REMARKS ON SOME CARDINAL INVARIANTS AND PARTITION RELATIONS

ASHUTOSH KUMAR AND SAHARON SHELAH

Abstract. We answer some questions about two cardinal invariants associated with separating and almost disjoint families and a partition relation involving indecomposable countable linear orderings.

1. Introduction

The aim of this paper is to prove some results about two cardinals invariants and partition relations. These cardinal invariants (denoted $\mathfrak{ls}(\mathfrak{c})$ and $\mathfrak{la}(\mathfrak{c})$) were first introduced by Higuchi, Lempp, Raghavan and Stephan in [2]. The primary motivation behind studying them was to shed some light on the order dimension of the Turing degrees.

Definition 1.1 ([2]). *Let κ be an infinite cardinal.*

(1) *$\mathfrak{ls}(\kappa)$ is the least cardinality of a family $\mathcal{F} \subseteq \mathcal{P}(\kappa)$ that separates countable subsets of κ from points in the following sense: For every countable $A \subseteq \kappa$ and $\alpha \in \kappa \setminus A$, there exists $X \in \mathcal{F}$ such that $\alpha \in X$ and $A \cap X = \emptyset$.*
(2) *$\mathfrak{la}(\kappa)$ is the least cardinal λ such that $\mathsf{cf}(\lambda) \geq \omega_1$ and there exists an almost disjoint family $\mathcal{A} \subseteq [\lambda]^{\mathsf{cf}(\lambda)}$ with $|\mathcal{A}| \geq \kappa$. Here $X, Y \in \mathcal{A}$ are almost disjoint iff $|X \cap Y| < \mathsf{cf}(\lambda)$.*

In [2], the authors showed $\omega_1 \leq \mathfrak{ls}(\kappa) \leq \mathfrak{la}(\kappa)$ for every uncountable cardinal κ and asked if this inequality could be strict when κ is the successor of a cardinal of uncountable cardinality. Another version of this question appears in [5] (Question 5.3) for the case $\kappa = \mathfrak{c} = \omega_3$. We positively answer both of them by showing the following.

Theorem 1.2. *Assume $V \models GCH$. Then there is a ccc forcing $\mathbb{P}$ such that $V^{\mathbb{P}} \models \mathfrak{ls}(\omega_3) = \omega_1 < \mathfrak{la}(\omega_3) = \omega_2 < \mathfrak{c} = \omega_3$.*

In the next section, we generalize our construction to separate $\mathfrak{la}(\kappa)$ and a stronger variant of $\mathfrak{ls}(\kappa)$ defined as follows.

Definition 1.3. *Let $\omega \leq \mu \leq \kappa \leq \lambda$ be cardinals. $\mathfrak{ls}(\lambda, \kappa, \mu)$ is the least cardinality of a family $\mathcal{F} \subseteq \mathcal{P}(\lambda)$ that is a (κ, μ)-separating family on λ which means the following: For every $A \in [\lambda]^{<\kappa}$ and $B \in [\lambda \setminus A]^{<\mu}$, there exists $X \in \mathcal{F}$ such that $B \subseteq X$ and $A \cap X = \emptyset$.*

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Observe that $\mathfrak{ls}(\lambda, \omega_1, \omega) = \mathfrak{ls}(\lambda)$. While $\mathfrak{ls}(\lambda) \le \mathfrak{la}(\lambda)$ holds for every uncountable λ, we will show that there is no such relation between $\mathfrak{la}(\lambda)$ and $\mathfrak{ls}(\lambda, \kappa, \mu)$ for arbitrary $\mu, \kappa < \lambda$.

Theorem 1.4. *Suppose $\mu < \kappa$ are infinite regular cardinals, κ is Mahlo and $\lambda > 2^\kappa$. There is a $< \mu$-closed κ-cc forcing $\mathbb{P}$ such that the following hold in $V^{\mathbb{P}}$.*

1. *$\kappa = \mu^+$.*
2. *$\mathfrak{ls}(\lambda, \kappa, \mu) = \kappa$.*
3. *If $\omega_1 \le \mu = 2^{<\mu}$, then $\mathfrak{la}(\lambda) = \mu < \mathfrak{ls}(\lambda, \kappa, \mu)$.*
4. *If $\mu = \omega$, then $\mathfrak{ls}(\lambda, \kappa, \mu) < \mathfrak{la}(\lambda)$.*

An order-theoretic variant of $\mathfrak{ls}(\kappa)$ studies in [5] is $\mathfrak{los}(\kappa)$. It equals the order dimension of the Turing degrees when $\kappa = \mathfrak{c}$ (see Corollary 2.11 in [5]).

Definition 1.5. *Let κ be an infinite cardinal. Define $\mathfrak{los}(\kappa)$ to be the least cardinality of a family $\mathcal{F}$ of linear orders on κ that separates countable subsets of κ from points in the following sense: For every countable $A \subseteq \kappa$ and $\alpha \in \kappa \setminus A$, there exists $\prec$ in $\mathcal{F}$ such that for every $\beta \in A$, $\beta \prec \alpha$.*

Note that $\mathfrak{los}(\kappa) \le \mathfrak{ls}(\kappa) \le \mathfrak{la}(\kappa)$ and each of these two inequalities can be strict at $\kappa = \omega_3$ (by Theorem 1.2 above and Lemma 5.1 in [5]). So we ask the following.

Question 1.6. *Is it consistent to have $\mathfrak{los}(\kappa) < \mathfrak{ls}(\kappa) < \mathfrak{la}(\kappa)$ for some infinite cardinal κ? What if $\kappa = \omega_3$?*

1.1. Partition relations.

Definition 1.7. *An order type φ is unionwise indecomposable iff for every linear ordering $(L, \prec)$ of type φ and a partition $L = A \sqcup B$, at least one of $(A, \prec)$ and $(B, \prec)$ contains a subordering of type φ.*

Let $\varphi, \psi, \varphi_0, \varphi_1, \psi_0, \psi_1$ be order types. Recall that we write

$$\binom{\psi}{\varphi} \longrightarrow \begin{pmatrix} \psi_0 & \psi_1 \\ \varphi_0 & \varphi_1 \end{pmatrix}$$

to denote the following statement: Whenever $(X, \prec_0)$ and $(Y, \prec_1)$ are linear orderings of type ψ and φ respectively and $c : X \times Y \to 2$, there exist $A \subseteq X$ and $B \subseteq Y$ such that one of the following holds.

(a) $(A, \prec_0)$ has type ψ_0, $(B, \prec_1)$ has type φ_0 and $c \restriction (A \times B)$ is constantly 0.
(b) $(A, \prec_0)$ has type ψ_1, $(B, \prec_1)$ has type φ_1 and $c \restriction (A \times B)$ is constantly 1.

The following questions were raised by Klausner and Weinert (Questions (C) and (D) in [4]).

Question 1.8 ([4])**.** *Does the following hold for all countable ordinals α and unionwise indecomposable countable order types φ?*

$$\binom{\omega_1}{\varphi} \longrightarrow \begin{pmatrix} \alpha & \alpha \\ \varphi & \varphi \end{pmatrix}$$

Question 1.9 ([4])**.** *Is it consistent to have the following for all countable ordinals α and unionwise indecomposable countable order types φ?*

$$\binom{\omega_1}{\varphi} \longrightarrow \begin{pmatrix} \omega_1 & \alpha \\ \varphi & \varphi \end{pmatrix}$$

In the final section, we will show that the answer to both of these questions is yes. In fact, we have the following.

Theorem 1.10. *Let $c : \omega_1 \times L \to K$ where $K < \omega$ and $(L, \prec_L)$ is a unionwise indecomposable countable linear order. Then for each $\alpha < \omega_1$, there exist $A \in [\omega_1]^\alpha$ and $B \subseteq L$ such that $(B, \prec_L) \cong (L, \prec_L)$ and $c \restriction (A \times B)$ is constant.*

Theorem 1.11. *Assume Martin's axiom plus $\mathfrak{c} > \omega_1$. Let $c : \omega_1 \times L \to K$ where $K < \omega$ and $(L, \prec_L)$ is a unionwise indecomposable countable linear order. Then there exist $A \in [\omega_1]^{\omega_1}$ and $B \subseteq L$ such that $(B, \prec_L) \cong (L, \prec_L)$ and $c \restriction (A \times B)$ is constant.*

In both of these theorems, if $L = \mathbb{Q}$ is the rationals, then we can also get that B is somewhere dense in $\mathbb{Q}$ (Lemma 4.8).

