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Infinite stable graphs with large chromatic number II

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Abstract. We prove a version of the strong Taylor's conjecture for stable graphs: if G is a stable graph whose chromatic number is strictly greater than $\beth_2(\aleph_0)$ then G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$ and thus has elementary extensions of unbounded chromatic number. This completes the picture from our previous work. The main new model-theoretic ingredient is a generalization of the classical construction of Ehrenfeucht–Mostowski models to an infinitary setting, giving a new characterization of stability.

Keywords. Chromatic number, stable graphs, Taylor's conjecture, EM-models

1. Introduction

The *chromatic number* $\chi(G)$ of a graph $G = (V, E)$ is the minimal cardinal $\varkappa$ for which there exists a vertex coloring with $\varkappa$ colors. There is a long history of structure theorems deriving from large chromatic number assumptions; see, e.g., [8]. The main topic of this paper will be the following conjecture proposed by Erdős–Hajnal–Shelah [5, Problem 2] and Taylor [1, Problem 43, p. 508].

Conjecture (Strong Taylor's Conjecture). For any graph G with $\chi(G) > \aleph_0$ there exists an $n \in \mathbb{N}$ such that G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$.

Here, for a cardinal κ, the *shift graph* $\mathrm{Sh}_n(\kappa)$ is the graph whose vertices are increasing n-tuples of ordinals less than κ, and we put an edge between s and t if for every $1 \le i \le n-1$, $s(i) = t(i-1)$ or vice versa. The shift graphs $\mathrm{Sh}_n(\kappa)$ have large chromatic numbers depending on κ; see Fact 2.4 below. Consequently, if the strong Taylor's conjecture holds for a graph G, then the graph has elementary extensions of unbounded chromatic number (having the same family of finite subgraphs).

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The strong Taylor's conjecture was refuted in [6, Theorem 4]. See [8] and the introduction of [7] for more historical information.

In [7] we initiated the study of variants of the strong Taylor's conjecture for some classes of graphs with *stable* first order theory (stable graphs). Stability theory, which is the study of stable theories and originated in the works of the third author in the 60s and 70s, is one of the most influential and important subjects in modern model theory. Examples of stable theories include abelian groups, modules, algebraically closed fields, graph-theoretic trees, or more generally superflat graphs [13]. Stability also had an impact in combinatorics, e.g. [2, 10] to name a few.

More precisely, in [7] we proved the strong Taylor's conjecture for ω-stable graphs and variants of the conjecture for superstable graphs (replacing $\aleph_0$ by $2^{\aleph_0}$) and for stable graphs which are interpretable in a stationary stable theory (replacing $\aleph_0$ by $\beth_2(\aleph_0)$). As there exist stable graphs that are not interpretable in a stationary stable structure (see [7, Proposition 5.22, Remark 5.23]), we asked what is the situation in general stable graphs and in this paper we answer it with the following theorem.

Theorem (Corollary 6.2). *Let $G = (V, E)$ be a stable graph. If $\chi(G) > \beth_2(\aleph_0)$ then G contains all finite subgraphs of $\mathrm{Sh}_n(\omega)$ for some $n \in \mathbb{N}$.*

The key tool in proving the results for ω-stable graphs and superstable graphs is that every large enough saturated model is an Ehrenfeucht–Mostowski model (EM-model) in some bounded expansion of the language.

An EM-model is a model which is the definable closure of an indiscernible sequence and was originally used by Ehrenfeucht–Mostowski in order to find models with many automorphisms [3]. It was shown by Lascar [9, Section 5.1] that every saturated model of cardinality $\aleph_1$ in an ω-stable theory is an EM-model in some countable expansion of the language; this was later generalized to any cardinality by Mariou [12, Theorem C] and to superstable theories by Mariou [11, Theorem 3.B] and by the third author [15].

It was shown by Mariou [11, Theorem 3.A] that in a certain sense the existence of such saturated EM-models for a stable theory necessarily implies that the theory is superstable. Consequently, a different tool is needed in order to prove the theorem for general stable theories.

In the stationary stable case, we use a variant of representations of structures in the sense of [16]. However, this method does not seem to easily adjust to the general stable case.

In this paper we resolve this problem by generalizing the notion of EM-models to *infinitary EM-models* and show in Theorem 3.7 that such saturated models exist for any stable theory. The definition is a bit technical, so here we will settle with an informal description:

In an EM-model every element is given by a term and a finite sequence of elements from the generating indiscernible sequence. Analogously, in an infinitary EM-model every element is given by some "term" with infinite (but bounded) arity and a suitable sequence of elements from an indiscernible sequence.

We prove that the existence of saturated infinitary EM-models characterizes stability.

Theorem (Theorem 3.7). *The following are equivalent for a complete $\mathcal{L}$-theory T:*

(1) *T is stable.*

(2) *Let κ, μ and λ be cardinals satisfying $\kappa = \mathrm{cf}(\kappa) \geq \min\{\kappa(T), |T|^+\} + \aleph_1$, $\mu^{<\kappa} = \mu \geq 2^{\kappa+|T|}$ and $\lambda = \lambda^{<\kappa} \geq \mu$, and let $T \subseteq T^{\mathrm{sk}}$ be an expansion with definable Skolem functions such that $|T| = |T^{\mathrm{sk}}|$ in a language $\mathcal{L} \subseteq \mathcal{L}^{\mathrm{sk}}$. Then there exists an infinitary EM-model $M^{\mathrm{sk}} \models T^{\mathrm{sk}}$ based on (α, λ), where $\alpha \in \kappa^U$ for some set U of cardinality at most μ, such that $M = M^{\mathrm{sk}} \restriction \mathcal{L}$ is saturated of cardinality λ.*

See the paragraph before Lemma 3.4 for the definition of $\kappa(T)$.

Section 3 is the only purely model-theoretic section and is the only place where stability is used. The results of this section (more specifically Theorem 3.7) are only used in Section 6. In Section 4 we study graphs on (perhaps infinite) increasing sequences whose edge relation is determined by the order type. Aiming to prove that if the chromatic number is large, then one can embed shift graphs, we analyze several different cases. The last case we deal with in Section 4 turns out to be rather complicated, so we devote all of Section 5 to it. There, we employ ideas inspired by PCF theory to get a coloring of small cardinality. Section 6 concludes the paper.

2. Preliminaries

We use small Latin letters a, b, c for tuples and capital letters A, B, C for sets. We also employ the standard model-theoretic abuse of notation and write $a \in A$ even for tuples when the length of the tuple is immaterial or understood from context.

For any two sets A and J, let $A^{\underline{J}}$ be the set of injective functions from J to A (where the notation is taken from the falling factorial notation), and if $(A, <)$ and $(J, <)$ are both linearly ordered sets, let $(A^{\underline{J}})_<$ be the subset of $A^{\underline{J}}$ consisting of strictly increasing functions. If we want to emphasize the order on J we will write $(A^{\underline{(J,<)}})_<$.

Throughout this paper, we interchangeably use sequence notation and function notation for elements of $A^{\underline{J}}$, e.g. for $f \in A^{\underline{J}}$, $f(i) = f_i$. For any sequence η we denote by $\mathrm{Range}(\eta)$ the underlying set of the sequence (i.e. its image). If $(A, <^A)$ and $(B, <^B)$ are linearly ordered sets, then the most significant coordinate of the lexicographic order on $A \times B$ is the left one.

2.1. *Stability*

We use fairly standard model-theoretic terminology and notation; see for example [17, 18]. We gather some of the needed notions. For stability, the reader can also consult [14].

We denote by $\mathrm{tp}(a/A)$ the complete type of a over A. Let $(I, <)$ be a linearly ordered set. A sequence $\langle a_i : i \in I \rangle$ inside a first order structure is *indiscernible* if for any $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_k$ in I,

$$\mathrm{tp}(a_{i_1}, \ldots, a_{i_k}) = \mathrm{tp}(a_{j_1}, \ldots, a_{j_k}).$$

A structure M is *κ-saturated*, for a cardinal κ, if any type p over A with $|A| < \kappa$ is realized in M. The structure M is *saturated* if it is $|M|$-saturated. A *monster model* for T, usually denoted by $\mathbb{U}$, is a large saturated model containing all sets and models (as elementary substructures) we will encounter.[1] All subsets and models will be *small*, i.e. of cardinality $< |\mathbb{U}|$.

A first order theory T is *stable* if there does not exist a model $M \models T$, a formula $\varphi(x, y)$ and elements $\langle a_i \in M : i < \omega \rangle$ such that $M \models \varphi(a_i, a_j) \Leftrightarrow i < j$. An equivalent definition is that there exists some $\kappa \geq |T|$ such that for all $M \models T$ with $|M| \leq \kappa$ the cardinality of complete types over M is at most κ. For any such κ, T has a saturated model of cardinality of κ [14, Theorem VIII.4.7].

Every indiscernible sequence in a stable theory is *totally indiscernible*, i.e. in the notation above, for any $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ in I,

$$\mathrm{tp}(a_{i_1}, \ldots, a_{i_k}) = \mathrm{tp}(a_{j_1}, \ldots, a_{j_k}).$$

Other than these notions, we will also require basic understanding of forking. See the above references for more information.

2.2. *Graph theory*

Here we gather some facts on graphs and the chromatic number of graphs (all can be found in [7]).

By a *graph* we mean a pair $G = (V, E)$ where $E \subseteq V^2$ is symmetric and irreflexive. A *graph homomorphism* between $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is a map $f : V_1 \to V_2$ such that $f(e) \in E_2$ for every $e \in E_1$. If f is injective we will say that f *embeds* G_1 *into* G_2 *as a subgraph*. If in addition we require that $f(e) \in E_2$ if and only if $e \in E_1$ we will say that f *embeds* G_1 *into* G_2 *as an induced subgraph*.

Definition 2.1. Let $G = (V, E)$ be a graph.

(1) For a cardinal $\varkappa$, a *vertex coloring* (or just coloring) of cardinality $\varkappa$ is a function $c : V \to \varkappa$ such that $x \; E \; y$ implies $c(x) \neq c(y)$ for all $x, y \in V$.

(2) The *chromatic number* $\chi(G)$ is the minimal cardinality of a vertex coloring of G.

These are the basic properties of $\chi(G)$ that we will require:

Fact 2.2 ([7, Lemma 2.3]). *Let $G = (V, E)$ be a graph.*

(1) *If $V = \bigcup_{i \in I} V_i$ then $\chi(G) \leq \sum_{i \in I} \chi(V_i, E \restriction V_i)$.*

(2) *If $E = \bigcup_{i \in I} E_i$ (with each E_i being symmetric) then $\chi(G) \leq \prod_{i \in I} \chi(V, E_i)$.*

(3) *If $\varphi : H \to G$ is a graph homomorphism then $\chi(H) \leq \chi(G)$.*

[1] There are set-theoretic issues in assuming that such a model exists, but these are overcome by standard techniques from set theory that ensure the generalized continuum hypothesis from some point on while fixing a fragment of the universe. The reader can just accept this or alternatively assume that $\mathbb{U}$ is merely κ-saturated and κ-strongly homogeneous for large enough κ.

(4) *If* $\varphi : (H, E^H) \to (G, E^G)$ *is a surjective graph homomorphism with*

$$e \in E^H \iff \varphi(e) \in E^G$$

then $\chi(H) = \chi(G)$.

Example 2.3. For any finite number $r \geq 1$ and any linearly ordered set $(A, <)$, let $\mathrm{Sh}_r(A)$ (or $\mathrm{Sh}_r(A, <)$ if we want to emphasize the order), *the shift graph on* A, be the following graph: its set of vertices is the set $(A^{\underline{r}})_<$ of increasing r-tuples $s_0, \dots, s_{r-1}$, and we put an edge between s and t if for every $1 \leq i \leq r-1$, $s(i) = t(i-1)$ or vice versa. It is an easy exercise to show that $\mathrm{Sh}_r(A)$ is a connected graph. If $r = 1$ this gives K_A, the complete graph on A.

Fact 2.4 ([7, Fact 2.6], [4, proof of Theorem 2]). *Let* $2 \leq r < \omega$ *be a natural number and* $\varkappa$ *be a cardinal. Then*

$$\chi(\mathrm{Sh}_r(\beth_{r-1}(\varkappa)^+)) \geq \varkappa^+.$$

Finally, the following fact is a very useful tool.

Fact 2.5 ([7, Proposition 3.2]). *Let* $G = (V, E)$ *be a graph and assume there exists a graph homomorphism* $t : \mathrm{Sh}_k(\omega) \to G$. *Then there exists* $n \leq k$ *such that*

(†) *G contains all finite subgraphs of* $\mathrm{Sh}_n(\omega)$.

Consequently, if H *is a graph that contains all finite subgraphs of* $\mathrm{Sh}_k(\omega)$ *for some* k, *and* $t : H \to G$ *is a graph homomorphism, then there exists some* $n \leq k$ *such that* G *satisfies* (†).

3. Infinitary EM-models and stability

Let T be a first order theory and $\mathbb{U}$ a monster model for T.

An *EM-model* for T is a model that is the definable closure of an indiscernible sequence (possibly in some expansion of the theory which admits Skolem functions).

Every element in an EM-model is of the form $t(a_{i_1}, \dots, a_{i_n})$, where t is a term (in the expanded language) and $a_{i_1}, \dots, a_{i_n}$ are elements of the indiscernible sequence. In other words, to any element we may associate a pair (i, η), where $i < |T|$ (this codes the term $t_i(\bar{x}_i)$) and η is an increasing sequence of cardinality $|\bar{x}_i|$.

Mariou [11, 12] and Shelah [15] proved that if T is ω-stable or even superstable then it has an EM-model in some expansion of the language whose restriction to the original language is saturated. For general stable theories, as we will see in this section, one needs to allow "terms" with possibly infinite arity to get a parallel result.

Let $\kappa \geq \aleph_0$ be a regular cardinal (which we think of as a bound on the arity) and let μ be a cardinal (which we think of as a bound on the number of terms). Let $\alpha \in \kappa^\mu$ be a function assigning to each function its arity.

Definition 3.1. Let κ be a cardinal, $(I, <)$ a linearly ordered set, U a set and $\alpha \in \kappa^U$. Let $a = \langle a_{i,\eta} : i \in U, \eta \in (I^{\underline{\alpha_i}})_< \rangle$ be a sequence of tuples from $\mathbb{U}$.

We say that a is (α, I)-*indiscernible* if for every $\langle i_j \in U : j < k\rangle$, $\langle \eta_j \in (I^{\underline{\alpha_{i_j}}})_< : j < k\rangle$ and $\langle \rho_j \in (I^{\underline{\alpha_{i_j}}})_< : j < k\rangle$ if there exists a partial isomorphism of $(I, <)$ mapping $\langle \eta_j : j < k\rangle$ to $\langle \rho_j : j < k\rangle$ then $\langle a_{i_j,\eta_j} : j < k\rangle$ and $\langle a_{i_j,\rho_j} : j < k\rangle$ have the same type.

Recall that given a subset $A \subseteq \mathbb{U}$ and an ultrafilter $\mathcal{D}$ on A we may define the global *average type* $p_{\mathcal{D}} = \mathrm{Av}(\mathcal{D}, \mathbb{U})$ by

$$p_{\mathcal{D}} \vdash \varphi(x, b) \iff \varphi(A, b) \in \mathcal{D}.$$

Obviously, $p_{\mathcal{D}}$ is finitely satisfiable in A.

Remark 3.2. If $\mathcal{D}$ is an ultrafilter on A and $A \subseteq B$ then $\{U \subseteq B : (\exists V \in \mathcal{D})(V \subseteq U)\}$ is the unique ultrafilter $\mathcal{D}'$ on B containing $\mathcal{D}$ and $p_{\mathcal{D}} = p_{\mathcal{D}'}$.

For any linearly ordered set $(I, <)$ and $A \subseteq B$, we say that $\langle a_i : i \in I\rangle$ *realizes* $(p_{\mathcal{D}})^{\otimes I}|B$ if for any $k \in I$, $a_k \models p_{\mathcal{D}}|B\langle a_i : i < k \in I\rangle$.

Proposition 3.3. *Assume that $\mathbb{U}$ has definable Skolem functions. Let $\kappa \geq \aleph_0$ be a regular cardinal and $\mu^{<\kappa} = \mu \geq 2^{\kappa+|\mathcal{L}|}$ a cardinal. Let $\alpha \in \kappa^\mu$ be any function and let $(I, <)$ be any infinite linear order.*

(1) *There exist $U \subseteq \mu \times \mu$ and a non-constant (α', I)-indiscernible sequence*

$$a = \langle a_{i,k,\eta} : (i, k) \in U,\ \eta \in (I^{\underline{\alpha_i}})_<\rangle,$$

where $\alpha' \in \kappa^U$ is defined by $\alpha'_{(i,k)} = \alpha_i$, for $(i, k) \in U$, such that $\mathbb{U} \restriction \mathrm{dcl}(\mathrm{Range}(a)) \prec \mathbb{U}$.

(2) *For $j < \mu$ and $\eta \in (I^{\underline{\alpha_j}})_<$, if*

$$A \subseteq A_{j,\eta} = \mathrm{dcl}\big(\{a_{i,k,\nu} : (i, k) \in U,\ i < j,\ \nu \in (\mathrm{Range}(\eta)^{\underline{\alpha_i}})_<\}\big)$$

with $|A| < \kappa$ and non-algebraic $p \in S(A)$ then there exists $k < \mu$ with $(j, k) \in U$ such that $a_{j,k,\eta} \models p$. Moreover, if p is finitely satisfiable in A then so is $\mathrm{tp}(a_{j,k,\eta}/A_{j,\eta})$.

(3) *If in addition $(I, <)$ is well-ordered and α satisfies $\alpha_i = (i \bmod \kappa)$ then:*
   - (a) *For any $A \subseteq \mathrm{dcl}(\mathrm{Range}(a))$ with $|A| < \kappa$ there exist $i < \mu$ and $\eta \in (I^{\underline{\alpha_i}})_<$ satisfying $A \subseteq A_{i,\eta}$.*
   - (b) *$\mathrm{dcl}(\mathrm{Range}(a))$ is κ-saturated.*
   - (c) *Assume that $(I, <)$ is a cardinal with $\mathrm{cf}(I) \geq \kappa$. For any infinite $A \subseteq B \subseteq \mathrm{dcl}(\mathrm{Range}(a))$ with $|B| < \kappa$, there is a non-principal ultrafilter $\mathcal{D}$ on A such that $(p_{\mathcal{D}})^{\otimes I}|B$ is realized in $\mathrm{dcl}(\mathrm{Range}(a))$.*

Proof. Since $\mathbb{U}$ has definable Skolem functions, for any $A \subseteq \mathbb{U}$, $\mathrm{dcl}(A)$ is an elementary substructure of $\mathbb{U}$.

First we pick, once and for all, for any small $A \subseteq \mathbb{U}$ and any $p \in S(A)$ finitely satisfiable in A, a non-principal ultrafilter $\mathcal{U}_p$ on A extending the filter $\{\varphi(A, a) : \varphi(x, a) \in p\}$.

