DIAMOND ON KUREPA TREES

ZIEMOWIT KOSTANA, ASSAF RINOT, AND SAHARON SHELAH

Abstract. We introduce a new weak variation of diamond that is meant to only guess the branches of a Kurepa tree. We demonstrate that this variation is considerably weaker than diamond by proving it is compatible with Martin's axiom. We then prove that this principle is nontrivial by showing it may consistently fail.

1. Introduction

Kurepa [Kur35] initiated the systematic study of uncountable trees through his attempt to solve Souslin's problem [Sou20] that asks whether the real line is the unique dense complete linear order with no endpoints satisfying that every pairwise disjoint collection of open intervals is countable. Kurepa showed that a counterexample is equivalent to the existence of what he called a *Souslin tree*. After sharing his findings with Aronszajn, the latter was able to construct a poor man's version of a Souslin tree that Kurepa then named an *Aronszajn tree*. A third type of an uncountable tree is now known as a *Kurepa tree*. These trees and their higher analogs were central in the development of set theory, where quite a few key concepts and techniques in forcing theory and large cardinals were discovered through solving problems concerning these objects. A milestone work here is Jensen's study [Jen72] of the fine structure of Gödel's inner model L [Göd40], where it is proved that a Souslin tree exists in L [Jen68]. Soon after, Solovay proved that a Kurepa tree exists in L and Silver [Sil71] extended it to outer universes under an optimal anti-large cardinal assumption. Then, Jensen (with a touch by Kunen) formulated the axiom $\diamondsuit$ as a *guessing principle* that holds in L but is sufficient for the construction of a Souslin tree in any universe of set theory, and then Jensen, Kunen and Silver devised the stronger variation $\diamondsuit^+$ as an axiom sufficient for the construction of a Kurepa tree.

Uncountable trees and diamond principles found many applications in infinite Ramsey theory, topology, measure theory and algebra. To give just one example, we mention that in [She74], the third author proved that $\diamondsuit^+$ is sufficient to imply that all Whitehead groups of size $\aleph_1$ are free. And there are in addition some indirect applications, e.g., the Higman-Stone construction of an uncountable inverse system of rings whose limit is empty [HS54] which involves the rediscovery of an Aronszajn tree.

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By now, a good understanding of diamond principles has been reached. For instance, the connection to other postulates such as the continuum hypothesis and forcing axioms has been determined [ST71, DJ74, DS78, SK80, Bau84, She10, GR12] and pump up lemmas were discovered showing that weaker diamonds can be made stronger by various combinatorial maneuvers [Dev78, Dev79, Kun80, Mat87, She05].

Motivated by the current state of this understanding and by the fact that $\diamondsuit^+$ is actually equivalent to the existence of a particular sort of a Kurepa tree [Kun80, Exercise VI.9], in this paper we revisit the very basics by proposing a generalization of diamond that becomes most interesting when tested against Kurepa trees and prove that all the familiar features of diamonds break down in the new context. Unlike the usual diamond, our new diamond does not imply the continuum hypothesis and it is compatible with forcing axioms. Furthermore, it is indestructible under various notions of forcing and can be made indestructible under all *proper* forcings. Three additional features that the classical diamond possesses and our new diamond lacks are: guessing uncountably often is equivalent to guessing stationarily often, guessing correctly modulo a countable error is equivalent to guessing correctly with no errors, and the feature that diamond cannot be added by a ccc forcing.

Next we make things precise. For this, let us first recall the definition of the trees under discussion.

Definition 1. A *binary tree of height* ω_1 is a set T satisfying the following three requirements:

- $T \subseteq {}^{<\omega_1}2$, i.e., T consists of binary sequences of countable length;
- T is downward-closed, i.e., for all $t \in T$ and $\alpha < \mathrm{dom}(t)$, $t \restriction \alpha \in T$;
- for every $\alpha < \omega_1$, the α^{th} level of T, $T_\alpha := \{t \in T \mid \mathrm{dom}(t) = \alpha\}$, is nonempty.

We denote by $\mathcal{B}(T) := \{f \in {}^{\omega_1}2 \mid \forall \alpha < \omega_1\, (f \restriction \alpha \in T_\alpha)\}$ the collection of all *cofinal branches* through T.

We now arrive at the main definition of this paper.

Definition 2. For a binary tree T of height ω_1, the axiom $\diamondsuit(T)$ asserts the existence of a sequence $\langle t_\alpha \mid \alpha < \omega_1\rangle$ such that:

- for every $\alpha < \omega_1$, t_α is a function from α to 2;
- for every $f \in \mathcal{B}(T)$, the set $\{\alpha < \omega_1 \mid f \restriction \alpha = t_\alpha\}$ is stationary.

Modulo the identification of sets with their characteristic functions, the classical $\diamondsuit$ is by definition equivalent to $\diamondsuit({}^{<\omega_1}2)$, where ${}^{<\omega_1}2$ is the full binary tree of height ω_1. In particular, $\diamondsuit$ implies $\diamondsuit(T)$ for every T.

A decomposition theorem of Ulam [Ula30] easily implies that $\diamondsuit(T)$ holds for every T such that $|\mathcal{B}(T)| \leq \aleph_1$. In addition, if $\diamondsuit(T)$ holds then so does $\diamondsuit(T')$ for some tree T' of size $\aleph_1$ with $\mathcal{B}(T') = \mathcal{B}(T)$. When put together, this shows that the study of $\diamondsuit(T)$ should focus on those T of size $\aleph_1$ such that $|\mathcal{B}(T)| > \aleph_1$. This is a well-known class of trees. Indeed, it is the first class of the following definition.

Definition 3. A binary tree T of height ω_1 is:

(1) *weak Kurepa* iff $|T| = \aleph_1 < |\mathcal{B}(T)|$;
(2) *Kurepa* iff $|T_\alpha| < \aleph_1$ for all $\alpha < \omega_1$ and $|\mathcal{B}(T)| > \aleph_1$;
(3) *Aronszajn* iff $|T_\alpha| < \aleph_1$ for all $\alpha < \omega_1$ and $\mathcal{B}(T) = \emptyset$;
(4) *Souslin* iff it is Aronszajn and for every uncountable $X \subseteq T$ there are $x, y \in X$ with $x \subsetneq y$.

Note that by Cantor's theorem, ${}^{<\omega_1}2$ happens to be a weak Kurepa tree iff the continuum hypothesis holds. Also note that it follows from the previous discussion and the trivial fact that every Kurepa tree is a weak Kurepa tree that the failure of diamond over a Kurepa tree would constitute the ultimate failure of our new principle, hence also of the classical one.

1.1. **Main results.** We will establish that $\diamondsuit \implies \clubsuit \implies \clubsuit_w \implies \diamondsuit(T)$ for every T of size $\aleph_1$, and show that our new principle is considerably weaker than the other ones. Moreover, it will be shown that Martin's axiom is compatible with $\diamondsuit(T)$ holding over a binary Kurepa tree T. This is a corollary to any of the following two results:

Theorem A. *If T and S are two binary Kurepa trees and $|\mathcal{B}(T)| < |\mathcal{B}(S)|$, then $\diamondsuit(T)$ holds.*

Theorem B. *It is consistent that there exists a binary Kurepa tree T such that $\diamondsuit(T)$ holds and cannot be killed by a proper forcing.*[1]

There are additional results that we already hinted upon earlier on, but the main result of this paper is the finding that this very weak variation of diamond is nevertheless nontrivial:

Theorem C. *It is consistent that $\diamondsuit(T)$ fails for some binary Kurepa tree T. Furthermore, such a tree T can be chosen to be either rigid or homogeneous. Furthermore, the failure of $\diamondsuit(T)$ can be arranged together with $\diamondsuit(T')$ holding for some binary Kurepa tree T' having the same number of branches as that of T.*

The model witnessing the preceding is obtained by a countable support iteration of proper notions of forcing for adding a branch that evades a potential diamond sequence using models as side conditions. Curiously, the said Kurepa tree T will start its life in the ground model as a particular Aronszajn tree obtained from $\diamondsuit^+$.

1.2. **Organization of this paper.** In Section 2, we provide some preliminaries on trees, justify our focus on Kurepa trees that are binary and compare the principle $\diamondsuit(T)$ with other standard set-theoretic hypotheses. The proof of Theorem A will be found there.

[1] A proper forcing may kill the Kurepa-ness of T, but then $\diamondsuit(T)$ will hold trivially.

In Section 3, we study indestructible forms of $\diamondsuit(T)$. The proof of Theorem B will be found there.

In Section 4, we present a notion of forcing $\mathbb{Q}(T,\vec{t})$ to add a branch through an $\aleph_1$-tree T that evades a given potential diamond sequence $\vec{t} = \langle t_\alpha \mid \alpha < \omega_1\rangle$. We give a sufficient condition for $\mathbb{Q}(T,\vec{t})$ to be proper, and then turn to iterate it. The proof of Theorem C will be found there.