2. Consistency of $\mathfrak{ls}(\omega_3) < \mathfrak{la}(\omega_3)$

A natural attempt to get a model of $\mathfrak{ls}(\omega_3) = \omega_1 < \mathfrak{la}(\omega_3) = \omega_2$ would be to start with a model of GCH and add ω_3 subsets of ω_1 using countable or finite conditions. Both of these fail.

Fact 2.1. *Assume $V \models GCH$. Let $\mathbb{P}$ consist of all partial functions from ω_3 to 2 such that either $(\forall p \in \mathbb{P})(|\mathsf{dom}(p)| < \omega)$ or $(\forall p \in \mathbb{P})(|\mathsf{dom}(p)| < \omega_1)$. Then $V^{\mathbb{P}} \models \mathfrak{ls}(\omega_3) = \mathfrak{la}(\omega_3)$.*

Proof. Note that both of these forcings preserve all cofinalities (and hence cardinals). If $\mathbb{P}$ consists of all finite partial functions from ω_3 to 2, then Lemma 5.1 in [5] implies that $V^{\mathbb{P}} \models \mathfrak{la}(\omega_3) = \mathfrak{ls}(\omega_3) = \omega_2$. So assume that $\mathbb{P}$ consists of all countable partial functions from ω_3 to 2. Then $\mathbb{P}$ does not add any new countable set of ordinals. Since $V \models 2^\omega = \omega_1$, it follows that $V \cap 2^{<\omega_1} = V^{\mathbb{P}} \cap 2^{<\omega_1}$ has size ω_1 in $V^{\mathbb{P}}$. Furthermore, as $V^{\mathbb{P}} \models 2^{\omega_1} \geq \omega_3$ we can find a family $\mathcal{F} \in V^{\mathbb{P}}$ consisting of ω_3 distinct subsets of ω_1. For each $A \in \mathcal{F}$, define $S_A = \{1_A \restriction \alpha : \alpha < \omega_1\}$. Observe that $\{S_A : A \in \mathcal{F}\}$ is a mod countable almost disjoint family of subsets of $2^{<\omega_1}$. Since $V^{\mathbb{P}} \models |2^{<\omega_1}| = \omega_1$, it follows that $V^{\mathbb{P}} \models \mathfrak{la}(\omega_3) \leq \omega_1$. As $\omega_1 \leq \mathfrak{ls}(\omega_3) \leq \mathfrak{la}(\omega_3)$, it follows that $V^{\mathbb{P}} \models \mathfrak{ls}(\omega_3) = \mathfrak{la}(\omega_3) = \omega_1$. □

An infinite ordinal δ is indecomposable iff for each $X \subseteq \delta$, either $\mathsf{otp}(X) = \delta$ or $\mathsf{otp}(\delta \setminus X) = \delta$. Recall that the following are equivalent.

(i) δ is infinite and indecomposable.
(ii) Whenever A, B are sets of ordinals of order type $< \delta$, $\mathsf{otp}(A \cup B) < \delta$.
(iii) $\delta = \omega^\alpha$ (ordinal exponentiation) for some $\alpha \geq 1$.

Definition 2.2. *For an uncountable cardinal κ and an indecomposable ordinal $\delta < \kappa$, define the forcing $\mathbb{Q}_{\kappa,\delta}$ as follows. $p \in \mathbb{Q}_{\kappa,\delta}$ iff the following hold.*

(i) *p is a function, $\mathsf{dom}(p) \subseteq \kappa$ and $\mathsf{range}(p) \subseteq 2$.*
(ii) *$\mathsf{otp}(\mathsf{dom}(p)) < \delta$.*
(iii) *$\{\alpha \in \mathsf{dom}(p) : p(\alpha) = 1\}$ is finite.*

For $p, q \in \mathbb{Q}_{\kappa,\delta}$, define $p \leq q$ iff $q \subseteq p$.

The following lemma shows that if $\delta \geq \omega_1$, then $\mathbb{Q}_{\kappa,\delta}$ collapses ω_1.

Lemma 2.3. *Let $\omega_1 \leq \delta < \kappa$ where δ is indecomposable. Then $V^{\mathbb{Q}_{\kappa,\delta}} \models |\delta| = \omega$.*

Proof. Suppose $V \models |\delta| = \theta \geq \omega_1$. Choose $\alpha < \theta^+$ such that $\delta = \omega^{\theta+\alpha} = \theta \cdot \omega^\alpha = \theta \cdot \gamma$ where $\gamma = \omega^\alpha$. Let G be $\mathbb{Q}_{\kappa,\delta}$-generic over V and $F = \bigcup G$. Then $F : \kappa \to 2$. Define $W = \{\beta < \delta : F(\beta) = 1\}$. An easy density argument shows that $\mathsf{otp}(W) = \omega$. For each $k < \omega$, let α_k be the kth member of W. Choose $\xi_k < \theta$ and $j_k < \gamma$ such that $\alpha_k = \theta \cdot j_k + \xi_k$. Define $h : \omega \to \theta$ by $h(k) = \xi_k$. Another density argument shows that for every $X \in V \cap [\theta]^\theta$, $\mathsf{range}(h) \cap X \neq \emptyset$. It follows that $V[G] \models |\delta| = |\theta| = \omega$. □

Recall that a forcing $\mathbb{Q}$ has ω_1 as a precaliber iff for every uncountable $A \subseteq \mathbb{Q}$, there exists an uncountable $B \subseteq A$ such that every finite set of conditions in B has a common extension in $\mathbb{Q}$. It is easy to see that if $\mathbb{Q}$ has ω_1 as a precaliber, then it satisfies ccc.

Lemma 2.4. *Suppose κ is uncountable, $\delta < \omega_1$ is indecomposable and $\mathbb{Q}_{\kappa,\delta}$ is as in Definition 2.2. Then $\mathbb{Q}_{\kappa,\delta}$ has ω_1 as a precaliber.*

Proof. Let $\langle p_i : i < \omega_1 \rangle$ be a sequence of conditions in $\mathbb{Q}$. Put $D_i = \mathsf{dom}(p_i)$, $A_i = \{\alpha \in D_i : p_i(\alpha) = 0\}$, $B_i = \{\alpha \in D_i : p_i(\alpha) = 1\}$ and $D = \bigcup\{D_i : i < \omega_1\}$. Let $\gamma = \mathsf{otp}(D)$. Clearly, $\gamma < \omega_2$. Let $h : \gamma \to D$ be the order preserving bijection. By replacing each D_i with $h^{-1}[D_i]$, WLOG, we can assume that $D_i \subseteq \gamma$. By induction on γ, we will show that there exists $X \in [\omega_1]^{\omega_1}$ such that for every $i, j \in X$, p_i and p_j are compatible (and hence $p_i \cup p_j \in \mathbb{Q}_{\kappa,\delta}$ as $\mathsf{otp}(D_i \cup D_j) < \delta$). This suffices since any finite set S of conditions in $\langle p_i : i \in X \rangle$ will have a common extension (namely its union) in $\mathbb{Q}_{\kappa,\delta}$.

Case 1: γ is a successor ordinal. Let $\gamma = \xi + 1$. Applying the inductive hypothesis to the sequence $\langle p_i \restriction \xi : i < \omega_1 \rangle$, we can find $Y \in [\omega_1]^{\omega_1}$ such that for every $i, j \in Y$, $p_i \restriction \xi$ and $p_j \restriction \xi$ are compatible. Choose $X \in [Y]^{\omega_1}$ and $k < 2$ such that either $(\forall i \in X)(\xi \notin D_i)$ or $(\forall i \in X)(\xi \in D_i \wedge p_i(\xi) = k)$. Then X is as required.