The construction is by induction on $j < \mu$. Let $j < \mu$ and assume we found $\{U_i \subseteq \mu : i < j\}$ and non-constant $a_{<j} = \langle a_{i,k,\eta} : i < j,\ k \in U_i,\ \eta \in (I^{\underline{\alpha_i}})_<\rangle$ such that $a_{<j}$ is $((\alpha')^{<j}, I)$-indiscernible, where $((\alpha')^{<j})_{i,k} = \alpha_i$ for $i < j$ and $k \in U_i$.

If $(I^{\underline{\alpha_j}})_< = \emptyset$ then there is nothing to do. Otherwise, fix some $\eta^* \in (I^{\underline{\alpha_j}})_<$ and let

$$A_{j,\eta^*} = \operatorname{dcl}\big(\{a_{i,k,\nu} : (i,k) \in U,\ i < j,\ \nu \in (\operatorname{Range}(\eta^*)^{\underline{\alpha_i}})_<\}\big).$$

Note that $|A_{j,\eta}| \leq \mu$. Indeed, this follows from the inequalities

$$\mu \cdot |\alpha_j|^{<\kappa} \leq \mu \cdot \kappa^{<\kappa} \leq \mu.$$

For any $A \subseteq A_{j,\eta^*}$ with $|A| < \kappa$ and a non-algebraic type $p \in S_1(A)$ we choose an extension $\tilde{p}$ of p to A_{j,η^*} and a non-principal ultrafilter $\mathcal{D}(p)$ on A_{j,η^*} such that $\tilde{p} = p_{\mathcal{D}(p)}|A_{j,\eta^*}$, in the following way:

- If p is finitely satisfiable in A then let $\mathcal{D}(p)$ be the unique ultrafilter (on A_{j,η^*}) extending the ultrafilter $\mathcal{U}_p$ on A (which extended $\{\varphi(A,a) : \varphi(x,a) \in p\}$, as chosen above). We let $\tilde{p} = p_{\mathcal{D}(p)}|A_{j,\eta^*}$.
- Otherwise, let $\tilde{p}$ be any non-algebraic extension of p to A_{j,η^*}. Since A_{j,η^*} is a model, $\tilde{p}$ is finitely satisfiable in A_{j,η^*}. Let $\mathcal{D}(p) = \mathcal{U}_{\tilde{p}}$ be the non-principal ultrafilter extending $\{\varphi(A_{j,\eta^*}, b) : \varphi(x,b) \in \tilde{p}\}$ from above.

We note that there are at most $\mu^{<\kappa} \leq \mu$ subsets $A \subseteq A_{j,\eta^*}$ with $|A| < \kappa$ and for all such A there are at most $2^{\kappa+|\mathcal{L}|} \leq \mu$ types on A.

Let $U_j \subseteq \mu$ be such that $\langle (p_{j,k,\eta^*}, \mathcal{D}_{j,k,\eta^*}) : k \in U_j\rangle$ enumerates the set of pairs $(\tilde{p}, \mathcal{D}(p))$ for non-algebraic $p \in S_1(A)$ and any A as above.

By the induction hypothesis, any partial order isomorphism π of I induces a partial elementary map $\hat{\pi}$ whose domain is

$$\operatorname{dcl}\big(\{a_{i,k,\nu} : i < j,\ k \in U_i,\ \operatorname{Range}(\nu) \subseteq \operatorname{Dom}(\pi)\}\big),$$

mapping $a_{i,k,\nu} \mapsto a_{i,k,\pi(\nu)}$, where $\pi(\nu) = \pi \circ \nu$. Note that for any π_1, π_2, if $\pi_1 \circ \pi_2$ makes sense then $\widehat{\pi_1 \circ \pi_2} = \widehat{\pi_1} \circ \widehat{\pi_2}$.

Note that, by the induction hypothesis on j, for any order-preserving partial isomorphism π of I whose domain contains $\operatorname{Range}(\eta^*)$, $\hat{\pi}(A_{j,\eta^*}) = A_{j,\pi(\eta^*)}$.

As a result, for any $\rho \in (I^{\underline{\alpha_j}})_<$ the unique order isomorphism $\pi_{\eta^*,\rho} : \eta^* \to \rho$ induces an elementary map $\widehat{\pi_{\eta^*,\rho}}$ whose domain is A_{j,η^*} and whose range is precisely $A_{j,\rho}$, which is given by $a_{i,k,\nu} \mapsto a_{i,k,\pi_{\eta^*,\rho}(\nu)}$. For every $k \in U_j$ let $p_{j,k,\rho} = \widehat{\pi_{\eta^*,\rho}}(p_{j,k,\eta^*}) \in S(A_{j,\rho})$ and let $\mathcal{D}_{j,k,\rho} = \widehat{\pi_{\eta^*,\rho}}(\mathcal{D}_{j,k,\eta^*})$.

Claim 3.3.1. *For any $\rho \in (I^{\underline{\alpha_j}})_<$ and π a partial isomorphism of I whose domain contains* $\operatorname{Range}(\rho)$, $\hat{\pi}(A_{j,\rho}) = A_{j,\pi(\rho)}$, $\hat{\pi}(p_{j,k,\rho}) = p_{j,k,\pi(\rho)}$ *and* $\hat{\pi}(\mathcal{D}_{j,k,\rho}) = \mathcal{D}_{j,k,\pi(\rho)}$.

Proof. There is no harm in restricting π to $\operatorname{Range}(\rho)$. Let $\pi_{\eta^*,\rho} : \eta^* \to \rho$ be the unique order isomorphism, so $\pi \circ \pi_{\eta^*,\rho}$ is the unique isomorphism from η^* to $\pi(\rho)$ and thus equal to $\pi_{\eta^*,\pi(\rho)}$. Hence $\pi = \pi_{\eta^*,\pi(\rho)} \circ \pi^{-1}_{\eta^*,\rho}$. The result follows. $\blacksquare_{\text{claim}}$

Let $((I^{\underline{\alpha_j}})_<, <^{\text{lex}})$ be the lexicographic ordering and let $(U_j, <)$ be the order induced from μ. By induction on $k \in U_j$, by compactness we may find a sequence $\langle a_{j,k,\eta} : \eta \in (I^{\underline{\alpha_j}})_< \rangle$ such that for any $\eta \in (I^{\underline{\alpha_j}})_<$,

$$a_{j,k,\eta} \models p_{\mathfrak{D}_{j,k,\eta}} | B_{j,k,\eta},$$

where

$$B_{j,k,\eta} = C_j \cup \{a_{j,l,\rho} : l < k,\ l \in U_j,\ \rho \in (I^{\underline{\alpha_j}})_<\} \cup \{a_{j,k,\rho} : \rho <^{\text{lex}} \eta,\ \rho \in (I^{\underline{\alpha_j}})_<\},$$

where $C_j = \{a_{i,l,\rho} : i < j,\ l \in U_i,\ \rho \in (I^{\underline{\alpha_i}})_<\}$.

We show $((\alpha')^{\leq j}, I)$-indiscernibility by induction on $\{(i, l) : i \leq j,\ l \in U_i\}$ (with the lexicographic ordering). In other words, given $k \in U_j$, we assume that for any $\langle (i_r, l_r) : r < n \rangle$ with $(i_r, l_r) < (j, k)$ and $l_r \in U_r$, any $\langle \eta_r \in (I^{\underline{\alpha_{i_r}}})_< : r < n \rangle$ and any partial isomorphism π of I whose domain contains $\bigcup \{\text{Range}(\eta_r) : r < n\}$,

$$\text{tp}(a_{i_0,l_0,\eta_0}, \ldots, a_{i_{n-1},l_{n-1},\eta_{n-1}}) = \text{tp}(a_{i_0,l_0,\pi(\eta_0)}, \ldots, a_{i_{n-1},l_{n-1},\pi(\eta_{n-1})}).$$

We wish to show the same statement for $(i_r, l_r) \leq (j, k)$.

We prove by induction on n that for any $\bar{b} \subseteq \{a_{i,l,\eta} : (i, l) < (j, k),\ \eta \in (I^{\underline{\alpha_i}})_<,\ l \in U_i\}$, any $\eta_{n-1} <^{\text{lex}} \cdots <^{\text{lex}} \eta_0 \in (I^{\underline{\alpha_j}})_<$ and any partial isomorphism π of $(I, <)$ whose domain contains

$$\text{Range}(\eta_0) \cup \cdots \cup \text{Range}(\eta_{n-1}) \cup \bigcup \{\text{Range}(\eta) : a_{i,l,\eta} \in \bar{b}\},$$

we have

$$\text{tp}(a_{j,k,\eta_0}, \ldots, a_{j,k,\eta_{n-1}}, \bar{b}) = \text{tp}(a_{j,k,\pi(\eta_0)}, \ldots, a_{j,k,\pi(\eta_{n-1})}, \hat{\pi}(\bar{b})).$$

Let $\varphi(x_0, \ldots, x_{n-1}, \bar{b})$ be some formula, where $\bar{b}$ is as above. We show that

$$\varphi(x_0, \ldots, x_{n-1}, \bar{b}) \in p_{\mathfrak{D}_{j,k,\eta_0}} \otimes \cdots \otimes p_{\mathfrak{D}_{j,k,\eta_{n-1}}} \iff \varphi(x_0, \ldots, x_{n-1}, \hat{\pi}(\bar{b})) \in p_{\mathfrak{D}_{j,k,\pi(\eta_0)}} \otimes \cdots \otimes p_{\mathfrak{D}_{j,k,\pi(\eta_{n-1})}}.$$

Indeed, if $\varphi(x_0, \ldots, x_{n-1}, \bar{b}) \in p_{\mathfrak{D}_{j,k,\eta_0}} \otimes \cdots \otimes p_{\mathfrak{D}_{j,k,\eta_{n-1}}}$ then by the choice of the $a_{j,k,\eta}$'s, $\varphi(a_{j,k,\eta_0}, \ldots, a_{j,k,\eta_{n-1}}, \bar{b})$ holds and thus $X = \varphi(A_{j,\eta_0}, a_{j,k,\eta_1}, \ldots, a_{j,k,\eta_{n-1}}, \bar{b}) \in \mathfrak{D}_{j,k,\eta_0}$. By Claim 3.3.1, $\hat{\pi}(X) \in \mathfrak{D}_{j,k,\pi(\eta_0)}$. By the induction hypothesis (on n), $\hat{\pi}$ is elementary on $a_{j,k,\eta_1} \cup \cdots \cup a_{j,k,\eta_{n-1}} \cup \bar{b}$, and consequently

$$\hat{\pi}(X) = \varphi(A_{j,\pi(\eta_0)}, a_{j,k,\pi(\eta_1)}, \ldots, a_{j,k,\pi(\eta_{n-1})}, \hat{\pi}(\bar{b})) \in \mathfrak{D}_{j,k,\pi(\eta_0)},$$

and as π preserves $<^{\text{lex}}$,

$$\varphi(x_0, a_{j,k,\pi(\eta_1)}, \ldots, a_{j,k,\pi(\eta_{n-1})}, \hat{\pi}(\bar{b})) \in p_{\mathfrak{D}_{j,k,\pi(\eta_0)}} | B_{j,k,\pi(\eta_0)}.$$

As a result, by the choice of the $a_{j,k,\eta}$'s,

$$\varphi(a_{j,k,\pi(\eta_0)}, a_{j,k,\pi(\eta_1)}, \ldots, a_{j,k,\pi(\eta_{n-1})}, \hat{\pi}(\bar{b})) \text{ holds,}$$

and thus

$$\varphi(x_0, \dots, x_{n-1}, \hat{\pi}(\bar{b})) \in p_{\mathcal{D}_{j,k,\pi(\eta_0)}} \otimes \cdots \otimes p_{\mathcal{D}_{j,k,\pi(\eta_{n-1})}}.$$

This proves (1), i.e. $a = \langle a_{i,k,\eta} : i < \mu, k \in U_i, \eta \in (I^{\underline{\alpha_i}})_< \rangle$ is (α', I)-indiscernible (with $U = \{(i,k) \in \mu \times \mu : k \in U_i\}$).

(2) follows immediately from the construction in (1).

(3) Now assume that $(I, <)$ is well-ordered and α satisfies $\alpha_i = (i \bmod \kappa)$ and let $A \subseteq a$ with $|A| < \kappa$. Since $(I, <)$ is well-ordered, there exist an ordinal β and an order isomorphism $\eta : \beta \to \bigcup_{a_{j,l,\nu} \in A} \mathrm{Range}(\nu)$. Since κ is a regular cardinal and for every $a_{j,l,\nu} \in A$ we have $|\mathrm{Range}(\nu)| = |\alpha_j| < \kappa$, it follows that $\beta < \kappa$.

Since $\kappa < 2^\kappa \leq \mu$ and $\mu^{<\kappa} = \mu$ (so $\mathrm{cf}(\mu) \geq \kappa$), we have $\hat{j} = \sup_{a_{j,l,\nu} \in A} j < \mu$. Let $i = \hat{j} \cdot \kappa + \beta < \mu$. By the choice of α, $\alpha_i = \beta$ and $\eta \in (I^{\underline{\alpha_i}})_<$. This implies that $A \subseteq A_{i,\eta}$. This gives (3.a). For (3.b), since $\mathrm{dcl}(\mathrm{Range}(a))$ is a model, it is enough to deal with non-algebraic types, which is exactly (2).

Item (3.c) follows from the construction; we give the details. Let $A \subseteq B$ with $|B| < \kappa$. By (3.a) there exist $j < \mu$ and $\eta \in (I^{\underline{\alpha_j}})_<$ such that $B \subseteq A_{j,\eta}$. As $\mathrm{cf}(I) \geq \kappa$, there is some $\mathrm{Range}(\eta) < \gamma^* \in I$. Let $p \in S(A)$ be a non-algebraic type that is finitely satisfiable in A and $k \in U_{j+1}$ be such that $(p_{j+1,k,\eta^\frown\gamma^*}, \mathcal{D}_{j+1,\eta^\frown\gamma^*})$ are the type (over $A_{j+1,\eta^\frown\gamma^*}$) and non-principal ultrafilter (on $A_{j+1,\eta^\frown\gamma^*}$) corresponding to p and $A \subseteq A_{j+1,\eta^\frown\gamma^*}$ as above.

By Claim 3.3.1, for any $\mathrm{Range}(\eta) < \gamma \in I$,

$$A_{j+1,\eta^\frown\gamma} = \hat{\pi}_{\eta^\frown\gamma^*,\eta^\frown\gamma}(A_{j+1,\eta^\frown\gamma^*}), \quad p_{j+1,k,\eta^\frown\gamma} = \hat{\pi}_{\eta^\frown\gamma^*,\eta^\frown\gamma}(p_{j+1,\eta^\frown\gamma^*}).$$

Observe that since $A \subseteq A_{j,\eta}$, $\hat{\pi}_{\eta^\frown\gamma^*,\eta^\frown\gamma}$ fixes A pointwise. By Remark 3.2 and the choice of the ultrafilters above, for every $\mathrm{Range}(\eta) < \gamma \in I$, $p_{\mathcal{D}_{j+1,k,\eta^\frown\gamma^*}} = p_{\mathcal{D}_{j+1,k,\eta^\frown\gamma}}$. Let $p_{\mathcal{D}} := p_{\mathcal{D}_{j+1,k,\eta^\frown\gamma^*}}$. It is finitely satisfiable in A.

By the choice of elements, for any $\mathrm{Range}(\eta) < \gamma \in I$,

$$a_{j+1,k,\eta^\frown\gamma} \models p_{\mathcal{D}}|A_{j,\eta}\langle a_{j+1,k,\eta^\frown\delta} : \mathrm{Range}(\eta) < \delta < \gamma\rangle.$$

We end by noting that since we are assuming that $(I, <)$ is a cardinal and $\mathrm{cf}(I) \geq \kappa > \alpha_j$, it follows that $|\{\gamma : \mathrm{Range}(\eta) < \gamma \in I\}| = |I|$. ∎

In stable theories, for any infinite indiscernible sequence I over some set A one may take the limit type defined by

$$\lim(I) = \{\varphi(x, c) : \varphi(a, c) \text{ holds for cofinitely many } a \in I\}.$$

It is a consistent complete type by stability. It is obviously finitely satisfiable in I. Moreover, if $\mathcal{D}$ is a non-principal ultrafilter on I, then $p_{\mathcal{D}} = \lim(I)$. We often write $\lim(I/A) = \lim(I)|A$.

Recall that for a theory T, $\kappa(T)$ is the least cardinal κ such that for all B and type $p \in S(B)$ there exists $A \subseteq B$ with $|A| < \kappa$ such that p does not fork over A, if such a cardinal exists, and ∞ otherwise. By, e.g., [18, Proposition 7.2.5], if $\kappa(T) < \infty$ then

$\kappa(T) \leq |T|^{+}$. For stable theories, this agrees with [14, Definition III.3.1] and [14, Corollary III.3.3]

The following is [14, Lemma III.3.10]; we give a proof for completeness.

Lemma 3.4. *Let T be a stable theory and $M \models T$. If M is $(\kappa(T) + \aleph_1)$-saturated and every countable indiscernible sequence over $A \subseteq M$, with $|A| < \kappa(T)$, in M can be extended to one of cardinality λ then M is λ-saturated.*

Proof. We may assume that $\lambda > \kappa(T) + \aleph_1$. By passing to M^{eq} (and T^{eq}) there is no harm in assuming that T eliminates imaginaries. Let $p \in S(C)$ with $C \subseteq M$ and $|C| < \lambda$. Let $B \subseteq C$ with $|B| < \kappa(T)$ be such that p does not fork over B. Let $q \supseteq p$ be a non-forking global extension. Since M is $\kappa(T)$-saturated, we may find a sequence $S = \langle b_i : i < \omega \rangle \subseteq M$ satisfying $b_i \models q|B\langle b_j : j < i\rangle$. Note that $q|BS$ is stationary by [14, Corollary III.2.11].

Since M is $(\kappa(T) + \aleph_1)$-saturated, we may find a Morley sequence $I = \langle a_i : i < \omega \rangle$ of q over SB, i.e. $a_i \models q|SBa_{<i}$ and $a_i \in M$. It follows that I is also a Morley sequence of q over $\mathrm{acl}(B)$.[2] Let $I \subseteq J \subseteq M$ be an indiscernible sequence (over B) of cardinality λ. As a result, J is also a Morley sequence of q over $\mathrm{acl}(B)$.

By [14, Lemma III.1.10 (2)], $\lim(J/M) = q|M$ and in particular $\lim(J/C) = p$. By [14, Corollary III.3.5 (1)], there is $J_0 \subseteq J$ with $J \setminus J_0$ indiscernible over C and $|J_0| \leq \kappa(T) + |C| < \lambda$. In particular, $|J \setminus J_0| \geq \aleph_0$ and thus for every $c \in J \setminus J_0$, $p = \mathrm{tp}(c/C)$. ■

Definition 3.5. Let T be a theory. We say that $M \models T$ is an *infinitary EM-model based on* (α, I) if $M = \mathrm{dcl}(a)$, where a is a non-constant (α, I)-indiscernible sequence for (α, I) as in Definition 3.1.