2. Warm up

2.1. **Abstract, Hausdorff and binary trees.** A *tree* is a partially ordered set $\mathbf{T} = (T, <_T)$ such that, for every $x \in T$, the cone $x_\downarrow := \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$; its order type is denoted by $\mathrm{ht}(x)$. For any ordinal α, the α^{th}-level of the tree is the collection $T_\alpha := \{x \in T \mid \mathrm{ht}(x) = \alpha\}$. The *height* of the tree is the first ordinal α for which $T_\alpha = \emptyset$. The tree is *normal* iff for every $t \in T$ and every ordinal α in-between $\mathrm{ht}(x)$ and the height of the tree, there exists some $y \in T_\alpha$ with $x <_T y$. The tree is *Hausdorff* iff for every limit ordinal α and all $x, y \in T_\alpha$, if $x_\downarrow = y_\downarrow$, then $x = y$. In particular, a (nonempty) Hausdorff tree has a unique root.

An $\aleph_1$-*tree* is a tree $\mathbf{T}$ of height ω_1 all of whose levels are countable. A *Kurepa tree* (resp. *Aronszajn tree*) is an $\aleph_1$-tree $\mathbf{T}$ satisfying that the set $\mathcal{B}(\mathbf{T})$ of all uncountable maximal chains in $\mathbf{T}$ has size $\geq \aleph_2$ (resp. is empty). Definition 2 generalizes to abstract trees of height ω_1 by interpreting $f \restriction \alpha$ (for $f \in \mathcal{B}(\mathbf{T})$ and $\alpha < \omega_1$) as the unique element of T_α that belongs to f.

Hereafter, whenever we talk about a *binary* $\aleph_1$-*tree*, we mean a set T as in Definition 1 such that T_α is countable for every $\alpha < \omega_1$, and we shall freely identify it with the Hausdorff $\aleph_1$-tree $\mathbf{T} := (T, \subsetneq)$.

Lemma 2.1. *For every Hausdorff $\aleph_1$-tree $\mathbf{T} = (T, <_T)$, there exists a binary $\aleph_1$-tree S that is club-isomorphic to $\mathbf{T}$, i.e., for some club $D \subseteq \omega_1$, $(\bigcup_{\alpha\in D} T_\alpha, <_T)$ and $(\bigcup_{\alpha\in D} S_\alpha, \subsetneq)$ are order-isomorphic.*

In particular, if $\mathbf{T}$ is Kurepa, then S is Kurepa and $\diamondsuit(S)$ iff $\diamondsuit(\mathbf{T})$.

Proof. Let $\mathbf{T} = (T, <_T)$ be a given Hausdorff $\aleph_1$-tree. For every $\alpha < \omega_1$, fix an injection $\varphi_\alpha : T_{\alpha+1} \to \omega$. Also fix an injective sequence $\langle r_m \mid m < \omega\rangle$ of functions from ω to 2. We shall define an injection $\psi : T \to {}^{<\omega_1}2$ satisfying the following two requirements:

(1) for every $t \in T$, $\mathrm{dom}(\psi(t)) = \omega \cdot \mathrm{ht}(t)$;
(2) for all $t', t \in T$, $t' <_T t$ iff $\psi(t') \subsetneq \psi(t)$.

The definition of ψ is by recursion on the heights of the nodes in $\mathbf{T}$. By Clause (1) we are obliged to send the unique root of $\mathbf{T}$ to $\emptyset$ and by Clauses (1) and (2), for every $\alpha \in \mathrm{acc}(\omega_1)$ and every $t \in T_\alpha$,[2] we are obliged to set $\psi(t) := \bigcup\{\psi(t') \mid t' <_T t\}$.[3] Thus, the only freedom we have is at nodes of successor levels. Here, for every $\alpha < \omega_1$ such that $\psi \restriction T_\alpha$ has already been

[2]For a set of ordinals A, we write $\mathrm{acc}(A) := \{\alpha \in A \mid \sup(\alpha) = \alpha > 0\}$.
[3]The injectivity here follows from Hausdorff-ness.

defined, and for every $t \in T_{\alpha+1}$, let t^- denote the immediate predecessor of t, and set

$$\psi(t) := \psi(t^-)^\frown r_{\varphi_\alpha(t)}.$$

Finally, consider $S := \{s \restriction \alpha \mid s \in \mathrm{Im}(\psi),\ \alpha \le \mathrm{dom}(s)\}$. It is clear that S is a downward-closed subfamily of ${}^{<\omega_1}2$. Thus, to see that S is a binary $\aleph_1$-tree, it suffices to prove that all of its levels are countable. However, by Clauses (1) and (2), more is true, namely, for every $\alpha < \omega_1$,

$$S_\alpha = \{\psi(t) \restriction \alpha \mid t \in T_\alpha\}.$$

Consider the club $D := \{\alpha < \omega_1 \mid \omega \cdot \alpha = \alpha\}$. Evidently, ψ witnesses that $(\bigcup_{\alpha \in D} T_\alpha, <_T)$ and $(\bigcup_{\alpha \in D} S_\alpha, \subsetneq)$ are order-isomorphic. □

Any tree $\mathbf{T} = (T, <_T)$ can be cofinally-embedded in a Hausdorff tree of the form $(S, \subsetneq)$ by letting S be the downward closure of the collection of all $s : \alpha \to T$ that are order-preserving maps from *a successor* ordinal $(\alpha, \in)$ onto some downward-closed subset of $(T, <_T)$. The map that sends each $t \in T$ to the unique $s \in S$ with $\max(\mathrm{Im}(s), <_T) = t$ embeds $\mathbf{T}$ to the successor levels of S. In case that $\mathbf{T}$ is non-Hausdorff, there is no better embedding. Since non-Hausdorff trees lack genuine limit levels and since our definition of diamond on trees has to do with stationary sets, the study here will be focused on the theory of diamond on Hausdorff Kurepa trees. By Lemma 2.1, then, we may moreover focus on binary Kurepa trees.

2.2. Diamonds and clubs.

As said before, $\diamondsuit \implies \clubsuit \implies \clubsuit_w \implies \diamondsuit(T)$ for those T of size $\aleph_1$. The first two implications are well-known. We now give the details of the last one.

Proposition 2.2. *If $\clubsuit_w$ holds, then so does $\diamondsuit(T)$ for every binary tree T of height ω_1 and size $\aleph_1$.*

Proof. Suppose that $\clubsuit_w$ holds. By [FSS97, p. 61], this means that we may fix a sequence $\langle B_\alpha \mid \alpha \in \mathrm{acc}(\omega_1)\rangle$ such that each B_α is a cofinal subset of α of order-type ω, and, for every uncountable $B \subseteq \omega_1$, the following set is stationary:

$$G(B) := \{\alpha \in \mathrm{acc}(\omega_1) \mid B_\alpha \setminus B \text{ is finite}\}.$$

Now, suppose T is a binary tree of height ω_1 and size $\aleph_1$. Fix a bijection $\pi : \omega_1 \leftrightarrow T$. For every $\alpha \in \mathrm{acc}(\omega_1)$, if there are $t_\alpha \in T_\alpha$ and a finite $F_\alpha \subseteq B_\alpha$ such that

$$t_\alpha = \bigcup\{\pi(\beta) \mid \beta \in B_\alpha \setminus F_\alpha\},$$

then we keep this unique t_α; otherwise (including the case $\alpha \notin \mathrm{acc}(\omega_1)$), we let t_α be an arbitrary choice of an element of T_α. To see that $\langle t_\alpha \mid \alpha < \omega_1\rangle$ witnesses $\diamondsuit(T)$, let $f \in \mathcal{B}(T)$. Consider the following club:

$$D := \{\delta \in \mathrm{acc}(\omega_1) \mid \forall \gamma < \delta\, [\pi^{-1}(f \restriction \gamma) < \delta]\}.$$

Note that for every $\delta \in D$, $\delta \leq \pi^{-1}(f \restriction \delta) < \min(D \setminus (\delta+1))$. In particular, the following set is uncountable:

$$B := \{\pi^{-1}(f \restriction \delta) \mid \delta \in D\}.$$

We claim that for every $\alpha \in G(B)$, it is the case that $f \restriction \alpha = t_\alpha$. Indeed, the set $F_\alpha := B_\alpha \setminus B$ is finite, and for every $\beta \in B_\alpha \setminus F_\alpha$, there exists a unique $\delta \in D$ such that $\delta \leq \beta = \pi^{-1}(f \restriction \delta) < \min(D \setminus (\delta+1))$. So, since $\sup(B_\alpha \setminus F_\alpha) = \alpha$, it is the case that

$$t_\alpha = \bigcup\{\pi(\beta) \mid \beta \in B_\alpha \setminus F_\alpha\} = f \restriction \alpha,$$

as sought. □

It follows from the preceding that $\diamondsuit(T)$ is compatible with $\neg\mathsf{CH}$. An alternative reasoning goes through the next proposition. Namely, add $\aleph_2$ many Cohen reals to a model of $\diamondsuit^+$.

Proposition 2.3. *Suppose that* $\mathbf{T}$ *is a Kurepa tree such that* $\diamondsuit(\mathbf{T})$ *holds, and that* $\mathbb{P}$ *is a notion of forcing. If* $\mathbb{P}$ *is* σ*-closed or if* $\mathbb{P}^2$ *satisfies the ccc, then* $\diamondsuit(\mathbf{T})$ *remains to hold in* $V^{\mathbb{P}}$.

