REVISED VERSION OF NONTRIVIAL AUTOMORPHISMS OF $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ FROM VARIANTS OF SMALL DOMINATING NUMBER

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ABSTRACT. This version provides details to fill a gap in a previous version of this article. It is shown that if various cardinal invariants of the continuum related to $\mathfrak{d}$ are equal to $\aleph_1$ then there is a nontrivial automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$. Some of these results extend to automorphisms of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ if κ is inaccessible.

1. Introduction

This is a revised version of [14]. The revision is required to address a gap in the proof of Lemma 3.2 of [14] in which it is claimed that a "standard diagonalization" yields the limit case. Enormous thanks are owed to the long suffering referee for carefully reading various early versions of this revision. While the general structure of the proof is the same, the use of the Lemma 3.1 now allows the argument to be presented without gaps, although adding the existence of square sequences as a hypothesis. A byproduct of this reorganization is that Lemma 3.3 now applies to both cases, κ inaccessible or ω. This improvement has allowed the proof of Lemma 3.3, which deals with the important case $\kappa = \omega$ to be considerably simplified.

A fundamental result in the study of the Čech–Stone compactification, due to W. Rudin [8, 9], is that, assuming the Continuum Hypothesis, there are $2^{\mathfrak{c}}$ autohomeomorphisms of $\beta\mathbb{N} \setminus \mathbb{N}$ and, hence, there are some that are non-trivial in the sense that they are not induced by any one-to-one function on $\mathbb{N}$. While Rudin established his result by showing that for any two P-points of weight $\aleph_1$ there is an autohomeomorphism sending one to the other, Parovičenko [7] showed that non-trivial autohomeomorphisms could be found by exploiting the countable saturation of the Boolean algebra of clopen subsets of $\beta\mathbb{N} \setminus \mathbb{N}$ — this is isomorphic to the algebra $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$. Indeed, the duality between Stone spaces of Boolean algebras and algebras of regular open sets shows that the existence of non-trivial autohomeomorphisms of $\beta\mathbb{N}\setminus\mathbb{N}$ is equivalent to the existence of non-trivial isomorphisms of the Boolean algebra $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ to itself.

Notation 1.1. If A and B are subsets of κ let $\equiv_\kappa$ denote the equivalence relation defined by $A \equiv_\kappa B$ if and only if $|A \triangle B| < \kappa$ and $A \subseteq_\kappa B$ will denote the assertion that $|A \setminus B| < \kappa$. Let $[A]_\kappa$ denote the equivalence class of A modulo $\equiv_\kappa$ and let $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ denote the quotient algebra of the $\mathcal{P}(\kappa)$ modulo the congruence relation $\equiv_\kappa$. If $\kappa = \omega$ it is customary to use $\equiv^*$ instead of $\equiv_\omega$ and $\subseteq^*$ instead of $\subseteq_\omega$.

Notation 1.2. If f is a function defined on the set A and $X \subseteq A$ then the notation $f(X)$ will be used to denote $\{f(x) \mid x \in X\}$ in spite of the potential for ambiguity.

Definition 1.1. An isomorphism $\Phi : \mathcal{P}(\kappa)/[\kappa]^{<\kappa} \to \mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ will be said to be *somewhere trivial* if there is some $B \in [\kappa]^\kappa$ and a one-to-one function $\varphi : B \to \kappa$ such that $\Phi([A]_\kappa) = [\varphi(A)]_\kappa$ for each $A \subseteq B$. The isomorphism Φ will be said to be *trivial* if $|\kappa \setminus B| < \kappa$ and Φ will be said to be *nowhere trivial* if it is not somewhere trivial.

The question of whether the Continuum Hypothesis, or some other hypothesis, is needed in order to find a non-trivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ to itself was settled in the affirmative by S. Shelah in [10]. The argument of [10] relies on an iterated oracle chain condition forcing to obtain a model where $2^{\aleph_0} = \aleph_2$ and every isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ to itself is induced by a one-to-one function from $\mathbb{N}$ to $\mathbb{N}$. The oracle chain condition requires the addition of cofinally many Cohen reals and so $\mathfrak{d} = \aleph_2$ in this model. Subsequent work has shown that it is also possible to obtain that every isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ is trivial by other approaches [16, 11, 2] but these have always required $\mathfrak{d} > \aleph_1$ as well. However, it was shown in Theorem 3.1 of [13] that this cardinal inequality is not entailed by the non existence of nowhere trivial isomorphisms from $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ to itself — in the model obtained by iterating ω_2 times Sacks reals there are no nowhere trivial isomorphisms yet $\mathfrak{d} = \aleph_1$; but there are non-trivial automorphisms in this model. The fact that it is possible to have a non-trivial automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ while also having that every automorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ is somewhere trivial is Theorem 2.2 of [12].