Lemma 3.6. *Let T be any theory. Let $\kappa \geq \aleph_0$ be a cardinal, $(I, <)$ a linearly ordered set, and $\alpha \in \kappa^U$, where U is a set. If $a = \langle a_{i,\eta} : i \in U, \eta \in (I^{\underline{\alpha_i}})_< \rangle$ is an (α, I)-indiscernible sequence, in some model $M \models T$, then there exists some set $\widehat{U}$, with $|\widehat{U}| \leq |T| \cdot |U| \cdot \kappa^{<\kappa}$, and $\widehat{\alpha} \in \kappa^{\widehat{U}}$ and an $(\widehat{\alpha}, I)$-indiscernible sequence b whose underlying set is* $\mathrm{dcl}(a)$.

Proof. For any $p \subseteq \kappa$ let $\varphi_p : \mathrm{otp}(p) \to p$ be the unique order isomorphism. Let $\mathcal{F}$ be the collection of all $\emptyset$-definable functions. We consider the family $\widehat{U}$ of tuples

$$(f(\bar{v}), s_0, p_0, \ldots, s_{|\bar{v}|-1}, p_{|\bar{v}|-1})$$

satisfying

- $f(\bar{v}) \in \mathcal{F}$,
- $s_0, \ldots, s_{|\bar{v}|-1} \in U$,
- for any $i < |\bar{v}|$, $p_i \subseteq \kappa$ with $\mathrm{otp}(p_i) = \alpha_{s_i}$,
- $\bigcup_{i<|\bar{v}|} p_i \in \mathrm{Ord}$.

[2] It is standard to see that I is independent and indiscernible over $\mathrm{acl}(B)$. On the other hand, since $q|BS$ is stationary, it isolates a complete type over $\mathrm{acl}(BS)$.

We note that $|\widehat{U}| \leq |T| \cdot |U|^{<\aleph_0} \cdot (\kappa^{<\kappa})^{<\aleph_0} \leq |T| \cdot |U| \cdot \kappa^{<\kappa}$. Let $\widehat{\alpha} \in \kappa^{\widehat{U}}$ be the function mapping $x = (f(\bar{v}), s_0, p_0, \dots, s_{|\bar{v}|-1}, p_{|\bar{v}|-1})$ to $\bigcup_{i<|\bar{v}|} p_i < \kappa$. For any such $x \in \widehat{U}$ and $\eta \in (I^{\underline{\widehat{\alpha}_x}})_<$ set

$$b_{x,\eta} = f(a_{s_0,\eta \restriction p_0 \circ \varphi_{p_0}}, \dots, a_{s_{|\bar{v}|-1},\eta \restriction p_{|\bar{v}|-1} \circ \varphi_{p_{|\bar{v}|-1}}}).$$

Note that for any $j < |\bar{v}|$, $(\eta \restriction p_j) \circ \varphi_{p_j} \in (I^{\underline{\alpha_{s_j}}})_<$.

Let $b = \langle b_{x,\eta} : x \in \widehat{U},\ \eta \in (I^{\underline{\widehat{\alpha}_x}})_< \rangle$. We will show that b is $(\widehat{\alpha}, I)$-indiscernible.

Consider $\langle x_j \in \widehat{U} : j < k \rangle$, $\langle \eta_j \in (I^{\underline{\widehat{\alpha}_{x_j}}})_< : j < k \rangle$ and let π be a partial isomorphism of $(I, <)$ whose domain contains $\bigcup_{j<k} \mathrm{Range}(\eta_j)$. For $j < k$, we write

$$b_{x_j,\eta_j} = f_j(a_{s_{j,0},\eta_j \restriction p_{j,0} \circ \varphi_{p_{j,0}}}, \dots, a_{s_{j,|\bar{v}_j|-1},\eta_j \restriction p_{j,|\bar{v}_j|-1} \circ \varphi_{p_{j,|\bar{v}_j|-1}}}).$$

Since a is (α, U)-indiscernible, the type of

$$\langle a_{s_{j,0},\eta_j \restriction p_{j,0} \circ \varphi_{p_{j,0}}}, \dots, a_{s_{j,|\bar{v}|-1},\eta_j \restriction p_{j,|\bar{v}_j|-1} \circ \varphi_{p_{j,|\bar{v}_j|-1}}} : j < k \rangle$$

is equal to the type of

$$\langle a_{s_{j,0},\pi(\eta_j \restriction p_{j,0} \circ \varphi_{p_{j,0}})}, \dots, a_{s_{j,\bar{v}|-1},\pi(\eta_j \restriction p_{j,|\bar{v}_j|-1} \circ \varphi_{p_{j,|\bar{v}_j|-1}})} : j < k \rangle,$$

and consequently the type of $\langle b_{x_j,\eta_j} : j < k \rangle$ is equal to the type of $\langle b_{x_j,\pi(\eta_j)} : j < k \rangle$.

Finally, let $c \in \mathrm{dcl}(a)$, that is, there exists a definable function $f(\bar{v}) \in \mathcal{F}$ and $a_{i_0,\eta_0}, \dots a_{i_{|\bar{v}|-1},\eta_{|\bar{v}|-1}} \in a$ such that $c = f(a_{i_0,\eta_0}, \dots a_{i_{|\bar{v}|-1},\eta_{|\bar{v}|-1}})$. Let $r = \bigcup_{i<|\bar{v}|} \mathrm{Range}(\eta_i)$ and $\psi : r \to \mathrm{otp}(r)$ be the unique order isomorphism. For any $j < |\bar{v}|$ set $p_j = \psi(\mathrm{Range}(\eta_j))$. For $x = (f(\bar{v}), i_0, p_0, \dots, i_{|\bar{v}|-1}, p_{|\bar{v}|-1})$ and $\eta = \psi^{-1}$, $c = b_{x,\eta}$ (because e.g. $\eta \restriction p_0 \circ \varphi_{p_0} = \psi^{-1} \restriction \psi(\mathrm{Range}(\eta_0)) \circ \varphi_{\psi(\mathrm{Range}(\eta_0))} = \psi^{-1} \restriction \psi(\mathrm{Range}(\eta_0)) \circ \psi \circ \eta_0 = \eta_0$). ■

Theorem 3.7. *The following are equivalent for a complete $\mathcal{L}$-theory T:*

(1) *T is stable.*

(2) *Let κ, μ and λ be cardinals satisfying $\kappa = \mathrm{cf}(\kappa) \geq \min\{\kappa(T), |T|^+\} + \aleph_1$, $\mu^{<\kappa} = \mu \geq 2^{\kappa+|T|}$ and $\lambda = \lambda^{<\kappa} \geq \mu$, and let $T \subseteq T^{\mathrm{sk}}$ be an expansion with definable Skolem functions such that $|T| = |T^{\mathrm{sk}}|$ in a language $\mathcal{L} \subseteq \mathcal{L}^{\mathrm{sk}}$. Then there exists an infinitary EM-model $M^{\mathrm{sk}} \models T^{\mathrm{sk}}$ based on (α, λ), where $\alpha \in \kappa^U$ for some set U of cardinality at most μ, such that $M = M^{\mathrm{sk}} \restriction \mathcal{L}$ is saturated of cardinality λ.*

(3) *There exists a saturated model of singular cardinality.*

Remark 3.8. For example, assuming T is stable, the assumptions in (2) hold for $\lambda = \mu = 2^{\kappa+|T|}$ for any $\kappa = \mathrm{cf}(\kappa) \geq \kappa(T) + \aleph_1$.

Proof. (1)⇒(2). Since T is stable, $\kappa(T) \leq |T|^+$ by [14, Corollary III.3.3] and thus $\min\{\kappa(T), |T|^+\} = \kappa(T)$.

In the following, the superscript $^{\mathrm{sk}}$ means that we work in T^{sk}.

We apply Proposition 3.3 (1, 3) with $\mathbb{U}$ there being a monster model for T^{sk} and $(I, <) = (\lambda, <)$. Consequently, there exists an (α', λ)-indiscernible sequence a, where α' is as in the proposition. Let $M^{\mathrm{sk}} = \mathrm{dcl}^{\mathrm{sk}}(a)$ and $M = M^{\mathrm{sk}} \restriction \mathcal{L}$. Note that $|M^{\mathrm{sk}}| = |M| = \mu \cdot \lambda^{<\kappa} = \lambda$.

Towards applying Lemma 3.4, note that M is indeed $(\kappa(T) + \aleph_1)$-saturated by Proposition 3.3 (3.b) and the assumption on κ. Let $I \subseteq M$ be an infinite countable indiscernible sequence over some $B \subseteq M$ with $|B| < \kappa(T) \leq \kappa$.

Since $\lambda < \lambda^{\mathrm{cf}(\lambda)}$, necessarily $\mathrm{cf}(\lambda) \geq \kappa$, so by Proposition 3.3 (3.c) there is a non-principal ultrafilter $\mathcal{D}$ on I and elements $\langle a_i \in \mathrm{dcl}(\mathrm{Range}(a)) : i < \lambda \rangle$ satisfying

$$a_i \models p^{\mathrm{sk}}_{\mathcal{D}} | BI \langle a_k : k < i \rangle$$

for any $i < \lambda$. Let $p_{\mathcal{D}}$ be the restriction of $p^{\mathrm{sk}}_{\mathcal{D}}$ to $\mathcal{L}$. Thus $p_{\mathcal{D}} = \lim(I)$ and for every $i < \lambda$,

$$a_i \models \lim(I) | BI \langle a_k : k < i \rangle.$$

By stability, $I + \langle a_i : i < \lambda \rangle$ is indiscernible over B (see also [17, Exercise 2.25] and [14, Lemma III.1.7 (2)]). By Lemma 3.4, M is saturated.

(2)⇒(3). Let $\kappa = |T|^+$ and let $\lambda = \mu = \beth_\kappa(\kappa)$. Then $\lambda^{<\kappa} = \lambda$ because κ is regular. Indeed, any function from some $\xi < \kappa$ to λ is a function to $\beth_\alpha(\kappa)$ for some $\alpha < \kappa$. So $\lambda^\xi = \sup_{\alpha<\kappa}(\beth_\alpha(\kappa)^\xi)$. But $\sup_{\alpha<\kappa}(\beth_\alpha(\kappa)^\xi) = \sup_{\alpha<\kappa}(\beth_{\alpha+1}(\kappa)^\xi)$, and $\beth_{\alpha+1}(\kappa)^\xi = (2^{\beth_\alpha(\kappa)})^\xi = 2^{\beth_\alpha(\kappa)\cdot\xi} = 2^{\beth_\alpha(\kappa)} = \beth_{\alpha+1}(\kappa)$ because $\kappa > \xi$. Consequently, $\lambda^\xi = \lambda$ and $\lambda^{<\kappa} = \lambda$.

Hence, by (2), there is a saturated model of size λ (note that λ is singular of cofinality $\kappa < \lambda$).

(3)⇒(1). By (3), there is a saturated model of size λ with λ singular. Hence, $\lambda^{<\lambda} > \lambda$. As a result, by [14, Theorem VIII.4.7], T is λ-stable (and hence stable). ■

4. Order-type graphs with large chromatic number

In this section we discuss graphs whose vertices are (possibly infinite) increasing sequences, where the edge relation is determined by the order type. More specifically, our main interest in this section is the following type of graphs.

Definition 4.1. Let $(I, <)$ and $(J, <)$ be linearly ordered sets and $\bar{a} \neq \bar{b} \in (I^{\underline{J}})_<$ be increasing sequences. We define a graph $E^J_{\bar{a},\bar{b}}$ and a directed graph $D^J_{\bar{a},\bar{b}}$ on $(I^{\underline{J}})_<$ by:

- $\bar{c}\ E^J_{\bar{a},\bar{b}}\ \bar{d} \Leftrightarrow \mathrm{otp}(\bar{c}, \bar{d}) = \mathrm{otp}(\bar{a}, \bar{b}) \vee \mathrm{otp}(\bar{d}, \bar{c}) = \mathrm{otp}(\bar{a}, \bar{b})$.
- $\bar{c}\ D^J_{\bar{a},\bar{b}}\ \bar{d} \Leftrightarrow \mathrm{otp}(\bar{c}, \bar{d}) = \mathrm{otp}(\bar{a}, \bar{b})$.

We omit J from $E^J_{\bar{a},\bar{b}}$ and $D^J_{\bar{a},\bar{b}}$ when it is clear from the context.

We call these graphs the (*directed*) *order-type graphs*.

Remark 4.2. Although it will not define a graph, we sometimes use the notation $D_{\bar{a},\bar{b}}$ and $E_{\bar{a},\bar{b}}$ even if $\bar{a} = \bar{b}$.

In Section 4.1 we isolate a family of order-type graphs whose members contain all finite subgraphs of $\mathrm{Sh}_m(\omega)$ for a certain integer m (Corollary 4.7). In Section 4.2 we show that order-type graphs with large chromatic number fall into this family (Theorem 4.8).

4.1. Embedding shift graphs into order-type graphs

Definition 4.3. Let $(I,<)$ and $(J,<)$ be linearly ordered sets, $\bar{a},\bar{b}\in(I^{\underline{J}})_<$ be increasing sequences and $0<k<\omega$.

We say that $\langle\bar{a},\bar{b}\rangle$ is *k-orderly* if there exists a finite partition $\mathrm{Conv}(\mathrm{Im}(\bar{a})\cup\mathrm{Im}(\bar{b}))=C_0\cup\cdots\cup C_k$ into convex increasing subsets such that for every $n<k$ and $i\in J$ we have $a_i\in C_n\Leftrightarrow b_i\in C_{n+1}$.

Recall the following from [7].

Definition 4.4. For any linearly ordered set $(A,<)$ and $k\geq 1$, let $\mathrm{LSh}_k(A)$ be the directed graph $((A^{\underline{k}})_<,D)$, were $(\eta,\rho)\in D$ if and only if $\eta(i)=\rho(i-1)$ for $0<i<k$ (if $k>1$) and $\eta(0)<\rho(0)$ (if $k=1$).

Lemma 4.5. *Let $0<k<\omega$ be an integer, α,δ be ordinals, and $(I,<)$ any infinite linearly ordered set satisfying $(\delta\times(2\cdot\alpha+1)^k,<^{\mathrm{lex}})\subseteq(I,<)$. Let $\bar{a},\bar{b}\in(I^{\underline{\alpha}})_<$. If $\langle\bar{a},\bar{b}\rangle$ is k-orderly then there exists a function $\varphi:\mathrm{LSh}_k(\delta)\to(I^{\underline{\alpha}})_<$ such that for any $\eta,\rho\in\mathrm{LSh}_k(\delta)$, if $(\eta,\rho)\in D$ then $\varphi(\eta)\ D_{\bar{a},\bar{b}}\ \varphi(\rho)$.*

Proof. Assume that $\mathrm{Conv}(\mathrm{Im}(\bar{a}),\mathrm{Im}(\bar{b}))=C_0\cup\cdots\cup C_k$, as in the definition.

Let $\alpha^*=\alpha\cup\{\beta^-:\beta<\alpha\}\cup\{\infty\}$, where the β^-'s are immediate predecessors and ∞ is a maximal element, i.e. for any $\beta<\gamma<\alpha$,

- $\gamma<\beta^-<\beta$,
- $\gamma^-<\beta^-$ if and only $\gamma<\beta$,
- $\gamma<\infty$.

For any $S\subseteq\alpha$, let S^* be $S\cup\{s^-:s\in S\}\cup\{0^-,\infty\}$.

For any $x\in(\alpha^*)^n$, we denote by x^- the immediate predecessor of x in the lexicographic order if it exists, and otherwise let $x^-=x$. Note that for any $x=(x_0,\ldots,x_{n-1})\neq(0^-,\ldots,0^-)\in(\alpha^*)^n$, if the maximal $l<n$ with $x_l\neq 0^-$ satisfies $x_l<\alpha$ then x has an immediate predecessor.

Note that the order type of α^* is $2\cdot\alpha+1$, so by the assumption on I we may replace I by an isomorphic copy to get that $(\delta\times(\alpha^*)^k,<^{\mathrm{lex}})\subseteq(I,<)$.

For any $0\leq i\leq k-1$ let $S_i=\{\beta<\alpha:a_\beta\in C_i\}$ and let $\mathcal{G}=\{\bar{g}=\langle g_i:S_i\cup\{\infty\}\to(S^*_{k-1}\times\cdots\times S^*_0,<^{\mathrm{lex}}):i<k\rangle:\ g_i\text{ increasing}\}$.

For any $\bar{g}\in\mathcal{G}$ and $\eta\in(\delta^{\underline{k}})_<$ let $f_{\eta,\bar{g}}\in(I^{\underline{\alpha}})_<$ be defined by

$$f_{\eta,\bar{g}}(\beta)=(\eta(n_\beta),g_{n_\beta}(\beta))\in\delta\times(S^*_{k-1}\times\cdots\times S^*_0)\subseteq I,$$

where $\beta\in S_{n_\beta}$. We note that $f_{\eta,\bar{g}}$ is increasing: if $n_{\beta_1}<n_{\beta_2}$ then $\eta(n_{\beta_1})<\eta(n_{\beta_2})$. If $n_{\beta_1}=n_{\beta_2}$ then the result follows since $g_{n_{\beta_1}}=g_{n_{\beta_2}}$ is increasing.

Claim 4.5.1. *There exists $\bar{g} \in \mathcal{G}$ such that for any $\eta, \rho \in \mathrm{Sh}_k(\delta)$ satisfying $\eta(i) = \rho(i-1)$ for $0 < i < k$ (if $k > 1$) or $\eta(0) < \rho(0)$ (if $k = 1$), $f_{\eta,\bar{g}}\ D_{\bar{a},\bar{b}}\ f_{\rho,\bar{g}}$.*

Proof. For the purpose of this proof, for $1 \le i \le k$ let $\pi_i : S^*_{k-1} \times \cdots \times S^*_0 \to S^*_{k-1} \times \cdots \times S^*_{k-i}$ be the projection on the first i coordinates. We choose increasing functions $g_i : S_i \cup \{\infty\} \to S^*_{k-1} \times \cdots \times S^*_{k-i} \times \{0^-\} \times \cdots \times \{0^-\}$ by downward induction on $i < k$. Define g_{k-1} by setting $g_{k-1}(\beta) = (\beta, 0^-, \ldots, 0^-)$ for $\beta \in S_{k-1} \cup \{\infty\}$.

Assume that g_i has been defined and we want to define g_{i-1}.

For any $\beta \in S_{i-1}$ if there is $\gamma \in S_i$ minimal such that $b_\beta \le a_\gamma$ then define

$$g_{i-1}(\beta) = \begin{cases} g_i(\gamma) = (\pi_{k-i}(g_i(\gamma)), 0^-, \ldots, 0^-) & \text{if } a_\gamma = b_\beta, \\ (\pi_i(g_i(\gamma))^-, \beta, 0^-, \ldots, 0^-) & \text{otherwise.} \end{cases}$$

If such a minimal $\gamma \in S_i$ does not exist then we define

$$g_{i-1}(\beta) = (\pi_i(g_i(\infty)), \beta, 0^-, \ldots, 0^-).$$

Lastly,

$$g_{i-1}(\infty) = (\pi_i(g_i(\infty)), \infty, 0^-, \ldots, 0^-).$$

Subclaim. *For any $0 \le i \le k-1$, and for every $\beta \in S_i$, $\pi_i(g_i(\beta))$ has an immediate predecessor, i.e., for every $\gamma < \beta \in S_i$, $\pi_i(g_i(\beta)) > \pi_i(g_i(\beta))^-$.*

For any $0 \le i \le k-1$, g_i is increasing.