Proof. By [Sil71], a σ-closed forcing does not add new branches to $\aleph_1$-trees. By [Ung13, Lemma 2.2], a forcing notion whose square satisfies the ccc does not add new branches to $\aleph_1$-trees. In addition, stationary sets are preserved by σ-closed notions of forcing and by ccc notions of forcing. So, in both cases, the ground model witness to $\diamondsuit(\mathbf{T})$ will survive as a witness in $V^{\mathbb{P}}$. □

Another simple argument shows that an instance of the parameterized diamond principle from [JTM03] is enough.

Proposition 2.4. *If* $\Phi(\omega,=)$ *holds, then so does* $\diamondsuit(T)$ *for every binary Kurepa tree* T.

Proof. Suppose that $\Phi(\omega,=)$ holds. This means that for every function $F : {}^{<\omega_1}2 \to \omega$, there exists a function $g : \omega_1 \to \omega$ such that, for every function $f : \omega_1 \to \omega$, the following set is stationary:

$$G(f) := \{\alpha < \omega_1 \mid F(f \restriction \alpha) = g(\alpha)\}.$$

Now, suppose T is a binary Kurepa tree. For every $\alpha < \omega_1$, fix an enumeration $\langle t^n_\alpha \mid n < \omega\rangle$ of T_α. Fix a function $F : {}^{<\omega_1}2 \to \omega$ such that for all $\alpha < \omega_1$ and $t \in T_\alpha$,

$$F(t) = \min\{n < \omega \mid t = t^n_\alpha\},$$

and then let $g : \omega_1 \to \omega$ be the corresponding function given by $\Phi(\omega,=)$. A moment's reflection makes it clear that the transversal $\langle t^{g(\alpha)}_\alpha \mid \alpha < \omega_1\rangle$ witnesses $\diamondsuit(T)$. □

Recall that a *Souslin tree* is an Aronszajn tree with no uncountable antichains. Every Souslin tree contains a Souslin subtree that is normal. As established in [JTM03], forcing with such trees gives an instance of $\Phi(\omega,=)$. A variant of that argument gives the following.

Proposition 2.5. *Forcing with a normal Souslin tree adds a $\diamondsuit(T)$-sequence for every ground model binary $\aleph_1$-tree T.*

Proof. Working in V, suppose that $\mathbf{S} = (S, <_S)$ is a normal Souslin tree, and that T is a binary $\aleph_1$-tree. As $\mathbf{S}$ is normal and $\mathcal{B}(\mathbf{S}) = \emptyset$, for each $s \in S$, we may let $\langle s^n \mid n < \omega\rangle$ be a bijective enumeration of some infinite maximal antichain above s. For every $\alpha < \omega_1$, fix an enumeration $\langle t^n_\alpha \mid n < \omega\rangle$ of T_α. Given a generic G for $\mathbf{S}^* := (S, >_S)$, for every $\alpha < \omega_1$, denote by s_α the unique element of S_α that belongs to G, and then let $g(\alpha)$ denote the unique integer $n < \omega$ such that $(s_\alpha)^n$ belongs to G. We claim that the transversal $\langle t^{g(\alpha)}_\alpha \mid \alpha < \omega_1\rangle$ witnesses $\diamondsuit(T)$.

To see this, back in V, fix a condition $s \in S$, a name $\dot f$ for a cofinal branch through T, and a club $C \subseteq \omega_1$; we shall find an $\alpha \in C$ and an extension of s forcing that $\dot f \restriction \alpha$ coincides with $t^{\dot g(\alpha)}_\alpha$. Note that since $\mathbf{S}^*$ is ccc, we can indeed restrict our attention to ground model clubs.

Fix a countable $M \prec \mathrm{H}_{\omega_2}$, containing $\{\dot f, s, \mathbf{S}^*, T, C\}$. Write $\alpha := M \cap \omega_1$, and note that $\alpha \in C$, since $C \in M$. As $\mathbf{S}$ is normal, fix $s' \in S_\alpha$ extending s. By a folklore fact, s' is an $\mathbf{S}^*$-generic branch over M, so in particular it determines $\dot f \restriction \alpha$, and therefore there is an integer n such that

$$s' \Vdash \dot f \restriction \alpha = t^n_\alpha.$$

Set $s'' := (s')^n$, so that $s'' \Vdash \dot g(\alpha) = n$. Then $s'' >_S s' >_S s$, and

$$s'' \Vdash \dot f \restriction \alpha = t^{\dot g(\alpha)}_\alpha,$$

as sought. □

Corollary 2.6. *If T is a normal binary almost-Kurepa Souslin tree, then in some ccc forcing extension, T is a binary Kurepa tree and $\diamondsuit(T)$ holds.*

Proof. By definition, as T is almost-Kurepa, in the forcing extension by $\mathbb{P} := (T, \supseteq)$, T is a Kurepa tree. By Proposition 2.5, $\diamondsuit(T)$ holds in $V^{\mathbb{P}}$. □

2.3. **Weak variations.** Devlin [Dev79, §2] showed that if there exists a sequence $\langle t_\alpha \mid \alpha < \omega_1\rangle \in \prod_{\alpha<\omega_1} {}^\alpha 2$ such that, for every function $f : \omega_1 \to 2$, there is an infinite ordinal $\alpha < \omega_1$ with $f \restriction \alpha = t_\alpha$, then $\diamondsuit$ holds. When combined with Theorem C, the next proposition shows that this does not generalize to our context.

Proposition 2.7. *Every $\aleph_1$-tree $\mathbf{T}$ admits a transversal $\langle t_\alpha \mid \alpha < \omega_1\rangle$ such that $G(f) := \{\alpha < \omega_1 \mid f \restriction \alpha = t_\alpha\}$ is uncountable for every $f \in \mathcal{B}(\mathbf{T})$.*

Proof. Suppose that $\mathbf{T} = (T, <_T)$ is a an $\aleph_1$-tree. Recall that for all $f \in \mathcal{B}(\mathbf{T})$ and $\alpha < \omega_1$, $f \restriction \alpha$ denotes the unique element of T_α that belongs to f. Likewise, for every $t \in T$, and $\alpha \leq \mathrm{ht}(t)$, we denote by $t \restriction \alpha$ the unique element of T_α that is $<_T$-comparable with t.

Now, for every limit ordinal $\beta < \omega_1$, fix a surjection $\phi_\beta : \omega \to T_{\beta+\omega}$, and then for every $n < \omega$, let $t_{\beta+n} := \phi_\beta(n) \restriction (\beta + n)$. To see that $\langle t_\alpha \mid \alpha < \omega_1\rangle$

is as sought, let $f \in \mathcal{B}(T)$. For every limit $\beta < \omega_1$, as $f \restriction (\beta + \omega)$ is in T, we may find some $n < \omega$ such that $\phi_\beta(n) = f \restriction (\beta + \omega)$. Consequently,

$$f \restriction (\beta + n) = \phi_\beta(n) \restriction (\beta + n) = t_{\beta+n}.$$

It follows that for some stationary $B \subseteq \omega_1$ and some $n' < \omega$, $G(f) \supseteq \{\beta + n' \mid \beta \in B\}$. In particular, $G(f)$ is uncountable. □

In [Kun80, Theorem II.7.14], Kunen proved that $\diamondsuit^-$ is equivalent to $\diamondsuit$. This means that if there exists a sequence $\langle T_\alpha \mid \alpha < \omega_1 \rangle \in \prod_{\alpha<\omega_1} [{}^\alpha 2]^{\aleph_0}$ such that, for every function $f : \omega_1 \to 2$, the set $\{\alpha < \omega_1 \mid f \restriction \alpha \in T_\alpha\}$ is stationary, then there exists a sequence $\langle t_\alpha \mid \alpha < \omega_1 \rangle \in \prod_{\alpha<\omega_1} {}^\alpha 2$ such that, for every function $f : \omega_1 \to 2$, the set $\{\alpha < \omega_1 \mid f \restriction \alpha = t_\alpha\}$ is stationary.

In contrast, if $\mathbf{T}$ is an $\aleph_1$-tree, then for every $\alpha < \omega_1$, T_α is a countable subset of ${}^\alpha 2$ and for every $f \in \mathcal{B}(\mathbf{T})$, the set $\{\alpha < \omega_1 \mid f \restriction \alpha \in T_\alpha\}$ is whole of ω_1. By Theorem C, this shows that in our context, guessing correctly modulo a countable error is not equivalent to guessing correctly.

2.4. **Additional ways to get diamond.** Kunen proved that $\diamondsuit$ cannot be introduced by a ccc forcing, whereas Proposition 2.5 demonstrates that this is not the case with $\diamondsuit(T)$. Another small ccc forcing that adds a diamond sequence for every ground model Kurepa tree is Cohen's forcing $\mathrm{Add}(\omega, 1)$. The core of this fact can be restated combinatorially, as follows.

