ell < \lg(\bar{t}^r_{i_r})$ implies $t^{r-1}_{(i_{r-1},\ell)} \leqslant_T t^r_{(i_r,\ell)}$. So $(\lg(\bar{t}^r_{i_r}) : r < \omega)$ is constant, say constantly k, and if $\ell < k$, then $(t^r_{(i_r,\ell)} : r < \omega)$ is a $\leqslant_T$-sequence. But $x_r \notin X_{n_r}$ and so $t^r_{(i_r,\ell)} \notin T_n$, hence $(t^r_{(i_r,\ell)} : r < \omega)$ is $<_T$-increasing, and so we reach a contradiction. This concludes the proof of Case 1.

Case 2: *The set* $Y = \{x/E_1 :$ *for some* $y \in x/E_1, \pi(y) \notin \mathbb{Q}y\}$ *is finite and* $\neq \emptyset$. Choose $x_0 \in X$ such that $\pi(x_0) \notin \mathbb{Q}x_0$. Let $n < \omega$ be such that $x_0 \in X_n$, and choose x_1 such that

$(\oplus_1)$ (a) $x_1 \in X \setminus \bigcup\{y/E_1 : y \in X_n\}$;
(b) $\pi(x_1) \in \mathbb{Q}x_1$.

[Why possible? By the assumption in Case 2.]

Notice now that

$(\oplus_2)$ For $\ell \in \{1, 2\}$, $\mathrm{supp}(x_\ell) \subseteq x_\ell/E_1$.

$(\oplus_3)$ By 5.15(6), there are $(\bar{t}_j, q_j : j < j_*)$ such that
(a) $\bar{t}_j$ $(j < j_*)$ are with no repetitions and $q_j \in \mathbb{Q}^+$;
(b) $\pi(x_0 + x_1) = \sum\{q_j f_{\bar{t}_j}(x_0 + x_1) : j < j_*\}$.

$(\oplus_4)$ We have

(a) $\pi(x_0) = \sum\{q_j f_{\bar{t}_j}(x_0) : j < j_*\}$;

(b) $\pi(x_1) = \sum\{q_j f_{\bar{t}_j}(x_1) : j < j_*\}$.

[Why? $\pi(x_0) - \sum\{q_j f_{\bar{t}_j}(x_0) : j < j_*\} = -\pi(x_1) + \sum\{q_j f_{\bar{t}_j}(x_1) : j < j_*\}$. Now the left-hand side has support $\subseteq x_0/E_1$ and right-hand side has support $\subseteq x_1/E_1$. As $x_0/E_1 \cap x_1/E_1 = \emptyset$, both the left-hand side and the right-hand side are 0, and so we are done.]

However $\pi(x_1) \in \mathbb{Q}x_1$ by choice, and so

$(\oplus_5)$ (a) for some $j < j_*$, $\bar{t}_j = (\,)$, without loss of generality, for $j = 0$;

(b) for $0 < j < j_*$, $\lg(\bar{t}_j) > 0$ (by $(\oplus_3)$(a)).

$(\oplus_6)$ For $i = 0, 1$, let $\mathcal{E}_i$ be the following equivalence relation on j_*:

$$\{(j_1, j_2) : f_{\bar{t}_{j_1}}(x_i) = f_{\bar{t}_{j_2}}(x_i)\}.$$

$(\oplus_7)$ $0/\mathcal{E}_1 = \{0\}$ and if $0 < j < j_*$, then

(a) $\sum\{q_\ell : \ell \in j/\mathcal{E}_1\} = 0$;

(b) $\sum\{q_\ell f_{\bar{t}_\ell}(x_1) : \ell \in j/\mathcal{E}_1\} = 0$.

[Why? Note that

$$\sum\{q_\ell f_{\bar{t}_\ell}(x_1) : \ell \in j/\mathcal{E}_1\} = \sum\{q_\ell : \ell \in j/\mathcal{E}_1\} f_{\bar{t}_j}(x_1).$$

So if $\sum\{q_\ell : \ell \in j/\mathcal{E}_1\} \neq 0$, then $f_{\bar{t}_j}(x_1)$ belongs to the support of the right-hand side of $(\oplus_4)$(b) but the support of this object is $\{x_1\}$ (by $(\oplus_1)$(b)) and $x_1 \neq f_{\bar{t}_j}(x_1)$, as $\bar{t}_j \neq (\,)$, together we reach a contradiction, and so we have $(\oplus_7)$(a), (b).]