Proof. It follows by downward induction that for any $\beta \in S_i$, if $g_i(\beta) = (x_0, \ldots, x_{k-1})$ then the maximal $l < k$ such that $x_l \ne 0^-$ satisfies $x_l < \alpha$.

The fact that the g_is are increasing now follows by downward induction. $\blacksquare_{\text{subclaim}}$

The main observation is that for any $1 \le i < k$,

(†) $\mathrm{otp}(\langle a_\beta : \beta \in S_i \rangle, \langle b_\beta : \beta \in S_{i-1} \rangle) = \mathrm{otp}(\langle g_i(\beta) : \beta \in S_i \rangle, \langle g_{i-1}(\beta) : \beta \in S_{i-1} \rangle)$.

To see this, let $1 \le i < k$. Since g_i and g_{i-1} are increasing, it is enough to compare a_{β_1} and b_{β_2}, where $\beta_1 \in S_i$ and $\beta_2 \in S_{i-1}$.

- If $a_{\beta_1} = b_{\beta_2}$ then $\beta_1 \in S_i$ is minimal such that $b_{\beta_2} \le a_{\beta_1}$ and thus by definition $g_{i-1}(\beta_2) = g_i(\beta_1)$.
- Assume $a_{\beta_1} < b_{\beta_2}$. If there does not exist a minimal $\gamma \in S_i$ with $b_{\beta_2} \le a_\gamma$ then

$$\begin{aligned} g_{i-1}(\beta_2) &= (\pi_i(g_i(\infty)), \beta_2, 0^-, \ldots, 0^-) \\ &> (\pi_i(g_i(\beta_1)), 0^-, \ldots, 0^-) = g_i(\beta_1). \end{aligned}$$

  Otherwise, let $\beta_1 < \gamma \in S_i$ be minimal such that $b_{\beta_2} \le a_\gamma$. If $b_{\beta_2} = a_\gamma$ then $g_{i-1}(\beta_2) = g_i(\gamma) > g_i(\beta_1)$. If $b_{\beta_2} < a_\gamma$ then

$$\begin{aligned} g_{i-1}(\beta_2) &= (\pi_i(g_i(\gamma))^-, \beta_2, 0^-, \ldots, 0^-) \\ &> (\pi_i(g_i(\beta_1)), 0^-, 0^-, \ldots, 0^-) = g_i(\beta_1). \end{aligned}$$

- Assume $a_{\beta_1} > b_{\beta_2}$ and let $\gamma \in S_i$ be minimal such that $a_\gamma \geq b_{\beta_2}$, so $\gamma \leq \beta_1$. If $a_\gamma = b_{\beta_2}$ then $\gamma < \beta_1$ and $g_{i-1}(\beta_2) = g_i(\gamma) < g_i(\beta_1)$. If $a_\gamma > b_{\beta_2}$ then

$$\begin{aligned} g_{i-1}(\beta_2) &= (\pi_i(g_i(\gamma))^-, \beta_2, 0^-, \dots, 0^-) \\ &< (\pi_i(g_i(\beta_1)), 0^-, \dots, 0^-) = g_i(\beta_1). \end{aligned}$$

This proves (†). Let $\eta, \rho \in \mathrm{LSh}_k(\delta)$ be as in the statement of the lemma. We proceed to prove that $f_{\eta,\bar{g}}\ D_{\bar{a},\bar{b}}\ f_{\rho,\bar{g}}$.

Let $\beta_1, \beta_2 < \alpha$ and assume that $\beta_1 \in S_{n_1}$ and $\beta_2 \in S_{n_2}$ for some $0 \leq n_1, n_2 \leq k-1$. Note that if $b_{\beta_2} \in C_n$ for some $0 < n \leq k$, then $n_2 = n-1$. Assume that $k > 1$.

- Assume that $0 < n_1 < k$, $b_{\beta_2} \in C_{n_1}$. So $n_2 = n_1 - 1$. Assume that $a_{\beta_1} \square b_{\beta_2}$, where $\square \in \{<, >, =\}$. By (†), $g_{n_1}(\beta_1) \square g_{n_2}(\beta_2)$ and as a result

$$\begin{aligned} f_{\eta,\bar{g}}(\beta_1) &= (\eta(n_1), g_{n_1}(\beta_1)) = (\rho(n_1 - 1), g_{n_1}(\beta_1)) \square (\rho(n_1 - 1), g_{n_2}(\beta_2)) \\ &= (\rho(n_2), g_{n_2}(\beta_2)) = f_{\rho,\bar{g}}(\beta_2). \end{aligned}$$

- If $b_{\beta_2} \in C_n$ for some $n_1 < n < k$ then necessarily $a_{\beta_1} < b_{\beta_2}$ and $n_2 = n - 1 \geq n_1$. Consequently,

$$\begin{aligned} f_{\eta,\bar{g}}(\beta_1) &= (\eta(n_1), g_{n_1}(\beta_1)) < (\eta(n_1 + 1), g_{n_2}(\beta_2)) = (\rho(n_1), g_{n_2}(\beta_2)) \\ &\leq (\rho(n_2), g_{n_2}(\beta_2)) = f_{\rho,\bar{g}}(\beta_2). \end{aligned}$$

- If $b_{\beta_2} \in C_n$ for some $n < n_1$ then necessarily $0 < n < n_1$, $a_{\beta_1} > b_{\beta_2}$ and $n_2 = n - 1 < n_1 - 1$. Hence

$$\begin{aligned} f_{\eta,\bar{g}}(\beta_1) &= (\eta(n_1), g_{n_1}(\beta_1)) = (\rho(n_1 - 1), g_{n_1}(\beta_1)) \\ &> (\rho(n_2), g_{n_2}(\beta_2)) = f_{\rho,\bar{g}}(\beta_2). \end{aligned}$$

- If $b_{\beta_2} \in C_k$ then necessarily $n_2 = k - 1$ and $a_{\beta_1} < b_{\beta_2}$. As a result

$$\begin{aligned} f_{\eta,\bar{g}}(\beta_1) &= (\eta(n_1), g_{n_1}(\beta_1)) \leq (\eta(k-1), g_{n_1}(\beta_1)) = (\rho(k-2), g_{n_1}(\beta_1)) \\ &< (\rho(k-1), g_{n_2}(\beta_2)) = (\rho(n_2), g_{n_2}(\beta_2)) = f_{\rho,\bar{g}}(\beta_2). \end{aligned}$$

If $k = 1$ then $n_1 = n_2 = 0$ and

$$f_{\eta,\bar{g}}(\beta_1) = (\eta(0), g_{n_1}(\beta_1)) < (\rho(0), g_{n_2}(\beta_2)) = f_{\rho,\bar{g}}(\beta_2). \qquad \blacksquare_{\text{claim}}$$

We may now define a map $\varphi : \mathrm{LSh}_k(\delta) \to (I^{\underline{\alpha}})_<$ by letting for $\eta \in \mathrm{LSh}_k(\delta)$, $\varphi(\eta) = f_{\eta,\bar{g}} \in (I^{\underline{\alpha}})_<$. This maps satisfies the requirements by the previous claim. $\blacksquare$

Definition 4.6. Let $(I, <)$ and $(J, <)$ be linearly ordered sets and $\bar{a}, \bar{b} \in (I^{\underline{J}})_<$ be increasing sequences. We say that $\{\bar{a}, \bar{b}\}$ is *k-orderly covered* if there exists an increasing ordered partition $\langle J_\varepsilon : \varepsilon \in S \rangle$ of J into convex sets for some $S \subseteq J$ such that for every $\varepsilon \in S$, exactly one of the following holds:

(1) $\langle \bar{a} \restriction J_\varepsilon, \bar{b} \restriction J_\varepsilon \rangle$ is k_ε-orderly for some $0 < k_\varepsilon \leq k$;

(2) $\langle \bar{b}\restriction J_\varepsilon, \bar{a}\restriction J_\varepsilon\rangle$ is k_ε-orderly for some $0 < k_\varepsilon \le k$;

(3) $|J_\varepsilon| = 1$ and $\bar{a}\restriction J_\varepsilon = \bar{b}\restriction J_\varepsilon$.

Moreover, for every $\varepsilon < \varepsilon' \in S$, $\mathrm{Im}(\bar{a}\restriction J_\varepsilon) < \mathrm{Im}(\bar{b}\restriction J_{\varepsilon'})$ and $\mathrm{Im}(\bar{b}\restriction J_\varepsilon) < \mathrm{Im}(\bar{a}\restriction J_{\varepsilon'})$.

Corollary 4.7. *Let α be an ordinal, and $(I,<)$ any infinite linearly ordered set with $(|\alpha|^+ + \aleph_0, <) \subseteq (I,<)$. Let $\bar{a} \neq \bar{b} \in (I^{\underline{\alpha}})_<$ be some fixed sequences. If $\{\bar{a}, \bar{b}\}$ is k-orderly covered then $((I^{\underline{\alpha}})_<, E_{\bar{a},\bar{b}})$ contains all finite subgraphs of $\mathrm{Sh}_m(\omega)$ for some $m \le k$.*

Proof. Let $\langle J_\varepsilon : \varepsilon \in S\rangle$ be an increasing partition of α as in Definition 4.6, where $S \subseteq \alpha$. Since $\bar{a} \neq \bar{b}$, there exists $\varepsilon \in S$ such that $|J_\varepsilon| > 1$.

For any $\varepsilon \in S$ with $|J_\varepsilon| > 1$, we say that J_ε is

- *of type A* if $\langle \bar{a}\restriction J_\varepsilon, \bar{b}\restriction J_\varepsilon\rangle$ is k_ε-orderly,
- *of type B* if $\langle \bar{b}\restriction J_\varepsilon, \bar{a}\restriction J_\varepsilon\rangle$ is k_ε-orderly.

Let $N < \omega$ be some natural number. By replacing I with an isomorphic copy, we may assume that $(\alpha \times (N \times (2\alpha+1)^k), <^{\mathrm{lex}}) \subseteq (I,<)$. Let $\varepsilon \in S$ and let $I_\varepsilon = \{\varepsilon\} \times (N \times (2\alpha+1)^k)$.

If $|J_\varepsilon| = 1$ then we let $\varphi_\varepsilon : \mathrm{LSh}_1(N) \to ((I_\varepsilon)^{\underline{J_\varepsilon}})_<$ be such that $\varphi_\varepsilon(\eta)$ is the constant function giving $(\varepsilon, 0, \dots, 0)$.

For any $\varepsilon \in S$ let $E^\varepsilon_{\bar{a},\bar{b}} = E_{\bar{a}\restriction J_\varepsilon, \bar{b}\restriction J_\varepsilon}$ and $D^\varepsilon_{\bar{a},\bar{b}} = D_{\bar{a}\restriction J_\varepsilon, \bar{b}\restriction J_\varepsilon}$ and similarly $E^\varepsilon_{\bar{b},\bar{a}}$ and $D^\varepsilon_{\bar{b},\bar{a}}$.

If $|J_\varepsilon| > 1$ and J_ε is of type A then let $\varphi_\varepsilon : \mathrm{LSh}_{k_\varepsilon}(N) \to (((I_\varepsilon)^{\underline{J_\varepsilon}})_<, D^\varepsilon_{\bar{a},\bar{b}})$ be as supplied by Lemma 4.5, i.e., for any $\eta, \rho \in (N^{\underline{k_\varepsilon}})_<$, if $\eta(i) = \rho(i-1)$ for $0 < i < k_\varepsilon$ (if $k_\varepsilon > 1$) and $\eta(0) < \rho(0)$ (if $k_\varepsilon = 1$) then $\mathrm{otp}(\varphi_\varepsilon(\eta), \varphi_\varepsilon(\rho)) = \mathrm{otp}(\bar{a}\restriction J_\varepsilon, \bar{b}\restriction J_\varepsilon)$.

If $|J_\varepsilon| > 1$ and J_ε is of type B then let $\widehat{\varphi_\varepsilon} : \mathrm{LSh}_{k_\varepsilon}(N) \to (((I_\varepsilon)^{\underline{J_\varepsilon}})_<, D^\varepsilon_{\bar{b},\bar{a}})$ be as supplied by Lemma 4.5, i.e, for any $\eta, \rho \in (N^{\underline{k_\varepsilon}})_<$, if $\eta(i) = \rho(i-1)$ for $0 < i < k_\varepsilon$ (if $k_\varepsilon > 1$) and $\eta(0) < \rho(0)$ (if $k_\varepsilon = 1$) then $\mathrm{otp}(\widehat{\varphi_\varepsilon}(\eta), \widehat{\varphi_\varepsilon}(\rho)) = \mathrm{otp}(\bar{b}\restriction J_\varepsilon, \bar{a}\restriction J_\varepsilon)$.

By composing with the isomorphism $\mathrm{RSh}_{k_\varepsilon}(N) \to \mathrm{LSh}_{k_\varepsilon}(N)$ mapping the tuple $(x_0, \dots, x_{k_\varepsilon - 1})$ to $(N-1-x_{k_\varepsilon-1}, \dots, N-1-x_0)$, we arrive at a directed graph homomorphism $\varphi_\varepsilon : \mathrm{RSh}_{k_\varepsilon}(N) \to (((I_\varepsilon)^{\underline{J_\varepsilon}})_<, D^\varepsilon_{\bar{b},\bar{a}})$. By definition this map can be seen as a directed graph homomorphism $\varphi_\varepsilon : \mathrm{LSh}_{k_\varepsilon}(N) \to (((I_\varepsilon)^{\underline{J_\varepsilon}})_<, D^\varepsilon_{\bar{a},\bar{b}})$.

For $1 \le m \le k$ let $\pi_m : (N^{\underline{k}})_< \to (N^{\underline{m}})_<$ be the projection on the first m coordinates. Note that it is a directed graph homomorphism $\mathrm{LSh}_k(N) \to \mathrm{LSh}_m(N)$. We now define $\varphi : \mathrm{LSh}_k(N) \to ((I^{\underline{\alpha}})_<, D_{\bar{a},\bar{b}})$. For any $\eta \in (N^{\underline{k}})_<$, $\varepsilon \in S$ and $\beta < \alpha$, let $\varepsilon(\beta) \in S$ be such that $\beta \in J_\varepsilon$. We define

$$\varphi(\eta)(\beta) = (\varepsilon(\beta), \varphi_{\varepsilon(\beta)}(\pi_{k_\varepsilon}(\eta))(\beta)).$$

Since, for any $\varepsilon \in S$ and $\eta \in (N^{\underline{k}})_<$, $\varphi_\varepsilon(\pi_{k_\varepsilon}(\eta))$ is increasing, it is clear that $\varphi(\eta)$ is increasing as well.

Assume that $\eta, \rho \in \mathrm{LSh}_k(N)$ are connected, i.e., $\eta(i) = \rho(i-1)$ for $0 < i < k$ (if $k > 1$) and $\eta(0) < \rho(0)$ (if $k = 1$). It is routine to check that $\mathrm{otp}(\varphi(\eta), \varphi(\rho)) = \mathrm{otp}(\bar{a}, \bar{b})$. As a result, φ is also a graph homomorphism from $\mathrm{Sh}_k(N)$ to $((I^{\underline{\alpha}})_<, E_{\bar{a},\bar{b}})$.

We have proved that for every $N < \omega$ there exists a graph homomorphism φ_N : $\mathrm{Sh}_k(N) \to ((I^{\underline{\alpha}})_<, E_{\bar a,\bar b})$. By compactness, we may find a graph homomorphism $\mathrm{Sh}_k(\omega) \to \mathscr{H}$ for some elementary extension $((I^{\underline{\alpha}})_<, E_{\bar a,\bar b}) \prec (\mathscr{H}, E)$. By Fact 2.5, there exists $m \leq k$ such that $((I^{\underline{\alpha}})_<, E_{\bar a,\bar b})$ contains all finite subgraphs of $\mathrm{Sh}_m(\omega)$. ∎

4.2. Analyzing order-type graphs with large chromatic number

The main goal of this section is to prove that every order-type graph of large enough chromatic number is k-orderly covered for some k, i.e. we will prove the following.

Theorem 4.8. *Let α be an ordinal, and $(\theta, <)$ an infinite ordinal with $|\alpha|^+ + \aleph_0 < \theta$. Let $\bar a \neq \bar b \in (\theta^{\underline{\alpha}})_<$ be some fixed sequences. Let $G = ((\theta^{\underline{\alpha}})_<, E_{\bar a,\bar b})$. If $\chi(G) > \beth_2(\aleph_0)$ then G contains all finite subgraphs of* $\mathrm{Sh}_m(\omega)$ *for some $m \in \mathbb{N}$.*

In order to achieve this we will need to analyze the order-type of two infinite sequences. The tools developed here, we believe, may be useful in their own right.

We fix some ordinals α and θ with θ infinite and $\bar a \neq \bar b \in (\theta^{\underline{\alpha}})_<$ increasing sequences.

We partition $\alpha = J_0 \cup J_+ \cup J_-$, where

$$J_0 = \{\beta < \alpha : a_\beta = b_\beta\}, \quad J_+ = \{\beta < \alpha : a_\beta < b_\beta\}, \quad J_- = \{\beta < \alpha : b_\beta < a_\beta\}.$$

Let R be the minimal convex equivalence relation on α containing

$$\{(\beta,\gamma) : a_\beta = b_\gamma\}, \quad \{(\beta,\gamma) : a_\beta < a_\gamma \leq b_\beta\} \quad \text{and} \quad \{(\beta,\gamma) : b_\beta < b_\gamma \leq a_\beta\}.$$

Lemma 4.9. *Let $A, B \in \alpha/R$ and assume that $A < B$. Then $\mathrm{Im}(\bar a \restriction A) < \mathrm{Im}(\bar b \restriction B)$ and $\mathrm{Im}(\bar b \restriction A) < \mathrm{Im}(\bar a \restriction B)$.*

Proof. We will show that $\mathrm{Im}(\bar a \restriction A) < \mathrm{Im}(\bar b \restriction B)$; the other assertion follows similarly. Let $\beta \in A$ and $\gamma \in B$, so $\beta < \gamma$. If $a_\beta \geq b_\gamma$ then $b_\beta < b_\gamma \leq a_\beta$ and hence $\beta \mathrel{R} \gamma$, contradiction. ∎

Lemma 4.10. (1) *For any $\beta \in J_0$, $[\beta]_R \subseteq J_0$.*

(2) *For any $\beta \in J_+$, $[\beta]_R \subseteq J_+$.*

(3) *For any $\beta \in J_-$, $[\beta]_R \subseteq J_-$. Moreover, $[\beta]_R = \{\beta\}$ for $\beta \in J_0$.*

Proof. To prove (1)–(3) it is sufficient to prove a weaker version where we assume that $\beta = \min\,[\beta]_R$.

