# ABSTRACT ELEMENTARY CLASSES NEAR $\aleph_1$
# SH88R

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ABSTRACT. We prove, in ZFC, that no $\psi \in \mathbb{L}_{\omega_1,\omega}[\mathbf{Q}]$ have unique models of uncountable cardinality; this confirms the Baldwin conjecture. But we analyze this in more general terms. We introduce and investigate AECs and also versions of limit models, and prove some basic properties like representation by a PC class, for any AEC.

For $\mathsf{PC}_{\aleph_0}$-representable AECs we investigate the conclusion of having not too many non-isomorphic models in $\aleph_1$ and $\aleph_2$, but we have to assume $2^{\aleph_0} < 2^{\aleph_1}$ and even $2^{\aleph_1} < 2^{\aleph_2}$.

**[2024-08-31: Finally done. The only way to track down all the indentation and botched spacing was to read all 8k lines, and I found plenty of other stuff that needs fixing. If your reaction to a line of red text is 'fine as-is,' it's probably a grammatical issue. Just give me an alternate phrasing and I'll patch in something acceptable. In addition to the marked stuff, semicolons are used in types and formulas throughout, but very inconsistently. E.g. $\text{gtp}(-,-,-)$ and $\text{gtp}(-;-;-)$ are used interchangeably, but in other places it seems to be very deliberate. In §4, starting around 4.10, you introduce '$\exists\bar{x} \wedge p(\bar{x})$;' I haven't seen it in any other papers and it doesn't look well-formed. Is this $(\exists\bar{x})p(\bar{x})$, $(\exists\bar{x})\bigwedge_p p(\bar{x})$, or something else entirely?]**

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## § 0. Introduction

In [She75a], proving a conjecture of Baldwin, we show that

$(*)_1$ No $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ has a unique uncountable model up to isomorphism. ($\mathbf{Q}$ here stands for the quantifier $\mathbf{Q}^{\rm car}_{\geq \aleph_1}$, "there are uncountably many.")

by showing that

$(*)_2$ Categoricity (of $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$) in $\aleph_1$ implies the existence of a model of $\psi$ of cardinality $\aleph_2$ (so $\psi$ has $\geq 2$ non-isomorphism models).

Unfortunately, both $(*)_1$ and $(*)_2$ were not proved in ZFC because diamond on $\aleph_1$ was assumed. In [She83a] and [She83b] this set-theoretic assumption was weakened to $2^{\aleph_0} < 2^{\aleph_1}$; here we shall prove it in ZFC (see §3). However, for getting the conclusion from the weaker model-theoretic assumption $\dot{I}(\aleph_1, \psi) < 2^{\aleph_1}$ as in those papers, we still need $2^{\aleph_0} < 2^{\aleph_1}$.

The main result of [She83a], [She83b] was:

$(*)_3$ If $n > 0$, $2^{\aleph_0} < 2^{\aleph_1} < \ldots < 2^{\aleph_n}$, $\psi \in \mathbb{L}_{\omega_1,\omega}$, $1 \leq \dot{I}(\aleph_\ell, \psi) < \mu_{\rm wd}(\aleph_\ell)$ for $1 \leq \ell \leq n$, (where $\mu_{\rm wd}(\aleph_\ell)$ is usually $2^{\aleph_\ell}$ and always $> 2^{\aleph_{\ell-1}}$; see 0.6 below) <u>then</u> $\psi$ has a model of cardinality $\aleph_{n+1}$.

$(*)_4$ If $2^{\aleph_0} < 2^{\aleph_1} < \ldots < 2^{\aleph_n} < 2^{\aleph_{n+1}} < \ldots$, $\psi \in \mathbb{L}_{\omega_1,\omega}$, and

$$1 \leq \dot{I}(\aleph_\ell, \psi) < \mu_{\rm wd}(\aleph_\ell)$$

for $\ell < \omega$, <u>then</u> $\psi$ has a model in every infinite cardinal (and satisfies Łos' Conjecture).

(Note that $(*)_3$ was proved in [She75a] for $n = 1$, assuming $\diamondsuit_{\aleph_1}$.)

In $(*)_4$, it is proved that without loss of generality $\mathfrak{k}$ is *excellent*; this means, in particular, that $K$ is the class of atomic models of some countable first-order $T$. The point is that an excellent class $\mathfrak{k}$ is similar to the class of models of an $\aleph_0$-stable first-order $T$. In particular, the set of relevant types $\mathbf{S}_{\mathfrak{k}}(A, M)$ is defined as the set of complete types $p(x)$ over $A$ in $M$ (in the first-order sense) such that $p \restriction B$ is isolated for every finite $B \subseteq A$.

<u>However</u>, we'd better restrict ourselves to "nice" $A$; that is, $A$ which are the universe of some $N \prec M$, or $A = N_1 \cup N_2$ where $N_0, N_1, N_2$ are in stable amalgamation, or $\bigcup\{N_u : u \in \mathscr{P} \subseteq \mathcal{P}(n)\}$ for some (so-called) stable system $\langle N_u : u \in \mathscr{P}\rangle$. (On such stable systems, in the stable first-order case, see [She90, XII,§5].)

So types are quite like the first-order case. In particular, we say $M \in \mathfrak{k}$ is $\lambda$-*full* <u>when</u> if $p \in \mathbf{S}_{\mathfrak{k}}(A, M)$ with $A$ as above, $|A| < \lambda$ implies $p$ is realized in $M$; this is the replacement for '$\lambda$-saturated' for that context.

In [She83a] and [She83b], why was $\psi$ assumed to be just in $\mathbb{L}_{\omega_1,\omega}$ and not more generally in $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$? Mainly because we feel that in [She75a], the logic $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ was incidental. We delay the search for the right context to this sequel.

So here we are working in an AEC, (an "abstract elementary class," so no logic is present in the context) which are formally like elementary classes; i.e. $(\mathrm{Mod}_T, \prec)$ with $T$ first-order. Note the absence of amalgamation, but they still have closure under unions of increasing chains. They are of the form $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$, where $\leq_{\mathfrak{k}}$ is the "abstract" notion of elementary submodel. So if $\mathcal{L}$ is a fragment of $\mathbb{L}_{\infty,\omega}(\tau)$ (for a fixed vocabulary), $T \subseteq \mathcal{L}$ a theory included in $\mathcal{L}$, and we let $K \coloneqq \{M : M \models T\}$

and $M \leq_{\mathfrak{k}} N$ if and only if $M \prec_{\mathcal{L}} N$, we get such a class; if $\mathcal{L}$ is countable then $\mathfrak{k}$ has LST number $\aleph_0$.

So the class of models of $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ is not represented directly, but can be with minor adaptation; see 3.19(2). Surprisingly (and by a not-so-hard proof), every AEC $\mathfrak{k}$ can be represented as a pseudo-elementary class <u>if</u> we allow omitting types (see 1.11). We introduce a relative of saturated models (for stable first-order $T$) and full models (for excellent classes, see [She83a] and [She83b]). That is, we are talking about limit models (really, several variants of this notion; see Definition 3.3.)

The strongest and most important variant is "$M \in K_\lambda$ superlimit," which means $M$ is universal (under $\leq_{\mathfrak{k}}$),

$$(\exists N)[M \leq_{\mathfrak{k}} N \wedge M \neq N],$$

and if $\langle M_i : i < \delta \leq \|M\| \rangle$ is $\leq_{\mathfrak{k}}$-increasing with each $M_i \cong M$ then $\bigcup\limits_{i<\delta} M_i \cong M$. If we restrict ourselves to $\delta$-s of cofinality $\kappa$, we get $(\lambda, \kappa)$-superlimit. Such $M$ exists for a first-order $T$ for some pairs $\lambda, \kappa$. In particular,[1]

$(*)_5$ For every $\lambda \geq 2^{|T|} + \beth_\omega$, a superlimit model of $T$ of cardinality $\lambda$ exists if and only if $T$ is superstable (by [She12, 3.1]).

Moreover,

$(*)_6$ "Almost always:" for $\lambda \geq 2^{|T|} + \kappa$ and $\kappa = \mathrm{cf}(\kappa)$ (for simplicity), we have that a $(\lambda, \kappa)$-superlimit model exists iff "$T$ is stable in $\lambda$" $\wedge\ \kappa \geq \kappa(T)$ <u>or</u> $\lambda = \lambda^{<\kappa}$.

But we can prove something under those circumstances: if $K$ is categorical in $\lambda$ (or we just have a superlimit model $M^*$ in $\lambda$, but the $\lambda$-amalgamation property fails for $M^*$) and $2^\lambda < 2^{\lambda^+}$, <u>then</u> $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$ (see 3.9). With some reasonable restrictions on $\lambda$ and $K$, we can prove that (e.g.)

$$\dot{I}(\lambda, K) = \dot{I}(\lambda^+, K) = 1 \Rightarrow \dot{I}(\lambda^{++}, K) \geq 1$$

(see 3.12, 3.14).

However, our long-term main aim was to do the parallel of [She83a] and [She83b] in the present context; i.e. for an AEC $\mathfrak{k}$ (and it is natural to assume $\mathfrak{k}$ is $\mathsf{PC}_{\aleph_0}$). Here we prepare the ground.

Sections 4 and 5 presently work toward this goal (§5 assuming $2^{\aleph_0} < 2^{\aleph_1}$, §4 without it). We should note that dealing with superlimit models rather than full ones causes problems, as well as the fact that the class is not necessarily elementary in some reasonable logics. Because of the second issue we were driven to use formulas which hold "generically", are "forced" instead of are satisfied, say "the type $\bar{a}$ is materialized" instead of realized, and use $\mathrm{gtp}(\bar{a}, N, M)$ instead of $\mathrm{tp}(\bar{a}, N, M)$.

We also (necessarily) encounter the case "$\mathbf{D}(N)$ of cardinality $\aleph_1$ for $N \in K_{\aleph_0}$" (see 5.2, 5.4(6)). Because of the first issue, the scenario for getting a full model in $\aleph_1$ (which can be adapted to $(\aleph_1, \{\aleph_1\})$-superlimit: see 5.18) does not seem to be enough for getting superlimit models in $\aleph_1$ (see 5.45).

We had felt that arriving at enough conclusions on the models of cardinality $\aleph_1$ to start dealing with models of cardinality $\aleph_2$ will be a strong indication that we can complete the generalization of [She83a] and [She83b], so getting superlimits in

[1]See more in [She12].

$\aleph_1$ is the culmination of this paper and a natural stopping point. Trying to do the rest (of the parallel to [She83a] and [She83b]) was delayed.

Much remains to be done.

**Problem 0.1.** 1) Prove $(*)_3, (*)_4$ in our context.

2) Parallel results in ZFC; e.g. prove $(*)_3$ for $n = 1$, $2^{\aleph_0} = 2^{\aleph_1}$.

Note that if $2^{\aleph_0} = 2^{\aleph_1}$, assuming $1 \leq \dot{I}(\aleph_1, K) < 2^{\aleph_1}$ really gives fewer model-theoretic consequences, as new phenomena arise (see §6). See §4 (and its concluding remarks).

3) Construct examples; e.g. (an AEC) $\mathfrak{k}$ (or $\psi \in \mathbb{L}_{\omega_1,\omega}$), categorical in $\aleph_0, \aleph_1, \ldots, \aleph_n$ but not in $\aleph_{n+1}$.

4) If $\mathfrak{k}$ is a $\mathsf{PC}_\lambda$ class, categorical in $\lambda$ and $\lambda^+$, does it necessarily have a model in $\lambda^{++}$?

See the book's introduction [Sheb] on the progress on those problems — in particular in [She01], redone here in [She75b]. The direct motivation for [She01] was that Grossberg asked me (in October 1994) some questions in this neighborhood (mainly 0.1(4)).

In particular:

- $(*)$ Assume $K = \mathrm{Mod}(T)$ (i.e. $K$ is the class of models of $T$), $T \subseteq \mathbb{L}_{\omega_1,\omega}$, $|T| = \lambda$, $I(\lambda, K) = 1$ and $1 \leq I(\lambda^+, K) < 2^{\lambda^+}$. Does it follow that $I(\lambda^{++}, K) > 0$?

We think of this as a test problem, and would much prefer a model-theoretic to a set-theoretic solution. This is closely related to 0.1(4) above and to 3.12 (where we assume categoricity in $\lambda^+$ and do not require $2^\lambda < 2^{\lambda^+}$, but take $\lambda = \aleph_0$ or some similar cases) and 5.30(4) (and see 5.2 and 4.8 on the assumptions) (there we require $2^\lambda < 2^{\lambda^+}$, $1 \leq I(\lambda^+, K) < 2^{\lambda^+}$ and $\lambda = \aleph_0$).

[She01, Problem 0.1] was stated *a posteriori* but is, I think, the real problem. It says:

- $(**)$ Can we have some (not necessarily much) classification theory for reasonable non-first-order classes $\mathfrak{k}$ of models, with no use of even traces of compactness and only mild set-theoretic assumptions?

This is a revised version of [She87] which continues [She83a], [She83b] but do not use them. The paper [She87] and the present chapter relies on [She75a] only when deducing results on $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$; it improves some of its early results and extends the context. The work on [She87] was done in 1977, and a preprint was circulated. Before the paper had appeared, a user-friendly expository article of Makowsky [Mak85] represented, gave background and explained the easy parts of the paper. In [She87] the author has corrected and replaced some proofs and added mainly §6. See more in [S$^+$].

We thank Rami Grossberg for lots of work in the early eighties on previous versions (i.e. [She87]) which improved this paper, and the writing up of an earlier version of §6 and Assaf Hasson on helpful comments in 2002 and Alex Usvyatsov for very careful reading, corrections and comments and Adi Jarden and Alon Siton on help in the final stages.

* * *

On history and background on $\mathbb{L}_{\omega_1,\omega}, \mathbb{L}_{\infty,\omega}$ and the quantifier $\mathbf{Q}$ see [Kei71]. On $(D,\lambda)$-sequence-homogeneous (which 2.2 - 2.5 here has generalized) see Keisler-Morley [KM67]: this is defined in 2.3(5), and 2.5 is from there. Theorem 3.9 is similar to [She83a, 2.7] and [She83b, 6.3].

*Remark* 0.2. On non-splitting (used here in 5.6) see [She71], [She90, Ch.I,Def.2.6,p.11] or [She75a].

We finish §0 by some necessary quotation.

By [Kei70] and [Mor70],

**Claim 0.3.** 1) *Assume that $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ has a model $M$ in which*

$$\{\mathrm{tp}_\Delta(\bar a, \varnothing, M) : \bar a \in M\}$$

*is uncountable, where $\Delta \subseteq \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ is countable. Then $\psi$ has $2^{\aleph_1}$ pairwise non-isomorphic models of cardinality $\aleph_1$. In fact, we can find models $M_\alpha$ of $\psi$ of cardinality $\aleph_1$ for $\alpha < 2^{\aleph_1}$ such that $\{\mathrm{tp}_\Delta(a;\varnothing, M_\alpha) : a \in M_\alpha\}$ are pairwise distinct, where*

$$\mathrm{tp}_\Delta(\bar a, A, M) := \big\{\varphi(\bar x, \bar b) : \varphi(\bar x, \bar y) \in \Delta,\ M \models \varphi[\bar a, \bar b],\ \text{and } \bar b \in {}^{\omega>}A\big\}.$$

2) *If $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$, $\Delta \subseteq \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ is countable, and*

$$\{\mathrm{tp}_\Delta(\bar a, \varnothing, M) : \bar a \in {}^{\omega>}M \text{ and } M \text{ is a model of } \psi\}$$

*is uncountable, then it has cardinality $2^{\aleph_0}$.*

Also note

**Observation 0.4.** *Assume ($\tau$ is a vocabulary and)*

- (a) *$K$ is a family of $\tau$-models of cardinality $\lambda$.*
- (b) *$\mu > \lambda^\kappa$*
- (c) *$\{(M, \bar a) : M \in K \text{ and } \bar a \in {}^\kappa M\}$ has $\geq \mu$ members up to isomorphism.*

*Then $K$ has $\geq \mu$ models up to isomorphism (similarly for $= \mu$).*

*Proof.* See [She78, VIII,1.3] or just check by cardinal arithmetic. $\square_{0.4}$

Furthermore,

**Claim 0.5.** 1) *Assume $\lambda$ is regular uncountable, $M_0$ is a model with countable vocabulary and $T = \mathrm{Th}_{\mathbb{L}}(M_0)$, $<$ a binary predicate from $\tau(T)$ and $(P^{M_0}, <^{M_0}) = (\lambda, <)$. Then every countable model $M$ of $T$ has an end-extension; i.e. $M \prec N$ and $P^M \neq P^N$ and $a \in P^N \wedge b \in P^M \wedge a <^N b \Rightarrow a \in M$.*

2) *Moreover, we can further demand $(P^N, <^N)$ is non-well ordered and we can demand $|P^N| = \aleph_1$ and $(P^N, <^N)$ is $\aleph_1$-like (which means that it has cardinality $\aleph_1$ but every (proper) initial segment has cardinality $< \aleph_1$); and we can demand $N$ is countable.*

3) *Moreover, we can add the demand that in $(P^N, <^N)$ there is a first element in $P^N \setminus P^M$, or that there is* no *first element in $P^N \setminus P^M$.*

*Proof.* 1,2) By Keisler [Kei70].

3) By [She75c], and independently Schmerl [Sch76]. $\square_{0.5}$

By Devlin-Shelah [DS78], and [She98, Ap,§1] (the so-called weak diamond).

**Theorem 0.6.** *Assume that* $2^\lambda < 2^{\lambda^+}$.

1) *There is a normal ideal* $\mathrm{WDmId}_{\lambda^+}$ *on* $\lambda^+$ *(and* $\lambda^+ \notin \mathrm{WDmId}_{\lambda^+}$*, of course — the members are called 'small sets') such that: if* $S \in (\mathrm{WDmId}_{\lambda^+})^+$ *(e.g.,* $S = \lambda^+$*) and* $\mathbf{c} : {}^{\lambda^+ >}(\lambda^+) \to \{0,1\}$*, <u>then</u> there is* $\bar\ell = \langle \ell_\alpha : \alpha < \lambda^+ \rangle \in {}^{\lambda^+}2$ *such that for every* $\eta \in {}^{\lambda^+}(\lambda^+)$ *the set* $\{\delta \in S : \mathbf{c}(\eta \restriction \delta) = \ell_\alpha\}$ *is stationary.*

*We call* $\bar\ell$ *a* weak diamond sequence *(for the colouring* $\mathbf{c}$ *and the stationary set* $S$*).*

2) $\mu_* = \mu_{\mathrm{wd}}(\lambda^+)$*, the cardinal defined by* $(*)$ *below, is* $> 2^\lambda$ *(we do* not *say '*$\geq 2^{\lambda^+}$*!')*

- $(*)$ $(\alpha)$ *If* $\mu < \mu_*$ *and* $\mathbf{c}_\varepsilon$ *for* $\varepsilon < \mu$ *is as above then we can find* $\bar\ell$ *as in part* (1) *for all the* $\mathbf{c}_\varepsilon$*-s simultaneously.*
  - $(\beta)$ $\mu_*$ *is maximal such that clause* $(\alpha)$ *holds.*

3) $\mu_* = \mu_{\mathrm{unif}}(\lambda^+, 2^\lambda)$ *satisfies* $\mu_*^{\aleph_0} = 2^{\lambda^+}$*; and moreover* $\lambda \geq \beth_\omega \Rightarrow \mu_* = 2^\lambda$*, where* $\mu_{\mathrm{unif}}(\lambda^+, \chi)$ *is the first cardinal* $\mu$ *such that we can find* $\langle \mathbf{c}_\alpha : \alpha < \mu \rangle$ *such that:*

- $(a)$ $\mathbf{c}_\alpha$ *is a function from* ${}^{\lambda^+ >}(\lambda^+)$ *to* $\chi$.
- $(b)$ *There is no* $\rho \in {}^{\lambda^+}\chi$ *such that for every* $\alpha < \mu$*, for some* $\eta \in {}^{\lambda^+}(\lambda^+)$*, the set* $\{\delta < \lambda : \mathbf{c}_\alpha(\eta \restriction \delta) \neq \rho(\delta)\}$ *is stationary (so* $\mu_{\mathrm{wd}}(\lambda^+) = \mu_{\mathrm{unif}}(\lambda^+, 2)$*).*

*See more in* [She09b, §0,§9] *and hopefully in* [**?**].

The following are used in §2.

**Definition 0.7.** 1) For a regular uncountable cardinal $\lambda$, let

$$\check{I}[\lambda] = \{S \subseteq \lambda : \text{some pair } (E, \bar a) \text{ witnesses } S \in \check{I}(\lambda)\}$$

(see below).

2) We say that $(E, \bar u)$ is a witness for $S \in \check{I}[\lambda]$ <u>if</u>:

- (A) $E$ is a club of the regular cardinal $\lambda$.
- (B) $\bar u = \langle u_\alpha : \alpha < \lambda \rangle$, $a_\alpha \subseteq \alpha$, and $\beta \in a_\alpha \Rightarrow a_\beta = \beta \cap a_\alpha$.
- (C) For every $\delta \in E \cap S$, $u_\delta$ is an unbounded subset of $\delta$ of order-type $< \delta$ (and $\delta$ is a limit ordinal).

By [She93] and [Shea]:

**Claim 0.8.** *Let* $\lambda$ *be regular uncountable.*

1) *If* $S \in \check{I}[\lambda]$ *<u>then</u> we can find a witness* $(E, \bar a)$ *for* $S \in \check{I}[\lambda]$ *such that:*

- $(a)$ $\delta \in S \cap E \Rightarrow \mathrm{otp}(a_\delta) = \mathrm{cf}(\delta)$
- $(b)$ *If* $\alpha \notin S$ *then* $\mathrm{otp}(a_\alpha) < \mathrm{cf}(\delta)$ *for some* $\delta \in S \cap E$.

2) $S \in \check{I}[\lambda]$ *<u>iff</u> there is a pair* $(E, \overline{\mathscr{P}})$ *such that:*

(a) *$E$ is a club of the regular uncountable $\lambda$.*

(b) *$\overline{\mathscr{P}} = \langle \mathscr{P}_\alpha : \alpha < \lambda \rangle$, where $\mathscr{P}_\alpha \subseteq \mathcal{P}(\alpha)$ has cardinality $< \lambda$.*

(c) *If $\alpha < \beta < \lambda$ and $\alpha \in u \in \mathscr{P}_\beta$ then $u \cap \alpha \in \mathscr{P}_\alpha$.*

(d) *If $\delta \in E \cap S$ then some $u \in \mathscr{P}_\delta$ is an unbounded subset of $\delta$ (and $\delta$ is a limit ordinal).*

## § 1. Axioms and simple properties for classes of models

*Context* 1.1. 1) Here in §1-§5, $\tau$ is a vocabulary, $K$ will be a class of $\tau$-models, and $\leq_{\mathfrak{k}}$ a two-place relation on the models in $K$. We do not always strictly distinguish between $K$ and $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$. We shall assume that $K, \leq_{\mathfrak{k}}$ are fixed; and usually we assume that $\mathfrak{k}$ is an AEC (abstract elementary class) which means that the following axioms hold.

2) For a logic $\mathcal{L}$ let $M \prec_{\mathcal{L}} N$ mean $M$ is an elementary submodel of $N$ for the language $\mathcal{L}(\tau_M)$ and $\tau_M \subseteq \tau_N$; i.e. if $\varphi(\bar{x}) \in \mathcal{L}(\tau_M)$ and $\bar{a} \in {}^{\ell g(\bar{x})}M$ then

$$M \models \varphi[\bar{a}] \Leftrightarrow N \models \varphi[\bar{a}].$$

Similarly, $M \prec_L N$ for $L$ a language; i.e. a set of formulas in some $\mathcal{L}(\tau_M)$. So $M \prec N$ in the usual sense means $M \prec_{\mathbb{L}} N$ as $\mathbb{L}$ is first-order logic and $M \subseteq N$ means $M$ is a submodel of $N$.

**Definition 1.2.** 1) We say $\mathfrak{k}$ is a AEC with LST number $\lambda(\mathfrak{k}) = \mathrm{LST}_{\mathfrak{k}}$ <u>if</u>:

**Ax. 0**: The truth of $M \in K$ and $N \leq_{\mathfrak{k}} M$ depends on $N, M$ only up to isomorphism; i.e.

$$M \in K \wedge M \cong N \Rightarrow N \in K$$

and 'if $N \leq_{\mathfrak{k}} M$ and $f$ is an isomorphism from $M$ onto the $\tau$-model $M'$, $f \restriction N$ is an isomorphism from $N$ onto $N'$ <u>then</u> $N' \leq_{\mathfrak{k}} M'$.'

**Ax. I**: if $M \leq_{\mathfrak{k}} N$ then $M \subseteq N$ (i.e. $M$ is a submodel of $N$).

**Ax. II**: $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_2$ implies $M_0 \leq_{\mathfrak{k}} M_2$ and $M \leq_{\mathfrak{k}} M$ for $M \in K$.

**Ax. III**: If $\lambda$ is a regular cardinal, $M_i$ is $\leq_{\mathfrak{k}}$-increasing (i.e. $i < j < \lambda$ implies $M_i \leq_{\mathfrak{k}} M_j$) and continuous (i.e. for $\delta < \lambda$, $M_\delta = \bigcup_{i<\delta} M_i$) for $i < \lambda$ <u>then</u>

$$M_0 \leq_{\mathfrak{k}} \bigcup_{i<\lambda} M_i.$$

**Ax. IV**: If $\lambda$ is a regular cardinal and $M_i$ (for $i < \lambda$) is $\leq_{\mathfrak{k}}$-increasing continuous and $M_i \leq_{\mathfrak{k}} N$ for $i < \lambda$ <u>then</u> $\bigcup_{i<\lambda} M_i \leq_{\mathfrak{k}} N$.

**Ax. V**: If $N_0 \subseteq N_1 \leq_{\mathfrak{k}} M$ and $N_0 \leq_{\mathfrak{k}} M$ <u>then</u> $N_0 \leq_{\mathfrak{k}} N_1$.

**Ax. VI**: If $A \subseteq N \in K$ and $|A| \leq \mathrm{LST}_{\mathfrak{k}}$, then for some $M \leq_{\mathfrak{k}} N$, we have $A \subseteq |M|$ and $\|M\| \leq \mathrm{LST}_{\mathfrak{k}}$ (and $\mathrm{LST}_{\mathfrak{k}}$ is the minimal infinite cardinal satisfying this axiom which is $\geq |\tau|$; the $\geq |\tau|$ is for notational simplicity).

2) We say $\mathfrak{k}$ is a weak[2] AEC <u>if</u> above we omit clause IV.

*Remark* 1.3. Note that **Ax.V** holds for $\prec_{\mathcal{L}}$ for any logic $\mathcal{L}$.

[2]This is not really investigated here.

*Notation* 1.4. Let $K_\lambda := \{M \in K : \|M\| = \lambda\}$, $K_{<\lambda} := \bigcup_{\mu<\lambda} K_\mu$, and

$$\mathfrak{k}_\lambda := (K_\lambda, \leq_{\mathfrak{k}} \restriction K_\lambda)$$

(and similarly for $\mathfrak{k}_{<\lambda}, K_{\leq\lambda}, \mathfrak{k}_{\geq\lambda}, K_{\geq\lambda}$). Recall that $\mathbb{L}$ denotes first-order logic.

**Definition 1.5.** The embedding $f : N \to M$ is called a $\leq_{\mathfrak{k}}$-embedding <u>if</u> the range of $f$ is the universe of a model $N' \leq_{\mathfrak{k}} M$ (so $f : N \to N'$ is an isomorphism onto).

**Definition 1.6.** Let $T_1$ be a theory in $\mathcal{L}(\tau_1)$, $\Gamma$ a set of types in $\mathcal{L}(\tau_1)$ for some logic $\mathcal{L}$, usually first-order.

1) $\mathrm{EC}(T_1, \Gamma) = \{M : M$ an $\tau_1$-model of $T_1$ which omits every $p \in \Gamma\}$.

We implicitly use the fact that $\tau_1$ is reconstructible from $T_1$ and $\Gamma$. A problem may arise only if some symbols from $\tau_1$ are not mentioned in $T_1$ or $\Gamma$, so we may write $\mathrm{EC}(T_1, \Gamma, \tau_1)$, but usually we ignore this point.

2) For $\tau \subseteq \tau_1$ we let

$$\mathsf{PC}(T_1, \Gamma, \tau) = \mathsf{PC}_\tau(T_1, \Gamma) = \big\{M : M \text{ is a } \tau\text{-reduct of some } M_1 \in \mathrm{EC}(T_1, \Gamma)\big\}.$$

3) We say that a class of $\tau$-models $K$ is a $\mathsf{PC}^\mu_\lambda$ (or $\mathsf{PC}_{\lambda,\mu}$) class <u>when</u>

$$K = \mathsf{PC}_\tau(T_1, \Gamma_1)$$

for some $\tau_1 \supseteq \tau$, $T_1$ a first-order theory in the vocabulary $\tau_1$ and $\Gamma_1$ a set of types in $\mathbb{L}(\tau_1)$, with $|T_1| \leq \lambda$ and $|\Gamma_1| \leq \mu$.

4) We say $\mathfrak{k}$ is $\mathsf{PC}^\mu_\lambda$ or $\mathsf{PC}_{\lambda,\mu}$ <u>if</u> for some $(T_1, \Gamma_1, \tau_1), (T_2, \Gamma_2, \tau_2)$ as in part (3) we have $K = \mathsf{PC}(T_1, \Gamma_1, \tau)$ and

$$\big\{(M, N) \in K \times K : M \leq_{\mathfrak{k}} N\big\} = \mathsf{PC}(T_2, \Gamma_2, \tau'),$$

where $\tau' = \tau \cup \{P\} \subseteq \tau_2$ with $P$ a new one-place predicate. (So $|\tau_\ell| \leq \lambda$ and $|\Gamma_\ell| \leq \mu$ for $\ell = 1, 2$.)

If $\mu = \lambda$ we may omit $\mu$.

5) In (4) we may say "$\mathfrak{k}$ is $(\lambda, \mu)$-presentable," and if $\lambda = \mu$ we may say "$\mathfrak{k}$ is $\lambda$-presentable".

**Example 1.7.** If $T \subseteq \mathbb{L}(\tau)$, $\Gamma$ a set of types in $\mathbb{L}(\tau)$, then $K := \mathrm{EC}(T, \Gamma)$ and $\leq_{\mathfrak{k}} := \prec_{\mathbb{L}}$ form an AEC with LST-number $\leq |T| + |\tau| + \aleph_0$; that is, it satisfies the Axioms from 1.2 (for $\mathrm{LST}_{\mathfrak{k}} := |\tau| + \aleph_0$).

**Observation 1.8.** *Let $I$ be a directed set (i.e. partially ordered by $\leq$, such that any two elements have a common upper bound).*

1) *If $M_t$ is defined for $t \in I$ and $t \leq s \in I$ implies $M_t \leq_{\mathfrak{k}} M_s$, <u>then</u> $\bigcup_{s\in I} M_s \in K$ and $M_t \leq_{\mathfrak{k}} \bigcup_{s\in I} M_s$ for every $t \in I$.*

2) *If in addition $(\forall t \in I)[M_t \leq_{\mathfrak{k}} N]$, <u>then</u> $\bigcup_{s\in I} M_s \leq_{\mathfrak{k}} N$.*

*Proof.* By induction on $|I|$ (simultaneously for (1) and (2)).

If $I$ is finite, then $I$ has a maximal element $t(0)$, hence $\bigcup_{t\in I} M_t = M_{t(0)}$, so there is nothing to prove.

So suppose $|I| = \mu$ and we have proved the assertion when $|I| < \mu$. Let $\lambda = \mathrm{cf}(\mu)$ so $\lambda$ is a regular cardinal; hence we can find $I_\alpha$ (for $\alpha < \lambda$) such that $|I_\alpha| < |I|$, $\alpha < \beta < \lambda$ implies $I_\alpha \subseteq I_\beta \subseteq I$, $\bigcup_{\alpha<\lambda} I_\alpha = I$ and $I_\delta = \bigcup_{\alpha<\delta} I_\alpha$ for limit $\delta < \lambda$, and each $I_\alpha$ is directed and non-empty. This is trivial when $\lambda > \aleph_0$ and obvious otherwise. Let $M^\alpha := \bigcup_{t\in I_\alpha} M_t$; so by the induction hypothesis on (1) we know that $t \in I_\alpha$ implies $M_t \le_{\mathfrak{k}} M^\alpha$. If $\alpha < \beta$ then $t \in I_\alpha$ implies $t \in I_\beta$, hence $M_t \le_{\mathfrak{k}} M^\beta$; hence by the induction hypothesis on (2) applied to $\langle M_t : t \in I_\alpha\rangle$ and $M_\beta$ we have $M^\alpha = \bigcup_{t\in I_\alpha} M_t \le_{\mathfrak{k}} M^\beta$.

So by **Ax.III** applied to $\langle M^\alpha : \alpha < \lambda\rangle$, we have $M^\alpha \le_{\mathfrak{k}} \bigcup_{\beta<\lambda} M^\beta = \bigcup_{t\in I} M_t$, and as $t \in I_\alpha$ implies $M_t \le_{\mathfrak{k}} M^\alpha$, by **Ax.II**, $t \in I$ implies $M_t \le_{\mathfrak{k}} \bigcup_{s\in I} M_s$. So we have finished proving part (1) for the case $|I| = \mu$.

To prove (2) in this case, note that for each $\alpha < \lambda$, $\langle M_t : t \in I_\alpha\rangle$ is $\le_{\mathfrak{k}}$-directed and $t \in I_\alpha \Rightarrow M_t \le_{\mathfrak{k}} N$, so clearly by the induction hypothesis for (2) we have $M^\alpha \le_{\mathfrak{k}} N$. So

$$\alpha < \lambda \Rightarrow M^\alpha \le_{\mathfrak{k}} N,$$

and as proved above $\langle M^\alpha : \alpha < \lambda\rangle$ is $\le_{\mathfrak{k}}$-increasing and obviously it is continuous, hence by **Ax.IV**, $\bigcup_{s\in I} M_s = \bigcup_{\alpha<\lambda} M^\alpha \le_{\mathfrak{k}} N$. $\square_{1.8}$

**Lemma 1.9. [Lemma/Definition]**

1) *Let*

$$\tau_1 = \tau_{\mathfrak{k}}(+) := \tau \cup \{F^n_i : i < \mathrm{LST}_{\mathfrak{k}},\ n < \omega\}$$

*with $F^n_i$ an $n$-place function symbol (assuming, of course, $F^n_i \notin \tau$).*

*Every model $M$ (in $K$) can be expanded to an $\tau_1$-model $M_1$ such that:*

- (A) *$M_{\bar a} \le_{\mathfrak{k}} M$, where for $n < \omega$ and $\bar a \in {}^n|M|$, $M_{\bar a}$ is the submodel of $M$ with universe $\{F^n_i(\bar a) : i < \mathrm{LST}_{\mathfrak{k}}\}$.*
- (B) *If $\bar a \in {}^n|M|$ then $\|M_{\bar a}\| \le \mathrm{LST}_{\mathfrak{k}}$.*
- (C) *If $\bar b$ is a subsequence of a permutation of $\bar a$, then $M_{\bar b} \le_{\mathfrak{k}} M_{\bar a}$.*
- (D) *For every $N_1 \subseteq M_1$ we have $N_1 \restriction \tau \le_{\mathfrak{k}} M$.*

2) *We say $\overline{M}^+ = \langle M^+_s : s \in I\rangle$ is a $\mathfrak{k}$-SE (a* suitable expansion*) of*

$$\overline{M} = \langle M_s : s \in I\rangle$$

<u>when</u>:

- (A) *$M^+_s$ is a $\tau_{\mathfrak{k}}(+)$-expansion of $M_s$, where $\tau_{\mathfrak{k}}(+)$ is defined as above.*
- (B) *$M_s \le_{\mathfrak{k}} M_t \Rightarrow M^+_s \subseteq M^+_t$.*

3) *Given $\overline{M} = \langle M_s : s \in I\rangle$ with $M_s \in K_{\mathfrak{k}}$ and $\langle s_\alpha : \alpha < \alpha_*\rangle$ an enumeration of $I$, there is a $\mathfrak{k}$-SE $\overline{M}^+$ such that:*

- *For every $\alpha$ there is a finite $u \subseteq M_{s_\alpha}$ such that $\beta < \alpha \Rightarrow u \not\subseteq M_{s_\beta}$.*

*Proof.* We define, by induction on $n$, the values of $M_{\bar a}$ and of $F^n_i(\bar a)$ for every $i < \mathrm{LST}_{\mathfrak k}$, $\bar a \in {}^n|M|$ such that $F^n_i$ is symmetric (i.e. preserved under permutation of its variables). Arriving to $n$, for each $\bar a \in {}^nM$ by **Ax.VI** there is an $M_{\bar a} \le_{\mathfrak k} M$ such that $\|M_{\bar a}\| \le \mathrm{LST}_{\mathfrak k}$, $|M_{\bar a}|$ includes

$$\bigcup\{M_{\bar b} : \bar b \text{ a subsequence of } \bar a \text{ of length} < n\} \cup \bar a$$

and $M_{\bar a}$ does not depend on the order of $\bar a$. Let $|M_{\bar a}| = \{c_i : i < i_0 \le \mathrm{LST}_{\mathfrak k}\}$ and define $F^n_i(\bar a) = c_i$ for $i < i_0$ and $c_0$ for $i_0 \le i < \mathrm{LST}_{\mathfrak k}$.

Clearly our conditions are satisfied; in particular, if $\bar b$ is a subsequence of $\bar a$ then $M_{\bar b} \le_{\mathfrak k} M_{\bar a}$ by **Ax.V**, and clause (D) holds by 1.8 and **Ax.IV**. $\square_{1.9}$

*Remark* 1.10. 1) This is the "main" place we use **Axs.V,VI**; it seems that we use it rarely; e.g. in 2.12, which is not used later. It is clear that we can omit **Ax.V** if we strengthen somewhat **Ax.VI** for the proofs above.

2) Note that in 1.9, we do not require that $M_{\bar a}$ is closed under the functions $(F^n_i)^{M_1}$. By a different bookkeeping we can have this: renaming

$$\tau_{1,\varepsilon} = \tau \cup \{F^n_i : i < \mathrm{LST}_{\mathfrak k} \times \varepsilon,\ n < \omega\}$$

for $\varepsilon \le \omega$ and we choose a $\tau_{1,n}$-expansion $M_{1,n}$ of $M$ such that

$$m < n \Rightarrow M_{1,n} \restriction \tau_{1,m} = M_{1,m}.$$

Let $M_{1,0} := M$, and if $M_{1,n}$ is defined, choose a (non-empty) subset $A^{1,n}_{\bar a}$ of $M_{1,n}$ of cardinality $\le \mathrm{LST}_{\mathfrak k}$ for every $\bar a \in {}^{\omega>}(M_{1,n})$, such that $A^{1,n}_{\bar a}$ is closed under the functions of $M_{1,n}$ and $M \restriction A^{1,n}_{\bar a} \le_{\mathfrak k} M$. Concretely, let

$$A^{1,n}_{\bar a} := \big\{c_{\bar a,i} : i \in \big[\mathrm{LST}_{\mathfrak k} \cdot n,\ \mathrm{LST}_{\mathfrak k} \cdot (n+1)\big)\big\}$$

and define $M_{1,n+1}$ by letting $(F^m_i)^{M_{1,n+1}}(\bar a) = c_{\bar a,i}$. Let $M_1 = M_{1,\omega}$ be the $\tau_\omega$-model with the universe of $M$ such that $n < \omega \Rightarrow M_1 \restriction \tau_{1,n} = M_{1,n}$.

3) Actually, $M_{1,1}$ suffices if we expand it by making every term $\tau(\bar x)$ equal to some function $F(\bar x)$.

4) Alternatively, for $n > 0$ demand that $F^n_i(\bar a)$ is $F^{|u|}_i(\bar a \restriction u)$, where

$$u = \big\{i < n : (\forall j < i)[a_i \neq a_j]\big\}.$$

**Lemma 1.11.** 1) $\mathfrak k$ *is* $(\mathrm{LST}_{\mathfrak k}, 2^{\mathrm{LST}_{\mathfrak k}})$*-presentable.*

2) *There is a set* $\Gamma$ *of types in* $\mathbb L(\tau_1)$ *(where* $\tau_1$ *is from Lemma 1.9) — in fact, complete [and] quantifier-free — such that* $K = \mathsf{PC}_\tau(\varnothing, \Gamma)$.

3) *For the* $\Gamma$ *from part* (2), *if* $M_1 \subseteq N_1 \in \mathrm{EC}(\varnothing, \Gamma)$ *and* $M, N$ *are the* $\tau$*-reducts of* $M_1$ *and* $N_1$, *respectively, then* $M \le_{\mathfrak k} N$.

4) *For the* $\Gamma$ *from part* (2), *we have*

$$\big\{(M,N) \in K^2 : M \le_{\mathfrak k} N\big\} = \big\{(M_1 \restriction \tau, N_1 \restriction \tau) : M_1 \subseteq N_1 \text{ are both from } \mathsf{PC}_\Gamma(\varnothing, \Gamma)\big\}.$$

*Proof.* 1) By part (2) the first half of "$\mathfrak{k}$ is $(\mathrm{LST}_{\mathfrak{k}}, 2^{\mathrm{LST}_{\mathfrak{k}}})$-presentable" holds. The second part will be proved with part (4).

2) Let $\Gamma_n$ be the set of complete quantifier-free $n$-types $p(x_0, \dots, x_{n-1})$ in $\mathbb{L}(\tau_1)$ such that if $M_1$ is a $\tau_1$-model, $\bar{a}$ realizes $p$ in $M_1$, and $M$ is the $\tau$-reduct of $M_1$, then $M_{\bar{a}} \in K$ and $M_{\bar{b}} \le_{\mathfrak{k}} M_{\bar{a}}$ for any subsequence $\bar{b}$ of any permutation of $\bar{a}$.

Recall that $M_{\bar{c}}$ (for $\bar{c} \in {}^m|M_1|$) is the submodel of $M$ whose universe is $\{F^m_i(\bar{c}) : i < \mathrm{LST}_{\mathfrak{k}}\}$. Clearly there are such submodels (when $K \neq \varnothing$).

Let $\Gamma$ be the set of $p$ which, for some $n$, are complete quantifier-free $n$-types (in $\mathbb{L}(\tau_1)$) which do not belong to $\Gamma_n$. By 1.8(1) we have $\mathsf{PC}_\tau(\varnothing, \Gamma) \subseteq K$ and by 1.9 $K \subseteq \mathsf{PC}_\tau(\varnothing, \Gamma)$.

3) Similar to the proof of (2) using 1.8(2).

4) The inclusion $\supseteq$ holds by part (3); so let us prove the other direction. Given $N \le_{\mathfrak{k}} M$ we apply the proof of 1.9 to $M$, but demand further $\bar{a} \in {}^nN \Rightarrow M_{\bar{a}} \subseteq N$; simply add this demand to the choice of the $M_{\bar{a}}$-s (hence of the $F^n_i$-s). We still have a debt from part (1).

We let $\Gamma'_n$ be the set of complete quantifier-free $n$-types in $\tau'_1 := \tau_1 \cup \{P\}$ ($P$ a new unary predicate), $p(x_0, \dots, x_{n-1})$ such that:

$(*)$ If $M_1$ is an $\tau'_1$-model, $\bar{a}$ realizes $p$ in $M_1$, and $M$ is the $\tau$-reduct of $M_1$, then
  - $(\alpha)$ $M_{\bar{b}} \le_{\mathfrak{k}} M_{\bar{a}}$ for any subsequence $\bar{b}$ of $\bar{a}$.
  - $(\beta)$ $\bar{b} \subseteq P^{M_1} \Rightarrow M_{\bar{b}} \subseteq P^{M_1}$ for $\bar{b} \subseteq \bar{a}$.

We leave the rest to the reader. (Alternatively, use $\mathsf{PC}_{\tau'_1}(T', \Gamma)$, with $T'$ saying "$P$ is closed under all the functions $F^n_i$.) $\square_{1.11}$

By the proof of 1.11(4), we conclude:

**Conclusion 1.12.** *The $\tau_1$ and $\Gamma$ from 1.11 (so $|\tau_1| \le \mathrm{LST}_{\mathfrak{k}}$) satisfy the following, for any $M \in K$ and any $\tau_1$-expansion $M_1$ of $M$ which is in $\mathrm{EC}_{\tau_1}(\varnothing, \Gamma)$.*

- (*a*) $N_1 \prec_{\mathbb{L}} M_1 \Rightarrow N_1 \subseteq M_1 \Rightarrow N_1 \restriction \tau \le_{\mathfrak{k}} M$
- (*b*) $N_1 \prec_{\mathbb{L}} N_2 \prec_{\mathbb{L}} M_1 \Rightarrow N_1 \subseteq N_2 \subseteq M_1 \Rightarrow N_1 \restriction \tau \le_{\mathfrak{k}} N_2 \restriction \tau$
- (*c*) *If $M \le_{\mathfrak{k}} N$ then there is a $\tau_1$-expansion $N_1$ of $N$ from $\mathrm{EC}_{\tau_1}(\varnothing, \Gamma)$ which extends $M_1$.*

**Conclusion 1.13.** *If $\mathfrak{k}$ has a model of cardinality $\ge \beth_\alpha$ for every $\alpha < (2^{\mathrm{LST}_{\mathfrak{k}}})^+$, then $K$ has a model in every cardinality $\ge \mathrm{LST}_{\mathfrak{k}}$.*

*Proof.* Use 1.11 and the classical upper bound on the value of the Hanf number for first-order theory and omitting any set of types, for languages of cardinality $\mathrm{LST}_{\mathfrak{k}}$ (see e.g. [She90, VII,5.3,5.5]). $\square_{1.13}$

*Notation* 1.14. 1) If $M \in \mathfrak{A}$ then $M \restriction \mathfrak{A}$ is the submodel of $M$ with universe $|M| \cap |\mathfrak{A}|$.

2) If $\mathfrak{B} \models$ "$M \in \mathfrak{k}$" then $M[\mathfrak{B}]$ is the following $\tau_K$-model:

(A) it has universe $\{b \in \mathfrak{B} : \mathfrak{B} \models$ "$b$ an element of the model $M$"$\}$.

(B) for any $m$-place predicate $Q$ of $\tau$,

$$Q^M = \big\{\langle b_0,\dots,b_{m-1}\rangle : \mathfrak{B} \models \text{“}M \models Q[b_0,\dots,b_{m-1}]\text{”}\big\}.$$

(C) Similarly for any $m$-place function symbol $G$ of $\tau$.

**Conclusion 1.15.** *Assume that $\mathfrak{k}$ is an AEC, $\mu = |\tau_{\mathfrak{k}}| + \mathrm{LST}_{\mathfrak{k}}$, and for simplicity $\tau_{\mathfrak{k}} \subseteq \mu$ or just $\tau_{\mathfrak{k}} \subseteq \mathbf{L}_\mu$, recalling $\mathbf{L}$ is the constructible universe of Göbel.*

*If $\lambda > \mu$, $\mathfrak{A} \prec (\mathcal{H}(\chi), \in)$, $\mu + 1 \subseteq \mathfrak{A}$, and $\mathfrak{k} \in \mathfrak{A}$*

*(which means $\{(M,N) : M \leq_{\mathfrak{k}} N$ has universe $\subseteq \mu\} \in \mathfrak{A}$)*

*<u>then</u>:*

(*A*) $M \in \mathfrak{k} \cap K \Rightarrow M \restriction \mathfrak{A} \leq_{\mathfrak{k}} M$
(*B*) *If $M \leq_{\mathfrak{k}} N$ (so both belong to $K$) and $M, N \in \mathfrak{A}$ then $M \restriction \mathfrak{A} \leq_{\mathfrak{k}} N \restriction \mathfrak{A}$.*
(*C*) *If $\mathfrak{A} \prec \mathfrak{B}$, $[b <_{\mathfrak{B}} \mu \Rightarrow b \in \mathfrak{A}]$, and $\mathfrak{B} \models$ “$M \in K$” <u>then</u> $M[\mathfrak{B}] \in K$.*
(*D*) *Similarly for $\mathfrak{B} \models$ “$M \leq_{\mathfrak{k}} N$”.*

*Proof.* Should be clear. $\square_{1.15}$

*Remark* 1.16. 1) Clearly $\{\mu \geq \mathrm{LST}_{\mathfrak{k}} : K_\mu \neq 0\}$ is an initial segment of the class of cardinals $\geq \mathrm{LST}_{\mathfrak{k}}$.

2) For every cardinal $\kappa$ ($\geq \aleph_0$) and ordinal $\alpha < (2^\kappa)^+$, there is an AEC $\mathfrak{k}$ such that $\mathrm{LST}_{\mathfrak{k}} = \kappa = |\tau_{\mathfrak{k}}|$ and $\mathfrak{k}$ has a model of cardinality in the interval $\big[\kappa, \beth_\alpha(\kappa)\big)$. This follows by [She90, VII, §5, p.432] (in particular, [She90, VII, 5.5(6)]) because

(A) If $\tau$ is a vocabulary of cardinality $\leq \kappa$, $T \subseteq \mathbb{L}(\tau)$, and $\Gamma$ a set of $(\mathbb{L}(\tau), {<}\omega)$-types, then $K = \{M : M$ a $\tau$-model of $T$ omitting every $\in \Gamma\}$ and $\leq_{\mathfrak{k}} = \prec\restriction K$ form an AEC (we can use $\Gamma$ a set of quantifier-free types and $T = \varnothing$), with $\mathrm{LST}(\mathfrak{k}, \leq_{\mathfrak{k}}) \leq \kappa$.
(B) If $\{c_i \neq c_j : i < j < \kappa\} \subseteq T$ then $K$ above has no model of cardinality $< \kappa$.

3) For more on such theorems, see [She99].

4) We can phrase 1.15 as “for any $\mathfrak{B}$ in appropriate $\mathrm{EC}(T_1, \Gamma_1)$”, but the present formulation is the way we use it.

## § 2. Amalgamation properties and homogeneity

*Context* 2.1. $\mathfrak{k}$ is an AEC.

The main theorem 2.9, the existence and uniqueness of the model-homogeneous models, is a generalization of Jonsson [Jón56], [Jón60] to the present context. The result on the upper bound $2^{2^{\aleph_0+|\tau|}}$ for the number of $D$-sequence homogeneous universal-models of cardinality is from Keisler-Morley [KM67]. Earlier there were serious good reasons to concentrate on sequence-homogeneous models, but here we deal with the model-homogeneous case. From 2.14 to the end we consider what we can say when we omit smoothness (i.e. **Ax.IV** of Definition 1.2).

**Definition 2.2.** 1) $\mathbb{D}(M) := \{N/\cong : N \leq_{\mathfrak{k}} M, \ \|N\| \leq \mathrm{LST}_{\mathfrak{k}}\}$.

2) $\mathbb{D}(\mathfrak{k}) := \{N/\cong : N \in K, \ \|N\| \leq \mathrm{LST}_{\mathfrak{k}}\}$.

3) $D(M) = \{\mathrm{tp}_{\mathbb{L}(\tau_M)}(\bar{a}, \varnothing, M) : \bar{a} \in {}^{\omega>}M\}$.

**Definition 2.3.** Let $\lambda > \mathrm{LST}_{\mathfrak{k}}$.

1) A model $M$ is $\lambda$-model-homogeneous when: <u>if</u> $N_0 \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} M$ and $\|N_1\| < \lambda$, <u>then</u> any $\leq_{\mathfrak{k}}$-embedding of $N_0$ into $M$ can be extended to a $\leq_{\mathfrak{k}}$-embedding $N_1 \to M$.

1A) A model $M$ is $(\mathbb{D}, \lambda)$-model-homogeneous <u>if</u> $\mathbb{D} = \mathbb{D}(M)$ and $M$ is a $\lambda$-model homogeneous.