Case 2: $\mathsf{cf}(\gamma) = \omega$. Since each B_i is a finite subset of γ and $\mathsf{cf}(\gamma) = \omega$, we can choose $Y \in [\omega_1]^{\omega_1}$ and $\gamma' < \gamma$ such that for every $i \in Y$, $B_i \subseteq \gamma'$. Applying the inductive hypothesis to $\langle p_i \restriction \gamma' : i \in Y \rangle$, we can find $X \in [Y]^{\omega_1}$ such that for every $i, j \in X$, $p_i \restriction \gamma'$ and $p_j \restriction \gamma'$ are compatible. Since for every $i \in X$, $p_i \restriction [\gamma', \gamma)$ is constantly 0, it follows that for every $i, j \in X$, p_i and p_j are compatible.

Case 3: $\mathsf{cf}(\gamma) = \omega_1$. Let $\langle \gamma_\xi : \xi < \omega_1 \rangle$ be a continuously increasing cofinal sequence in γ. Since D_i's are countable subsets of γ, we can choose a club $E \subseteq \omega_1$ consisting of limit ordinals such that for every $\xi \in E$ and $i < \xi$, $D_i \subseteq \gamma_\xi$.

Let $F = \{\xi \in E : (\forall i > \xi)(D_i \cap \gamma_\xi \text{ is unbounded in } \gamma_\xi)\}$. We claim that F is countable. Suppose not and fix a strictly increasing sequence $\langle \xi(i) : i < \omega_1 \rangle$ in F. Choose j such that $\xi(\delta) < j < \omega_1$. Then $\sup(D_j \cap \gamma_{\xi(i)}) = \gamma_{\xi(i)}$ for every $i < \delta$. Define $f : \delta \to D_j$ by $f(i) = \min([\gamma_{\xi(i)}, \gamma_{\xi(i+1)}) \cap D_j)$. Then f is strictly increasing and hence $\mathsf{otp}(\mathsf{range}(f)) = \delta$. But this implies that $\mathsf{otp}(D_j) \geq \mathsf{otp}(\mathsf{range}(f)) = \delta$ which is impossible. So F must be countable.

Next fix a club $C \subseteq E \setminus F$ and a function $h : C \to \omega_1$ such that for every $\xi \in C$, $h(\xi) > \xi$ and $\sup(D_{h(\xi)} \cap \gamma_\xi) < \gamma_\xi$. It follows that the function $g : C \to \omega_1$ defined by $g(\xi) = \min(\{\xi' < \xi : \sup(D_{h(\xi)} \cap \gamma_\xi) < \gamma_{\xi'}\})$ is regressive on C. By Fodor's lemma, we can find a stationary $S \subseteq C$ and $\xi_\star < \omega_1$ such that $\min(S) > \xi_\star$ and for every $\xi \in S$, $D_{h(\xi)} \cap \gamma_\xi \subseteq \gamma_\star = \gamma_{\xi_\star}$. Let $T \in [S]^{\omega_1}$ be such that for every $\xi_1 < \xi_2$ in T, $\xi_1 < h(\xi_1) < \xi_2 < h(\xi_2)$. Put $Y = h[T]$ and note that for every $i < j$ in Y, $D_i \cap D_j \subseteq \gamma_\star$. Applying the inductive hypothesis to $\langle p_i \restriction \gamma_\star : i \in Y \rangle$, choose

$X \in [Y]^{\omega_1}$ such that for every $i, j \in X$, $p_i \restriction \gamma_\star$ and $p_j \restriction \gamma_\star$ are compatible. Since for every $i < j$ in X, $D_i \cap D_j \cap [\gamma_\star, \gamma) = \emptyset$, it follows that $\langle p_i : i \in X\rangle$ has pairwise compatible functions.

As $\gamma < \omega_2$, there are no more cases and we are done. □

Lemma 2.5. *Let κ be an uncountable cardinal. Let $\mathbb{P}$ be the finite support product of $\mathbb{Q}_{\kappa,\delta}$'s where δ runs over the set of all countable indecomposable ordinals. Then the following hold.*

(1) *$\mathbb{P}$ has ω_1 as a precaliber.*
(2) $V^{\mathbb{P}} \models \mathfrak{ls}(\kappa) = \omega_1$.

Proof. (1) Let $\langle p_i : i < \omega_1\rangle$ be a sequence in $\mathbb{P}$. By the Δ-system lemma, we can find $Y \in [\omega_1]^{\omega_1}$ and a finite set R of indecomposable countable ordinals such that for every $i < j$ in Y, $\mathsf{dom}(p_i) \cap \mathsf{dom}(p_j) = R$. Using Lemma 2.4, we can choose $X \in [Y]^{\omega_1}$ such that for every $i, j \in X$ and $\delta \in R$, $p_i(\delta)$ and $p_j(\delta)$ are compatible in $\mathbb{Q}_{\kappa,\delta}$. It follows any finite set of conditions in $\langle p_i : i \in X\rangle$ has a common extension in $\mathbb{P}$. Therefore $\mathbb{P}$ has ω_1 as a precaliber.

(2) Let G be $\mathbb{P}$-generic over V. By Clause (1), all cofinalities (and hence cardinals) from V are preserved in $V[G]$. Since $\kappa \geq \omega_1$, it is easy to see that $\mathfrak{ls}(\kappa) \geq \omega_1$ – For any countable $\mathcal{F} \subseteq \mathcal{P}(\kappa)$, consider $A = \{\min(X) : X \in \mathcal{F}\}$ and $\alpha \in \kappa \setminus A$. For the other inequality, we'll show that in $V[G]$, there is a family $\mathcal{F}$ of size ω_1 that separates countable subsets of κ from points. For each indecomposable $\delta < \omega_1$, define $X_\delta = \{\alpha < \kappa : (\exists p \in G)(\delta \in \mathsf{dom}(p) \wedge \alpha \in \mathsf{dom}(p(\delta)) \wedge p(\delta)(\alpha) = 1)\}$. Put $\mathcal{F} = \{X_\delta : \delta < \omega_1 \text{ is indecomposable}\}$. We claim that $V[G] \models \mathcal{F}$ separates countable subsets of κ from points. For suppose $A \subseteq \kappa$ is countable and $\alpha \in \kappa \setminus A$. Since $\mathbb{P}$ satisfies ccc, there exists $B \in V \cap [\kappa]^{<\omega_1}$ such that $A \subseteq B$ and $\alpha \notin B$. Now a simple density argument shows that there exists $p \in G$ and $\delta \in \mathsf{dom}(p)$ with $\delta > \mathsf{otp}(B)$ such that $B \cup \{\alpha\} \subseteq \mathsf{dom}(p(\delta))$, $(\forall \beta \in B)(p(\delta)(\beta) = 0)$ and $p(\delta)(\alpha) = 1$. This means that $\alpha \in X_\delta$ and $X_\delta \cap A = \emptyset$. Hence $\mathcal{F}$ separates countable subsets of κ from points. □

Theorem 2.6. *Suppose $\omega_1 \leq \theta$ and $2^\theta < \kappa = \kappa^\omega$. Let $\mathbb{P}$ be the finite support product of $\mathbb{Q}_{\kappa,\delta}$'s where δ runs over the set of all countable indecomposable ordinals. Then $V^{\mathbb{P}} \models \mathfrak{c} = \kappa$, $\mathfrak{ls}(\kappa) = \omega_1$ and $\theta < \mathfrak{la}(\kappa)$.*

Proof. Since $\mathbb{P}$ satisfies ccc and $|\mathbb{P}| = \kappa^\omega = \kappa$, a standard name counting argument shows that $V^{\mathbb{P}} \models \mathfrak{c} = |\mathcal{P}(\omega)| \leq \kappa$. To see that $V^{\mathbb{P}} \models \mathfrak{c} \geq \kappa$, just note that $\mathbb{Q}_{\kappa,\omega} \lessdot \mathbb{P}$ and $\mathbb{Q}_{\kappa,\omega}$ is the forcing for adding κ Cohen reals. Furthermore, Lemma 2.5 implies that $V^{\mathbb{P}} \models \mathfrak{ls}(\kappa) = \omega_1$. So we only need to check that $V^{\mathbb{P}} \models \theta < \mathfrak{la}(\kappa)$.