We show (2); items (1) and (3) are proved similarly. Assume that $[\beta]_R \nsubseteq J_+$. Let $X = \{\delta < \alpha : \delta \in [\beta]_R \wedge (\forall \beta \leq x \leq \delta)(a_x < b_x)\}$ (in (1) we replace $a_x < b_x$ by $a_x = b_x$ and in (3) by $a_x > b_x$). By the assumptions, X is a non-empty initial segment of $[\beta]_R$ and $Y = [\beta]_R \setminus X$ is non-empty convex.

We will show that both X and Y are closed under the relations defining R and thus derive a contradiction to the minimality of R.

Assume that $a_y = b_z$ with $y \in X$ and $z \in Y$. Since X is an initial segment, $y < z$. Consequently, $a_y < b_y < b_z$, contradiction. Now assume that $z \in X$ and $y \in Y$, so there exists $z < x \leq y$ with $a_x \geq b_x$ and as a result $a_z < b_z < b_x \leq a_x \leq a_y = b_z$, contradiction.

Assume that $a_y < a_z \leq b_y$ with $y \in X$ and $z \in Y$, so $y < z$. Hence there is some $y < x \leq z$ with $a_x \geq b_x$, hence $a_y < a_z \leq b_y < b_x \leq a_x \leq a_z$, contradiction. Now assume that $z \in X$ and $y \in Y$, so $z < y$. This implies that $a_z < a_y < a_z$, contradiction.

Assume that $b_y < b_z \leq a_y$ with $y \in X$ and $z \in Y$. Consequently, $a_y < b_y < b_z \leq a_y$, contradiction. Now assume that $z \in X$ and $y \in Y$, so $z < y$. As a result, $b_z < b_y < b_z$, contradiction.

Finally, we show the "moreover" part. Assume it is not true; then by (1) it is easy to see that both $\{\beta\}$ and $[\beta]_R \setminus \{\beta\}$ are closed under the relations generating R. This contradicts the minimality of R. ■

By Lemma 4.10, $R \restriction J_+$ is an equivalence relation on J_+. For any $A \in J_+/R$ we construct a set $Z_A \subseteq A$. We construct a sequence δ_n^A for $n < \omega$ as follows. Let $\delta_0^A = \min A$ and assume that δ_n^A has been chosen. Let $\delta_{n+1}^A \in A$ be the minimal index satisfying $b_{\delta_n^A} \leq a_{\delta_{n+1}^A}$ if such exists, otherwise stop. Let $Z_A = \langle \delta_n^A : n < n_A \rangle$, where $n_A \leq \omega$. Note that Z_A is a strictly increasing sequence because $A \in J_+/R$. Furthermore, set $C^A = \mathrm{Conv}(\mathrm{Im}(\bar{a} \restriction A) \cup \mathrm{Im}(\bar{b} \restriction A))$ and

- $C_0^A = [a_{\delta_0^A}, b_{\delta_0^A})$;
- if $n_A = \omega$ then for any $0 < n < \omega$ set $C_n^A = [b_{\delta_{n-1}^A}, b_{\delta_n^A})$;
- if $n_A < \omega$ then for any $0 < n < n_A$ set $C_n^A = [b_{\delta_{n-1}^A}, b_{\delta_n^A})$ and $C_{n_A}^A = \{x \in C^A : b_{\delta_{n_A-1}^A} \leq x\}$.

Lemma 4.11. *Let $A \in J_+/R$.*

(1) *If $n_A = \omega$ then $A = \bigcup_{n<\omega} [\delta_0^A, \delta_n^A]$.*

(2) *If $n_A = \omega$ then $C^A = \bigcup_{n<\omega} C_n^A$.*

(3) *For every $\beta \in A$ and $0 \leq n < n_A$, $a_\beta \in C_n^A \Leftrightarrow b_\beta \in C_{n+1}^A$.*

Proof. (1) Let $X = \bigcup_{n<\omega} [\delta_0^A, \delta_n^A]$. Since the δ_n^A's are chosen from A, and A is convex, $X \subseteq A$. As in the proof of Lemma 4.10, it is enough to show that both X and $Y = A \setminus X$ are closed under the relations defining R.

If $x, y \in A$ satisfy $a_x = b_y$ then since $b_y = a_x < b_x$ we conclude that $y < x$ and thus if $x \in X$ then $y \in X$. Now if we assume that $y \in X$, e.g. $y < \delta_n^A$, then $a_x = b_y < b_{\delta_n^A} \leq a_{\delta_{n+1}^A}$, so $x < \delta_{n+1}^A$.

Assume that $x, y \in A$ satisfy $a_x < a_y \leq b_x$. If $x \in X$, e.g. $x < \delta_n^A$, then $a_x < a_y \leq b_x < b_{\delta_n^A} \leq a_{\delta_{n+1}^A}$, so $y < \delta_{n+1}^A$. If $y \in X$ then since $x < y$ we conclude that $x \in X$ as well.

Assume that $x, y \in A$ satisfy $b_x < b_y \leq a_x$. If $y \in X$ then since $x < y$ it follows that $x \in X$ as well. Assume that $y \in Y$, i.e. $y \geq \delta_n^A$ for all n. But then $b_{\delta_n^A} \leq b_y \leq a_x < b_x$ for all n. This implies that $\delta_n^A < x$ for all n and hence $x \notin X$ as well.

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(2) The right-to-left inclusion is straightforward. For the other inclusion, let $x \in C^A$. Since $\delta_0^A = \min A$ and $A \in J_+/R$, it follows that $a_{\delta_0^A} = \min C^A$ and hence $a_{\delta_0^A} \leq x$. If there exists $n < \omega$ with $x < b_{\delta_n^A}$ then for the minimal such n, $x \in C_n^A$. Otherwise, since $a_\beta < b_\beta$ for any $\beta \in A$, we may assume that $x \leq b_\beta$ for some $\beta \in A$. Hence $x \leq b_{\delta_n^A}$ for some $n < \omega$ by (1).

(3) Let $\beta \in A$ and n be as in the statement. Assume that $n_A = \omega$ is infinite ($n_A < \omega$ is similar).

Let $a_\beta \in C_n^A$. First assume $n = 0$, i.e. $a_{\delta_0^A} \leq a_\beta < b_{\delta_0^A}$. It is always true that $b_{\delta_0^A} \leq b_\beta$. If $\beta \geq \delta_1^A$ then $a_{\delta_1^A} \leq a_\beta < b_{\delta_0^A}$, contradicting the choice of δ_1^A.

Now, if $n > 0$ then $b_{\delta_{n-1}^A} \leq a_\beta < b_{\delta_n^A}$ and thus by definition of δ_n^A, we have $\delta_n^A \leq \beta$ so $b_{\delta_n^A} \leq b_\beta$. If, on the other hand, $\beta \geq \delta_{n+1}^A$ then $a_{\delta_{n+1}^A} \leq a_\beta < b_{\delta_n^A}$, contradiction. Hence $b_{\delta_n^A} \leq b_\beta < b_{\delta_{n+1}^A}$.

Let $b_\beta \in C_{n+1}^A$. By (2), $a_\beta \in C_k^A$ for some $k < \omega$. Using the above we conclude that $b_\beta \in C_{k+1}^A$ and thus $k + 1 = n + 1$, i.e. $k = n$. ∎

Lemma 4.12. *For any $A \in J_+/R$ there exists an increasing sequence $\langle \zeta_n^A \in A : n < n_A \rangle$ such that for every n with $n + 1 < n_A$,*

$$a_{\zeta_{n+1}^A} \leq b_{\zeta_n^A},$$

and for every n with $n + 2 < n_A$,

$$b_{\zeta_n^A} < a_{\zeta_{n+2}^A}.$$

Proof. Let n be such that $n + 1 < n_A$. Assume for now that $b_{\delta_n} \neq a_{\delta_{n+1}}$ (and hence $b_{\delta_n} < a_{\delta_{n+1}}$) and assume towards a contradiction that

(∗) for any $\varepsilon \in (\delta_n, \delta_{n+1})$,

$$b_\varepsilon < a_{\delta_{n+1}}.$$

Note that this implies that for any such ε, $a_{\delta_n} < a_\varepsilon < b_{\delta_n} < b_\varepsilon < a_{\delta_{n+1}}$. Let $X = \{\beta \in A : \beta < \delta_{n+1}\}$ and $Y = A \setminus X$. This gives a convex partition of A, and we will show that both X and Y are closed under the relations defining R.

Let $\beta, \gamma \in A$ with $a_\beta = b_\gamma$. If $\beta < \delta_{n+1}$ and $\gamma \geq \delta_{n+1}$ then $b_{\delta_{n+1}} \leq b_\gamma = a_\beta < a_{\delta_{n+1}}$, contradiction. Now assume that $\gamma < \delta_{n+1}$ and $\beta \geq \delta_{n+1}$. If $\gamma \leq \delta_n$ then $a_{\delta_{n+1}} \leq a_\beta = b_\gamma \leq b_{\delta_n}$, contradiction. If $\gamma > \delta_n$ then $a_{\delta_{n+1}} \leq a_\beta = b_\gamma < a_{\delta_{n+1}}$ since $\gamma \in (\delta_n, \delta_{n+1})$ and by (∗), contradiction.

Let $\beta, \gamma \in A$ with $a_\beta < a_\gamma \leq b_\beta$. Assume that $\beta < \delta_{n+1}$ and $\gamma \geq \delta_{n+1}$. If $\beta \leq \delta_n$ then $a_{\delta_{n+1}} \leq a_\gamma \leq b_\beta \leq b_{\delta_n}$, contradiction. If $\beta \in (\delta_n, \delta_{n+1})$ then $b_\beta < a_{\delta_{n+1}} \leq a_\gamma \leq b_\beta$ by (∗), contradiction. Note that we cannot have $\gamma < \delta_{n+1}$ and $\beta \geq \delta_{n+1}$ since $\beta < \gamma$ by assumption.

Let $\beta, \gamma \in A$ with $b_\beta < b_\gamma \leq a_\beta$. If $\beta < \delta_{n+1}$ and $\gamma \geq \delta_{n+1}$ then $b_{\delta_{n+1}} \leq b_\gamma \leq a_\beta < a_{\delta_{n+1}} < b_{\delta_{n+1}}$, contradiction. As before, $\gamma < \delta_{n+1}$ and $\beta \geq \delta_{n+1}$ is not possible since $\beta < \gamma$.

As a result, we may conclude that for all n such that $n+1 < n_A$ we may find $\gamma_n \in (\delta_n, \delta_{n+1}]$ satisfying $a_{\delta_n} < a_{\gamma_n} \leq b_{\delta_n} \leq a_{\delta_{n+1}} \leq b_{\gamma_n}$ (if $b_{\delta_n} = a_{\delta_{n+1}}$ choose $\gamma_n = \delta_{n+1}$, otherwise use the above).

Let $I = \{\delta_n, \gamma_n : n+1 < n_A\}$. The crucial property is that for every $\gamma \in I \setminus \{\sup I\}$ there is some $\beta \in I$ satisfying $a_\gamma < a_\beta \leq b_\gamma$. We note that for every n such that $n+1 < n_A$, if $\gamma \leq \delta_n$ then $\beta \leq \delta_{n+1}$. Indeed, otherwise $a_{\delta_{n+1}} < a_\beta \leq b_\gamma \leq b_{\delta_n}$, contradiction.

We construct a sequence $\langle \zeta_n : n < k \rangle$ for some $k \leq \omega$ as follows. Define $\zeta_0 = \delta_0$ and for every n let $\zeta_{n+1} \in I$ be maximal[3] with $a_{\zeta_n} < a_{\zeta_{n+1}} \leq b_{\zeta_n}$, if it exists. Obviously, this is an increasing sequence. We claim that $k \geq n_A$. By induction on $n < k$ with $n < n_A$, $\zeta_n \leq \delta_n$. In particular, if $n+1 < n_A$ then ζ_{n+1} exists. Finally, we note that by maximality, for all $n+2 < n_A$, $a_{\zeta_{n+1}} \leq b_{\zeta_n} < a_{\zeta_{n+2}}$. ∎

For $A \in J_-/R$ we make dual (i.e. exchanging the roles of $\bar{a}$ and $\bar{b}$) constructions and similar properties hold.

Corollary 4.13. *Let α be an ordinal, and $(\theta, <)$ an infinite ordinal with $|\alpha|^+ + \aleph_0 < \theta$. Let $\bar{a} \neq \bar{b} \in (\theta^{\underline{\alpha}})_<$ be some fixed sequences. Let $G = ((\theta^{\underline{\alpha}})_<, E_{\bar{a},\bar{b}})$. If there exists $0 < k < \omega$ with $n_A \leq k$ for all $A \in (J_+ \cup J_-)/R$ then G contains all finite subgraphs of $\mathrm{Sh}_m(\omega)$ for some $m \leq k$.*

Proof. By Lemmas 4.9 and 4.11 (3), $\{\bar{a}, \bar{b}\}$ is k-orderly covered in the sense of Definition 4.6. Now apply Corollary 4.7. ∎

The aim of the rest of this section is to prove that $\{n_A : A \in (J_+ \cup J_-)/R\}$ has a finite bound. From now on we will only need the sequences defined in Lemma 4.12.

Lemma 4.14. *If $\chi(G) > 2^{\aleph_0}$ then for any $A \in (J_+ \cup J_-)/R$, $n_A < \omega$.*

Proof. We assume that $A \in J_+/R$; the proof for $A \in J_-/R$ is similar. Assume towards a contradiction that $n_A = \omega$. We will show that $\chi(G) \leq 2^{\aleph_0}$.

Let $S = \{\beta \leq \theta : \mathrm{cf}(\beta) = \aleph_0\}$. Let $\langle \zeta_l = \zeta_l^A \in A : l < \omega \rangle$ be the sequence supplied by Lemma 4.12.

For any $\gamma \in S$ choose an increasing sequence $\langle \alpha_{\gamma,n} : n < \omega \rangle \subseteq \gamma$ of ordinals with limit γ. We define a coloring map $c : (\theta^{\underline{\alpha}})_< \to 2^{\aleph_0 \times \aleph_0}$. For any $\bar{f} \in (\theta^{\underline{\alpha}})_<$ let $\gamma(\bar{f}) = \sup\{f_{\zeta_l} : l < \omega\} \in S$ and

$$c(\bar{f}) = \{(l,n) : l,n < \omega,\ f_{\zeta_l} < \alpha_{\gamma(\bar{f}),n}\}.$$

To show that it is a legal coloring, let $\bar{f}, \bar{g} \in (\theta^{\underline{\alpha}})_<$ be such that $\mathrm{otp}(\bar{f}, \bar{g}) = \mathrm{otp}(\bar{a}, \bar{b})$. By assumption $f_{\zeta_l} < f_{\zeta_{l+1}} \leq g_{\zeta_l} \leq f_{\zeta_{l+2}}$ for $l < \omega$, and hence $\gamma(\bar{f}) = \gamma(\bar{g})$. By definition, there is some $n < \omega$ such that $f_{\zeta_0} < \alpha_{\gamma(\bar{f}),n}$ and let k be the minimal such that $f_{\zeta_{k+1}} \geq \alpha_{\gamma(\bar{f}),n}$. So by minimality of k,

$$f_{\zeta_k} < \alpha_{\gamma(\bar{f}),n} \leq f_{\zeta_{k+1}} \leq g_{\zeta_k},$$

and hence $(k,n) \in c(\bar{f})$ but $(k,n) \notin c(\bar{g})$, so $c(\bar{f}) \neq c(\bar{g})$. ∎

[3] If n_A is finite then such a maximal element clearly exists. Otherwise, for $\zeta \in I$ there is some $n < \omega$ such that $b_\zeta < a_{\delta_n}$, and hence $\zeta \in \{\xi \in I : a_\xi \leq b_\zeta\}$ is finite.

The next lemma requires a more complicated argument: see Section 5 below. Let us introduce some notation.

Fix some sequence $\langle A_\varepsilon \in (J_+/R : \varepsilon < \omega\rangle$. For any $\varepsilon < \omega$, let $J_\varepsilon = \{\zeta_n^{A_\varepsilon} \in A_\varepsilon : n < n_{A_\varepsilon}\}$ be the sequence supplied by Lemma 4.12 applied to A_ε. Let $J = \bigcup_{\varepsilon<\omega} J_\varepsilon$, $\Omega = ({}^{J}\theta)_<$,

$$R' = \{(\bar{c}, \bar{d}) \in \Omega^2 : \text{otp}(\bar{c}, \bar{d}) = \text{otp}(\bar{a} \restriction J, \bar{b} \restriction J)\} \quad \text{and} \quad \chi = \beth_2(\aleph_0).$$

Note that R' is an irreflexive relation on Ω such that if $f_1, f_2 \in \Omega$ and $f_1 \; R' \; f_2$, then for every $\varepsilon < \omega$ and $i \in J_\varepsilon$, the following hold:

$$f_1(i) < f_2(i), \tag{1}$$

and for any $i \in J_\varepsilon$ with $\text{Suc}(i) \in J_\varepsilon$,

$$f_1(\text{Suc}(i)) \leq f_2(i), \tag{2}$$

and for any $i \in J_\varepsilon$ with $\text{Suc}(\text{Suc}(i)) \in J_\varepsilon$,

$$f_2(i) < f_1(\text{Suc}(\text{Suc}(i))), \tag{3}$$

where $\text{Suc}(i)$ is the successor of i in J_ε.

Under these assumptions (or more generally under Assumption 5.1), we will prove in Conclusion 5.9 that

(∗) If $n_{A_\varepsilon} < \omega$ for all $\varepsilon < \omega$, then there exists a function $c : \Omega \to \chi$ such that if $f_1, f_2 \in \Omega$ and $f_1 \; R' \; f_2$ then $c(f_1) \neq c(f_2)$. In other words, there exists a coloring of the directed graph (Ω, R') of cardinality χ.

Lemma 4.15. *If $\chi(G) > \beth_2(\aleph_0)$ then the set $\{n_A : A \in (J_+ \cup J_-)/R\}$ is bounded.*

Proof. By Lemma 4.14, for any $A \in (J_+ \cup J_-)/R$, $n_A < \omega$. We will show that $\{n_A : A \in J_+/R\}$ and $\{n_A : A \in J_-/R\}$ are both bounded.

Assume that $\{n_A : A \in J_+/R\}$ is unbounded. Let $\{A_\varepsilon \in J_+/R : \varepsilon < \omega\}$ be a family of convex equivalence classes such that $\varepsilon < n_{A_\varepsilon}$.