1B) Adding "above $\mu$" means in $\mathfrak{k}_{\geq \mu}$.

2) $M$ is $\lambda$-strongly model-homogeneous <u>if</u>: for every $N \in K_{<\lambda}$ such that $N \leq_{\mathfrak{k}} M$, any $\leq_{\mathfrak{k}}$-embedding of $N$ into $M$ can be extended to an automorphism of $M$.

3) $M$ is $\lambda$-universal model-homogeneous (for $\mathfrak{k}$) <u>when</u>: $\lambda > \mathrm{LST}_{\mathfrak{k}}$, every[3] $N \in K_{\mathrm{LST}_{\mathfrak{k}}}$ is $\leq_{\mathfrak{k}}$-embeddable into $M$ and for every $N_0, N_1 \in K_{<\lambda}$ such that $N_0 \leq_{\mathfrak{k}} N_1$ and $\leq_{\mathfrak{k}}$-embedding $f : N_0 \to M$ there exists a $\leq_{\mathfrak{k}}$-embedding $g : N_1 \to M$ extending $f$.

Unlike (1), we do not demand that $N_1$ is $\leq_{\mathfrak{k}}$-embeddable into $M$.

**[That sounds *exactly* like what you're demanding. I don't know how else to interpret 'there exists a $\leq_{\mathfrak{k}}$-embedding $g : N_1 \to M$.']**

(The universal is related to $\lambda$, it does not imply $M$ is universal).

4) For each of the above three properties and the one below, if $M$ has cardinality $\lambda$ and has the $\lambda$-property then we may say for short that $M$ has the property (i.e. omitting $\lambda$).

5) $M$ is $(D, \lambda)$-sequence-homogeneous <u>if</u>:

(A) $D = D(M) = \{\mathrm{tp}_{\mathbb{L}(\tau_M)}(\bar{a}, \varnothing, M) : \bar{a} \in |M|\}$ (i.e. $\bar{a}$ a finite sequence from $M$).

[3] In fact, $N \in K_{\leq \lambda}$ is okay by 2.5(2).

(B) If $a_i \in M$ for $i \leq \alpha < \lambda$, $b_j \in M$ for $j < \alpha$, and

$$\mathrm{tp}_{\mathbb{L}(\tau M)}(\langle a_i : i < \alpha\rangle, \varnothing, M) = \mathrm{tp}_{\mathbb{L}(\tau_M)}(\langle b_i : i < \alpha\rangle, \varnothing, M),$$

<u>then</u> for some $b_\alpha \in M$,

$$\mathrm{tp}_{\mathbb{L}(\tau_M)}(\langle a_i : i < \alpha\rangle\hat{\ }\langle a_\alpha\rangle, \varnothing, M) = \mathrm{tp}_{\mathbb{L}(\tau_M)}(\langle b_i : i < \alpha\rangle\hat{\ }\langle b_\alpha\rangle, \varnothing, M).$$

5A) In (5) we omit $D$ when

$$D = \{\mathrm{tp}_{\mathbb{L}(\tau_K)}(\bar{a}, \varnothing, N) : \bar{a} \in {}^n N,\ n < \omega,\ \text{and } M \prec_{\mathbb{L}} N\}.$$

6) We omit the "model" or "sequence" when it is clear from the context; i.e. if $D$ is as in 2.2(3) = 2.3(5)(a), $(D, \lambda)$-homogeneous means $(D, \lambda)$-sequence-homogeneous. If $\mathbb{D}$ is as in Definition 2.2(1), $(\mathbb{D}, \lambda)$-homogeneous means $(\mathbb{D}, \lambda)$-model-homogeneous. If not obvious, we mean the model version.

7) $M$ is $\lambda$-universal <u>when</u> every $N \in K_\lambda$ can be $\leq_{\mathfrak{k}}$-embedded into it. Similarly for $(< \lambda)$-universal and $(\leq \lambda)$-universal.

**Claim 2.4.** *Assume $N$ is $\lambda$-model-homogeneous and $\mathbb{D}(M) \subseteq \mathbb{D}(N)$ (and $\mathrm{LST}_{\mathfrak{k}} < \lambda$, of course).*

1) *If $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M$, $\|M_0\| < \lambda$, $\|M_1\| \leq \lambda$, and $f$ is a $\leq_{\mathfrak{k}}$-embedding of $M_0$ into $N$, <u>then</u> we can extend $f$ to a $\leq_{\mathfrak{k}}$-embedding of $M_1$ into $N$.*

2) *If $M_1 \leq_{\mathfrak{k}} M$ and $\|M_1\| \leq \lambda$ <u>then</u> there is a $\leq_{\mathfrak{k}}$-embedding of $M_1$ into $N$.*

*Proof.* We prove simultaneously, by induction on $\mu \leq \lambda$, that:

$(i)_\mu$ For every $M_1 \leq_{\mathfrak{k}} M$ with $\|M_1\| \leq \mu$ (Yes! *Not* '$< \mu$!'), there is a $\leq_{\mathfrak{k}}$-embedding of $M_1$ into $N$.

$(ii)_\mu$ If $M_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M$, $\|M_1\| \leq \mu$, and $\|M_0\| < \lambda$, then any $\leq_{\mathfrak{k}}$-embedding of $M_0$ into $N$ can be extended to a $\leq_{\mathfrak{k}}$-embedding of $M_1$ into $N$.

Clearly $(i)_\lambda$ is part (2) and $(ii)_\lambda$ is part (1), so this is enough.

*Proof.* **<u>Proof of $(i)_\mu$</u>:**

If $\mu \leq \mathrm{LST}_{\mathfrak{k}}$, this follows by $\mathbb{D}(M) \subseteq \mathbb{D}(N)$.

If $\mu > \mathrm{LST}_{\mathfrak{k}}$, then by 1.12 we can find $\overline{M}_1 = \langle M_1^\alpha : \alpha < \mu\rangle$ such that $M_1 = \bigcup_{\alpha<\mu} M_1^\alpha$, $M_1^\alpha$ is $\leq_{\mathfrak{k}}$-increasing continuous with $\alpha$, and

$$\alpha < \mu \Rightarrow \|M_1^\alpha\| < \mu \wedge M_1^\alpha \leq_{\mathfrak{k}} M_1.$$

We define a $\leq_{\mathfrak{k}}$-embedding $f_\alpha : M_1^\alpha \to N$ by induction on $\alpha$ such that $f_\alpha$ extends $f_\beta$ for $\beta < \alpha$. For $\alpha = 0$ we can define $f_\alpha$ by clause $(i)_{\chi(0)}$ (the base case of the induction hypothesis), where $\chi(\beta) := \|M_1^\beta\|$.

Next we define $f_\alpha$ for $\alpha = \gamma + 1$: by $(ii)_{\chi(\alpha)}$ (which holds by the induction hypothesis) there is a $\leq_{\mathfrak{k}}$-embedding $f_\alpha$ of $M_1^\alpha$ into $N$ extending $f_\gamma$.

Lastly, for limit $\alpha$ we let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$; it is a $\leq_{\mathfrak{k}}$-embedding into $N$ by 1.8. So we finish the induction and $\bigcup_{\alpha<\mu} f_\alpha$ is as required. $\square_{(\mathrm{i})_\mu}$

*Proof.* **Proof of $(ii)_\mu$:**

**[I have a lot of doubts about this proof. I'm not qualified to judge it on its merits, but a proof by induction on $\mu \leq \lambda$ should *end* with 'now take $\mu = \lambda$.' If it *starts* with 'assume that $\mu = \lambda$,' you're skipping the bit where the actual proof should go.]**

First, assume that $\mu = \lambda$ so we have proved $(ii)_\theta$ for $\theta < \lambda$ and $\|M_1\| = \lambda > \|M_0\|$, so $\mathrm{LST}_{\mathfrak{k}} < \mu = \lambda$ hence we can find $\langle M_1^\alpha : \alpha < \mu\rangle$ as in the proof of $(i)_\mu$ such that $M_1^0 = M_0$ and let $\chi(\beta) = \|M_1^\beta\|$. Now we define $f_\beta$ by induction on $\beta \leq \mu$ such that $f_\beta$ is a $\leq_{\mathfrak{k}}$-embedding of $M_\beta^1$ into $N$ and $f_\beta$ is increasing continuous in $\beta$ and $f_0 = f$. We can do this as in the proof of $(i)_\mu$ by $(ii)_{\chi(\alpha)}$ for $\alpha < \mu$.

Second, assume $\|M_1\| < \lambda$. Let $g$ be a $\leq_{\mathfrak{k}}$-embedding of $M_1$ into $N$; it exists by $(i)_\mu$, which we have just proved. Let $g$ be onto $N_1' \leq_{\mathfrak{k}} N$, and let $g \restriction M_0$ be onto $N_0' \leq_{\mathfrak{k}} N_1'$, and let $f$ be onto $N_0 \leq_{\mathfrak{k}} N$. So clearly $h : N_0' \to N_0$ defined by $h(g(a)) = f(a)$ for $a \in |M_0|$, is an isomorphism from $N_0'$ onto $N_0$. So $N_0, N_0', N_1' \leq_{\mathfrak{k}} N$. As $\|M_1\| < \lambda$ clearly $\|N_1'\| < \lambda$ so (by the assumption "$N$ is $\lambda$-model-homogeneous" — see Definition 2.3(1)) we can extend $h$ to an isomorphism $h'$ from $N_1'$ onto some $N_1 \leq_{\mathfrak{k}} N$, so $h' \circ g : M_1 \to N$ is as required. $\square_{(\mathrm{ii})_\mu}$

**[Also, I don't see why $(i)_\mu$ needs to be its own clause. If $f : M_0 \to N$ can be extended to $M_1 \to N$ for any $M_1$ up to some cardinality, then it would trivially follow that a map from $M_1 \to N$ *exists*. The fact that it's written like this is making me intensely suspicious.]** $\square_{2.4}$

**Conclusion 2.5.** 1) *If $M, N$ are model-homogeneous, of the same cardinality ($> \mathrm{LST}_{\mathfrak{k}}$), and $\mathbb{D}(M) = \mathbb{D}(N)$ then $M, N$ are isomorphic. Moreover, if $M_0 \leq_{\mathfrak{k}} M$ and $\|M_0\| < \|M\|$, then any $\leq_{\mathfrak{k}}$-embedding of $M_0$ into $N$ can be extended to an isomorphism from $M$ onto $N$.*

2) *The number of model-homogeneous models from $\mathfrak{k}$ of cardinality $\lambda$ is $\leq 2^{2^{\mathrm{LST}_{\mathfrak{k}}}}$.*

*If in the definition of $\mathrm{LST}_{\mathfrak{k}}$ (in Definition 1.2, **Ax.VI**) we omit '$|\tau| \leq \mathrm{LST}_{\mathfrak{k}}$,' the bound is $2^{2^{\mathrm{LST}_{\mathfrak{k}} + |\tau(\mathfrak{k})|}}$.*

3) *If $M$ is $\lambda$-model-homogeneous and $\mathbb{D}(M) = \mathbb{D}(\mathfrak{k})$ then $M$ is $(\leq \lambda)$-universal; i.e. every model $N$ (in $K$) of cardinality $\leq \lambda$ has a $\leq_{\mathfrak{k}}$-embedding into $M$.*

*So if $\mathbb{D}(M) = \mathbb{D}(\mathfrak{k})$ then: $M$ is $\lambda$-model universal homogeneous (see Definition 2.3(3)) iff $M$ is a $\lambda$-model-homogeneous iff $M$ is $(\lambda, \mathbb{D}(\mathfrak{k}))$-homogeneous.*

4) *If $M$ is $\lambda$-model-homogeneous then it is $\lambda$-universal for*

$$\{N \in K_\lambda : \mathbb{D}(N) \subseteq \mathbb{D}(M)\}.$$

5) *If $M$ is $(D, \lambda)$-sequence-homogeneous, (and $\lambda > \mathrm{LST}_{\mathfrak{k}}$) then $M$ is a $\lambda$-model homogeneous.*

6) *For $\lambda > \mathrm{LST}_{\mathfrak{k}}$, $M$ is $\lambda$-model universal homogeneous iff $M$ is $\lambda$-model-homogeneous and $(\leq \mathrm{LST}_{\mathfrak{k}})$-universal.*

*Proof.* 1) Immediate by 2.4(1), using the standard hence-and-forth argument.

2) The number of models (in $K$) of power $\leq \mathrm{LST}_{\mathfrak{k}}$ is, up to isomorphism, $\leq 2^{\mathrm{LST}_{\mathfrak{k}}}$ (recalling that we are assuming $|\tau(\mathfrak{k})| \leq \mathrm{LST}_{\mathfrak{k}}$). Hence the number of possible $\mathbb{D}(M)$ is $\leq 2^{2^{\mathrm{LST}_{\mathfrak{k}}}}$. So by 2.5(1) we are done.

3-5) Immediate. $\square_{2.5}$

*Remark* 2.6. The results parallel to 2.5(1)-(4) for $\lambda$-sequence homogeneous models and $D(M)$ also hold.

**Definition 2.7.** 1) A model $M$ has the $(\lambda, \mu)$-amalgamation property or *am.p.* (in $\mathfrak{k}$, of course) <u>if</u> for every $M_1, M_2$ such that $\|M_1\| = \lambda$, $\|M_2\| = \mu$, $M \leq_{\mathfrak{k}} M_1$, and $M \leq_{\mathfrak{k}} M_2$, there is a model $N$ and $\leq_{\mathfrak{k}}$-embeddings $f_1 : M_1 \to N$ and $f_2 : M_2 \to N$ such that $f_1 \restriction |M| = f_2 \restriction |M|$.

Now the meaning of (e.g.) the $(\leq \lambda, < \mu)$-amalgamation property should be clear. Always $\lambda, \mu \geq \mathrm{LST}_{\mathfrak{k}}$ (and, of course, if we use '$< \mu$' then $\mu > \mathrm{LST}_{\mathfrak{k}}$).

1A) In part (1) we add the adjective "disjoint" <u>when</u> $f_1(M_1) \cap f_2(M_2) = M$. Similarly in (2) below.

2) $\mathfrak{k}$ has the $(\kappa, \lambda, \mu)$-amalgamation property <u>if</u> every model $M$ (in $K$) of cardinality $\kappa$ has the $(\lambda, \mu)$-amalgamation property. The $(\kappa, \lambda)$-amalgamation property for $\mathfrak{k}$ means just the $(\kappa, \kappa, \lambda)$-amalgamation property. The $\kappa$-amalgamation property for $\mathfrak{k}$ is just the $(\kappa, \kappa, \kappa)$-amalgamation property.

3) $\mathfrak{k}$ has the $(\lambda, \mu)$-JEP (joint embedding property) if for any $M_1, M_2 \in K$ of cardinality $\lambda$ and $\mu$, respectively, there is an $N \in K$ into which $M_1$ and $M_2$ are $\leq_{\mathfrak{k}}$-embeddable.

4) The $\lambda$-JEP is the $(\lambda, \lambda)$-JEP.

5) The amalgamation property means the $(\kappa, \lambda, \mu)$-amalgamation property for every $\lambda, \mu \geq \kappa$ $(\geq \mathrm{LST}_{\mathfrak{k}})$.

6) The JEP means the $(\lambda, \mu)$-JEP for every $\lambda, \mu \geq \mathrm{LST}_{\mathfrak{k}}$.

*Remark* 2.8. Clearly, the roles of $\lambda$ and $\mu$ are symmetric in 2.7.

**Theorem 2.9.** 1) *If* $\mathrm{LST}_{\mathfrak{k}} < \kappa \leq \lambda = \lambda^{<\kappa}$, $K_\lambda \neq \varnothing$, *and* $\mathfrak{k}$ *has the* $(<\kappa, \lambda)$-*amalgamation property* <u>*then*</u> *for every model* $M$ *of cardinality* $\lambda$, *there is a* $\kappa$-*model-homogeneous model* $N$ *of cardinality* $\lambda$ *satisfying* $M \leq_{\mathfrak{k}} N$. *If* $\kappa = \lambda$, *then alternatively the* $(<\kappa, <\lambda)$-*amalgamation property suffices.*

2) *So in* (1), *if* $\kappa = \lambda$ *then there is a universal, model-homogeneous model of cardinality* $\lambda$, *provided that for some* $M \in K_{\leq\lambda}$, $\mathbb{D}(M) = \mathbb{D}(\mathfrak{k})$ *or just* $\mathfrak{k}$ *has the* $\mathrm{LST}_{\mathfrak{k}}$-JEP.

3) *If* $\mathfrak{k}$ *has the amalgamation property and the* $\mathrm{LST}_{\mathfrak{k}}$-JEP, <u>*then*</u> $\mathfrak{k}$ *has the* JEP.

*Remark* 2.10. 1) The last assumption of 2.9(2) holds; e.g. if the $(\leq \mathrm{LST}_{\mathfrak{k}}, < 2^{\mathrm{LST}_{\mathfrak{k}}})$-JEP holds and $|\mathbb{D}(\mathfrak{k})| \leq \lambda$.

2) If $\mathbb{D}(M) = \mathbb{D}(\mathfrak{k})$ for some $M \in K$, <u>then</u> we can have such $M$ of cardinality $\leq 2^{\mathrm{LST}_{\mathfrak{k}}}$.

3) In 2.9, we can replace the assumption "$(<\kappa, \lambda)$-amalgamation property" by "$(<\kappa, <\lambda)$-amalgamation property" if, e.g., no $M \in K_{<\lambda}$ is maximal.

*Proof.* Immediate; in (1), note that if $\kappa$ is singular then necessarily

$$\kappa < \lambda = \lambda^\kappa = \lambda^{<\kappa^+},$$

so we can replace $\kappa$ by $\kappa^+$. $\square_{2.9}$

*Remark* 2.11. Also, the corresponding converses hold.

**Lemma 2.12.** 1) *If* $\mathrm{LST}_{\mathfrak{k}} \leq \kappa$ *and* $\mathfrak{k}$ *has the* $\kappa$*-amalgamation property* *<u>then</u>* $\mathfrak{k}$ *has the* $(\kappa, \kappa^+)$*-amalgamation property, and even the* $(\kappa, \kappa^+, \kappa^+)$*-amalgamation property.*

2) *If* $\kappa \leq \mu \leq \lambda$ *and* $\mathfrak{k}$ *has the* $(\kappa, \mu)$*-amalgamation property and the* $(\mu, \lambda)$*-amalgamation property* *<u>then</u>* $\mathfrak{k}$ *has the* $(\kappa, \lambda)$*-amalgamation property. If* $\mathfrak{k}$ *has the* $(\kappa, \mu, \mu)$ *and the* $(\mu, \lambda)$*-amalgamation property,* *<u>then</u>* $\mathfrak{k}$ *has the* $(\kappa, \lambda, \mu)$*-amalgamation property.*

3) *If* $\lambda_i$ *is increasing and continuous for* $i \leq \alpha$, $\mathrm{LST}_{\mathfrak{k}} \leq \lambda_0$, *and* $\mathfrak{k}$ *has the* $(\lambda_i, \mu + \lambda_i, \lambda_{i+1})$*-amalgamation property for every* $i < \alpha$, *<u>then</u>* $\mathfrak{k}$ *has the* $(\lambda_0, \mu + \lambda_0, \lambda_\alpha)$*-amalgamation property.*

4) *If* $\kappa \leq \mu_1 \leq \mu$, *and for every* $M$ *with* $\|M\| = \mu_1$ *there is* $N$ *such that* $M \leq_{\mathfrak{k}} N$ *and* $\|N\| = \mu$, *<u>then</u> the* $(\kappa, \mu, \lambda)$*-amalgamation property (for* $\mathfrak{k}$*) implies the* $(\kappa, \mu_1, \lambda)$*-amalgamation property (for* $\mathfrak{k}$*).*

5) *Similarly with the disjoint amalgamation version.*

*Proof.* Straightforward, e.g.

3) So assume $M_0 \in K_{\lambda_0}$, $M_0 \leq_{\mathfrak{k}} M_1 \in K_{\mu+\lambda_0}$, $M_0 \leq_{\mathfrak{k}} M_2 \in K_{\lambda_\alpha}$, and for variety we prove the disjoint amalgamation version (see part (5)). By (e.g.) 1.12 we can find an $\leq_{\mathfrak{k}}$-increasing continuous sequence $\langle M_{2,i} : i \leq \alpha \rangle$ such that $M_{2,0} = M_0$, $M_{2,\alpha} = M_2$, and $M_{2,i} \in K_{\lambda_i}$ for $i \leq \alpha$.

Without loss of generality $M_1 \cap M_2 = M_0$. We now choose $M_{1,i}$ by induction on $i \leq \alpha$ such that:

(∗) (a) $\langle M_{1,j} : j \leq i \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous.
 (b) $M_{1,0} = M_1$
 (c) $M_{1,i} \in K_{\mu+\lambda_i}$
 (d) $M_{2,i} \leq_{\mathfrak{k}} M_{1,i}$
 (e) $M_{2,i} \cap M_{1,\alpha} = M_{1,i}$.

For $i = 0$ see clause (b); for $i$ limit take unions; for $i = j + 1$ apply the disjoint $(\lambda_j, \mu + \lambda_j, \lambda_i)$-amalgamation to $M_{2,j}, M_{1,j}, M_{2,j+1}$. For $i = \alpha$ we are done. $\square_{2.12}$

**Conclusion 2.13.** *If* $\mathrm{LST}_{\mathfrak{k}} \leq \chi_1 < \chi_2$ *and* $\mathfrak{k}$ *has the* $\kappa$*-amalgamation property whenever* $\kappa \in [\chi_1, \chi_2)$ *<u>then</u>* $\mathfrak{k}$ *has the* $(\kappa, \lambda, \mu)$*-amalgamation property for all* $\lambda, \mu \in [\kappa, \chi_2]$.

* * *

It may be interesting to note that we can say something even when we waive **Ax.IV**.

*Context* 2.14. For the remainder of this section $\mathfrak{k}$ is just a weak AEC; i.e. **Ax.IV** is not assumed.

**Definition 2.15.** Let $M \in K$ have cardinality $\lambda > \mathrm{LST}_{\mathfrak{k}}$, with $\lambda$ a regular uncountable cardinal. We say $M$ is *smooth* if there is a $\leq_{\mathfrak{k}}$-increasing continuous sequence $\langle M_i : i < \lambda \rangle$ with $M = \bigcup\limits_{i<\lambda} M_i$, $M_i \leq_{\mathfrak{k}} M$, and $\|M_i\| < \lambda$ for $i < \lambda$.

*Remark* 2.16. We can define $S/\mathcal{D}$-*smooth* for $S$ a subset of $\mathcal{P}(\lambda)$ and $\mathcal{D}$ a filter on $\mathcal{P}(\lambda)$.

That is, $M \in K_\lambda$ is $(S/\mathcal{D})$-smooth when for every one-to-one function $f$ from $|M|$ onto $\lambda$, the set

$$\left\{u \in \mathcal{P}(\lambda) : M \restriction f^{-1}[u] \leq_{\mathfrak{k}} M\right\} \in \mathcal{D}.$$

Usually we demand that for every permutation $f$ on $\lambda$,

$$\{u \subseteq \lambda : u \text{ is closed under } f\} \in \mathcal{D},$$

and usually we demand that $\mathcal{D}$ is a normal $\mathrm{LST}^+_{\mathfrak{k}}$-complete filter).

**Claim 2.17.** *Assume that $\lambda = \lambda^{<\lambda} > |\tau_K|$, $\mathfrak{k}_{<\lambda}$ has no maximal member, $\mathfrak{k}$ has the $(<\lambda, <\lambda, <\lambda)$-amalgamation property, and $\mathrm{LST}_{\mathfrak{k}} < \lambda$ (or at least assume in the $(<\lambda, <\lambda, <\lambda)$-amalgamation demand that the resulting model has cardinality $< \lambda$). Then $\mathfrak{k}_\lambda$ has a smooth model-homogeneous member.*

*Proof.* Same proof. $\square_{2.17}$

**Lemma 2.18.** *If $M, N \in K_\lambda (\lambda > \mathrm{LST}_{\mathfrak{k}})$ are smooth, model-homogeneous, and $\mathbb{D}(M) = \mathbb{D}(N)$ then $M \cong N$.*

*Proof.* By the hence-and-forth argument, left to the reader.

(The set of approximations is

$$\{f : f \text{ is an isomorphism from some } M' \leq_{\mathfrak{k}} M$$
$$\text{of cardinality} < \lambda \text{ onto some } N' \leq_{\mathfrak{k}} N\},$$

but note that for an increasing continuous sequence of approximations, the union is [not always / never] an approximation.) $\square_{2.18}$

*Remark* 2.19. It is reasonable to consider

(∗) If $M \in K_\lambda$, is smooth and model-homogeneous and $N \in K_\lambda$ is smooth (with $\lambda > \mathrm{LST}_{\mathfrak{k}}$), and $\mathbb{D}(N) \subseteq \mathbb{D}(M)$ then $N$ can be $\leq_{\mathfrak{k}}$-embedded into $M$.

This can be proved in the context of universal classes (e.g. **Ax.**$\mathrm{Fr}_1$ from [She09d]).

**Fact 2.20.** 1) If $\mathfrak{k}_i = (K_i, <_i)$ is a [weak] AEC (i.e. with $\lambda_i = \text{LST}(K_i, \leq_i) \geq \aleph_0$ for $i < \alpha$), $\tau_{K_i} := \tau$ for $i < \alpha$, $K := \bigcap\limits_{i<\alpha} K_i$, and '$\leq$' is defined by

$$M \leq N \Leftrightarrow (\forall i < \alpha)[M \leq_i N],$$

then $\mathfrak{k} = (K, \leq)$ is a [weak] AEC with $\text{LST}_{\mathfrak{k}} \leq \sum\limits_{i<\alpha} \lambda_i$.

2) Concerning **Axs.I-V**, we can omit some of them in the assumption and still get the rest in the conclusion. But for **Ax.VI** we need in addition to assume **Ax.V** + $\mathbf{Ax.IV}_\theta$ for at least one $\theta = \text{cf}(\theta) \leq \sum\limits_{i<\alpha} \lambda_i$.

*Proof.* Easy. $\square_{2.20}$

**Example 2.21.** Consider the class $K$ of norm[ed] spaces over the reals with $M \leq_{\mathfrak{k}} N$ iff $M \subseteq N$ and $M$ is complete inside $N$. Now $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$ is a weak AEC with $\text{LST}_{\mathfrak{k}} = 2^{\aleph_0}$ and it is as required in 2.17.

## § 3. Limit models and other results

In this section we introduce various variants of limit models (the most important are the superlimit ones). We prove that if $\mathfrak{k}$ has a superlimit model $M^*$ of cardinality $\lambda$ for which the $\lambda$-amalgamation property fails and $2^\lambda < 2^{\lambda^+}$, <u>then</u> $\dot{I}(\lambda, K) = 2^\lambda$ (see 3.9). We later prove that if $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ is categorical in $\aleph_1$ then it has a model in $\aleph_2$ (see 3.19(2)). This finally solves Baldwin's problem (see §0). In fact, we prove an essentially more general result on AECs and $\lambda$ (see 3.12, 3.14).

The reader can read 3.3(1),(1A),(1B) ignore the other definitions, and continue with 3.8(2),(5) and everything from 3.9 (interpreting all variants as superlimits).

You may wonder if can we prove the parallel to Baldwin conjecture in $\lambda^+$ if $\lambda > \aleph_0$. It would be:

- $\circledast_\lambda$ If $\mathfrak{k}$ is a $\lambda$-presentable AEC (where $\mathrm{LST}_\mathfrak{k} = \lambda$), categorical in $\lambda^+$, then $K_{\lambda^{++}} \neq \varnothing$.

This is **false** when $\mathrm{cf}(\lambda) > \aleph_0$.

*Context* 3.1. $\mathfrak{k}$ is an AEC.

**Example 3.2.** Let $\lambda$ be given and $\mathfrak{k} = (K, \leq_\mathfrak{k})$ be defined by

$$K = \big\{(A,<) : (A,<) \text{ a well-order of order type} \leq \lambda^+\big\}$$
$$\leq_\mathfrak{k} = \big\{(M,N) \in K \times K : N \text{ is an end-extension of } M\big\}.$$

Now

- (A) $\mathfrak{k}$ is an abstract elementary class with $\mathrm{LST}_\mathfrak{k} = \lambda$ and $\mathfrak{k}$ categorical in $\lambda^+$.
- (B) If $\lambda$ has cofinality $\geq \aleph_1$ <u>then</u> $\mathfrak{k}$ is $\lambda$-presentable (see e.g. [She90, VII,§5] and history there); by clause (a) it is always $(\lambda, 2^\lambda)$-presentable.
- (C) $\mathfrak{k}$ has no model of cardinality $> \lambda^+$.

Note that if we are dealing with classes which are categorical (or just simple in some sense), we have a good chance to find limit models and they are useful in constructions.

**Definition 3.3.** Let $\lambda$ be a cardinal $\geq \mathrm{LST}_\mathfrak{k}$. For parts (3)–(7) (but not (8)), for simplifying the presentation we assume the axiom of global choice (alternatively, we restrict ourselves to models with universe an ordinal $< \lambda^+$).

1) $M \in K_\lambda$ is *locally superlimit* (for $\mathfrak{k}$) <u>if</u>:

- (a) For every $N \in K_\lambda$ such that $M \leq_\mathfrak{k} N$, there is $M' \in K_\lambda$ isomorphic to $M$ such that $N \leq_\mathfrak{k} M'$ and $N \neq M'$.
- (b) If $\delta < \lambda^+$ is a limit ordinal, $\langle M_i : i < \delta\rangle$ is $\leq_\mathfrak{k}$-increasing sequence, and $M_i \cong M$ for $i < \delta$ then $\bigcup_{i<\delta} M_i \cong M$.

1A) $M \in K_\lambda$ is *globally* superlimit if (a)+(b) hold and

- (c) $M$ is universal in $\mathfrak{k}_\lambda$; i.e. any $N \in K_\lambda$ can be $\leq_\mathfrak{k}$-embedded into $M$.

1B) When we just say *superlimit*, we mean globally. Similarly with the other notions below; we define the global version as adding clause (1A)(c), and the default version will be the global one.

(Note that in the local version we can restrict our class to

$$\{N \in K_\lambda : M \text{ can be } \leq_{\mathfrak{k}}\text{-embedded into } N\}$$

and get the global one.)

2) For $\Theta \subseteq \{\mu \in [\aleph_0, \lambda) : \mu \text{ regular}\}$, $M \in K_\lambda$ is *locally* $(\lambda, \Theta)$*-superlimit* if:

(a) As in part (1) above.
(b) If $\langle M_i : i \leq \mu \rangle$ is $\leq_{\mathfrak{k}}$-increasing, $M_i \cong M$ for $i < \mu$, and $\mu \in \Theta$ then $\bigcup_{i<\mu} M_i \cong M$.

2A) If $\Theta$ is a singleton (say, $\Theta = \{\theta\}$) we may say that $M$ is locally $(\lambda, \theta)$-superlimit.

3) Let $S \subseteq \lambda^+$ be stationary. $M \in K_\lambda$ is called *locally* $S$*-strong limit* or locally $(\lambda, S)$-strong limit when for some function $\mathbf{F} : K_\lambda \to K_\lambda$, we have:

(a) $N \leq_{\mathfrak{k}} \mathbf{F}(N)$ for $N \in K_\lambda$.
(b) If $\delta \in S$ is a limit ordinal, $\langle M_i : i < \delta \rangle$ is a $\leq_{\mathfrak{k}}$-increasing continuous sequence[4] in $K_\lambda$, $M_0 \cong M$, and
$$i < \delta \Rightarrow \mathbf{F}(M_{i+1}) \leq_{\mathfrak{k}} M_{i+2},$$
then $M \cong \bigcup_{i<\delta} M_i$.
(c) If $M \leq_{\mathfrak{k}} M_1 \in K_\lambda$ then there is $N$ such that $M_1 <_{\mathfrak{k}} N \in K_\lambda$.

4) Let $S \subseteq \lambda^+$ be stationary. $M \in K_\lambda$ is called locally $S$-limit or locally $(\lambda, S)$-limit if for some function $\mathbf{F} : K_\lambda \to K_\lambda$ we have:

(a) $N \leq_{\mathfrak{k}} \mathbf{F}(N)$ for $N \in K_\lambda$.
(b) If $\langle M_i : i < \lambda^+ \rangle$ is a $\leq_{\mathfrak{k}}$-increasing continuous sequence of members of $K_\lambda$, $M_0 \cong M$, and $\mathbf{F}(M_{i+1}) \leq_{\mathfrak{k}} M_{i+2}$ then for some closed unbounded[5] subset $C$ of $\lambda^+$,
$$\delta \in S \cap C \Rightarrow M_\delta \cong M.$$
(c) If $M \leq_{\mathfrak{k}} M_1 \in K_\lambda$ then there is $N$ such that $M_1 <_{\mathfrak{k}} N \in K_\lambda$.

5) We define "locally $S$-weak limit" and "locally $S$-medium limit" like "locally $S$-limit", "locally $S$-strong limit" respectively, by demanding that the domain of $\mathbf{F}$ is the family of $\leq_{\mathfrak{k}}$-increasing continuous sequence of members of $\mathfrak{k}_{<\lambda}$ of length $< \lambda$ and replacing "$\mathbf{F}(M_{i+1}) \leq_{\mathfrak{k}} M_{i+2}$" by

$$\text{“}M_{i+1} \leq_{\mathfrak{k}} \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \leq_{\mathfrak{k}} M_{i+2}\text{”}.$$

We replace "limit" by "limit$^-$" if

$$\text{“}\mathbf{F}(M_{i+1}) \leq_{\mathfrak{k}} M_{i+2}\text{” and “}M_{i+1} \leq_{\mathfrak{k}} \mathbf{F}(\langle M_j : j \leq i+1 \rangle) \leq_{\mathfrak{k}} M_{i+2}\text{”}$$

are replaced by "$\mathbf{F}(M_i) \leq_{\mathfrak{k}} M_{i+1}$" and "$M_i \leq_{\mathfrak{k}} \mathbf{F}(\langle M_j : j \leq i \rangle) \leq_{\mathfrak{k}} M_{i+1}$", respectively.

6) If $S = \lambda^+$ then we omit $S$ (in parts (3)-(5)).

---

[4]No loss if we add $M_{i+1} \cong M$, so this simplifies the demand on $\mathbf{F}$; i.e. only $\mathbf{F}(M')$ for $M' \cong M$ are required.

[5]We can use a filter as a parameter.

7) For $\Theta \subseteq \{\mu \in [\aleph_0, \lambda] : \mu \text{ is regular}\}$, $M$ is locally $(\lambda, \Theta)$-strong limit if $M$ is locally $\{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$-strong limit. Similarly for the other notions (where $\Theta \subseteq \{\mu \leq \lambda : \mu \text{ regular}\}$). If we do not write $\lambda$ we mean $\lambda = \|M\|$.

8) We say that $M \in K_\lambda$ is *invariantly strong limit* when in part (3) we demand that $\mathbf{F}$ is just a subset of $\{(M,N)/\cong\, : M \leq_{\mathfrak{k}} N \text{ are from } K_\lambda\}$ and in (3)(b) we replace "$\mathbf{F}(M_{i+1}) \leq_{\mathfrak{k}} M_{i+2}$" by

$$\text{“}(\exists N)\big[M_{i+1} \leq_{\mathfrak{k}} N \leq_{\mathfrak{k}} M_{i+2} \wedge (M_{i+1}, N)/\cong\, \in \mathbf{F}\big],\text{”}$$

but (abusing notation) we still write $N = \mathbf{F}(M)$ instead of $((M,N)/\cong) \in \mathbf{F}$. Similarly with the other notions, so if $\mathbf{F}$ acts on suitable $\leq_{\mathfrak{k}}$-increasing sequence of models then we use the isomorphism type of $\overline{M}\hat{\ }\langle N\rangle$.

*Remark* 3.4. **[Obvious implication diagram:]**

For $\Theta, S_1$ as in 3.3(7) and $S_1 \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) \in \Theta\}$ a stationary subset of $\lambda^+$:

superlimit $= (\lambda, \{\mu \in \mathrm{Reg} : \mu \leq \lambda\})$-superlimit
↓
$(\lambda, \Theta)$-superlimit
↓
$S_1$-strong limit
↙ ↘
$S_1$-medium limit    $S_1$-limit
↘ ↙
$S_1$-weak limit

**Lemma 3.5.** 0) *All the properties are preserved if $S$ is replaced by a subset. and if $\mathfrak{k}$ has the $\lambda$-JEP then the local and global version in Definition 3.3 are equivalent.*

1) *If $S_i \subseteq \lambda^+$ for $i < \lambda^+$, $S := \big\{\alpha < \lambda^+ : (\exists i < \alpha)[\alpha \in S_i]\big\}$, and $S_i \cap i = \varnothing$ for $i < \lambda$, <u>then</u> $M$ is $S_i$-strong limit for each $i < \lambda$ if and only if $M$ is $S$-strong limit.*

2) *Suppose $\kappa \leq \lambda$ is regular, $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa\}$ is a stationary set and $M \in K_\lambda$ <u>then</u> the following are equivalent:*

- (a) *$M$ is $S$-strong limit.*
- (b) *$M$ is $(\lambda, \{\kappa\})$-strong limit.*
- (c) *$M \in \mathfrak{k}_\lambda$ is $\leq_{\mathfrak{k}}$-universal but not $<_{\mathfrak{k}}$-maximal, and there is a function $\mathbf{F} : K_\lambda \to K_\lambda$ satisfying $(\forall N \in K_\lambda)[N \leq_{\mathfrak{k}} \mathbf{F}(N)]$ such that if $M_i \in K_\lambda$ for $i < \kappa$,*
$$i < j \Rightarrow M_i \leq_{\mathfrak{k}} M_j,$$
*$\mathbf{F}(M_{i+1}) \leq_{\mathfrak{k}} M_{i+2}$ and $M_0 \cong M$ <u>then</u> $\bigcup_{i<\kappa} M_i \cong M$.*

2A) *If $S \subseteq \lambda^+$ and $\Theta = \{\mathrm{cf}(\delta) : \delta \in S\}$, <u>then</u> $M$ is $S$-strong limit iff clause (2)(c) above holds for every $\kappa \in \Theta$.*

3) *In part (1) we can replace "strong limit" by "limit", "medium limit" and "weak limit".*

4) *Suppose $\kappa \le \lambda$ is regular, $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \kappa\}$ is a stationary set which belongs to $\check{I}[\lambda]$ (see 0.7, 0.8 above) and $M \in K_\lambda$.*

*The following are equivalent:*

- (a) *$M$ is $S$-medium limit in $\mathfrak{k}_\lambda$.*
- (b) *$M \in K_\lambda$ is $\le_{\mathfrak{k}}$-universal not maximal and there is a function*
$$\mathbf{F} : {}^{\kappa>}K_\lambda \to K_\lambda$$
*such that*
  - ($\alpha$) *For any $\le_{\mathfrak{k}}$-increasing $\langle M_i : i \le \alpha\rangle$, if $M_0 = M$, $\alpha < \kappa$, and $M_i \in K_\lambda$, then $M_\alpha \le_{\mathfrak{k}} \mathbf{F}(\langle M_i : i \le \alpha\rangle)$.*
  - ($\beta$) *If $\langle M_i : i < \kappa\rangle$ is $\le_{\mathfrak{k}}$-increasing, $M_0 = M$, $M_i \in K_\lambda$, and for $i < \kappa$ we have $M_{i+1} \le_{\mathfrak{k}} \mathbf{F}(\langle M_j : j \le i+1\rangle) \le_{\mathfrak{k}} M_{i+2}$ <u>then</u> $\bigcup_{i<\kappa} M_i \cong M$.*

*Proof.* 0) Trivial.

1) Recall that in Definition 3.3(3), clause (b), we use $\mathbf{F}$ only on $M_{i+1}$. (See the proof of (2A) below, second part.)

2) For (c) $\Rightarrow$ (a) note that the demands on the sequence are "local:"
$$M_{i+1} \le_{\mathfrak{k}} \mathbf{F}(M_{i+1}) \le_{\mathfrak{k}} M_{i+2}$$
(whereas in part (4) they are "global").

2A) First assume that $M$ is $S$-strong limit and let $\mathbf{F}$ witness it. Suppose $\kappa \in \Theta$, so we choose $\delta_\kappa \in S$ with $\mathrm{cf}(\delta_\kappa) = \kappa$ and let $\langle \alpha_i : i < \kappa\rangle$ be increasing continuous with limit $\delta$, $\alpha_0 = 0$, and $\alpha_{i+1}$ a successor of a successor ordinal for each $i < \kappa$. We now define $\mathbf{F}_\kappa$ as follows: first we will define $\mathbf{F}_{\kappa,\alpha}$ by induction on $\alpha \le \delta$.

- (a) If $\alpha = 0$ then $\mathbf{F}_{\kappa,0}(M) := M$.
- (b) If $\alpha = \beta + 1$ then $\mathbf{F}_{\kappa,\alpha}(M) := \mathbf{F}(\mathbf{F}_{\kappa,\beta}(M))$.
- (c) If $\alpha \le \delta$ a limit ordinal then $\mathbf{F}_{\kappa,\alpha}(M) := \bigcup_{\beta<\alpha} \mathbf{F}_{\kappa,\beta}(M)$.

Lastly, let $\mathbf{F}_\kappa(M) := \mathbf{F}_{\kappa,\delta}(M)$.

Now suppose $\langle N_i : i \le \kappa\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous, $N_i \in K_\lambda$ and
$$\mathbf{F}_\kappa(N_{i+1}) \le_{\mathfrak{k}} N_{i+2}$$
for $i < \kappa$, and we should prove $N_\kappa \cong M$. Now we can find $\langle M_j : j < \lambda^+\rangle$ such that it obeys $\mathbf{F}$ and $M_{\alpha_i} = N_i$ for $i < \kappa$; so clearly we are done.

Second, assume that for each $\kappa \in \Theta$, clause (c) of 3.5(2) holds and let $\mathbf{F}_\kappa$ exemplify this. Let $\langle \kappa_\varepsilon : \varepsilon < \varepsilon_*\rangle$ list $\Theta$ (so $\varepsilon_* < \lambda^+$) and define $\mathbf{F}$ as follows. For any $M \in \mathfrak{k}$ choose $M_{[\varepsilon]}$ by induction on $\varepsilon \le \varepsilon_*$ as follows:

- $M_{[0]} := M$
- $M_{[\varepsilon+1]} := \mathbf{F}_{\kappa_\varepsilon}(M_{[\varepsilon]})$
- For $\varepsilon$ limit let $M_{[\varepsilon]} := \bigcup_{\zeta<\varepsilon} M_{[\zeta]}$.

Lastly, let $\mathbf{F}[M] := M_{[\varepsilon_*]}$. Now check.

3) No new point.

4) First note that $(a) \Rightarrow (b)$ should be clear. Second, we prove that $(b) \Rightarrow (a)$ so let $\mathbf{F}$ witness that clause (b) holds. Let $E$, $\langle u_\alpha : \alpha < \lambda\rangle$ witness that $S \in \check{I}[\lambda]$; i.e.

$(*)_1$ (a) $E$ is a club of $\lambda$.
(b) $u_\alpha \subseteq \alpha$ and $\mathrm{otp}(u_\alpha) \le \kappa$ for $\alpha < \lambda$.
(c) If $\alpha \in S \cap E$ then $\sup(u_\alpha) = \alpha$ and $\mathrm{otp}(u_\alpha) = \kappa$.
(d) If $\alpha \in \lambda \setminus (S \cap E)$ then $\mathrm{otp}(u_\alpha) < \kappa$.
(e) If $\alpha \in u_\beta$ then $u_\alpha = u_\beta \cap \alpha$.

We can add

$(*)_2$ (f) If $\beta \in u_\alpha$ then $\beta$ is of the form $3\gamma + 1$.

Let $\langle \alpha_\varepsilon : \varepsilon < \lambda \rangle$ list $E$ in increasing order; without loss of generality, $\alpha_0 = 0$ and $\alpha_{1+\varepsilon}$ is a limit ordinal (note that only the limit ordinals of $S$ count).

To define $\mathbf{F}'$ as required we shall deal with the requirement according to whether $\delta \in S$ is "easy" (i.e. $\delta \notin E$, so $\delta \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}]$ for some $\varepsilon < \lambda^+$, so after $\alpha_\varepsilon$ we can "take care of it"), or $\delta$ is "hard" (i.e. $\delta \in E$) so we use the $\alpha \in u_\delta$.

We choose $\langle e_\delta : \delta \in S \setminus E \rangle$ such that $\delta \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}] \cap S$ implies $e_\delta \subseteq \delta = \sup(e_\delta)$ and $\min(e_\delta) > \alpha_\varepsilon$, $\mathrm{otp}(e_\delta) = \kappa$, $e_\delta$ is closed, and

$$\alpha \in e_\delta \Rightarrow \sup(e_\delta \cap \alpha) = \alpha \vee \alpha \in \{3\gamma + 2 : \gamma < \delta\}.$$

If $\delta \in S \cap E$ let $e_\delta$ be the closure of $u_\delta$. Let $\langle \gamma_{\delta,\zeta} : \zeta < \kappa \rangle$ list $e_\delta$ in increasing order.

We now define a function $\mathbf{F}'$; so let $\langle M_j : j \le i + 1 \rangle$ be given and let $\alpha_\varepsilon \le i < \alpha_{\varepsilon+1}$. We fix $\varepsilon$ ([so fixing the interval] $(\alpha_\varepsilon, \alpha_{\varepsilon+1})$) and now define $\mathbf{F}'(\langle M_j : j \le i+1 \rangle)$ by induction on $i \in [\alpha_\varepsilon, \alpha_{\varepsilon+1})$, assuming that if $\alpha_\varepsilon \le j' + 1 < i + 1$ then $\mathbf{F}'(\langle M_j : j \le j' + 1 \rangle) \le_{\mathfrak{k}} M_{j'+2}$. Furthermore, there is

$$\overline{N}^{j'+1} = \langle N_{j'+1,\xi} : \xi < \alpha_{\varepsilon+1} \rangle$$

such that the following holds:

$(*)_3$ $\overline{N}^{j'+1}$ is $\le_{\mathfrak{k}_\lambda}$-increasing continuous, $M_{j'+1} \le_{\mathfrak{k}} N_{j'+1,0}$, and

$$N_{j'+1,\xi} \le_{\mathfrak{k}_\lambda} M_{j'+2}.$$

$(*)_4$ If $\delta \in (S \setminus E) \cap (\alpha_{\varepsilon+1} \setminus \alpha_\varepsilon)$ and $j' + 1 = \gamma_{\delta,\zeta}$ (so necessarily

$$j' + 1 \in (\alpha_\varepsilon, \alpha_{\varepsilon+1}) \cap \{3\gamma + 2 : \gamma < \lambda\}$$

and $\zeta$ is a successor ordinal) then let $\overline{N}^*_{\delta,j'} = \langle N^*_{\delta,j',\zeta'} : \zeta' \le \zeta \rangle$ be the following sequence of length $\zeta + 1$:

$$N^*_{\delta,j',\zeta'} := \begin{cases} N_{\gamma_{\delta,\zeta'},\zeta'} & \text{if } \zeta' \text{ is a successor ordinal} \\ M_{\gamma_{\delta,\zeta'}} & \text{if } \zeta' \text{ is limit or zero.} \end{cases}$$

We demand $\mathbf{F}(\langle N^*_{\delta,j',\zeta'} : \zeta' \le \zeta \rangle) \le_{\mathfrak{k}} N_{j'+1,\zeta+1}$.

$(*)_5$ If $j' + 1 \in u_\delta$ for some $\delta \in S \cap E$ (hence $j' + 1 \in \{3\gamma + 1 : \gamma < \delta\}$ and $\zeta = \mathrm{otp}(u_{j'+1}) < \kappa$), $f_\varepsilon$ is the one-to-one order-preserving function from $\zeta + 1$ onto $c\ell(u_{j'+1} \cup \{j'+1\})$, and $\zeta'$ is a successor, then

$$\mathbf{F}(\langle M_{\alpha_{f_\varepsilon(\zeta')}} : \zeta' \le \zeta \rangle) \le_{\mathfrak{k}} M_{\alpha_\varepsilon + 1}.$$

This implicitly defines $\mathbf{F}'$. Now $\mathbf{F}'$ is as required: $M_i \cong M$ when $i < \lambda$, $\mathrm{cf}(i) = \kappa$ by $(*)_4$ when $(\exists \varepsilon)[\alpha_\varepsilon < i < \alpha_{\varepsilon+1}]$ and by $(*)_5$ when $(\exists \varepsilon)[i = \alpha_\varepsilon]$. $\square_{3.5}$

**Lemma 3.6.** *Let $T$ be a first-order complete theory, $K$ its class of models, and $\leq_{\mathfrak{k}} = \prec_{\mathbb{L}}$.*

1) *If $\lambda$ is regular and $M$ a saturated model of $T$ of cardinality $\lambda$, then $M$ is $(\lambda, \{\lambda\})$-superlimit.*

2) *If $T$ is stable and $M$ is a saturated model of $T$ of cardinality $\lambda$, then $M$ is $(\lambda, [\kappa(T), \lambda] \cap \mathrm{Reg})$-superlimit.*[6] *(Note that by* [She90], *if $\lambda$ is singular and $T$ has a saturated model of cardinality $\lambda$ then $T$ is stable and $\mathrm{cf}(\lambda) \geq \kappa(T)$.)*

3) *If $T$ is stable, $\lambda$ singular $> \kappa(T)$, $M$ a special model of $T$ of cardinality $\lambda$, $S \subseteq \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \mathrm{cf}(\lambda)\}$ is stationary and $S \in \check{I}[\lambda]$ (see 0.7, 0.8) then $M$ is $(\lambda, S)$-medium limit.*

*Remark* 3.7. See more in [She12].

*Proof.* 1) Because if $M_i$ is a $\lambda$-saturated model of $T$ for $i < \delta$ and $\mathrm{cf}(\delta) \geq \lambda$, then $\bigcup\limits_{i<\delta} M_i$ is $\lambda$-saturated. Remembering that a $\lambda$-saturated model of $T$ of cardinality $\lambda$ is unique, we finish.

2) Use [She90, III,3.11]: if $M_i$ is a $\lambda$-saturated model of $T$, $\langle M_i : i < \delta \rangle$ increasing, and $\mathrm{cf}(\delta) \geq \kappa(T)$ then $\bigcup\limits_{i<\delta} M_i$ is $\lambda$-saturated.

3) Should be clear by now. $\square_{3.6}$

**Claim 3.8.** 1) *If $M_\ell \in K_\lambda$ are $S_\ell$-weak limit and $S_0 \cap S_1$ is stationary, then $M_0 \cong M_1$, provided $\kappa$ has $(\lambda, \lambda)$-JEP.*

2) *$K$ has at most one locally weak limit model of cardinality $\lambda$, provided $K$ has the $(\lambda, \lambda)$-JEP.*

3) *If $M \in K_\lambda$ then $\{S \subseteq \lambda^+ : M$ is $S$-weak limit or $S$ not stationary$\}$ is a normal ideal over $\lambda^+$.*

*Instead of "$S$-weak limit", we may use "$S$-medium limit", "$S$-limit", or "$S$-strong limit."*

4) *In Definition 3.3, without loss of generality $\mathbf{F}(N) \cong M$ or $\mathbf{F}(\overline{M}) \cong M$ according to the case (and we can add $N <_{\mathfrak{k}} \mathbf{F}(N)$, etc.)*

5) *If $K$ is categorical in $\lambda$ then the $M \in K_\lambda$ is superlimit, provided that $K_{\lambda^+} \neq \varnothing$ (or equivalently, $M$ has a proper $\leq_{\mathfrak{k}}$-extension).*

*Proof.* Easy.