Towards a contradiction, assume $V^{\mathbb{P}} \models \mathfrak{la}(\kappa) \leq \theta$. Then we can find $p \in \mathbb{P}$, $\lambda \leq \theta$ and $\langle \mathring{A}_i : i < \kappa\rangle$ such that the following hold.

(i) $\mathsf{cf}(\lambda) = \mu \geq \omega_1$.
(ii) For every $i < \kappa$, $p \Vdash \mathring{A}_i \in [\lambda]^\mu$.
(iii) For all $i < j < \kappa$, $p \Vdash |\mathring{A}_i \cap \mathring{A}_j| < \mu$.

For each $i < \kappa$, define $B_i = \{\xi < \lambda : (\exists q \leq p)(q \Vdash \xi \in \mathring{A}_i)\}$. Since $\mathbb{P}$ satisfies ccc, it is easy to see that for every $i < \kappa$, $|B_i| = \mu$. Furthermore, each $B_i \in V \cap \mathcal{P}(\lambda)$ and $p \Vdash \mathring{A}_i \subseteq B_i$. As $V \models |\mathcal{P}(\lambda)| = 2^\lambda \leq 2^\theta < \kappa$, we can find $X \in [\kappa]^\kappa$ and $B_\star \subseteq \lambda$ such that for every $i \in X$, $B_i = B_\star$. Fix a bijection $h : B_\star \to \mu$. Since $\mathbb{P}$ satisfies ccc and μ is regular uncountable, for every $i < j$ in X, we can choose $\xi(i,j) < \mu$ such that $p \Vdash h[\mathring{A}_i \cap \mathring{A}_j] < \xi(i,j)$. As $V \models \kappa > 2^\theta \geq 2^\mu$, by the Erdős-Rado theorem, we can find $Y \in [X]^{\mu^+}$ and $\xi_\star < \mu$ such that for every $i < j$ in Y, $\xi(i,j) = \xi_\star$. Choose $\alpha \in B_\star$ such that $\xi_\star < h(\alpha) < \mu$. For each $i \in Y$, choose $q_i \leq p$ such that $q_i \Vdash \alpha \in \mathring{A}_i$. Now note that no two conditions in $\{q_i : i \in Y\}$ are compatible. Since $|Y| = \mu^+ > \mu \geq \omega_1$, this contradicts the fact that $\mathbb{P}$ satisfies ccc. Hence $V^{\mathbb{P}} \models \mathfrak{la}(\kappa) > \theta$. □

Corollary 2.7. *Assume $V \models 2^{\omega_k} = \omega_{k+1}$ for $k < 3$. Let $\mathbb{P}$ be the finite support product of $\mathbb{Q}_{\omega_3,\delta}$'s where δ runs over the set of all countable indecomposable ordinals. Then $V^{\mathbb{P}} \models \mathfrak{c} = \omega_3$, $\mathfrak{ls}(\omega_3) = \omega_1$ and $\mathfrak{la}(\omega_3) = \omega_2$.*

Proof. By Theorem 2.6, $V^{\mathbb{P}} \models \mathfrak{c} = \omega_3$, $\mathfrak{ls}(\omega_3) = \omega_1$ and $\mathfrak{la}(\omega_3) > \omega_1$. Since there is an almost disjoint family in $[\omega_2]^{\omega_2}$ of size ω_3, we must have $V^{\mathbb{P}} \models \mathfrak{la}(\omega_3) = \omega_2$. □

3. Stronger separating families

Definition 3.1. *Let $\mu < \delta \leq \lambda$ be infinite cardinals such that μ, δ are regular and $\delta = \delta^{<\mu}$. Define a forcing $\mathbb{Q}_{\lambda,\delta,\mu}$ as follows. $p \in \mathbb{Q}_{\lambda,\delta,\mu}$ iff the following hold.*

(i) *p is a function, $\mathsf{dom}(p) \subseteq \lambda$ and $\mathsf{range}(p) \subseteq 2$.*
(ii) *$|\mathsf{dom}(p)| < \delta$.*
(iii) *$|\{\xi \in \mathsf{dom}(p) : p(\xi) = 1\}| < \mu$.*

For $p, q \in \mathbb{Q}_{\lambda,\delta,\mu}$, define $p \leq q$ iff $q \subseteq p$.

Lemma 3.2. *Let $\mathbb{Q} = \mathbb{Q}_{\lambda,\delta,\mu}$ be as in Definition 3.1. Then the following hold.*

(1) *$\mathbb{Q}$ is $< \mu$-closed.*
(2) *For every $X \in [\mathbb{Q}]^{\delta^+}$, there exists $Y \in [X]^{\delta^+}$ such that for any $F \in [Y]^{<\mu}$, there exists $p \in \mathbb{Q}$ such that $(\forall q \in F)(p \leq q)$. So $\mathbb{Q}$ satisfies δ^+-cc.*
(3) *$V^{\mathbb{Q}} \models |\delta| = \mu$.*
(4) *Forcing with $\mathbb{Q}$ preserves all cardinals $\leq \mu$ and $\geq \delta^+$ and collapses every cardinal in (μ, δ^+) to μ.*

Proof. That $\mathbb{Q}$ is $< \mu$-closed is easy to see. This implies that all cardinals $\leq \mu$ are preserved.

Next, suppose $\langle p_i : i < \delta^+ \rangle$ is a sequence of conditions in $\mathbb{Q}$. Put $A_i = \mathsf{dom}(p_i)$, $B_i = \{\xi \in \mathsf{dom}(p_i) : p_i(\xi) = 1\}$ and $A = \bigcup\{A_i : i < \delta^+\}$. Then $|A| \leq \delta^+$. WLOG, we can assume $A \subseteq \delta^+$. Fix a club $E \subseteq \delta^+$ such that for each $\gamma \in E$ and $i < \gamma$, $A_i \subseteq \gamma$. Let $S = \{\gamma \in E : \mathsf{cf}(\gamma) = \delta\}$. Then S is stationary in δ^+ and the function $h : S \to \delta^+$ defined by $h(\gamma) = \sup(A_\gamma \cap \gamma)$ is regressive. By Fodor's lemma, we can find $T \subseteq S$ and $\gamma_\star < \delta^+$ such that T is stationary in δ^+ and $h \restriction T$ is constantly $\gamma_\star$. Observe that, as $T \subseteq E$, for every $i < j$ in T, $A_i \cap A_j \subseteq \gamma_\star$. Since $|\gamma_\star| \leq \delta$ and $\delta^{<\mu} = \delta$, we can find $B_\star \in [\gamma_\star]^{<\mu}$ and $W \subseteq T$ such that W is stationary in δ^+ and for every $i \in W$, $B_i \cap \gamma_\star = B_\star$. It follows that $\langle p_i : i \in W \rangle$ consists of pairwise compatible functions. Clause (2) follows.

To see (3), suppose G is $\mathbb{Q}$-generic over V. Put $F = \bigcup G$. Then $F : \lambda \to 2$. Let $W = \{\xi < \delta : F(\xi) = 1\}$. Fix a partition $\delta = \bigsqcup\{W_i : i < \delta\}$ in V such that each $W_i \in [\delta]^\delta$. An easy density argument shows that $\mathsf{otp}(W) = \mu$ and for every $i < \delta$, $W \cap W_i \neq \emptyset$. It follows that $V[G] \models |\delta| = \mu$.

By Clause (2), all cardinals $\geq \delta^+$ are preserved and by Clause (1) all cardinals $\leq \mu$ are preserved. Hence Clause (4) follows from Clauses (1)-(3). □

Definition 3.3. *Suppose $\mu < \kappa < \lambda$ are infinite cardinals and μ is regular. Suppose $S_\star = \{\delta : \mu < \delta < \kappa$ and δ is inaccessible$\}$ is stationary in κ (so κ is Mahlo). Let $\mathbb{P}_{\lambda,\kappa,\mu}$ be the Easton-support product of $\langle \mathbb{Q}_{\lambda,\delta,\mu} : \delta \in S_\star \rangle$. So $p \in \mathbb{P}_{\lambda,\kappa,\mu}$ iff*

(a) *p is a function with $\mathsf{dom}(p) \subseteq S_\star$,*
(b) *for every $\delta \in S_\star \cup \{\kappa\}$, $\sup(\mathsf{dom}(p) \cap \delta) < \delta$ and*
(c) *for every $\delta \in \mathsf{dom}(p)$, $p(\delta) \in \mathbb{Q}_{\lambda,\delta,\mu}$.*

For $p, q \in \mathbb{P}_{\lambda,\kappa,\mu}$, define $p \leq q$ iff $\mathsf{dom}(q) \subseteq \mathsf{dom}(p)$ and for every $\delta \in \mathsf{dom}(q)$, $p(\delta) \leq_{\mathbb{Q}_{\lambda,\delta,\mu}} q(\delta)$.