Proposition 2.8. *Suppose that $\mathbf{T}$ is a Kurepa tree. If $\mathrm{cov}(\mathcal{M}) > |\mathcal{B}(\mathbf{T})|$,*[4] *then $\diamondsuit(\mathbf{T})$ holds.*

Proof. For every $\alpha < \omega_1$, fix an enumeration $\langle t^n_\alpha \mid n < \omega \rangle$ of T_α. Fix a partition $\langle S_i \mid i < \omega \rangle$ of ω_1 into stationary sets, and let $\pi : \omega_1 \to \omega$ be such that $\pi[S_i] = \{i\}$ for every $i < \omega$. For each $f \in \mathcal{B}(\mathbf{T})$, define a map $r_f : \omega \to \omega$ via:

$$r_f(i) := \min\{n < \omega \mid \{\alpha \in S_i \mid f \restriction \alpha = t^n_\alpha\} \text{ is stationary}\}.$$

Finally, assuming $\mathrm{cov}(\mathcal{M}) > |\mathcal{B}(\mathbf{T})|$, by [BJ95, Lemma 2.4.2], we may fix a function $g : \omega \to \omega$ such that $r_f \cap g \neq \emptyset$ for every $f \in \mathcal{B}(\mathbf{T})$. Then the transversal $\langle t^{g(\pi(\alpha))}_\alpha \mid \alpha < \omega_1 \rangle$ witnesses $\diamondsuit(\mathbf{T})$. □

Remark 2.9. $\diamondsuit^+$ implies that $\diamondsuit(\mathbf{T})$ holds for some Kurepa tree $\mathbf{T}$ with $\mathrm{cov}(\mathcal{M}) < |\mathcal{B}(\mathbf{T})|$. In Corollary 3.5 below, we get a model in which $\diamondsuit(\mathbf{T})$ holds for some Kurepa tree $\mathbf{T}$ with $\mathrm{cov}(\mathcal{M}) = |\mathcal{B}(\mathbf{T})|$.

Remark 2.10. A proof similar to that of Proposition 2.8 shows that $\diamondsuit(\mathbf{T})$ holds for every Kurepa tree $\mathbf{T}$ with $\mathfrak{d}_{\omega_1} > |\mathcal{B}(\mathbf{T})|$.

Our next task is proving Theorem A. For this, let us recall the following definition.

Definition 2.11. A tree $\mathbf{T} = (T, <_T)$ of height ω_1 is:

- a *weak Kurepa tree* iff $|T| = \aleph_1 < |\mathcal{B}(\mathbf{T})|$;

[4] That is, if the real line cannot be covered by $|\mathcal{B}(\mathbf{T})|$-many meager sets.

- *thick* iff $|\mathcal{B}(\mathbf{T})| = 2^{\aleph_1}$.

As mentioned earlier, CH implies that $({}^{<\omega_1}2, \subsetneq)$ is a thick weak Kurepa tree.

Proposition 2.12. *Suppose that* $\mathbf{T}$ *is a Kurepa tree,* $\mathbf{S}$ *is a weak Kurepa tree, and* $|\mathcal{B}(\mathbf{T})| < |\mathcal{B}(\mathbf{S})|$*. Then* $\diamondsuit(\mathbf{T})$ *holds.*

Proof. By [IR24, Corollary 7.18], if κ is a regular uncountable cardinal and there is a Kurepa tree with (at least) κ-many branches, then $\mathsf{onto}(\{\aleph_1\}, J^{\mathrm{bd}}[\kappa], \aleph_0)$ holds. The same proof works equally well against weak Kurepa trees. Therefore, $\kappa := |\mathcal{B}(\mathbf{T})|^+$ is a regular uncountable cardinal such that $\mathsf{onto}(\{\aleph_1\}, J^{\mathrm{bd}}[\kappa], \aleph_0)$ holds. This means that we may fix a map $c : \omega_1 \times \kappa \to \omega$ such that for every $B \in [\kappa]^\kappa$, there exists $i < \omega_1$ such that $c[\{i\} \times B] = \omega$. For each $\beta < \kappa$, define $g_\beta : \omega_1 \to \omega$ via $g_\beta(i) := c(i, \beta)$. It is easy to check that for every $\mathcal{R} \in [{}^{\omega_1}\omega]^{<\kappa}$, there exists some $\beta < \kappa$ such that $r \cap g_\beta \neq \emptyset$ for every $r \in \mathcal{R}$. From this point on, the proof continues the same way as that of Proposition 2.8, where the only change is that we fix a partition $\langle S_i \mid i < \omega_1 \rangle$ into $\aleph_1$-many stationary sets and so each of the r_f's is now a function from ω_1 to ω. □

Corollary 2.13. CH *implies* $\diamondsuit(\mathbf{T})$ *for every non-thick Kurepa tree* $\mathbf{T}$. □

3. Indestructible diamonds

We would like to show that if the universe is close to being constructible, then there exists a Kurepa tree on which $\diamondsuit$ holds. The construction of such a tree relies on the concept of a *sealed* Kurepa tree:

Definition 3.1 (Hayut-Müller, [HM23])**.** A Kurepa tree $\mathbf{T}$ is *sealed* if for every notion of forcing $\mathbb{P}$ that preserves both ω_1 and ω_2, $\mathbb{P}$ does not add a new branch through $\mathbf{T}$.

Fact 3.2 (Poór-Shelah, [PS21, §4])**.** *If* $\omega_1 = \omega_1^{\mathsf{L}[A]}$ *and* $\omega_2 = \omega_2^{\mathsf{L}[A]}$ *for some* $A \subseteq \omega_1$*, then there exists a Kurepa tree* $\mathbf{T}$ *such that* $\mathcal{B}(\mathbf{T}) \subseteq \mathsf{L}[A]$*.*[5]

Corollary 3.3 (Giron-Hayut, [GH23])**.** *If* $\omega_1 = \omega_1^{\mathsf{L}[A]}$ *and* $\omega_2 = \omega_2^{\mathsf{L}[A]}$ *for some* $A \subseteq \omega_1$*, then there exists a sealed Kurepa tree.* □

Corollary 3.4. *Suppose that* $V = \mathsf{L}[A]$ *for some* $A \subseteq \omega_1$*. Then there exists a binary Kurepa tree* T *such that* $\diamondsuit(T)$ *holds in any forcing extension preserving* ω_1*,* ω_2*, and the stationary subsets of* ω_1*.*

Proof. Let $\mathbf{T}$ be a tree as in Fact 3.2. It can be verified that $\mathbf{T}$ is Hausdorff, but, regardless, as described right after Lemma 2.1, there is a Hausdorff tree $\mathbf{S}$ and an order-preserving injection π from $\mathbf{T}$ to $\mathbf{S}$ such that $\pi[T_\alpha] = S_{\alpha+1}$ for every $\alpha < \omega_1$. In particular, π induces a bijective correspondence between $\mathcal{B}(\mathbf{T})$ and $\mathcal{B}(\mathbf{S})$, meaning that we may as well assume that $\mathbf{T}$ is Hausdorff.

[5]This was generalized by Hayut and Müller [HM23, Lemma 15] to any successor of a regular uncountable cardinal κ such that $\kappa^+ = (\kappa^+)^{\mathsf{L}}$.

Next, as $A \subseteq \omega_1$, it is the case that $\diamondsuit$ holds in $\mathsf{L}[A]$, so $\diamondsuit(\mathbf{T})$ holds as well. Suppose $\mathbb{P}$ is a notion of forcing preserving ω_1 and ω_2. The construction of $\mathbf{T}$ takes place inside $\mathsf{L}[A]$, so

$$V^{\mathbb{P}} \models \mathcal{B}(\mathbf{T}) \subseteq \mathsf{L}[A].$$

As the definition of $\mathsf{L}[A]$ is absolute, all the branches through $\mathbf{T}$ from $V^{\mathbb{P}}$ are already in V. Now, if $\mathbb{P}$ also preserves stationary subsets of ω_1, then the witness to $\diamondsuit(\mathbf{T})$ in V will still witness $\diamondsuit(\mathbf{T})$ in $V^{\mathbb{P}}$. Finally, by running the translation procedure of Lemma 2.1, we obtain a binary Kurepa tree with the same key features as the Hausdorff tree $\mathbf{T}$. □

Corollary 3.5. *It is consistent that $2^{\aleph_0} = \aleph_2$, Martin's axiom holds, and $\diamondsuit(T)$ holds for some binary Kurepa tree T.*

Proof. Work in L. By Corollary 3.4, we may fix a binary Kurepa tree T such that $\diamondsuit(T)$ holds in any forcing extension preserving ω_1, ω_2, and the stationary subsets of ω_1. Now, let $\mathbb{P}$ be some ccc notion of forcing such that in $\mathsf{L}^{\mathbb{P}}$, Martin's axiom holds and $2^{\aleph_0} = \aleph_2$. As $\mathbb{P}$ preserves ω_1, ω_2, and the stationary subsets of ω_1, $\diamondsuit(T)$ holds in $\mathsf{L}^{\mathbb{P}}$. □

A Kurepa tree that is sealed for all proper forcings can also be added by forcing, and in fact such a notion of forcing was already devised by D. H. Stewart in his 1966 Master's thesis.

Definition 3.6 (Stewart (see [Jec71]))**.** For a cardinal $\kappa > \aleph_1$, the forcing $\mathbb{S}_\kappa$ consisting of all triples (T_p, ϵ_p, b_p) such that:

(1) $\epsilon_p \in \mathrm{acc}(\omega_1)$;
(2) T_p is a countable downward-closed subfamily of ${}^{\leq \epsilon_p}2$ such that $(T_p, \subsetneq)$ is a normal tree of height $\epsilon_p + 1$;
(3) b_p is an injection from a countable subset of κ to the top level of T_p,

and the ordering is given by $q \leq p$ iff

- $\epsilon_q \geq \epsilon_p$;
- $T_q \supseteq T_p$ and $T_q \restriction (\epsilon_p + 1) = T_p$;
- $\mathrm{dom}(b_q) \supseteq \mathrm{dom}(b_p)$;
- for every $\xi \in \mathrm{dom}(b_p)$, $b_q(\xi) \supseteq b_p(\xi)$.