1) E.g. let $\mathbf{F}_\ell$ witness that $M_\ell$ is $S_\ell$-weak limit. We can choose $(M^0_\alpha, M^1_\alpha)$ by induction on $\alpha$ such that $\langle M^\ell_\beta : \beta \leq \alpha \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous for $\ell = 0, 1$, $M^0_\alpha \leq_{\mathfrak{k}} M^1_{\alpha+1}$, $M^1_\alpha \leq_{\mathfrak{k}} M^0_{\alpha+1}$, and $\mathbf{F}_\ell(\langle M^\ell_\beta : \beta \leq \alpha + 1 \rangle) \leq M^\ell_{\alpha+2}$. So for some club $E_\ell$ of $\lambda^+$, $\delta \in S_\ell \cap E_\ell \Rightarrow M^\ell_\delta \cong M_\ell$ for $\ell = 0, 1$. But $S_0 \cap S_1$ is stationary hence there is a limit ordinal $\delta \in S_0 \cap S_1 \cap E_0 \cap E_1$, hence $M_0 \cong M^0_\delta = M^1_\delta \cong M_1$ as required. $\square_{3.8}$

[6] On $\kappa(T)$, see [She90, III,§3].

**Theorem 3.9.** *If $2^\lambda < 2^{\lambda^+}$, $M \in K_\lambda$ superlimit, $S = \lambda^+$ or $M$ is $S$-weak limit, $S$ is not small (see Definition 0.6) and $M$ does not have the $\lambda$-amalgamation property (in $\mathfrak{k}$) then $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$. Moreover, there is no universal member in $\mathfrak{k}_{\lambda^+}$ and $(2^\lambda)^+ < 2^{\lambda^+} \Rightarrow \dot{I}\dot{E}(\lambda^+, K) = 2^{\lambda^+}$ (that is, there are $2^{\lambda^+}$-many models $M \in K_{\lambda^+}$, no one of them $\leq_{\mathfrak{k}}$-embeddable into another).*

*Remark* 3.10. 0) So in 3.9, if $K$ is categorical in $\lambda$ then it has $\lambda$-amalgamation.

1) We can define a superlimit for a family of models; i.e. when

$$\mathbf{N} := \{N_t : t \in I\} \subseteq \mathfrak{k}_\lambda$$

is superlimit (i.e. if $\langle M_i : i < \delta\rangle$ is $\leq_{\mathfrak{k}}$-increasing, $i < \delta \Rightarrow M_i \in \mathfrak{k}_\lambda$, $\delta < \lambda^+$ a limit ordinal, and $M_\delta = \bigcup\limits_{i<\delta} M_i$ then $\bigwedge\limits_{i<\delta} \bigvee\limits_{t\in I} [M_i \cong N_t] \Rightarrow \bigvee\limits_{t \in I} [M_\delta \cong N_t]$ — and similarly for the other variants).

Of course, the family is contained $K_\lambda$ and non-empty. Essentially, everything generalizes, but in 3.9 the hypothesis should be stronger: the family should satisfy that any member does not have the amalgamation property. (E.g. $\mathbf{N} = \mathfrak{k}_\lambda$ — and we can reduce the general case to this by changing $\mathfrak{k}$). But this complicates the situation and the gain is unclear, so we do not elaborate on this.

2) We can many times (and in particular in 3.9) strengthen "there is no $\leq_{\mathfrak{k}}$-universal $M \in K_{\lambda^+}$" to "there is no $M \in K_\mu$ into which every $N \in K_{\lambda^+}$ can be $\leq_{\mathfrak{k}}$-embedded" for $\mu$ not too large. We need[7] $\neg\mathrm{unif}(\lambda^+, S, 2, \mu)$.

*Proof.* Let $\mathbf{F}$ be as in Definition 3.3(5) for $M$. We now choose by induction on $\alpha < \lambda^+$, models $M_\eta$ for $\eta \in {}^\alpha 2$ such that:

$\circledast_1$ (i) $M_\eta \in K_\lambda$, $M_{\langle\ \rangle} = M$
(ii) If $\beta < \alpha$ and $\eta \in {}^\alpha 2$ then $M_{\eta\restriction\beta} \leq_{\mathfrak{k}} M_\eta$.
(iii) If $i + 2 \leq \alpha$ and $\eta \in {}^\alpha 2$, then $(\mathbf{F}(\langle M_{\eta\restriction j} : j \leq i+1\rangle)) \leq_{\mathfrak{k}} M_{\eta\restriction(i+2)}$.
(iv) If $\alpha = \beta + 1$ and $\beta$ non-limit, $\eta \in {}^\alpha 2$, then $M_{\eta\restriction\beta} \neq M_\eta$.
(v) If $\alpha < \lambda$ is a limit ordinal and $\eta \in {}^\alpha 2$ then:
  (a) $M_\eta = \bigcup\{M_{\eta\restriction\beta} : \beta < \ell g(\eta)\}$
  (b) If $M_\eta$ fails the $\lambda$-amalgamation property then $M_{\eta\hat{}\langle 0\rangle}$, $M_{\eta\hat{}\langle 1\rangle}$ cannot be amalgamated over $M_\eta$; i.e. for no $N \in K$ do we have $M_\eta \leq_{\mathfrak{k}} N$ and $M_{\eta\hat{}\langle 0\rangle}$, $M_{<\eta\hat{}\langle 1\rangle}$ can be $\leq_{\mathfrak{k}}$-embedded into $N$ over $M_\eta$.

For $\alpha = 0$ or $\alpha$ limit we have no problem. For $\alpha + 1$ with $\alpha$ limit: if $M_\eta$ fails the $\lambda$-amalgamation property, use its definition; otherwise, let $M_{\eta\hat{}\langle 1\rangle} = M_\eta = M_{\eta\hat{}\langle 0\rangle}$. For $\alpha + 1$ with $\alpha$ non-limit, use $\mathbf{F}$ to guarantee clause (iii) and then for clause (iv) use Definition 3.3(5) (i.e. 3.3(4)(c)).

For $\eta \in {}^{\lambda^+}2$, let $M_\eta = \bigcup\limits_{\alpha<\lambda^+} M_{\eta\restriction\alpha}$. By changing names we can assume that

$\circledast_1$ (vi) For $\eta \in {}^\alpha 2$ (with $\alpha < \lambda^+$), the universe of $M_\eta$ is an ordinal $< \lambda^+$ (or even $\subseteq \lambda \times (1 + \ell g(\eta))$, and we could even demand equality).

So (by clause (iv)) for $\eta \in {}^{\lambda^+}2$, $M_\eta$ has universe $\lambda^+$.

First, why is there no universal member in $\mathfrak{k}_{\lambda^+}$? If $N \in K_{\lambda^+}$ is universal (by $\leq_{\mathfrak{k}}$, of course), without loss of generality its universe is $\lambda^+$. For $\eta \in {}^{\lambda^+}2$, as $M_\eta \in K_{\lambda^+}$,

[7] See [She98, AP,§1].

there is a $\le_{\mathfrak{k}}$-embedding $f_\eta$ of $M_\eta$ into $N$. So $f_\eta$ is a function from $\lambda^+$ to $\lambda^+$. Let $\eta \in {}^{\lambda^+}2$, so by the choice of $\mathbf{F}$ and of $\langle M_{\eta\restriction\alpha} : \alpha < \lambda^+\rangle$ there is a closed unbounded $C_\eta \subseteq \lambda^+$ such that $\alpha \in S \cap C_\eta \Rightarrow M_{\eta\restriction\alpha} \cong M$, hence $M_{\eta\restriction\alpha}$ fails the $\lambda$-amalgamation property. Without loss of generality, $M_{\eta\restriction\delta}$ has universe $\delta$ for each $\delta \in C_\eta$.

Now by 0.6, if $\langle (f_\rho, C_\rho) : \rho \in {}^{\lambda^+}2\rangle$ is such that $f_\rho : \lambda^+ \to \lambda^+$ and $C_\rho \subseteq \lambda^+$ is closed and unbounded for each $\rho \in {}^{\lambda^+}2$, <u>then</u> for some $\eta \neq \nu \in {}^{\lambda^+}2$ and $\delta \in C_\eta \cap S$, we have $\eta \restriction \delta = \nu \restriction \delta$, $\eta(\delta) \neq \nu(\delta)$, and $f_\eta \restriction \delta = f_\nu \restriction \delta$.

[Why? For every $\delta < \lambda^+$, $\rho \in {}^{\delta}2$, and $f : \delta \to \lambda^+$, we define $\mathbf{c}(\rho, f) \in 2$ as follows: it is 1 <u>iff</u> there is $\nu \in {}^{\lambda^+}2$ such that $\rho = \nu \restriction \delta \wedge f = f_\nu \restriction \delta \wedge \nu(\delta) = 0$ and is 0 otherwise. So some $\eta \in {}^{\lambda^+}2$ is a weak diamond sequence for the colouring $\mathbf{c}$ and the stationary set $S$. Now $C_\eta, f_\eta$ are well defined and

$$S' := \{\delta \in S : \delta \text{ limit and } \eta(\delta) = \mathbf{c}(\eta \restriction \delta, f \restriction \delta)\}$$

is a stationary subset of $\lambda^+$, so we can choose $\delta \in S' \cap C_\eta$. If $\eta(\delta) = 0$, then $\mathbf{c}(\eta \restriction \delta, f \restriction \delta) = 0$ by the choice of $S'$ but $\eta$ witnesses that $\mathbf{c}(\eta \restriction \delta, f \restriction \delta)$ is 1, standing for $\nu$ there. If $\eta(\delta) = 1$ there is $\nu$ witnessing $\mathbf{c}(\eta \restriction \delta, f_\eta \restriction \delta) = 1$; in particular, $\nu(\delta) = 0$ so $\eta, \nu$, and $\eta \restriction \delta$ are as required.]

Now as $\delta \in S \cap C_\eta \subseteq C_\eta$ it follows that $M_{\eta\restriction\delta} \cong M$ hence $M_{\eta\restriction\delta}$ fails the $\lambda$-amalgamation property. Also, $M_{\eta\restriction\delta}$ has universe $\delta$ as $\delta \in C_\eta$, and $M_{\eta\restriction\delta} = M_{\nu\restriction\delta}$ as $\eta \restriction \delta = \nu \restriction \delta$.

So $f_\eta \restriction M_{\eta\restriction\delta} = f_\eta \restriction \delta = f_\nu \restriction \delta = f_\nu \restriction M_{\nu\restriction\delta}$. So $f_\eta \restriction M_{\eta\restriction(\delta+1)}, f_\nu \restriction M_{\nu\restriction(\delta+1)}$ show that $M_{\eta\restriction(\delta+1)}, M_{\nu\restriction(\delta+1)}$ can be amalgamated over $M_{\eta\restriction\delta}$, contradicting clause (v)(b) of the construction (i.e. of $\circledast$). So there is no $\le_{\mathfrak{k}}$-universal $N \in \mathfrak{k}_{\lambda^+}$.

It takes some more effort to get $2^{\lambda^+}$ pairwise non-isomorphic models (rather than just quite many).

<u>**Case A**</u>:[8] There is $M^* \in K_\lambda$ with $M \le_{\mathfrak{k}} M^*$ such that for every $N$ satisfying $M^* \le_{\mathfrak{k}} N \in K_\lambda$, there are $N^1, N^2 \in K_\lambda$ such that $N \le_{\mathfrak{k}} N^1$, $N \le_{\mathfrak{k}} N^2$, and $N^2, N^1$ cannot be $\le_{\mathfrak{k}}$-amalgamated over $M^*$ (not just $N$).

In this case we do not need "$M$ is $S$-weak limit".

We redefine $M_\eta$, $\eta \in {}^{\alpha}2$, $\alpha < \lambda^+$ so that:

$\circledast_2$
- (a) $\nu \lhd \eta \in {}^{\alpha}2 \Rightarrow M_\nu \le_{\mathfrak{k}} M_\eta \in K_\lambda$
- (b) If $\alpha = 0$ then $M_{\langle\,\rangle} = M^*$.
- (c) If $\alpha$ limit and $\eta \in {}^{\alpha}2$ <u>then</u> $M_\eta = \bigcup_{\beta<\alpha} M_{\eta\restriction\beta}$.
- (d) If $\eta \in {}^{\beta}2$ and $\alpha = \beta + 1$, use the assumption for $N = M_\eta$. Now obviously the $(N^1, N^2)$ there satisfies $N^1 \neq N$ and $N^2 \neq N$, so we can have $M_\eta <_{\mathfrak{k}} M_{\eta\hat{}\langle 1\rangle} \in K_\lambda$, $M_\eta <_{\mathfrak{k}} M_{\eta\hat{}\langle 0\rangle} \in K_\lambda$ such that $M_{\eta\hat{}\langle 0\rangle}, M_{\eta\hat{}\langle 1\rangle}$ cannot be amalgamated over $M^*$.

Obviously, the models $M_\eta = \bigcup_{\alpha<\lambda^+} M_{\eta\restriction\alpha}$ for $\eta \in {}^{\lambda^+}2$ are pairwise non-isomorphic over $M^*$, and by 0.4 (as $2^\lambda < 2^{\lambda^+}$) we finish proving $\dot{I}(\lambda^+, \mathfrak{k}) = 2^{\lambda^+}$.

Note also that for each $\eta \in {}^{\lambda^+}2$ the set

$$\{\nu \in {}^{\lambda^+}2 : M_\nu \text{ can be } \le_{\mathfrak{k}}\text{-embedded into } M_\eta\}$$

[8] We can make it a separate claim.

has cardinality $\leq \left|\{f : f \text{ a } \leq_{\mathfrak{k}}\text{-embedding of } M^* \text{ into } M_\eta\}\right| \leq 2^\lambda$. So if $(2^\lambda)^+ < 2^{\lambda^+}$, then by the Hajnal free subset theorem [Haj62] there are $2^{\lambda^+}$-many models $M_\eta \in K_{\lambda^+}$ ($\eta \in {}^{\lambda^+}2$), no one $\leq_{\mathfrak{k}}$-embeddable into another.

**<u>Case B</u>**: Not Case A.

Now we return to the first construction, but we can add

(vii) If $\eta \in {}^{(\alpha+1)}2$ and $M_\eta \leq_{\mathfrak{k}} N^1, N^2$ (both in $K_\lambda$), <u>then</u> $N^1, N^2$ can be $\leq_{\mathfrak{k}}$-amalgamated over $M_{\eta\restriction\alpha}$.

As $\{W \subseteq \lambda^+ : W \text{ is small}\}$ is a normal ideal (see 0.6) and it is on a successor cardinal, it is well known that we can find $\lambda^+$ pairwise disjoint non-small $S_\zeta \subseteq S$ for $\zeta < \lambda^+$. We define a colouring (= function) $\mathbf{c}$:

$\circledast_3$ (a) $\mathbf{c}(\eta,\nu,f)$ will be defined <u>iff</u> $\eta,\nu \in {}^\delta 2$ for some limit ordinal $\delta < \lambda^+$, and $f$ is a function from $\delta$ to $\lambda^+$.

(b) $\mathbf{c}(\eta,\nu,f) = 1$ <u>iff</u> the triple $(\eta,\nu,f)$ belongs to the domain of $\mathbf{c}$ (i.e. is as in (a)) and $M_\eta, M_\nu$ have universe $\delta$, $f$ is a $\leq_{\mathfrak{k}}$-embedding of $M_\eta$ into $M_\nu$, and for some $\rho$ with $\nu{}^\frown\langle 0\rangle \vartriangleleft \rho \in {}^{\lambda^+}2$ the function $f$ can be extended to a $\leq_{\mathfrak{k}}$-embedding of $M_{\eta^\frown\langle 0\rangle}$ into $M_\rho$.

(c) $\mathbf{c}(\eta,\nu,f)$ is zero <u>iff</u> it is defined but is $\neq 1$.

For each $\zeta < \lambda^+$, as $S_\zeta$ is not small, by a simple coding there is $h_\zeta : S_\zeta \to \{0,1\}$ such that:

$(*)_\zeta$ For every $\eta,\nu \in {}^{\lambda^+}2$ and $f : \lambda^+ \to \lambda^+$, for a stationary set of $\delta \in S_\zeta$,
$$\mathbf{c}(\eta\restriction\delta, \nu\restriction\delta, f\restriction\delta) = h_\zeta(\delta).$$

Now, for every $W \subseteq \lambda^+$ we define $\eta_W \in {}^{\lambda^+}2$ as follows:
$$\eta_W(\alpha) := \begin{cases} h_\zeta(\alpha) & \text{if } \zeta \in W \text{ and } \alpha \in S_\zeta, \\ 0 & \text{if there is no such } \zeta. \end{cases}$$

(Note that there is at most one $\zeta$.)

Now we can show (chasing the definitions) that

$\circledast_4$ If $W_1, W_2 \subseteq \lambda^+$ and $W_1 \nsubseteq W_2$, <u>then</u> $M_{\eta_{W_1}}$ cannot be $\leq_{\mathfrak{k}}$-embedded into $M_{\eta_{W_2}}$.

This clearly suffices.

Why is $\circledast_4$ true? Suppose $W_1 \nsubseteq W_2$; let $\zeta \in W_1 \setminus W_2$, and toward contradiction let $f$ be a $\leq_{\mathfrak{k}}$-embedding of $M_{\eta_{W_1}}$ into $M_{\eta_{W_2}}$, so
$$E := \{\delta : M_{\eta_{W_1}\restriction\delta} \text{ and } M_{\eta_{W_2}\restriction\delta} \text{ have universe } \delta, \text{ and } f\restriction\delta \text{ is a } \leq_{\mathfrak{k}}\text{-embedding of } M_{\eta_{W_1}\restriction\delta} \text{ into } M_{\eta_{W_2}\restriction\delta}\}$$
is a club of $\lambda^+$. Hence by the choice of $\mathbf{c}$ and $h_\zeta$ there is $\delta \in E \cap S_\zeta$ such that

$\boxtimes$ $\mathbf{c}(\eta_{W_1}\restriction\delta, \eta_{W_2}\restriction\delta, f\restriction\delta) = h_\zeta(\delta)$ and $M_{\eta_{w(1)}\restriction\delta}$ is not an amalgamation base.

Now the proof splits to two cases.

**Case 1**: $h_\zeta(\delta) = 0$.

So $\eta_{W_1}(\delta) = \eta_{W_2}(\delta) = 0$, and by clause (b) of $\circledast_3$ above (i.e. the definition of $\mathbf{c}$) we have the objects $\eta_{W_1}, \eta_{W_2}$, and $f \restriction M_{\eta_{W_1}\char`^\langle 0\rangle} = f \restriction M_{\eta_{W_1}\restriction(\delta+1)}$ witness that $\mathbf{c}(\eta_{W_1} \restriction \delta, \eta_{W_2} \restriction \delta, f \restriction \delta) = 1$, a contradiction.

**Case 2**: $h_\zeta(\delta) = 1$.

So $\eta_{W_1}(\delta) = 1$, $\eta_{W_2}(\delta) = 0$, $\mathbf{c}(\eta_{W_1} \restriction \delta, \eta_{W_2} \restriction \delta, f \restriction \delta) = 1$. By the definition of $\mathbf{c}$, we can find $\nu \in {}^{\lambda^+}2$ such that $(\eta_{W_2} \restriction \delta)\char`^\langle 0\rangle \trianglelefteq \nu$ and a $\leq_{\mathfrak{k}}$-embedding $g$ of $M_{(\eta_{W_1}\restriction\delta)\char`^\langle 0\rangle}$ into $M_\nu$.

For some $\alpha \in (\delta, \lambda^+)$, $f$ embeds $M_{\eta_{W_1}\restriction(\delta+1)} = M_{(\eta_{W_1}\restriction\delta)\char`^\langle 1\rangle}$ into $M_{\eta_{W_2}\restriction\alpha}$ and $g$ embeds $M_{(\eta_{W_1}\restriction\delta)\char`^\langle 0\rangle}$ into $M_{\nu\restriction\alpha}$.

As $\eta_{W_2} \restriction \delta\char`^\langle 0\rangle \triangleleft \nu \restriction \alpha$ and $\eta_{W_2} \restriction \delta\char`^\langle 0\rangle \triangleleft \eta_{W_2} \restriction \alpha$ by clause (vii) above, there are $f_1, g_1$ and $N \in K_\lambda$ such that

(a) $M_{\eta_{W_2}\restriction\delta} \leq_{\mathfrak{k}} N$
(b) $f_1$ is a $\leq_{\mathfrak{k}}$-embedding of $M_{\eta_{W_2}\restriction\alpha}$ into $N$ over $M_{\eta_{W_2}\restriction\delta}$.
(c) $g_1$ is a $\leq_{\mathfrak{k}}$-embedding of $M_{\nu\restriction\alpha}$ into $N$ over $M_{\eta_{W_2}\restriction\delta}$.

So **[I don't understand the numbering here]**

(b)* $f_1 \circ f$ is a $\leq_{\mathfrak{k}}$-embedding of $M_{(\eta_{W_1}\restriction\delta)\char`^\langle 1\rangle}$ into $N$
(c)* $g_1 \circ g$ is a $\leq_{\mathfrak{k}}$-embedding of $M_{(\eta_{W_1}\restriction\delta)\char`^\langle 0\rangle}$ into $N$
(d)* $f_1 \circ f$ and $g_1 \circ g$ both extend $f \restriction \delta : M_{\eta_{W_1}\restriction\delta} \to N$.

So together we get a contradiction to assumption $(*)_1(\mathrm{d})$. $\square_{3.9}$

**[There is no $(*)_1(d)$. There's a $\circledast_1$(iv); maybe that's it?]**

**Theorem 3.11.** 1) *Assume one of the following cases occurs:*

$(a)_1$ $\mathfrak{k}$ *is* $\mathsf{PC}_{\aleph_0}$ *(hence* $\mathrm{LST}_{\mathfrak{k}} = \aleph_0$*) and* $1 \leq \dot{I}(\aleph_1, \mathfrak{k}) < 2^{\aleph_1}$
*<u>or</u>*
$(a)_2$ $\mathfrak{k}$ *has models of arbitrarily large cardinality,* $\mathrm{LST}_{\mathfrak{k}} = \aleph_0$, *and* $\dot{I}(\aleph_1, \mathfrak{k}) < 2^{\aleph_1}$.

*<u>Then</u> there is an AEC* $\mathfrak{k}_1$ *such that*

(A) $M \in K_1 \Rightarrow M \in K$, $M \leq_{\mathfrak{k}_1} N \Rightarrow M \leq_{\mathfrak{k}} N$, *and* $\mathrm{LST}_{\mathfrak{k}_1} = \mathrm{LST}_{\mathfrak{k}} = \aleph_0$.
(B) *If* $K$ *has models of arbitrarily large cardinality <u>then</u> so does* $K_1$.
(C) $\mathfrak{k}_1$ *is* $\mathsf{PC}_{\aleph_0}$.
(D) $(K_1)_{\aleph_1} \neq \varnothing$
(E) *All models of* $K_1$ *are* $\mathbb{L}_{\infty,\omega}$*-equivalent,*

$$M \leq_{\mathfrak{k}_1} N \Leftrightarrow M \prec_{\mathbb{L}_{\infty,\omega}} N \wedge M \leq_{\mathfrak{k}} N,$$

$K_1$ *is categorical in* $\aleph_0$, *and*

$$M_* \in (K_1)_{\aleph_0} \Rightarrow K_1 = \{N \in K : N \equiv_{\mathbb{L}_{\infty,\omega}(\tau_K)} M_*\}.$$

(F) *if* $\mathfrak{k}$ *is categorical in* $\aleph_1$ *<u>then</u>* $(K_1)_\lambda = K_\lambda$ *for every* $\lambda > \aleph_0$*; moreover,* $\leq_{\mathfrak{k}_1} = \leq_{\mathfrak{k}} \restriction (K_1)_{\geq\aleph_1}$.

2) *If in* (1) *we add* $\mathrm{LST}_{\mathfrak{k}}$ *names to formulas in* $\mathbb{L}_{\infty,\omega}$ *(i.e. to a set of representations up to equivalence)* *then* *we can assume each member of* $K$ *is* $\aleph_0$*-sequence-homogeneous. The vocabulary remains countable; in fact, for some countable first-order theory* $T$*, the models of* $K$ *are the atomic models of* $T$ *(in the first-order sense) and* $\le_{\mathfrak{k}}$ *becomes* $\subseteq$ *(being a submodel).*

*Proof.* Like [She75a, 2.3,2.5] (using 2.20 here for $\alpha = 2$). E.g. why, if $K$ is categorical in $\aleph_1$ then $\le_{\mathfrak{k}_1} = \le_{\mathfrak{k}} \restriction (K_1)_{\ge\aleph_1}$? We have to prove that if $M \le_{\mathfrak{k}} N$ are uncountable then $M \prec_{\mathbb{L}_{\infty,\omega}(\tau_K)} N$. But there is $M_* \in K_{\aleph_0}$ such that

$$K_1 = \{M' \in K : M' \equiv_{\mathbb{L}_{\infty,\omega}} M_*\}$$

and $(K_1)_{\aleph_1} = K_{\aleph_1} \neq \varnothing$, so it suffices to prove $M \prec_{\mathbb{L}_{\omega_1,\omega}(T)} N$, so assume this is a counterexample.

So for some $\varphi(x,\bar{y}) \in \mathbb{L}_{\omega_1,\omega}(\tau)$, $\bar{a} \in {}^{\ell g(\bar{y})}M$, and $b \in N$ we have $N \models \varphi[b,\bar{a}]$ but for no $b' \in M$ do we have $N \models \varphi[b',\bar{a}]$. Without loss of generality the quantifier depth of $\varphi(x,\bar{y})$ (call it $\gamma$) is minimal, for all such pairs $(M,N)$. Let

$$\Delta_\gamma := \{\psi(\bar{z}) \in \mathbb{L}_{\omega_1,\omega}(\tau_K) : \psi \text{ has quantifier depth } \le \gamma\}$$

hence $M' \le_{\mathfrak{k}} N' \wedge M' \in K_{>\aleph_0} \Rightarrow M' \prec_{\Delta_\gamma} N'$. Also without loss of generality, $\|M\| = \|N\| = \aleph_1$. Now choose $M_\alpha \in K_{\aleph_1}$ by induction on $\alpha < \omega_2$ to be $\le_{\mathfrak{k}}$-increasing continuous (hence $\prec_{\Delta_\gamma}$-increas[ing]) and for each $\alpha$ there is an isomorphism $f_\alpha$ from $N$ onto $M_{\alpha+1}$ mapping $M$ onto $M_\alpha$, recalling the categoricity. By Fodor's lemma, for some $\alpha < \beta$ we have $f_\alpha(\bar{a}) = f_\beta(\bar{a})$, so $f_\beta^{-1}(f_\alpha(b))$ contradicts the choice of $\varphi(x,\bar{y})$, $b$, and $\bar{a}$. $\square_{3.11}$

We arrive to the main theorem of this section.

**Theorem 3.12.** *Suppose* $\mathfrak{k}$ *and* $\lambda$ *satisfy the following conditions:*

- $(A)$ $\mathfrak{k}$ *has a superlimit member* $M^*$ *of cardinality* $\lambda \ge \mathrm{LST}_{\mathfrak{k}}$.
  *(If* $K$ *is categorical in* $\lambda$*, then by assumption* $(B)$ *below there is such* $M^*$*; really, 'invariantly* $\lambda^+$*-strong limit' suffices if* $(*)(d)$ *of* 3.13(2) *below holds.*[9]*)*
- $(B)$ $\mathfrak{k}$ *is categorical in* $\lambda^+$.
- $(C)$ $(\alpha)$ $\mathfrak{k}$ *is* $\mathsf{PC}_{\aleph_0}$ *and* $\lambda = \aleph_0$,
  *or*
  $(\beta)$ $\mathfrak{k} = \mathsf{PC}_\lambda$, $\lambda = \beth_\delta$, $\mathrm{cf}(\delta) = \aleph_0$,
  *or*
  $(\gamma)$ $\lambda = \aleph_1$ *and* $\mathfrak{k}$ *is* $\mathsf{PC}_{\aleph_0}$,
  *or*
  $(\delta)$ $\mathfrak{k}$ *is* $\mathsf{PC}_\mu$ *and* $\lambda \ge \beth_{(2^\mu)^+}$. *(This is not useful for 3.12; still, it too implies* $(*)_{\lambda,\mu}$ *in 3.13.)*

*Then* $K$ *has a model of cardinality* $\lambda^{++}$.

*Remark* 3.13. 1) If $\lambda = \aleph_0$ we can waive hypothesis (A) by the previous theorem (3.11).

2) Hypothesis (C) can be replaced by the following (giving a stronger theorem):

$(*)_{\lambda,\mu}$ (a) $\mathfrak{k}$ is $\mathsf{PC}_\mu$.

[9] See Definition 3.3.

(b) Any $\psi \in \mathbb{L}_{\mu^+,\omega}$ which has a model $M$ of order-type $\lambda^+$ [and] $|P^M| = \lambda$, has a non-well-ordered model $N$ of cardinality $\lambda$.

(c) $\{M \in K_\lambda : M \cong M^*\}$ is $\mathsf{PC}_\mu$ (among models in $K_\lambda$).

(d) for some $\mathbf{F}$ witnessing "$M^*$ is invariantly $\lambda$-strong limit," that is the class $\big\{(M, \mathbf{F}(M)) : M \in K_\lambda\big\}$ is $\mathsf{PC}_\mu$. (If $M^*$ is superlimit this clause is not required, as $\mathbf{F} = \mathrm{id}_{K_\lambda}$ is okay.)

3) It is well known, see e.g. [She90, VII,§5] that hypothesis (C) implies $(*)_{\lambda,\mu}$ from part (2), see more [GS].

*Proof.* By 3.13(3) we can assume $(*)_{\lambda,\mu}$ from 3.13(2).

**Stage A:** It suffices to find $N_0 \le_{\mathfrak{k}} N_1$, $\|N_0\| = \lambda^+$, $N_0 \ne N_1$.

Why? We define a model $N_\alpha \in K_{\lambda^+}$ by induction on $\alpha < \lambda^{++}$ such that $\beta < \alpha$ implies $N_\beta \le_{\mathfrak{k}} N_\alpha$ and $N_\beta \ne N_\alpha$. Clearly $N_0, N_1$ are defined (without loss of generality $\|N_1\| = \lambda^+$ as $\lambda \ge \mathrm{LST}_{\mathfrak{k}}$, as otherwise we already have the desired conclusion). For limit $\delta < \lambda^{++}$, the model $\bigcup\limits_{\alpha<\delta} N_\alpha$ is as required. For $\alpha = \beta + 1$, by the $\lambda^+$-categoricity, $N_0$ is isomorphic to $N_\beta$ (say, by $f$) and we define $N_{\beta+1}$ such that $f$ can be extended to an isomorphism from $N_1$ onto $N_{\beta+1}$, so clearly $N_{\beta+1}$ is as required. Now $\bigcup\limits_{\alpha<\lambda^{++}} N_\alpha \in K_{\lambda^{++}}$ is as required. Hence the following theorem will complete the proof of 3.12 (use $\mathbf{F}$ = the identity for the superlimit case). $\square_{\mathrm{A}}$

We can find $N_0, N_1 \in K^{\mathbf{F}}_{\lambda^+}$ such that $N_0 \le_{\mathfrak{k}} N_1$ and $N_0 \ne N_1$ when the following clauses hold:

**Theorem 3.14.** *Suppose the following:*

*(A) $\mathfrak{k}$ has an invariantly $\lambda$-strong limit member $M^*$ of cardinality $\lambda$, as exemplified by $\mathbf{F} : K_\lambda \to K_\lambda$, and $\mathfrak{k}_\lambda$ has the* JEP *(see Definition 3.3).*

*(B) $\dot{I}(\lambda^+, K_{\lambda^+}) < 2^{\lambda^+}$ or even just $\dot{I}(\lambda^+, K^{\mathbf{F}}_{\lambda^+}) < 2^{\lambda^+}$ (or just $\dot{I}\dot{E}(\lambda^+, K^{\mathbf{F}}_{\lambda^+}) < 2^{\lambda^+}$; see below).*

*(C) $\mathfrak{k}$ is a $\mathsf{PC}_\mu$ class, as well as $\mathbf{F}$; i.e. $K'$ is $\mathsf{PC}_\mu$ where $K'$ is a class closed under an isomorphism of $(\tau_{\mathfrak{k}} \cup \{P\})$-models and $P$ a unary predicate such that $K'_\lambda = \{(N, M) : N = \mathbf{F}(M)\}$.*

*(D) $\mu = \lambda = \aleph_0$, or $\mu = \lambda = \beth_\delta$ with $\mathrm{cf}(\delta) = \aleph_0$, or $\mu = \aleph_0$ and $\lambda = \aleph_1$, or just $(*)_{\lambda,\mu}(c)$ from 3.13(2).*

*(E) $K$ is categorical in $\lambda$, or at least there is $\psi \in \mathbb{L}_{\omega_1,\omega}(\tau^+)$ such that*
$$(M^*/\cong) = \{M \restriction \tau_{\mathfrak{k}} : M \models \psi,\ \|M\| = \lambda\}.$$

Here we define

**Definition 3.15.** Assume $\mathbf{F} : K_\lambda \to K_\lambda$ satisfies $M \le_{\mathfrak{k}} \mathbf{F}(M)$ for $M \in K_\lambda$; or more generally, $\mathbf{F} \subseteq \{(M, N) : M \le_{\mathfrak{k}} N$ are from $K_\lambda\}$ satisfies
$$(\forall M \in K_\lambda)(\exists N)\big[(M, N) \in \mathbf{F}\big]$$
or just
$$(\forall M \in K_\lambda)(\exists N_0, N_1)\big[(N_0, N_1) \in \mathbf{F} \wedge M \le_{\mathfrak{k}} N_0 \le_{\mathfrak{k}} N_1\big].$$

Then we let

$$K^{\mathbf{F}}_{\lambda^+} := \Big\{ \bigcup_{i<\lambda^+} M_i : M_i \in K_\lambda,\ \langle M_i : i < \lambda^+\rangle \text{ is } \le_{\mathfrak{k}}\text{-increasing continuous and not eventually constant, and } \mathbf{F}(M_{i+1}) \le_{\mathfrak{k}} M_{i+2} \text{ or } (M_{i+1}, M_{i+2}) \in \mathbf{F}\Big\}$$

for $i < \lambda$.

*Remark* 3.16. 1) As the sequence in the definition of $K^{\mathbf{F}}_{\lambda^+}$ is $\le_{\mathfrak{k}}$-increasing and not eventually constant (which follows if $(M,N) \in \mathbf{F} \Rightarrow M \neq N$), necessarily $K^{\mathbf{F}}_{\lambda^+} \subseteq \mathfrak{k}_{\lambda^+}$.

2) Theorem 3.14 is good for classes which are not exactly AEC; see (e.g.) 3.19.

Considering $K^{\mathbf{F}}_{\lambda^+}$, we may note that the proofs of some earlier claims give us more. In particular (before proving 3.14), similarly to 3.9:

**Claim 3.17.** *Assume that*

- $(a)$ $2^\lambda < 2^{\lambda^+}$
- $(b)$ $\mathfrak{k}$ *is an AEC and* $\mathrm{LST}_{\mathfrak{k}} \le \lambda$.
- $(c)$ $M \in K_\lambda$ *is $S$-weak limit, $S$ not small (see Definition 0.6).*
- $(d)$ *$M$ does not have the amalgamation property in $\mathfrak{k}$ (= 'is an amalgamation base').*
- $(e)$ $\mathbf{F}$ *is as in 3.15.*

*Then* $\dot{I}(\lambda^+, K^{\mathbf{F}}_{\lambda^+}) = 2^{\lambda^+}$.

*Proof.* To avoid confusion, rename $\mathbf{F}$ of clause (e) as $\mathbf{F}_1$, and choose $\mathbf{F}_2$ which exemplifies "$M$ is $S$-weak limit" (i.e. as in Definition 3.3(5)). Now we define $\mathbf{F}'$ with the same domain as $\mathbf{F}_2$ by

$$\mathbf{F}'(\langle M_j : j \le i\rangle) := \mathbf{F}_1\big(\mathbf{F}_2(\langle M_j : j \le i\rangle)\big),$$

and continue as in the proof of 3.9 (noting that $\mathbf{F}'$ works there as well).

The sequence of models $\langle M_\eta : \eta \in {}^{\lambda^+}2\rangle$ we got there are from $K^{\mathbf{F}_1}_{\lambda^+}$ (so they witness that $\dot{I}(\lambda^+, K^{\mathbf{F}_1}_{\lambda^+}) = 2^{\lambda^+}$) because:

- $(*)$ If the sequence $\langle M_\alpha : \alpha < \lambda^+\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous with $M_\alpha \in \mathfrak{k}_\lambda$ for $\alpha < \lambda^+$ and $\mathbf{F}'(\langle M_j : j \le i+1\rangle) \le_{\mathfrak{k}} M_{i+2}$, then $\bigcup_{\alpha<\lambda^+} M_\alpha \in K^{\mathbf{F}_1}_{\lambda^+}$.

$\square_{3.17}$

Also similarly to 3.11, we can prove:

**Claim 3.18.** *Assume $\mathfrak{k}$ is a $\mathsf{PC}_{\aleph_0}$ and $\mathbf{F}$ a $\mathsf{PC}_{\aleph_0}$ is as in 3.15. If*

$$1 \le \dot{I}(\aleph_1, K^{\mathbf{F}}_{\aleph_1}) < 2^{\aleph_1}$$

*then the conclusion of 3.11 above holds.*

*Proof.* **[Proof of 3.14]** (Hence of 3.12.)

The reader may do well to read it with '$\mathbf{F}$ = the identity' in mind.

**Stage B:** We now try to find $N_0, N_1$ as mentioned in Stage A above by approximations of cardinality $\lambda$. A triple will denote here $(M, N, a)$ satisfying $M, N \cong M^*$ (see hypothesis 3.14(A)), $M \leq_{\mathfrak{k}} N$ and $a \in N \setminus M$. Let $<$ be the following partial order among this family of triples: $(M, N, a) < (M', N', a')$ if $a = a'$, $N \leq_{\mathfrak{k}} N'$, $M \leq_{\mathfrak{k}} M'$, $M \neq M'$, and moreover $(\exists N'')[N \leq_{\mathfrak{k}} N'' \wedge \mathbf{F}(N'') \leq_{\mathfrak{k}} N']$ and

$$(\exists M'')[M \leq_{\mathfrak{k}} M'' \wedge \mathbf{F}(M'') \leq_{\mathfrak{k}} M'].$$

(It is tempting to omit $a$ and require $M = M' \cap N$, but this apparently does not work as we do [not] know if disjoint amalgamation $\mathfrak{k}_{\aleph_0}$ exists).

We first note that there is at least one triple (as $M^*$ has a proper elementary extension which is isomorphic to it, because it is a limit model by clause (A) of the assumption).

**Stage C:** We show that if there is no maximal triple, our conclusion follows.

We choose a triple $(M_\alpha, N_\alpha, a)$ by induction on $\alpha$, increasing by $<$. For $\alpha = 0$ see the end of previous stage; for $\alpha = \beta + 1$, we can define $(M_\alpha, N_\alpha, a)$ by the hypothesis of this stage. For limit $\delta < \lambda^+$, $(M_\delta, N_\delta, a)$ will be $\big(\bigcup_{\alpha<\delta} M_\alpha, \bigcup_{\alpha<\delta} N_\alpha, a\big)$. (Notice $M_\delta \leq_{\mathfrak{k}} N_\delta$ by **Ax.IV** of 1.2 and $M_\delta, N_\delta$ are isomorphic to $M^*$ by the choice of $\mathbf{F}$ and the definition of order on the family of triples.) Now similarly $M := \bigcup_{\alpha<\lambda^+} M_\alpha \leq_{\mathfrak{k}} N := \bigcup_{\alpha<\lambda^+} N_\alpha$ are both from $\mathfrak{k}^{\mathbf{F}}_{\lambda^+}$ and the element $a$ exemplifies $M \neq N$, so by Stage A we finish.

Recall

⊛ If $(M, N, a)$ is a maximal triple <u>then</u> there is no triple $(M', N', a)$ such that $M' \leq_{\mathfrak{k}} N'$, $M <_{\mathfrak{k}} M'$, $N \leq_{\mathfrak{k}} N'$, $a \in N' \setminus M'$,
$$(\exists M'')\big[M \leq_{\mathfrak{k}} M'' \leq_{\mathfrak{k}} \mathbf{F}(M'') \leq_{\mathfrak{k}} M'\big],$$
and $(\exists N'')\big[N \leq_{\mathfrak{k}} N'' \leq_{\mathfrak{k}} \mathbf{F}(N'') \leq_{\mathfrak{k}} N'\big]$.

**Stage D:** There are $M_i \cong M^*$ for $i \leq \omega$ such that

$$i < j \leq \omega \Rightarrow M_j <_{\mathfrak{k}} M_i \wedge \mathbf{F}(M_{i+1}) \leq_{\mathfrak{k}} M_i$$

and $|M_\omega| = \bigcap_{n<\omega} |M_n|$ (and note that $M_i$ is $\lambda^+$-strong limit).

This stage is dedicated to proving this statement. As $M^*$ is superlimit (or just strong limit), there is an $\leq_{\mathfrak{k}}$-increasing continuous sequence $\langle M_i : i < \lambda^+ \rangle$ with $M_i \cong M^*$ and $\mathbf{F}(M_{i+1}) \leq_{\mathfrak{k}} M_{i+2}$. (Note that this is true also for limit models as we can restrict ourselves to a club of $i$-s). So without loss of generality $\bigcup_{i<\lambda^+} M_i$ has universe $\lambda^+$ and $M_0$ has universe $\lambda$.

Define a model $\mathfrak{B}$; first, its universe will be $\lambda^+$.

<u>Relations and Functions</u>:

(a) Those of $\bigcup_{i<\lambda^+} M_i$.

(b) $R$, a two-place relation: $a \, R \, i$ if and only if $a \in M_i$.

(c) $P$ (a monadic relation): $P = \lambda$, which is the universe of $M_0$.

(d) $g$, a two-place function such that for each $i$, $g(i, -)$ is an isomorphism from $M_0$ onto $M_i$.

(e) $<$ (a two-place relation) — the usual ordering on the ordinals $< \lambda^+$.

(f) Relations with parameter $i$ witnessing $M_i \le_{\mathfrak{k}} \bigcup_{j<\lambda^+} M_j$. (We can instead make functions witnessing $M \in K$ as in 1.11 (the strong version) and have that each $M_i$ is closed under them.)

(g) Relations with parameter $i$ witnessing each $\mathbf{F}(M_{i+1}) \le_{\mathfrak{k}} M_{i+2}$ and $M_{i+1} \neq M_{i+2}$ (including $(M_{i+1}, \mathbf{F}(M_{i+1})) \in \mathbf{F}$).

(h) If $\mu = \lambda$, then also individual constants for each $a \in M_0$.

Let $\psi \in \mathbb{L}_{\mu^+,\omega}$ describe this. In particular, for clauses (f), (g) use clause (C) of the assumptions. So $\psi$ has a non-well ordered model $\mathfrak{B}^*$ with $|P^{\mathfrak{B}^*}| = \lambda$ by clause (D) of the assumption (see 3.13(2),(3)). So let

$$\mathfrak{B}^* \models \text{“}a_{n+1} < a_n\text{” for } n < \omega.$$

For $a \in \mathfrak{B}^*$, let $A_a := \{x \in \mathfrak{B}^* : \mathfrak{B}^* \models x\, R\, a\}$ and

$$M_a := (\mathfrak{B}^* \restriction \tau_{\mathfrak{k}}) \restriction A_a.$$

Easily, $M_a \le_{\mathfrak{k}} (\mathfrak{B}^* \restriction \tau_{\mathfrak{k}})$ (use clause (f)) and $\|M_a\| = \lambda$. In fact, $M_a$ is superlimit (or just isomorphic to $M^*$) if $\mu = \lambda$, as $\psi$ includes the diagram of $M_0 = M^*$, having names for all members. If $\mu < \lambda$, see assumption (E). So $M_{a_n} \le_{\mathfrak{k}} \mathfrak{B}^* \restriction \tau_{\mathfrak{k}}$ and $M_{a_{n+1}} \subseteq M_{a_n}$, hence $M_{a_{n+1}} \le_{\mathfrak{k}} M_{a_n}$ by **Ax.V**. Let $M_n := M_{a_n}$. Let

$$I := \big\{b \in \mathfrak{B}^* : \bigwedge_{n<\omega} [\mathfrak{B}^* \models b < a_n]\big\}.$$

Also as $M_b <_{\mathfrak{k}} \mathfrak{B}^* \restriction \tau_{\mathfrak{k}}$ for $b \in I$ and $M_{b_1} <_{\mathfrak{k}} M_{b_2}$ for $b_1 <^{\mathfrak{B}^*} b_2$, by **Ax.IV** clearly $M_\omega := (\mathfrak{B}^* \restriction \tau_{\mathfrak{k}}) \restriction \bigcup_{b\in I} A_b$ satisfies $M_\omega \le_{\mathfrak{k}} \mathfrak{B}^* \restriction \tau_{\mathfrak{k}}$, hence $M_\omega \le_{\mathfrak{k}} M_n$ for $n < \omega$. Obviously $M_\omega \subseteq \bigcap_{n<\omega} M_n$, and equality holds as $\psi$ guarantees

(∗) For every $y \in \mathfrak{B}^*$ there is a minimal $x \in \mathfrak{B}^*$ such that $y \in M_x$.

As each $M_b$ is isomorphic to $M^*$ and of cardinality $\lambda$, $M_\omega$ must be as well.

**Stage E:** Suppose that there is a maximal triple, then we shall show $\dot{I}(\lambda^+, K) = 2^{\lambda^+}$ and moreover $\dot{I}(\lambda^+, K^{\mathbf{F}}_{\lambda^+}) = 2^{\lambda^+}$, and so we shall get a contradiction to assumption (B).

So there is a maximal triple $(M^0, N^0, a)$. Hence by the uniqueness of the limit model for each $M \in K_\lambda$ which is isomorphic to $M^*$ hence to $M^0$ there are $N, a$ satisfying $M \le_{\mathfrak{k}} N \cong M^* \in K_\lambda$ and $a \in N \setminus M$ such that if $M <_{\mathfrak{k}} M' \le_{\mathfrak{k}} N' \in \mathfrak{k}_\lambda$, $N <_{\mathfrak{k}} N'$,

$$(\exists M'')[M \le_{\mathfrak{k}} M'' \le_{\mathfrak{k}} \mathbf{F}(M'') \le_{\mathfrak{k}} M' \cong M^*],$$

and

$$(\exists N'')[N \le_{\mathfrak{k}} N'' \le_{\mathfrak{k}} \mathbf{F}(N'') \le_{\mathfrak{k}} N' \cong M^*]$$

then $a \in M'$. (That is, in some sense $a$ is algebraic over $M$). We can waive the latter, as by the definition of strong limit there is $N'_* \cong M^*$ such that $\mathbf{F}(N') \le_{\mathfrak{k}} N'_*$. On the other hand, by Stage **D**:

$(*)_1$ For each $M \in K_\lambda$ isomorphic to $M^*$ there are $M'_n$ (for $n < \omega$) such that

$$M \leq_{\mathfrak{k}} M'_{n+1} <_{\mathfrak{k}} M'_n \in K_\lambda,$$

$$M'_n \cong M^*, \mathbf{F}(M'_{n+1}) \leq_{\mathfrak{k}} M'_n, \text{ and } \bigcap_{n<\omega} M'_n = M.$$

For notational simplicity: for $M \in K_\lambda$, $|M|$ an ordinal $\Rightarrow |\mathbf{F}(M)|$ an ordinal.

Now for each $S \subseteq \lambda^+$ we define $M^S_\alpha$ by induction on $\alpha \leq \lambda^+$, increasing (by $<_{\mathfrak{k}}$) and continuous with universe an ordinal $< \lambda^+$ such that $M^S_\alpha \cong M^*$ and if $\beta+2 \leq \alpha$ then $\mathbf{F}(M_{\beta+1}) \leq_{\mathfrak{k}} M_{\beta+1}$. Let $M^S_0 = M^*$, and for limit $\delta < \lambda^+$ let $M^S_\delta = \bigcup_{\alpha<\delta} M^S_\alpha$; by the induction assumption and the choice of $M^*$ and $\mathbf{F}$, clearly $M^S_\delta$ is isomorphic to $M^*$. For $\alpha = \beta+1$ with $\beta$ successor, let $M^S_\alpha$ be such that $\mathbf{F}(M^S_\beta) <_{\mathfrak{k}} M^S_\alpha \cong M^*$. So we are left with the case $\alpha = \delta+1$, with $\delta$ limit or zero.

Now if $\delta \in S$ hence $M^S_\delta \cong M^*$, choose $M_{\delta+1}, a^S_\delta$ such that $(M^S_{\delta+1}, M^S_\delta, a^S_\delta)$ is a maximal triple (possible as by the hypothesis of this case there is a maximal triple, and there is a unique strong limit model). If $\delta \notin S$ we choose $M^{S,n}_\delta \in K_\lambda$ for $n < \omega$ (not used) such that $M^S_\delta <_{\mathfrak{k}} M^{S,n+1}_\delta \leq_{\mathfrak{k}} M^{S,n}_\delta$ and $\mathbf{F}(M^{S,n+1}_\delta) \leq_{\mathfrak{k}} M^{S,n}_\delta$ for $n < \omega$ and $M^S_\delta = \bigcap_{n<\omega} M^{S,n}_\delta$ and $M^{S,n}_\delta \cong M^*$; and let $M^S_{\delta+1} = M^{S,0}_\delta$ (again possible as $M_\delta \cong M^*$ and an $(*)_1$ above).

Lastly, let $M^S = \bigcup_\alpha M^S_\alpha$.

Now clearly it suffices to prove that if $S^0, S^1 \subseteq \lambda^+$ [and] $S^1 \setminus S^0$ is stationary, then $M^{S^1} \ncong M^{S^0}$. So suppose $f$ is a $\leq_{\mathfrak{k}}$-embedding from $M^{S^1}$ onto $M^{S^0}$ (or just into $M^{S^0}$). Then

$$E^2 := \left\{\delta < \lambda^+ : M^{S^1}_\delta, M^{S^0}_\delta \text{ each have universe } \delta \text{ and } [i < \delta \Leftrightarrow f(i) < \delta]\right\}$$

is a closed unbounded subset of $\lambda^+$, hence there is a limit ordinal $\delta \in (S^1 \setminus S^0) \cap E^2$. Let us look at $f(a^{S^1}_\delta)$; as $\delta \in S^1$, $a^{S^1}_\delta$ is well defined and [a member of] $M^{S^1}_{\delta+1} \setminus M^{S^1}_\delta$. As $\delta \in E^2$, it follows that $f(a^{S^1}_\delta)$ ess $\delta$ hence $f(a^{S^1}_\delta)$ belongs to $M^{S^0} \setminus M^{S^0}_\delta$ but $M^{S^0}_\delta = \bigcap_{n<\omega} M^{S^0,n}_\delta$ (as $\delta \notin S^0$).

Hence $f(a^{S^1}_\delta) \notin M^{S^0,n}_\delta$ for some $n$. Let $\beta \in (\delta, \lambda^+)$ be large enough such that $f(M^{S^1}_{\delta+1}) \subseteq M^{S^0}_\beta$. But then $f(M^{S^1}_\delta) \leq_{\mathfrak{k}} M^{S^0,n}_\delta \leq_{\mathfrak{k}} M^{S^0}_\beta$ and $f(M^{S^1}_{\delta+1}) \leq_{\mathfrak{k}} M^{S^0}_\beta$ and $a^{S^1}_\delta \notin f^{-1}(M^{S^0,n}_\delta)$.