Lemma 3.4. *Let $\mu, \kappa, \lambda, S_\star$ and $\mathbb{P} = \mathbb{P}_{\lambda,\kappa,\mu}$ be as above. Then the following hold.*

(1) *Forcing with $\mathbb{P}$ collapses all cardinals in the interval (μ, κ) to μ.*
(2) *$\mathbb{P}$ is $< \mu$-closed and κ-cc. So all cardinals $\leq \mu$ are preserved and $V^{\mathbb{P}} \models \kappa = \mu^+$.*

Proof. For each $\delta \in S_\star$, $\mathbb{Q}_{\lambda,\delta,\mu} \lessdot \mathbb{P}$. Therefore Clause (1) follows from Lemma 3.2. It is also clear that $\mathbb{P}$ is $< \mu$-closed. Therefore, it suffices to check that $\mathbb{P}$ satisfies the κ-cc.

Towards a contradiction, let $\langle p_i : i < \kappa \rangle$ be a sequence of pairwise incompatible conditions in $\mathbb{P}$. Choose a club $E \subseteq \kappa$ such that for every $\gamma \in E$ and $i < \gamma$, $\sup(\mathsf{dom}(p_i)) < \gamma$. Since the function $h : E \cap S_\star \to \kappa$ defined by $h(\delta) = \sup(\mathsf{dom}(p_\delta) \cap \delta)$ is regressive, by Fodor's lemma, we can find a stationary subset $T \subseteq E \cap S_\star$ and $\gamma_\star < \kappa$ such that for every $\delta \in T$, $h(\delta) < \gamma_\star$. Note that for any $\delta_1 < \delta_2$ in T, $\mathsf{dom}(p_{\delta_1}) \cap \mathsf{dom}(p_{\delta_2}) \subseteq \gamma_\star$. Define a coloring $c : [T]^2 \to \gamma_\star$ by $c(\{\delta_1, \delta_2\})$ is the least $\gamma \in \mathsf{dom}(p_{\delta_1}) \cap \mathsf{dom}(p_{\delta_2})$ such that $p_{\delta_1}(\gamma)$ and $p_{\delta_2}(\gamma)$ are incompatible in $\mathbb{Q}_{\lambda,\gamma,\mu}$. Put $\theta = |\gamma_\star|^{++}$. Since $|T| = \kappa$ is inaccessible and $\theta < \kappa$, using Erdős-Rado theorem, we can find $X \in [T]^\theta$ and $\gamma < \gamma_\star$ such that $c \restriction [X]^2$ takes the constant value γ. But this means that $\{p_\delta(\gamma) : \delta \in X\}$ is an antichain of size $\theta > \gamma^+$ in $\mathbb{Q}_{\lambda,\gamma,\mu}$ which is impossible by Lemma 3.2. □

Lemma 3.5. *Let $\mu, \kappa, \lambda, S_\star$ and $\mathbb{P} = \mathbb{P}_{\lambda,\kappa,\mu}$ be as above. Then the following hold in $V^{\mathbb{P}}$.*

(1) *There is a family $\mathcal{F} \subseteq \mathcal{P}(\lambda)$ such that $|\mathcal{F}| = \kappa$ and for any $A \in [\lambda]^{<\kappa}$ and $B \in [\lambda]^{<\mu}$, if $A \cap B = \emptyset$, then there exists $X \in \mathcal{F}$ such that $B \subseteq X$ and $A \cap X = \emptyset$.*
(2) *If $\mu = \omega$, then $\mathfrak{ls}(\lambda) = \omega_1$. If $\mu \geq \omega_1$, then $\mathfrak{ls}(\lambda) = \mu$.*
(3) *If $\lambda > 2^\kappa$, then there is no family $\mathcal{A} \subseteq [\kappa]^\kappa$ such that $|\mathcal{A}| = \lambda$ and for every $X \neq Y$ in $\mathcal{A}$, $|X \cap Y| < \kappa$.*
(4) *If $\mu = \omega$, then $\mathfrak{la}(\lambda) \geq \kappa^+$. If $\omega_1 \leq \mu = 2^{<\mu}$, then $\mathfrak{la}(\lambda) = \mu$.*

Proof. (1) Let G be $\mathbb{P}$-generic over V. For each $\delta \in S_\star$, define

$$X_\delta = \{\xi < \lambda : (\exists p \in G)(p(\delta)(\xi) = 1)\}.$$

Put $\mathcal{F} = \{X_\delta : \delta \in S_\star\}$. Then $|\mathcal{F}| = \kappa$. We claim that $\mathcal{F}$ is as required. For suppose $A \in [\lambda]^{<\kappa}$ and $B \in [\lambda]^{<\mu}$. Since $\mathbb{P}$ is $< \mu$-closed, $B \in V$. Since $\mathbb{P}$ satisfies κ-cc, we can find $C \in V \cap [\lambda \setminus B]^{<\kappa}$ such that $A \subseteq C$. Now observe that the set of conditions $p \in \mathbb{P}$ satisfying the following is dense in $\mathbb{P}$: There exists $\delta \in \mathsf{dom}(p)$ such that (a)-(c) below hold.

(a) $|C| < \delta$.
(b) $(\forall \xi \in B)(p(\delta)(\xi) = 1)$.
(c) $(\forall \xi \in C)(p(\delta)(\xi) = 0)$.

Choose such a $p \in G$ and a witnessing $\delta \in \mathsf{dom}(p)$. It follows that $B \subseteq X_\delta$ and $C \cap X_\delta = A \cap X_\delta = \emptyset$.

(2) Recall that forcing with $\mathbb{P}$ preserves all cardinals $\leq \mu$ and $\geq \kappa$ and collapses all cardinals in the interval (μ, κ) to μ. So $V^{\mathbb{P}} \models \kappa = \mu^+$. First suppose $\mu = \omega$. Then by (1), $V^{\mathbb{P}} \models \mathfrak{ls}(\lambda) \leq \kappa = \mu^+ = \omega_1$. Since λ is uncountable, we also have $\mathfrak{ls}(\lambda) \geq \omega_1$. Therefore $V^{\mathbb{P}} \models \mathfrak{ls}(\lambda) = \omega_1$.

Next assume $\mu \geq \omega_1$. Fix $\delta \in S_\star$ and a bijection $h : \lambda \times \delta \to \lambda$ such that $h \in V$. Let G be $\mathbb{P}$-generic over V. For each $i < \delta$, define

$$X_i = \{\xi < \lambda : (\exists p \in G)(p(\delta)(h(\xi, i)) = 1)\}.$$

Put $\mathcal{F} = \{X_i : i < \delta\}$. An easy density argument shows that for every $A \in [\lambda]^{<\delta} \cap V$ and $B \in [\lambda \setminus A]^{<\mu} \cap V$, there exists $i < \delta$ such that $A \cap X_i = \emptyset$ and $B \subseteq X_i$. Since $\mathbb{P}$ is $< \mu$-closed, forcing with $\mathbb{P}$ does not add new countable subsets of λ. Hence $V[G] \models \mathcal{F}$ separates countable subsets of λ from points. Since $V[G] \models |\mathcal{F}| = |\delta| = \mu$, it follows that $\mathfrak{ls}(\lambda) \leq \mu$.