By (∗), there exists a function $c : \Omega \to \beth_2(\aleph_0)$ such that if $f_1, f_2 \in \Omega$ and $f_1 \; R' \; f_2$ then $c(f_1) \neq c(f_2)$. Let $H = (\Omega, (R')^{\text{sym}})$ be the graph induced by R' (i.e. $(\bar{c}, \bar{d}) \in (R')^{\text{sym}} \Leftrightarrow (\bar{c}, \bar{d}) \in R' \vee (\bar{d}, \bar{c}) \in R'$). The map c induces a coloring on H and hence $\chi(H) \leq \beth_2(\aleph_0)$. Since the map $({}^{\alpha}\theta)_< \to \Omega$ given by $\eta \mapsto \eta \restriction J$ is a graph homomorphism, $\chi(G) \leq \beth_2(\aleph_0)$, and this contradicts the assumption.

If on the other hand $\{n_A : A \in J_-/R\}$ is unbounded then we proceed as above but using

$$R'' = \{(\bar{c}, \bar{d}) \in \Omega^2 : \text{otp}(\bar{c}, \bar{d}) = \text{otp}(\bar{b} \restriction J, \bar{a} \restriction J)\}$$

and the dual construction (replacing R' by R'' in (∗)) mentioned above instead and arrive at a similar contradiction. ∎

Finally, we may conclude:

Proof of Theorem 4.8. This is a direct consequence of Corollary 4.13 and Lemma 4.15. ∎

5. Coloring increasing functions

This section's main result is Conclusion 5.9, used in the final stage of the previous section. We prove that under mild conditions on a directed graph, namely Assumption 5.1, on a family of strictly increasing functions there exists a coloring of small cardinality.

Let $\kappa = \mathrm{cf}(\kappa)$ be a regular cardinal and $(J, <)$ a well-order of cofinality κ. Let $\sigma = (2^\kappa)^+$, θ be an ordinal and $\chi = \chi^{<\sigma}$ a cardinal.

Let $\langle J_\varepsilon : \varepsilon < \kappa \rangle$ be an increasing partition of J into finite convex sets. Assume that $\sup_{\varepsilon<\kappa} |J_\varepsilon| = \omega$. Let $\mathcal{D}$ be a non-principal ultrafilter on κ containing the filter generated by $\{\{\varepsilon < \kappa : |J_\varepsilon| \geq n\} : n < \omega\}$.

Let Ω be the set of functions from J to θ that are strictly increasing on each J_ε $(\varepsilon < \kappa)$. Let $\mathcal{H} = (\theta + 1)^\kappa$.

Assumption 5.1. *R is an irreflexive relation on Ω such that if $f_1, f_2 \in \Omega$ and $f_1 \; R \; f_2$, then for every $\varepsilon < \kappa$ and $i \in J_\varepsilon$,*

$$f_1(i) < f_2(i), \tag{1}$$

and for any $i \in J_\varepsilon$ with $\mathrm{Suc}(i) \in J_\varepsilon$,

$$f_1(\mathrm{Suc}(i)) \leq f_2(i), \tag{2}$$

and for any $i \in J_\varepsilon$ with $\mathrm{Suc}(\mathrm{Suc}(i)) \in J_\varepsilon$,

$$f_2(i) < f_1(\mathrm{Suc}(\mathrm{Suc}(i))), \tag{3}$$

where $\mathrm{Suc}(i)$ is the successor of i in the finite set J_ε.

We say a subset X of Ω is *trivial* if $f_1 \not R \; f_2$ for any $f_1, f_2 \in X$.

Definition 5.2. An *approximation* $\mathbf{a}$ is a partition $\Omega = \bigcup_{s \in S_\mathbf{a}} \Omega^\mathbf{a}_s$ (so all the $\Omega^\mathbf{a}_s$'s are non-empty), $\rho^\mathbf{a}_s \in \chi^{<\sigma}$ and $h^\mathbf{a}_s \in \mathcal{H}$ $(s \in S_\mathbf{a})$ satisfying

(1) for every $s \in S_\mathbf{a}$ and $f \in \Omega^\mathbf{a}_s$, $\{\varepsilon < \kappa : \mathrm{Range}(f \restriction J_\varepsilon) \subseteq h^\mathbf{a}_s(\varepsilon)\} \in \mathcal{D}$;

(2) if $s \neq t \in S_\mathbf{a}$ and $\rho^\mathbf{a}_s = \rho^\mathbf{a}_t$ then for every $f_1 \in \Omega^\mathbf{a}_s$ and $f_2 \in \Omega^\mathbf{a}_t$, $f_1 \not R \; f_2$.

We want to define when one approximation is better than the other.

Definition 5.3. For two approximations $\mathbf{a}$ and $\mathbf{b}$ we will say that $\mathbf{a} \trianglelefteq_g \mathbf{b}$ if there exists a surjective function $g : S_\mathbf{b} \to S_\mathbf{a}$ satisfying

(1) for any $s \in S_\mathbf{a}$, $\{\Omega^\mathbf{b}_t : t \in g^{-1}(s)\}$ is a partition of $\Omega^\mathbf{a}_s$;

(2) if $s \in S_\mathbf{a}$ and $\Omega^\mathbf{a}_s$ is trivial then $g^{-1}(s)$ is a singleton $t \in S_\mathbf{b}$ satisfying $h^\mathbf{a}_s = h^\mathbf{b}_t$ and $\Omega^\mathbf{a}_s = \Omega^\mathbf{b}_t$ (in particular, $\Omega^\mathbf{a}_t$ is also trivial);

(3) for $t \in S_\mathbf{b}$, $\{\varepsilon < \kappa : h^\mathbf{b}_t(\varepsilon) \leq h^\mathbf{a}_{g(t)}(\varepsilon)\} \in \mathcal{D}$;

(4) for $t \in S_\mathbf{b}$, $\rho^\mathbf{a}_{g(t)}$ is an initial segment of $\rho^\mathbf{b}_t$.

We will say that $\mathbf{a} \triangleleft_g \mathbf{b}$ if $\mathbf{a} \trianglelefteq_g \mathbf{b}$ and in addition for every $t \in S_\mathbf{b}$, either $\Omega^\mathbf{b}_t$ is trivial or $\{\varepsilon < \kappa : h^\mathbf{b}_t(\varepsilon) < h^\mathbf{a}_{g(t)}(\varepsilon)\} \in \mathcal{D}$.

The following is clear.

Lemma 5.4. *Let* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *be approximations. If* $\mathbf{a} \trianglelefteq_g \mathbf{b}$ *and* $\mathbf{b} \trianglelefteq_h \mathbf{c}$ *then* $\mathbf{a} \trianglelefteq_{g \circ h} \mathbf{c}$*. If in addition either* $\mathbf{a} \triangleleft_g \mathbf{b}$ *or* $\mathbf{b} \triangleleft_h \mathbf{c}$ *then* $\mathbf{a} \triangleleft_{g \circ h} \mathbf{c}$.

Proposition 5.5. *Let* $\mathbf{a}$ *be an approximation. Then there exists an approximation* $\mathbf{a} \trianglelefteq_g \mathbf{b}$ *satisfying the following:*

(1) *If* $t \in S_{\mathbf{b}}$, $\Omega^{\mathbf{a}}_{g(t)}$ *is non-trivial and* $\{\varepsilon < \kappa : 0 < \mathrm{cf}(h^{\mathbf{a}}_{g(t)}(\varepsilon)) \leq \chi\} \in \mathcal{D}$ *then either* $\Omega^{\mathbf{b}}_t$ *is trivial or* $\{\varepsilon < \kappa : h^{\mathbf{b}}_t(\varepsilon) < h^{\mathbf{a}}_{g(t)}(\varepsilon)\} \in \mathcal{D}$.

(2) *If* $t \in S_{\mathbf{b}}$ *is such that* $\{\varepsilon < \kappa : 0 < \mathrm{cf}(h^{\mathbf{a}}_{g(t)}(\varepsilon)) \leq \chi\} \notin \mathcal{D}$ *then* $h^{\mathbf{b}}_t = h^{\mathbf{a}}_{g(t)}$ *and* $\Omega^{\mathbf{b}}_t = \Omega^{\mathbf{a}}_{g(t)}$.

Lastly, for $t \in S_{\mathbf{b}}$*, if* $\rho^{\mathbf{a}}_{g(t)} \in \chi^{\xi}$ *with* $\xi < \sigma$ *then* $\rho^{\mathbf{b}}_t \in \chi^{\xi+1}$.

Proof. We partition $S_{\mathbf{a}}$ into $S_1 = \{s \in S_{\mathbf{a}} : (\forall^{\mathcal{D}} \varepsilon < \kappa)(0 < \mathrm{cf}(h^{\mathbf{a}}_s(\varepsilon)) \leq \chi)$ and $\Omega^{\mathbf{a}}_s$ is non-trivial$\}$ and $S_0 = S_{\mathbf{a}} \setminus S_1$.

Fix any $s \in S_1$. For any $\varepsilon < \kappa$, if $0 < \mathrm{cf}(h^{\mathbf{a}}_s(\varepsilon)) \leq \chi$ we choose an unbounded subset $C_{s,\varepsilon} \subseteq h^{\mathbf{a}}_s(\varepsilon)$ of order type $\mathrm{cf}(h^{\mathbf{a}}_s(\varepsilon))$, and we set $C_{s,\varepsilon} = \{h^{\mathbf{a}}_s(\varepsilon)\}$ otherwise.

Set $A_s = \{\varepsilon < \kappa : 0 < \mathrm{cf}(h^{\mathbf{a}}_s(\varepsilon)) \leq \chi\}$. Note that $A_s \in \mathcal{D}$.

Let $H_s = \{h \in \mathcal{H} :$ if $\varepsilon \in A_s$ then $h(\varepsilon) \in C_{s,\varepsilon}$, and $h(\varepsilon) = h^{\mathbf{a}}_s(\varepsilon)$ otherwise$\}$. Since $\chi^\kappa = \chi$, $|H_s| \leq \chi$ and hence there is some $\xi_s \leq \chi$ and an enumeration $\langle h_{s,\xi} : \xi < \xi_s \rangle$ of H_s.

By induction on $\xi < \xi_s$ we define

$$\Omega_{s,\xi} = \{f \in \Omega^{\mathbf{a}}_s : (\forall^{\mathcal{D}} \varepsilon < \kappa)(\mathrm{Range}(f \restriction J_\varepsilon) \subseteq h_{s,\xi}(\varepsilon))\} \setminus \bigcup_{\alpha<\xi} \Omega_{s,\alpha},$$

and for $\xi = \xi_s$,

$$\Omega_{s,\xi} = \{f \in \Omega^{\mathbf{a}}_s : (\forall^{\mathcal{D}} \varepsilon < \kappa)(h^{\mathbf{a}}_s(\varepsilon) \text{ is a successor and } h^{\mathbf{a}}_s(\varepsilon) - 1 = \max \mathrm{Range}(f \restriction J_\varepsilon))\}.$$

We claim that $\Omega^{\mathbf{a}}_s = \bigsqcup_{\xi \leq \xi_s} \Omega_{s,\xi}$. Let $f \in \Omega^{\mathbf{a}}_s$. Note that for every $\varepsilon \in A_s$ either (a) there is an ordinal $\gamma \in C_{s,\varepsilon}$ such that $\mathrm{Range}(f \restriction J_\varepsilon) \subseteq \gamma$ or (b) there is no such γ. We may find $A'_s \subseteq A_s$ such that $A'_s \in \mathcal{D}$ and either (a) holds for all $\varepsilon \in A'_s$, or (b) holds for all $\varepsilon \in A'_s$.

Assume that (a) holds for all $\varepsilon \in A'_s$ and let $\langle \gamma_\varepsilon : \varepsilon \in A'_s \rangle$ witness this. Define a function $h \in H_s$ by setting $h(\varepsilon) = \gamma_\varepsilon$ for all $\varepsilon \in A'_s$. For $\varepsilon \notin A'_s$ choose arbitrary $h(\varepsilon)$ as long as $h \in H_s$. Let $\xi < \xi_s$ be minimal such that $(\forall^{\mathcal{D}} \varepsilon < \kappa)(\mathrm{Range}(f \restriction J_\varepsilon) \subseteq h_{s,\xi}(\varepsilon))$, so $f \in \Omega_{s,\xi}$.

Now, assume that (b) holds for all $\varepsilon \in A'_s$. As $\mathbf{a}$ is an approximation (see Definition 5.2 (1)) we may assume that for all $\varepsilon \in A'_s$, $\mathrm{Range}(f \restriction J_\varepsilon) \subseteq h^{\mathbf{a}}_s(\varepsilon)$ but we cannot find any $\gamma \in C_{s,\varepsilon}$ satisfying $\mathrm{Range}(f \restriction J_\varepsilon) \subseteq \gamma$. For any $\varepsilon \in A'_s$, because J_ε is finite this implies that $\mathrm{cf}(h^{\mathbf{a}}_s(\varepsilon)) = 1$, i.e. $h^{\mathbf{a}}_s(\varepsilon)$ is a successor ordinal and $h^{\mathbf{a}}_s(\varepsilon) - 1 = \max \mathrm{Range}(f \restriction J_\varepsilon)$. Hence $f \in \Omega_{s,\xi_s}$.

Let $S_{\mathbf{b}} = \{(s, \xi) : s \in S_1, \xi \leq \xi_s, \Omega_{s,\xi} \neq \emptyset\} \cup S_0$ and let $g : S_{\mathbf{b}} \to S_{\mathbf{a}}$ be the function defined by $g(s, \xi) = s$ for $s \in S_1$ and $g(s) = s$ otherwise. For any $s \in S_0$ let $\Omega^{\mathbf{b}}_s = \Omega^{\mathbf{a}}_s$,

$h^{\mathbf{b}}_s = h^{\mathbf{a}}_s$ and $\rho^{\mathbf{b}}_s = \rho^{\mathbf{a}}_s{}^\frown\langle 0\rangle$. For $s \in S_1$, if $\xi \leq \xi_s$ we set $\Omega^{\mathbf{b}}_{(s,\xi)} = \Omega_{s,\xi}$ and $\rho^{\mathbf{b}}_{(s,\xi)} = \rho^{\mathbf{a}}_s{}^\frown\langle\xi\rangle$. Finally, for $\xi < \xi_s$ we set $h^{\mathbf{b}}_{(s,\xi)} = h_{s,\xi}$ and for $\xi = \xi_s$ we set $h^{\mathbf{b}}_{(s,\xi)} = h^{\mathbf{a}}_s$.

Claim 5.5.1. $\mathbf{b}$ *is an approximation and* $\mathbf{a} \trianglelefteq_g \mathbf{b}$.

Proof. We first show that $\mathbf{b}$ is an approximation. Items (1) and (2) from the definition follow since $\mathbf{a}$ is an approximation and in view of the construction above. For example, if $(s_1, \xi_1) \neq (s_2, \xi_2) \in S_{\mathbf{b}}$ and $\rho^{\mathbf{b}}_{(s_1,\xi_1)} = \rho^{\mathbf{b}}_{(s_2,\xi_2)}$ then since $\xi_1 = \xi_2$ necessarily $s_1 \neq s_2$ and $\rho^{\mathbf{a}}_{s_1} = \rho^{\mathbf{a}}_{s_2}$, so we may use the fact that $\mathbf{a}$ is an approximation.

Finally, $\mathbf{a} \trianglelefteq_g \mathbf{b}$ by construction. $\blacksquare_{\text{claim}}$

Showing (1) from the statement of the proposition boils down to showing that $\Omega^{\mathbf{b}}_{(s,\xi_s)} = \Omega_{s,\xi_s}$ is trivial. This follows from Assumption 5.1 (1). $\blacksquare$

Proposition 5.6. *Let* $\mathbf{a}$ *be an approximation. Then there exists an approximation* $\mathbf{a} \trianglelefteq_g \mathbf{b}$ *satisfying the following:*

(1) *If* $t \in S_{\mathbf{b}}$ *with* $\{\varepsilon < \kappa : \mathrm{cf}(h^{\mathbf{a}}_{g(t)}(\varepsilon)) > \chi\} \in \mathcal{D}$ *then either* $\Omega^{\mathbf{b}}_t$ *is trivial or* $\{\varepsilon < \kappa : h^{\mathbf{b}}_t(\varepsilon) < h^{\mathbf{a}}_{g(t)}(\varepsilon)\} \in \mathcal{D}$.

(2) *If* $t \in S_{\mathbf{b}}$ *is such that* $\{\varepsilon < \kappa : \mathrm{cf}(h^{\mathbf{a}}_{g(t)}(\varepsilon)) > \chi\} \notin \mathcal{D}$ *then* $h^{\mathbf{b}}_t = h^{\mathbf{a}}_{g(t)}$ *and* $\Omega^{\mathbf{b}}_t = \Omega^{\mathbf{a}}_{g(t)}$.

Lastly, for $t \in S_{\mathbf{b}}$, *if* $\rho^{\mathbf{a}}_{g(t)} \in \chi^{\xi}$ *with* $\xi < \sigma$, *then* $\rho^{\mathbf{b}}_t \in \chi^{\xi+1}$.

Proof. Let $S_1 = \{s \in S_{\mathbf{a}} : \Omega^{\mathbf{a}}_s$ is non-trivial and $(\forall^{\mathcal{D}}\varepsilon < \kappa)(\mathrm{cf}(h^{\mathbf{a}}_s(\varepsilon)) > \chi\}$ and $S_0 = S_{\mathbf{a}} \setminus S_1$. Fix any $s \in S_1$. Let $A_s = \{\varepsilon < \kappa : \mathrm{cf}(h^{\mathbf{a}}_s(\varepsilon)) > \chi\}$, so $A_s \in \mathcal{D}$.

Let $\mathcal{D}_s = \{D \in \mathcal{D} : D \subseteq A_s\}$ be the induced ultrafilter on A_s. Consider the ultraproduct $\prod_{\varepsilon\in A_s} h^{\mathbf{a}}_s(\varepsilon)/\mathcal{D}_s$. We may consider it as a linearly ordered set, ordered by $<_{\mathcal{D}_s}$.

Claim 5.6.1. *There exists a sequence* $H_s = \langle h_{s,\beta} \in \mathcal{H} : \beta < \beta_s\rangle$ *satisfying*

(1) *for all* $\varepsilon \in A_s$ *and* $\beta < \beta_s$, $h_{s,\beta}(\varepsilon) < h^{\mathbf{a}}_s(\varepsilon)$;

(2) *for all* $\varepsilon \in \kappa \setminus A_s$ *and* $\beta < \beta_s$, $h_{s,\beta}(\varepsilon) = h^{\mathbf{a}}_s(\varepsilon)$;

(3) $\langle (h_{s,\beta} \restriction A_s)/\mathcal{D}_s : \beta < \beta_s\rangle$ *is* $<_{\mathcal{D}_s}$ *increasing and cofinal in* $\prod_{\varepsilon\in A_s} h^{\mathbf{a}}_s(\varepsilon)/\mathcal{D}_s$;

(4) *for any* $f \in \Omega^{\mathbf{a}}_s$ *there exists* $\beta < \beta_s$ *such that* $\{\varepsilon < \kappa : \mathrm{Range}(f \restriction J_\varepsilon) \subseteq h_{s,\beta}(\varepsilon)\} \in \mathcal{D}$.