Now $\big(f(M^{S^1}_\delta), f(M^{S^1}_{\delta+1}), f(a^{S^1}_\delta)\big)$ has the same properties as $\big(M^{S^1}_\delta, M^{S^1}_{\delta+1}, a^{S^1}_\delta\big)$ because if $f$ is an isomorphism from $M'$ onto $M'' \in K_\lambda$ then we can extend $f$ to an isomorphism from $\mathbf{F}(M')$ onto $\mathbf{F}(M'')$ (i.e. the "invariant"). But

$$\big(f(M^{S^1}_\delta), f(M^{S^1}_{\delta+1}), f(a^{S^1}_\delta)\big) < \big(M^{S^0,n}_\delta, M^{S^0}_\beta, f(a^{S^1}_\delta)\big),$$

a contradiction.

So we are done. $\square_{3.14}$

**Conclusion 3.19.** 1) *If* $\mathrm{LST}_{\mathfrak{k}} = \aleph_0$, $K$ *is* $\mathsf{PC}_{\aleph_0}$, *and* $\dot{I}(\aleph_1, K) = 1$, *then* $K$ *has a model of cardinality* $\aleph_2$.

2) *If* $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ *(*$\mathbf{Q}$ *is the quantifier "there are uncountably many") has one and only one model of cardinality* $\aleph_1$ *up to isomorphism then* $\psi$ *has a model in* $\aleph_2$.

*Proof.* 1) By 3.11 we get suitable $\mathfrak{k}_1$ (as in its conclusion) and by 3.12 the class $\mathfrak{k}_1$ has a model in $\aleph_2$, hence $\mathfrak{k}$ has a model in $\aleph_2$.

2) We can replace $\psi$ by a countable theory $T \subseteq \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$.

Let $L$ be a fragment of $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau)$ in which $T$ is included. (E.g. $L$ is the closure of $T \cup$ (the atomic formulas) under subformulas, $\neg, \wedge, (\exists x)$, and $(\mathbf{Q}x)$. In particular, $L$ includes (of course) first-order logic).

By [She75a], without loss of generality $T$ "says" that every formula $\varphi(x_0, \ldots, x_{n-1})$ of $L$ is equivalent to an atomic formula (i.e. $P(x_0, \ldots, x_{n-1})$ with $P$ a predicate), every type realized in a model of $T$ is isolated (i.e. every model is atomic), and $T$ is complete in $L$. Let

$$\begin{aligned} K := \big\{M : \; & M \text{ an atomic } \tau(T)\text{-model of } T \cap \mathbb{L}, \text{ and if } M \models P[\bar a] \\ & \text{and } (\forall \bar x)\big[P(\bar x) \equiv \neg(\mathbf{Q}y)R(y,\bar x)\big] \in T \\ & \text{then } \{b : M \models R[b,\bar a]\} \text{ is countable}\big\}. \end{aligned}$$

So $\mathfrak{k} = (K, \leq_{\mathfrak{k}})$ is categorical in $\aleph_0$, is an AEC, and is $\mathsf{PC}_{\aleph_0}$. Let $\mathbf{F}$ (see 3.3(8)) be such that for $M \in K_{\aleph_0}$, $N = \mathbf{F}(M)$ <u>iff</u> $M <^{**} N$. By this we mean $M \leq_{\mathfrak{k}} N \in K_{\aleph_0}$ and if $\bar a \in M$, $M \models P[\bar a]$, and $(\forall \bar x)[P(\bar x) \equiv (\mathbf{Q}y)R(y,\bar x)] \in T$, <u>then</u> for some $b \in N \setminus M$ we have $N \models R[b,\bar a]$. So $\mathbf{F}$ is invariant.

Note that every $M \in K^{\mathbf{F}}_{\aleph_1}$ is a model of $\psi$. So 3.14 gives that some $M \in K^{\mathbf{F}}_{\aleph_1}$ has a proper extension in $K^{\mathbf{F}}_{\aleph_1}$.

The rest should be easy, just as in Stage **A** of the proof of 3.12. $\square_{3.19}$

*Question* 3.20. Under the assumptions of 3.19(2), can we get $M \in K_{\aleph_2}$ such that if $M \models P[\bar a]$ and $(\forall \bar x)\big[P(\bar x) \equiv (\mathbf{Q}y)R(y,\bar x)\big] \in T$, then $\big\{b \in M : M \models R[b,\bar a]\big\}$ has cardinality $\aleph_2$? Note that in the proof of 3.14 we show that no triple is maximal.

*Remark* 3.21. 1) We could have used multi-valued $\mathbf{F}$; then in the proof above $N = \mathbf{F}(M)$ just means the demand there.

2) To answer 3.20 (i.e. to prove the existence of $M \in K_{\aleph_2}$ as above) we have to prove:

- $(*)_1$ There are $N_i \in K^{\mathbf{F}}_{\aleph_1}$ for $i < \omega_1$ and $N \leq_{\mathfrak{k}} N_i$ such that if $N \models P[\bar a]$ and the sentence $(\forall \bar x)\big[P(\bar x) \equiv (\mathbf{Q}y)R(y,\bar x)\big]$ belongs to $T$, <u>then</u> for some $i < \omega_1$ there is $b_* \in N_i \setminus N$ such that $N_i \models R[b_*,\bar a]$.

Clearly

- $(*)_2$ The existence of $N, N_i$ as in $(*)_1$ is equivalent to "$\psi^*$ has a model" for some $\psi^* \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ which is defined from $T, \leq_{\mathfrak{k}}$.

Hence

- $(*)_3$ It is enough to prove that for some forcing notion $\mathbb{P}$ in $\mathbf{V}^{\mathbb{P}}$ there are $N, N_i$ as in $(*)_1$.

There are some natural ccc forcing notions tailor-made for this.

- $(*)_4$ Consider the class of triples $(M,N,a)$ such that $M \leq_{\mathfrak{k}} N \in K_{\aleph_0}$, $\bar a \in {}^{\omega>}N$, and $\ell < \ell g(\bar a) \Rightarrow a_\ell \notin M$, ordered as in the proof of 3.14. By the same proof there is no maximal triple.

3) We can restrict ourselves in $(*)_2$ to

$$\{R(y, \bar{a}) : \bar{a} \in {}^{\ell g(\bar{x})}N \text{ and } \bar{a} \text{ realizes a type } p(\bar{x})\}.$$

Also, we may demand $i < \omega_1 \Rightarrow N_i = N_0$ and we may try to force such a sequence of models (or pairs), and there is a natural forcing. By absoluteness it is enough to prove that it satisfies the ccc.

**Problem 3.22.** If $\mathfrak{k}$ is $\mathsf{PC}_\lambda$ and $K$ is categorical in $\lambda$ and $\lambda^+$, does it necessarily have a model in $\lambda^{++}$?

*Remark* 3.23. The problem is proving $(*)$ of 3.13.

*Question* 3.24. Assume $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau)$ is complete in $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau)$, is categorical in $\aleph_1$, has an uncountable model $M$, $\bar{a} \in {}^nM$ and $\varphi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau)$ axiomatizes the $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau)$-theory of $(M, \bar{a})$. Is $\varphi$ categorical in $\aleph_1$?

*Question* 3.25. Can we weaken the demand on $M^*$ in 3.14 to "$M^*$ is a $\lambda^+$-limit model"?

## § 4. Forcing and categoricity

The main aim in this section is, for $\mathfrak{k}$ as in §1 with $\mathrm{LST}_{\mathfrak{k}} = \aleph_0$, to find what we can deduce from $1 \le \dot{I}(\aleph_1, K) < 2^{\aleph_1}$, first without assuming $2^{\aleph_0} < 2^{\aleph_1}$.

We can build a model of cardinality $\aleph_1$ by an $\omega_1$-sequence of countable approximations. Among those, there are models which are the union of a quite generic $<_{\mathfrak{k}}$-increasing sequence $\langle N_i : i < \omega_1 \rangle$ of countable models, so it is natural to look at them (e.g. if $\mathfrak{k}$ is categorical in $\aleph_1$, every model in $K_{\aleph_1}$ is like that). We say of such models that they are quite generic. More exactly, we look at countable models and figure out properties of the quite generic models in $\mathfrak{k}_{\aleph_1}$. The main results are 4.13(a),(f). Note that the case $2^{\aleph_0} = 2^{\aleph_1}$, though in general making our work harder, can be utilized positively — see 4.11.

A central notion is (e.g.) "the type which $\bar{a} \in {}^{\omega>}(N_1)$ materializes in $(N_1, N_0)$", for $N_0 \le_{\mathfrak{k}} N_1 \in K_{\aleph_0}$. This is (as the name indicates) the type materialized in $N_1^+$, which is $N_1$ expanded by $P^{N_1^+} = N_0$; it consists of the set of formulas forced (in the model-theoretic sense started by Robinson) to satisfy; here 'forced' is defined thinking on $(K_{\aleph_0}, \le_{\aleph_0})$, so models in $K_{\aleph_1}$ can be constructed as the union of quite generic $<_{\mathfrak{k}}$-increasing $\omega_1$-sequences. As we would like to build models of cardinality $\aleph_1$ by such sequences, the "materialize" in $(N_1, N_0)$ becomes realized in the (quite generic) $N \in K_{\aleph_1}$; but most of our work is in $K_{\aleph_0}$. This is also a way to express $\mathbf{Q}$ speaking on countable models.

By the hypothesis 4.8 justified by §3, the $\mathbb{L}_{\infty,\omega}(\tau_{\mathfrak{k}})$-theory of $M \in K$ is clear; in particular, it has elimination of quantifiers hence $M \le_{\mathfrak{k}} N \Rightarrow M \prec_{\mathbb{L}_{\infty,\omega}} N$. But for $\overline{N} = \langle N_\alpha : \alpha < \omega_1 \rangle$ as above we would like to understand $(N_\beta, N_\alpha)$ for $\alpha < \beta$. (From the point of view of $N$, $\overline{N}$ is not reconstructible, but its behaviour on a club is.) Toward a parallel analysis of such pairs we again analyze them by $\langle L^0_\alpha : \alpha < \omega_1 \rangle$ (similarly to [Mor70]).

**Convention 4.1.** We fix $\lambda > \mathrm{LST}_{\mathfrak{k}}$ as well as the AEC $\mathfrak{k}$.

The main case below is here $\lambda = \aleph_1$, $\kappa = \aleph_0$.

**Definition 4.2.** For $\lambda > \mathrm{LST}_{\mathfrak{k}}$, $N_* \in K_{<\lambda}$, and $\mu, \kappa$ satisfying $\lambda \ge \kappa \ge \aleph_0$, $\mu \ge \kappa$:

1) Let $\mathbb{L}^0_{\mu,\kappa}$ be first-order logic enriched by conjunctions (and disjunctions) of length $< \mu$, homogeneous strings of existential quantifiers or of universal quantifiers of length $< \kappa$, and the cardinality quantifier $\mathbf{Q}$ interpreted as $\exists^{\ge\lambda}$. But we apply those operations such that any formula has $< \kappa$ free variables and the non-logical symbols are from $\tau(\mathfrak{k})$, so actually we should write $\mathbb{L}^0_{\mu,\kappa}(\tau_{\mathfrak{k}})$ but we may omit this when clear; the syntax does not depend on $\lambda$ but we shall mention it in the definition of satisfaction.

2) For a logic $\mathcal{L}$ and $A_i$, $A \subseteq N_*$ for $i < \alpha < \lambda$, let $\mathcal{L}(N_*, A_i; A)_{i<\alpha}$ be the language with the logic $\mathcal{L}$ and the vocabulary $\tau_{N_*,\bar{A},A}$, where $\bar{A} = \langle A_i : i < \alpha \rangle$ and $\tau_{N_*,\bar{A};A}$ consists of $\tau(K)$, the predicates $x \in N_*$ and $x \in A_i$ for $i < \alpha$, and the individual constants $c$ for $c \in A$. (If $A = \varnothing$ we may omit the $A$; if we omit $N_*$ then "$x \in N_*$" is omitted; if the sequence of the $A_i$ is omitted then the "$x \in A_i$" are omitted, so $\mathcal{L}(\ )$ means having the vocabulary $\tau(K)$). So $\mathcal{L}(N_*, A_i; A)_{i<\alpha}$ formally should have been written $\mathcal{L}(\tau_{N_*,\bar{A};A})$.

3) $\mathbb{L}^1_{\mu,\kappa}$ is defined is as in part (1), but we have also variables (and quantification) over relations of cardinality $< \lambda$. Let $\mathbb{L}^{-1}_{\mu,\kappa}$ be as in part (1) but not allowing the cardinality quantifier $\mathbf{Q}$; this is the classical logic $\mathbb{L}_{\mu,\kappa}$.

4) $(N, N_*, A_i; A)_{i<\alpha}$ is the model $N$ expanded to a $\tau_{N_*,\bar{A};A}$-model by monadic predicates for $N_*$, $A_i$, and individual constants for every $c \in A$.

5) For "$x \in N_*$" and "$x \in A_i$" we use the predicates $P$ and $P_i$, respectively, so we may write $\mathcal{L}(\tau + P)$ instead of $\mathcal{L}(N_*)$. But [when] writing $\mathcal{L}(N_*)$, we fix the interpretation of $P$.

Let $\tau^{+\alpha} := \tau \cup \{P, P_\beta : \beta < \alpha\}$. If $L = \mathcal{L}(\tau^{+0})$ (i.e. for $\alpha = 0$) then $L(N)$ means $L$ but we fix the interpretation of $P$ as $N$; i.e. $|N|$, the set of elements of $N$.

Let $L(N_*, N_i)_{i \in u}$, where $u$ is a set of $< \kappa$ ordinals, mean the language $L$ in the vocabulary $T \cup \{P, P_i : i \in u\}$ when we fix the interpretation of $P$ as $N_*$ and of $P_{\mathrm{otp}(u \cap \alpha)}$ as $N_\alpha$.

**Definition 4.3.** 1) For $N_* \in K_{<\lambda}$ and $\varphi(x_0, \ldots) \in \mathbb{L}^1_{\mu,\kappa}(N_*, \bar{A}; A)$, we define when $N_0 \Vdash^\lambda_{\mathfrak{k}} \varphi[a_0, \ldots]$ by induction on $\varphi$, where $N_* \le_{\mathfrak{k}} N_0 \in K_{<\lambda}$ and $a_0, \ldots$ are elements of $N_0$ or appropriate relations over it, depending on the kind of $x_i$. (Pedantically, we should write '$(N_0, N_*, \bar{A}; A) \Vdash^\lambda_{\mathfrak{k}} \varphi[a_0, \ldots]$', and we may do it when not clear from the context.)

For $\varphi$ atomic this means $N_0 \models \varphi[a_0, \ldots]$. For $\varphi = \bigwedge\limits_i \varphi_i$ this means

$$N_0 \Vdash^\lambda_{\mathfrak{k}} \varphi_i[a_0, \ldots] \text{ for each } i.$$

For $\varphi = (\exists \bar{x})\psi(\bar{x}, a_0, \ldots)$, this means that for every $N_1$ satisfying $N_0 \le_{\mathfrak{k}} N_1 \in K_{<\lambda}$ there is $N_2$ satisfying $N_1 \le_{\mathfrak{k}} N_2 \in K_{<\lambda}$ and $\bar{b}$ from $N_2$ of the appropriate length (and kind) such that $N_2 \Vdash^\lambda_{\mathfrak{k}} \psi[\bar{b}, a]$.

For $\varphi = \neg\psi$ this means that for no $N_1$ do we have $N_0 \le_{\mathfrak{k}} N_1 \in K_{<\lambda}$ and $N_1 \Vdash^\lambda_{\mathfrak{k}} \psi[a_0, \ldots]$.

For $\varphi(x_0, \ldots) = (\mathbf{Q}y)\psi(y, x_0, \ldots)$ this means that for every $N_1$ satisfying $N_0 \le_{\mathfrak{k}} N_1 \in K_{<\lambda}$ there is $N_2$ satisfying $N_0 \le_{\mathfrak{k}} N_2 \in K_{<\lambda}$ and $a \in N_2 \setminus N_1$ such that $N_2 \Vdash^\lambda_{\mathfrak{k}} \psi[a, a_0, \ldots]$.

2) In part (1) if $\varphi \in \mathbb{L}^1_{\mu,\kappa}(N_*)$ we can omit the demand "$N_* \le_{\mathfrak{k}} N$" similarly below.

3) For a language $L \subseteq \mathbb{L}^1_{\mu,\kappa}(N_*, \bar{A}; A)$ and a model $N$ satisfying $N_* \le_{\mathfrak{k}} N \in K_{<\lambda}$ and a sequence $\bar{a} \in {}^{\lambda>}N$ the $L$-generic type of $\bar{a}$ in $N$ is

$$\mathrm{gtp}(\bar{a}; N_*, \bar{A}; A; N) = \big\{\varphi(\bar{x}) \in L : N \Vdash^\lambda_{\mathfrak{k}} \varphi[\bar{a}]\big\}.$$

4) For $N_* \le_{\mathfrak{k}} N \in K_\lambda$ and $L \subseteq \mathcal{L}(N_*, \bar{A}; A)$, let $\mathrm{gtp}^\lambda_L(\bar{a}; N_*, \bar{A}; A; N)$ be

$$\big\{\varphi(\bar{x}) : \varphi \in \mathcal{L}(N_*, \bar{A}; A), \text{ and for some } N' \in K_{<\lambda} \text{ we have } N \le_{\mathfrak{k}} N' \le_{\mathfrak{k}} N \text{ and } N' \Vdash^\lambda_{\mathfrak{k}} \varphi[\bar{a}]\big\}.$$

We may omit $\bar{A}, A$ (and omit $\lambda$ if clear from the context) and may write $\mathcal{L}$ instead of $L = \mathcal{L}(N_*, \bar{A}; A)$ (but note Definition 4.4).

5) We say "$\bar{a}$ *materializes* $p$ (or $\varphi$)" if $p$ (or $\{\varphi\}$) is a subset of the $L$-generic type of $\bar{a}$ in $N$.

**Definition 4.4.** Let $\langle N_i : i < \lambda\rangle$ be an increasing (by $\leq_{\mathfrak{k}}$) continuous sequence, $N = \bigcup_{i<\lambda} N_i$, $\|N_i\| < \lambda$, and $L^* \subseteq \bigcup_{\alpha<\kappa} \mathbb{L}^1_{\infty,\kappa}(\tau^{+\alpha})$.

1) $N$ is $L^*$-generic, if for any formula $\varphi(x_0,\ldots) \in L^* \cap \mathbb{L}^1_{\infty,\kappa}(\tau_{\mathfrak{k}})$ and $a_0,\ldots \in N$ we have:

$$N \models \varphi[a_0,\ldots] \Leftrightarrow N_\alpha \Vdash^\lambda_{\mathfrak{k}} \varphi[a_0,\ldots] \text{ for some } \alpha < \lambda.$$

2) The $\leq_{\mathfrak{k}}$-presentation $\langle N_i : i < \lambda\rangle$ of $N$ is $L^*$-generic <u>when</u> for any $\alpha < \lambda$ of cofinality $\geq \kappa$ and $\psi(x_0,\ldots) \in L^*(N_\alpha, N_i)_{i\in I}$ with $I \in [\alpha]^{<\kappa}$ and $a_0,\ldots \in N$ we have:

$$N \models \psi[a_0,\ldots] \Leftrightarrow N_\gamma \Vdash^\lambda_{\mathfrak{k}} \psi[a_0,\ldots] \text{ for some } \gamma < \lambda$$

and for each $\beta \geq \alpha$ with cofinality $\geq \kappa$, $N_\beta$ is almost $L^*(N_\alpha, N_i)_{i\in I}$-generic (see part (5)).

3) $N$ is *strongly* $L^*$-generic <u>if</u> it has an $L^*$-generic presentation. (In this case, if $\lambda$ is regular, then for any presentation $\langle N_i : i < \lambda\rangle$ of $N$ there is a closed unbounded $E \subseteq \lambda$ such that $\langle N_i : i \in E\rangle$ is an $L^*$-generic presentation.)

4) We say that $N \in K_{<\lambda}$ is pseudo $L^*$-generic <u>if</u>

- (a) For every $\varphi(\bar{x}) = (\exists \bar{y})\psi(\bar{x},\bar{y}) \in L^*$, if $N \Vdash^\lambda_{\mathfrak{k}} \varphi(\bar{a})$ then $N \Vdash^\lambda_{\mathfrak{k}} \psi(\bar{a},\bar{b})$ for some $\bar{b}$.
- (b) For every $\bar{a} \in N$, $\bar{a}$ materializes some complete $L^*$-type in $N$.

5) We add "almost" to any of the notions defined above <u>when</u> for $\Vdash^\lambda_{\mathfrak{k}}$, the inductive definition of satisfaction works (except possibly for $\mathbf{Q}$.) E.g. $N \Vdash^\lambda_{\mathfrak{k}} (\exists x)\varphi(x,\ldots)$ iff $N \Vdash^\lambda_{\mathfrak{k}} \varphi(a,\ldots)$ for some $a \in N$.

*Remark* 4.5. 1) Notice we can choose $N_i = N_0 = N$, so $\|N\| < \lambda$. In particular, almost (and pseudo-) $L^*$-generic models of cardinality $< \lambda$ may well exist.

2) Here we concentrate on $\lambda = \aleph_1$ and fragments of $\mathbb{L}^0_{\infty,\omega}$ (mainly $\mathbb{L}^0_{\omega_1,\omega}$ and its countable fragments).

3) There are obvious implications, and forcing is preserved by isomorphism and replacing $N$ ($\in K_{<\lambda}$) by $N'$ with $N \leq_{\mathfrak{k}} N' \in K_{<\lambda}$.

There are obvious theorems on the existence of generic models; e.g.

**Theorem 4.6.** 1) *Assume $N_0 \in K_{<\lambda}$, $\lambda = \mu^+$, $\mu^{<\kappa} = \mu$, $L \subseteq \bigcup_{\alpha<\kappa} \mathbb{L}_{\infty,\kappa}(\tau^{+\alpha})$, $L$ is closed under subformulas, and $|L| < \lambda$. <u>Then</u> there are $N_i$ ($i < \lambda$) such that $\langle N_i : i < \lambda\rangle$ is an $L$-generic representation of $N = \bigcup_{i<\lambda} N_i$, (hence $N$ is strongly $L$-generic).*

2) *In part* (1), *$N \in K_\lambda$ if no $N'$ with $N_0 \leq_{\mathfrak{k}} N' \in K_{<\lambda}$ is $\leq_{\mathfrak{k}}$-maximal.*

*Proof.* Straightforward. $\square_{4.6}$

*Remark* 4.7. 1) If $L = \bigcup_{i<\lambda} L_i$, $|L_i| < \lambda$, then we can get "$\langle N_i : j < i < \lambda \rangle$ is an $L_j$-generic representation of $N$ for each $j < \lambda$".

2) When we speak on a "complete $L$-type $p$," we mean $p = p(x_0, \dots, x_{n-1})$ for some $n$.

From time to time we add some hypotheses and prove a series of claims; such that the hypothesis holds (at least without loss of generality) in the case we are interested in. We are mainly interested in the case $\dot{I}(\aleph_1, \mathfrak{k}) < 2^{\aleph_1}$, etc., so by 3.11, 3.18 it is reasonable to state the following:

**Hypothesis 4.8.** $\mathfrak{k}$ is $\mathsf{PC}_{\aleph_0}$, $\leq_{\mathfrak{k}}$ refines $\mathbb{L}_{\infty,\omega}$, $\mathfrak{k}$ is categorical in $\aleph_0$, $1 \leq \dot{I}(\aleph_1, K)$, and $\dot{I}(\aleph_1, K^{\mathbf{F}}_{\aleph_1}) < 2^{\aleph_1}$ (where $K^{\mathbf{F}}_{\aleph_1}$ is as in Definition 3.15 and is $\mathsf{PC}_{\aleph_0}$ or just $\mathbf{K}^{\mathbf{F}}_{\aleph_1} = \{M \restriction \tau_{\mathfrak{k}} : M \models \psi\}$ for some $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ — if $\mathbf{F}$ is invariant, this follows).

*Remark* 4.9. 0) We can add 'every $M \in K_{\aleph_0}$ is atomic' (an atomic model of $\mathrm{Th}_{\mathbb{L}}(M)$).

1) Usually below we ignore the case $\dot{I}(\aleph_1, \mathfrak{k}) < 2^{\aleph_0}$ as the proof is the same.

2) We can deal similarly with the case $1 \leq \dot{I}(\aleph_1, K') < 2^{\aleph_0}$, where

$$\mathfrak{k}_{\aleph_1} \subseteq K'_{\aleph_1} \subseteq \{M \in \mathfrak{k}_{\aleph_1} : M \text{ is strongly } L_*\text{-generic}\}$$

and $K'$ is $\mathsf{PC}_{\aleph_0}$ (or less: $\{M \restriction \tau_{\mathfrak{k}} : M \text{ a model of } \psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})(\tau^*)\}$).

3) Can we use $\mathbf{F}$ a function with domain $K_{\aleph_0}$ such that $M \leq_{\mathfrak{k}} \mathbf{F}(M_0) \in K_{\aleph_0}$ for $M \in K_{\aleph_0}$, without the extra assumptions, or even

$$\mathbf{F} : \{\overline{M} = \langle M_i : i \leq \alpha \rangle : \overline{M} \text{ is } \leq_{\mathfrak{k}_{\aleph_0}}\text{-increasing continuous}\} \to \mathfrak{k}_{\aleph_0}$$

such that $M_\alpha \leq_{\mathfrak{k}} \mathbf{F}(\langle M_i : i \leq \alpha \rangle)$? We cannot use the non-definability of well ordering (see 3.11(3), as in the proof of (f) of 4.13).

**Claim 4.10.** 1) *If $\bar{a} \in N \in K_{\aleph_0}$ and $\varphi(\bar{x}) \in \mathbb{L}^0_{\infty,\omega}(\tau^{+0})$ (so $\bar{a}$ is a finite sequence) <u>then</u> $(N, N) \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi[\bar{a}]$ or $(N, N) \Vdash^{\aleph_1}_{\mathfrak{k}} \neg\varphi[\bar{a}]$ (i.e. $P$ is interpreted as $N$).*

2) *If $(N, N) \Vdash^{\aleph_1}_{\mathfrak{k}} \exists \bar{x} \wedge p(\bar{x})$, where $p(\bar{x})$ is a not necessarily complete $n$-type in $L$ (here $n = \ell g(\bar{x})$), where $L \subseteq \mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$ is countable, <u>then</u> for some complete $n$-type $q$ in $L$ extending $p$ we have $(N, N) \Vdash^{\aleph_1}_{\mathfrak{k}} \exists \bar{x} \wedge q(\bar{x})$.*

**[I don't recognize this notation. Is it $(\exists \bar{x}) p(\bar{x})$, $(\exists \bar{x}) \bigwedge_p p(\bar{x})$, or something different?]**

*Proof.* 1) Suppose not. Then for each $S \subseteq \omega_1$, we define $N^S_\alpha \in K_{\aleph_0}$ by induction on $\alpha < \omega_1$, increasing (by $\leq_{\mathfrak{k}}$) and continuous.

$N^S_0 := N$ and $N^S_\alpha := \bigcup_{\beta<\alpha} N^S_\beta$ for limit $\alpha$. For $\alpha = 2\beta + 1$, remember that $(N^S_\beta, \bar{a}) \cong (N, \bar{a})$ because $N = N_0 \leq_{\mathfrak{k}} N^S_\beta$, hence $N_0 \prec_{\mathbb{L}_{\infty,\omega}} N^S_\beta \in K_{\aleph_0}$ hence $(N^S_\beta, \bar{a}) \equiv_{\mathbb{L}_{\infty,\omega}} (N, \bar{a})$ hence they are isomorphic. So $(N^S_\beta, N^S_\beta)$ forces ($\Vdash^{\aleph_1}_{\mathfrak{k}}$) neither $\varphi[\bar{a}]$ nor $\neg\varphi[\bar{a}]$. So there are $M_\ell$ (for $\ell = 0, 1$) such that $N^S_\beta \leq_{\mathfrak{k}} M_\ell \in K_{\aleph_0}$ and

$(M_0, N^S_\beta) \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi[\bar{a}]$ but $(M_1, N^S_\beta) \Vdash^{\aleph_1}_{\mathfrak{k}} \neg\varphi[\bar{a}]$. Now if $\beta \in S$ we let $N^S_\alpha = M_0$, and if $\beta \notin S$ we let $N^S_\alpha = M_1$.

Lastly, $M_{2\beta+2} = \mathbf{F}(M_{2\beta+1})$, recalling $\mathbf{F}$ is from 4.8. Let $N^S := \bigcup_{\alpha<\omega_1} N^S_\alpha$. Now if $S_0 \setminus S_1$ is stationary then $(N^{S_0}, \bar{a}) \ncong (N^{S_1}, \bar{a})$.

Why? Because if $f : N^{S_0} \to N^{S_1}$ is an isomorphism from $N^{S_0}$ onto $N^{S_1}$ mapping $\bar{a}$ to $\bar{a}$, then for some closed unbounded set $E \subseteq \omega_1$, we have: 'if $\alpha \in E$ then $f$ maps $N^{S_0}_\alpha$ onto $N^{S_1}_\alpha$.' So choose some $\alpha \in E \cap S_0 \setminus S_1$ and choose $\beta \in E \setminus (\alpha + 1)$. Now $(N^{S_0}_{\alpha+1}, N^{S_0}_\alpha) \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi[\bar{a}]$ hence $(N^{S_0}_\beta, N^{S_0}_\alpha) \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi[\bar{a}]$, and similarly $(N^{S_1}_\beta, N^{S_1}_\alpha) \Vdash^{\aleph_1}_{\mathfrak{k}} \neg\varphi(\bar{a})$, but $f \restriction N^{S_0}_\beta$ is an isomorphism from $N^{S_0}_\beta$ onto $N^{S_1}_\beta$ mapping $N^{S_0}_\alpha$ onto $N^{S_1}_\alpha$ and $\bar{a}$ to itself, and we get a contradiction. By 0.4, we get $\dot{I}(\aleph_1, K) = 2^{\aleph_1}$, a contradiction.

2) Easy, by 4.6 and part (1). In detail: if $N \leq_{\mathfrak{k}} M_1 \in \mathfrak{k}_{\aleph_0}$ then by the definition of $\Vdash^{\aleph_1}_{\mathfrak{k}}$ and the assumption we can find $(M_2, \bar{a})$ satisfying $M_1 \leq_{\mathfrak{k}} M_2 \in \mathfrak{k}_{\aleph_0}$ and $\bar{a} \in M_2$ such that $(M_2, N) \Vdash^{\aleph_1}_{\mathfrak{k}} \wedge p(\bar{a})$. As $L$ is countable and the definition of $\Vdash^{\aleph_1}_{\mathfrak{k}}$, without loss of generality $(M_2, N) \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi[\bar{a}]$ or $(M_2, N) \Vdash^{\aleph_1}_{\mathfrak{k}} \neg\varphi[\bar{a}]$ for every formula $\varphi(\bar{x}) \in L$.

[Why? Simply let $\langle \varphi_n(\bar{x}) : n < \omega \rangle$ list the formulas $\varphi(\bar{x}) \in L$ and choose $M_{2,n} \in \mathfrak{k}_{\aleph_0}$ by induction on $n$ with $M_{2,0} = M_2$ and $M_{2,n} \leq_{\mathfrak{k}} M_{2,n+1}$ such that

$$(M_{2,n+1}, N) \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi_n(\bar{x}) \text{ or } (M_{2,n+1}, N) \Vdash^{\aleph_1}_{\mathfrak{k}} \neg\varphi_n(\bar{x});$$

now replace $M_2$ by $\bigcup_{n<\omega} M_{2,n}$.]

Recalling Definition 4.3(4), let $q := \mathrm{gtp}_{L(N)}(\bar{a}, N, M_2)$; it is a complete $(L(N), n)$-type. So clearly $(M_2, N) \Vdash^{\aleph_1}_{\mathfrak{k}} (\exists\bar{x}) \wedge q(\bar{x})$. Now apply the proof of part (1) to the formula $(\exists\bar{x}) \wedge q(\bar{x})$, so we are done. $\square_{4.10}$

**Claim 4.11.** *For each countable $L \subseteq \mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$ and $N \in K_{\aleph_0}$, the number of complete $L(N)$-types $p$ (with no parameters) such that $N \Vdash^{\aleph_1}_{\mathfrak{k}} (\exists\bar{x}) \wedge p(\bar{x})$ is countable.*

*Proof.* At first glance it seemed that 0.3 would imply this trivially. However, here we need the parameter $N$ as an interpretation of the predicate $P$, and if $2^{\aleph_0} = 2^{\aleph_1}$ then there are too many choices. So we shall deal with "every $N_\alpha$ in some presentation." Suppose the conclusion fails. First we choose $N_\alpha$ by induction on $\alpha < \omega_1$ such that:

⊛ (i) $N_\alpha \in K_{\aleph_0}$ is $\leq_{\mathfrak{k}}$-increasing and $\langle N_\alpha : \alpha < \omega_1 \rangle$ is $L$-generic.
(ii) For each $\beta < \alpha$, there is $a^\beta_\alpha \in N_{\alpha+1} \setminus N_\alpha$ materializing an $L(N_\beta)$-type not materialized in $N_\alpha$, (i.e. in[10] $(N_\alpha, N_\beta)$; possible by 4.10 and our assumption toward contradiction).
(iii) $|N_\alpha| = \omega \cdot \alpha$
(iv) For $\alpha < \beta$, $N_\beta$ is pseudo-$L(N_\alpha)$-generic and $\mathbf{F}(N_{2\beta+1}) \leq_{\mathfrak{k}} N_{2\beta+2}$.

Now let $N := \bigcup_{\alpha<\omega_1} N_\alpha$, and we expand $N$ by all relevant information: the order $<$ on the countable ordinals, $c \in N_0$, enough "set theory," "witnesses" for $N_\beta \leq_{\mathfrak{k}} N_\alpha$ for $\beta < \alpha$, the 2-place functions $F(\beta, \alpha) := a^\beta_\alpha$; and lastly, witnesses of

[10] see Definition 4.3(2) on 'materialize.'

$\mathbf{F}(N_{2\beta+1}) \le_{\mathfrak{k}} N_{2\beta+2}$ (recalling $\mathbf{F}$ is quite definable by Definition 4.8) and names for all formulas in $L(N_\alpha)$ (with $\alpha$ as a parameter); i.e. the relations

$$R_{\varphi(\bar{x})} := \big\{\langle\alpha\rangle\hat{\ }\bar{a} : \alpha < \omega_1,\ \bar{a} \in {}^{\ell g(x)}N, \text{ and } (N_\beta, N_\alpha) \Vdash^{\aleph_1}_{\mathfrak{k}} \text{“}\varphi(\bar{a})\text{”} \text{ for every } \beta < \omega_1 \text{ large enough}\big\}$$

for $\varphi(\bar{x}) \in L$.

Clearly for every $\alpha < \omega_1$, every $\varphi(\bar{x}) \in L(N_\alpha)$, and $\bar{a} \in {}^{\ell g(\bar{x})}N$, we have $(N, N_\alpha) \models \varphi[\bar{a}]$ iff for every $\beta < \omega_1$ large enough we have $(N_\beta, N_\alpha) \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi[\bar{a}]$. We get a model $\mathfrak{B}$ with countable vocabulary and $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ expressing all this. By 0.3(1) applied to the case $\Delta = L$, there are models $\mathfrak{B}_i$ (for $i < 2^{\aleph_1}$) of cardinality $\aleph_1$ (note $N_0 \le_{\mathfrak{k}} \mathfrak{B} \restriction \tau_{\mathfrak{k}}$), so that the set of $L(N_0)$-types realizes in $N^i$ (the $\tau(K)$-reduct of $\mathfrak{B}_i$) are distinct for distinct $i$-s. So $(N^i, c)_{c \in N_0}$ are pairwise non-isomorphic. If $2^{\aleph_0} < 2^{\aleph_1}$ we finish by 0.4.

So we can assume $2^{\aleph_0} = 2^{\aleph_1}$. In $N$, uncountably many complete $L(N_0)$-n-types are realized, hence by 0.3(2) the set

$$\{p : p \text{ a complete } L(N_0)\text{-}m\text{-type for some } m < \omega \text{ realized in some } N' \in \mathfrak{k}_{\aleph_1} \text{ with } N_0 \le_{\mathfrak{k}} N'\}$$

has cardinality continuum, hence by 4.10 the set of complete $L(N_0)$-types $p = p(x)$ such that $(N_0, N_0) \Vdash^{\aleph_1}_{\mathfrak{k}} \exists \bar{x} \wedge p(\bar{x})$ has cardinality $2^{\aleph_0}$. So we choose a sequence $\langle N^\alpha_i, a^\alpha_i : i < \omega_1\rangle$ by induction on $\alpha < 2^{\aleph_0}$ such that:

(a) $N^\alpha_i \in \mathfrak{k}_{\aleph_0}$

(b) $N^\alpha_{i_0} \le_{\mathfrak{k}} N^\alpha_i$ for $i_0 < i < \omega_1$.

(c) $a^\alpha_i \in N^\alpha_{i+1} \setminus N^\alpha_i$ materializes a complete $L(N^\alpha_i)$-type $p^\alpha_i$.

(d) If $j < \omega_1$ is a limit ordinal then $N^\alpha_j := \bigcup_{i<j} N^\alpha_i$.

(e) $p^\alpha_i \notin \{\mathrm{gtp}(\bar{a}; N^\beta_{j_1}; N^\beta_{j_2}) : j_1 < j_2 < \omega_1,\ \bar{a} \in {}^{\omega>}(N^\beta_{j_2}) \text{ and } \beta < \alpha\}$ (See Definition 4.3(4).)

(f) $\mathbf{F}(N_{2\beta+1}) \le_{\mathfrak{k}} N_{2\beta+2}$.

As $\aleph_1 < 2^{\aleph_1} = 2^{\aleph_0}$ this is possible; i.e. in clause (e) we should find a type which is not in a set of $\le \aleph_1 \times |\alpha| < 2^{\aleph_0}$ types, as the number of possibilities is $2^{\aleph_0}$. Let $N_\alpha := \bigcup_{i<\omega_1} N^\alpha_i$ for $\alpha < 2^{\aleph_0}$; clearly $N_\alpha \in K_{\aleph_1}$.

Now toward contradiction, if $\beta < \alpha < 2^{\aleph_0}$ and $N_\alpha \cong N_\beta$ then there is an isomorphism $f$ from $N_\alpha$ onto $N_\beta$; necessarily $f$ maps $N^\alpha_i$ onto $N^\beta_i$ for a club of $i$. For any such $i$, $p^\alpha_i \in \mathrm{gtp}_L(f(\bar{a}^\alpha_i); N^\beta_i; N^\beta_j)$ for $j$ large enough, a contradiction. $\square_{4.11}$

*Remark* 4.12. In the proof of 4.11(2), we can fix $m$ and we can combine the two cases, when for $N \in K^{\mathbf{F}}_{\aleph_1}$ represent by $\langle N_\alpha : \alpha < \omega_1\rangle$ we consider

$$\mathbf{P}_N := \big\{p : p \text{ a complete } L\text{-}m\text{-type such that for a club of } \alpha < \omega_1 \text{ and some } \beta \in (\alpha, \omega_1) \text{ and } \bar{a} \in {}^m(N_\beta) \text{ materialize } p \text{ in } (N_\beta, N_\alpha)\big\}.$$

We can replace “club” by “stationarily many”. That is, we can prove that $\{\mathbf{P}_N : N \in K^{\mathbf{F}}_{\aleph_1}\}$ has cardinality $2^{\aleph_1}$.

**Lemma 4.13.** 1) *There are countable $L^0_\alpha \subseteq \mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$ for $\alpha < \omega_1$ increasing continuous in $\alpha$, closed under finitary operations and subformulas such that, letting $L^0_{<\omega_1} := \bigcup_{\alpha<\omega_1} L^0_\alpha$, we have (some clauses do not mention the $L^0_\alpha$-s):*

(a) *For each $N \in K_{\aleph_0}$ and every complete $L^0_\alpha(N)$-type $p(\bar{x})$, we have*

$$N \Vdash^{\aleph_1}_{\mathfrak{k}} (\exists \bar{x}) \wedge p(\bar{x}) \Rightarrow \wedge p \in L^0_{\alpha+1}(N).$$

*Hence for every $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$-formula $\psi(\bar{x})$ there are formulas $\varphi_n(\bar{x}) \in L^0_{<\omega_1}$ for $n < \omega$ such that $(N, N) \Vdash^{\aleph_1}_{\mathfrak{k}} (\forall \bar{x})[\psi(\bar{x}) \equiv \bigvee_n \varphi_n(\bar{x})]$.*

(b) *For every $N_0 \leq_{\mathfrak{k}} N_1 \in K_{\aleph_0}$ there is $N_2$ with $N_1 \leq_{\mathfrak{k}} N_2 \in K_{\aleph_0}$ such that for every $\bar{a} \in N_2$ and $\varphi(\bar{x}) \in \mathbb{L}^0_{\omega_1,\omega}(N_0)$ (with $\ell g(\bar{a}) = \ell g(\bar{x}) < \omega$, of course), we have $(N_2, N_0) \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi[\bar{a}]$ or $(N_2, N_0) \Vdash^{\aleph_1}_{\mathfrak{k}} \neg\varphi[\bar{a}]$.*

(c) *If $N \leq_{\mathfrak{k}} N_\ell \in K_{\aleph_0}$ and $\bar{a}_\ell \in N_\ell$ (for $\ell = 1, 2$), and the $L^0_{<\omega_1}(N)$-generic types of $\bar{a}_\ell$ in $N_\ell$ are equal,*[11] *<u>then</u> so are the $\mathbb{L}^0_{\infty,\omega}(N)$-generic types. In fact, there is $M \geq_{\mathfrak{k}} N$ and $\leq_{\mathfrak{k}}$-embeddings $f_\ell : N_\ell \to M$ such that $f_\ell$ maps $N$ onto itself and $f_1(\bar{a}_1) = f_2(\bar{a}_2)$ (though we do not claim $f_1 \restriction N = f_2 \restriction N$). Also, if $N_1 = N_2$ then there is $M \in K_{\aleph_0}$ which $\leq_{\mathfrak{k}}$-extends $N_1$ and an automorphism $f$ of $M$ mapping $N$ onto itself and $\bar{a}_1$ to $\bar{a}_2$.*

(d) *For each $N \in K_{\aleph_0}$ and complete $\mathbb{L}^0_{\omega_1,\omega}(N)$-type $p(\bar{x})$, the class*

$$K^1 := \big\{(N, M, \bar{a}) : M \in K_{\aleph_0},\ N \leq_{\mathfrak{k}} M \text{ and } M \leq_{\mathfrak{k}} M' \text{ for some } M' \in K_{\aleph_0} \text{ and } \bar{a} \text{ materializes } p \text{ in } (M; N)\big\}$$

*is a $\mathsf{PC}_{\aleph_0}$-class.*

(e) *For any complete $\mathbb{L}^{-1}_{\omega_1,\omega}(N)$-type $p(\bar{x})$, for some complete $\mathbb{L}^0_{\omega_1,\omega}(N)$-type $q_p$, if $N \leq_{\mathfrak{k}} M \in K_{\aleph_0}$, $\bar{a} \in M$, and $\bar{a}$ materializes $p$ in $(M, N)$, <u>then</u> $\bar{a}$ materializes $q_p$ in $(M, N)$. (On $\mathbb{L}^0, \mathbb{L}^{-1}$, see Definition 4.2(1),(3).)*

(f) *The number of complete $\mathbb{L}^0_{\omega_1,\omega}(N)$-types $p$ which are materialized in $(M, N)$ by $\bar{a}$ (for some $M \in K_{\aleph_0}$ and $\bar{a} \in {}^{\omega>}M$ with $N \leq_{\mathfrak{k}} M$) is $\leq \aleph_1$.*

(g) *If in clause (f) we get that there are $\aleph_1$ such types then $\dot{I}(\aleph_1, K) \geq \aleph_1$.*

(h) *Let $L^{-1}_\alpha := L^0_\alpha \cap \mathbb{L}^{-1}_{\omega_1,\omega}(\tau^{+0})$. Then the parallel clauses to (a)-(g) hold.*

2) *Clause (e) means that*

(i) *Assume further that $N_0 \leq_{\mathfrak{k}} N_\ell \in K_{\aleph_0}$ and $\bar{a}_\ell \in N_\ell$ for $\ell = 1, 2$, and the $L^{-1}_{<\omega_1}(N)$-type which $\bar{a}_1$ materializes in $N_1$ is equal to the $L^{-1}_{<\omega_1}(N)$-type which $\bar{a}_2$ materializes in $N_2$. <u>Then</u> we can find $N^+_1, N^+_2$ such that $N_\ell \leq_{\mathfrak{k}} N^+_\ell \in K_{\aleph_0}$ for $\ell = 1, 2$ and an isomorphism $f$ from $N^+_1$ onto $N^+_2$ mapping $N$ onto itself and $\bar{a}_1$ to $\bar{a}_2$.*

*Remark* 4.14. 1) We cannot get rid of the case of $\aleph_1$ types (but see 5.23, 5.30) by the following variant of a well known example of Morley [Mor70] for $\dot{I}(\aleph_0, K) = \aleph_2$. Let

$$K := \big\{(A, E, <) : E \text{ an equivalence relation on } A, \text{ each } E\text{-equivalence class is countable, } x < y \Rightarrow x\,E\,y, \text{ and } x\,E\,y \Rightarrow (x/E, <, x) \cong (y/E, <, y)\big\}.$$

[11] Though they are not necessarily complete; i.e. for every $\varphi(\bar{x}) \in L^0_{<\omega_1}(N)$ we have $N_1 \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi(\bar{a}_1)$ <u>iff</u> $N_2 \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi[\bar{a}_2]$.

(That is, $<$ is a 1-*transitive linear order* on each $E$-equivalence class.) Let $M \leq_{\mathfrak{k}} N$ if $M \subseteq N$ and

$$x \in M \wedge y \in N \wedge x\,E\,y \Rightarrow y \in M.$$

By the analysis of such countable linear orders, each $(a/E^M, <)$ is determined up to isomorphism by $(\alpha, \ell) \in \omega_1 \times 2$. For appropriate $\mathbf{F}$, if $M = \mathbf{F}(N)$, $a \in N$, and $I$ is an interval of $(a/E^N, <^N)$ which is 1-transitive then for some $b \in M \setminus N$, $(b/E^M, <^M)$ is isomorphic to $(I, <^N)$. This is enough.

2) In clauses (c),(i) of 4.13, the mappings are not necessarily the identity on $N$. In clause (i) the assumption is apparently weaker (tho[ugh] by its conclusion the assumption of (c) holds).

3) Note that clause (f) of 4.13 does not follow from clause (a) as there may be $\aleph_1$-Kurepa trees.

4) In clause (c) of 4.13 for the second sentence we can weaken the assumption: if $\varphi(\bar{x}) \in L^0_{<\omega_1}(N)$ and $(N_1; N) \nVdash^{\aleph_1}_{\mathfrak{k}} \varphi(\bar{a}_1)$ then $(N_2, N) \nVdash^{\aleph_1}_{\mathfrak{k}} \varphi(\bar{a}_2)$. This is enough to get the $M_{1,\alpha}, M_{2,\alpha}$ from the proof.

[Why? For each $\alpha < \omega_1$, there are $M_{1,\alpha}$ such that $N_1 \leq_{\mathfrak{k}} M_{1,\alpha} \in K_{\aleph_0}$ and a complete $L^0_\alpha$-$\ell g(\bar{a}_i)$-type $p_*(\bar{x})$ such that $(M_{1,\alpha}, N) \Vdash \wedge p_*(\bar{a}_1)$. But $\neg \wedge p_1(\bar{x}) \in L_{\alpha+1}$ and obviously $(N_1, N) \nVdash \neg \wedge p_*(\bar{a}_1)$ hence $(N_2, N) \nVdash^{\aleph_1}_{\mathfrak{k}} \neg \wedge p_*(\bar{a}_2)$ hence there is $M_{2,\alpha}$ such that $N_2 \leq_{\mathfrak{k}} M_{2,\alpha} \in K_{\aleph_0}$ and $(M_{2,\alpha}; N) \Vdash^{\aleph_1}_{\mathfrak{k}} \wedge p_*(\bar{a}_2)$. Now continue as in the proof below.]

*Remark* 4.15. We can prove clause (b) (and the last sentence in clause (c) of 4.13) directly, not mentioning the $L^0_\alpha$-s.

*Proof.* Note that proving clause (e) we just need to say "repeat the proof of clauses (a)-(d) for $L^{-1}_{\omega,\omega}$".

**Clause (a):** We choose $L^0_\alpha$ by induction on $\alpha$ using 4.11. The second phrase is proved by induction on the depth of the formula using 4.10.

**Clause (b):** By iterating $\omega$ times, it suffices to prove this for each $\bar{a} \in N_1$, so again by iterating $\omega$ times it suffices to prove this for a fixed $\bar{a} \in N_1$. If the conclusion fails we can define, by induction on $n < \omega$, a model $M_\eta$ and $\varphi_\eta(\bar{x}) \in \mathbb{L}^0_{\omega_1,\omega}(N)$ for every $\eta \in {}^n2$ such that:

- (*i*) $M_{\langle\,\rangle} = N_1$
- (*ii*) $M_\eta \leq_{\mathfrak{k}} M_{\eta\hat{}\langle\ell\rangle} \in K_{\aleph_0}$ for $\ell = 0, 1$.
- (*iii*) $(M_\eta, N) \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi_\eta(\bar{a})$
- (*iv*) $\varphi_{\eta\hat{}\langle 1\rangle}(\bar{x}) = \neg\varphi_{\eta\hat{}\langle 0\rangle}(\bar{x})$.

Now for $\eta \in {}^\omega 2$, let $M_\eta = \bigcup\limits_{n<\omega} M_{\eta\restriction n}$. Clearly for $\eta \in {}^\omega 2$ we have

$$M_\eta \Vdash^{\aleph_1}_{\mathfrak{k}} (\exists \bar{x})\Big[\bigwedge_{n<\omega} \varphi_{\eta\restriction n}(\bar{x})\Big]$$

and after slight work, we get a contradiction to 4.11 + 4.10.

**Clause (c):** In general, by clause (a) we can find $M^\alpha_\ell \in K_{\aleph_1}$ for $\ell = 1, 2$ and $\alpha < \omega_1$ such that $N_\ell \leq_{\mathfrak{k}} M^\alpha_\ell$, $(M^\alpha_1, \bar{a}_1), (M^\alpha_2, \bar{a}_2)$ are $L^0_\alpha(N)$-equivalent, and without loss of generality each of $N, N_\ell, M^\alpha_\ell$ have universe an ordinal $< \omega_1$. Let

$$\mathfrak{A} := \big(\mathcal{H}(\aleph_2), N, N_1, N_2, \langle M^\alpha_1 : \alpha < \omega_1\rangle, \langle M^\alpha_2 : \alpha < \omega_1\rangle\big).$$

Let $\mathfrak{A}_1 \prec \mathfrak{A}$ be countable, and (recalling 0.5(3)) find a non-well ordered countable model $\mathfrak{A}_2$ which is an end-extension of $\mathfrak{A}_1$ for $\omega_1^{\mathfrak{A}_1}$. Hence $\omega^{\mathfrak{A}_2} = \omega$, so $N^{\mathfrak{A}_2} = N$ and $N_\ell^{\mathfrak{A}_2} = N_\ell$ for $\ell = 1, 2$. For $x \in (\omega_1)^{\mathfrak{A}_2} \setminus \mathfrak{A}_1$, let $M_\ell^x := (M_\ell^x)^{\mathfrak{A}_2}$ so $N_\ell \leq_{\mathfrak{k}} M_\ell^x \in K_{\aleph_0}$. Now there are $x_n$ such that $\mathfrak{A}_2 \models$ "$x_{n+1} < x_n$ are countable ordinals", so using the hence-and-forth argument

$$(M_1^{x_0}, \bar{a}_1, N) \cong (M_2^{x_0}, \bar{a}_2, N).$$

[Why? Let

$$\mathcal{F}_n := \big\{(\bar{b}^1, \bar{b}^2) : \bar{b}^\ell \in {}^n(M_\ell^{x_0}) \text{ and } \mathfrak{A}_2 \models \text{"}\mathrm{gtp}_{L^0_{x_n}}(\bar{a}^1{}^\frown\bar{b}^1, N; M_1^{x_0}) = \mathrm{gtp}_{L^0_{x_n}}(\bar{a}^2{}^\frown\bar{b}^2; N; M_2^{x_0})\text{"}\big\}.$$

Clearly $(\langle\ \rangle, \langle\ \rangle) \in \mathcal{F}_0$ and if $(\bar{b}^1, \bar{b}^2) \in \mathcal{F}_n$, $\ell \in \{1, 2\}$, and $b_n^\ell \in M_\ell^{x_0}$ <u>then</u> there is $b_n^{3-\ell} \in M_{3-\ell}^{x_0}$ such that $\big(\bar{b}^1{}^\frown\langle b_n^1\rangle, \bar{b}^2{}^\frown\langle b_n^2\rangle\big) \in \mathcal{F}_{n+1}$. As $M_1^{x_0}, M_2^{x_0}$ are countable, we can find an isomorphism.]