Finally, to see that $V^{\mathbb{P}} \models \mathfrak{ls}(\lambda) \geq \mu$, towards a contradiction, fix $p \in \mathbb{P}$, $\theta < \mu$ and $\langle \mathring{A}_i : i < \theta \rangle$ such that $(\forall i < \theta)(p \Vdash \mathring{A}_i \in \mathcal{P}(\lambda))$ and $p \Vdash \{\mathring{A}_i : i < \theta\}$ separates countable subsets of λ from points. Define $\mathring{B}_\xi = \{i < \theta : \xi \in \mathring{A}_i\}$. As $\mathbb{P}$ is $< \mu$-closed, $p \Vdash \mathring{B}_\xi \in \mathcal{P}(\theta) \cap V$. Since $V \models 2^\theta < \kappa < \lambda$, we can choose $\mathring{X} \in [\lambda]^\lambda \cap V^{\mathbb{P}}$, $q_0 \in \mathbb{P}$ and $B_\star \in \mathcal{P}(\theta) \cap V$ such that $q_0 \leq p$ and $q_0 \Vdash (\forall \xi \in \mathring{X})(\mathring{B}_\xi = B_\star)$. Choose $q \leq q_0$ and $\xi_1 < \xi_2 < \lambda$ such that $q \Vdash \{\xi_1, \xi_2\} \subseteq \mathring{X}$. Then $q \Vdash \mathring{B}_{\xi_1} = \mathring{B}_{\xi_2} = B_\star$. Now observe that for every $i < \theta$,

$$q \Vdash (\xi_1 \in \mathring{A}_i \iff i \in \mathring{B}_{\xi_1} \iff i \in B_\star \iff i \in \mathring{B}_{\xi_2} \iff \xi_2 \in \mathring{A}_i).$$

Therefore $q \Vdash \{\mathring{A}_i : i < \theta\}$ does not separate countable subsets of λ from points. A contradiction. Hence $V^{\mathbb{P}} \models \mathfrak{ls}(\lambda) = \mu$.

(3) Towards a contradiction, fix $p \in \mathbb{P}$ and $\langle \mathring{A}_i : i < \lambda \rangle$ such that

$$p \Vdash (\forall i < \lambda)(\mathring{A}_i \in [\kappa]^\kappa) \text{ and } (\forall i < j < \lambda)(|\mathring{A}_i \cap \mathring{A}_j| < \kappa).$$

Since $\mathbb{P}$ satisfies the κ-cc, we can find $c : [\lambda]^2 \to \kappa$ in V such that for every $i < j < \lambda$, $p \Vdash \sup(\mathring{A}_i \cap \mathring{A}_j) < c(\{i, j\})$. Since $V \models \lambda > 2^\kappa$, by the Erdős-Rado theorem, there are $H \in [\lambda]^{\kappa^+}$ and $\gamma < \kappa$ such that for every $i < j$ in H, $c(\{i, j\}) = \gamma$. It now follows that $p \Vdash \{\mathring{A}_i \setminus \gamma : i \in H\}$ is a family of κ^+ pairwise disjoint sets in $[\kappa]^\kappa$ which is impossible since all cardinals $\geq \kappa$ are preserved in $V^{\mathbb{P}}$.

(4) First assume $\mu = \omega$ and $\mathfrak{la}(\lambda) \leq \kappa$. Then by the previous clause, $\mathfrak{la}(\lambda) < \kappa$ and therefore $\mathfrak{la}(\lambda) \leq \mu = \omega$ (as $V^{\mathbb{P}} \models \kappa = \mu^+$) which is impossible.

Next, suppose $\omega_1 \leq \mu = 2^{<\mu}$. Fix some $\delta \in S_\star$ (so δ is an inaccessible between μ and κ). Observe that $\mathbb{Q}_{\lambda,\delta,\mu} \lessdot \mathbb{P}$ and forcing with $\mathbb{Q}_{\lambda,\delta,\mu}$ adds at least λ new subsets of δ. Thus $V^{\mathbb{P}} \models 2^\mu = |2^\delta| \geq \lambda$. Furthermore, since $\mathbb{P}$ is $< \mu$-closed, $V^{\mathbb{P}} \models 2^{<\mu} = \mu$. It follows that in $V^{\mathbb{P}}$, the family $\mathcal{F} = \{\{x \restriction \xi : \xi < \mu\} : x \in {}^\mu 2\}$ satisfies $\mathcal{F} \subseteq [{}^{<\mu}2]^\mu$, $|\mathcal{F}| \geq \lambda$ and for every $A \neq B \in \mathcal{F}$, $|A \cap B| < \mu$. So $\mathfrak{la}(\lambda) \leq \mu$. The other inequality $\mathfrak{la}(\lambda) \geq \mu$ follows from the fact that $\mathbb{P}$ is $< \mu$-closed and $\lambda > 2^{<\mu} = \mu$. $\square$

Proof of Theorem 1.2: Let $\mathbb{P} = \mathbb{P}_{\lambda,\kappa,\mu}$ be as in Definition 3.3. Then by Lemma 3.4, $\mathbb{P}$ is $< \mu$-closed and κ-cc and $V^{\mathbb{P}} \models \kappa = \mu^+$. Clause (2) follows from Lemma 3.5(1) and $\mathfrak{ls}(\lambda, \kappa, \mu) \geq \kappa$. The last three clauses follow from Lemma 3.5(3)-(4). $\square$

4. Partition relations

Let $(L, \prec)$ be a linear ordering. Recall that (C_0, C_1) is a cut in $(L, \prec)$ iff C_0 is downward closed in L, C_1 is upward closed in L and $L = C_0 \sqcup C_1$ (disjoint union).

Definition 4.1. *A linear ordering $(L, \prec)$ is additively indecomposable iff for every cut (C_0, C_1) in $(L, \prec)$, at least one of $(C_0, \prec)$ and $(C_1, \prec)$ contains an isomorphic copy of $(L, \prec)$.*

In the remainder of this section, we will write "indecomposable" for "unionwise indecomposable" (Definition 1.7). It is easy to see that indecomposable linear orderings are also additively indecomposable but the converse is false (consider $\omega \times \mathbb{Z}$ with lexicographic ordering). Recall that a linear ordering is scattered iff it does not contain a copy of the rationals $(\mathbb{Q}, <)$. The following fact appears in [3] (also Exercise 10.4.10 in [8]).

Fact 4.2. *Let $(L, \prec)$ be a scattered additively indecomposable linear ordering. Then one of the following holds.*

(a) *For every cut (C_0, C_1) in L, if $C_0 \neq \emptyset$, then L embeds into C_0. In this case, we say that $(L, \prec)$ is indecomposable to the left.*
(b) *For every cut (C_0, C_1) in L, if $C_1 \neq \emptyset$, then L embeds into C_1. In this case, we say that $(L, \prec)$ is indecomposable to the right.*

Definition 4.3. *Let $(L, \prec)$ be an indecomposable linear ordering. An ultrafilter $\mathcal{U}$ on L is uniform iff for every $A \in \mathcal{U}$, L embeds into A.*

Definition 4.4. *Let $(L, \prec)$ be an indecomposable linear ordering and $\mathcal{U}_L$ be a uniform ultrafilter on L. We say that $\mathcal{U}_L$ is nice iff one of the following holds.*

(a) *$(L, \prec_L)$ is indecomposable to the left and for every cut (C_0, C_1) in L, if $C_0 \neq \emptyset$, then $C_0 \in \mathcal{U}_L$. Moreover, for every family $\mathcal{F} \subseteq \mathcal{U}_L$, if $|\mathcal{F}| \leq \omega_1$, then there exists $X \in \mathcal{U}_L$ such that for every $A \in \mathcal{F}$, $X \setminus A$ is bounded from below in $(L, \prec_L)$.*
(b) *$(L, \prec_L)$ is indecomposable to the right and for every cut (C_0, C_1) in L, if $C_1 \neq \emptyset$, then $C_1 \in \mathcal{U}_L$. Moreover, for every family $\mathcal{F} \subseteq \mathcal{U}_L$, if $|\mathcal{F}| \leq \omega_1$, then there exists $X \in \mathcal{U}_L$ such that for every $A \in \mathcal{F}$, $X \setminus A$ is bounded from above in $(L, \prec_L)$.*

The following should be compared with the results of Section 13.3 in [9].