Proof. First we choose a well-ordered increasing cofinal sequence in $\prod_{\varepsilon\in A_s} h^{\mathbf{a}}_s(\varepsilon)/\mathcal{D}_s$ and then choose a sequence of representatives $\langle h_{s,\beta} \restriction A_s : \beta < \beta_s\rangle$. To get (2), set $h_{s,\beta}(\varepsilon) = h^{\mathbf{a}}_s(\varepsilon)$ for any $\varepsilon \in \kappa \setminus A_s$. This gives us (1)–(3).

We show (4). Let $f \in \Omega^{\mathbf{a}}_s$. Since $\mathbf{a}$ is an approximation, the set $X_{s,f} = \{\varepsilon \in A_s : \mathrm{Range}(f \restriction J_\varepsilon) \subseteq h^{\mathbf{a}}_s(\varepsilon)\}$ is in $\mathcal{D}$. Let $h_f : A_s \to \mathrm{Ord}$ be the function defined by mapping $\varepsilon \in X_{s,f}$ to $\max \mathrm{Range}(f \restriction J_\varepsilon) + 1$ and $\varepsilon \in A_s \setminus X_{s,f}$ to 0. Note that for any $\varepsilon \in A_s$, $h_f(\varepsilon) < h^{\mathbf{a}}_s(\varepsilon)$. Indeed, if $h_f(\varepsilon) = h^{\mathbf{a}}_s(\varepsilon)$ for some $\varepsilon \in X_{s,f}$, then $h^{\mathbf{a}}_s(\varepsilon)$ is a successor, contradicting $\varepsilon \in A_s$. Similarly (and even easier), this holds if $\varepsilon \in A_s \setminus X_{s,f}$. It follows that for some $\beta < \beta_s$, $h_f/\mathcal{D}_s \leq_{\mathcal{D}_s} (h_{s,\beta} \restriction A_s)/\mathcal{D}_s$ and it is easy to check that such a β satisfies (4). $\blacksquare_{\text{claim}}$

For any $f \in \Omega^{\mathbf{a}}_s$, $n < \omega$ and $\beta < \beta_s$ let $B_n(f, h_{s,\beta}) = \{\varepsilon < \kappa : |\{i \in J_\varepsilon : f(i) \geq h_{s,\beta}(\varepsilon)\}| \leq n\}$. By Claim 5.6.1 (4), we may set $\beta_{s,n}(f) = \min\{\beta : B_n(f, h_{s,\beta}) \in \mathcal{D}\}$. Note that $\beta_{s,n}(f) \geq \beta_{s,n+1}(f)$. Let $\beta_s(f) = \min\{\beta_{s,n}(f) : n < \omega\}$ and let $n_s(f) = \min\{n < \omega : (\forall k \geq n)(\beta_{s,k}(f) = \beta_{s,n}(f))\}$.

Claim 5.6.2. $\{\varepsilon < \kappa : |\{i \in J_\varepsilon : f(i) \geq h_{s,\beta_s(f)}(\varepsilon)\}| = n_s(f)\} \in \mathcal{D}$.

Proof. Call this set $Y_{s,f}$. Note that $Y_{s,f} \subseteq B_{n_s(f)}(f, h_{s,\beta_s(f)})$.

If $n_s(f) = 0$ then $Y_{s,f} = B_0(f, h_{s,\beta_{s,0}(f)}) \in \mathcal{D}$. Assume $n_s(f) > 0$. If $Y_{s,f} \notin \mathcal{D}$ then $\{\varepsilon < \kappa : |\{i \in J_\varepsilon : f(i) \geq h_{s,\beta_s(f)}(\varepsilon)\}| \leq n_s(f) - 1\} \in \mathcal{D}$. So $\beta_{s,n_s(f)-1} \leq \beta_s(f) = \beta_{s,n_s(f)}(f)$, contradiction. $\blacksquare_{\text{claim}}$

For $s \in S_1$, $\beta < \beta_s$ and $n < \omega$, let $\Omega_{(s,\beta,n)} = \{f \in \Omega^{\mathbf{a}}_s : \beta_s(f) = \beta,\ n_s(f) = n\}$. Let $S_{\mathbf{b}} = \{(s, \beta, n) : s \in S_1,\ \beta < \beta_s,\ n < \omega,\ \Omega_{(s,\beta,n)} \neq \emptyset\} \cup S_0$ and let $g : S_{\mathbf{b}} \to S_{\mathbf{a}}$ be the function defined by $g(s, \beta, n) = s$ for $s \in S_1$ and $g(s) = s$ otherwise. For any $s \in S_0$ let $\Omega^{\mathbf{b}}_s = \Omega^{\mathbf{a}}_s$, $h^{\mathbf{b}}_s = h^{\mathbf{a}}_s$ and $\rho^{\mathbf{b}}_s = \rho^{\mathbf{a}}_s{}^\frown\langle 0\rangle$. For $s \in S_1$, $\beta < \beta_s$ and $n < \omega$, we set $\Omega^{\mathbf{b}}_{(s,\beta,n)} = \Omega_{(s,\beta,n)}$, $\rho^{\mathbf{b}}_{(s,\beta,n)} = \rho^{\mathbf{a}}_s{}^\frown\langle n\rangle$ and

$$h^{\mathbf{b}}_{(s,\beta,n)} = \begin{cases} h_{s,\beta}, & n = 0, \\ h^{\mathbf{a}}_s, & n > 0. \end{cases}$$

Claim 5.6.3. $\mathbf{b}$ *is an approximation and* $\mathbf{a} \trianglelefteq_g \mathbf{b}$.

Proof. We check that $\mathbf{b}$ satisfies (1) and (2) from the definition. (1) follows by the choice of $h^{\mathbf{b}}_{(s,\beta,n)}$.

We are left with (2). Let $t \in S_0$ and (s, β, n) with $s \in S_1$. If $\rho^{\mathbf{b}}_t = \rho^{\mathbf{b}}_{(s,\beta,n)}$ then $\rho^{\mathbf{a}}_t = \rho^{\mathbf{a}}_s$ so the result follows since $\mathbf{a}$ is an approximation. Let $(s_1, \beta_1, n_1) \neq (s_2, \beta_2, n_2) \in S_{\mathbf{b}}$. If $\rho^{\mathbf{b}}_{(s_1,\beta_1,n_1)} = \rho^{\mathbf{b}}_{(s_2,\beta_2,n_2)}$ then $\rho^{\mathbf{a}}_{s_1} = \rho^{\mathbf{a}}_{s_2}$. If $s_1 \neq s_2$ then the result follows since $\mathbf{a}$ is an approximation. So assume that $s = s_1 = s_2$ and $n = n_1 = n_2$. Assume that $\beta_1 < \beta_2 < \beta_s$ and let $f_1 \in \Omega^{\mathbf{b}}_{(s,\beta_1,n)}$ and $f_2 \in \Omega^{\mathbf{b}}_{(s,\beta_2,n)}$. We need to show that $f_1 \not\mathrel{R} f_2$ and $f_2 \not\mathrel{R} f_1$.

By choice of $\beta_1 = \beta_s(f_1)$ and $n = n_s(f_1)$, $B_n(f_1, h_{s,\beta_1}) \in \mathcal{D}$. On the other hand, since $\beta_1 < \beta_2 = \beta_s(f_2) \leq \beta_{s,n+2}(f_2)$, it follows that $B_{n+2}(f_2, h_{s,\beta_1}) \notin \mathcal{D}$, i.e. $\kappa \setminus B_{n+2}(f_2, h_{s,\beta_1}) \in \mathcal{D}$. Let $\varepsilon \in B_n(f_1, h_{s,\beta_1}) \cap (\kappa \setminus B_{n+2}(f_2, h_{s,\beta_1}))$ and let i_l be the $(n + l)$-th element of J_ε from the end, for $l = 1, 2, 3$. As a result,

$$f_1(i_3) < f_1(i_1) < h_{s,\beta_1}(\varepsilon) \leq f_2(i_3).$$

Consequently, $f_1 \not\mathrel{R} f_2$ by Assumption 5.1 (3) (since $\mathrm{Suc}(\mathrm{Suc}(i_3)) = i_1$) and $f_2 \not\mathrel{R} f_1$ by Assumption 5.1 (1).

$\mathbf{a} \trianglelefteq_g \mathbf{b}$ by construction. $\blacksquare_{\text{claim}}$

To complete the proof, we note that for $(s, \beta, n) \in S_{\mathbf{b}}$ with $n > 0$, $\Omega^{\mathbf{b}}_{(s,\beta,n)}$ is trivial. Let $f_1, f_2 \in \Omega^{\mathbf{b}}_{(s,\beta,n)}$. By Claim 5.6.2, and the assumptions on $\mathcal{D}$, we may find $\varepsilon < \kappa$ such that for $l = 1, 2$ the following holds:

(1) $|\{i \in J_\varepsilon : f_l(i) \geq h_{s,\beta}(\varepsilon)\}| = n$,

(2) $|J_\varepsilon| > n$.

Since $n > 0$, letting i be the $n+1$-th element from the end of J_ε we have $f_l(i) < h_{s,\beta}(\varepsilon) \leq f_l(\mathrm{Suc}(i))$ for $l = 1, 2$. If $f_1 \; R \; f_2$ then $f_1(\mathrm{Suc}(i)) \leq f_2(i) < h_{s,\beta}(\varepsilon)$ by Assumption 5.1 (2), contradiction. ∎

Proposition 5.7. *Let* $\mathbf{a}$ *be an approximation. Then there exists an approximation* $\mathbf{c}$ *and a surjective function* $r : S_{\mathbf{c}} \to S_{\mathbf{a}}$ *such that* $\mathbf{a} \lhd_r \mathbf{c}$. *Moreover, for* $t \in S_{\mathbf{c}}$, *if* $\rho^{\mathbf{a}}_{r(t)} \in \chi^{\xi}$, *with* $\xi < \sigma$, *then* $\rho^{\mathbf{c}}_t \in \chi^{\xi+2}$.

Proof. Let $\mathbf{a} \trianglelefteq_g \mathbf{b}$ be the approximation supplied by Proposition 5.5 and let $\mathbf{b} \trianglelefteq_f \mathbf{c}$ be the approximation supplied by Proposition 5.6. Note that $\mathbf{a} \trianglelefteq_{g \circ f} \mathbf{c}$ by Lemma 5.4; we claim that $\mathbf{a} \lhd_{g \circ f} \mathbf{c}$.

Let $t \in S_{\mathbf{c}}$. Since $\mathbf{a}$ is an approximation and $\sup_{\varepsilon<\kappa} |J_\varepsilon| = \omega$, we cannot have that $\{\varepsilon < \kappa : \mathrm{cf}(h^{\mathbf{a}}_{gf(t)}(\varepsilon)) = 0\} \in \mathcal{D}$; see Definition 5.2 (1).

Assume that $\{\varepsilon < \kappa : 0 < \mathrm{cf}(h^{\mathbf{a}}_{gf(t)}(\varepsilon)) \leq \chi\} \in \mathcal{D}$. If $\Omega^{\mathbf{a}}_{gf(t)}$ is trivial then so is $\Omega^{\mathbf{c}}_t$, so assume not. By Proposition 5.5 (1) applied to $f(t) \in S_{\mathbf{b}}$, either $\Omega^{\mathbf{b}}_{f(t)}$ is trivial (and thus so is $\Omega^{\mathbf{c}}_t$) or $\{\varepsilon < \kappa : h^{\mathbf{b}}_{f(t)}(\varepsilon) < h^{\mathbf{a}}_{gf(t)}(\varepsilon)\} \in \mathcal{D}$. If it is the latter then, since $\{\varepsilon < \kappa : h^{\mathbf{c}}_t(\varepsilon) \leq h^{\mathbf{b}}_{f(t)}(\varepsilon)\} \in \mathcal{D}$, we conclude that $\{\varepsilon < \kappa : h^{\mathbf{c}}_t(\varepsilon) < h^{\mathbf{a}}_{gf(t)}(\varepsilon)\} \in \mathcal{D}$.

Assume that $\{\varepsilon < \kappa : \mathrm{cf}(h^{\mathbf{a}}_{gf(t)}(\varepsilon)) > \chi\} \in \mathcal{D}$. In particular, since by Proposition 5.5 (2), $h^{\mathbf{a}}_{gf(t)} = h^{\mathbf{b}}_{f(t)}$, it follows that $\{\varepsilon < \kappa : \mathrm{cf}(h^{\mathbf{b}}_{f(t)}(\varepsilon)) > \chi\} \in \mathcal{D}$. Assuming that $\Omega^{\mathbf{c}}_t$ is not trivial, by Proposition 5.6 (1), $\{\varepsilon < \kappa : h^{\mathbf{c}}_t(\varepsilon) < h^{\mathbf{b}}_{f(t)}(\varepsilon)\} \in \mathcal{D}$. Since $\{\varepsilon < \kappa : h^{\mathbf{b}}_{f(t)}(\varepsilon) \leq h^{\mathbf{a}}_{gf(t)}(\varepsilon)\} \in \mathcal{D}$ we conclude that $\{\varepsilon < \kappa : h^{\mathbf{c}}_t(\varepsilon) < h^{\mathbf{a}}_{gf(t)}(\varepsilon)\} \in \mathcal{D}$, as needed.

The “moreover” part follows immediately from the construction. ∎

Lemma 5.8. *Let* $\delta < \sigma$ *be a limit ordinal and* $\langle \mathbf{a}_\alpha : \alpha < \delta \rangle$ *a sequence of approximations. Assume that* $\mathbf{a}_\alpha \trianglelefteq_{g_{\alpha,\beta}} \mathbf{a}_\beta$ *for* $\alpha < \beta < \delta$, *and* $g_{\alpha,\beta} \circ g_{\beta,\gamma} = g_{\alpha,\gamma}$ *for* $\alpha < \beta < \gamma < \delta$. *Then the inverse limit exists, i.e. there are* $(\mathbf{a}_\delta, \langle g_{\alpha,\delta} : \alpha < \delta \rangle)$ *such that* $\mathbf{a}_\alpha \trianglelefteq_{g_{\alpha,\delta}} \mathbf{a}_\delta$ *and* $g_{\alpha,\beta} \circ g_{\beta,\delta} = g_{\alpha,\delta}$ *for* $\alpha < \beta < \delta$.

In particular, if $\mathbf{a}_\alpha \lhd_{g_{\alpha,\beta}} \mathbf{a}_\beta$ *for some* $\alpha < \beta < \delta$ *then* $\mathbf{a}_\alpha \lhd_{g_{\alpha,\delta}} \mathbf{a}_\delta$.

Furthermore, for any $t \in S_{\mathbf{a}_\delta}$, *if* $\xi = \sup\{\zeta : \rho^{\mathbf{a}_\alpha}_{g_{\alpha,\delta}(t)} \in \chi^{\zeta}, \alpha < \delta\}$ *then* $\rho^{\mathbf{a}_\delta}_t \in \chi^{\xi+1}$.

Proof. For every $f \in \Omega$ let $t_f \in \prod_{\alpha<\delta} S_{\mathbf{a}_\alpha}$ be the function defined by $t_f(\alpha) = s$ if and only if $f \in \Omega^{\mathbf{a}_\alpha}_s$. Note that for $\alpha < \beta < \delta$, since for any $s \in S_{\mathbf{a}_\alpha}$, $\{\Omega^{\mathbf{a}_\beta}_t : t \in g^{-1}_{\alpha,\beta}(s)\}$ is a partition of $\Omega^{\mathbf{a}_\alpha}_s$, necessarily $g_{\alpha,\beta}(t_f(\beta)) = t_f(\alpha)$.

Let $S_* = \{t_f : f \in \Omega\}$ and for any $t \in S_*$, let $\Omega_t = \{f \in \Omega : t_f = t\}$. Clearly, it is a partition of Ω. Furthermore, note that if $t_1, t_2 \in S_*$ and $\alpha < \delta$ is such that $t_1(\alpha) = t_2(\alpha)$ then $t_1(\alpha') = t_2(\alpha')$ for any $\alpha' \leq \alpha$.

Let $S_0 = \{t \in S_* : (\exists \alpha < \delta)(\Omega^{\mathbf{a}_\alpha}_{t(\alpha)} \text{ is trivial})\}$ and $S_1 = S_* \setminus S_0$.

For any $t \in S_1$ and $\varepsilon < \kappa$, let $A_{t,\varepsilon} = \{h^{\mathbf{a}_\alpha}_{t(\alpha)}(\varepsilon) : \alpha < \delta\}$. Obviously, $1 \leq |A_{t,\varepsilon}| \leq |\delta| < \sigma \leq \chi$.

For every $(t, h/\mathcal{D}) \in S_1 \times \prod_{\varepsilon<\kappa} A_{t,\varepsilon}/\mathcal{D}$ let $\Omega_{(t,h/\mathcal{D})} = \{f \in \Omega_t : (\forall^{\mathcal{D}} \varepsilon < \kappa)(h(\varepsilon) = \min\{x \in A_{t,\varepsilon} : \mathrm{Range}(f \restriction J_\varepsilon) \subseteq x\})\}$. Let $S_{\mathbf{a}_\delta} = \{(t, h/\mathcal{D}) \in S_1 \times \prod_{\varepsilon<\kappa} A_{t,\varepsilon}/\mathcal{D} : \Omega_{(t,h/\mathcal{D})} \neq \emptyset\} \cup S_0$. For $t \in S_0$, set $\Omega^{\mathbf{a}_\delta}_t = \Omega^{\mathbf{a}_\alpha}_{t(\alpha)}$ and $h^{\mathbf{a}_\delta}_t = h^{\mathbf{a}_\alpha}_{t(\alpha)}$, where $\alpha < \delta$ is minimal such that $\Omega^{\mathbf{a}_\alpha}_{t(\alpha)}$ is trivial. For $(t, h/\mathcal{D}) \in S_{\mathbf{a}_\delta} \setminus S_0$, set $\Omega^{\mathbf{a}_\delta}_{(t,h/\mathcal{D})} = \Omega_{(t,h/\mathcal{D})}$.

Note that if $t \in S_1$ then $\{\varepsilon < \kappa : (\exists x \in A_{t,\varepsilon})(\mathrm{Range}(f \restriction J_\varepsilon) \subseteq x)\} \in \mathcal{D}$ because this set contains $\{\varepsilon < \kappa : \mathrm{Range}(f \restriction J_\varepsilon) \subseteq h^{\mathbf{a}_0}_{t(0)}(\varepsilon)\}$, which is in $\mathcal{D}$ since $\mathbf{a}_0$ is an approximation and $f \in \Omega^{\mathbf{a}_0}_{t(0)}$. Thus for any $t \in S_1$ and for every $f \in \Omega_t$ there is a unique $h/\mathcal{D} \in \prod_{\varepsilon<\kappa} A_{t,\varepsilon}/\mathcal{D}$ such that $f \in \Omega^{\mathbf{a}_\delta}_{(t,h/\mathcal{D})}$. We choose $h^{\mathbf{a}_\delta}_{(t,h/\mathcal{D})}$ to be any representative of the class $h/\mathcal{D}$.