But this is as required in the second phrase of (c).

We still have to prove the first phrase. For this we prove by induction on the ordinal $\alpha$ that

$\circledast^1_\alpha$ Let $\ell = 1, 2$. If $\bar{a}_\ell \in {}^{\omega>}(N_\ell)$ materializes a complete $L^0_{<\alpha}$-type $p(\bar{x})$ in $(N_\ell, N_*)$ not depending on $\ell$, and $\varphi(\bar{x}) \in \mathbb{L}^0_{\infty,\omega}(N_*)$ has quantifier depth $< \alpha$, then $(N_\ell, N_*) \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi(\bar{a}_\ell)$ <u>or</u> $(N_\ell, N_*) \Vdash^{\aleph_1}_{\mathfrak{k}} \neg\varphi(\bar{a}_\ell)$.

For countable $N \leq_{\mathfrak{k}} M$ and $\bar{a} \in {}^{\omega>}N$,

$\odot_1$ Let $\mathbf{P}_\alpha(N, M, \bar{a}) :=$

$\big\{\mathrm{gtp}_{L^0_{<\alpha}}(\bar{a}; N; M^+) : M \leq_{\mathfrak{k}} M^+ \in K_{\aleph_0}$ and $\mathrm{gtp}_{L^0_\alpha}(\bar{a}; N; M^+)$ is a complete $L^0_\alpha$-type$\big\}$.

Now

$\odot_2$ For $\beta < \alpha < \omega_1$, we can complete $\mathbf{P}_\beta(N, M, \bar{a})$ from $\mathrm{gtp}_{L^0_\alpha}(\bar{a}; N; M)$.

$\odot_3$ For $\alpha < \omega_1$, from $\mathbf{P}_\beta(N, M, \bar{a})$ we can compute $\mathrm{gtp}_{L^0_\alpha}(\bar{a}; N; M)$.

$\odot_4$ Assume $N \leq_{\mathfrak{k}} M$ are countable and $\bar{a} \in {}^{\omega>}M$. For $\varphi(\bar{x}) \in L^0_{\omega_1,\omega}(N)$ of quantifier depth $< \alpha$ we have

$$\varphi(\bar{x}) \in \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}(N)}(\bar{a}; N; M)$$

<u>iff</u> for every $q(\bar{x}) \in \mathbf{P}_\alpha(N, M, \bar{a})$, $\varphi(\bar{x})$ belongs to the type computed implicitly in $\circledast_\alpha$; i.e. if $q(\bar{x}) = \mathrm{gtp}_{L^0_{<\alpha}}(\bar{a}'; N'; M')$ then $(N', M') \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi(\bar{x})$.

Those three should be clear, and give the desired conclusion. Also, the last sentence is easy.

<u>**Clause (d):**</u> Let $N_0 \leq_{\mathfrak{k}} M_0 \in K_{\aleph_0}$ and $\bar{a}_0 \in M_0$ be such that

$$(M_0, N_0) \Vdash^{\aleph_1}_{\mathfrak{k}} \bigwedge_{\varphi(\bar{x}) \in p} \varphi[\bar{a}_0]$$

(if it does not exist, the set of triples is empty). Let

$$K'' := \big\{(N, M, \bar{a}) : M, N \in K_{\aleph_0},\ N \leq_{\mathfrak{k}} M, \text{ and there are } M'' \in K_{\aleph_0} \text{ with } M \leq_{\mathfrak{k}} M'' \text{ and a } \leq_{\mathfrak{k}}\text{-embedding } f : M_0 \to M'' \text{ such that } f(N_0) = N,\ g(\bar{a}_0) = \bar{a}\big\}.$$

**[What's $g$?]**

Clearly it is a $\mathsf{PC}_{\aleph_0}$ class. Also,

$$M_0 \leq_{\mathfrak{k}} M' \in K_{\aleph_0} \Rightarrow \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}(N_0)}(\bar{a}; N_0; M_0) = \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}(N_0)}(\bar{a}; N_0, M').$$

Now first, if $(N, M, \bar{a}) \in K''$ let $(M'', f)$ witness this; so by applying clause (b) of 4.13,

$$\begin{aligned} \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}}(\bar{a}; N; M) \subseteq \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}}(\bar{a}; N; M'') &= \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}}(\bar{a}; N; f(M_0)) \\ &= \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}}(a_0; N_0; M_0) = p, \end{aligned}$$

so $(N, M, \bar{a}) \in K^1$.

Second, if $(N, M, \bar{a}) \in K^1$ let $f_0$ be an isomorphism from $M_0$ onto $M_0$. Let $(M_1, f_1)$ be such that $N_0 \leq_{\mathfrak{k}} M_1 \in K_{\aleph_0}$, $f_1 \supseteq f_0$ is an isomorphism from $M_1$ onto $M$, and $\bar{a}_1 = f_i^{-1}(\bar{a})$. Hence $p = \mathrm{gtp}_{\mathbb{L}^0_{\omega_1,\omega}}(\bar{a}_1; N_0; M_1)$ and we apply clause (c) of 4.13, with $N_0, M_0, \bar{a}_0, M_1, \bar{a}_1$ here standing in for $N, M_1, \bar{a}_1, M_2, \bar{a}_2$ there, and can finish easily.

<u>**Clause (e):**</u> We can define $\langle L^{-1}_\alpha : \alpha < \omega_1\rangle$ satisfying the parallel of Clause (a) and repeat the proofs of clauses (b),(c), and we are done.

<u>**Clause (f):**</u> Suppose this fails. The proof splits to two cases.

**Case A**: $2^{\aleph_0} = 2^{\aleph_1}$.

We shall prove $\dot{I}(\aleph_1, K) \geq 2^{\aleph_0}$, thus contradicting Hypothesis 4.8 (as $2^{\aleph_0} = 2^{\aleph_1}$).

Let $p_i$ (for $i < \omega_2$) be distinct complete $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$-types such that for each $i$, $p_i$ is materialized in some pair $(M, N)$ (so $N \leq_{\mathfrak{k}} M \in K_{\aleph_0}$; they exist by the assumption that (f) fails). For each $i < \omega_2$ and $\alpha < \omega_1$, we define $N_{i,\alpha}, \xi_{i,\alpha}$, and $\bar{a}_{i,\alpha}$ such that:

$\boxtimes_1$
- (i) $N_{i,\alpha} \in K_{\aleph_0}$ has universe $\omega \cdot (1 + \alpha)$ and $N_{0,0} := N$.
- (ii) $\langle N_{i,\alpha} : \alpha < \omega_1\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous.
- (iii) $\bar{a}_{i,\alpha} \in N_{i,\alpha+1}$ materializes $p_i$ in $(N_{i,\alpha+1}, N_{i,\alpha})$.
- (iv) For every $\alpha < \beta < \omega_1$ and $\bar{a} \in {}^{\omega>}(N_{i,\beta})$, the sequence $\bar{a}$ materializes a complete $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$-type in $(N_{i,\beta}, N_{i,\alpha})$.
- (v) $\xi_{i,\alpha} < \omega_1$ is strictly increasing continuous in $\alpha$.
- (vi) For $\alpha < \beta$, $N_{i,\beta}$ is pseudo-$L^0_\beta(N_{i,\alpha})$-generic (see 4.4(4)) and 'takes care of' $\mathbf{Q}$.
  I.e. if $\gamma < \beta$, $p(y, \bar{x})$ is a complete $L^0_\gamma$-type and
  $$(N_{i,\beta}, N_{i,\alpha}) \Vdash^{\aleph_1}_{\mathfrak{k}} (\mathbf{Q}y) \wedge p(y, \bar{a})$$
  then for some $b \in N_{i,\beta+1} \setminus N_{i,\beta}$ we have $(N_{i,\beta+1}, N_{i,\alpha}) \Vdash^{\aleph_1}_{\mathfrak{k}} \wedge p(b, \bar{a})$.
- (vii) If $\alpha < \beta$ and $\bar{a}, \bar{b} \in N_{\beta-1}$ materialize different $\mathbb{L}^0_{\omega_1,\omega}(N_{i,\alpha})$-types in $N_{i,\beta}$, <u>then</u> $\bar{a}$ and $\bar{b}$ realize different $\big(\mathbb{L}_{\omega_1,\omega}(\tau^{+0}) \cap L^{-1}_{\xi_{i,\beta+1}}\big)(N_\alpha)$-types in $N_{i,\beta}$.
- (viii) $N_i = \bigcup\limits_{\alpha<\omega_1} N_{i,\alpha}$
- (ix) If $\alpha_\ell < \beta$ for $\ell = 1, 2$, $\gamma < \beta$, $n < \omega$, and $\bar{a}_1 \in {}^n(N_{i,\beta})$ then for some $\bar{a}_2 \in {}^n(N_{i,\beta})$ we have
  $$\mathrm{gtp}_{L^0_\gamma}(\bar{a}_1; N_{i,\alpha_1}; N_{i,\beta}) = \mathrm{gtp}_{L^0_\gamma}(\bar{a}_2; N_{i,\alpha_2}; N_{i,\beta}).$$

(ix)$^+$ Moreover, if $n < \omega$, $\gamma_1 < \gamma_2 < \beta$, $\alpha_\ell < \beta$, $\bar{a}_\ell \in {}^n(N_{i,\beta})$ for $\ell = 1, 2$,

$$\mathrm{gtp}_{L^0_{\gamma_2}}(\bar{a}_1; N_{i,\alpha_1}; N_{i,\beta}) = \mathrm{gtp}_{L^0_{\gamma_2}}(\bar{a}_2; N_{i,\alpha_2}; N_{i,\beta}),$$

and $b_1 \in N_{i,\beta}$ <u>then</u> for some $b_2 \in N_{i,\beta}$ we have

$$\mathrm{gtp}_{L^0_{\gamma_1}}(\bar{a}_1 {}^\frown \langle b_1 \rangle; N_{i,\alpha_1}; N_{i,\beta}) = \mathrm{gtp}_{L^0_{\gamma_1}}(\bar{a} {}^\frown \langle b_2 \rangle; N_{i,\alpha_2}; N_{i,\beta}).$$

This is possible by the earlier claims. By clause (e) of 4.13, clearly

$\boxtimes_2$ The pair $(N_i, N_0)$ is $L^{-1}_{<\omega_1}(\tau^{+0})$-homogeneous.

Below we could use $D_i$ a set of complete $L^0_{\delta_i}$-types; the only problem is that the countable $(D_i, \aleph_0)$-homogeneous models have to be redefined using "materialized" instead of "realized". As it is, we need to use clause (e) to translate the results on $L^0_{\delta_i}$ to $L^{-1}_{\delta_i}$.

Let $\tau^* := \{\in, Q_1, Q_2\} \cup \{c_\ell : \ell < 5\}$, with each $c_\ell$ an individual constant, and $\mathfrak{A}^*_i$ be $(\mathcal{H}(\aleph_2), \in)$ expanded to a $\tau^*$-model, by predicates for $K$ **[and]** $\leq_{\mathfrak{k}}$, with

$$Q_1^{\mathfrak{A}^*_i} := K \cap \mathcal{H}(\aleph_2)$$
$$Q_2^{\mathfrak{A}^*} := \big\{(M, N) : M \leq_{\mathfrak{k}} N \text{ both in } \mathcal{H}(\aleph_2)\big\},$$

and $c_0^{\mathfrak{A}^*_i}, \ldots, c_4^{\mathfrak{A}^*_i}$ being $\{\langle N_{i,\alpha} : \alpha < \omega_1 \rangle\}$, $\langle \xi_{i,\alpha} : \alpha < \omega_1 \rangle$, $\{\langle \bar{a}_{i,\alpha} : \alpha < \omega_1 \rangle\}$, $N_i$, and $\{i\}$, respectively.

Let $\mathfrak{A}_i$ be a countable elementary submodel of $\mathfrak{A}^*_i$, so $|\mathfrak{A}_i| \cap \omega_1$ is an ordinal $\delta_i < \omega_1$. It is also clear that $c_3^{\mathfrak{A}_i}$ is $N_{i,\delta_i}$ as $c_3^{\mathfrak{A}^*_i} = N_i$. As $\mathfrak{A}_i$ is defined for $i < \omega_2$, for some unbounded $S \subseteq \omega_2$ and $\delta < \omega_1$, $\delta_i = \delta$ for every $i \in S$. For $i, j \in S$, we know that some sequence from $N_j$ materializes $p_i$ in the pair $(N_j, N_{j,\delta(j)})$ <u>iff</u> $i = j$. For $i \in S$, let $D_i$ be the set of complete $L^{-1}_{\delta_i}$-types materialized in $(N_{i,\delta_i}, N_{i,0})$. Because of the choice of $\xi_{i,\alpha}$-s and $\boxtimes_2$, the pair $(N_{i,\delta}, N_0)$ is $(D_i, \aleph_0)$-homogeneous and $D_i$ is a countable set of complete $L^{-1}_\delta$-types. Note that by the choice of $S$,

$$i \neq j \in S \Rightarrow D_i \neq D_j.$$

Let

$$\Gamma := \Big\{ D : D \text{ a countable set of complete } L^{-1}_\delta\text{-types, such that for some model}$$
$$\mathfrak{A} = \mathfrak{A}_D \text{ of } \bigcap_{i \in S} \mathrm{Th}_{\mathbb{L}_{\omega,\omega}}(\mathfrak{A}_i), \text{ with } \{a : \mathfrak{A}_D \models \text{“}a \text{ a countable ordinal”}\} = \delta$$
$$\text{we have } D = \big\{ \{\varphi(\bar{x}) \in L^{-1}_\delta : \mathfrak{A}_D \models (N; N_0) \Vdash^{\aleph_1}_{\mathfrak{k}} \varphi[\bar{a}]\} : \bar{a} \in N \big\} \Big\}$$

(where $N = c_3^{\mathfrak{A}_D}$).

So $D_i \in \Gamma$ for $i < \omega_2$, hence $\Gamma$ is uncountable.

By standard descriptive set theory $\Gamma$ (is an analytic set, hence) has cardinality continuum. So let $D_\zeta \in \Gamma$ be distinct for $\zeta < 2^{\aleph_0}$. For each $\zeta$, let $\mathfrak{A}^0_{D_\zeta}$ be as in the definition of $\Gamma$. We define $\mathfrak{A}^\alpha_{D_\zeta}$ by induction on $\alpha < \omega_1$ such that

(A) $\mathfrak{A}^\alpha_{D_\zeta}$ is countable.

(B) $\alpha < \beta \Rightarrow \mathfrak{A}^\alpha_{D_\zeta} \prec_{\mathbb{L}_{\omega,\omega}} \mathfrak{A}^\beta_{D_\zeta}$

(C) For limit $\alpha$m we have $\mathfrak{A}^\alpha_{D_\zeta} = \bigcup_{\beta < \alpha} \mathfrak{A}^\beta_{D_\zeta}$.

(D) If $d \in \mathfrak{A}^{\alpha+1}_{D_\zeta} \setminus \mathfrak{A}^\alpha_{D_\zeta}$ and $\mathfrak{A}^{\alpha+1}_{D_\zeta} \models$ "$d$ a countable ordinal", <u>then</u> for $a \in \mathfrak{A}^\alpha_{D_\zeta}$ we have $\mathfrak{A}^{\alpha+1}_{D_\zeta} \models$ "if $a$ is a countable ordinal then $a < d$".

(E) For $\alpha = 0$, there is no minimal such $d$ in clause (D).

(F) For every $\alpha$ there is $d_{\zeta,\alpha} \in \mathfrak{A}^{\alpha+1}_{D_\zeta} \setminus \mathfrak{A}^\alpha_{D_\zeta}$ satisfying $\mathfrak{A}^{\alpha+1}_{D_\zeta} \models$ "$d_{\zeta,\alpha}$ a countable ordinal", and for $\alpha \neq 0$ it is minimal.

Without loss of generality

(∗) $\left(\mathcal{H}(\aleph_1)^{\mathfrak{A}^0_{D_\zeta}}, \in^{\mathfrak{A}^0_{D_\zeta}}\right)$ is equal to its Mostowski collapse (and $\mathbb{L}_{\omega_1,\omega}(N) \subseteq \mathcal{H}(\aleph_1)$).

(We could have also fixed $\mathrm{otp}(\mathfrak{A}_i \cap \omega_2)$, and hence ensure that $(\mathfrak{A}^0_{D_\zeta}, \in^{\mathfrak{A}^0_{D_\zeta}})$ is also equal to its Mostowski collapse).

Let $M_{\zeta,\alpha}$ be the $d_{\zeta,\alpha}$-th member of the $\omega_1$-sequence of models in $\mathfrak{A}^\beta_{D_\zeta}$ for $\beta > \alpha$ (remember $c_0^{\mathfrak{A}^*_i} = \langle N_{i,\alpha} : \alpha < \omega_1\rangle$). Let $M_\zeta = \bigcup\limits_{\alpha<\omega_1} M_{\zeta,\alpha}$. By absoluteness from $\mathfrak{A}^\beta_{D_\zeta}$ we have $M_{\zeta,\alpha} \le_{\mathfrak{k}} M_{\zeta,\beta} \in K_{\aleph_0}$. Now,

(∗) $(M_{\zeta,\beta}, M_{\zeta,\alpha})$ is $(D_\zeta, \aleph_0)$-homogeneous for $0 < \alpha < \beta$.

[Why? Assume $\mathfrak{A}^\alpha_{D_\zeta} \models$ "$d_1 < d_2$ are countable ordinals $> \gamma$" when $\gamma < \delta$. Now if $\bar a, \bar b \in {}^{\omega>}(N_{d_2}^{\mathfrak{A}^\alpha_{D_\zeta}})$ and

$$\gamma < \delta \Rightarrow \mathrm{gtp}_{L^0_\gamma}\left(\bar a; N_{d_1}^{\mathfrak{A}^\alpha_{D_\zeta}}; N_{d_2}^{\mathfrak{A}^\alpha_{D_\zeta}}\right) = \mathrm{gtp}_{L^0_\gamma}\left(\bar b; N_{d_1}^{\mathfrak{A}^\alpha_{D_\zeta}}; N_{d_2}^{\mathfrak{A}^\alpha_{D_\zeta}}\right)$$

then $\mathfrak{A}^\alpha_{D_\zeta}$ also satisfies this. But $\mathfrak{A}^\alpha_{D_\zeta}$ "thinks that" the countable ordinals are well-ordered <u>hence</u> for some $d$, $\mathfrak{A}^\alpha_{D_\zeta} \models$ "$d$ is a countable ordinal $> \gamma$" for each $\gamma < \delta$, and we have

$$\mathfrak{A}^\alpha_{D_\zeta} \models \text{“}\mathrm{gtp}_{L^0_d}(\bar a; N_{d_1}; N_{d_2}) = \mathrm{gtp}_{L^0_d}(\bar a; N_{d_1}; N_{d_2})\text{”}.$$

Hence if $\mathfrak{A}^\alpha_{D_\zeta} \models$ "$d' < d$" then for every $a \in N_{d_2}^{\mathfrak{A}^\alpha_{D_\zeta}}$, for some $b \in N_{d_2}^{\mathfrak{A}^\alpha_{D_\zeta}}$, we have

$$\mathfrak{A}^\alpha_{D_\zeta} \models \text{“}\mathrm{gtp}_{L^0_d}(\bar a\hat{\ }\langle a\rangle; N_{d_1}; N_{d_2}) = \mathrm{gtp}_{L^0_d}(\bar b\hat{\ }\langle b\rangle; N_{d_1}; N_{d_2})\text{”}$$

hence $\mathrm{gtp}_{L^0_\gamma}(\bar a\hat{\ }\langle a\rangle; N^{\mathfrak{A}^\alpha_{D_\zeta}}; N_{d_2}^{\mathfrak{A}^\alpha_{D_\zeta}}) = \mathrm{gtp}(\bar b\hat{\ }\langle b\rangle; N_{d_1}^{\mathfrak{A}^\alpha_{D_\zeta}}; N_{d_2}^{\mathfrak{A}^\alpha_{D_\zeta}})$.

Also, we can replace $L^0_\delta$ by $L^{-1}_\delta$. By clause $(\mathrm{ix})^+$ of $\boxtimes_1$, the set

$$\left\{\mathrm{gtp}_{L^0_\delta}(\bar a; N_{d_1}^{\mathfrak{A}^\alpha_{D_\zeta}}; N_{d_2}^{\mathfrak{A}^\alpha_{D_\zeta}}) : \bar a \in {}^{\omega>}\left(N_{d_2}^{\mathfrak{A}^\alpha_{D_\zeta}}\right)\right\} = D_i.$$

So $\left(N_{d_2}^{\mathfrak{A}^\alpha_{D_\zeta}}, N_{d_2}^{\mathfrak{A}^\alpha_{D_\zeta}}\right)$ is $(D_i, \aleph_0)$-homogeneous.

So from the isomorphism type of $M_\zeta$ we can compute $D_\zeta$. So $\zeta \neq \xi \Rightarrow M_\zeta \ncong M_\xi$. As $M_\zeta \in K_{\aleph_1}$ we finish.

**Case B**: $2^{\aleph_0} < 2^{\aleph_1}$.

By 3.9, $\mathfrak{k}$ has the $\aleph_0$-amalgamation property. So clearly if $N \le_{\mathfrak{k}} M \in K_{\aleph_0}, \bar a \in M$, <u>then</u> $\bar a$ materializes a complete $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$-type in $(M, N)$. We would now like to use descriptive set theory.

We represent a complete $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$-type materialized in some $(N, M)$ by a real, by representing the isomorphism type of some $(N, M, \bar a)$ with $N \le_{\mathfrak{k}} M \in K_{\aleph_0}$

and $\bar{a} \in M$. The set of representatives is analytic, recalling $\mathfrak{k}$ is $\mathsf{PC}_{\aleph_0}$, and the equivalence relation is $\Sigma^1_1$.

[As $(N_1, M_1, \bar{a}_1), (N_2, M_2, \bar{a}_2)$ represent the same type if and only if for some $(N, M)$ with $N \le_{\mathfrak{k}} M \in K_{\aleph_0}$, there are $\le_{\mathfrak{k}}$-embeddings $f_1 : M_1 \to M$ and $f_2 : M_2 \to M$ such that $f_1(N_1) = f_2(N_2) = N$ and $f_1(\bar{a}) = f_2(\bar{a})$.]

By Burgess [Bur78][12] as there are $> \aleph_1$ equivalence classes, there is a perfect set of representation, pairwise representing different types.

**[". . . set of representatives?"]**

From this we easily get that without loss of generality, their restrictions to some $L^0_\alpha$ are distinct, contradicting clause (a).

**Clause (g):** Easy, by the proof of Case A of clause (f) above, but much simpler as in 4.12.

**Clause (h):** As in the proof of clause (e).

2) Should by clear by now. $\square_{4.13}$

*Remark* 4.16. 1) Note that in the proof of 4.13(f), in Case A we also get many types, but it was not clear whether we can make the $N_\zeta$ to be generic enough to get the contradiction we got in Case B (but this is not crucial here).

2) We may like to replace $\mathbb{L}^0_{\omega_1,\omega}$ by $\mathbb{L}^1_{\omega_1,\omega}$ in 4.10, 4.11 and 4.13 (except that for our benefit, we may retain the definition of $L^1(N)$ in 4.13(e)). We lose the ability to build $L$-generic models in $K_{\aleph_1}$ (as the number of relations (even unary) on $N \in K_{\aleph_0}$ is $2^{\aleph_0}$, which may be $> \aleph_1$). However, we can say "$\bar{a}$ materializes the type $p = p(\bar{x})$ in $N \in K_{\aleph_0}$ which is a complete type in $\mathbb{L}^1_{\omega_1,\omega}(N_n, N_{n-1}, \ldots, N_0)$; where $N_0 \le_{\mathfrak{k}} \ldots \le_{\mathfrak{k}} N_n \le_{\mathfrak{k}} N$ with $N_\ell$ countable)".

[Why? Let some $N^1, \bar{a}^1$ be as above and $\bar{a}^1$ materializes $p$ in $(N^1, N_n, \ldots, N_0)$. Then this holds for $(N, \bar{a})$ iff for some $N'$ and $f$ we have $N \le_{\mathfrak{k}} N' \in K_{\aleph_1}$ and $f$ is an isomorphism from $N^1$ onto $N''$ mapping $\bar{a}^1$ to $\bar{a}$ and $N_\ell$ to $N_\ell$ for $\ell \le n$. If there is no such pair $(N^1, \bar{a}^1)$, this is trivial.]

We can get something on formulas.

This suffices for 4.10.

## Concluding remarks for §4:

*Remark* 4.17. 0) We can get more information on the case $1 \le \dot{I}(\aleph_1, K) < 2^{\aleph_1}$ (and the case $1 \le \dot{I}(\aleph_1, K^{\mathbf{F}}_{\aleph_1}) < 2^{\aleph_1}$, etc.).

1) As in 3.9, there is no difficulty in getting the results of this section for the class of models of $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$; because using $(K, \le_{\mathfrak{k}})$ from the proof of 3.19(2) in all constructions, we get many non-isomorphic models for appropriate $\mathbf{F}$ (as in 4.9(2)).

2) For generic enough $N \in K_{\aleph_1}$ with a $\le_{\mathfrak{k}}$-representation $\langle N_\alpha : \alpha < \omega_1 \rangle$, we have determined the $N_\alpha$-s (by having that without loss of generality $K$ is categorical in $\aleph_0$). In this section we have shown that for some club $E$ of $\omega_1$, for all $\alpha < \beta$ from $E$, the isomorphism type of $(N_\beta, N_\alpha)$ is essentially[13] unique. We can continue the

[12] Or see [She84].

[13] Why only essentially? As the number of relevant complete types can be $\aleph_1$; we can get rid of this by shrinking $\mathfrak{k}$.

analysis; e.g. deal with sequences $N_0 \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} \ldots \leq_{\mathfrak{k}} N_k \in K_{\aleph_0}$ such that $N_{\ell+1}$ is pseudo-$L^0_\alpha(N_\ell, N_{\ell-1}, \ldots, N_0)$-generic. We can prove by induction on $k$ that for any countable $L \subseteq \mathbb{L}^0_{\omega_1,\omega}(\tau^{+k})$ and some $\alpha$, any strong $L$-generic $N \in K_{\aleph_1}$ is $L$-determined. That is, for any $\leq_{\mathfrak{k}}$-increasing continuous $\langle N_\alpha : \alpha < \omega_1\rangle$ with union $N$ and $N_\alpha \leq_{\mathfrak{k}} N$ countable, for some club $E$, for all $\alpha_0 < \ldots < \alpha_k$ from $N$, the isomorphic type of $\langle N_{\alpha_k}, N_{\alpha_k}, \ldots, N_{\alpha_0}\rangle$ is the same; i.e. determining for $\mathbb{L}_{\infty,\omega}(\mathsf{aa})$.

3) We can do the same for stronger logics: let us elaborate.

Let us define a logic $\mathcal{L}^*$. It has variables for elements $x_1, x_2 \ldots$ and variables for filters $\mathcal{Y}_1, \mathcal{Y}_2 \ldots$

The atomic formulas are:

(i) The usual ones.
(ii) '$x \in \mathrm{dom}(\mathcal{Y})$'.

The logical operations are:

(a) $\wedge$ conjunction, $\neg$ negation.
(b) $(\exists x)$ existential quantification, where $x$ is an individual variable.
(c) the quantifier $\mathsf{aa}$ acting on variables $\mathcal{Y}$ (so we can form $(\mathsf{aa}\,\mathcal{Y})\varphi$).
(d) the quantification $(\exists x \in \mathrm{dom}(\mathcal{Y}))\varphi$.
(e) the quantification $(\exists^f x \in \mathrm{dom}(\mathcal{Y}))\varphi$.

**[I'm guessing $f$ stands for 'filter?' Can I change it to $\exists^{\rm fil}$ instead? I had assumed there should be some function $f$ in the definition.]**

It should be clear what are the free variables of a formula $\varphi$. The variable $\mathcal{Y}$ varies on pairs (a countable set, a filter on the set). Now in $(\exists x)[\varphi, \mathcal{Y}]$, $(\exists x \in \mathrm{dom}(\mathcal{Y}))\varphi$, and $(\exists^f x \in \mathrm{dom}(\mathcal{Y}))\varphi$, $x$ is bounded but not $\mathcal{Y}$; and in $(\mathsf{aa}\,\mathcal{Y})$, $\mathcal{Y}$ is bounded.

The satisfaction relation is defined as usual, plus

($\alpha$) $M \models \big(\exists x \in \mathrm{dom}(\mathcal{Y})\big)\varphi(x, \mathcal{Y}, \bar{a})$ iff for some $b$ from the domain of $\mathcal{Y}$, we have $M \models \varphi[b, \mathcal{Y}, \bar{a}]$.
($\beta$) $M \models \big(\exists^f x \in \mathrm{dom}(\mathcal{Y})\big)\varphi(x, \mathcal{Y}_{\bar{a}})$ iff $\big\{x \in \mathrm{dom}(\mathcal{Y}) : M \models \varphi(x, \mathcal{Y}, \bar{a})\big\} \in \mathcal{Y}$.
($\gamma$) $M \models (\mathsf{aa}\,\mathcal{Y}, \bar{a})\varphi(\mathcal{Y})$ iff there is a function

$$\mathbf{F} : {}^{\omega>}([M]^{<\aleph_1}) \to [M]^{<\aleph_1}$$

such that if $\bar{A} = \langle A_n : n < \omega\rangle$ is $\subseteq$-increasing with $A_n \in [M]^{<\aleph_0}$ and $\mathbf{F}(A_0, \ldots, A_n) \subseteq A_{n+1}$ then

$$M \models \varphi[\mathcal{Y}_{\bar{A}}, \bar{a}]$$

where $\mathcal{Y}_{\bar{A}}$ is the filter on $\bigcup\limits_{n<\omega} A_n$ generated by $\big\{\bigcup\limits_{n<\omega} A_n \setminus A_\ell : \ell < \omega\big\}$.

**[I'm not sure what this function $\mathbf{F}$ adds. It's not used in the conclusion, and choosing $\mathbf{F}(A_0, \ldots, A_n) := A_0$ would always satisfy the condition trivially.]**

**[Also, should that be $M \models (\mathsf{aa}\,\mathcal{Y})\varphi(\mathcal{Y}, \bar{a})$ at the top?]**

4) We can, of course, define $\mathcal{L}^*_{\mu,\kappa}$ (extending $\mathbb{L}_{\mu,\ell}$). As we would like to analyze models in $\aleph_1$, it is most natural to deal with $\mathcal{L}^*_{\omega_1,\omega}$.

We can prove that (if $1 \leq \dot{I}(\aleph_1, \mathfrak{k}) < 2^{\aleph_1}$) the quantifier $\mathsf{aa}\,\mathcal{Y}$ is determined on $K_{\aleph_1}$ (i.e. we have $\varphi(\mathcal{Y})$ for almost all $\mathcal{Y}$ iff we do *not* have $\neg\varphi(\mathcal{Y})$ for almost all $\mathcal{Y}$.

5) The logic from (3) strengthens the stationary logic $\mathbb{L}(\mathsf{aa})$ (see [She75c] and [BKM78]).

Not so strongly: looking at $\mathsf{PC}_{\aleph_0}$ class for $\mathbb{L}_{\omega_1,\omega}(\mathsf{aa})$

$$\text{(i.e. } \{M \restriction \tau : M \text{ a model of } \psi \text{ of cardinality } \aleph_1\}),$$

we can assume that $\psi \vdash$ "$<$ is an $\aleph_1$-like order". Now we can express $\varphi \in \mathcal{L}^*_{\omega_1,\omega}$, but the determinacy tells us more. Also, we can continue to define higher variables $\mathcal{Y}$.

## § 5. There is a superlimit model in $\aleph_1$

Here we make the following change:

**Hypothesis 5.1.** Like 4.8, but also $2^{\aleph_0} < 2^{\aleph_1}$.

(Note that we can assume that $K_{\aleph_0}$ is the class of atomic models of a first-order complete countable theory).

This section is the deepest (of this paper = chapter). The main difficulties are proving the facts which are obvious in the context of [She75a]. So while it was easy to show that every $p \in \mathbf{D}^*(N)$ is definable over a finite set,[14] it was not clear to me how to prove that if you extend the type $p$ to $q \in \mathbf{D}^*(M)$, where $N \leq_{\mathfrak{k}} M \in K_{\aleph_0}$ by the same definition, then $q \models p$. (Remember, $p$ and $q$ are types materialized but not realized, and at this point in the paper we still do not have the tools to replace the models by uncountable generic enough models.) So rather, we have to show that failure is a non-structure property; i.e. it implies existence of many models.

Also, symmetry of stable amalgamation becomes much more complicated. We prove existence of stable amalgamation by four stages (5.29, 5.30(3), 5.34, 5.37). The symmetry is proved as a consequence of uniqueness of one-sided amalgamation (so it cannot be used in its proof). Originally, the intention was for the culmination of the section to be the existence of a superlimit models in $\aleph_1$ (5.45). This seems to be a natural stopping point, as it seems reasonable to expect that the next step should be phrasing the induction on $n$; i.e. dealing with $\aleph_n$ and $\mathcal{P}(n-\ell)$-diagrams of models of power $\aleph_\ell$ as in [She83a], [She83b] (so this is done in [She09c]).

But less is needed in [She09a].

**Definition 5.2.** We define functions $\mathbf{D}, \mathbf{D}^*$ with domain $K_{\aleph_0}$.

1) For $N \in K_{\aleph_0}$ let

$$\mathbf{D}(N) := \big\{p : p \text{ is a complete } \mathbb{L}^0_{\omega_1,\omega}(N)\text{-type over } N \text{ such that for some } \bar{a} \in M \in K_{\aleph_0},\ N \leq_{\mathfrak{k}} M \text{ and } \bar{a} \text{ materializes } p \text{ in } (M,N)\big\}.$$

(I.e. the members of $p$ have the form $\varphi(\bar{x}, \bar{a})$, where $\bar{x}$ is finite and fixed for each $p$, $\bar{a}$ is a finite sequence from $N$, and $\varphi \in \mathbb{L}^0_{\omega_1,\omega}(N)$.)

2) For $N \in K_{\aleph_0}$, let

$$\mathbf{D}^*(N) := \big\{p : p \text{ is a complete } \mathbb{L}^0_{\omega_1,\omega}(N;N)\text{-type such that for some } \bar{a} \in M \in K_{\aleph_0},\ N \leq_{\mathfrak{k}} M \text{ and } \bar{a} \text{ materializes } p \text{ in } (M,N;N)\big\}.$$

3) For $p(\bar{x}, \bar{y}) \in \mathbf{D}(N)$, let $p(\bar{x}, \bar{y}) \restriction \bar{x} \in \mathbf{D}(N)$ be defined naturally. I.e. if for some $M \in K_{\aleph_0}$ with $N \leq_{\mathfrak{k}} M$ and $\bar{a} \restriction \bar{b} \in {}^{\ell g(\bar{x}\hat{}\bar{y})}M$ materializing $p(\bar{x}, \bar{y})$ such that $\ell g(\bar{x}) = \ell g(\bar{a})$, the sequence $\bar{a}$ materializes $p(\bar{x}, \bar{y}) \restriction x \in \mathbf{D}(N)$. Similarly for permuting the variables.

**Explanation 5.3.** 0) Recall that any formula in $\mathbb{L}^0_{\omega_1,\omega}(N)$ has finitely many free variables.

1) So for every finite $\bar{b} \in N$ and $\varphi(\bar{x}, \bar{y}) \in \mathbb{L}^0_{\omega_1,\omega}(N)$, if $p \in \mathbf{D}(N)$, then $\varphi(\bar{x}, \bar{b}) \in p$ or $\neg\varphi(\bar{x}, \bar{b}) \in p$.

[14] $\mathbf{D}^*(N)$ is defined below.

2) But a formula from $p \in \mathbf{D}^*(N)$ may have all $c \in N$ as parameters, whereas a formula from $p \in \mathbf{D}(N)$ can mention only finitely many members of $N$.

**Lemma 5.4.** 1) $\mathfrak{k}$ *has the* $\aleph_0$*-amalgamation property.*

2) *If* $N_* \leq_{\mathfrak{k}} N \in K_{\aleph_0}$ *and* $A_i \subseteq N_*$ *for* $i \leq n$, *then for every sentence* $\psi \in \mathbb{L}^1_{\infty,\omega}(N_*, A_n, \dots, A_1; A_0)$ *we have*

$$N \Vdash^{\aleph_1}_{\mathfrak{k}} \psi \text{ or } N \Vdash^{\aleph_1}_{\mathfrak{k}} \neg\psi.$$

3) *If* $N \leq_{\mathfrak{k}} M \in K_{\aleph_0}$ *then every* $\bar{a} \in M$ *materializes in* $(M, N; N)$ *one and only one type from* $\mathbf{D}^*(N)$ *and also materializes in* $(M, N)$ *one and only one type from* $\mathbf{D}(N)$. *Also, for every* $N \leq_{\mathfrak{k}} M \in K_{\aleph_0}$ *and* $q \in \mathbf{D}^*(N)$, *for some* $M'$, $M \leq_{\mathfrak{k}} M' \in K_{\aleph_0}$ *and some* $\bar{b} \in M'$ *materializes* $q$ *in* $(M; N)$.

4) *For every* $N \in K_{\aleph_0}$ *and countable* $L \subseteq \mathbb{L}^0_{\omega_1,\omega}(N; N)$, *the number of complete* $L(N; N)$*-types* $p$ *such that* $N \Vdash^{\aleph_1}_{\mathfrak{k}}$ "$(\exists\bar{x}) \wedge p$" *is countable; note that, pedantically,* $L \subseteq \mathbb{L}_{\omega_1,\omega}(\tau^+ \cup \{c : c \in N\})$ *and we restrict ourselves to models* $M$ *such that* $P^M = |N|$ *and* $c^M = c$.

5) *For* $N \in K_{\aleph_0}$ *there are countable* $L^0_\alpha \subseteq \mathbb{L}^0_{\omega_1,\omega}(N; N)$ *for* $\alpha < \omega_1$ *increasing continuous in* $\alpha$, *closed under finitary operations (and subformulas) such that:*

- $(*)$ *For each complete* $L^0_\alpha$*-type* $p$ *we have*
$$N \Vdash^{\aleph_1}_{\mathfrak{k}} \exists\bar{x} \wedge p \Rightarrow \wedge p \in L^0_{\alpha+1}.$$

*Hence for every* $\mathbb{L}^0_{\omega_1,\omega}(N; N)$*-formula* $\psi(\bar{x})$, *for some* $\varphi_n(\bar{x}) \in \bigcup_{\alpha<\omega} L^0_\alpha$ *with* $n < \omega$, *for every* $N \in K_{\aleph_0}$,

$$(N, N) \Vdash^{\aleph_1}_{\mathfrak{k}} (\forall\bar{x})\big[\psi(\bar{x}) \equiv \bigvee_{n<\omega} \varphi_n(\bar{x})\big].$$

6) *For* $N \in K_{\aleph_0}$ *we have* $|\mathbf{D}^*(N)| \leq \aleph_1$ *and* $|\mathbf{D}(N)| \leq \aleph_1$.

7) *If* $p \in \mathbf{D}^*(N)$ *then there is* $q$ *such that if* $N \leq_{\mathfrak{k}} M \in K_\lambda$ *and* $\bar{a} \in M$ *materializes* $p$ *in* $(M; N)$, *then the complete* $\mathbb{L}^0_{\infty,\omega}(N)$*-type which* $\bar{a}$ *realizes in* $M$ *over* $N$ *is* $q$; *also,* $q$ *belongs to* $\mathbf{D}(N)$ *and is unique. Moreover, we can replace* $q$ *by the complete* $\mathbb{L}^{-1}_{\omega_1,\omega}(N)$*-type which* $\bar{a}$ *materializes in* $M$. *Similarly for* $\mathbf{D}(N)$, $\mathbb{L}^0_{\infty,\omega}(N)$, $\mathbb{L}^{-1}_{\omega_1,\omega}(N)$.

8) *If* $n < \omega$ *and* $\bar{b}, \bar{c} \in {}^n N$ *realize the same* $\mathbb{L}_{\omega_1,\omega}(\tau)$*-type in* $N$, *then they materialize the same* $\mathbb{L}^1_{\omega_1,\omega}(\tau^{+0})$*-type in* $(N, N)$.

9) *If* $f$ *is an isomorphism from* $N_1 \in K_{\aleph_0}$ *onto* $N_2 \in K_{\aleph_0}$ *then* $f$ *induces a one-to-one function from* $\mathbf{D}(N_1)$ *onto* $\mathbf{D}(N_2)$ *and from* $\mathbf{D}^*(N_1)$ *onto* $\mathbf{D}^*(N_2)$.

*Proof.* 1) By 3.9.

2) By 1).

3) By (2) and (1).

4) Like the proof of 4.11 (just easier).

5) Like the proof of 4.13(a).

6) Like the proof of 4.13(f) (recalling 0.4).

7) Clear, as in $p \in \mathbf{D}^*(N)$ we allow more formulas than for $q \in \mathbf{D}(N)$.

8,9) Easy as well. $\square_{5.4}$

From now on, we will use a variant of gtp. (In Definition 4.3(4) we defined $\mathrm{gtp}_L(\bar a; N_*, \bar A; A; N)$.)

**Definition 5.5.** 1) If $N_0 \le_{\mathfrak{k}} N_1 \in K_{\aleph_0}, \bar a \in N_1$, $\mathrm{gtp}(\bar a, N_0, N_1)$ is the $p \in \mathbf{D}(N_0)$ such that $(N_1, N_0) \Vdash^{\aleph_1}_{\mathfrak{k}} \wedge p[\bar a]$. So $\bar a$ materializes (but does not necessarily realize) $\mathrm{gtp}(\bar a, N_0, N_1)$. We may omit $N_1$ when clear from context. We define $\mathrm{gtp}^*(\bar a, N_0, N_1) \in \mathbf{D}^*(N_0)$ similarly.

2) We say $p = \mathrm{gtp}^*(\bar b, N_0, N_1)$ is definable over $\bar a \in N_0$ <u>if</u>

$$\mathrm{gtp}(\bar b, N_0, N_1) = p^- :=$$
$$\big\{\varphi(\bar x, \bar a) \in p : \varphi(\bar x, \bar y) \in \mathbb{L}^0_{\omega_1,\omega}(N_0) \text{ and } \bar a \in {}^{\ell g(\bar y)}(N_0) \subseteq {}^{\omega>}(N_0)\big\}$$

is definable over $\bar a$.

**[Nothing here depends on $\bar b$, and there appear to be too many $\bar a$-s.]**

(See Definition 5.7 below; note that $p \mapsto p^-$ is a one-to-one mapping from $\mathbf{D}^*(N_0)$ onto $\mathbf{D}(N_0)$ by 5.9(1) below.) So stationarization is defined for $p \in \mathbf{D}^*(N_0)$ as well, after we know 5.9(1).

**Claim 5.6.** 1) *Each $p \in \mathbf{D}(N)$ does not $\big(\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0}), \mathbb{L}_{\omega_1,\omega}(\tau)\big)$-split (see Definition 5.7 below) over some finite subset $C$ of $N$, hence $p$ is definable over it.*

*Moreover, letting $\bar c$ list $C$, there is a function $g_p$ satisfying $g_p(\varphi(\bar x, \bar y))$ is $\psi_{p,\varphi}(\bar y, \bar z) \in \mathbb{L}_{\omega_1,\omega}(\tau)$ such that for each $\varphi(\bar x, \bar y) \in \mathbb{L}^0_{\omega_1,\omega}(N)$ and $\bar a \in N$, we have*

$$\varphi(\bar x, \bar a) \in p \Leftrightarrow N \models \psi_{p,\varphi}(\bar a, \bar c).$$

*(In particular, $\mathbf{Q}$ is "not necessary.")*

2) *Every automorphism of $N$ maps $\mathbf{D}(N)$ onto itself and each $p \in \mathbf{D}(N)$ has at most $\aleph_0$ possible images (we may also call them* conjugates*). So if $g$ is an isomorphism from $N_0 \in K_{\aleph_0}$ onto $N_1 \in K_{\aleph_0}$ then $g(\mathbf{D}(N_0)) = \mathbf{D}(N_1)$.*

3) *If $N_0 \le_{\mathfrak{k}} N_1 \le_{\mathfrak{k}} N_2 \in K_{\aleph_0}$ and $\bar a \in N_1$, then $\mathrm{gtp}(\bar a, N_0, N_1) = \mathrm{gtp}(\bar a, N_0, N_2)$.*

Before we prove 5.6:

**Definition 5.7.** Assume

(a) $N$ is a model.
(b) $\Delta_1$ is a set of formulas (possibly in a vocabulary $\not\subseteq \tau_N$) closed under negation.
(c) $\Delta_2$ is a set of formulas in the vocabulary $\tau = \tau_N$.
(d) $p$ is a $(\Delta_1, n)$-type over $N$.
 (I.e. each member has the form $\varphi(\bar x, \bar a)$ with $\bar a$ from $N$, $\varphi(\bar x, \bar y)$ from $\Delta_1$, and $\bar x = \langle x_\ell : \ell < n\rangle$; no more is required. We may allow other formulas, but they are irrelevant.)
(e) $A \subseteq N$.

0) We say $p$ is a complete $\Delta_1$-type over $B$ when:

- $(i)$ $B \subseteq N$
- $(ii)$ $\varphi(\bar{x}, \bar{b}) \in p \Rightarrow \bar{b} \subseteq A \wedge \varphi(\bar{x}, \bar{y}) \in \Delta_1$
- $(iii)$ if $\varphi(\bar{x}, \bar{y}) \in \Delta_1$ and $\bar{b} \in {}^{\ell g(\bar{y})}A$, then $\varphi(\bar{x}, \bar{b}) \in p$ or $\neg\varphi(\bar{x}, \bar{b}) \in p$.

The default value here for $\Delta_1$ is $\mathbb{L}_{\omega_1,\omega}(\tau_{\mathfrak{k}})$.

1) We say that $p$ does $(\Delta_1, \Delta_2)$-split over $A$ when there are $\varphi(\bar{x}, \bar{y}) \in \Delta_1$ and $\bar{b}, \bar{c} \in {}^{\ell g(\bar{y})}N$ such that

- $(\alpha)$ $\varphi(\bar{x}, \bar{b}), \neg\varphi(\bar{x}, \bar{c}) \in p$
- $(\beta)$ $\bar{b}$ and $\bar{c}$ realize the same $\Delta_2$-type over $A$.

2) We say that $p$ is $(\Delta_1, \Delta_2)$-definable over $A$ <u>when</u> for every formula $\varphi(\bar{x}, \bar{y}) \in \Delta_1$ there is a formula $\psi(\bar{y}, \bar{z}) \in \Delta_2$ and $\bar{c} \in {}^{\ell g(\bar{z})}A$ such that

$$\varphi(\bar{x}, \bar{b}) \in p \Rightarrow N \models \psi[\bar{b}, \bar{c}]$$

$$\neg\varphi(\bar{x}, \bar{b}) \in p \Rightarrow N \models \neg\psi[\bar{b}, \bar{c}].$$

(In the case $p$ is complete over $B, \bar{b} \subseteq B$ we get "iff.")

3) Above, we may write $\Delta_2$ instead of $(\Delta_1, \Delta_2)$ when this holds for every $\Delta_1$ (equivalently, $\Delta_1$ is $\{\varphi(\bar{x}, \bar{y}) : \varphi(\bar{x}, \bar{a}) \in p\}$).

**Observation 5.8.** *Assume*

$(a)$-$(e)$ *As in 5.7.*

*In addition:*

$(d)^+$ *$p$ is a complete $(\Delta_1, n)$-type over $N$.*
*I.e. if $\varphi(\bar{x}, \bar{y}) \in \Delta_1$, $\bar{d} \in {}^{\ell g(\bar{y})}N$, and $\bar{x} = \langle x_\ell : \ell < n\rangle$, then $\varphi(\bar{x}, \bar{d}) \in p$ or $\neg\varphi(\bar{x}, \bar{d}) \in p$.*

<u>*Then*</u> *the following conditions are equivalent:*

- $(\alpha)$ *$p$ does not $(\Delta_1, \Delta_2)$-split over $A$.*
- $(\beta)$ *There is a sequence of $\langle g_{\varphi(\bar{x}, \bar{y})} : \varphi(\bar{x}, \bar{y}) \in \Delta_1\rangle$ of functions such that:*
  - $(i)$ $\mathrm{dom}(g_{\varphi(\bar{x}, \bar{y})}) \supseteq \{\mathrm{tp}_{\Delta_2}(\bar{b}, A, N) : \bar{b} \in {}^{\ell g(\bar{y})}N\}$.
  - $(ii)$ *the values of $g_{\varphi(\bar{x}, \bar{y})}$ are truth values.*
  - $(iii)$ *If $\varphi(\bar{x}, \bar{y}) \in \Delta_1$, $\bar{b} \in {}^{\ell g(\bar{y})}N$, and $q = \mathrm{tp}_{\Delta_2}(\bar{b}, A, N)$, <u>then</u>*
    $\varphi(\bar{x}, \bar{b}) \in p \Rightarrow g_{\varphi(\bar{x}, \bar{y})}(q) = \mathsf{true}$ *and* $\neg\varphi(\bar{x}, \bar{b}) \in p \Rightarrow g_{\varphi(\bar{x}, \bar{y})}(q) = \mathsf{false}$.

*Proof.* **[Proof of 5.8:]**

Reflect on the definitions. $\square_{5.8}$

*Proof.* **[Proof of 5.6:]**

1) Clearly the second sentence follows from the first, so we shall prove the first. Assume this fails. Let $(M, \bar{a})$ be such that $N \leq_{\mathfrak{k}} M \in K_{\aleph_0}$ and the sequence $\bar{a} \in M$ materializes $p$. Clearly, for every $\bar{b} \in M$, $(M, N) \Vdash \wedge q[\bar{b}]$ for some $q(\bar{x}) \in \mathbf{D}(N)$, and let $\langle b^*_\ell : \ell < \omega\rangle$ list $N$. We choose $\langle C^0_\eta, C^1_\eta, f_\eta, \bar{a}^0_\eta, \bar{a}^1_\eta : \eta \in {}^n 2\rangle$ by induction on $n$ such that

(a) For $\ell < 2$ and $\eta \in {}^n 2$, $C^\ell_\eta$ is a finite subset of $N$.

(b) $f_\eta$ is an automorphism of $N$ mapping $C^0_\eta$ onto $C^1_\eta$.