Theorem 4.5. *Assume $\mathsf{MA} + \mathfrak{c} > \omega_1$. Then every countably infinite scattered indecomposable linear ordering admits a nice uniform ultrafilter.*

Proof. Let $\mathcal{C}$ be the class of all countable scattered indecomposable linear orderings. Laver [7] showed that there is a rank function $r : \mathcal{C} \to \omega_1$ such that the following hold.

(i) $r(L) = 0$ iff $|L| = 1$ iff L is finite.
(ii) For every infinite $L \in \mathcal{C}$, either $r(L) = 1$ and $L \in \{\omega, \omega^\star\}$ or L is the sum of an ω or $\omega^\star$-sequence of members of $\mathcal{C}$ of strictly smaller ranks.
(iii) If $L_1, L_2 \in \mathcal{C}$ and L_1 embeds into L_2, then $r(L_1) \leq r(L_2)$.

We will construct $\mathcal{U}_L$ by induction on the rank of L. As L is countably infinite, $r(L) \geq 1$. If $r(L) = 1$, this is clear: Say $(L, \prec_L) = (\omega, <)$ (the case when L is $\omega^\star$ is similar). Using MA, build a sequence $\langle A_i : i < \mathfrak{c} \rangle$ of $\subseteq^\star$-descending sequence in $[\omega]^\omega$ such that for every $X \subseteq \omega$, there exists $i < \mathfrak{c}$ such that either $A_i \subseteq X$ or

$X \cap A_i = \emptyset$. Take $\mathcal{U}_\omega$ to be the filter generated by $\{A_i : i < \mathfrak{c}\}$. It is clear that $\mathcal{U}_\omega$ is a nice uniform ultrafilter on ω.

Now suppose $r(L) > 1$. Let us consider the case when L is the sum of an ω-sequence of members of $\mathcal{C}$ of strictly smaller ranks. The case when L is the sum of an $\omega^\star$-sequence is similar. Fix $\{(L_n, \prec_n) : n < \omega\}$ such that L_n's are pairwise disjoint, $r(L_n) < r(L)$, $L = \bigcup\{L_n : n < \omega\}$ and for every $a, b \in L$,

$$a \prec_L b \iff a \in L_m \text{ and } b \in L_n \text{ and } (m < n \text{ or } (m = n \text{ and } a \prec_n b)).$$

Note that $(L, \prec_L)$ must be indecomposable to the right. Otherwise, it would embed into $\sum_{k \le N} L_k$ for some $N < \omega$. As $(L, \prec_L)$ is indecomposable, we can find some $k \le N$ such that $(L, \prec_L)$ embeds into $(L_k, \prec_k)$ which is impossible since $r(L) > r(L_k)$.

Claim 4.6. *There exists a strictly increasing sequence $\langle n_k : k < \omega \rangle$ such that each L_{n_k} is infinite and the following hold.*

- (i) *Either for every k, there is a nice uniform ultrafilter $\mathcal{U}_k$ on L_{n_k} satisfying Clause (a) of Definition 4.4 or for every k, there is a nice uniform ultrafilter $\mathcal{U}_k$ on L_{n_k} satisfying Clause (b) in Definition 4.4.*
- (ii) *For every $n \le k < \omega$, L_n embeds into L_{n_k}.*

Proof. Let $X_0 = \{n : |L_n| = 1\}$. Let X_1 (resp. X_2) be the set of all $n \in \omega \setminus X_0$ such that there is a nice uniform ultrafilter on L_n satisfying Clause (a) (resp. (b)) of Definition 4.4. Then, by the inductive hypothesis, $X_1 \cup X_2 = \omega \setminus X_0$. Since L is indecomposable, we can fix some $i \in \{1, 2\}$ ($i = 0$ is impossible because $r(L) > r(\omega) = 1$) such that L embeds into $\sum_{n \in X_i} L_n$. Put $X = X_i$.

We will recursively choose a strictly increasing sequence $\langle n_k : k < \omega \rangle$ in X. Clause (i) will automatically hold since each $n_k \in X$. For Clause (ii), it suffices to show that for every $N < \omega$, there exists $n > N$ such that $n \in X$ and for every $k \le N$, L_k embeds into L_n.

Towards a contradiction, suppose this fails for some N. For each $k \le N$, define

$$X_k = \{n \in X : n > N \text{ and } L_k \text{ does not embed into } L_n\}.$$

Then $\bigcup_{k \le N} X_k = X \cap (N, \omega)$. As L is indecomposable to the right, it embeds into $\sum_{n \in X \cap (N, \omega)} L_n$. Since L is indecomposable, we can fix some $k_\star \le N$ such that L embeds into $\sum_{n \in X_{k_\star}} L_n$. It follows that $L_{k_\star}$ embeds into a proper initial segment of $\sum_{n \in X_{k_\star}} L_n$. As $L_{k_\star}$ is indecomposable, it follows that it embeds into L_n for some $n \in X_{k_\star}$: A contradiction. ◻

Fix $\langle n_k : k < \omega \rangle$ as in Claim 4.6. Furthermore, let us assume that for every $k < \omega$, there is a nice uniform ultrafilter $\mathcal{U}_k$ on L_{n_k} satisfying Clause (a) of Definition 4.4. The argument in the other case is almost identical. Note that each L_{n_k} is indecomposable to the left. Let $\mathcal{U}_\omega$ be the nice ultrafilter on ω defined above. Define

$$\mathcal{U}_L = \{X \subseteq L : \{k < \omega : X \cap L_{n_k} \in \mathcal{U}_k\} \in \mathcal{U}_\omega\}.$$

Claim 4.7. *$\mathcal{U}_L$ is a nice uniform ultrafilter on L.*

It is clear that $\mathcal{U}_L$ is an ultrafilter on L. To see that it is uniform, fix $A \in \mathcal{U}_L$ and define $S = \{k < \omega : A \cap L_{n_k} \in \mathcal{U}_k\}$. Then $S \in \mathcal{U}_\omega$ is infinite. Let $\langle k_j : j < \omega \rangle$ list S in increasing order. By Claim 4.6(b), L_j embeds into $L_{n_{k_j}}$ and the latter embeds into $A \cap L_{n_{k_j}}$ (as $\mathcal{U}_{k_j}$ is a uniform ultrafilter on $L_{n_{k_j}}$). Composing these, we get an embedding $f_j : L_j \to A \cap L_{n_{k_j}}$. It follows that $f = \bigcup_j f_j$ is an embedding from L to A. Thus $\mathcal{U}_L$ is uniform.

Next, suppose $\mathcal{F} \subseteq \mathcal{U}_L$ and $|\mathcal{F}| \leq \omega_1$. We can assume that $L \in \mathcal{F}$. For each $A \in \mathcal{F}$, define $S_A = \{k < \omega : A \cap L_{n_k} \in \mathcal{U}_k\}$. Then $\{S_A : A \in \mathcal{F}\} \in [\mathcal{U}_\omega]^{\leq \omega_1}$. As $\mathcal{U}_\omega$ is nice, we can choose $B \in \mathcal{U}_\omega$ such that $B \setminus S_A$ is finite for every $A \in \mathcal{F}$. Let $\mathcal{F}_k = \{A \cap L_{n_k} : A \in \mathcal{F}\} \cap \mathcal{U}_{n_k}$. Since $\mathcal{U}_k$ is nice, we can choose $X_k \in \mathcal{U}_{n_k}$ such that $X_k \setminus W$ is bounded from below in L_{n_k} for every $W \in \mathcal{A}_k$. Put $Y = \bigcup\{X_k : k \in B\}$. It is clear that $Y \in \mathcal{U}_L$.

For each $k < \omega$, fix a strictly decreasing left-cofinal sequence $\langle x_k(j) : j < \omega\rangle$ in $(L_{n_k}, \prec_{n_k})$. For each $A \in \mathcal{F}$, fix $N_A < \omega$ such that $B \setminus N_A \subseteq S_A$. Choose $f_A : \omega \to \omega$ such that for every $k \in B \setminus N_A$, $(Y \setminus A) \cap L_{n_k}$ is bounded from below by $x_k(f_A(k))$. Using MA_{ω_1}, fix $f_\star : \omega \to \omega$ dominating every function in $\{f_A : A \in \mathcal{F}\}$. Put $Z = \bigcup\{Y \cap W_k : k < \omega\}$ where $W_k = \{y \in L_{n_k} : y \prec_{n_k} x_k(f_\star(k))\}$. Then $Z \in \mathcal{U}_L$ and for every $A \in \mathcal{F}$, $Z \setminus A$ is bounded from above in $(L, \prec_L)$. So Claim 4.7 holds and we are done.