$(\oplus_8)$ E_1 refines E_0.

[Why? Assume that $j_1, j_2 < j_*$ and $j_1 \mathcal{E}_1 j_2$. This means that $f_{\bar{t}_{j_1}}(x_1) = f_{\bar{t}_{j_2}}(x_1)$. By 5.2(j), as $x_1 \notin X_n$, we have that $X_n \subseteq \mathrm{dom}(f_{\bar{t}_{j_1}}) \cap \mathrm{dom}(f_{\bar{t}_{j_2}})$ and $f_{\bar{t}_{j_1}} \restriction X_n = f_{\bar{t}_{j_2}} \restriction X_n$. As $x_0 \in X_n$, we get that $f_{\bar{t}_{j_1}}(x_0) = f_{\bar{t}_{j_2}}(x_0)$, which means $j_1 \mathcal{E}_0 j_2$, as desired.]

$(\oplus_9)$ $0/E_0 = \{0\}$ and if $0 < j < j_*$, then

(a) $\sum\{q_\ell : \ell \in j/\mathcal{E}_0\} = 0$;

(b) $\sum\{q_\ell f_{\bar{t}_0}(x_\ell) : \ell \in j/\mathcal{E}_0\} = 0$.

[Why? By $(\oplus_7)$+$(\oplus_8)$, recalling 5.2(j).]

$(\oplus_{10})$ $\pi(x_0) = q_0 x_0$ (follows by $(\oplus_9)$(b)).

But $(\oplus_{10})$ contradicts our choice of x_0, as $\pi(x_0) \notin \mathbb{Q}x_0$.

Case 3: *The set* $Y = \{x/E_1 :$ *for some* $y \in x/E_1, \pi(y) \notin \mathbb{Q}y\}$ *is empty.* For $x \in X$, let $\pi(x) = q_x x$. Now, first of all we claim

$(\star_{0.1})$ If $a \in G_1^+$, $\pi(a) = 0$ and $x/E_1 \cap \mathrm{supp}(a) = \emptyset$, then $\pi(x) = 0$.

[Why? Let $p = p_{x+a}$. Firstly, notice that $\pi(x+a) = \pi(x) + \pi(a) = q_x x$. Secondly, recalling that $x/E_1 \cap \mathrm{supp}(a) = \emptyset$, notice that by 5.15(2), we have that $x \notin G_{(1,p)}$, but this contradicts that $x + a \in G_{(1,p)}$, as $\pi(x+a) = q_x x$.]

$(\star_{0.2})$ If π is not one-to-one, then π of the form $a \mapsto 0$ for all $a \in G_1$.

[Why? Let $a \in G_1^+$ be such that $\pi(a) = 0$. If $y \in X \setminus \mathrm{supp}(a)$, then we get that $\pi(y) = 0$, by applying $(\star_{0.1})$ to (a, y). If $y \in \mathrm{supp}(a)$, choose $x \in X \setminus \mathrm{supp}(a)$ and apply $(\star_{0.1})$ to (x, y).]

$(\star_{0.3})$ Without loss of generality, π is one-to-one.

[Why? Otherwise, by $(\star_{0.2})$, π is multiplication by an integer, and so we are done.]

$(\star_1)$ $(q_x : x \in X)$ is constant.

Why $(\star_1)$? Choose $x_0, x_1 \in X$ such that $q_{x_0} \neq q_{x_1}$ and, if possible, they are both $\neq 0$. Let $n < \omega$ be such that $x_0, x_1 \in X_n$ and choose a $<_X$-minimal $x_2 \in X_{n+1} \setminus X_n$, possible by 5.9(2). Let $a = x_0 + x_1 + x_2$, $p = p_a$ (cf. 5.13) and $\bar{x} = (x_0, x_1, x_2)$. As $a \in G_{(1,p)}$ and $\pi \in \mathrm{End}(G_1)$, clearly $\pi(a) = b \in G_{(1,p)}$ and so by 5.15(6) there are $j < \omega$ and, for $i < j$, $\bar{y}^i \in \mathrm{seq}_3(X)$ and $q^i \in \mathbb{Q}^+$ such that $\bar{x} \leqslant^3_X \bar{y}^i$ and

$(\star_{1.1})$ $b = \sum_{i<j} q^i (\sum_{\ell<3} y^i_\ell)$.