(c) $\{b^*_{\ell g(\eta)}\} \cup C^0_\eta \cup C^1_\eta \subseteq C^0_{\eta\hat{}\langle\ell\rangle} \cap C^1_{\eta\hat{}\langle\ell\rangle}$ for $\ell = 0, 1$.

(d) $\bar a^0_\eta, \bar a^1_\eta \in N$ realize in $N$ the same $\mathbb{L}_{\omega_1,\omega}(\tau)$-type over $C^0_\eta \cup C^1_\eta \cup \{b^*_{\ell g(\eta)}\}$ in $(M, N)$,

**[Which is it? In $N$ or in $(M, N)$?]**

but $\bar a\hat{}\bar a^0_\eta$, $\bar a\hat{}\bar a^1_\eta$ do not materialize the same $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$ in $(M, N)$ (this exemplifies splitting). So $\varphi_\eta(\bar x, \bar y_\eta)$ belongs to the first and $\neg\varphi_\eta(\bar x, \bar y_\eta)$ belongs to the second (where $\ell g(\bar x) = \ell g(\bar a)$ and $\ell g(\bar y_\eta) = \ell g(\bar a^0_\eta)$).

(e) $f_{\eta\hat{}\langle 0\rangle}(\bar a^0_\eta) = \bar a^1_\eta$, $f_{\eta\hat{}\langle 1\rangle}(\bar a^1_\eta) = \bar a^1_\eta$

**[This isn't symmetric. (Could still be correct, tho.)]**

(f) $f_\eta \restriction C^0_\eta \subseteq f_{\eta\hat{}\langle\ell\rangle}$ for $\ell = 0, 1$.

(g) $\bar a^0_\eta\hat{}\bar a^1_\eta \subseteq C^0_{\eta\hat{}\langle\ell\rangle} \cap C^1_{\eta\hat{}\langle\ell\rangle}$.

For $n = 0$ let $C^0_\eta, C^1_\eta = \varnothing$ and $f_\eta = \mathrm{id}_N$. Recall that $K_{\aleph_0}$ is categorical in $\aleph_0$ and $N$ is countable, hence if $n < \omega$ and $\bar b', \bar b'' \in {}^n N$ realize the same $\mathbb{L}_{\omega_1,\omega}(\tau)$-type over a finite subset $B$ of $N$, then some automorphism of $N$ over $B$ maps $\bar b'$ to $\bar b''$ by a theorem of Scott (see [Kei71]). If $(C^0_\eta, C^1_\eta, f_\eta)$ are defined and satisfies clauses (a)+(b), we recall that by our assumption toward contradiction, as

$$C^0_\eta \cup C^1_\eta \cup \{b^*_{\ell g(\eta)}\}$$

is a finite subset of $N$, there are $\bar a^0_\eta, \bar a^1_\eta \in {}^{\omega>}N$ as required in clause (d) again. So clearly there are automorphisms $f_{\eta\hat{}\langle 0\rangle}$, $f_{\eta\hat{}\langle 1\rangle}$ extending $f_\eta \restriction C^0_\eta$ such that $f_{\eta\hat{}\langle 0\rangle}(\bar a^0_\eta) = \bar a^1_\eta$ and $f_{\eta\hat{}\langle 1\rangle}(\bar a^1_\eta) = \bar a^1_\eta$ as required in clause (e), (f).

Lastly, choose

$$C^0_{\eta\hat{}\langle\ell\rangle} := C^0_\eta \cup C^1_\eta \cup f^{-1}_{\eta\hat{}\langle\ell\rangle}(C^0_\eta) \cup \left\{b^*_{\ell g(\eta)}, f^{-1}_{\eta\hat{}\langle\ell\rangle}(b^*_{\ell g(\eta)}), \bar a^0_\eta\hat{}\bar a^1_\eta, f^{-1}_{\eta\hat{}\langle\ell\rangle}(\bar a^0_\eta\hat{}\bar a^1_\eta)\right\}$$

and $C^1_{\eta\hat{}\langle\ell\rangle} := f_{\eta\hat{}\langle\ell\rangle}(C^0_{\eta\hat{}\langle\ell\rangle})$.

Having carried the induction, for every $\eta \in {}^\omega 2$ clearly $f_\eta = \bigcup\limits_{n<\omega} \left(f_{\eta\restriction n} \restriction C^0_\eta\right)$ is an automorphism of $N$.

[Why? As $\langle f_{\eta\restriction n} \restriction C^0_{\eta\restriction n} : n < \omega\rangle$ is an increasing sequence of functions by clauses (b)+(c)+(f), the union $f_\eta$ is a partial function; as, in addition, each $f_{\eta\restriction n}$ is an automorphism of $N$ by clause (b), $f_\eta$ is also a partial automorphism of $N$. Recalling that $\langle b^*_\ell : \ell < n\rangle$ lists $N$, clearly $f_\eta$ have domain $N$ by clause (c). And as $f_{\eta\restriction n}(C^0_{\eta\restriction n}) = C^1_{\eta\restriction n}$, the union $f_\eta$ has range $N$ by clause (c).]

Hence for some $M_\eta \in K_{\aleph_0}$ there is an isomorphism $f^+_\eta$ from $M$ onto $M_\eta$ extending $f$. Now for some $p_\eta \in \mathbf{D}(N)$, $f_\eta(\bar a)$ materializes $p_\eta$ in $(M_\eta, N)$. Choose a countable $L \subseteq \mathbb{L}^0_{\omega_1,\omega}(\tau^+)$ which includes $\{\varphi_\eta(\bar x, \bar y_\eta) : \eta \in {}^{\omega>}2\}$. Easily, if $\eta\hat{}\langle\ell\rangle \triangleleft \eta_\ell \in {}^\omega 2$ for $\ell = 0, 1$, then $\varphi(\bar x, \bar a^1_\eta) \in p_0$ and $\neg\varphi(\bar x, \bar a^1_\eta) \in p_1$. So

$$\eta \neq \nu \in {}^\omega 2 \Rightarrow p_\eta \cap L \neq p_\nu \cap L$$

by clauses (d)+(e), in contradiction to 5.4(4) (as we can use $\leq \aleph_0$ formulas to distinguish them).

2) Follows.

3) Trivial. $\square_{5.6}$

**Claim 5.9.** 1) *Suppose $N_0 \leq_{\mathfrak{k}} N_1 \in K_{\aleph_0}$ and $N_1$ forces that $\bar a, \bar b$ (in $N_1$) realize the same $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type over $N_0$, then $N_1$ forces that they realize the same $\mathbb{L}^0_{\omega_1,\omega}(N_0;N_0)$-type (the inverse is trivial).*

1A) *Suppose $N_0 \subseteq_{\mathfrak{k}} N_\ell \in K_{\aleph_0}$, $\bar a_\ell \in {}^{\omega>}(N_\ell)$ for $\ell = 1,2$, and*

$$\mathrm{gtp}(\bar a_1, N_0, N_1) = \mathrm{gtp}(\bar a_2, N_0, N_1).$$

*Then we can find $(N_1^+, N_2^+, f)$ such that $N_1 \leq_{\mathfrak{k}} N_1^+ \in K_{\aleph_0}$, $N_2 \leq_{\mathfrak{k}} N_2^+ \in K_{\aleph_0}$, and $f$ is an isomorphism from $N_1^+$ onto $N_2^+$ over $N_0$ mapping $\bar a_1$ to $\bar a_2$.*

2) *If $N_0 \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} N_2 \in K_{\aleph_0}$ and $\bar a, \bar b \in N_2$ then*[15] *we can compute the $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-generic type of $\bar a$ over $N_0$ from the $\mathbb{L}^0_{\omega_1,\omega}(N_1)$-generic type of $\bar a$ over $N_1$.*

*(Hence if the $\mathbb{L}^0_{\omega_1,\omega}(N_1)$-generic types of $\bar a, \bar b$ over $N_1$ are equal, then so are the $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-generic types of $\bar a, \bar b$ over $N_0$.)*

3) *For every $N_a \in K_{\aleph_0}$ there is a one-to-one function $f$ from $\mathbf{D}(N)$ onto $\mathbf{D}^*(N)$ such that if $N \subseteq_{\mathfrak{k}} M \in K_{\aleph_0}$ and $\bar a \in {}^{\omega>}M$, then*

$$f(\mathrm{gtp}(\bar a, N, M)) = \mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(N;N)}(\bar a; N; N; M).$$

*Remark* 5.10. 1) So there is no essential difference between $\mathbf{D}(N)$ and $\mathbf{D}^*(N)$.

2) Recall that in a formula of $\mathbb{L}^0_{\omega_1,\omega}(N_0;N_0)$, all $c \in N_0$ may appear as individual constants.

*Proof.* 1) We shall prove there are $N_2$ such that $N_1 \leq_{\mathfrak{k}} N_2 \in K_{\aleph_0}$ and an automorphism of $N_2$ over $N_0$ taking $\bar a$ to $\bar b$. This clearly suffices, and we prove the existence of such $N_2$ by hence-and-forth arguments (of course). We shall use 5.4(2) freely. So by renaming and symmetry, it suffices to prove that

(∗) If $m < \omega$, $N_0 \leq_{\mathfrak{k}} N_0$, and $\bar a, \bar b \in {}^m(N_1)$ materialize the same $\mathbb{L}^0_{\infty,\omega}(N_0)$-type over $N_0$, then for every $c \in N_1$, there are $N_2$ and $d \in N_2$ such that $\bar a {}^\frown \langle c\rangle, \bar b {}^\frown \langle d\rangle$ materialize the same $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type over $N_0$.

However, by the previous Claim 5.4, for some $\bar a^* \in {}^{\omega>}(N_0)$, the $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type over $N_0$ that $\bar a {}^\frown \langle c\rangle$ materializes in $(N_1, N_0)$ does not $\mathbb{L}^0_{\omega_1,\omega}(\tau^{+0})$-split over $\bar a^*$. Now $\bar a, \bar b$ materialize the same $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type over $N_0$ in $(N_1, N_0)$, hence $\bar a^* {}^\frown \bar a$, $\bar a^* {}^\frown \bar b$ materialize the same $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type in $(N_1, N_0)$. Hence there is $N_2 \in K_0$ with $N_1 \leq_{\mathfrak{k}} N_2$ and an automorphism $f$ of $N_2$ mapping $N_0$ onto $N_1$ and mapping $\bar a^* {}^\frown \bar a$ to $\bar a^* {}^\frown \bar b$ (but possibly $f \restriction N_0 \neq \mathrm{id}_{N_0}$). This holds by the last sentence in 4.13(c). Let $d := f(c)$; hence if $\bar a {}^\frown \langle c\rangle$ and $\bar b {}^\frown \langle d\rangle$ materialize the same $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type in $(N_2, N_0)$ then they materialize the same $\mathbb{L}^0_{\omega_1,\omega}(N_0)$-type over $N_0$ in $(N_2, N_0)$.

1A) Similarly to part (1).

2) Clearly it suffices to prove the "hence" part. By the assumption and proof of 5.9(1) there are $N_3$ satisfying $N_2 \leq_{\mathfrak{k}} N_3 \in K_{\aleph_0}$ and $f$ an automorphism of $N_3$ over $N_1$ taking $\bar a$ to $\bar b$. Now the conclusion follows.

3) Should be clear. $\square_{5.9}$

**Definition 5.11.** 1) We say that $\mathbf{D}_*$ is a $\mathfrak{k}$-*diagram function* when

[15] Remember, $N_2$ determines the complete $\mathbb{L}^0_{\omega_1,\omega}(N_1)$-generic types of $\bar a, \bar b$.

(a) $\mathbf{D}_*$ is a function with domain $K_{\aleph_0}$. (Later we shall lift it to $K$.)
(b) $\mathbf{D}_*(N) \subseteq \mathbf{D}(N)$, and has at least one non-algebraic member, for $N \in K_{\aleph_0}$.
(c) If $N_1, N_2 \in K_{\aleph_0}$ and $f$ is an isomorphism from $N_1$ onto $N_2$, then $f$ maps $\mathbf{D}_*(N_1)$ onto $\mathbf{D}_*(N_2)$; in particular, this applies to an automorphism of $N \in K_{\aleph_0}$.

1A) Such $\mathbf{D}_*$ is called *weakly good* <u>when</u>:

(d) ($\alpha$) $\mathbf{D}_*(N)$ is closed under subtypes: that is, if $p(\bar{x}) \in \mathbf{D}_*(N)$,

$$\bar{x} = \langle x_\ell : \ell < m \rangle,$$

and $\pi$ is a function from $\{0, \ldots, m-1\}$ into $\{0, \ldots, n-1\}$, then some (necessarily unique) $\bar{q}(\langle x_0, \ldots, x_{n-1} \rangle) \in \mathbf{D}_*(N)$ is equal to

$$\{\varphi(\langle x_0, \ldots, x_{n-1} \rangle) : \varphi(x_{\pi(0)}, \ldots, x_{\pi(m-1)}) \in p(\bar{x})\}.$$

($\beta$) If $N \leq_{\mathfrak{k}} M \in K_{\aleph_0}$, $\bar{a}_1, \bar{b}_1 \in {}^{\omega>}N$, $\bar{a}_2 \in {}^{\ell g(\bar{a}_1)}M$, $(M, \bar{a}_1) \cong (M, \bar{a}_2)$, and $\mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{a}_2; N; M) \in \mathbf{D}(N)$, <u>then</u> for some $M^+, \bar{b}_2$ we have $M \leq_{\mathfrak{k}} M^+ \in K_{\aleph_0}$, $\bar{b}_2 \in {}^{\ell g(\bar{b}_1)}(M^+)$, $(M^+, \bar{a}_1{}^\frown\bar{b}_1) \cong (M^+, \bar{a}_2{}^\frown\bar{b}_2)$, and $\mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{a}_2{}^\frown\bar{b}_2; N; M^+) \in \mathbf{D}(N)$.

($\gamma$) If $N \leq_{\mathfrak{k}} M \in K_{\aleph_0}$, $\bar{a} \in {}^{\omega>}M$, $\bar{b} \in {}^{\omega>}N$, and

$$\mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{a}; N; M) \in \mathbf{D}(N)$$

then $\mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{a}{}^\frown\bar{b}; N; M) \in \mathbf{D}(N)$.

2) Such $\mathbf{D}_*$ is called *countable* if $N \in K_{\aleph_0} \Rightarrow |\mathbf{D}_*(N)| \leq \aleph_0$.

3) Such $\mathbf{D}_*$ is called *good* when it is weakly good (i.e. clause (d) holds) and

(e) $\mathbf{D}_*(N)$ has amalgamation.
(I.e. if $p_0(\bar{x}), p_1(\bar{x}, \bar{y}), p_2(\bar{x}, \bar{z}) \in \mathbf{D}_*(N)$ and $p_0 \subseteq p_1 \cap p_2$ then there is $q(\bar{x}, \bar{y}, \bar{z}) \in \mathbf{D}_*(N)$ which includes $p_1(\bar{x}, \bar{y}) \cup p_2(\bar{x}, \bar{z})$.)

4) Such $\mathbf{D}_*$ is called *very good* if it is good and:

(f) If $N_0 \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} N_2 \in K_{\aleph_0}$, $\bar{a}_0 \subseteq \bar{a}_1 \subseteq \bar{a}_2$, $\bar{a}_\ell \subseteq N_\ell$ for $\ell = 0, 1, 2$, and $\mathrm{gtp}(\bar{a}_{\ell+1}, N_\ell, N_{\ell+1})$ is definable over $\bar{a}_\ell$ and belongs to $\mathbf{D}_*(N_\ell)$ for $\ell = 0, 1$ <u>then</u> $\mathrm{gtp}(\bar{a}_2, N_0, N_2)$ belongs to $\mathbf{D}_*(N_0)$ and is definable over $\bar{a}_0$.

*Remark* 5.12. 1) Note that if $\mathbf{D}$ is a weakly good $\mathfrak{k}$-diagram function, $N \in K_{\aleph_0}$, and $p \in \mathbf{D}(N)$ <u>then</u> we can find $(M, \bar{a})$ such that $N \leq_{\mathfrak{k}} M \in K_{\aleph_0}$, $\bar{a} \in {}^{\omega>}M$, $p = \mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{a}; N; M)$, and for every $\bar{b} \in {}^{\omega>}M$ the type $\mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{b}; N; M)$ belongs to $\mathbf{D}(N)$.

2) Moreover, if $\mathbf{D}$ is a good $\mathfrak{k}$-diagram function then we can demand above that $M$ is $(\mathbf{D}(N), \aleph_0)^*$-homogeneous (see Definition 5.15(1) below).

3) On 'very good' $\mathbf{D}$, see 5.13(2).

4) The $\mathbf{D}_\alpha$-s in 5.13 below are very good $\mathfrak{k}$-diagrams, and for us it suffices to then have the properties mentioned above, so we do not elaborate.

**Fact 5.13.** 1) for $\alpha < \omega_1$ there are $\mathbf{D}_\alpha, \mathbf{D}^*_\alpha$, functions with domain $K_{\aleph_0}$, such that:

(a) For $N \in K_{\aleph_0}$, $\mathbf{D}_\alpha(N)$ and $\mathbf{D}^*_\alpha(N)$ are countable subsets of $\mathbf{D}(N)$ and $\mathbf{D}^*(N)$, respectively.

(b) For each $N \in K_{\aleph_0}$, $\langle \mathbf{D}_\alpha(N) : \alpha < \omega_1 \rangle$ and $\langle \mathbf{D}^*_\alpha(N) : \alpha < \omega_1 \rangle$ are increasing continuous.

(c) $\mathbf{D}(N) = \bigcup_{\alpha<\omega_1} \mathbf{D}_\alpha(N)$ and $\mathbf{D}^*(N) = \bigcup_{\alpha<\omega_1} \mathbf{D}^*_\alpha(N)$.

(d) If $N_1, N_2 \in K_{\aleph_0}$, $f$ is an isomorphism from $N_1$ onto $N_2$ <u>then</u> $f$ maps $\mathbf{D}_\alpha(N_1)$ onto $\mathbf{D}_\alpha(N_2)$ and $\mathbf{D}^*_\alpha(N_1)$ onto $\mathbf{D}^*_\alpha(N_2)$ for $\alpha < \omega_1$.

(e) For every $\alpha < \omega_1$ and $N \in K_{\aleph_0}$, there is a $(\mathbf{D}_\alpha(N), \aleph_0)^*$-homogeneous model (see Definition 5.15(1) below; obviously, it is unique up to isomorphism over $N$).

(f) If $N_0 \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} N_2 \in K_{\aleph_0}$, $N_2$ is $(\mathbf{D}_\alpha(N_1), \aleph_0)^*$-homogeneous, and $N_1$ is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$-homogeneous[16] <u>then</u> $N_2$ is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$-homogeneous.

(f)$^+$ If $\langle \alpha_\varepsilon : \varepsilon \leq \zeta \rangle$ is an increasing continuous sequence of countable ordinals, $\langle N_\varepsilon : \varepsilon \leq \zeta \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous with $N_\varepsilon \in \mathfrak{k}_{\aleph_0}$,

$$\mathrm{gtp}(\bar{a}, N_\varepsilon, N_{\varepsilon+1}) \in \mathbf{D}_\alpha(N_\varepsilon)$$

for every $\bar{a} \in N_{\varepsilon+1}$, and for every $\xi < \zeta$, for some $\varepsilon \in [\xi, \zeta)$, $N_{\varepsilon+1}$ is $(\mathbf{D}_{\alpha_\varepsilon}(N_\varepsilon), \aleph_0)^*$-homogeneous <u>then</u> $N_\zeta$ is $(\mathbf{D}_{\alpha_\zeta}(N_0), \aleph_0)^*$-homogeneous.

(g) $N_1$ is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$-homogeneous <u>iff</u> $N_1$ is $(\mathbf{D}^*_\alpha(N_0), \aleph_0)^*$-homogeneous, where $N_0 \leq_{\mathfrak{k}} N_1 \in K_{\aleph_0}$.

(h) $\mathbf{D}_\alpha$ is a very good countable $\mathfrak{k}$-diagram function.

2) If $\mathbf{D}$ is very good then clauses (d),(e),(f),(f)$^+$ hold for it (and also (g), defining $\mathbf{D}^*$ as $f''(\mathbf{D})$, $f$ from 5.17(3)).

*Remark* 5.14. 1) We can add

(i) If $\mathfrak{k}, <^*$ are as derived from the $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ in the proof of 3.19(2), then we can add: if $N_0 \leq_{\mathfrak{k}} N_1 \in K_{\aleph_0}$ and every $p \in \mathbf{D}_0(N_0)$ is materialized in $N_1$, then $N_0 <^* N_1$.

2) So our results apply to $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ as well.

3) So it follows that if $\langle N_i : i \leq \alpha \rangle$ is $\leq_{\mathfrak{k}}$-increasing in $K_{\aleph_0}$, $N_{i+1}$ is $(\mathbf{D}_{\beta_i}(N_0), \aleph_0)^*$-homogeneous, and $\langle \beta_i : i < \alpha \rangle$ is non-decreasing with supremum $\beta$, then $N_\alpha$ is $(\mathbf{D}_\beta, \aleph_0)^*$-homogeneous.

4) So by 5.13(1)(h), each $\mathbf{D}_\alpha$ is very good and countable.

*Proof.* **[Proof of 5.13:]**

First, $\mathbf{D}$ is a $\mathfrak{k}$-diagram function by Definition 5.2 and 5.4(9). As $\mathbf{D}(N)$ has cardinality $\leq \aleph_1$ by 5.4(6) we can find a sequence $\langle \mathbf{D}_\alpha : \alpha < \omega_1 \rangle$ such that

⊛ (a) $\mathbf{D}_\alpha$ is a countable $\mathfrak{k}$-diagram function.

(b) For every $N \in K_{\aleph_0}$ the sequence $\langle \mathbf{D}_\alpha(N) : \alpha < \omega_1 \rangle$ is increasing continuous with union $\mathbf{D}(N)$.

[16] Or just $(\mathbf{D}_\beta(N_0), \aleph_0)^*$-homogeneous for some $\beta \leq \alpha$, <u>or</u> just

$$\bar{b} \in {}^{\omega>}(N_1) \Rightarrow \mathrm{gtp}_{\mathbb{L}_{\omega_1,\omega}(\tau^+)}(\bar{b}; N_0; N_1) \in \mathbf{D}(N_0).$$

Second, $\mathbf{D}$ is very good. (Clause (f) of 5.11 obviously holds, but to prove that it reflects to $\mathbf{D}_\alpha$ for a club of $\alpha < \omega_1$ we need 5.23 below. There is no vicious circle; the other way is easier.)

Third, note that for each of the demands (d),(e),(f) from Definition 5.11, for a club of $\delta < \omega_1$, $\mathbf{D}_\delta$ satisfies it. So without loss of generality each $\mathbf{D}_\alpha$ is very good.

The parts on $\mathbf{D}^*_\alpha$ follow by 5.9. See 5.17(1) below, which does not rely on 5.13–5.16 (and see proof of 5.19). $\square_{5.13}$

**Definition 5.15.** Assume $N_0 \leq_{\mathfrak{k}} N_1 \in K_{\aleph_0}$ and $\mathbf{D}_*$ is a $\mathfrak{k}$-diagram.

1) We say that $(N_1, N_0)$, or just $N_1$, is $(\mathbf{D}_*(N_0), \aleph_0)^*$-homogeneous over $N_0$ (but we may omit the "over $N_0$") if:

- (a) Every $\bar{a} \in N_1$ materializes some $p \in \mathbf{D}_*(N_0)$ in $(N_1, N_0)$ over $N_0$, and every $q \in \mathbf{D}_\alpha(N_0)$ is materialized in $(N_0, N_1)$ by some $\bar{b} \in N_1$.
- (b) If $\bar{a}, \bar{b} \in N_1$ materialize the same type over $N_0$ in $(N_1, N_0)$ and $c \in N_1$, <u>then</u> for some $d \in N_1$ the sequences $\bar{a}\hat{\ }\langle c\rangle$, $\bar{b}\hat{\ }\langle d\rangle$ materialize the same type from $\mathbf{D}_*(N_0)$ in $(N_1, N_0)$.

2) Similarly for $(\mathbf{D}^*_*(N_0), \aleph_0)^*$-homogeneity. Pedantically, we have to say $(N_1, N_0; N_0)$ is $(\mathbf{D}^*(N), \aleph_0)^*$-homogeneous, but normally we just say $N_1$ is.

*Remark* 5.16. 1) Now this is meaningful only for $N \leq_{\mathfrak{k}} M \in K_{\aleph_0}$, but later it becomes meaningful for any $N \leq_{\mathfrak{k}} M \in K$.

2) Uniqueness for such countable models hold in this context as well.

Now by 5.9:

**Conclusion 5.17.** *If $(N_1, N_0)$ is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$-homogeneous <u>then</u> $N_1$ (i.e. $(N_1, N_0, c)_{c \in N_0}$) is $(\mathbf{D}^*_\alpha(N_0), \aleph_0)^*$-homogeneous.*

*Proof.* This is easy by 5.9(1) and clause (g) of 5.13. $\square_{5.17}$

**Lemma 5.18.** *There is $N^* \in K_{\aleph_1}$ such that $N^* = \bigcup_{\alpha<\omega_1} N_\alpha$, $N_\alpha \in K_{\aleph_0}$ is $\leq_{\mathfrak{k}}$-increasing continuous with $\alpha$, and $N_{\alpha+1}$ is $(\mathbf{D}_{\alpha+1}(N_\alpha), \aleph_0)^*$-homogeneous for $\alpha < \omega_1$.*

*Proof.* Should be clear. $\square_{5.18}$

**Theorem 5.19.** *The $N^* \in K_{\aleph_1}$ from 5.18 is unique (not even depending on the choice of $\mathbf{D}_\alpha(N)$-s), is universal, and is $(\mathbb{D}(\mathfrak{k}), \aleph_1)$-model-homogeneous (hence model-homogeneous for $\mathfrak{k}$).*

*Proof.* <u>**Uniqueness:**</u> For $\ell = 0, 1$ and $\alpha < \omega_1$, let $N^\ell_\alpha, \mathbf{D}^\ell_\alpha$ be as in 5.13, 5.18, and we should prove $\bigcup_{\alpha<\omega_1} N^0_\alpha \cong \bigcup_{\alpha<\omega_1} N^1_\alpha$; because of 5.13(1)(g), it does not matter if we use the $\mathbf{D}$ or $\mathbf{D}^*$ version.

As $\mathbf{D}^\ell_\alpha$ is increasing and continuous for $\alpha < \omega_1$, $|\mathbf{D}^\ell_\alpha(N)| \le \aleph_0$,

$$\bigcup_{\alpha<\omega_1} \mathbf{D}^\ell_\alpha(N) = \mathbf{D}(N)$$

for every $N \in K_{\aleph_0}$, and the $\mathbf{D}^\ell_\alpha$-s commute with isomorphisms, clearly there is a closed unbounded $E \subseteq \omega_1$ consisting of limit ordinals such that

$$\alpha \in E \Rightarrow \mathbf{D}^0_\alpha = \mathbf{D}^1_\alpha.$$

Let $E := \{\alpha(i) : i < \omega_1\}$ with $\alpha(i)$ increasing and continuous. Now we define, by induction on $i < \omega_1$, an isomorphism $f_i$ from $N^0_{\alpha(i)}$ onto $N^1_{\alpha(i)}$ increasing with $i$. For $i = 0$ use the $\aleph_0$-categoricity of $K$, and for limit $i$ let $f_i := \bigcup_{j<i} f_j$.

Suppose $f_i$ is defined; then by 5.13(1)(d) the function $f_i$ maps $\mathbf{D}^0_{\alpha(i+1)}(N^0_{\alpha(i)})$ onto $\mathbf{D}^0_{\alpha(i+1)}(N^1_{\alpha(i)})$, and by the choice of $E$, $\mathbf{D}^0_{\alpha(i+1)} = \mathbf{D}^1_{\alpha(i+1)}$. By the assumption on the $N^\ell_\alpha$ and clause 5.13(1)(f)$^+$, $N^\ell_{\alpha(i+1)}$ is $(\mathbf{D}^\ell_{\alpha(i+1)}(N^\ell_{\alpha(i)}), \aleph_0)^*$-homogeneous. Summing up those facts and 5.13(e) we see that we can extend $f_i$ to an isomorphism $f_{i+1}$ from $N^0_{\alpha(i+1)}$ onto $N^1_{\alpha(i+1)}$.

Now $\bigcup_{i<\omega_1} f_i$ is the required isomorphism.

**Universality:** Let $M \in K_{\aleph_1}$, so $M = \bigcup_{\alpha<\omega_1} M_\alpha$ with $M_\alpha$ is $\le_{\mathfrak{k}}$-increasing continuous and $\|M_\alpha\| \le \aleph_0$. We now define $f_\alpha, N_\alpha, \gamma_\alpha$ by induction on $\alpha < \omega_1$ such that $\gamma_\alpha \in [\alpha, \omega_1)$ is increasing continuous with $\alpha$, $f_\alpha$ is a $\le_{\mathfrak{k}}$-embedding of $M_\alpha$ into $N_\alpha \in K_{\aleph_0}$, $N_\alpha$ is $\le_{\mathfrak{k}}$-increasing continuous, $f_\alpha$ is increasing and continuous, and $N_{\beta+1}$ is $(\mathbf{D}_{\gamma_{\beta+1}}(N_\beta), \aleph_0)^*$-homogeneous for $\beta < \alpha$.

For $\alpha = 0$ let $N_\alpha := M_\alpha$ and $f_\alpha := \mathrm{id}_{N_\alpha}$. For $\alpha$ limit use unions. For $\alpha$ successor, let $\alpha = \beta + 1$ and we use the $\aleph_0$-amalgamation property (which holds by 3.9,4.8). So there is a pair $(f_\alpha, N'_\alpha)$ such that $N_\beta \le_{\mathfrak{k}} N'_\alpha \in K_{\aleph_0}$ and $f_\alpha$ is a $\le_{\mathfrak{k}}$-embedding of $M_\alpha$ into $N'_\alpha$ extending $f_\beta$. The set

$$\left\{\mathrm{gtp}(\bar{a}, N_\beta, N'_\alpha) : \bar{a} \in {}^{\omega>}(N'_\alpha)\right\}$$

is a countable subset of $\mathbf{D}(N_\beta)$ hence is $\subseteq \mathbf{D}_{\gamma_\alpha}(N_\beta)$ for some $\gamma \in (\gamma_\beta, \omega_1)$. By 5.13(1)(c) there is $N_\alpha$ which $\le_{\mathfrak{k}}$-extends $N'_\alpha$ and is $(\mathbf{D}_{\gamma_\alpha}(N'_\alpha), \aleph_0)^*$-homogeneous; by 5.13(1)(f) we are done. So $f := \bigcup_{\alpha<\omega_1} f_\alpha$ embeds $M$ into $N := \bigcup_{\alpha<\omega_1} N_\alpha$, which is isomorphic to $N^*$ by the uniqueness. So the universality follows from the uniqueness.

**$(\mathbb{D}(\mathfrak{k}), \aleph_1)$-Model-homogeneity:** So let $\langle N_\alpha : \alpha < \omega_1\rangle$, $\mathbf{D}_\alpha, N^*$ be as in 5.13, 5.18, and we are given $(M_0, M_1, M_0^+, f)$ such that $M_0 \le_{\mathfrak{k}} M_0^+ \in K_{\aleph_0}$, $M_1 \le_{\mathfrak{k}} N^*$, and $f$ an isomorphism from $M_0$ onto $M_1$. For some $\gamma < \omega_1$ we have $M_1 \le_{\mathfrak{k}} N_\gamma$.

Now $\left\{\mathrm{gtp}(\bar{a}, M_0, M_0^+) : \bar{a} \in {}^{\omega>}(M_0^+)\right\}$ is a countable subset of $\mathbf{D}(M_0)$, hence $\subseteq \mathbf{D}_{\gamma_0}(M_0)$ for some $\gamma_0 < \omega_1$; also, $\{\mathrm{gtp}(\bar{a}, M_1, N_\gamma) : \bar{a} \in {}^{\omega>}(N_\gamma)\}$ is a countable subset of $\mathbf{D}(M_1)$ and hence $\subseteq \mathbf{D}_{\gamma_1}(M_1)$ for some $\gamma_1 < \omega_1$.

Let $\beta := \max\{\gamma, \gamma_0, \gamma_1\}$ and let $M_0^* \in K_{\aleph_0}$ be $(\mathbf{D}_\beta(M_0^+), \aleph_0)^*$-homogeneous, so $M_0^+ \le_{\mathfrak{k}} M_0^*$ exists by 5.13(1)(e), hence $M_0^* \in K_{\aleph_0}$ is $(\mathbf{D}_\beta(M_0), \aleph_0)^*$-homogeneous by 5.13(1)(f) because $\beta \ge \gamma_0$. Now $N_\beta$ is $(\mathbf{D}(N_\gamma), \aleph_0)^*$-homogeneous by 5.13(1), so as $\beta \ge \gamma_1$ is follows that $N_\beta$ is $(\mathbf{D}_\gamma(M_1), \aleph_0)^*$-homogeneous.

By 5.13(1)(d),(e) we can extend $f$ to an isomorphism $g$ from $M_0^*$ onto $N_\beta$, so $g \restriction M_0^+$ is a $\leq_{\mathfrak{k}}$-embedding of $M_0^+$ into $N$.

We can deduce "$N^*$ is a model-homogeneous" directly: let $M_0, M_1 \leq_{\mathfrak{k}} N^*$ be countable and $f$ is an isomorphism from $M_0$ onto $M_1$. Let $\gamma < \omega_1$ be such that $M_0, M_1 \leq_{\mathfrak{k}} N_\gamma$, let $\gamma_\ell$ be such that

$$\{\operatorname{gtp}(\bar{a}, M_\ell, N_\gamma) : \bar{a} \in {}^{\omega>}(N_\gamma)\} \subseteq \mathbf{D}_{\gamma_\ell}(M_\ell)$$

for $\ell = 0, 1$, and let $\beta := \max\{\gamma, \gamma_0, \gamma_1\} + 1$. As above, $N_\beta$ is $(\mathbf{D}_\beta(M_\ell), \aleph_0)^*$-homogeneous, and now we choose an automorphism $f_\alpha$ of $N_\alpha$ increasing with $\alpha \in [\beta, \omega_1)$ and extending $f$ by induction. Now $\bigcup\{f_\alpha : \alpha \in (\beta, \omega_1)\}$ is an automorphism of $N^*$ extending $f$. $\square_{5.19}$

**Definition 5.20.** 1) If $N_0 \leq_{\mathfrak{k}} N_1 \in K_{\aleph_0}$, $p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 0, 1$, and they are definable in the same way,[17] <u>then</u> we call $p_1$ the stationarization of $p_0$ over $N_1$.

2) For $\ell = 0, 1$, $N_0 \leq_{\mathfrak{k}} N_1$, and $p_\ell \in \mathbf{D}(N_\ell)$, let $p_1 \models p_0$ mean that if $N_1 \leq_{\mathfrak{k}} N_2 \in K_{\aleph_0}$ and $\bar{a} \in N_2$ materializes $p_1$, then it materializes $p_0$.

*Remark* 5.21. It is easy to justify the uniqueness implied by "<u>the</u> stationarization".

Observe

**Claim 5.22.** *If $p_\ell = \operatorname{gtp}(\bar{a}, N_\ell, N_2)$ for $\ell = 0, 1$ and $N_0 \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} N_2 \in K_{\aleph_0}$, <u>then</u> $p_1 \models p_0$.*

*Proof.* Easy. $\square_{5.22}$

**Claim 5.23.** 1) *Suppose $N_0 \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} N_2 \in K_{\aleph_0}$, $\bar{a}_\ell \in N_\ell$ for $\ell = 0, 1, 2$, $\bar{a}_0 \subseteq \bar{a}_1 \subseteq \bar{a}_2$ (i.e. the ranges increase), $\operatorname{gtp}(\bar{a}_1, N_0, N_1)$ is definable over $\bar{a}_0$, and $\operatorname{gtp}(\bar{a}_2, N_1, N_2)$ is definable over $\bar{a}_1$. <u>Then</u> $\operatorname{gtp}(\bar{a}_2, N_0, N_2)$ is definable over $\bar{a}_0$. Moreover, the definition depends only on the definitions mentioned previously.*

2) *If $N_0 \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} N_2$, $p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 0, 1, 2$, and $p_{\ell+1}$ is the stationarization of $p_\ell$ over $N_{\ell+1}$ for $\ell = 0, 1$, <u>then</u> $p_2$ is the stationarization of $p_0$ over $N_2$.*

*Proof.* 1) So we have to prove that $\operatorname{gtp}(\bar{a}_2, N_0, N_2)$ does not split over $\bar{a}_0$. Let $n < \omega$ and $\bar{b}, \bar{c} \in {}^n N_0$ realize the same type in $N_0$ over $\bar{a}_0$. (That is, in the logic $\mathbb{L}_{\omega_1,\omega}(\tau_{\mathfrak{k}})$, or even first-order logic when every $N \in K_{\aleph_0}$ is atomic.) Now $\bar{b}\hat{\ }\bar{a}_1, \bar{c}\hat{\ }\bar{a}_1$ also materialize the same $\mathbb{L}_{\omega_1,\omega}(N_0)$-type in $N_1$, hence they realize the same $\mathbb{L}_{\omega_1,\omega}(\tau_{\mathfrak{k}})$-type (recall 5.4(8)). Hence $\bar{b}, \bar{c}$ realize the same $\mathbb{L}_{\omega_1,\omega}(\tau_{\mathfrak{k}})$-type in $N_1$ over $\bar{a}_1$ in $N_1$. But $\operatorname{gtp}(\bar{a}_2, N_0, N_2)$ does not split over $\bar{a}_1$, so by the previous sentence we get that $\bar{b}\hat{\ }\bar{a}_2$ and $\bar{c}\hat{\ }\bar{a}_2$ materialize the same $\mathbb{L}_{\omega_1,\omega}(N_0)$-type in $N_2$.

2) Easy. The "moreover" is proved similarly. $\square_{5.23}$

**Lemma 5.24.** *Suppose $N_0 \leq_{\mathfrak{k}} N_1 \in K_{\aleph_0}$, $p_\ell \in \mathbf{D}(N_\ell)$, and $p_1$ is a stationarization of $p_0$ over $N_1$. <u>Then</u> $p_1 \models p_0$; i.e. every sequence materializing $p_1$ materializes $p_0$ in any $N_2$ such that $N_1 \leq_{\mathfrak{k}} N_2$.*

[17] See Definition 5.7 and 5.6; so in particular, they do not both split over the same finite subset of $N_0$.

*Remark* 5.25. 1) In [She75a], [She83a], [She83b], and [She90], the parallel proof of the claims were totally trivial, but here we need to invoke $\dot{I}(\aleph_1, K) < 2^{\aleph_1}$.

2) A particular case can be proved in the context of §4.

*Proof.* Suppose $N_0, N_1, p_0, p_1$ contradict the claim, and let $\bar{a}^* \in N_0$ be such that $p_0$ is definable over $\bar{a}^*$ (so $p_1$ is as well). By 5.13(e)+(f) there are $\delta < \omega_1$ and $N_2 \in K_{\aleph_0}$ satisfying $N_1 \leq_{\mathfrak{k}} N_2$ such that $N_2$ is $(\mathbf{D}^*_\delta(N_\ell), \aleph_0)^*$-homogeneous for $\ell = 0, 1$. We can find $p_2 \in \mathbf{D}(N_2)$ which is the stationarization of $p_0$ and $p_1$. It is enough to prove that $p_2 \models p_1$.

[Why? First, note that there is an automorphism $f$ of $N_2$ which maps $N_1$ onto $N_0$ and $f(\bar{a}^*) = \bar{a}^*$, hence $f(p_2) = p_2$ and $f(p_1) = p_0$, hence $p_2 \models p_0$. Now assume that $N_1 \leq_{\mathfrak{k}} N_1^+ \in K_{\aleph_0}$ and $\bar{a}_1 \in N_1^+$ materializes $p_1$. Clearly we can find $N_2^+$ and $\bar{a}_2$ such that $N_2 \leq_{\mathfrak{k}} N_2^+ \in K_{\aleph_0}$ and $\bar{a}_2 \in N_2^+$ materializes $p_2$. As we are assuming $p_2 \models p_1$ it also materializes $p_1$, hence there are $N_3, f$ such that $N_1^+ \leq_{\mathfrak{k}} N_3 \in K_{\aleph_0}$ and $f$ is a $\leq_{\mathfrak{k}}$-embedding of $N_2^+$ into $N_3$ over $N_1$ mapping $\bar{a}_2$ to $\bar{a}_1$. But $p_2 \models p_0$ (see above) hence $f(\bar{a}_2) = \bar{a}_1$ materializes $p_0$ and $p_1$ as well.]

So without loss of generality for some $\delta$,

⊛ $N_1$ is $(\mathbf{D}^*_\delta(N_0), \aleph_0)^*$-homogeneous over $N_0$.

For $N \in K_{\aleph_0}$ with $N_0 \leq_{\mathfrak{k}} N$, let $p_N$ be the stationarization of $p$ over $N$, so

$\boxtimes_1$ If $N_0 \leq_{\mathfrak{k}} N \in K_{\aleph_0}$ then $p_N$ is definable over $\bar{a}^*$.

Without loss of generality the universes of $N_0, N_1$ are $\omega$ and $\omega \times 2$, respectively.

Now we choose models $N_\alpha \in K_{\aleph_0}$ for $\alpha < \omega_1$, with $|N_\alpha| = \omega \times (1 + \alpha)$ and $\beta < \alpha \Rightarrow N_\beta \leq_{\mathfrak{k}} N_\alpha$. $N_0$ and $N_1$ are the ones mentioned in the claim, and $\bar{a}_\alpha \in N_{\alpha+1}$ materializes the stationarization $p_\alpha \in \mathbf{D}^*_\delta(N_\alpha)$ of $p_0$ over $N_\alpha$. For $\beta > \alpha$, $N_\beta$ is $(\mathbf{D}^*_\delta(N_\alpha), \aleph_0)$-homogeneous (see 5.13(f),(f)$^+$). Recalling that $\mathfrak{k}$ is categorical in $\aleph_0$ (and the uniqueness over $N_0$ of $(\mathbf{D}_\delta(N_0), \aleph_0)^*$-homogeneous models) we have

$$\alpha > \beta \Rightarrow (N_\alpha, N_\beta) \cong (N_1, N_0).$$

So recalling ⊛, clearly $\bar{a}_\alpha$ does not materialize $p_{N_\beta}$ (in $N_{\alpha+1}$).

Let $N := \bigcup_{\alpha<\omega_1} N_\alpha$. Let $\mathfrak{B}$ be $(\mathcal{H}(\aleph_2), \in)$ expanded by $N$, $K \cap \mathcal{H}(\aleph_2)$, $\leq_{\mathfrak{k}} \restriction \mathcal{H}(\aleph_2)$, and anything else which is necessary. Let $\mathfrak{B}^-$ be a countable elementary submodel of $\mathfrak{B}$ to which $\langle N_\alpha : \alpha < \omega_1 \rangle$ and $N$ belong, and let $\delta(*) := \mathfrak{B}^- \cap \omega_1$. For any stationary co-stationary $S \subseteq \omega_1$, let $\mathfrak{B}_S$ be a model satisfying the following.

- $\bullet_1$ $\mathfrak{B}_S$ an elementary extension of $\mathfrak{B}^-$.
- $\bullet_2$ $\mathfrak{B}_S$ is an end-extension of $\mathfrak{B}^-$ for $\omega_1$.
  (That is, if $\mathfrak{B}_S \models$ "$s < t$ are countable ordinals" and $t \in \mathfrak{B}^-$ then $s \in \mathfrak{B}^-$.)
- $\bullet_3$ Among the $\mathfrak{B}_S$-countable ordinals not in $\mathfrak{B}^-$, there is no first one.
- $\bullet_4$ "The set of countable ordinals" of $\mathfrak{B}_S$ is $I_S = \bigcup_{\alpha<\omega_1} I^S_\alpha$, even $I^S_0$ is not well ordered, each $I_\alpha$ a countable initial segment of $I_S$, and
$$\alpha < \beta \Rightarrow I^S_\alpha \subsetneq I^S_\beta.$$
- $\bullet_5$ $I_S \setminus I^S_\alpha$ has a first element if and only if $\alpha \in S$ (in which case we call it $s(\alpha)$).

In particular, $\omega$ and finite sets are standard in $\mathfrak{B}_S$. For $s \in I_S$, $N_s[\mathfrak{B}_s] := N_s^{\mathfrak{B}_S}$ is defined naturally, and so is $N^S = N^{\mathfrak{B}_S}$. Clearly $N_s^{\mathfrak{B}_S} \in K_{\aleph_0}$ is $\leq_{\mathfrak{k}}$-increasing with $s \in I$, as those definitions are $\Sigma^1_1$ (as $\mathfrak{k}$ is $\mathsf{PC}_{\aleph_0}$). Let $N^S_\alpha := \bigcup_{s \in I_\alpha} N_s^{\mathfrak{B}_S}$ and let $s+1$ be the successor of $s$ in $I_S$.

So

⊞ If $\mathfrak{B}_S \models$ "$s < t$ are countable ordinals" then $(N_t^{\mathfrak{B}_S}, N_s^{\mathfrak{B}_S})$ is $(\mathbf{D}^*_\delta(N_s^{\mathfrak{B}_S}), \aleph_0)^*$-homogeneous, and if $s \in I_\alpha$ then $N^S_\alpha$ is $(\mathbf{D}^*_\delta(N_1^{\mathfrak{B}_S}), \aleph_0)^*$-homogeneous.

If $\alpha \in S$ then clearly the type $p = p_{N^S_\alpha}$ satisfies the following (using absoluteness from $\mathfrak{B}_S$ because $N^S_\alpha$ is definable in $\mathfrak{B}_S$ as $N^{\mathfrak{B}_S}_{s(\alpha)}$).

(A) $p$ is materialized in $N^S$ (i.e. in $N^S_\beta$ for a club of $\beta \in S$).

But by the assumption toward contradiction

(B) For a closed unbounded $E \subseteq \omega_1$, for no $\beta \in E \cap S$ with $\beta > \alpha^*$ and $\gamma \in (\beta, \omega_1)$, does a sequence from $N^S$ materialize both $p = p_{N^S_\alpha}$ and its stationarization $p_{N^S_\beta}$ over $N^S_\beta$ in $N^S_\gamma$. (Again, remember $N^S_\alpha = N^{\mathfrak{B}_S}_{s(\alpha)}$ because $\alpha \in S$.)

and similarly

(C) For a closed unbounded set of $\beta > \alpha$, $N^S_\beta$ is $(\mathbf{D}^*_\delta(N^S_\alpha), \aleph_0)^*$-homogeneous.

We shall prove that every $\alpha < \omega_1$,

⊡ If $\alpha \notin S$ then $\alpha$ cannot satisfy the statement (C) above.

This is sufficient because if $S_1, S_2 \subseteq \omega_1$ are stationary and co-stationary and $f$ is an isomorphism from $N^{S_1}$ onto $N^{S_2}$ mapping $\bar{a}^*$ to itself, then for a closed unbounded set $E \subseteq \omega_1$, for each $\alpha < \omega_1$

**[This has to be 'for each $\alpha \in E$,' right? Otherwise nothing you wrote depends on $E$.]**

the function $f$ maps $N^{S_1}_\alpha$ onto $N^{S_2}_\alpha$, hence the property above is preserved, hence $S_1 \cap E = S_2 \cap E$. But there is a sequence $\langle S_i : i < 2^{\aleph_1}\rangle$ of subsets of $\omega_1$ such that for $i \neq j$ the set $S_i \setminus S_j$ is stationary. So by 0.4 we have $\dot{I}(\aleph_1, K) = 2^{\aleph_1}$, a contradiction.

So suppose $\alpha \in \omega_1 \setminus S$, $p = p_{N^S_\alpha}$, and clause (C) above holds. But obviously (C) $\Rightarrow$ (A), recalling $p_0 \in \mathbf{D}_\delta(N_0)$, hence $p_{N^S_\alpha} \in \mathbf{D}_\delta(N^S_\alpha)$. So let $\bar{a} \in N^S$ materialize $p$ in $N^S$ and we shall get a contradiction.

There are elements $0 = t(0) < t(1) < \ldots < t(k)$ of $I^S$ and $\bar{a}_0 \in N_0 = N^{\mathfrak{B}_S}_{t(0)}$, $\bar{a}_{\ell+1} \in N^{\mathfrak{B}_S}_{t(\ell)+1}$ such that $\bar{a} \subseteq \bar{a}_k$, $\bar{a}^* \subseteq \bar{a}_0$, $\bar{a}_\ell \subseteq \bar{a}_{\ell+1}$, and $\mathrm{gtp}(\bar{a}_{\ell+1}, N^{\mathfrak{B}_S}_{t(\ell)}, N^{\mathfrak{B}_S}_{t(\ell+1)})$ is definable over $\bar{a}_\ell$. Furthermore, if $t(\ell+1)$ is a successor (in $I_S$) then it is the successor of $t(\ell)$, and if limit in $I^S$ then $\bar{a}_\ell = \bar{a}_{\ell+1}$.

[Why do they exist? Because of the sentence saying that for every $\bar{a}$ we can find such $k, t(\ell)$, and $\bar{a}_\ell$ as above (for $\ell \leq k$) satisfied by $\mathfrak{B}$ and involve parameters which belong to $\mathfrak{B}^-$ hence to $\mathfrak{B}_S$, etc., so $\mathfrak{B}_S$ inherits it (and finiteness is absolute from $\mathfrak{B}_S$).]

It follows that $\mathrm{gtp}(\bar{a}, N^{\mathfrak{B}_S}_{t(\ell)}, N^{\mathfrak{B}_S}_{t(k)})$ is definable over $\bar{a}_\ell$ for each $\ell < k$.

Clearly $t(0) = 0 \in I_\alpha$ but $t(k) \notin I_\alpha$. (Otherwise $t(k)+1 \in I_\alpha$ hence $\bar a \in N^{\mathfrak{B}_S}_{t(k)+1} \le_{\mathfrak{k}} N^S_\alpha$, which is impossible as $p$ is a non-algebraic type over $N^{\mathfrak{B}_S}_\alpha$.) Hence for some $\ell$ we have $t(\ell) \in I_\alpha$ and $t(\ell+1) \notin I_\alpha$. By the construction $t(\ell+1)$ is limit (in $I^S$) hence $\bar a_{\ell+1} = \bar a_\ell$. As $\alpha \notin S$ we can choose $t(*) \in I_S \setminus I^S_\alpha$ with $t(*) < t(\ell+1)$. As we are assuming (toward contradiction) that $\alpha, p$ satisfy clause (C), for some $\beta \in S$, $s(\beta)$ is well defined, $s(\beta) > t(k)$, and[18] $N^S_\beta$ is $(\mathbf{D}^*_\delta(N^S_\alpha), \aleph_0)^*$-homogeneous. Now $N^{\mathfrak{B}_S}_{s(\beta)} = N^S_\beta$ and $N^{\mathfrak{B}_S}_{t(\ell+1)}$ are isomorphic over $N_{t(*)}$ (being both $(\mathbf{D}^*_\delta(N^{\mathfrak{B}_S}_{t(*)}), \aleph_0)^*$-homogeneous by the choice of $\mathfrak{B}_S$; see ⊞ above).