Note that $\mathbb{S}_\kappa$ is σ-closed, and that, assuming CH, it has the $\aleph_2$-cc. The following establishes Theorem B.

Proposition 3.7. *Suppose $\kappa > \aleph_1$ is some cardinal. In the forcing extension by $\mathbb{S}_\kappa$, there exists a binary Kurepa tree T such that $\diamondsuit(T)$ holds and no proper forcing adds a new branch through T, let alone kills diamond over it.*

Proof. Suppose G is $\mathbb{S}_\kappa$-generic. Let $T := \bigcup\{T_p \mid p \in G\}$ denote the generic binary tree added by $\mathbb{S}_\kappa$. For each $\xi < \kappa$, let

$$f_\xi := \{t \in T \mid \exists p \in G\,(\xi \in \mathrm{dom}(b_p) \wedge t \subseteq b_p(\xi)\}.$$

A density argument shows that T is an $\aleph_1$-tree and that $\langle f_\xi \mid \xi < \kappa\rangle$ is an injective sequence of branches through it. By a theorem of Baumgartner, any σ-closed forcing that adds a new subset of ω_1 forces $\diamondsuit$ to hold (see [Sak06, Theorem 3.1]), hence $\diamondsuit(T)$ hold in $V[G]$.

In [Jec71, p. 10], Jech proved that $\{f_\xi \mid \xi < \kappa\}$ enumerates *all* cofinal branches through T. It is a folklore fact that this remains the case in any extension by a proper notion of forcing. Hints of this fact may be found in [Tod84, Lemma 8.14] and [Kos05, Proposition 37], but for completeness, we give the details here. Following [Kos05, Definition 1], we consider the following set:

$$\mathcal{S} := \{X \in [\kappa]^{\aleph_0} \mid X \cap \omega_1 \in \mathrm{acc}(\omega_1),\ T_{X\cap\omega_1} = \{f_\xi \restriction (X \cap \omega_1) \mid \xi \in X\}\}.$$

Claim 3.7.1. *$\mathcal{S}$ is stationary.*

Proof. Back in V, pick a name $\dot{C}$ for a club in $[\kappa]^{\aleph_0}$, and a condition p. Recursively construct a sequence $\langle (p_n, X_n) \mid n < \omega\rangle$ such that $(p_0, X_0) := (p, \omega)$ and, for every $n < \omega$:

- $p_{n+1} \leq p_n$;
- $\epsilon_{p_{n+1}} > \epsilon_{p_n}$;
- $\mathrm{dom}(b_{p_{n+1}}) \supseteq X_n$;
- every element of T_{p_n} is extended by some element of $\mathrm{Im}(b_{p_{n+1}})$;
- $X_{n+1} \in [\kappa]^{\aleph_0}$ with $X_{n+1} \supseteq X_n \cup \mathrm{dom}(b_{p_n}) \cup (\sup(X_n \cap \omega_1) + 1)$;
- $p_{n+1} \Vdash X_{n+1} \in \dot{C}$.

Set $X := \bigcup_{n<\omega} X_n$ and note that $X \cap \omega_1 \in \mathrm{acc}(\omega_1)$. Define a condition q by letting:

(1) $\mathrm{dom}(b_q) := \bigcup_{n<\omega} X_n$;
(2) for all $n < \omega$ and $\xi \in X_n$, $b_q(\xi) := \bigcup_{n<m<\omega} b_{p_m}(\xi)$;
(3) $\epsilon_q := \sup_{n<\omega} \epsilon_{p_n}$;
(4) $T_q := \bigcup_{n<\omega} T_{p_n} \cup \mathrm{Im}(b_q)$.

It is clear that $q \leq p_n$ for all $n < \omega$, and hence $q \Vdash X \in \dot{C}$. In addition, Clause (4) implies that $q \Vdash X \in \dot{S}$. □

Now, let $\mathbb{Q}$ be any proper forcing in $V[G]$, and work in $V[G][H]$, where H is $\mathbb{Q}$-generic. Towards a contradiction, suppose that f is a branch through T distinct from any of the f_ξ's. As $\mathbb{Q}$ is proper, $\mathcal{S}$ remains stationary, so we may fix an elementary submodel $M \prec \mathrm{H}_\theta$ (for a large enough regular cardinal θ) containing $\{T, f, \langle f_\xi \mid \xi < \kappa\rangle\}$ such that $X := M \cap \kappa$ is in $\mathcal{S}$. By elementarity,

$$M \models \forall \xi < \kappa\, (f \neq f_\xi).$$

Denote $\alpha := X \cap \omega_1$. As f is a branch through T, we get that $f \restriction \alpha \in T_\alpha$. As $X \in \mathcal{S}$, we may find a $\xi \in X$ such that $f \restriction \alpha = f_\xi \restriction \alpha$. But $\xi \in X = M \cap \kappa$, and then, by elementarity,

$$M \models f = f_\xi.$$

This is a contradiction. □

Despite the fact that the generic tree added by $\mathbb{S}_\kappa$ is sealed for proper forcings, we can still add new branches to it without collapsing cardinals or adding reals using the quotient forcing $\mathbb{S}_{\kappa^+}/\mathbb{S}_\kappa$. More generally, consider the countable support iteration $(\langle \mathbb{P}_\xi \mid \xi \leq \kappa\rangle, \langle \dot{\mathbb{Q}}_\xi \mid \xi < \kappa\rangle)$, for an uncountable cardinal κ, where $\mathbb{Q}_0$ is the Jech partial order for adding a Souslin tree T, and, for every nonzero $\xi < \kappa$,

$$\mathbb{P}_\xi \Vdash \dot{\mathbb{Q}}_\xi = T.$$

By Proposition 3.7 and the upcoming proposition, $\mathbb{P}_\kappa$ is proper (even σ-strategically closed) for any choice of κ, and yet the quotient forcings of the form $\mathbb{P}_\mu/\mathbb{P}_\lambda$ are not proper, whenever $\aleph_2 \leq \lambda < \mu$.

Proposition 3.8. *$\mathbb{P}_\kappa$ has a dense subset that is isomorphic to $\mathbb{S}_\kappa$.*

Proof. Let R denote the collection of all *rectangular* conditions in $\mathbb{P}_\kappa$, i.e, the collection of all conditions q for which there exists some $\delta < \omega_1$ such that:

- $q(0)$ is a tree of height $\delta + 1$,
- for every $\xi \in \kappa \setminus \{0\}$, $q(\xi)$ is either trivial or a node at the δ^{th}-level of $q(0)$,
- all maximal nodes of $q(0)$ which are of the form $q(\xi)$, for $\xi \in \kappa \setminus \{0\}$, are pairwise distinct.

To see that R is isomorphic to $\mathbb{S}_\kappa$, note that we can map a condition $q \in R$ to a condition $\phi(q)$ in $\mathbb{S}_\kappa$, by keeping the tree coordinate, and using the function $b_{\phi(q)}$ to record all maximal nodes of $q(0)$ that are of the form $q(\xi)$. More precisely, we put:

- $T_{\phi(q)}(0) := q(0)$,
- $\epsilon_{\phi(q)} := \mathrm{ht}(q(0)) - 1$,
- $b_{\phi(q)}(\xi) := q(\xi)$, whenever $q(\xi)$ belongs to the top level of $q(0)$.

This mapping is clearly order-preserving, with the image

$$\{p \in \mathbb{S}_\kappa \mid 0 \notin \mathrm{dom}(b_p)\},$$

which is obviously isomorphic to $\mathbb{S}_\kappa$. Moreover, it is straightforward to define an inverse of ϕ.

It remains to show that R is dense. Let p be an arbitrary condition in $\mathbb{P}_\kappa$. Fix a countable $M \prec \mathrm{H}_\theta$, for a sufficiently large regular cardinal θ, containing all relevant objects. Let $\delta := \omega_1 \cap M$, and let $G \subseteq P \cap M$ be a $\mathbb{P}_\kappa$-generic filter over M, containing p. For $\xi \in \kappa \cap M$, let us denote by $G(\xi)$ the projection of G to the coordinate ξ. Therefore $\bigcup G(0)$ is a tree of height δ, with countable levels, and for each $\xi \in \kappa \cap M \setminus \{0\}$, $G(\xi)$ determines a cofinal branch f_ξ through $G(0)$, given by the formula

$$f_\xi := \bigcup \{t \in {}^{<\omega_1}2 \mid \exists r \in G\, (r \restriction \xi \Vdash r(\xi) = t)\}.$$

The latter follows from the observation that for any $\epsilon < \delta$, the set

$$D_{\xi,\epsilon} := \{r \in \mathbb{P}_\kappa \mid \exists t \in {}^{<\omega_1}2\, (r \restriction \xi \Vdash r(\xi) = t \wedge \mathrm{dom}(t) \geq \epsilon)\},$$

is dense in $\mathbb{P}_\kappa$, and belongs to M.