For any $t \in S_0$ and $\alpha < \delta$, $g_{\alpha,\delta}(t) = t(\alpha)$, and for every $(t, h/\mathcal{D}) \in S_{\mathbf{a}_\delta} \setminus S_0$ and $\alpha < \delta$, $g_{\alpha,\delta}((t, h/\mathcal{D})) = t(\alpha)$. Note that it already follows that $g_{\alpha,\beta} \circ g_{\beta,\delta} = g_{\alpha,\delta}$ for $\alpha < \beta < \delta$.

For any $t \in S_0$ let

$$\rho^{\mathbf{a}_\delta}_t = \bigcup \{\rho^{\mathbf{a}_\alpha}_{t(\alpha)} : \alpha < \delta\}^\frown\langle 0\rangle.$$

Now let $(t, h/\mathcal{D}) \in S_{\mathbf{a}_\delta} \setminus S_0$. Since $\chi^\kappa = \chi$, there exists some $\gamma_t \leq \chi$ and an enumeration of $\prod_{\varepsilon<\kappa} A_{t,\varepsilon}/\mathcal{D}$ as $\langle h_{t,\gamma}/\mathcal{D} : \gamma < \gamma_t\rangle$. Now for $(t, h/\mathcal{D}) \in S_{\mathbf{a}_\delta}$ set

$$\rho^{\mathbf{a}_\delta}_{(t,h/\mathcal{D})} = \bigcup \{\rho^{\mathbf{a}_\alpha}_{t(\alpha)} : \alpha < \delta\}^\frown\langle \gamma\rangle,$$

where $h/\mathcal{D} = h_{t,\gamma}/\mathcal{D}$. Assume that $t = t_f$ for some $f \in \Omega$ and let $\alpha < \beta < \delta$. Since $g_{\alpha,\beta}(t(\beta)) = t(\alpha)$, $\rho^{\mathbf{a}_\alpha}_{t(\alpha)}$ is an initial segment of $\rho^{\mathbf{a}_\beta}_{t(\beta)}$. This implies that $\rho^{\mathbf{a}_\delta}_{(t,h/\mathcal{D})} \in \chi^{\xi+1}$, where $\xi = \sup\{\zeta : \rho^{\mathbf{a}_\alpha}_{g_{\alpha,\delta}(t)} \in \chi^\zeta, \alpha < \delta\}$.

We check that $\mathbf{a}_\delta$ is an approximation. Item (1) of Definition 5.2 follows from the definition of $\Omega^{\mathbf{a}_\delta}_t$ and the choice of $h^{\mathbf{a}_\delta}_t$, for $t \in S_{\mathbf{a}_\delta}$.

We show item (2). Let $(t_1, h_1/\mathcal{D}) \neq (t_2, h_2/\mathcal{D}) \in S_{\mathbf{a}_\delta} \setminus S_0$, assume that $\rho^{\mathbf{a}_\delta}_{(t_1,h_1/\mathcal{D})} = \rho^{\mathbf{a}_\delta}_{(t_2,h_2/\mathcal{D})}$ and let $f_1 \in \Omega^{\mathbf{a}_\delta}_{(t_1,h_1/\mathcal{D})}$ and $f_2 \in \Omega^{\mathbf{a}_\delta}_{(t_2,h_2/\mathcal{D})}$. If $t_1 \neq t_2$ then there exists some $\alpha < \delta$ such that $t_1(\alpha) \neq t_2(\alpha)$. But $\rho^{\mathbf{a}_\alpha}_{t_1(\alpha)} = \rho^{\mathbf{a}_\alpha}_{t_2(\alpha)}$ and hence $f_1 \not R f_2$. Assume that $t_1 = t_2$. Since $\rho^{\mathbf{a}_\delta}_{(t_1,h_1/\mathcal{D})} = \rho^{\mathbf{a}_\delta}_{(t_2,h_2/\mathcal{D})}$, it follows that $h_1/\mathcal{D} = h_2/\mathcal{D}$, which gives a contradiction. Let $(t, h/\mathcal{D}) \in S_{\mathbf{a}_\delta} \setminus S_0$ and $s \in S_0$. If $s \neq t$ then the same argument as above applies. On the other hand, it cannot be that $s = t$ by the definition of S_0. If $t_1 \neq t_2 \in S_0$ then the same argument as above applies.

Finally, we show that $\mathbf{a}_\alpha \trianglelefteq_{g_{\alpha,\delta}} \mathbf{a}_\delta$ for $\alpha < \delta$. Items (1), (2) and (4) are straightforward. We show item (3). Let $t \in S_0$ and let $\alpha' < \delta$ be minimal such that $\Omega^{\mathbf{a}_{\alpha'}}_{t(\alpha')}$ is trivial. If $\alpha' \leq \alpha$ then $h^{\mathbf{a}_\delta}_t = h^{\mathbf{a}_{\alpha'}}_{t(\alpha')} = h^{\mathbf{a}_\alpha}_{t(\alpha)}$. If $\alpha < \alpha'$ then $\{\varepsilon < \kappa : h^{\mathbf{a}_\delta}_t(\varepsilon) \leq h^{\mathbf{a}_\alpha}_{t(\alpha)}(\varepsilon)\} \in \mathcal{D}$ because $h^{\mathbf{a}_\delta}_t = h^{\mathbf{a}_{\alpha'}}_{t(\alpha')}$. Now let $(t, h/\mathcal{D}) \in S_{\mathbf{a}_\delta} \setminus S_0$. Since $\Omega^{\mathbf{a}_\delta}_{(t,h/\mathcal{D})}$ is non-empty, we may choose some function $f \in \Omega^{\mathbf{a}_\delta}_{(t,h/\mathcal{D})}$. On the one hand, since $f \in \Omega^{\mathbf{a}_\alpha}_{t(\alpha)}$ and since $\mathbf{a}_\alpha$ is an approximation, $\{\varepsilon < \kappa : \mathrm{Range}(f \restriction J_\varepsilon) \subseteq h^{\mathbf{a}_\alpha}_{t(\alpha)}(\varepsilon)\} \in \mathcal{D}$. On the other hand, since $f \in \Omega^{\mathbf{a}_\delta}_{(t,h/\mathcal{D})}$, we see that $\{\varepsilon < \kappa : h(\varepsilon) = \min\{x \in A_{t,\varepsilon} : \mathrm{Range}(f \restriction J_\varepsilon) \subseteq x\}\} \in \mathcal{D}$. Combining these observations with the fact that $h^{\mathbf{a}_\alpha}_{t(\alpha)} \in A_{t,\varepsilon}$, it follows that $\{\varepsilon < \kappa : h(\varepsilon) \leq h^{\mathbf{a}_\alpha}_{t(\alpha)}(\varepsilon)\} \in \mathcal{D}$ since it contains the intersection of the two sets. ∎

Conclusion 5.9. *There exists a function $c : \Omega \to \chi$ such that if $f_1, f_2 \in \Omega$ and $f_1 \; R \; f_2$ then $c(f_1) \neq c(f_2)$.*

Proof. We define $(\mathbf{a}_\xi, \langle g_{\zeta,\xi} : \zeta < \xi\rangle)$ such that $\mathbf{a}_\zeta \triangleleft_{g_{\zeta,\xi}} \mathbf{a}_\xi$ (for $\zeta < \xi$), by induction on $\xi < \sigma = (2^\kappa)^+$.

- If $\xi = 0$ then $\Omega^{\mathbf{a}_0} = \Omega$, $S_{\mathbf{a}_0} = \{0\}$, $\Omega_0^{\mathbf{a}_0} = \Omega^{\mathbf{a}_0}$, $\rho_0^{\mathbf{a}_0} = \emptyset$ and let $h_0^{\mathbf{a}_0} \in \mathcal{H}$ be the constant function θ.
- If $\xi = \alpha + 1$ for some $\alpha < \xi$ then let $\mathbf{a}_\alpha \triangleleft_{g_{\alpha,\xi}} \mathbf{a}_\xi$ be the approximation supplied by Proposition 5.7. For any $\zeta \leq \alpha$ we define $g_{\zeta,\xi} = g_{\zeta,\alpha} \circ g_{\alpha,\xi}$.
- If ξ is a limit ordinal we apply Lemma 5.8.

It follows by induction, and using Proposition 5.7 and Lemma 5.8, that for $\alpha < \beta < \sigma$ and $t \in S_{\mathbf{a}_\alpha}$ and $s \in S_{\mathbf{a}_\beta}$, if $\rho_t^{\mathbf{a}_\alpha} \in \chi^{\lambda_1}$ and $\rho_s^{\mathbf{a}_\beta} \in \chi^{\lambda_2}$ then $\lambda_1 < \lambda_2$ and hence $\rho_t^{\mathbf{a}_\alpha} \neq \rho_s^{\mathbf{a}_\beta}$.

Claim 5.9.1. (1) *For any $\rho \in \chi^{<\sigma}$,*

$$\Omega_\rho := \bigcup \{\Omega_s^{\mathbf{a}_\xi} : \xi < \sigma,\ s \in S_{\mathbf{a}_\xi},\ \Omega_s^{\mathbf{a}_\xi} \text{ is trivial and } \rho_s^{\mathbf{a}_\xi} = \rho\}$$

is trivial.

(2) $\Omega = \bigcup_{\rho \in \chi^{<\sigma}} \Omega_\rho$.

Proof. (1) Let $\rho \in \chi^{<\sigma}$ and $f_1, f_2 \in \Omega_\rho$. By definition, there exist $s_1 \in S_{\mathbf{a}_{\xi_1}}$ and $s_2 \in S_{\mathbf{a}_{\xi_2}}$ such that $f_1 \in \Omega_{s_1}^{\mathbf{a}_{\xi_1}}$ and $f_2 \in \Omega_{s_2}^{\mathbf{a}_{\xi_2}}$. As noted above, since $\rho_{s_1}^{\mathbf{a}_{\xi_1}} = \rho_{s_2}^{\mathbf{a}_{\xi_2}}$, necessarily $\xi = \xi_1 = \xi_2$. If $s_1 = s_2$ then $f_1 \not\mathrel{R} f_2$ since $\Omega_{s_1}^{\mathbf{a}_\xi} = \Omega_{s_2}^{\mathbf{a}_\xi}$ is trivial. If $s_1 \neq s_2$ then by the definition of approximation, since $\rho_{s_1}^{\mathbf{a}_\xi} = \rho_{s_2}^{\mathbf{a}_\xi}$, we have $f_1 \not\mathrel{R} f_2$.

(2) Assume that there exists some $f \in \Omega \setminus \bigcup_{\rho \in \chi^{<\sigma}} \Omega_\rho$. We construct a sequence $\langle h_\xi \in \mathcal{H} : \xi < \sigma \rangle$ of functions such that for any $\alpha < \beta < \sigma$, $h_\beta <_{\mathfrak{D}} h_\alpha$.

For any $\xi < \sigma$ let $h_\xi = h_t^{\mathbf{a}_\xi}$ for the unique $t \in S_{\mathbf{a}_\xi}$ such that $f \in \Omega_t^{\mathbf{a}_\xi}$. By assumption, $\Omega_t^{\mathbf{a}_\xi}$ is non-trivial for any such ξ (otherwise $f \in \Omega_\rho$ for $\rho = \rho_t^{\mathbf{a}_\xi}$). For any $\alpha < \beta < \sigma$, since $\mathbf{a}_\alpha \triangleleft_{g_{\alpha,\beta}} \mathbf{a}_\beta$, we have $h_\beta <_{\mathfrak{D}} h_\alpha$.

We color pairs $\{(h_\alpha, h_\beta) : \alpha < \beta < \sigma\}$ of functions by κ colors, by declaring that (h_α, h_β) has color $\varepsilon_{\alpha,\beta} < \kappa$ if $\varepsilon_{\alpha,\beta}$ is the minimal ε for which $h_\beta(\varepsilon) < h_\alpha(\varepsilon)$. We know that such an ε exists, since $h_\beta <_{\mathfrak{D}} h_\alpha$. By Erdős–Rado there exists a subset $A \subseteq \sigma$ of cardinality κ^+ and $\varepsilon < \kappa$ such that $h_\beta(\varepsilon) < h_\alpha(\varepsilon)$ for every $\alpha < \beta \in A$. This contradicts the fact that the ordinals are well-ordered. $\blacksquare_{\text{claim}}$

Recalling that $\chi^{<\sigma} = \chi$ (as cardinals), we may now define $c : \Omega \to \chi$ by choosing for every $f \in \Omega$ some $\rho \in \chi^{<\sigma}$ such that $f \in \Omega_\rho$ and setting $c(f) = \rho$. $\blacksquare$

6. Conclusion: stable graphs

We now combine the results of the previous sections.

Theorem 6.1. *Let $\mathfrak{L}$ be a first order language containing a binary relation E. Let T be an $\mathfrak{L}$-theory specifying that E is a symmetric and irreflexive relation. Let $G = (V; E, \dots) \models T$ be an infinitary EM-model based on (α, θ), where $\alpha \in \kappa^U$ for some set U, $\kappa \geq \aleph_0$ a cardinal and θ an ordinal with $\kappa < \theta$. Let $\varkappa > 2^{2^{<(\kappa+\aleph_1)}} + |T| \cdot |U|$ be a regular cardinal. If $\chi(G) \geq \varkappa$ then G contains all finite subgraphs of* $\mathrm{Sh}_n(\omega)$ *for some $n \in \mathbb{N}$.*

Proof. By Lemma 3.6 there exists some $(\hat{\alpha}, \theta)$-indiscernible sequence $b = \langle b_{i,\eta} : i \in \hat{U}, \eta \in (\theta^{\underline{\hat{\alpha}_i}})_< \rangle$ whose underlying set is V, where $\hat{\alpha} \in \kappa^{\hat{U}}$ and $\hat{U}$ is a set such that $|\hat{U}| \le |T| \cdot |U| \cdot \kappa^{<\kappa}$.

Let $B = \{(i, \eta) : i \in \hat{U}, \eta \in (\theta^{\underline{\hat{\alpha}_i}})_<\}$ and $R = \{((i_1, \eta_1), (i_2, \eta_2)) : b_{i_1,\eta_1} \; E \; b_{i_2,\eta_2}\}$. Since $(i, \eta) \mapsto b_{i,\eta}$ is surjective and $((i_1, \eta_1), (i_2, \eta_2)) \in R \Leftrightarrow (b_{i_1,\eta_1} b_{i_2,\eta_2}) \in E$, it follows that $\chi(B, R) = \chi(G) \ge \varkappa$ (by Fact 2.2 (4)). Moreover, by Fact 2.5 it is enough to prove the conclusion for the graph (B, R).

For any $i \in \hat{U}$ let $B_i = \{(i, \eta) : \eta \in (\theta^{\underline{\alpha_i}})_<\}$. By Fact 2.2 (1), since $B = \bigcup_{i \in \hat{U}} B_i$, it follows that $\varkappa \le \chi(B, R) \le \sum_{i \in \hat{U}} \chi(B_i, R \restriction B_i)$. By definition[4] $\kappa \le 2^{<\kappa}$, which implies $\kappa^{<\kappa} \le \kappa^\kappa = 2^\kappa \le 2^{2^{<\kappa}}$ and thus $\varkappa > |\hat{U}|$. Since $\varkappa$ is a regular cardinal, there exists $i \in \hat{U}$ with $\chi(B_i, R \restriction B_i) \ge \varkappa$. As a result, it is enough to prove the conclusion for the graph $((\theta^{\underline{\hat{\alpha}_i}})_<, S)$, where $S = \{(\eta_1, \eta_2) : (i, \eta_1) \; R \; (i, \eta_2)\}$.

For $P = \{\mathrm{otp}(\bar{a}, \bar{b}) : (\bar{a}, \bar{b}) \in S\}$, by $(\hat{\alpha}, \theta)$-indiscernibility,

$$S = \bigcup_{p \in P} \{(\bar{c}, \bar{d}) \in ((\theta^{\underline{\hat{\alpha}_i}})_<)^2 : \mathrm{otp}(\bar{c}, \bar{d}) = p \vee \mathrm{otp}(\bar{d}, \bar{c}) = p\}.$$

By Fact 2.2 (2),

$$\varkappa \le \chi((\theta^{\underline{\hat{\alpha}_i}})_<, S)) \le \prod_{p \in P} \chi((\theta^{\underline{\hat{\alpha}_i}})_<, P_p),$$

where $P_p = \{(\bar{c}, \bar{d}) \in ((\theta^{\underline{\hat{\alpha}_i}})_<)^2 : \mathrm{otp}(\bar{c}, \bar{d}) = p \vee \mathrm{otp}(\bar{d}, \bar{c}) = p\}$. Assume towards a contradiction that $\chi((\theta^{\underline{\hat{\alpha}_i}})_<, P_p) \le \beth_2(\aleph_0)$ for all $p \in P$. Hence $\varkappa \le \beth_2(\aleph_0)^{2^{|\alpha_i| + \aleph_0}} \le \beth_2(|\alpha_i| + \aleph_0)$. Since $|\alpha_i| + \aleph_0 < \kappa + \aleph_1$ and $\varkappa > 2^{2^{<(\kappa+\aleph_1)}}$, we derive a contradiction.

Consequently, there exists $p \in P$ with $\chi((\theta^{\underline{\hat{\alpha}_i}})_<, P_p) > \beth_2(\aleph_0)$ and we may conclude by Theorem 4.8. ∎

Corollary 6.2. *Let $G = (V, E)$ be a stable graph. If $\chi(G) > \beth_2(\aleph_0)$ then G contains all finite subgraphs of* $\mathrm{Sh}_n(\omega)$ *for some $n \in \mathbb{N}$.*

Proof. Let $T = \mathrm{Th}(G)$ and T^{sk} be a complete expansion of T with definable Skolem functions in the language $E \in \mathcal{L}^{\mathrm{sk}}$.

We apply Theorem 3.7 with $\kappa = \aleph_1$, $\mu = 2^{\aleph_1}$ and $\lambda = 2^{\max\{\mu, |G|\}}$. We get an infinitary EM-model $\mathcal{G}^{\mathrm{sk}} \models T^{\mathrm{sk}}$ based on (α, λ), where $\alpha \in \kappa^U$ for some set U of cardinality at most μ, such that $\mathcal{G} = \mathcal{G}^{\mathrm{sk}} \restriction \{E\}$ is saturated of cardinality λ. Since $\mathcal{G}$ is saturated of cardinality $> |G|$, we may embed G as an elementary substructure of $\mathcal{G}$. Since $\chi(\mathcal{G}) \ge \chi(G) > \beth_2(\aleph_0)$ and the conclusion is an elementary property, it is enough to show it for $\mathcal{G}$.

Since $2^{2^{<(\kappa+\aleph_1)}} + |T| + |U| \le 2^{2^{\aleph_0}} + \aleph_0 + \mu \le \beth_2(\aleph_0) + 2^{\aleph_1} = \beth_2(\aleph_0)$, Theorem 6.1 applies with $\theta = \lambda$ and $\varkappa = (\beth_2(\aleph_0))^+$. ∎

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[4] As $2^{<\kappa} = \sup\{2^\mu : \mu < \kappa\}$, if $2^{<\kappa} < \kappa$ then $2^{2^{<\kappa}} \le 2^{<\kappa}$.

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