So as $N^S_\alpha \le_{\mathfrak{k}} N^{\mathfrak{B}_S}_{t(\ell+1)} \le_{\mathfrak{k}} N^{\mathfrak{B}_S}_{s(\beta)} = N^S_\beta$ and (as said above) $N^S_\beta$ is $(\mathbf{D}^*_\delta(N^S_\alpha), \aleph_0)^*$-homogeneous (also, $N^{\mathfrak{B}_S}_{t(\ell+1)}$ is $(\mathbf{D}^*_\delta(N^S_\alpha), \aleph_0)^*$-homogeneous as well),

$$(N^{\mathfrak{B}_S}_{t(\ell+1)}, N^S_\alpha, \bar a^*) \cong (N_1, N_0, \bar a^*).$$

As by ⊞ above, clearly $N^S_\alpha, N^{\mathfrak{B}_S}_{t(*)}$ are $(\mathbf{D}^*_\delta(N^{\mathfrak{B}_S}_{t(\ell)+1}), \aleph_0)^*$-homogeneous, there is an isomorphism $f_0$ from $N^S_\alpha$ onto $N^{\mathfrak{B}_S}_{t(*)}$ over $N^{\mathfrak{B}_S}_{t(\ell)+1}$. As $N^{\mathfrak{B}_S}_{t(\ell+1)}$ is $(\mathbf{D}^*_\delta(N^{\mathfrak{B}_S}_{t(*)}), \aleph_0)^*$-homogeneous and $(\mathbf{D}^*_\delta(N^S_\alpha), \aleph_0)^*$-homogeneous by the previous paragraph (where we use $\beta$) we can extend $f_0$ to an automorphism $f_1$ of $N^{\mathfrak{B}_S}_{t(\ell+1)}$. Let $\gamma \in S \cap E$ satisfy $s(\gamma) \ge t(k)+1$. As $\mathrm{gtp}(\bar a_k, N^{\mathfrak{B}_S}_{t(\ell+1)}, N^S_\gamma)$ is definable over $\bar a_\ell = \bar a_{\ell+1}$ and $\bar a_\ell = f_0(\bar a_\ell) = f_1(\bar a_\ell)$ (as $\bar a_\ell \in N^{\mathfrak{B}_S}_{t(\ell)+1}$) and $N^S_{\gamma+1}$ is $(\mathbf{D}^*_\delta(N^{\mathfrak{B}_S}_{t(\ell+1)}), \aleph_0)^*$-homogeneous, we can extend $f_1$ to an automorphism $f_2$ of $N^S_\gamma$ satisfying $f_2(\bar a_k) = \bar a_k$.

Notice that by the choice of $\langle \bar a_\ell : \ell \le k\rangle$ and $\langle t(\ell) : \ell \le k\rangle$, it follows that $\mathrm{gtp}(\bar a_k, N_{t(m)}, N_{t(k)+1})$ does not split over $\bar a_m$ for any $m < k$, Hence is definable over [$\bar a_m$?] by 5.23, and recall that we know that $\bar a_\ell = \bar a_{\ell+1}$.

So there is in $N^S$ a sequence materializing both $\mathrm{gtp}(\bar a, N^S_\alpha, N^S_\gamma) = p_{N^S_\alpha}$ and its stationarization over $N^S_{t(\ell+1)}$: just $\bar a$ $(\subseteq \bar a_k)$ (so use $f_2$).

This contradicts the assumption as $(N_1, N_0, \bar a^*) \cong (N^{\mathfrak{B}_S}_{t(\ell+1)}, N^S_\alpha, \bar a^*)$. $\square_{5.24}$

Clauses (5)-(9) of the following claim are closely related to Definition 5.27.

**Claim 5.26.** 1) *If $\bar a \in N_0 \le_{\mathfrak{k}} N_1 \le_{\mathfrak{k}} N_2 \in K_{\aleph_0}$, $\bar b \in N_2$, and $p_1 = \mathrm{gtp}(\bar b, N_1, N_2)$ is definable over $\bar a \in N_0$, then $p_0 = \mathrm{gtp}(\bar b, N_0, N_2)$ is definable in the same way over $\bar a$, hence $\mathrm{gtp}(\bar b, N_1, N_2)$ is its stationarization.*

2) *For a fixed countable $M \in K_{\aleph_0}$, to have a common stationarization in $\mathbf{D}(N')$ for some $N'$ satisfying $M \le_{\mathfrak{k}} N'$ or $N' \le_{\mathfrak{k}} M$ is an equivalence relation on the set $\bigcup_{N \le_{\mathfrak{k}} M} \mathbf{D}(N)$ (and we can choose the common stationarization in $\mathbf{D}(M)$ as a representative). So if $N_0 \le_{\mathfrak{k}} N_1 \le_{\mathfrak{k}} N_2 \in K_{\aleph_0}$, $p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 0,1,2$, and $p_1, p_2$ are stationarizations of $p_0$ then $p_2 \models p_1$.*

3) *If $N_\alpha \in K_{\aleph_0}$ is $\le_{\mathfrak{k}}$-increasing and continuous (for $\alpha \le \omega+1$) and $\bar a \in N_{\omega+1}$ then for some $n < \omega$, for every $k$, if $n < k \le \alpha \le \omega$ then $\mathrm{gtp}(\bar a, N_\alpha, N_{\omega+1})$ is the stationarization of $\mathrm{gtp}(\bar a, N_k, N_{\omega+1})$.*

4) *If $N \le_{\mathfrak{k}} M \in K$, $N \in K_{\aleph_0}$ and $\bar a \in M$, then $\mathrm{gtp}(\bar a, N, M')$ is constant for all $M' \in K_{\aleph_0}$ satisfying $\bar a \in M'$ and $N \le_{\mathfrak{k}} M' \le_{\mathfrak{k}} M$. We will call it $\mathrm{gtp}(\bar a, N, M)$.*

[18] On the definition of $s(\gamma)$ for $\gamma \in S$, see $\bullet_5$ above.

*(The new point is that $M$ is not necessarily countable. This is compatible with Definition 5.27(c) being a special case.)*

5) *Suppose $N_0 \leq_{\mathfrak{k}} N_1$ (in $K$) and $\bar{a} \in N_1$. Then there is a countable $M \leq_{\mathfrak{k}} N_0$ such that for every countable $M'$ satisfying $M \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} N_0$, we have that $\mathrm{gtp}(\bar{a}, M', N_1)$ is the stationarization of $\mathrm{gtp}(\bar{a}, M, N_1)$. Moreover, there is a finite $A \subseteq N_0$ such that any countable $M \leq_{\mathfrak{k}} N_0$ which includes $A$ is okay. So $\mathrm{gtp}(\bar{a}, N_0, N_1)$ from 5.27(c) is well-defined, a member of $\mathbf{D}(N_0)$, and is definable over some finite $A \subseteq N_0$.*

6) *The parallel of part (3) holds for $N_\alpha \in K$ as well, and for any limit ordinal instead of $\omega$. That is, if $\langle N_\alpha : \alpha \leq \delta + 1\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous and $\bar{a} \in N_{\delta+1}$, then for some $\alpha < \delta$ and countable $M \leq_{\mathfrak{k}} N_\alpha$, we have*

$$M \leq_{\mathfrak{k}} M' \leq_{\mathfrak{k}} M_\delta \Rightarrow \mathrm{gtp}(\bar{a}, M', M_\delta) \text{ is the stationarization of } \mathrm{gtp}(\bar{a}, M, M_\delta).$$

*Similarly for every $p \in \mathbf{D}(N_\delta)$.*

7) *If $N_0 \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} N_2 \leq_{\mathfrak{k}} N_3 \leq_{\mathfrak{k}} N_4$, $\bar{a} \in N_4$, and $\mathrm{gtp}(\bar{a}, N_3, N_4)$ is the stationarization of $\mathrm{gtp}(\bar{a}, N_0, N_4)$, then $\mathrm{gtp}(\bar{a}, N_2, N_4)$ is the stationarization of $\mathrm{gtp}(\bar{a}, N_1, N_3)$. Also, if $\bar{b}$ satisfies $\mathrm{rang}(\bar{b}) \subseteq \mathrm{rang}(\bar{a})$ and $\mathrm{gtp}(\bar{a}, N_2, N_4)$ is the stationarization of $\mathrm{gtp}(\bar{a}, N_1, N_4)$, then this holds also for $\bar{b}$. We can replace $\mathrm{gtp}(\bar{a}, N_3, N_4)$ by $p \in \mathbf{D}(N_4)$.*

8) *If $N_0 \leq_{\mathfrak{k}} N_1 \leq_{\mathfrak{k}} N_2 \in K_{\aleph_0}$, $p_\ell \in \mathbf{D}(N_\ell)$ for $\ell = 0, 1, 2$, and $p_{\ell+1}$ is the stationarization of $p_\ell$ for $\ell = 0, 1$ then $p_2$ is the stationarization of $p_0$.*

9) *If $\langle M_\alpha : \alpha \leq \delta + 1\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous, $\delta$ a limit ordinal, and $\bar{a} \in {}^{\omega>}(M_{\delta+1})$ then*

- (a) *For some $\alpha < \delta$, for all $\beta \in [\alpha, \delta)$, we have $\mathrm{gtp}(\bar{a}, M_\beta, M_{\delta+1})$ is the stationarization of $\mathrm{gtp}(\bar{a}, M_\alpha, M_{\delta+1})$.*
- (b) *If $\mathrm{gtp}(\bar{a}, M_\alpha, M_{\delta+1})$ is the stationarization of $\mathrm{gtp}(\bar{a}, M_0, M_{\delta+1})$ for every $\alpha < \delta$ then this holds for $\alpha = \delta$ as well.*

10) *If $\langle M_\alpha : \alpha \leq \delta\rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous, $\delta$ a limit ordinal and $p_\delta \in \mathbf{D}(M_\delta)$, then for some $\alpha < \beta$ there is $p_\alpha \in \mathbf{D}(M_\alpha)$ such that $p_\delta$ is the stationarization of $p_\alpha$.*

11) *Those definitions in 5.27 are compatible with the ones for countable models.*

12) *$\mathrm{gtp}(\bar{a}, N, M)$ (where $\bar{a} \in M$ and $N \leq_{\mathfrak{k}} M$ are both in $K$) is the stationarization over $N$ of $\mathrm{gtp}(\bar{a}, N', M)$ for every large enough countable $N' \leq_{\mathfrak{k}} N$ (see 5.26(5)).*

*Proof.* 1) As we can replace $N_2$ by any $N_2'$ satisfying $N_2 \leq_{\mathfrak{k}} N_2' \in K_{\aleph_0}$, without loss of generality, $N_2$ is $(\mathbf{D}^*_\alpha(N_0), \aleph_0)^*$-homogeneous and $(\mathbf{D}^*_\alpha(N_1), \aleph_0)^*$-homogeneous for some $\alpha$. Let $p_2 \in \mathbf{D}(N_2)$ be the stationarization of $p_1$ over $N_2$.

So by 5.24 we get $p_2 \models p_1$. On the other hand, clearly there is an isomorphism $f_0$ from $N_0$ onto $N_1$ such that $f_0(\bar{a}) = \bar{a}$; and by the assumption above on $N_2$, $f_0$ can be extended to an automorphism $f_1$ of $N_2$.

Note that $f_1$ maps $p_0 = \mathrm{gtp}(\bar{b}, N_0, N_2)$ to $p_0' := \mathrm{gtp}(f_1(\bar{b}), f_1(N_0), N_2)$, and maps $p_2$ to itself as $f_0(\bar{a}) = \bar{a}$.

Now $p_1 \models p_0$ (by the choices of $p_1$ and $p_0$) and $p_2 \models p_1$ by 5.9(1), so together $p_2 \models p_0$. As $f_1(p_2) = p_2$ and $f_1(p_0) = p_0'$, it follows that $p_2 \models p_0'$. As also $p_2 \models p_1$ and $p_0', p_1 \in \mathbf{D}(N_1)$, it follows that $p_0' = p_1$ hence $p_1, p_0'$ have the same definition over $\bar{a}$. But now also $p_0 \in \mathbf{D}(N_0)$ and $p'_0 \in \mathbf{D}(N_1)$ have the same definition over $\bar{a}$ (using $f_1$); together, $p_1, p_0$ have the same definition over $\bar{a}$, which means that $p_1$ is the stationarization of $p_0$ over $N_1$ and we are done.

2) Trivial.

3) By part (1).

4) Easy.

5) By (3) and (4).

6)-12) Easy by now. $\square_{5.26}$

**Definition 5.27.** By 5.26(5) the type $\mathrm{gtp}(\bar{a}, M, N)$ can be reasonably defined when $M \le_{\mathfrak{k}} N$ and $\bar{a} \in {}^{\omega>}N$, and we can define $\mathbf{D}(N)$, $\mathbf{D}_*(N)$, $\mathrm{gtp}(\bar{a}, N, M)$ and stationarization for not necessarily countable $N$ with $N \le_{\mathfrak{k}} M \in K$. Everything still holds, except that maybe some $p$-s are not materialized in any $\le_{\mathfrak{k}}$-extension of $N$.

More formally,

(a) If $N \le_{\mathfrak{k}} M$, $N \in K_{\aleph_0}$, and $p \in \mathbf{D}(N)$ then the stationarization of $p$ over $M$ is

$$\bigcup \big\{q : N_1 \in K_{\aleph_0},\ N \le_{\mathfrak{k}} N_1 \le_{\mathfrak{k}} M \text{ and } q \text{ is the stationarization of } p \in \mathbf{D}(N_1)\big\}.$$

(b) If $M \in \mathfrak{k}$ then

$$\mathbf{D}(M) = \{q : \text{for some countable } N \le_{\mathfrak{k}} M \text{ and } p \in \mathbf{D}(N), \text{ the type } q \text{ is the stationarization of } p \text{ over } M\}.$$

Similarly for $\mathbf{D}_*$ a $\mathfrak{k}$-diagram.

(c) If $N \le_{\mathfrak{k}} M$ and $\bar{a} \in {}^{\omega>}M$ then $\mathrm{gtp}(\bar{a}, N, M)$ is defined as

$$\bigcup \big\{\mathrm{gtp}(\bar{a}, N', M') : N_0 \le_{\mathfrak{k}} N' \le_{\mathfrak{k}} M' \in K_{\aleph_0},\ M' \le_{\mathfrak{k}} M,\ N' \le_{\mathfrak{k}} N\big\}$$

for every countable $N_0 \le_{\mathfrak{k}} N$ large enough; it is well defined and belongs to $\mathbf{D}(N)$ by 5.26(5), and we say '$\bar{a}$ materializes $\mathrm{gtp}(\bar{a}, N, M)$ in $M$.'

(d) If $N \in \mathfrak{k}$, $N \le_{\mathfrak{k}} M$, and $p \in \mathbf{D}(N)$ is definable over the countable $N_0 \le_{\mathfrak{k}} N$ (equivalently, it is the stationarization of some $p' \in \mathbf{D}(N_0)$), then the stationarization of $p$ over $M$ is the stationarization of $p'$ over $M$ (see clause (a)). Equivalently,

$$\bigcup \big\{p_{M_0} : N_0 \le_{\mathfrak{k}} M_0 \le_{\mathfrak{k}} M,\ M_0 \text{ is countable}\big\}$$

**[What about it?]**

where $p_{M_0}$ is the stationarization of $p' \in \mathbf{D}(N_0)$ over $M_0$; it belongs to $\mathbf{D}(N_0)$.

(e) If $p(\bar{x}, \bar{y}) \in \mathbf{D}(M)$ then $p(\bar{x}, \bar{y}) \restriction \bar{x} \in \mathbf{D}(M)$ is naturally defined [as in] 5.2(3); similarly for permuting the variables.

(f) For $N \le_{\mathfrak{k}} M$, we say that $M$ is $(\mathbf{D}(N), \aleph_0)^*$-homogeneous when for every $p(\bar{x}, \bar{y}) \in \mathbf{D}(N)$ and $\bar{a} \in {}^{\ell g(\bar{x})}M$ materializing $p(\bar{x}, \bar{y}) \restriction x$ in $M$, there is $\bar{b} \in {}^{\ell g(\bar{y})}M$ such that $\bar{a}\hat{\ }\bar{b}$ materializes $p(\bar{x}, \bar{y})$ in $M$.

*Remark* 5.28. Claim 5.29 below strengthens 3.9; it is a step toward non-forking amalgamation.

**Claim 5.29.** *Suppose $N_0 \leq_{\mathfrak{k}} N_1 \in K_{\aleph_0}$, $N_0 \leq_{\mathfrak{k}} N_2 \in K_{\aleph_0}$, and $\bar{a} \in N_1$. Then we can find $M$ with $N_0 \leq_{\mathfrak{k}} M \in K_{\aleph_0}$ and $\leq_{\mathfrak{k}}$-embeddings $f_\ell$ of $N_\ell$ into $M$ over $N_0$ (for $\ell = 1, 2$) such that $\text{gtp}(f_1(\bar{a}), f_2(N_2), M)$ is a stationarization of $p_0 := \text{gtp}(\bar{a}, N_0, N_1)$ (so $f_1(\bar{a}) \notin f_2(N_2)$).*

*Proof.* Let $p_2 \in \mathbf{D}(N_2)$ be the stationarization of $p_0$. Clearly we can find an $\alpha < \omega_1$ (in fact, a closed unbounded set of $\alpha$-s), some $N'_1, N'_2$ from $K_{\aleph_0}$ which are $(D^*_\alpha(N_0), \aleph_0)^*$-homogeneous and $N_\ell \leq_{\mathfrak{k}} N'_\ell$ (for $\ell = 1, 2$), and some $\bar{b} \in N'_2$ materializing $p_2$. But by 5.24, $\bar{b}$ materializes $p_0$ hence there is an isomorphism $f$ from $N'_1$ onto $N'_2$ over $N_0$ satisfying $f(\bar{a}) = \bar{b}$, recalling 5.9(1A). Now let $M := N'_2$, $f_1 := f \restriction N_1$, $f_2 := \text{id}$. $\square_{5.29}$

**Claim 5.30.** 1) *For any $N_0 \leq_{\mathfrak{k}} N_1 \in K_{\aleph_1}$ so $N_0 \in K_{\leq \aleph_1}$, there is $N_2$ such that $N_1 \leq_{\mathfrak{k}} N_2 \in K_{\aleph_1}$ and $N_2$ is $(\mathbf{D}(N_0), \aleph_0)^*$-homogeneous.*

2) *Also, 5.29 holds for $N_2 \in K_{\aleph_1}$ (but still with $N_0, N_1 \in K_{\aleph_0}$).*

3) *If $N_0 \leq_{\mathfrak{k}} N_1 \in K_{\aleph_0}$ and $N_0 \leq_{\mathfrak{k}} N_2 \in K_{\leq \aleph_1}$, then we can find $M \in K_{\leq \aleph_1}$ and $\leq_{\mathfrak{k}}$-embeddings $f_1, f_2$ of $N_1$ and $N_2$ into $M$ over $N_0$, respectively, such that $\text{gtp}(f_1(\bar{c}), f_2(N_2), M)$ is a stationarization of $\text{gtp}(\bar{c}, N_0, N_1)$ for every $\bar{c} \in N_1$, hence $f_1(N_1) \cap f_2(N_2) = N_0$.*

4) *$K_{\aleph_2} \neq \emptyset$.*

*Remark* 5.31. 1) Note that 5.30(3) is another step toward stable amalgamation.

2) Note that 5.30(3) strengthens 5.30(2), and hence 5.29.

*Proof.* 1) As we can iterate $\leq_{\mathfrak{k}}$-increasing $N_1$ in $K_{\aleph_1}$, it is enough to prove that if $p(\bar{x}, \bar{y}) \in \mathbf{D}(N_0)$ and $\bar{a} \in N_1$ materializes $p(\bar{x}, \bar{y}) \restriction \bar{x}$ in $(N_1, N_0)$, then for some $N_2 \in K_{\aleph_1}$ with $N_1 \leq_{\mathfrak{k}} N_2$ and $\bar{b} \in N_2$, the sequence $\bar{a}\hat{\ }\bar{b}$ materializes $p(\bar{x}, \bar{y})$ in $(N_2, N_0)$. Let $M_0 \leq_{\mathfrak{k}} N_0$ be countable and $q \in \mathbf{D}(M_0)$ be such that $p(\bar{x}, \bar{y})$ a stationarization of $q$. Without loss of generality if $N_0$ is countable then $M_0 = N_0$. (Note that the case $N_0 = M_0$ is easier.)

Choose $M_i$ $(0 < i < \omega_1)$ such that $M_i \leq_{\mathfrak{k}} N_1$, $N_1 = \bigcup_{i<\omega_1} M_i$, $\langle M_i : i < \omega_1 \rangle$ is a $\leq_{\mathfrak{k}}$-increasing continuous sequence of countable models, and $M_0 \cup \bar{a} \subseteq M_1$. As $\langle M_i \cap N_0 : i < \omega_1 \rangle$ is an increasing continuous sequence of countable sets with union $N_0$, clearly for a club of $i < \omega_1$, $M_i \cap N_0 \leq_{\mathfrak{k}} N_0$ hence $M_i \cap N_0 \leq_{\mathfrak{k}} M_i$. So without loss of generality

$$i < \omega_1 \Rightarrow M_i \cap N_0 \leq_{\mathfrak{k}} N_0, M_i.$$

For every $\bar{c} \in N_1$ there is a countable $N_{0,\bar{c}}$ such that $M_0 \leq_{\mathfrak{k}} N_{0,\bar{c}} \leq_{\mathfrak{k}} N_0$ and if $N_{0,\bar{c}} \leq_{\mathfrak{k}} N' \leq_{\mathfrak{k}} N_0$ and $N' \in K_{\aleph_0}$ then $\text{gtp}(\bar{c}, N', N_1)$ is the stationarization of $\text{gtp}(\bar{c}, N_{0,\bar{c}}, N_1)$. Without loss of generality $\bar{c} \in M_i \Rightarrow N_{0,\bar{c}} \subseteq M_i$, hence

(∗) For every $\bar{c} \in M_i$, $\text{gtp}(\bar{c}, N_0, N_1)$ is a stationarization of $\text{gtp}(\bar{c}, N_0 \cap M_i, M_i)$.

We can find $M^*_1 \in K_{\aleph_0}$ satisfying $M_1 \leq_{\mathfrak{k}} M^*_1$ and $\bar{b} \in M^*_1$ such that $q = \text{gtp}(\bar{a}\hat{\ }\bar{b}, M_0, M^*_1)$. We can find $\bar{a}_2, \bar{a}_1, \bar{a}_0$ such that $\bar{a}_0 \in M_1 \cap N_0$, $\bar{a}_1 \in M_1$, $\bar{a}_2 \in M^*_1$, $\bar{b} \subseteq \bar{a}_2$, $\bar{a} \subseteq \bar{a}_1$, $\bar{a}_0 \trianglelefteq \bar{a}_1 \trianglelefteq \bar{a}_2$, and $\text{gtp}(\bar{a}_2, M_1, M^*_1)$ and $\text{gtp}(\bar{a}_1, M_1 \cap N_0, M_1)$ are definable over $\bar{a}_1$ and $\bar{a}_0$, respectively. Now we define $f_j, M^*_j$ by induction on $j < \omega_1$ such that:

(*i*) $\langle M^*_i : 1 \leq i \leq j \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous.

(*ii*) $M_j^*$ is countable ($M_1^*$ is already given).

(*iii*) $f_j$ is a $\leq_{\mathfrak{k}}$-embedding of $M_j$ into $M_j^*$.

(*iv*) $f_1$ is the identity on $M_1$.

(*v*) $f_j$ is increasing continuous with $j$.

(*vi*) $\text{gtp}(\bar{a}_2, f_j(M_j), M_j^*)$ is the stationarization of $\text{gtp}(\bar{a}_2, M_1, M_1^*)$ (so definable over $\bar{a}_1$).

For $j = 1$ we have it letting $f_j^* = \text{id}_{M_1}$.

For $j > 1$ successor, use 5.29 to define $(M_j, f_j)$ such that $\text{gtp}(\bar{a}_2, f_j(M_j), M_j^*)$ is the stationarization of $\text{gtp}(\bar{a}_2, f_{j-1}(M_{j-1}), M_{j-1}^*)$. So clauses (*i*)-(*v*) clearly hold. Clause (*vi*) follows by 5.26(8).

For $j$ limit: let $M_j^* := \bigcup_{1 \leq i < j} M_i^*$ and $f_j := \bigcup_{1 \leq i < j} f_i$. Condition (*vi*) holds by 5.26(3).

By renaming, without loss of generality $f_j = \text{id}_{M_j}$ for $j \in [1, \omega_1)$.

By $(*)$ we get that $\text{gtp}(\bar{a}_1, N_0 \cap M_j, M_j^*) = \text{gtp}(\bar{a}_1, N_0 \cap M_j, M_j)$ is definable over $\bar{a}_0$ (as this holds for $j = 1$). Combining this and clause (*vi*), by 5.23(1) we get that for every $j \geq 1$, $\text{gtp}(\bar{a}_2, N_0 \cap M_j, M_j^*)$ is the stationarization of $\text{gtp}(\bar{a}_2, N_0 \cap M_1, M_1^*)$. Hence by the choice of $\bar{a}_2, \bar{a}_1, a_0$ and 5.26(7), easily $\text{gtp}(\bar{a}\hat{\ }\bar{b}, N_0 \cap M_j, M_j^*)$ is the stationarization of $\text{gtp}(\bar{a}\hat{\ }\bar{b}, N_0 \cap M_1, M_1^*)$ hence of $\text{gtp}(\bar{a}\hat{\ }\bar{b}, M_0, M_1^*)$.

Let $N_2 := \bigcup_{j \in [1, \omega_1)} M_j^*$. Clearly $N_1 \leq_{\mathfrak{k}} N_2 \in K_{\aleph_1}$.

So by 5.26(9), clause (c), and the first sentence in the proof, we finish.

2) Similar proof[19] (or use the proof of part (3)).

3) Without loss of generality $N_2 \cong N^*$ from 5.18 (as we can replace $N_2$ by an extension — so use 5.19 and 5.26(7)).

Also (by 5.30(1)) there is $M$ with $N_2 \leq_{\mathfrak{k}} M \in K_{\aleph_1}$ such that $M$ is $(\mathbf{D}(N_2), \aleph_0)^*$-homogeneous. As $N_1$ is countable, there is $\alpha < \omega_1$ such that for every $\bar{c} \in N_1$, $\text{gtp}(\bar{c}, N_0, N_1) \in \mathbf{D}_\alpha(N_0)$. Let $M = \bigcup_{i < \omega_1} M_i$ with $M_i \in K_{\aleph_0}$ being $\leq_{\mathfrak{k}}$-increasing continuous. So for some $i \in (\alpha, \omega_1)$ we have $M_i \cap N_2 \leq_{\mathfrak{k}} M$ and (recalling 5.26(6)) for every $\bar{c} \in M_i$, $\text{gtp}(\bar{c}, N_2, M)$ is stationarization of $\text{gtp}(\bar{c}, N_2 \cap M_i, M_i)$ and $M_i$ is $(\mathbf{D}_i(N_2 \cap M_i), \aleph_0)^*$-homogeneous. Now we can find an isomorphism $f_0$ from $N_0$ onto $N_2 \cap M_i$ (as $K$ is $\aleph_0$-categorical) and extend it to an automorphism $f_2$ of $N_2$ (by 5.19-model homogeneity). Also, there is $N_1'$ such that $N_1 \leq_{\mathfrak{k}} N_1' \in K_{\aleph_0}$ and $N_1'$ is $(\mathbf{D}_i(N_1), \aleph_0)^*$-homogeneous, hence is $(\mathbf{D}_i(N_0), \aleph_0)^*$-homogeneous (by the choice of $\alpha$, as $\alpha < i$; see 5.13(f)). Hence there is an isomorphism $f_1'$ from $N_1'$ onto $M_i$ extending $f_0$. Now $f_0, f_1' \restriction N_1, f_2, M$ show that amalgamation as required exists (we just change names).

4) Immediate; use (1) or (2) or (3) $\omega_2$-many times. $\square_{5.30}$

**Definition 5.32.** For any $\mathbf{D}_* = \mathbf{D}_\alpha$ for some $\alpha < \omega_1$ (or just any very good $\mathfrak{k}$-diagram $\mathbf{D}_*$; i.e. satisfies the demands on each $\mathbf{D}_\alpha$ in 5.13 — see 5.11) we define:

[19] here $N_1 \in K_{\aleph_1}$ is okay; similar to 2.12(1)

1) $M \leq_{\mathbf{D}_*} N$ if $M \leq_{\mathfrak{k}} N$ and for every $\bar{a} \in N$,

$$\text{gtp}(\bar{a}, M, N) \in \mathbf{D}_*(M).$$

2) $K_{\mathbf{D}_*}$ is the class of $M \in K$ which are the union of a family of countable submodels which is directed by $\leq_{\mathbf{D}_*}$.

3) $\mathfrak{k}_{\mathbf{D}_*} = (K_{\mathbf{D}_*} \leq_{\mathbf{D}_*})$, or pedantically $(K_{\mathbf{D}_*}, \leq_{\mathbf{D}_*} \restriction K_{\mathbf{D}_*})$.

**Claim 5.33.** *Let $\mathbf{D}_*$ be countable and as in 5.32.*

1) *The pair $(K_{\mathbf{D}_*}, \leq_{\mathbf{D}_*})$ is an $\aleph_0$-presentable AEC; that is, it satisfies all the axioms from 1.2(1) and is $\mathsf{PC}_{\aleph_0}$.*

2) *Also for $(K_{\mathbf{D}_*}, \leq_{\mathbf{D}_*})$, we get $\mathbf{D}(N)$ countable and equal to $\mathbf{D}_*(N)$ for every countable $N \in K_{\mathbf{D}_*}$.*

*Proof.* 1) Obviously $K_{\mathbf{D}_*}$ is a class of $\tau$-models and $\leq_{\mathbf{D}_*}$ is a two-place relation on $K_{D_*}$; also they are preserved by isomorphisms. About being $\mathsf{PC}_{\aleph_0}$, note that

- $\circledast_1$ $M \in K_{\mathbf{D}_*}$ iff $M \in K$ and for some model $\mathfrak{B}$ with universe $|M|$ and countable vocabulary, for every countable $\mathfrak{B}_1 \subseteq \mathfrak{B}_2 \subseteq \mathfrak{B}$ we have

  $$M \restriction \mathfrak{B}_1 \leq_{\mathbf{D}_*} M \restriction \mathfrak{B}_2$$

  iff there is a directed partial order and $\langle M_t : t \in I \rangle$ such that $M_t \in K_{\aleph_0}$ and $s <_I t \Rightarrow M_s \leq_{\mathfrak{k}} M_t$ and $\bar{a} \subseteq M_t \Rightarrow \text{gtp}(\bar{a}, M_s, M_t) \in \mathbf{D}_*(M_s)$.
  **[You have two 'iff's here. Should I read this as $A \Leftrightarrow B \Leftrightarrow C$ or $A \Leftrightarrow (B \Leftrightarrow C)$?]**
- $\circledast_2$ similarly for $M \leq_{\mathbf{D}_*} N$.

**<u>Ax.I</u>:** If $M \leq_{\mathbf{D}_*} N$ then $M \leq_{\mathfrak{k}} N$ hence $M \subseteq N$.

**<u>Ax.II</u>:** The transitivity of $\leq_{\mathbf{D}_*}$ holds by 5.11(4), 5.23(1), and Definition 5.27 (this works as $\mathbf{D}_*$ is closed enough, or use clause (f) of 5.13). The demand $M \leq_{\mathbf{D}_*} M$ is trivial.[20]

**<u>Ax.III</u>:** Assume $\langle M_i : i < \lambda \rangle$ is $\leq_{\mathbf{D}_*}$-increasing continuous and $M = \bigcup_{i<\lambda} M_i$. As $\mathfrak{k}$ is an AEC, clearly $M \in K$ and $i < \lambda \Rightarrow M_i \leq_{\mathfrak{k}} M$. Also, for each $i < \lambda$ and $\bar{a} \in M$, for some $j \in (i, \lambda)$, we have $\bar{a} \in M_j$ hence $\text{gtp}(\bar{a}, M_i, M_j) \in \mathbf{D}_*(M_i)$. But recalling 5.26(7), it follows that $\text{gtp}(\bar{a}, M_i, M) = \text{gtp}(\bar{a}, M_i, M_j) \in \mathbf{D}_*(M_i)$. So $i < \lambda \Rightarrow M_i \leq_{\mathbf{D}_*} M$. By applying $\circledast_1$ to every $M_i$ and coding we can easily show that $M \in K_{\mathbf{D}_*}$ thus finishing.

**<u>Ax.IV</u>:** Assume $\langle M_i : i < \lambda \rangle$, $M$ are as above and $i < \lambda \Rightarrow M_i \leq_{\mathbf{D}_*} N$. To prove $M \leq_{\mathbf{D}_*} N$, note that as $\mathfrak{k}$ is an AEC we have $M \leq_{\mathfrak{k}} N$, and consider $\bar{a} \in N$. By 5.26(6), $\text{gtp}(\bar{a}, M, N)$ is the stationarization of $\text{gtp}(\bar{a}, M_i, N)$for some $i < \lambda$, but the latter belongs to $\mathbf{D}_*(M_i)$ hence $\text{gtp}(\bar{a}, M, N) \in \mathbf{D}_*(M)$ as required.

**<u>Ax.V</u>:** By $\circledast_2$ this is translated to the case $N_0, N_1, M \in K_{\aleph_0}$, but then it holds easily.

[20] Recall that $M \restriction \mathfrak{B} = M \restriction \{a \in M : a \in \mathfrak{B}\}$.

**Ax.VI:** By $\circledast_1 + \circledast_2 +$ **Ax.VI** for $\mathfrak{k}$.

2) So we replace $\mathfrak{k}$ by $\mathfrak{k}' = \mathfrak{k}_{\mathbf{D}_*}$, and easily all that we need for $\mathbf{D}$ is that $\mathfrak{k}'$ is satisfied by $\mathbf{D}_*$ (actually, repeating the work in §5 up to this point on $\mathfrak{k}'$, we get it) noting that

- $\circledast$ If $M_0 \le_{\mathbf{D}_*} M_\ell \in K_{\aleph_0}$ for $\ell = 1, 2$ and $\mathrm{gtp}(\bar a_1, M_0, M_1) = \mathrm{gtp}(\bar a_2, M_0, M_2)$, then there is a triple $(M_1^+, M_2^+, f)$ such that $M_\ell \le_{\mathbf{D}_*} M_\ell^+ \in K_{\aleph_0}$, $M_\ell^+$ is $(\mathbf{D}(M_i), \aleph_0)^*$-homogeneous for $i = 0, \ell$, and $f$ is an isomorphism from $M_1^+$ onto $M_2^+$ over $M_0$ mapping $\bar a_1$ to $a_2$.

This follows by:

- $\circledast_1$ If $M_0 \le_{\mathbf{D}_*} M_1 \le_{\mathbf{D}_*} M_2$ and $\bar a \in M_1$ then
$$\mathrm{gtp}(\bar a, M_0, M_1) = \mathrm{gtp}(\bar a, M_0, M_2) \in \mathbf{D}_*(M_0).$$
- $\circledast_2$ If $M_0 \in K_{\aleph_0}$, then for some $M_1 \in K_{\aleph_0}$ we have $M_0 \le_{\mathbf{D}_*} M_2$ and $M_1$ is $(\mathbf{D}_*(M_0), \aleph_0)^*$-homogeneous.
- $\circledast_3$ If $M_0 \le_{\mathbf{D}_*} M_1 \le_{\mathbf{D}_*} M_2$ and $M_2$ is $(\mathbf{d}_*(M_1), \aleph_0)^*$-homogeneous then $M_2$ is $(\mathbf{D}_*(M_0), \aleph_0)^*$-homogeneous.
- $\circledast_4$ If $M_0 \le_{\mathbf{D}_*} M_\ell \in K_{\aleph_0}$ and $\mathrm{gtp}(\bar a_1, M_0, M_1) = \mathrm{gtp}(\bar a_2, M_0, M_2)$, then there is an isomorphism from $M_1$ onto $M_2$ over $M_0$ mapping $\bar a_1$ to $\bar a_2$.

$\square_{5.33}$

**Claim 5.34.** *Suppose $N_0 \le_{\mathfrak{k}} N_\ell \in K_{\aleph_0}$ (for $\ell = 1, 2$) and $\bar c \in N_2$. Then there is $M$ such that $N_0 \le_{\mathfrak{k}} M$ and $\le_{\mathfrak{k}}$-embeddings $f_\ell$ of $N_\ell$ into $M$ over $N_0$ such that*

(*i*) *For every $\bar a \in N_1$, $\mathrm{gtp}(f_1(\bar a), f_2(N_2), M)$ is a stationarization of $\mathrm{gtp}(\bar a, N_0, N_1)$.*

(*ii*) *$\mathrm{gtp}(f_2(\bar c), f_1(N_1), M)$ is a stationarization of $\mathrm{gtp}(\bar c, N_0, N_2)$.*

*Remark* 5.35. This is one more step toward stable amalgamation: in 5.29 we have obtained it for one $\bar a \in N_1$ and in 5.30(3) for *every* $\bar a \in N_1$, which gives disjoint amalgamation.

*Proof.* Clearly, for $\ell = 1, 2$ we can replace $N_\ell$ by any $N'_\ell \in K_{\aleph_0}$ with $N_\ell \le_{\mathfrak{k}} N'_\ell$, and without loss of generality $N_0 = N_1 \cap N_2$. By 5.30(3) there is $N_3 \in K_{\aleph_0}$ such that $N_\ell \le_{\mathfrak{k}} N_3$ for $\ell < 3$ and

$$\bar a \in {}^{\omega>}(N_1) \Rightarrow \mathrm{gtp}(\bar a, N_2, N_3) \text{ is the stationarization of } \mathrm{gtp}(\bar a, N_0, N_1).$$

So we can assume that for some $\mathbf{D}_\alpha$ as in Definition 5.32 and $\ell = 1, 2$, $N_\ell$ is $(\mathbf{D}_\alpha(N_0), \aleph_0)^*$-homogeneous. As in the proof of 5.24, we can find a countable linear order $I$ such that every element $s \in I$ has an immediate successor $s + 1$, 0 is the first element, $I^*$ has a subset isomorphic to the rationals,[21] and models $M_s \in K_{\aleph_0}$ for $s \in I$ such that $s < t \Rightarrow M_s \le_{\mathfrak{k}} M_t$ and $M_t$ is $(\mathbf{D}_\alpha(M_s), \aleph_0)$-homogeneous, etc.

So by 5.26(3), for every initial segment $J$ of $I$ and $t \in I$ such that[22] $J < t$, if $J$ has no last element and $I \setminus J$ has no first element then $M_t$ is $(\mathbf{D}_\alpha(M_J), \aleph_0)^*$-homogeneous, where

$$M_J := \bigcup_{s \in J} M_s = \bigcap_{t \in I \setminus J} M_t.$$

[21] Really, this follows.
[22] That is, $(\forall s \in J)[s <_I t]$.

We let $N_0^J := M_J$, $N_1^J := M_I$, and $N_2^J$ be a $(\mathbf{D}_\alpha(N_0^J), \aleph_0)^*$-homogeneous model satisfying $N_0^J \le_{\mathfrak{k}} N_2^J$; without loss of generality $N_1^J \cap N_2^J = N_0^J$. Also easily, there is $N_0' <_{\mathfrak{k}} N_0$ such that $\mathrm{gtp}(\bar{c}, N_0, N_1)$ is definable over some $\bar{c}_0 \subseteq N_0'$ and $N_0$ is $(\mathbf{D}_\alpha(N_0'), \aleph_0)$-homogeneous. Clearly the triples $(N_0, N_1, N_2), (N_0^J, N_1^J, N_2^J)$ are isomorphic, and let $f_0^J, f_1^J, f_2^J$ be appropriate isomorphisms such that $f_0^J \subseteq f_1^J, f_2^J$. Without loss of generality $f_0^J(N_0') = M_0$. Now by 5.30(3), there is $M^J \in K_{\aleph_0}$ satisfying $N_\ell^J \le_{\mathfrak{k}} M^J$ for $\ell = 0, 1, 2$ such that for every $\bar{a} \in N_1^J$, $\mathrm{gtp}(\bar{a}, N_2^J, M^J)$ is the stationarization of $\mathrm{gtp}(\bar{a}, N_0^J, N_1^J)$ and there exist $N_3 \in K_{\aleph_0}$ with $N_\ell \le_{\mathfrak{k}} N_3$ for $\ell = 0, 1, 2$ and an isomorphism $f_3^J \supseteq f_1^J \cup f_2^J$ from $N_3$ onto $M^J$.

Suppose our conclusion fails. Then $\mathrm{gtp}(f_2^J(\bar{c}), N_1^J, M^J)$ is not the stationarization of $\mathrm{gtp}(f_2^J(\bar{c}), N_0^J, M^J)$. Moreover, as in the proof of 5.24,

$$t \in I \setminus J \Rightarrow M_I := N_1^J \text{ and } M_t \text{ are isomorphic over } N_0^J := M_J,$$

hence we can replace $N_1^J$ by $M_t$ for any $t \in I \setminus J$. So as we assume that our conclusion fails,

$$t \in I \setminus J \Rightarrow \mathrm{gtp}(f_2^J(\bar{c}), M_t, M^J) \text{ is not a stationarization of } \mathrm{gtp}(f_2^J(\bar{c}), N_0^J, M^J)$$

and the latter is the stationarization of $\mathrm{gtp}(f_2^J(\bar{c}), M_0, M^J)$. Let

$$p_J := \mathrm{gtp}(f_2^J(\bar{c}), N_1^J, M^J) = \mathrm{gtp}(\bar{c}, M_I, M^J);$$

all this was done for any appropriate $J$. So it is easy to check that

$$J_1 \neq J_2 \Rightarrow p_{J_1} \neq p_{J_2},$$

but as $I^* \subseteq I \wedge |I| = \aleph_0$, we have continuum many such $J$-s and hence that many $p_J$-s. If CH fails, we are done. Otherwise, note that we can ensure that for $J_1 \neq J_2$ as above there is an automorphism of $M_I$ taking $p_{J_1}$ to $p_{J_2}$, hence the set of such $p_J$-s is contained in $\mathbf{D}_\beta(M_I)$ for some $\beta < \omega_1$; i.e. $(f_1^{J_2}) \circ (f_1^{J_1})^{-1}$ maps one to the other, [giving a] contradiction by clause (d) of 5.13.

Alternatively, repeat the proof of 5.24. More elaborately, by the way $\mathbf{D}_\alpha$ was chosen, Claim 5.30(3) holds for $\mathfrak{k}_{\mathbf{D}_*}$ hence without loss of generality $M^J$ is $(\mathbf{D}_\alpha(N_1), \aleph_0)$-homogeneous. So without loss of generality for some $t_* \in I \setminus J$, $N_1^J = M_{t_*}$), and $N^J = M_{t_*+1}$, and we get a contradiction as in the proof of 5.24 (i.e. the choice of $\langle \bar{a}_\ell : \ell \le \ell(*) \rangle$ there.[23]) $\square_{5.34}$

**Definition 5.36.** 1) $\mathfrak{k}$ has the symmetry property when the following holds: if $N_0 \le_{\mathfrak{k}} N_\ell \le_{\mathfrak{k}} N_3$ for $\ell = 1, 2$ and $\mathrm{gtp}(\bar{a}, N_2, N_3)$ is the stationarization of $\mathrm{gtp}(\bar{a}, N_0, N_3)$ for every $\bar{a} \in N_1$, then for every $\bar{b} \in N_2$, $\mathrm{gtp}(\bar{b}, N_1, N_3)$ is the stationarization of $\mathrm{gtp}(\bar{b}, N_0, N_3)$.

2) If $N_0, N_1, N_2 \le_{\mathfrak{k}} N_3$ satisfies the assumption and conclusion of part (1) we say that $N_1, N_2$ are in *stable amalgamation* over $N_0$ inside $N_3$ (or in two-sided stable amalgamation over $N_0$ inside $N_3$). If only the hypothesis of (1) holds, we say they are in a *one-sided* stable amalgamation over $N_0$ inside $N_3$. (Then the order of $(N_1, N_2)$ is important.)

3) We say that $\mathfrak{k}$ has unique [one-sided] amalgamation when: if $N_0 \le_{\mathfrak{k}} N_\ell \in K_{\aleph_0}$ for $\ell = 1, 2$ then $N_1, N_2$ has unique [one-sided] stable amalgamation, see part (4).

4) We say $N_1, N_2$ have a unique [one-sided] stable amalgamation over $N_0$ (where for notational simplicity, $N_1 \cap N_2 = N_0$) provided that: if $(*)$ then $(**)$, where:

[23] A third way is to use forcing and absoluteness to use the case 'CH fails.'

(∗) (a) $N_1 \leq_{\mathfrak{k}} N_3$, $N_2 \leq_{\mathfrak{k}} N_3$, $(N_1, N_2)$ are in [one-sided] stable amalgamation inside $N_3$ over $N_0$, and $\|N_3\| \leq \|N_1\| + \|N_2\|$.
(b) $M_0 \leq_{\mathfrak{k}} M_\ell \leq_{\mathfrak{k}} M_3$ for $\ell = 1, 2$ and $(M_1, M_2)$ are in [one-sided] stable amalgamation inside $M_3$ over $M_0$ (hence $M_1 \cap M_2 = M_0$).
(c) $f_\ell$ is an isomorphism from $N_\ell$ onto $M_\ell$ for $\ell = 0, 1, 2$.
(d) $f_0 \subseteq f_1$ and $f_0 \subseteq f_2$.

(∗∗) We can find $M'_3$ with $M_3 \leq_{\mathfrak{k}} M'_3$, and $f_3$ a $\leq_{\mathfrak{k}}$-embedding of $N_3$ into $M'_3$ extending $f_1 \cup f_2$.

We at last get the existence of stable amalgamation (to which earlier we got approximations).

**Claim 5.37.** *For any $N_0 \leq_{\mathfrak{k}} N_1, N_2$, all from $K_{\aleph_0}$, we can find $M \in K_{\aleph_0}$ with $N_0 \leq_{\mathfrak{k}} M$ and $\leq_{\mathfrak{k}}$-embeddings $f_1, f_2$, of $N_1$ and $N_2$ respectively, over $N_0$ into $N$ such that $N_0, f_1(N_1), f_2(N_1)$ are in stable amalgamation.*

*Remark* 5.38. In the proof we could have "inverted the tables" and used $\bar{c}_\zeta$ in the $\omega_1$ direction.

*Proof.* We define $\langle M^\zeta_\alpha : \alpha < \omega_1 \rangle$ and $\bar{c}_\zeta$ by induction on $\zeta < \omega_1$ such that:

(*i*) $\langle M^\zeta_\alpha : \alpha < \omega_1 \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous and $M^\zeta_\alpha \in K_{\aleph_0}$.
(*ii*) For $\alpha < \zeta$, $M^\zeta_\alpha = M^\alpha_\alpha$ and $\xi < \zeta \wedge \alpha < \omega_1 \Rightarrow M^\xi_\alpha \leq_{\mathfrak{k}} M^\zeta_\alpha$.
(*iii*) For $\zeta$ limit, $M^\zeta_\alpha := \bigcup_{\xi < \zeta} M^\xi_\alpha$.
(*iv*) For $\zeta \leq \alpha < \omega_1$ and $\zeta$ non-limit, $M^\zeta_{\alpha+1}$ is $(\mathbf{D}_{\alpha+1}(M^\zeta_\alpha), \aleph_0)^*$-homogeneous.
(*v*) For every $\bar{c} \in M^\zeta_{\alpha+1}$, $\mathrm{gtp}(\bar{c}, M^{\zeta+1}_\alpha, M^{\zeta+1}_{\alpha+1})$ is a stationarization of $\mathrm{gtp}(\bar{c}, M^\zeta_\alpha, M^\zeta_{\alpha+1})$.
(*vi*) $\bar{c}_\zeta \in M^{\zeta+1}_{\zeta+1}$, and for $\alpha \in (\zeta + 1, \omega_1)$, $\mathrm{gtp}(\bar{c}_\zeta, M^\zeta_\alpha, M^{\zeta+1}_\alpha)$ is the stationarization of $\mathrm{gtp}(\bar{c}_\zeta, M^\zeta_{\zeta+1}, M^{\zeta+1}_{\zeta+1})$.
(*vii*) For every $p \in \mathbf{D}(M^\xi_\alpha)$, for some $\zeta \in (\xi + \alpha, \omega_1)$, we have $\mathrm{gtp}(\bar{c}_\zeta, M^\zeta_{\zeta+1}, M^{\zeta+1}_{\zeta+1})$ is a stationarization of $p$.

There is no problem doing this (by 5.34 and as in earlier constructions); in limit stages we use local character 5.26(3) and $\mathbf{D}_\alpha$ being closed under stationarization.

Now easily, for a thin enough closed unbounded set $E \subseteq \omega_1$, for every $\zeta \in E$, we have

$(*)_\zeta$ (a) $M^\zeta_\zeta$ is $(\mathbf{D}_\zeta(M^0_\zeta), \aleph_0)^*$-homogeneous.
(b) For every $\bar{c} \in M^\zeta_\zeta$, $\mathrm{gtp}\big(\bar{c}, \bigcup_{\alpha < \omega_1} M^0_\alpha, \bigcup_{\xi < \omega_1} M^\xi_\xi\big)$ is a stationarization of $\mathrm{gtp}(\bar{c}, M^0_\zeta, M^\zeta_\zeta)$.
(c) For every $\bar{c} \in M^0_{\zeta+1}$, $\mathrm{gtp}(\bar{c}, M^{\zeta+1}_\zeta, M^{\zeta+1}_{\zeta+1})$ is a stationarization of $\mathrm{gtp}(\bar{c}, M^0_\zeta, M^0_{\zeta+1})$.

[Why? Clause (c) holds by clause (*v*) of the construction (as $\langle M^\zeta_\varepsilon : \varepsilon \leq \zeta \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous). Clause (b) holds as $E$ is thin enough; i.e. is proved as in earlier constructions (i.e. see (∗) in the proof of 5.30(1)). As for Clause (a), first note that by clauses $(i)$-$(iii)$ the sequence $\langle M^\zeta_\varepsilon : \varepsilon \le \zeta\rangle$ is $\le_{\mathfrak{k}}$-increasing continuous. By clause $(vi)$ we have

$$\varepsilon < \zeta \Rightarrow \mathrm{gtp}(\bar{c}_\varepsilon, M^\varepsilon_\zeta, M^{\varepsilon+1}_\zeta) \text{ does not fork over } M^\varepsilon_\zeta.$$

By clause $(vii)$ of the construction we have: if $p \in \mathbf{D}_\zeta(M^\zeta_\varepsilon)$ with $\varepsilon < \zeta$, then for some $\xi \in (\varepsilon, \zeta)$, $\mathrm{gtp}(\bar{c}_\xi, M^\zeta_\xi, M^\zeta_{\xi+1})$ is a non-forking extension of $p$. As $E$ is thin enough we have $\bar{d} \in M^\zeta_\zeta \Rightarrow \mathrm{gtp}(\bar{d}, M^\zeta_0, M^\zeta_\zeta) \in \mathbf{D}_\zeta(M^\zeta_0)$. Together it is easy to get clause (a) (e.g. see 5.47).]

So as in the proof of 5.30(3) we can finish (choose $\zeta \in E$, $f_0$ an isomorphism from $N_0$ onto $M^0_\zeta$, $f_1 \supseteq f_0$ an $\le_{\mathfrak{k}}$-embedding of $N_1$ into $M^\zeta_\zeta$, and $f_2 \supseteq f_0$ a $\le_{\mathfrak{k}}$-embedding of $N_2$ into $M^0_{\zeta+1}$). $\square_{5.37}$

*Remark* 5.39. Note that in [She09a] we use only the results up to this point.