Finally, we define a condition q by the following considerations:

- $q(0) := G(0) \cup \{f_\xi \mid \xi \in M \cap \kappa\}$,
- $q(\xi) := f_\xi$ for every $\xi \in M \cap \kappa \setminus \{0\}$,
- $q(\xi) := \emptyset$ for every $\xi \in \kappa \setminus M$.

Then q is a rectangular condition extending p, as sought. $\square$

4. The failure of diamond on a Kurepa tree

The main result of this section is a consistent example of a binary Kurepa tree on which diamond fails. At the end of this section, we shall derive a few additional corollaries.

As a first step, we present a notion of forcing with side conditions for adding a branch through a given normal binary $\aleph_1$-tree T. The poset takes a transversal $\vec{t}$ of the full binary tree $({}^{<\omega_1}2, \subsetneq)$ as a second parameter, and ensures that the generic branch will disagree on a club with this transversal.

Definition 4.1. Let T be a normal binary $\aleph_1$-tree, and let $\vec{t} \in \prod_{\alpha<\omega_1} {}^{\alpha}2$.

The forcing $\mathbb{Q}(T, \vec{t})$ consists of all triples $p = (x_p, \mathcal{M}_p, f_p)$ that satisfy all of the following:

1. x_p is a node in T;
2. $\mathcal{M}_p$ is a finite, $\in$-increasing chain of countable elementary submodels of H_{ω_1};
3. f_p is a partial function from $\mathcal{M}_p$ to ω_1;
4. for every $M \in \mathcal{M}_p$:
   - $\mathrm{dom}(x_p) \geq M \cap \omega_1$,
   - $x_p \restriction (M \cap \omega_1) \neq \vec{t}(M \cap \omega_1)$;
   - $f_p \restriction M \in M$.

The ordering is defined by letting $q \leq p$ iff

1. $x_q \supseteq x_p$;
2. $\mathcal{M}_q \supseteq \mathcal{M}_p$;
3. for every $M \in \mathrm{dom}(f_p)$, $M \in \mathrm{dom}(f_q)$ and $f_q(M) \geq f_p(M)$.

Remark 4.2. $\mathbb{Q}(T, \vec{t}) \subseteq \mathrm{H}_{\omega_1}$, so that $\mathbb{Q}(T, \vec{t}) \in \mathrm{H}_{\omega_2}$.

Lemma 4.3. *Suppose that T is a normal binary $\aleph_1$-Souslin tree, and $\vec{t} \in \prod_{\alpha<\omega_1} {}^{\alpha}2$. Let $p \in \mathbb{Q}(T, \vec{t})$. Then for every countable $M^* \prec \mathrm{H}_{\omega_2}$ with $\mathbb{Q}(T, \vec{t}) \in M^*$ such that $M^* \cap \mathrm{H}_{\omega_1} \in \mathcal{M}_p$, p is M^*-generic.*

Proof. Fix a model M^* as above, and let $D \in M^*$ be a dense open subset of $\mathbb{Q}(T, \vec{t})$; we need to find an $r \in D \cap M^*$ compatible with p.

Fix a large enough $\gamma \in M^* \cap \omega_1$ such that $N \cap \omega_1 \subseteq \gamma$ for every $N \in \mathcal{M}_p \cap M^*$. Define a condition

$$\bar{p} := (x_p \restriction \gamma, \mathcal{M}_p \cap M^*, f_p \restriction M^*),$$

and note that $\bar{p} \in M^*$.

Claim 4.3.1. *The set $D' := \{x_q \mid q \in D,\ q \leq \bar{p}\}$ is dense in $(T, \supseteq)$ below $x_{\bar{p}}$.*

Proof. Let y be any extension of $x_{\bar{p}}$. Evidently, $(y, \mathcal{M}_{\bar{p}}, f_{\bar{p}})$ is a legitimate condition. Pick $q \in D$ such that $q \leq (y, \mathcal{M}_{\bar{p}}, f_{\bar{p}})$. Then x_q is an extension of y that lies in D', as sought. $\square$

Denote $\delta := M^* \cap \omega_1$. Since T is a Souslin tree lying in M^*, any node in T_δ is $(T, \supseteq)$-generic over M^*. In particular, $x_p \restriction \delta$ is $(T, \supseteq)$-generic over M^*, and it follows that there is an $x \in D' \cap M^*$ such that

$$x \subseteq x_p \restriction \delta.$$

Now, pick any q witnessing $x \in D'$, so that $x = x_q \subseteq x_p$. By elementarity, we can assume that $q \in M^*$.

Define a condition r by letting $x_r := x_p$, $\mathcal{M}_r := \mathcal{M}_p \cup \mathcal{M}_q$, and f_r be such that $\mathrm{dom}(f_r) = \mathrm{dom}(f_p) \cup \mathrm{dom}(f_q)$ and

$$f_r(\alpha) := \max(\{f_p(\alpha) \mid \alpha \in \mathrm{dom}(f_p)\} \cup \{f_q(\alpha) \mid \alpha \in \mathrm{dom}(f_q)\}).$$

It is not hard to see that r is a legitimate condition extending both p and q. $\square$

Lemma 4.4. *Suppose that T is a normal binary $\aleph_1$-Souslin tree, and $\vec{t} \in \prod_{\alpha<\omega_1} {}^{\alpha}2$. Then:*

1. *$\mathbb{Q}(T, \vec{t})$ is proper;*
2. *$\mathbb{Q}(T, \vec{t})$ adds a branch through T that evades $\vec{t}$ on a club.*

Proof.
(1) To verify properness, fix $p \in \mathbb{Q}(T, \vec{t})$, and a countable $M^* \prec \mathrm{H}_{\omega_2}$, containing everything relevant including p. Denote $\delta := M^* \cap \omega_1$. As T is normal and Souslin, we may be able to extend x_p to some node $x \in T_\delta$ distinct from $\vec{t}(\delta)$.

The triple $(x, \mathcal{M}_p \cup \{M^* \cap \mathrm{H}_{\omega_1}\}, f_p)$ is a condition in $\mathbb{Q}(T, \vec{t})$, and by Lemma 4.3, it is an M^*-generic condition.

(2) Let G be a generic filer. It suffices to verify that the following uncountable set is moreover closed:

$$D := \{M \cap \omega_1 \mid \exists p \in G\,(M \in \mathcal{M}_p)\},$$

that is, every $\gamma \in \kappa \setminus D$ is not an accumulation point of D.

Let $\dot{D}$ be a name for the above set, and suppose $p \Vdash \dot{\gamma} \in \kappa \setminus \dot{D}$. By extending p, we can assume that it forces that $\dot{\gamma} = \gamma$, and γ lies between two consecutive elements $N_0 \in N_1$ of $\mathcal{M}_p$. By extending p further, we can assume that $f_p(N_0) \geq \gamma$. Since γ is forced to not be the height of any element of $\mathcal{M}_p$, it follows that $N_0 \cap \omega_1 < \gamma$, and

$$p \Vdash \dot{D} \cap (N_0 \cap \omega_1, \gamma) = \emptyset,$$

as sought. $\square$

Definition 4.5. Given a sequence of binary $\aleph_1$-trees $T^0, \ldots, T^{n+1}$, the tree product $T^0 \otimes \cdots \otimes T^n$ is the collection of all $(x_0, \ldots, x_n) \in T^0 \times \cdots \times T^n$ such that $\mathrm{dom}(x_0) = \cdots = \mathrm{dom}(x_n)$, and the ordering is such that a node $(x_0, \ldots, x_n)$ is below a node $(y_0, \ldots, y_n)$ iff $x_i \subseteq y_i$ for all $i \leq n$.

In our context, it will be useful to know of the following fact.

Fact 4.6 ([BRY24, §5]). *If $\diamondsuit^+$ holds, then there is a binary $\aleph_1$-Aronszajn tree T and a sequence $\vec{T} = \langle T^\eta \mid \eta < \omega_2\rangle$ of normal binary $\aleph_1$-subtrees of T such that, for every nonempty $a \in [\omega_2]^{<\omega}$, the tree product $\bigotimes_{\eta\in a} T^\eta$ is an $\aleph_1$-Souslin tree.*

It is well-known that if the product $T \otimes S$ of two normal binary $\aleph_1$-trees is Souslin, then

$$(T, \supseteq) \Vdash \text{“}S \text{ is Souslin”}.$$

Thus, it is natural to expect that furthermore

$$\mathbb{Q}(T, \vec{t}) \Vdash \text{“}S \text{ is Souslin”}.$$

The next lemma shows that this is indeed the case.