Note that this proof also works under the assumptions: $\mathfrak{b} > \omega_1$ and there is a nice ultrafilter on ω and both of these hold under $\mathfrak{p} = \mathfrak{c} > \omega_1$ (see [6]). $\square$

Proof of Theorem 1.11: If $(L, \prec_L)$ contains a copy of the rationals, then this is Theorem 6.7 in [4] (For a slightly stronger result see Lemma 4.8 below). So assume it is scattered. Fix $\mathcal{U}_L$ as in Theorem 4.5 and WLOG assume that $\mathcal{U}_L$ satisfies Clause (a) of Definition 4.4. Suppose $c : \omega_1 \times L \to K$. For each $\alpha < \omega_1$, fix $A_\alpha \in \mathcal{U}_L$ and $k_\alpha < K$ such that for every $x \in A_\alpha$, $c(\alpha, x) = k_\alpha$. Fix $X \in [\omega_1]^{\omega_1}$ such that $k_\alpha = k_\star$ does not depend on $\alpha \in X$. Fix $B \in \mathcal{U}_L$ such that for each $\alpha \in X$, $B \setminus A_\alpha$ is bounded from below in $(L, \prec_L)$ by some $y_\alpha \in L$. Choose $Y \in [X]^{\omega_1}$ such that $y_\alpha = y_\star$ does not depend on $\alpha \in Y$. Put $D = \{x \in B : x \prec_L y_\star\}$. Then $D \in \mathcal{U}_L$ and $c \restriction (Y \times D)$ is constantly $k_\star$. As $\mathcal{U}_L$ is uniform, L embeds into D and the proof is complete. $\square$

Proof of Theorem 1.10: We use an absoluteness argument like the one in [1]. Let W be a ccc extension of V satisfying $\mathsf{MA} + \mathfrak{c} > \omega_1$. Let $\alpha < \omega_1$. Fix linear orders $\prec_1$ and $\prec_2$ on ω such that $\mathsf{otp}(\omega, \prec_1) = (\alpha, <)$ and $(\omega, \prec_2) \cong (L, \prec_L)$. Let T be the set of all pairs (s, t) such that

- s, t are functions, $\mathsf{dom}(s) = \mathsf{dom}(t) = N < \omega$,
- $\mathsf{range}(s) \subseteq \omega_1$, $\mathsf{range}(t) \subseteq L$,
- $c \restriction (\mathsf{range}(s) \times \mathsf{range}(t))$ is constant and
- for every $m, n < N$, $(m \prec_1 n \iff s(m) < s(n))$ and $(m \prec_2 n \iff t(m) \prec_L t(n))$.

Define $(s, t) \preceq_T (s', t')$ iff $s \subseteq s'$ and $t \subseteq t'$ and note that $(T, \preceq_T)$ is well-founded iff there is no c-homogeneous set of type $\alpha \times L$. But this is absolute between V and W since a tree is well-founded iff there is a rank function on it. So it suffices to construct such a homogeneous set in W. But this was already done in Theorem 1.11. $\square$

Lemma 4.8. *Let $f : \omega_1 \times \mathbb{Q} \to K$ where $K < \omega$.*

(1) *Assume MA_{ω_1} (or just $\mathfrak{p} > \omega_1$). Then there exist $X \in [\omega_1]^{\omega_1}$ and $Y \subseteq \mathbb{Q}$ such that Y is somewhere dense in $\mathbb{Q}$ and $f \restriction (X \times Y)$ is constant.*

(2) *For each $\alpha < \omega_1$, there exist $X \subseteq \omega_1$ and $Y \subseteq \mathbb{Q}$ such that $\mathsf{otp}(X) = \alpha$, Y is somewhere dense in $\mathbb{Q}$ and $f \restriction (X \times Y)$ is constant.*

Proof. (1) It is enough to show this for $K = 2$ for then we can argue by induction on K. For each $i < \omega_1$, put $A_i = \{x \in \mathbb{Q} : f(i, x) = 1\}$. The following is Lemma 6.11 in [4].

Fact 4.9 ([4]). *Let $\langle A_i : i < \omega_1 \rangle$ be a sequence of subsets of $\mathbb{Q}$. There exist $W \in [\omega_1]^{\omega_1}$, $c < 2$ and a rational interval J such that for every finite $F \subseteq W$, $\bigcap_{i \in F} A_i^c$ is dense in J. Here, $A_i^c = A_i$ if $c = 0$ and $\mathbb{Q} \setminus A_i$ otherwise.*

Using Fact 4.9, we can find a rational interval J and $W \in [\omega_1]^{\omega_1}$ such that either the intersection of any finite subfamily of $\{A_i : i \in W\}$ is dense in J or the intersection of any finite subfamily of $\{\mathbb{Q} \setminus A_i : i \in W\}$ is dense in J. WLOG, let us assume that the former situation holds. Define a forcing $\mathbb{P}$ as follows: $p \in \mathbb{P}$ iff $p = (u_p, v_p, I_p)$ where

(i) $u_p \in [\mathbb{Q}]^{<\omega}$ and $v_p \in [W]^{<\omega}$.
(ii) $\mathcal{I}_p$ is a finite family of rational subintervals of J.
(iii) For each $I \in \mathcal{I}_p$, $u_p \cap I \neq \emptyset$.
(iv) For $p, q \in \mathbb{P}$, define $p \leq q$ iff
 (a) $u_q \subseteq u_p$, $v_q \subseteq v_p$, $\mathcal{I}_q \subseteq \mathcal{I}_p$.
 (b) If $x \in u_p \setminus u_q$ and $i \in v_q$, then $x \in A_i$.

$\mathbb{P}$ is σ-centered (as there are countably many u_p's) and if $G \subseteq \mathbb{P}$ is sufficiently generic (use $\mathfrak{p} > \omega_1$) then $X = W = \bigcup\{v_p : p \in G\}$ and $Y = \bigcup\{u_p : p \in G\}$ are as claimed in (1).

(2) Fix a linear order $\prec_\alpha$ on ω such that $\mathsf{otp}(\omega, \prec_\alpha) = (\alpha, <)$. For each rational interval J, fix a computable enumeration $\langle J_n : n < \omega \rangle$ of all rational subintervals of J and define $X = X_J$ to be the set of all finite sequences $s = \langle (x_n, i_n) : n < N \rangle$ such that the following hold.

- For every $n < N$, $x_n \in J_n$.
- $\langle i_n : n < N \rangle$ is an injective sequence of countable ordinals.
- For every $m, n < N$, $(i_m < i_n \iff m \prec_\alpha n)$.
- $f \restriction (\{i_n : n < N\} \times \{x_n : n < N\})$ is constant.

Define a relation $R = R_J$ on X by sRt iff $t \subsetneq s$. Note that (X_J, R_J) is not well-founded iff there exist $X \subseteq \omega_1$ and $Y \subseteq \mathbb{Q}$ such that $\mathsf{otp}(X) = \alpha$, Y is dense in J and $f \restriction (X \times Y)$ is constant.

Now we can start repeating the proof of part (1). Choose a rational interval J and the forcing $\mathbb{P}$ as there and get a $\mathbb{P}$-generic filter G over V. In $V[G]$, (X_J, R_J) is not well-founded. By absoluteness, the same holds in V and we are done. □

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(Kumar) Department of Mathematics and Statistics, Indian Institute of Technology Kanpur, Kanpur 208016, UP, India.
Email address: krashu@iitk.ac.in
URL: https://home.iitk.ac.in/~krashu/

(Shelah) Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Edmond J Safra Campus, Givat Ram, Jerusalem 91904, Israel and Department of Mathematics, Rutgers, The State University of New Jersey, Hill Center-Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA.
Email address: shelah@math.huji.ac.il
URL: https://shelah.logic.at/