**Theorem 5.40.** 1) *Suppose, in addition to the hypothesis of this section, that $2^{\aleph_1} < 2^{\aleph_2}$ and the club ideal on $\aleph_1$ is not $\aleph_2$-saturated and $\dot{I}(\aleph_2, K) < 2^{\aleph_2}$ (or just $\dot{I}(\aleph_2, K(\aleph_1\text{-saturated})) < 2^{\aleph_2}$). Then $\mathfrak{k}$ has the symmetry property.*

2) *Assume $2^{\aleph_1} < 2^{\aleph_2}$ and $\dot{I}(\aleph_2, K(\aleph_1\text{-saturated})) < \mu_{\mathrm{unif}}(\aleph_2, 2^{\aleph_1})$ (this number is always $> 2^{\aleph_1}$, usually $2^{\aleph_2}$; see 0.6). Then $\mathfrak{k}$ has the symmetry property and stable amalgamation in $K_{\aleph_0}$ is unique (we know that it always exists, and it follows by (1)+(2) that one-sided amalgamation is unique).*

**Discussion 5.41.** 1) This certainly gives a desirable conclusion. However, part (2) is not used so we shall return to it in [She09b].

!! More elaborately, in [She09b, 4.1], in the 'lean version' of [She09b],[24] assuming the weak diamond ideal is not $\aleph_2$-saturated, we prove 5.40(2). Hence we also prove a slight weaker version of 5.40(1), replacing "$\dot{I}(\aleph_2, K)(\aleph_1\text{-saturated}) < 2^{\aleph_2}$" by

$$\dot{I}(\aleph_2, K(\aleph_1\text{-saturated})) < \mu_{\mathrm{unif}}(\aleph_2, 2^{\aleph_1}).$$

!! Better, in [She09b, 4.40] we prove 5.40(2) fully. Still, the proof of part (1) given below is not presently covered by [She09b], and it gives nicer reasons for non-isomorphisms (essentially different natural invariants).

2) As for part (1), we can avoid using it (except in 5.45 below). More fully, in
!! [She09a, §3] dealing with $\mathfrak{k}$ as here by [She09a, 3.4], for every $\alpha < \omega_1$ we derive a good $\aleph_0$-frame $\mathfrak{s}_\alpha$ with $\mathfrak{k}^{\mathfrak{s}_\alpha} = \mathfrak{k}_{\mathbf{D}_\alpha}$. (If we would have liked to derive a good $\aleph_1$-frame we would need 5.40.)

Then in [She09c] if $\mathfrak{s}$ is successful (holds, e.g., if $2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_2}$, $\dot{I}(\aleph_2, \mathfrak{k}^{\mathfrak{s}_\alpha}) < 2^{\aleph_2}$, and $\mathrm{WDmId}_{\aleph_1}$ is not $\aleph_2$-saturated) then we derive the successor $\mathfrak{s}^+_\alpha$, a good $\aleph_1$-frame with $K^{\mathfrak{s}^+_\alpha} \subseteq \{M \in K^{\mathfrak{s}_\alpha}_{\aleph_1} : M \text{ is } \aleph_1\text{-saturated for } K^{\mathfrak{s}_\alpha}\}$, and $\mathfrak{s}^+_\alpha$ is even
!! good$^+$ (see [She09c, Claim 1.6(2)] and [She09c, Definition 1.3]). This suffices for
!! the main conclusions of [She09a, §9] and end of [She09c, §12].

3) Still, we may wonder: is $\le_{\mathfrak{s}^+_\alpha}$ the same as $\le_{\mathfrak{k}} \restriction \mathfrak{k}_{\mathfrak{s}^+_\alpha}$? If $\mathfrak{s}_\alpha$ is good$^+$ then the answer is yes (see [She09c, 1.6(1)]). That is, the present theorem 5.40 is used in

[She09c, §1] to prove $\mathfrak{s}$ is "good$^+$;" really, this is proved in 5.45. In fact, part (1)
!! of 5.40 is enough to prove that $\mathfrak{s}_{\mathbf{D}_*}$ is good$^+$; see [She09c, 1.5](1A).

4) The proof of 5.40(1) gives that if $\mathfrak{k}$ fails the symmetry property then $\dot{I}(\aleph_2, K) \geq 2^{\aleph_1}$ even if $2^{\aleph_1} = 2^{\aleph_2}$, and do[es] not use $2^{\aleph_0} = 2^{\aleph_1}$ directly (but uses earlier results of §5). The case "$\mathcal{D}_{\aleph_1}$ is $\aleph_2$-saturated, $2^{\aleph_0} < 2^{\aleph_1} < 2^{\aleph_2}$, and $\dot{I}(\aleph_2, \aleph_2) < \mu_{\rm unif}(\aleph_2, 2^{\aleph_2})$" is covered in [She09b].

*Proof.* 1) So in the first part, towards contradiction we can assume that $K^4 \neq \varnothing$, where $K^4$ is the class of quadruples $\overline{N} = (N_0, N_1, N_2, N_3)$ such that $N_1, N_2$ are one-sided stably amalgamated over $N_0$ inside $N_3$ but $N_2, N_1$ are not. Hence there is $\bar{c} \in N_2$ such that $\operatorname{gtp}(\bar{c}, N_1, N_3)$ is not the stationarization of

$$\operatorname{gtp}(\bar{c}, N_0, N_2) = \operatorname{gtp}(\bar{c}, N_0, N_3).$$

We define a two-place relation $\leq$ on $K^4$ by $\overline{N}^1 \leq \overline{N}^2$ iff $N_0^1 = N_0^2$, $N_\ell^1 \leq_{\mathfrak{k}} N_\ell^2$ for $\ell = 0, 1, 2$, and

$$\bar{a} \in N_1^1 \Rightarrow \operatorname{gtp}(\bar{a}, N_2^2, N_3^2) \text{ is definable over some } \bar{b} \in N_0^1.$$

Easily, this is a partial order and $K^4$ is closed under unions of increasing countable sequences. Hence without loss of generality, for some $\mathbf{D}_*$ and $\overline{N}^*$,

(∗) (a) $\mathbf{D}_* \in \{\mathbf{D}_\alpha : \alpha < \omega_1\}$
(b) $\overline{N}^* \in K^4$
(c) $N_\ell^*$ is $(\mathbf{D}_*(N_0^*), \aleph_0)^*$-homogeneous over $N_0^*$ for $\ell = 1, 2$.
(d) $N_3^*$ is $(\mathbf{D}_*(N_\ell^*), \aleph_0)^*$-homogeneous over $N_\ell^*$ for $\ell = 1, 2$.

So we have established the following.

**Observation 5.42.** *To prove 5.40, we can assume that $\mathbf{D} = \mathbf{D}_\alpha$ for [some] $\alpha < \omega_1$; i.e. $\mathbf{D}$ is countable.*

**[Continuation of the proof of 5.40:]**

A problem is that we still have not proven the existence of a superlimit model of $K$ of cardinality $\aleph_1$, though we have a candidate $N^*$ from 5.18. So we use $N^*$, but to ensure we get it at limit ordinals (in the induction on $\alpha < \aleph_2$), we have to take a stationary $S_0 \subseteq \omega_1$ with $\omega_1 \setminus S_0$ not small. I.e. $\omega_1 \setminus S_0$ does not belong to the ideal $\mathrm{WDmId}_{\aleph_1}$ from Theorem 0.6 and "devote" it to ensure this, using 5.37.

The point of using $S_0$ is as follows (this is supposed to help to understand the quotation from [She09b]):

**Definition 5.43.** 1) Let

$$K^{\rm qt} := \big\{\overline{N} = \langle N_\alpha : \alpha < \omega_1\rangle : \overline{N} \text{ is } \leq_{\mathfrak{k}}\text{-increasing continuous, } N_\alpha \in K_{\aleph_0}, \text{ and } N_{\alpha+1} \text{ is } (\mathbf{D}_\alpha(N_\alpha), \aleph_0)^*\text{-homogeneous}\big\}.$$

2) On $K^{\rm qt}$ we define a two-place relation $<_S^a$ (for $S \subseteq \omega_1$) as follows.

$\overline{N}^1 <_S^a \overline{N}^2$ iff for some closed unbounded $E \subseteq \omega_1$:

[24] See Reading plan A in [She09b, §0].

(a) For every $\alpha \in C$, we have $N^1_\alpha \leq_{\mathfrak{k}} N^2_\alpha$ and $N^1_{\alpha+1} \leq_{\mathfrak{k}} N^2_{\alpha+1}$.

(b) For every $\alpha < \beta$ from $E$, we have $N^2_\beta \cap \bigcup\limits_{\alpha<\omega_1} N^1_\alpha = N^1_\beta$ and $N^1_\beta, N^2_\alpha$ are in one-sided stable amalgamation over $N^1_\alpha$ inside $N^2_\beta$. (I.e. if $\bar{a} \in N^1_\beta$ then $\mathrm{gtp}(\bar{a}, N^2_\alpha, N^2_\beta)$ is the stationarization of $\mathrm{gtp}(\bar{a}, N^1_\alpha, N^1_\beta)$.)

(c) If $\alpha \in S \cap C$ then $N^2_\alpha$ and $N^1_{\alpha+1}$ are in stable amalgamation over $N^1_\alpha$ inside $N^2_{\alpha+1}$.

**Fact 5.44.** 0) The two-place relation $<^a_S$ defined in 5.43 are partial orders on $K^{\mathrm{qt}}$ for $n < \omega$.

1) Suppose $\overline{N}^n \leq^a_{S_0} \overline{N}^{n+1}$ and let $E_n$ exemplify this (as in the Definition 5.43). Let $E_\omega := \bigcap\limits_{n<\omega} E_n$, $E'_\omega := \{\alpha, \alpha+1 : \alpha \in C_\omega\}$,

[undefined]

and let $N^\omega_\alpha := \bigcup\limits_{n<\omega} N^n_\beta$ when $\beta := \min(E'_\omega \setminus \alpha)$. Then $\langle N^\omega_\alpha : \alpha < \omega_1 \rangle \in K_{<\aleph_1}$ and $\overline{N}^n \leq^a_{S_0} \langle N^\omega_\alpha : \alpha < \omega_1 \rangle$ for $n < \omega$.

2) If $\langle \overline{N}^\varepsilon : \varepsilon < \omega_1 \rangle$ is $<^a_S$-increasing and $N^\varepsilon = \bigcup\limits_{\alpha<\omega_1} N^\varepsilon_\alpha \in K_{\aleph_1}$ is $\leq_{\mathfrak{k}}$-increasing continuous, [if] the club $E_{\varepsilon,\zeta}$ witnesses $\overline{N}^\varepsilon \leq \overline{N}^\zeta$ for $\varepsilon < \zeta < \aleph_1$ and $\langle N_\alpha : \alpha < \omega_1 \rangle$ a $\leq_{\mathfrak{k}}$-representation of $N$, and $N_\alpha = \bigcup\limits_{\varepsilon<\alpha} N^\varepsilon_\alpha$ and $N_{\alpha+1} = \bigcup\limits_{\varepsilon<\alpha} N^\varepsilon_{\alpha+1}$ for club-many $\alpha < \aleph_1$, then $\varepsilon < \omega_1 \Rightarrow \overline{N}^\varepsilon \leq^a_{S_0} \overline{N}$.

*Proof.* Should be easy by now. $\square_{5.44}$

**[Continuation of the proof of 5.40:]**

It is done as follows.

There is $\langle S_\varepsilon : \varepsilon < \omega_1 \rangle$ such that $S_\varepsilon \subseteq \omega_1$, $\zeta < \varepsilon \Rightarrow S_\zeta \cap S_\varepsilon$ countable and $S_0, S_{\varepsilon+1} \setminus S_\varepsilon \in (\mathcal{D}_{\omega_1})^+$ (this is possible by an assumption).

Now for any $u \subseteq \omega_2$ we choose $N^u_\varepsilon, N^u_\varepsilon$ by induction on $\varepsilon < \omega_2$ such that

$\circledast$ (a) $\overline{N}^u_\varepsilon = \langle N^u_{\varepsilon,\alpha} : \alpha < \omega_1 \rangle \in K^{\mathrm{qt}}$

(b) $N^u_\varepsilon = \bigcup\limits_{\alpha<\omega_1} N^u_{\varepsilon,\alpha} \in K_{\aleph_1}$

(c) For $\zeta < \varepsilon$ we have $\bar{N}^u_\zeta <^1_{S_\xi} \overline{N}^u_\varepsilon$ when $\xi \notin [\zeta, \varepsilon) \cap u$. (We can use $S'_{[\zeta,\varepsilon)}$, the complement of the diagonal union of $\{\langle S_\xi : \varepsilon \in [\zeta, \varepsilon) \rangle \cap u\}$.) [Not sure what those braces are doing.]

(d) We can demand continuity, as defined implicitly in Fact 5.44.

(e) For each $\varepsilon \in u$, for a club of $\alpha < \omega_1$, if $\alpha \in S_\varepsilon$ then $N^u_{\varepsilon+1,\alpha}, N^u_{\varepsilon,\alpha+1}$ are not in stable amalgamation over $N^u_{\varepsilon,\alpha}$ inside $N^u_{\varepsilon+1,\alpha+1}$ (though they are in one[-sided]).

Lastly, let $N^u := \bigcup\limits_{\varepsilon<\omega_1} N^u_\varepsilon \in K_{\aleph_2}$. Now we can prove that if $u, v \subseteq \omega_2$ and $N^u \approx N^v$ then for some club $C$ of $\omega_2$, $u \cap C = v \cap C$. So we can easily get $\dot{I}(\aleph_2, \mathfrak{k}) = 2^{\aleph_2}$ and even $\dot{I}(\aleph_2, \mathfrak{k}(\aleph_1\text{-saturated})) = 2^{\aleph_2}$. $\square_{5.40}$

**[$\approx$ isn't defined or used anywhere else in this paper. Did you mean $\cong$?]**

**Theorem 5.45.** *Suppose $\mathfrak{k}$ has the symmetry property (this holds if the assumption of 5.40(1) holds). Then $\mathfrak{k}$ has a superlimit model in $\aleph_1$.*

*Proof.* We have a candidate $N^*$ from 5.18. So let $\langle N_i : i < \delta \rangle$ be $\leq_{\mathfrak{k}}$-increasing with $N_i \cong N^*$, and without loss of generality $\delta = \mathrm{cf}(\delta)$. If $\delta = \omega_1$ this is very easy. If $\delta = \omega$, let $N_\omega = \bigcup_{i<\omega} N_i$ and for each $i \leq \omega$ let $\langle N_i^\alpha : \alpha < \omega_1 \rangle$ be $\leq_{\mathfrak{k}}$-increasing continuous with union $N_i$ and $N_i^\alpha \in K_{\aleph_0}$. Now by restricting ourselves to a club $E$ of $\alpha$-s and renaming it $E = \omega_1$, we get: $N_i^\alpha = N_i \cap N_j^\alpha$ for $i < j \leq \omega$ and

- $\circledast_1$ For any $\alpha < \beta < \omega_1$, $\bar{a} \in N_\omega^\alpha$, and $i < \omega$, the type $\mathrm{gtp}(\bar{a}, N_i^\beta, N_\omega^\beta)$ is a stationarization of $\mathrm{gtp}(\bar{a}, N_i^\alpha, N_\omega^\alpha)$.

To prove $N_\omega \cong N^*$ it is enough to prove:

- $\circledast_2$ If $\alpha < \omega_1$ and $p \in \mathbf{D}(N_\omega^\alpha)$ then some $\bar{b} \subseteq N_\omega$ realizes $p$ in $N_\omega$.

By 5.26(3) there is $i < \omega$ such that $p$ is the stationarization of $q := p \restriction N_i^\alpha \in \mathbf{D}(N_i^\alpha)$. As $N_i \cong N^*$, there is $\bar{b} \subseteq N_i$ which realizes $q$ and we can find $\beta \in (\alpha, \omega_1)$ such that $\bar{b} \subseteq N_i^\beta$. By $\circledast_1$, we have $N_\omega^\alpha, N_i^\beta$ are in one-sided stable amalgamation over $N_i^\alpha$ inside $N_\omega^\beta$ (see 5.36(2)).

As we assume $\mathfrak{k}$ has the symmetry property, $N_i^\beta, N_\omega^\alpha$ are also in stable amalgamation over $N_i^\alpha$ inside $N_\omega^\beta$. In particular, as $\bar{b} \subseteq N_i^\beta$, we have $\mathrm{gtp}(\bar{b}, N_\omega^\alpha, N_\omega^\beta)$ is the stationarization of $\mathrm{gtp}(\bar{b}, N_i^\alpha, N_i^\beta)$ but the latter is $p \restriction N_i^\alpha$. So by uniqueness of stationarization, $p = \mathrm{gtp}(\bar{b}, N_\omega^\alpha, N_\omega^\beta)$ which is $\mathrm{gtp}(\bar{b}, N_\omega^\alpha, N_\omega)$, so $p$ is realized in $N_\omega$ as required. $\square_{5.45}$

We have implicitly proved

**Claim 5.46.** *Assume that $N_0 \leq_{\mathfrak{k}} N_1 \in K_{\aleph_0}$ and $\bar{a}_\ell \in {}^{\omega>}(N_1)$ for $\ell = 1, 2$. Then $(*)_1 \Leftrightarrow (*)_2$, where: (for $\ell = 1, 2$)*

- $(*)_\ell$ *There are $M_1, M_2, \bar{b}_1, \bar{b}_2$ such that*
  - (a) $N_0 \leq_{\mathfrak{k}} M_1 \leq_{\mathfrak{k}} M_2 \in K_{\aleph_1}$
  - (b) $\bar{a}_k \in {}^{\omega>}(M_k)$ *for* $k = 1, 2$.
  - (c) $\mathrm{gtp}(\bar{b}_{3-\ell}, N_0, M_1) = \mathrm{gtp}(\bar{a}_{3-\ell}, N_0, N_1)$
    ***[Either one or both of those subscripts need to be an $\ell$.]***
  - (d) $\mathrm{gtp}(\bar{b}_\ell, M_1, M_2)$ *is the stationarization of* $\mathrm{gtp}(\bar{a}_\ell, N_0, N_1)$ *from* $\mathbf{D}(M_1)$.
  - (e) $\mathrm{gtp}(\bar{b}_1{}^\frown\bar{b}_2, N_0, M_2) = \mathrm{gtp}(\bar{a}_1{}^\frown\bar{a}_2, N_0, N_1)$.

*Proof.* We can deduce it from 5.34 (or imitate the proof of 5.24).

In detail: by symmetry it is enough to assume $(*)_2$ and prove $(*)_1$. So let $M_1, M_2, \bar{b}_1, \bar{b}_2$ witness $(*)_2$.

By 5.37 we can find $M_2', f$ such that $M_2 \leq_{\mathfrak{k}} M_2' \in K_{\aleph_0}$, $f$ is a $\leq_{\mathfrak{k}}$-embedding of $M_2$ into $M_2'$ over $N_0$ such that $M_1, f(M_2)$ is in stable amalgamation over $N_0$ inside $M_2'$. Now, as $f(M_2), M_1$ are in one-sided stable amalgamation over $N_0$ inside $M_2'$, by the choice of $(M_1, M_2, \bar{b}_1, \bar{b}_2)$, we get $\mathrm{gtp}(f(\bar{b}_2), M_1, M_2') = \mathrm{gtp}(\bar{b}_2, M_1, M_2')$ hence

$$\mathrm{gtp}(\bar{b}_1{}^\frown\bar{b}_2, N_0, M_2') = \mathrm{gtp}(\bar{b}_1{}^\frown f(\bar{b}_2), N_0, M_2').$$

By the choice of $M_1^2$ and $f$, $\mathrm{gtp}(\bar{b}_1, f(M_2), M_2')$ is the stationarization of

$$\mathrm{gtp}(\bar{b}_1, N_0, M_2) = \mathrm{gtp}(\bar{a}_1, N_0, N_1).$$

Now $(*)_1$ holds, as exemplified by $(f(M_2), M_2', f(\bar{b}_2), \bar{b}_1)$. $\square_{5.46}$

*Exercise* 5.47. Assume $\alpha \leq \omega_1$ and

- (a) $\langle M_i : i \leq \delta \rangle$ is $\leq_{\mathfrak{k}}$-increasing continuous, $\delta$ a limit ordinal.
- (b) If $p \in \mathbf{D}(M_i)$ is realized in $M_{i+1}$ then it is a member of $\mathbf{D}_\alpha(M_i)$ (or just $p \restriction M_0 \in \mathbf{D}(M_0)$).
- (c) If $i < \delta$ and $p \in \mathbf{D}_\alpha(M_i)$, then $p$ is materialized in $M_j$ for some $j \in (i, \delta)$.

Then $M_\delta$ is $(\mathbf{D}_\alpha(M_0), \aleph_0)^*$-homogeneous.

*Proof.* Easy. $\square_{5.47}$

**Discussion 5.48.** 1) Consider $\psi \in \mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$, $|\tau_\psi| \leq \aleph_0$, and $\dot{I}(\aleph_1, \psi) \in [1, 2^{\aleph_0})$. We translate it to $\mathfrak{k}$ and $<^{**}$ as earlier (see 3.19).

2) What if we waive categoricity in $\aleph_0$? Adopting this was okay, as we shrink $\mathfrak{k}$ but not too much. But without shrinking probably we still can say something on the models in

$$\mathfrak{k}^* := \left\{ M \in \mathfrak{k}_{\geq \aleph_0} : \text{if } N_0 \leq_{\mathfrak{k}} M,\ N_0 \in K_{\aleph_0} \text{ then } (\exists N_1)[N_0 <^* N_1 \leq_{\mathfrak{k}} M] \right\}$$

as there are good enough approximations.

## § 6. Counterexamples

In [She75a] the statement of Conclusion 3.9 was proved for the first time, where $K$ is the class of atomic models of a first order theory assuming Jensen's diamond $\diamondsuit_{\aleph_1}$ (taking $\lambda = \aleph_0$). In [She83a] and [She83b] the same theorem was proved using only $2^{\aleph_0} < 2^{\aleph_1}$ (using 0.6). Let us now concentrate on the case $\lambda = \aleph_0$. We asked whether the assumption $2^{\aleph_0} < 2^{\aleph_1}$ is necessary to get Conclusion 3.9. In this section we construct [four] classes of models $K^1, K^2, K^3, K^4$ failing amalgamation (i.e. failing the conclusion of 3.9). $K^2, K^3, K^4$ are AECs with LST-number $\aleph_0$ while $K^1$ satisfies all the axioms needed in the proof of Conclusion 3.9 (but it is not an abstract elementary class — it fails to satisfy **Axs. IV,V**).

$K^2$ is $\mathsf{PC}_{\aleph_0}$ and is axiomatizable in $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$.

$K^3$ is $\mathsf{PC}_{\aleph_0}$ and is axiomatizable in $\mathbb{L}(\mathbf{Q})$. Now the common phenomena to $K^1, K^2, K^3, K^4$ are that all of them satisfy the hypothesis of Conclusion 3.9; i.e. for $\ell = 1, 2, 3$ we have $\dot{I}(\aleph_0, K^\ell) = 1$ and the $\aleph_0$-amalgamation property fails in $K^\ell$, but assuming $\aleph_1 < 2^{\aleph_0}$ and $\mathsf{MA}_{\aleph_1}$ for $\ell = 1, 2, 3$ we have $\dot{I}(\aleph_1, K^\ell) = 1$.

**Definition 6.1.** Let $Y$ be an infinite set. For ease of notation, if $X \subseteq Y$ then we will denote $X^0 := X$ and $X^1 := Y \setminus X$.

A family $\mathscr{P}$ of infinite subsets of $Y$ is called *independent* if for every $\eta \in {}^{\omega>}2$ and pairwise distinct $X_0, X_1, \ldots, X_{\ell g(\eta)-1}$, the following set $\bigcap\limits_{k<\ell g(\eta)} X_k^{\eta[k]}$ is infinite.

**Definition 6.2.** 1) The class of models $K^0$ is defined by

$$\mathbb{P} = \big\{ f : f \text{ is a partial finite isomorphism from } M \text{ into } N \text{ satisfying} \\ (\forall \alpha < \omega_1)\big(\forall x \in \mathrm{dom}(f)\big)[x \in M_\alpha \Leftrightarrow f(x) \in N_\alpha]\big\},$$

2) For $M \in K^0$, let $A^M_y := \{x \in P^M : x\, R^M y\}$ for every $y \in Q^M$.

3) Let $K^1$ be the class of $M \in K^0$ such that

(a) The family $\{A^M_y : y \in Q^M\}$ is independent, which means that if $m < n$ and $y_0, \ldots, y_{n-1}$ are pairwise distinct members of $Q^M$, then the set
$$\big\{x \in P^M : x\, R^M y_\ell \equiv \ell < m \text{ for every } \ell < n\big\}$$
is infinite.

(b) For all disjoint finite subsets $u, w$ of $P^M$ we have $\|M\| = |A^M_{u,w}|$, where
$$A^M_{u,w} := \big\{y \in Q^M : a \in u \Rightarrow a\, R^M y, \text{ and } b \in w \Rightarrow \neg(b\, R^M\, y)\big\}.$$

4) The notion of (strict) substructure, denoted $\leq_{\mathfrak{k}^1}$, is defined as follows.

For $M_1, M_2 \in K^1$, $M_1 \leq_{\mathfrak{k}^1} M_2$ <u>iff</u> $M_1 \subseteq M_2$, $P^{M_1} = P^{M_2}$, and if $M_1 \neq M_2$ then for any finite disjoint $u, w \subseteq P^{M_2}$ the set $A^{M_2}_{u,w} \setminus M_1$ is infinite (equivalently, 'non-empty').

5) $\mathfrak{k}^1 = (K^1, \leq_{\mathfrak{k}^1})$.

**Lemma 6.3.** *The class* $(K^1, <_{\mathfrak{k}^1})$ *satisfies*

0) ***Ax.0***.

1) ***Ax.I***.

2) ***Ax.II***.

3) ***Ax.III***.

4) ***Ax.IV*** *fails even for* $\lambda = \aleph_0$*; but if* $\langle M_\alpha : \alpha \leq \delta \rangle$ *is* $\leq_{\mathfrak{k}}$*-increasing and*

$$\Big\| \bigcup_{\alpha<\delta} M_\alpha \Big\| < \|M_\delta\|$$

*then* $\bigcup\limits_{\alpha<\delta} M_\alpha <_{\mathfrak{k}^1} M_\delta$.

5) ***Ax.V*** *fails for countable models.*

6) ***Ax.VI*** *holds with* $\mathrm{LST}(\mathfrak{k}^1) = \aleph_0$*; in fact, it holds for every cardinal.*

7) *For every* $M \in K^1$, $\|M\| \leq 2^{\aleph_0}$.

*Proof.* 0-2) Follows trivially from the definition.

3) To prove that $M := \bigcup\limits_{i<\lambda} M_i \in K^1$, it is enough to verify that for every finite disjoint $u, w \subseteq P^M$, $|A^M_{u,w}| = \|M\|$. If $\langle M_i : i < \lambda \rangle$ is eventually constant we are done; hence without loss of generality $\langle M_i : i < \lambda \rangle$ is $<_{\mathfrak{k}^1}$-increasing. From the definition of $<_{\mathfrak{k}^1}$ it follows that for each $i$, $M_{i+1}$ has a new $y = y_i$ as above; i.e. $y_i \in A^{M_{i+1}}_{u,w} \setminus M_i$ for every $i < \lambda$. Also. for each $i$ there are at least $\|M_i\|$-many members in $A^{M_i}_{u,w} \subseteq A^M_{u,w}$. Together there are at least $\|M\|$ members in $A^M_{u,w}$.

4) Let $\{M_n : n < \omega\} \subseteq K^1_{\aleph_0}$ be an $<_{\mathfrak{k}^1}$-increasing chain and let $M := \bigcup\limits_{n<\omega} M_n$; by part (3) we have $M \in K^1_{\aleph_0}$. Since $|Q^M| = \aleph_0$ by Claim 6.5(a) below, there exists an infinite $A \subseteq P^M \setminus \{A^M_y : y \in Q^M\}$ such that $\{A_y : y \in Q^M\} \cup \{A\}$ is independent. Now define $N \in K^1$ by $P^N := P^M$, let $y_0 \notin M$ and take $Q^N := Q^M \cup \{y_0\}$, and finally let

$$R^N := R^M \cup \big\{ \langle a, y_0 \rangle : a \in P^N \wedge a \in A \big\}.$$

Clearly $M_n \leq_{\mathfrak{k}^1} N$ for every $n < \omega$, but $N$ is not an $\leq_{\mathfrak{k}^1}$-extension of $M = \bigcup\limits_{n<\omega} M_n$ because the second part in Definition 6.2(4) is violated.

5) Let $N_0 <_{\mathfrak{k}^1} N \in K^1$ be given. As in (4), define $N_1 \subseteq N$, $|N_1| \supseteq |N_0|$ by adding a single element to $Q^{N_0}$ (from the elements of $Q^N \setminus Q^{N_0}$). It is obvious that $N_0 \leq_{\mathfrak{k}^1} N$ and $N_1 \leq_{\mathfrak{k}^1} N$ but $N_0 \not\leq_{\mathfrak{k}^1} N_1$.

6) By closing the set under the second requirement in Definition 6.2(3).

7) Let $y_1 \neq y_2 \in Q^M$; we show that $A^M_{y_1} \neq A^M_{y_2}$. If $A^M_{y_1} \subseteq A^M_{y_2}$ then

$$A^M_{y_1} \cap (P^M \setminus A^M_{y_2}) = \varnothing,$$

in contradiction to the requirement that $\{A_y : y \in Q\}$ is independent. Hence $|Q^M| \leq 2^{|P^M|} = 2^{\aleph_0}$, and as $|P^M| = \aleph_0$ we are done. $\square_{6.3}$

**Theorem 6.4.** $\mathfrak{k}^1 = (K^1, <_{\mathfrak{k}^1})$ *satisfies the hypothesis of Conclusion 3.9. Namely*

1) $\dot{I}(\aleph_0, K^1) = 1$.

2) *Every* $M \in K^1_{\aleph_0}$ *has a proper* $\leq_{\mathfrak{k}^1}$*-extension in* $K^1_{\aleph_0}$.

3) $\mathfrak{k}^1$ *is closed under chains of length* $\leq \omega_1$.

4) $\mathfrak{k}^1$ *fails the* $\aleph_0$*-amalgamation property.*

*Proof.* 1) Let $M_1, M_2 \in K^1_{\aleph_0}$, pick the following enumerations $|M_1| = \{a_n : n < \omega\}$ and $|M_2| = \{b_n : n < \omega\}$. It is enough to define an increasing sequence of finite partial isomorphisms $\langle f_n : n < \omega\rangle$ from $M_1$ to $M_2$ such that for every $k < \omega$, for some $n(k) < \omega$, $a_k \in \mathrm{dom}(f_{n(k)})$ and $b_k \in \mathrm{rang}(f_{n(k)})$. Finally take $f := \bigcup_{n<\omega} f_n$, and this will be an isomorphism from $M_1$ onto $M_2$.

Define the sequence $\langle f_n : n < \omega\rangle$ by induction on $n < \omega$.

First, $f_0 := \varnothing$. If $n = 2m$ denote $k := \min\{k < \omega : a_k \notin \mathrm{dom}(f_n)\}$. Distinguish between the following two alternatives:

(A) If $a_k \in P^{M_1}$ let $\{a'_0, \dots, a'_{j-1}\} = Q^{M_1} \cap \mathrm{dom}(f_n)$. Without loss of generality there exists $i \leq j-1$ such that $a_k \, R^{M_1} a'_\ell$ for all $\ell < i$ and $\neg(a_k \, R \, a'_\ell)$ for all $i \leq \ell \leq j-1$. By 6.2(1), $P^{M_\ell}$ is infinite, hence by clause (b) of 6.2(2) $Q^{M_\ell}$ is also infinite. Hence by 6.2(3)(a) there are infinitely many $y \in P^{M_2}$ such that $y \, R^{M_2} f_n(a'_\ell)$ for all $\ell < i$ and $\neg(y \, R^{M_2} f_n(a'_\ell))$ for all $i \leq \ell < j-1$. But $\mathrm{rang}(f_n)$ is finite. Hence there is such $y \in P^{M_2} \setminus \mathrm{rang}(f_n)$. Finally, let $f_{n+1} := f_n \cup \{\langle a_k, y\rangle\}$.

(B) If $a_k \in Q^{M_1}$ let $\{a'_0, \dots, a'_{j-1}\} = P^{M_1} \cap \mathrm{dom}(f_n)$. As before we may assume that there exists $i \leq j-1$ such that $a'_\ell \, R^{M_1} a_k$ for all $\ell < i$ and $\neg(a'_\ell \, R^{M_1} a_k)$ for all $i \leq \ell < j-1$. By 6.2(3)(b) there exists $y \in Q^{M_2} \setminus \mathrm{dom}(f_n)$ such that $(\forall \ell < i)\big[f_n(a'_\ell) \, R^{M_2} \, y\big]$ and

$$\big(\forall \ell \in [i, j-1)\big) \, \neg\big[f_n(a'_\ell) \, R^{M_2} \, y\big].$$

Now define $f_{n+1} := f_n \cup \{\langle a_k, y\rangle\}$.

[$m$ **isn't used anywhere.]** $\square_{(1)}$

2) First we prove the following.

**Observation 6.5.** (a) *Let* $P$ *be a countable set. For every countable family* $\mathscr{P}$ *of infinite subsets of* $P$, *<u>if</u>* $\mathscr{P}$ *is independent <u>then</u> there exists an infinite* $A \subseteq P$ *such that* $A \notin \mathscr{P}$ *and* $\mathscr{P} \cup \{A\}$ *is independent.*

(b) *If* $A$ *and* $\mathscr{P}$ *are as in (a) <u>then</u> for every infinite* $B \subseteq P$ *satisfying*

$$|A \,\Delta\, B| < \aleph_0$$

*and* $B \notin \mathscr{P}$, $\mathscr{P} \cup \{B\}$ *is also independent.*

(c) *Moreover, in clause (a) we can additionally require that for any finite disjoint* $u, v \subseteq P$ *there exists* $A \subseteq P$ *as in (a) satisfying* $u \subseteq A$ *and* $A \cap v = \varnothing$.

*Proof.* **[Proof of Claim 6.5:]**

**Clause (a):** Let

$$\mathscr{P}^* := \Big\{X \subseteq P : (\exists n < \omega)(\exists X_0, \dots, X_{n-1} \in \mathscr{P})(\exists k \leq n) \\ \big[X \text{ or } P \setminus X \text{ is equal to } \bigcap_{i<k} X_i \cap \bigcap_{k \leq i < n} (P \setminus X_i)\big]\Big\}.$$

Clearly $|\mathscr{P}^*| = \aleph_0$, hence we can list them in a sequence $\langle A_n : n < \omega\rangle$ **[(where each set is repeated infinitely often)]** such that for every $k < \omega$ there exists $n > k$ satisfying $A_n = A_k$ (hence for some $m > k$, $A_m = P \setminus A_k$).

Let $P = \{a_n : n < \omega\}$ without repetition.

Now define $i(n) < \omega$ by induction on $n$. Let $i(0) = 0$.

If $n = k + 1$, let

$$i(n) := \min\big\{\ell < \omega : i(n-1) < \ell \text{ and } a_\ell \in (A_k \setminus \{a_{i(0)}, \dots, a_{i(n-1)}\}\big\}.$$

It is easy to verify that the construction is possible. Directly from the construction it follows that $A = \{a_{i(n)} : n < \omega\}$ is a set as required.

**Clause (b):** Easy.

**Clause (c):** Let $u, w \subseteq P$ be finite disjoint and $\mathscr{P}$ a countable family of subsets of $P$ which is independent.

Let $A' \subseteq P$ be as proved in clause (a). According to (b), $A = (A' \cup u) \setminus w$ also satisfies 'the family $\mathscr{P} \cup \{A\}$ is independent.' $\square_{6.5}$

*Proof.* **[Return to the proof of Theorem 6.4(2):]**

Let $\mathscr{P} := \{A^M_y \subseteq P^M : y \in Q^M\}$. Let $\langle s_n : n < \omega\rangle$ be an enumeration of $[P^M]^{<\aleph_0}$ (with repetition) such that $s_{2k} \cap s_{2k+1} = \varnothing$ for each $k < \omega$, and for every finite disjoint $u, w \subseteq P^M$ there exists $n < \omega$ such that $s_{2n} = u$ and $s_{2n+1} = w$.

It is enough to define an increasing chain $\{\mathscr{P}_n : n < \omega\}$ of countable independent families of subsets of $P^M$ such that $\mathscr{P}_0 = \mathscr{P}$ and for all $k < \omega$ and every finite disjoint $u, w \subseteq P^M$,

$$(\exists n < \omega)(\exists A \in \mathscr{P}_n \setminus \mathscr{P}_k)[u \subseteq A \wedge A \cap w = \varnothing]$$

because $\bigcup_{n<\omega} \mathscr{P}_n$ enables us to define $N \in K^1_{\aleph_0}$ such that $M \leq_{\mathfrak{k}^1} N$ as required. Assume $\mathscr{P}_n$ is defined; apply Claim 6.5(c) on $P = P^M$ and $\mathscr{P}_n$ when substituting $u = s_{2n}, w = s_{2n+1}$ let $A \subseteq P$ be supplied by the Claim and define $\mathscr{P}_{n+1} := \mathscr{P}_n \cup \{A\}$. It is easy to check that $\{\mathscr{P}_n : n < \omega\}$ satisfies our requirements.

3) This is a special case of **Ax.III** which we checked in Lemma 6.3(3).

4) Let $M \in K^1_{\aleph_0}$, and we shall find $M_\ell \in K^1_{\aleph_0}$ (for $\ell = 0, 1$) with $M \leq_{\mathfrak{k}^1} M_\ell$, which cannot be amalgamated over $M$. By part (2) we can find a model $M_1$ such that $M <_{\mathfrak{k}^1} M_1 \in K^1_{\aleph_0}$, and choose $y \in Q^{M_1} \setminus Q^M$. Define $M_2 \in K^1_{\aleph_0}$; its universe is $|M_1|$, $P^{M_2} := P^{M_1}$, $Q^{M_2} := Q^{M_1}$, and

$$R^{M_2} := \big\{(a, b) : a\, R^{M_1}\, b \wedge b \neq y \ \underline{\text{or}}\ a \in P^M \wedge b = y \wedge \neg(a\, R\, y)\big\}.$$

Clearly $M_1, M_2$ cannot be amalgamated over $M$ (since the amalgamation must contain a set and its complement). $\square_{6.4(2)\text{-}(4)}$

**Theorem 6.6.** *Assume* $\mathsf{MA}_{\aleph_1}$ *(hence* $2^{\aleph_0} > \aleph_1$*). The class* $(K^1, <_{\mathfrak{k}^1})$ *is categorical in* $\aleph_1$.

*Proof.* Let $M, N \in K^1_{\aleph_1}$ and we shall prove that they are isomorphic. By repeated use of Lemma 6.3(6),(4) for **Ax.VI** we get (strictly) $<_{\mathfrak{k}^1}$-increasing continuous chains $\{M_\alpha : \alpha < \omega_1\}, \{N_\alpha : \alpha < \omega_1\} \subseteq K^1_{\aleph_0}$ such that $M = \bigcup_{\alpha<\omega_1} M_\alpha$ and $N = \bigcup_{\alpha<\omega_1} N_\alpha$ (so $M_\alpha <_{\mathfrak{k}^1} M_\beta$ and $N_\alpha <_{\mathfrak{k}^1} N_\beta$ for $\alpha < \beta$).

Now define a forcing notion which supplies an isomorphism $g : M \to N$.

$$\begin{aligned}\mathbb{P} := \big\{ f : {} & f \text{ is a partial finite isomorphism from } M \text{ into } N \text{ satisfying} \\ & (\forall \alpha < \omega_1)\big(\forall x \in \operatorname{dom}(f)\big)[x \in M_\alpha \Leftrightarrow f(x) \in N_\alpha]\big\}\end{aligned}$$

The order is inclusion. It is trivial to check that if $G \subseteq \mathbb{P}$ is a directed subset then $g = \bigcup G$ is a partial isomorphism from $M$ to $N$. We show that $\operatorname{dom}(g) = |M|$ if $G$ is generic enough.

For every $a \in |M|$ define $\mathscr{J}_a = \big\{ f \in \mathbb{P} : a \in \operatorname{dom}(f) \big\}$, and we shall show that for all $a \in |M|$ the set $\mathscr{J}_a$ is dense. For $a \in M$ let

$$\alpha(a) := \min\{\alpha < \omega_1 : a \in M_\alpha\}.$$

Clearly it is zero or a successor ordinal. Let $f \in \mathbb{P}$ be a given condition; it is enough to find $h \in \mathscr{J}_a$ such that $f \subseteq h$ and $a \in \operatorname{dom}(h)$. Let $A := \operatorname{dom}(f)$ and let $B, C \subseteq A$ be disjoint sets such that $B \cup C = A$, $B = \operatorname{dom}(f) \cap P^M$, and $C = \operatorname{dom}(f) \cap Q^M$. Without loss of generality $a \notin B \cup C$. If $a \in P^M$ let

$$\varphi(x, \bar{c}) = \bigwedge \big\{ \pm\, x\, R\, c : c \in C,\ M \models \pm a\, R\, c \big\}.$$

From the definition of $K^1$ there exists $b \in P^N \setminus \operatorname{rang}(f)$ such that $N \models \varphi[b, f(\bar{c})]$. If $a \in Q^M$ let $\varphi(x, \bar{b}) := \bigwedge\{\pm b\, R\, x : b \in B,\ M \models \pm b\, R\, a\}$. We can find infinitely many $b \in Q^{N_{\alpha(a)}} \setminus \bigcup_{\beta<\alpha(a)} N_\beta$ satisfying $\varphi(x, f(\bar{b}))$.

Why? This is as $\bigcup\{N_\beta : \beta < \alpha(a)\} <_{\mathfrak{k}^1} N_{\alpha(a)}$ as $C$ is finite. Without loss of generality $b \notin f(C)$.

Finally, let $h = f \cup \{\langle a, b\rangle\}$.

The proof that $\operatorname{rang}(g) = |N|$ is analogous to the proof that $\operatorname{dom}(g) = |M|$. In order to use $\mathsf{MA}$ we just have to show that $R$ has the ccc. Let $\{f_\alpha : \alpha < \omega_1\} \subseteq R$ be given. It is enough to find $\alpha, \beta < \omega_1$ such that $f_\alpha, f_\beta$ have a common extension. Without loss of generality we may assume $|M| \cap |N| = \varnothing$. By the finitary $\Delta$-system lemma there exists $S \subseteq \omega_1$ with $|S| = \aleph_1$ such that $\{\operatorname{dom}(f_\alpha) \cup \operatorname{rang}(f_\alpha) : \alpha \in S\}$ is a $\Delta$-system with heart $A$. Let $B \subseteq |M|$, $C \subseteq |N|$ be such that $A = B \cup C$. Now without loss of generality, for every $\alpha \in S$, $f_\alpha$ maps $B$ into $C$.

[Why? If not,

$$S_1 := \big\{\alpha \in S : (\exists b_\alpha \in B)[f_\alpha(b_\alpha) \notin C]\big\}$$

is uncountable hence for some $b \in B$, $S_2 := \{\alpha \in S_1 : b_\alpha = b\}$ is uncountable; so $\langle f_\alpha(b) : \alpha \in S_2\rangle$ is without repetitions hence is uncountable. But

$$\{f(b) : f \in \mathbb{P} \text{ and } b \in \operatorname{dom}(f) \cap B\}$$

is countable because

$$f \in \mathbb{P} \wedge b \in \operatorname{dom}(f) \wedge \alpha < \omega_1 \Rightarrow [b \in M_\alpha \Leftrightarrow f(b) \in N_\alpha].$$

Similarly, $f_\alpha^{-1}$ maps $C$ into $B$, so necessarily $f_\alpha$ maps $B$ onto $C$; but the number of possible functions from $B$ to $C$ is $|C|^{|B|} < \aleph_0$. Hence there exists $S_1 \subseteq S$ with $|S_1| = \aleph_1$ such that for all $\alpha, \beta \in S_1$, $f_\alpha \restriction B = f_\beta \restriction B$. $\mathrm{dom}(f_\alpha) \cap M_0 \subseteq B$, and $\mathrm{rang}(f_\alpha) \cap N_0 \subseteq C$. As $P^{M_\alpha} = P^{M_0} \subseteq M_0$ and $P^{N_\alpha} = P^{N_0} \subseteq N_0$ for every $\alpha \in S_1$, we have $P^M \cap \mathrm{dom}(f_\alpha) \subseteq B$ and $P^N \cap \mathrm{rang}(f_\alpha) \subseteq C$. Therefore $f_\alpha \cup f_\beta \in \mathbb{P}$ for all $\alpha, \beta \in S_1$, and in particular there exists $\alpha \neq \beta < \omega_1$ such that $f_\alpha \cup f_\beta \in \mathbb{P}$.] $\square_{6.6}$

In the terminology of [GS83], Theorems 6.4 and 6.6 give us together:

**Conclusion 6.7.** *Assuming $2^{\aleph_0} > \aleph_1$ and $\mathsf{MA}_{\aleph_1}$, $\mathfrak{k}^1$ is a nice category which has a universal object in $\aleph_1$. Moreover, it is categorical in $\aleph_1$.*

**Definition 6.8.** 1) $K^2$ is the class of $M \in K^0$ (see Definition 6.2) satisfying:

- (a) $(\forall x \in Q^M)(\forall u \in [P^M]^{<\aleph_0})(\exists y \in Q)[A_x^M \,\Delta\, A_y^M = u]$
- (b) If $k < \omega$ and $y_0, \ldots, y_{k-1} \in Q$ satisfies $|A_{y_\ell} \,\Delta\, A_{y_m}| \geq \aleph_0$ for $\ell < m < k$ <u>then</u> the set $\{A_{y_\ell}^M : \ell < k\}$ is an independent family of subsets of $P^M$.
- (c) $Q(y) \wedge Q(z) \wedge (\forall x \in P)[x\,R\,y \Leftrightarrow x\,R\,z] \Rightarrow y = z$
- (d) For every $k < \omega$, for some $y_0, \ldots, y_k \in Q^M$, we have
$$\bigwedge_{\ell < m \leq k} \Big[|A_{y_\ell} \,\Delta\, A_{y_m}| \geq \aleph_0\Big].$$

2) For $M_1, M_2 \in K^2$,
$$M_1 \leq_{\mathfrak{k}^2} M_2 \Leftrightarrow^{\mathrm{df}} M_1 \subseteq M_2 \wedge P^{M_1} = P^{M_2}.$$

3) $\mathfrak{k}^2 = (K^2, \leq_{\mathfrak{k}^2})$.

4) $K^3$ is the class of models $M = \big(|M|, P^M, Q^M, R^M, E^M\big)$ such that

- (a) $(|M|, P^M, Q^M, R^M) \in K^1$
- (b) $E^M$ is an equivalence relation on $Q^M$.
- (c) $E^M$ has infinitely many equivalence classes.
- (d) Each equivalence class of $E^M$ is countable.
- (e) If $u, w \subseteq P^M$ are finite disjoint and $y \in Q^M$, then for some $y' \in y/E^M$ we have $a \in u \Rightarrow a\,R^M y'$ and $b \in w \Rightarrow \neg(b\,R^M y')$.

5) We define $\leq_{\mathfrak{k}^3}$ as follows:
$$M_1 \leq_{\mathfrak{k}^3} M_2 \Leftrightarrow^{\mathrm{df}} M_1 \subseteq M_2 \wedge (\forall a \in M_1)[a/E^{M_2} = a/E^{M_1}].$$

6) $\mathfrak{k}^3 = (K^3, \leq_{\mathfrak{k}^3})$.

If we would like to have a class defined by a sentence from $\mathbb{L}_{\omega_1,\omega}$ (rather than $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$), we can use an alternative.

**Definition 6.9.** 1) $\mathfrak{k}^4$ is defined as follows:

- (A) $\tau(\mathfrak{k}^4) = \{P, Q, R\} \cup \{P_n : n < \omega\}$, $R$ is a two-place predicate, and $P, Q, P_n$ are unary predicates.
- (B) $M \in K^4$ <u>iff</u> $M$ is a $\tau(\mathfrak{k}^4)$-model such that $M \restriction \{P, Q, R\} \in K^2$ and
  - (a) $\langle P_n^M : n < \omega\rangle$ is a partition of $P^M$.

(b) $P_n^M$ has exactly $2^n$ elements.

(c) $(\forall x \in Q)(\forall u \in [P^M]^{<\aleph_0})(\exists y \in Q^M)[A_x^M \,\Delta\, A_y^M = u]$

(d) If $k < \omega$ and $y_0, \ldots, y_{k-1} \in Q$ satisfies $|A_{y_\ell} \,\Delta\, A_{y_m}| \geq \aleph_0$ for $\ell < m < k$ <u>then</u> the set $\{A_{y_\ell}^M : \ell < k\}$ is an independent family of subsets of $P^M$. Moreover, for any $n$ large enough and any $\eta \in {}^k 2$, the set

$$P_n^M \cap \bigcap_{\eta(\ell)=1} A_{y_\ell}^M \setminus \bigcup_{\eta(\ell)=0} A_{y_\ell}^M$$

has exactly $2^{n-k}$ elements.

(e) $Q^M(y) \wedge Q^M(z) \wedge (\forall x \in P^M)[x\, R^M y \Leftrightarrow x\, R^M z] \Rightarrow y = z$

(f) For every $k < \omega$, for some $y_0, \ldots, y_k \in Q^M$, we have

$$\bigwedge_{\ell < m \leq k} \Big[|A_{y_\ell} \,\Delta\, A_{y_m}| \geq \aleph_0\Big].$$

(C) $M \leq_{\mathfrak{k}^4} N$ iff $M, N \in K^4$ and $M \subseteq N$ and $P^M = P^N$.

**Theorem 6.10.** 1) $(K^2, <_{\mathfrak{k}^2})$ *is an* $\aleph_0$*-presentable abstract elementary class which is categorical in* $\aleph_0$*.*

2) *Also,* $\mathfrak{k}^3$ *and* $\mathfrak{k}^4$ *are* $\aleph_0$*-presentable AECs categorical in* $\aleph_0$*.*

*Proof.* Similar to the proof for $\mathfrak{k}^1$. $\square_{6.10}$

**Theorem 6.11.** 1) $\mathfrak{k}^1_{\aleph_1}$ *has an axiomatization in* $\mathbb{L}(\mathbf{Q})$ *and* $\leq_{\mathfrak{k}^1}$ *is* $<^{**}$ *from the proof of* 3.19 *(this is* $<^{**}$ *from* [She83a] *and* [She83b]*).*

2) $\mathfrak{k}^2$ *has an axiomatization in* $\mathbb{L}_{\omega_1,\omega}(\mathbf{Q})$ *and* $\leq_{\mathfrak{k}^2}$ *is* $\leq^*$ *from the proof of* 3.19 *(this is* $<^*_{\omega_1,\omega}$ *from* [She83a] *and* [She83b]*).*

3) $\mathfrak{k}^3$ *has an axiomatization in* $\mathbb{L}(\mathbf{Q})$ *and* $\leq_{\mathfrak{k}^3}$ *is* $<^*$ *from* [She83a] *and* [She83b].

4) $\mathfrak{k}^4$ *has an axiomatization in* $\mathbb{L}_{\omega_1,\omega}$ *and* $\leq_{\mathfrak{k}^4}$ *is just being a submodel.*

5) $(\forall \ell \in \{1,2,3,4\})[K^\ell$ *is* $\mathsf{PC}_{\aleph_0}]$.

*Proof.* Should be clear. $\square_{6.11}$

**Theorem 6.12.** *If* $\mathsf{MA}_{\aleph_1}$ <u>*then*</u> $K^\ell$ *is categorical in* $\aleph_1$ *for* $\ell = 2, 3$*.*

*Proof.* Easy.[25] $\square_{6.12}$

**Conclusion 6.13.** *Assuming* $\mathsf{MA}_{\aleph_1}$*, there exists an abstract elementary class which is* $\mathsf{PC}_{\aleph_0}$*, categorical in* $\aleph_0$ *and* $\aleph_1$*, but without the* $\aleph_0$*-amalgamation property.*

[25] In the earlier version this was claimed also for $\ell = 4$, but, as Baldwin noted, this was wrong

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