Lemma 4.7. *Suppose T and S are normal binary $\aleph_1$-trees the product of which is Souslin, and let $\vec{t} \in \prod_{\alpha<\omega_1} {}^{\alpha}2$. Then*

$$\mathbb{Q}(T, \vec{t}) \Vdash \text{“}S \textit{ is Souslin”}.$$

Proof. Let $q \in \mathbb{Q}(T, \vec{t})$, and fix a $\mathbb{Q}(T, \vec{t})$-name $\dot{A}$ for a maximal antichain in S. Fix a countable $M^* \prec \mathrm{H}_{\omega_2}$ containing $\{\mathbb{Q}(T, \vec{t}), q, \dot{A}\}$. Denote $\delta := M^* \cap \omega_1$, and pick a node $x \in T_\delta \setminus \{\vec{t}(\delta)\}$ extending x_q. We define an extension $q' \leq q$ by letting:

$$q' := (x, \mathcal{M}_q \cup \{M^* \cap \mathrm{H}_{\omega_1}\}, f_q).$$

It is clear that q' is indeed a condition and $q' \leq q$. We claim that

$$q' \Vdash \text{“}\dot{A} \cap (S \restriction \delta) \text{ is a maximal antichain in } S\text{”}.$$

To see this, pick a condition $p \leq q'$ and a node $y \in S_\delta$; our aim is to find a condition $r \leq p$ that forces y to extend an element of $\dot{A}$.

Fix a large enough $\gamma < \delta$ such that $N \cap \omega_1 \subseteq \gamma$ for every $N \in \mathcal{M}_p \cap M^*$. Let $\bar{x} := x_p \restriction \gamma$ and $\bar{y} := y \restriction \gamma$. Define a condition

$$\bar{p} := (\bar{x}, \mathcal{M}_p \cap M^*, f_p \restriction M^*).$$

Now, we turn to run a recursion producing a family

$$\mathcal{E} = \{(q_\alpha, y_\alpha) \mid \alpha < \theta\},$$

satisfying that for each α:

(1) $q_\alpha \in \mathbb{Q}(T, \vec{t})$ with $q_\alpha \leq \bar{p}$;
(2) $y_\alpha \in S$ with $\bar{y} \subseteq y_\alpha$;
(3) $q_\alpha \Vdash \exists a \in \dot{A}\,(a \subseteq y_\alpha)$;
(4) $\mathrm{dom}(x_{q_\alpha}) = \mathrm{dom}(y_\alpha)$;
(5) for all $\beta < \alpha$, $(x_{q_\beta}, y_\beta) \perp (x_{q_\alpha}, y_\alpha)$ in $T \otimes S$.

We continue the construction until it is not possible to choose the next element. Because $T \otimes S$ is Souslin, Requirement (5) ensures that the construction terminates after countably many steps, so the ordinal θ ends up being countable.

Claim 4.7.1. *The set* $\{(x_{q_\alpha}, y_\alpha) \mid \alpha < \theta\}$ *is a maximal antichain above* $(\bar{x}, \bar{y})$ *in* $T \otimes S$.

Proof. Suppose not. Then we may pick $(x', y') \in T \otimes S$ extending $(\bar{x}, \bar{y})$ that is incompatible with (x_{q_α}, y_α) for every $\alpha < \theta$. By possibly extending both x' and y', we may assume that $x' = x_{q'}$ for some condition $q' \leq \bar{p}$ which moreover satisfies:

$$q' \Vdash \exists a \in \dot{A}\,(a \subseteq y').$$

But now we can add (q', y') to $\mathcal{E}$, contradicting its maximality. □

Since $q \in M^*$, by making canonical choices in the recursive construction we may secure that the countable family $\mathcal{E}$ be a subset of M^*. So, by Claim 4.7.1, we may find an $\alpha < \theta$ such that ($x_{q_\alpha} \subseteq x$ and $y_\alpha \subseteq y$). Since $\mathcal{E} \subseteq M^*$, we infer that $q_\alpha \in M^*$. Now, as in the proof of Lemma 4.3, we see that there exists a condition r extending both p and q_α. Recalling Requirement (3) in the choice of q_α, we get that

$$r \Vdash \exists a \in \dot{A}\,(a \subseteq y),$$

as sought. □

We are now in conditions to prove the core of Theorem C.

Theorem 4.8. *It is consistent that there exists a binary Kurepa tree* T *such that* $\diamondsuit(T)$ *fails.*

Proof. We start with a model of GCH and $\diamondsuit^+$. Using Fact 4.6, fix a binary $\aleph_1$-tree T and a sequence $\vec{T} = \langle T^\xi \mid \xi < \omega_2\rangle$ of normal binary $\aleph_1$-subtrees of T such that $\bigotimes_{\xi \in a} T^\xi$ is an $\aleph_1$-Souslin tree for every nonempty $a \in [\omega_2]^{<\omega}$.[6] Note that for all $\xi \neq \eta$, the trees T^ξ and T^η have a countable intersection.

Let $\mathcal{P}$ denote the collection of all pairs $(\mathbb{R}, \tau)$ such that $\mathbb{R}$ is a notion of forcing lying in H_{ω_2}, and τ is a nice $\mathbb{R}$-name for an element of $\prod_{\alpha<\omega_1} T_\alpha$. Using GCH, we may fix a (repetitive) enumeration $\langle(\mathbb{R}_\xi, \tau_\xi) \mid \xi < \omega_2\rangle$ of $\mathcal{P}$ in such a way that every pair is listed cofinally often.

Finally, we force with $\mathbb{P}_{\omega_2}$, where $(\langle\mathbb{P}_\xi \mid \xi \leq \omega_2\rangle, \langle\dot{\mathbb{Q}}_\xi \mid \xi < \omega_2\rangle)$ is the countable support iteration satisfying that for every $\xi < \omega_2$:

(i) $\mathbb{P}_\xi \Vdash$ "$\dot{\mathbb{Q}}_\xi = \dot{\mathbb{Q}}(T^\xi, \sigma_\xi)$", and
(ii) σ_ξ is a nice $\mathbb{P}_\xi$-name for an element of $\prod_{\alpha<\omega_1} T_\alpha$ such that if $\mathbb{R}_\xi = \mathbb{P}_\eta$ for some $\eta \leq \xi$, then σ_ξ is the lift of τ_ξ (from a $\mathbb{P}_\eta$-name to a $\mathbb{P}_\xi$-name).

Recalling Remark 4.2, we infer that $(\mathbb{P}_\xi, \tau_\xi) \in \mathcal{P}$ for all $\xi < \omega_2$.

Claim 4.8.1. *For every* $\xi < \omega_2$, $\mathbb{P}_\xi$ *forces that* $\dot{\mathbb{Q}}(T^\xi, \sigma_\xi)$ *is proper.*

[6]The tree T given by the fact is moreover Aronszajn, but this feature is not necessary for our application here.

Proof. By Lemma 4.4(1), it suffices to prove that for every $\xi < \omega_2$,
$$\mathbb{P}_\xi \Vdash \text{“}T^\xi \text{ is Souslin”}.$$

We shall prove by induction on $\xi < \omega_2$ a stronger claim, namely that for every finite tuple $\xi < \eta_0 < \ldots < \eta_n < \omega_2$,
$$\mathbb{P}_\xi \Vdash \text{“}T^\xi \otimes T^{\eta_0} \otimes \cdots \otimes T^{\eta_n} \text{ is Souslin”}.$$

There are three cases to consider:

- $\xi = 0$. This is since the sequence $\vec{T}$ was given by Fact 4.6.
- $\xi + 1$. Recall that
$$\mathbb{P}_{\xi+1} \simeq \mathbb{P}_\xi * \dot{\mathbb{Q}}(T^\xi, \sigma_\xi).$$
Let $\xi + 1 < \eta_0 < \ldots < \eta_n < \omega_2$ be a finite tuple. By the induction hypothesis,
$$\mathbb{P}_\xi \Vdash \text{“}T^\xi \otimes T^{\xi+1} \otimes T^{\eta_0} \otimes \cdots \otimes T^{\eta_n} \text{ is Souslin”}.$$
By invoking Lemma 4.7 in $V^{\mathbb{P}_\xi}$, we infer that
$$\mathbb{P}_\xi * \dot{\mathbb{Q}}(T^\xi, \sigma_\xi) \Vdash \text{“}T^{\xi+1} \otimes T^{\eta_0} \otimes \cdots \otimes T^{\eta_n} \text{ is Souslin”},$$
as sought.
- $\xi \in \mathrm{acc}(\omega_2)$. We apply a result from [AS93, §3] or [Miy93] stating that for any $\aleph_1$-tree S, the property “S *is Souslin*” is preserved at a limit stage of a countable support iteration of proper forcings. □

A standard name counting argument shows that CH is preserved in every intermediate stage, so $\mathbb{P}_{\omega_2}$ satisfies the $\aleph_2$-cc.[7] In addition, $\mathbb{P}_{\omega_2}$ is proper, so our forcing preserves all cardinals.