Notice that by $(\star_{0.2})$ we have $j > 0$, and without loss of generality, we can assume that for $i < j-1$, we have $\bar{y}^{j-1} \not\leqslant^3_X \bar{y}^i$ and also that $\bar{x}$ is reasonable (so the $\bar{y}^i$'s are also reasonable). Also,

$(\star_{1.2})$ $b = \pi(a) = q_{x_0} x_0 + q_{x_1} x_1 + q_{x_2} x_2$.

As $i < j-1$ implies $\bar{y}^{j-1} \not\leqslant^3_X \bar{y}^i$, clearly $y^{j-1}_2 \notin \{y^i_\ell : i < j-1, \ell \leqslant 2\} \cup \{y^{j-1}_0, y^{j-1}_1\}$ (by 5.9(9) and $\bar{y}^{j-1} \in \mathrm{seq}_3(X)$), and so y^{j-1}_2 appears exactly once in the right-hand side of equation $(\star_{1.1})$, and so it appears in left-hand side of $(\star_{1.1})$, so $y^{j-1}_2 \in \mathrm{supp}(b) \subseteq \{x_0, x_1, x_2\}$. But $x_2 \notin x_0/E_1 \cup x_1/E_1$, as $x_0, x_1 \in X_n$ and $x_2 \in X_{n+1} \setminus X_n$ is $<_X$-minimal. On the other hand, clearly $y^i_\ell \in x_\ell/E_1$ for $\ell \leqslant 2$ and $i < j$. Hence, necessarily $y^{j-1}_2 = x_2$. Finally, as x_2 is $<_X$-minimal and for some $\bar{t} \in T^{<\omega}$, $f_{\bar{t}}(\bar{x}) = \bar{y}^{j-1}$, necessarily, $f_{\bar{t}}(x_2) = y^{j-1}_2$, so clearly $\bar{t} = (\,)$. Hence, $\bar{y}^{j-1} = \bar{x}$ and of course $\bar{x} \leqslant^3_X \bar{y}$ implies $f_{\bar{t}}(\bar{x}) \leqslant^3_X \bar{y}$. Thus, by the statement after $(\star_{1.1})$, $j = 1$ and $\bar{y}^0 = \bar{x}$. So we have

$(\star_{1.3})$ $q_{x_0} x_0 + q_{x_1} x_1 + q_{x_2} x_2 = q^0(y^0_0 + y^0_1 + y^0_2) = q^0(x_0 + x_1 + x_2)$.

Thus, $q_{x_0} = q^0 = q_{x_1}$, contradicting our assumption that $q_{x_0} \neq q_{x_1}$.

$(\star_2)$ Let $q_x = q_*$ for $x \in X$ (recalling $(\star_1)$).

$(\star_3)$ q_* is an integer.

Why $(\star_3)$? Let $q_* = \frac{m}{n}$, with $m, n \in \mathbb{Z}^+$, m and n coprimes. Suppose that there is a prime p such that $p \,|\, n$. Then we easily reach a contradiction noticing that

(·) if $x \in X$ is $<_1$-minimal and r is a prime different from p_x, then $r \nmid x$;
(·) there are $<_1$-minimal $x, y \in X$ such that $x \neq y$.

It follows that $n = 1$ and so $(*_3)$ holds.

Hence, our proof is complete, as Cases 1 and 2 are contradictory, while in Case 3 we showed that the arbitrary $\pi \in \mathrm{End}(G_1)$ is indeed multiplication by an integer. □

Remark 5.17. Notice that in the proof of 5.16, Cases 2 and 3 do not use the assumption that T is well-founded and so for an arbitrary tree T (as in 5.1) and $\pi \in \mathrm{End}(G_1[T])$, we have

(a) Case 1 happens if only if T is not well-founded;
(b) Case 2 never happens;
(c) if Case 3 happens, then π is multiplication by an integer.

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