Claim 4.8.2. *$\mathbb{P}_{\omega_2}$ forces that T is Kurepa.*

Proof. For every $\xi < \omega_2$, $\mathbb{P}_{\omega_2}$ projects to a forcing of the form $\mathbb{Q}(T^\xi, \vec{t})$, and then Lemma 4.4(2) implies that in $V^{\mathbb{P}_{\omega_2}}$, there exists a branch f_ξ through T^ξ, and in particular, through T. As the elements of $\langle T^\xi \mid \xi < \omega_2\rangle$ have a pairwise countable intersection, in $V^{\mathbb{P}_{\omega_2}}$, $\langle f_\xi \mid \xi < \omega_2\rangle$ is injective. □

Claim 4.8.3. *$\mathbb{P}_{\omega_2}$ forces that $\diamondsuit(T)$ fails.*

Proof. Otherwise, as $\prod_{\alpha<\omega_1} T_\alpha$ lies in the ground model, we may fix a nice $\mathbb{P}_{\omega_2}$-name $\dot{t}$ for a transversal $\vec{t} \in \prod_{\alpha<\omega_1} T_\alpha$ that witnesses $\diamondsuit(T)$ in the extension. As $\mathbb{P}_{\omega_2}$ has the $\aleph_2$-cc and $\dot{t}$ is a nice name for an $\aleph_1$-sized set, there is a large enough $\eta < \omega_2$ such that all nontrivial conditions appearing in $\dot{t}$ belong to $\mathbb{P}_\eta$. It thus follows that $\vec{t}$ admits a nice $\mathbb{P}_\eta$-name, say, τ. Clearly, $(\mathbb{P}_\eta, \tau) \in \mathcal{P}$, so we may find a large enough $\xi \in [\eta, \omega_2)$ such that $(\mathbb{R}_\xi, \tau_\xi) = (\mathbb{P}_\eta, \tau)$. Recalling Clauses (i) and (ii) in the definition of our iteration, it is the case that $\mathbb{P}_{\xi+1} \simeq \mathbb{P}_\xi * \dot{\mathbb{Q}}(T^\xi, \sigma_\xi)$ where σ_ξ is a $\mathbb{P}_\xi$-name for $\vec{t}$. By Lemma 4.4(2), then, $\mathbb{P}_{\xi+1}$ introduces a branch through T that evades $\vec{t}$ on a club. So $\vec{t}$ cannot witness $\diamondsuit(T)$ in $\mathbb{P}_{\omega_2}$. This is a contradiction. □

[7] Recall [Bau83, Theorem 2.2].

This completes the proof. □

4.1. **Ramifications.** Proposition 2.12 may suggest that the validity of $\diamondsuit(T)$ for a Kurepa tree T only depends on the cardinal $|\mathcal{B}(T)|$. However, the next corollary shows that this is not the case:

Corollary 4.9. *It is consistent that there exist two binary Kurepa trees T' and T such that $|\mathcal{B}(T')| = |\mathcal{B}(T)| = 2^{\aleph_1}$, $\diamondsuit(T')$ holds, but $\diamondsuit(T)$ fails.*

Proof. Work in L. By Corollary 3.4, we may fix a binary Kurepa tree T' such that $\diamondsuit(T')$ holds in any forcing extension preserving ω_1, ω_2, and the stationary subsets of ω_1. As L satisfies GCH and $\diamondsuit^+$, the proof of Theorem 4.8 provides us with an $\aleph_2$-cc proper notion of forcing $\mathbb{P}_{\omega_2} \subseteq \mathrm{H}_{\omega_2}$ that introduces a binary Kurepa tree T on which $\diamondsuit$ fails. Altogether, $\mathsf{L}^{\mathbb{P}_{\omega_2}}$ is a model satisfying the desired configuration. □

Corollary 4.10. *It is consistent that there exists a binary Kurepa tree T such that $\diamondsuit(T)$ fails and T is uniformly homogeneous.*[8]

Proof. The proof is almost identical to that of Theorem 4.8. We start with a model of GCH and $\diamondsuit^+$. Using Fact 4.6, we fix a binary $\aleph_1$-tree T and a sequence $\langle T^\xi \mid \xi < \omega_2\rangle$ of normal binary $\aleph_1$-subtrees of T such that $\bigotimes_{\xi\in a} T^\xi$ is an $\aleph_1$-Souslin tree for every nonempty $a \in [\omega_2]^{<\omega}$. Now, let T' be the collection of all functions $t' \in {}^{<\omega_1}2$ for which there exists some $t \in T$ such that $\mathrm{dom}(t') = \mathrm{dom}(t)$ and $\{\alpha \in \mathrm{dom}(t) \mid t(\alpha) \neq t'(\alpha)\}$ is finite. Then T' is a uniformly homogeneous $\aleph_1$-tree having each of the T^ξ's as a normal binary $\aleph_1$-subtree. So, we can continue with the proof of Theorem 4.8 using T' instead of T. □

Corollary 4.11. *It is consistent that there exists a binary Kurepa tree S such that $\diamondsuit(S)$ fails and S is rigid.*

Proof. The proof is quite close to that of Theorem 4.8. We start with a model of GCH and $\diamondsuit^+$. Instead of using Fact 4.6, we appeal to [BRY24, §5] to obtain a downward closed subfamily $T \subseteq {}^{<\omega_1}\omega$ such that $(T, \subsetneq)$ is an Aronszajn tree, every node in T admits infinitely many immediate successors, and there exists a sequence $\langle T^\xi \mid \xi < \omega_2\rangle$ of normal downward-closed subtrees of T such that $\bigotimes_{\xi\in a} T^\xi$ is an $\aleph_1$-Souslin tree for every nonempty $a \in [\omega_2]^{<\omega}$. We force with $\mathbb{P}_{\omega_2}$, where $(\langle \mathbb{P}_\xi \mid \xi \leq \omega_2\rangle, \langle \dot{\mathbb{Q}}_\xi \mid \xi < \omega_2\rangle)$ is the countable support iteration satisfying that for every $\xi < \omega_2$, $\mathbb{P}_\xi \Vdash$ "$\dot{\mathbb{Q}}_\xi = \dot{\mathbb{Q}}(T^\xi, \sigma_\xi)$", where σ_ξ is a nice $\mathbb{P}_\xi$-name for an element of $\prod_{\alpha<\omega_1} T_\alpha$ obtained from some bookkeeping sequence. This time, a typical transversal $\vec{t} \in \prod_{\alpha<\omega_1} T_\alpha$ is an element of $\prod_{\alpha<\omega_1} {}^\alpha\omega$ instead of $\prod_{\alpha<\omega_1} {}^\alpha 2$, but everything goes through and we end up in a generic extension in which T is a Kurepa tree on which diamond fails.

Next, let S be the binary $\aleph_1$-tree produced by the proof of Lemma 2.1 when fed with the tree $\mathbf{T} := (T, \subsetneq)$. Since $\mathbf{T}$ is Kurepa on which diamond fails, S is a binary Kurepa tree and $\diamondsuit(S)$ fails.

[8]The definition of a *uniformly homogeneous* tree may be found in [BRY24, §4].

Recall that the proof of Lemma 2.1 made use of an injective sequence $\langle r_m \mid m < \omega\rangle$ of functions from ω to 2. For our purpose here, we shall moreover assume that for all $m \neq m'$, it is the case that $\Delta(r_m, r_{m'}) = \min\{m, m'\}$.[9]

To prove that S is rigid, we must establish the following.

Claim 4.11.1. *Suppose that $\pi : S \leftrightarrow S$ is an automorphism of $(S, \subsetneq)$. Then π is the identity map.*

Proof. Suppose not, so that $\pi(s) \neq s$ for some $s \in S$. By the definition of S and as π is order-preserving, it follows that there exists a $t \in T$ such that $\pi(\psi(t)) \neq \psi(t)$. Let $\beta < \omega_1$ be the least for which there exists a $t \in T_\beta$ with $\pi(\psi(t)) \neq \psi(t)$. Clearly, β is a successor ordinal, say, $\beta = \alpha + 1$. Fix $t_0 \neq t_1$ in $T_{\alpha+1}$ such that $\pi(\psi(t_0)) = \psi(t_1)$ and note that the minimality of β implies that $t_0 \restriction \alpha = t_1 \restriction \alpha$, which we hereafter denote by t. Now, since every node in T admits infinitely many immediate successors, we may fix $t_2, t_3 \in T_{\alpha+1}$ such that:

- t_2, t_3 are immediate successors of t,[10]
- $\pi(\psi(t_2)) = \psi(t_3)$, and
- $\min\{\varphi_\alpha(t_2), \varphi_\alpha(t_3)\} > \max\{\varphi_\alpha(t_0), \varphi_\alpha(t_1)\}$.

Recalling the proof of Lemma 2.1, for every $i < 4$, letting $m_i := \varphi_\alpha(t_i)$, it is the case that

$$\psi(t_i) = \psi(t)^\frown r_{m_i}.$$

As $m_0 < m_2$, $\Delta(r_{m_0}, r_{m_2}) = m_0$ and

$$\Delta(\psi(t_0), \psi(t_2)) = \omega \cdot \alpha + m_0.$$

Likewise,

$$\Delta(\pi(\psi(t_0)), \pi(\psi(t_2))) = \Delta(\psi(t_1), \psi(t_3)) = \omega \cdot \alpha + m_1.$$

As $m_0 \neq m_1$, we infer that

$$\Delta(\psi(t_0), \psi(t_2)) \neq \Delta(\pi(\psi(t_0)), \pi(\psi(t_2))),$$

contradicting the fact that π is an automorphism of S. □

This completes the proof. □

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[9] This can easily be arranged by defining $r_m : \omega \to 2$ via $r_m(n) := 1$ iff $n < m$.

[10] Possibly, $t_2 = t_3$.

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Institute of Mathematics of the Czech Academy of Sciences, Czech Republic

Email address: ziemek314@gmail.com

Department of Mathematics, Bar-Ilan University, Ramat-Gan 52900, Israel.
URL: http://www.assafrinot.com

Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Israel and Department of Mathematics, Rutgers University, U.S.A.
URL: http://shelah.logic.at