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On the other hand, while we now know that the Continuum Hypothesis cannot be completely eliminated from Rudin's result, perhaps it can be weakened to some other cardinal equality such as $\mathfrak{d} = \aleph_1$. It will be shown in this article that non-trivial isomorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ to itself can indeed be constructed from hypotheses on cardinal arithmetic weaker than $2^{\aleph_0} = \aleph_1$ and reminiscent of $\mathfrak{d} = \aleph_1$. However, it is shown in [3] that it is consistent with set theory that $\mathfrak{d} = \aleph_1$ yet all isomorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ are trivial so some modification of the equality $\mathfrak{d} = \aleph_1$ will be required.

It will also be shown that natural generalizations of the arguments can be applied to the same question for $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ where κ is inaccessible and $\square_\kappa$ holds . The chief interest here is that, unlike $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$, the algebra $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ is not countably saturated if $\kappa > \omega$— to see this, simply consider a family $\{A_n\}_{n\in\omega} \subseteq [\kappa]^\kappa$ such that $\bigcap_{n\in\omega} A_n = \emptyset$. In other words, Parovičenko's transfinite induction argument to construct non-trivial isomorphisms from $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ to itself is not available and some other technique is needed.

The statement and proof of Lemma 2.1 is provided for all κ and will apply both to the case that $\kappa = \omega$ and to the case that κ is inaccessible. However, the key pigeonhole argument in the case that κ is inaccessible requires a different, somewhat simpler, hypothesis than the case $\kappa = \omega$, which relies on some technical details not needed in the inaccessible case.

2. A sufficient condition for a non-trivial isomorphism

The following lemma provides sufficient conditions for the existence of a nontrivial isomorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ to itself. The set theoretic requirements for the satisfaction of these conditions will be examined later. The basic idea of the lemma is that an isomorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ can be approximated by partitioning κ into small sets I_ν and constructing isomorphisms from subalgebras of $\mathcal{P}(I_\nu)$ and taking the union of these. Unless the subalgebras of $\mathcal{P}(I_\nu)$ are all of $\mathcal{P}(I_\nu)$, this union will only be a partial isomorphism. Hence a κ^+ length sequence of ever larger families of subalgebras of $\mathcal{P}(I_\nu)$ is needed to obtain a full isomorphism. In order to guarantee that this isomorphism is not trivial, the prediction principles described in Hypothesis (4) and Hypothesis (5) of Lemma 2.1 are needed.

Lemma 2.1. *There is a non-trivial automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$ provided that κ is regular and there are $\{I_\nu\}_{\nu\in\kappa}$, $\{\mathfrak{B}_{\xi,\nu}\}_{\xi\in\kappa^+,\nu\in\kappa}$ and $\{\Phi_{\xi,\nu}\}_{\xi\in\kappa^+,\nu\in\kappa}$ such that:*
(1) $\{I_\nu\}_{\nu\in\kappa}$ is a partition of κ such that $|I_\nu| < \kappa$ for each $\nu \in \kappa$.
(2) $\mathfrak{B}_{\xi,\nu}$ is a Boolean subalgebra of $\mathcal{P}(I_\nu)$ and $\Phi_{\xi,\nu}$ is an automorphism of $\mathfrak{B}_{\xi,\nu}$ for each $\xi \in \kappa^+$ and $\nu \in \kappa$.
(3) If $\xi \in \eta \in \kappa^+$ then there is $\beta \in \kappa$ such that $\mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\eta,\nu}$ and $\Phi_{\xi,\nu} \subseteq \Phi_{\eta,\nu}$ for all $\nu \in \kappa \setminus \beta$.
(4) For any one-to-one $F : \kappa \to \kappa$ such that $F(I_\nu) \subseteq I_\nu$ for all but an initial segment of ν there are $\xi \in \kappa^+$ and cofinally many $\nu \in \kappa$ for which there is an $A \in \mathfrak{B}_{\xi,\nu}$ and $w \in A$ such that $F(w) \notin \Phi_{\xi,\nu}(A)$.
(5) For any $A \subseteq \kappa$ there are $\xi \in \kappa^+$ and β in κ such that $A \cap I_\nu \in \mathfrak{B}_{\xi,\nu}$ for all $\nu \in \kappa \setminus \beta$.

Proof. Define

$$\Phi([A]_\kappa) = \lim_{\xi\to\kappa^+}\left[\bigcup_{\nu\in\kappa}\Phi_{\xi,\nu}(A\cap I_\nu)\right]_\kappa$$

and begin by observing that this is well defined. To see this, it must first be observed that given A and B such that $A \equiv_\kappa B$ there is $\alpha \in \kappa^+$ such that for all ν in a final segment of κ the equation

$$\Phi_{\alpha,\nu}(A\cap I_\nu) = \Phi_{\alpha,\nu}(B\cap I_\nu)$$

is defined and valid by Hypothesis (5). From Hypothesis (3) it then follows that if $\xi \geq \alpha$ then

$$\bigcup_{\nu\in\kappa}\Phi_{\xi,\nu}(A\cap I_\nu) \equiv_\kappa \bigcup_{\nu\in\kappa}\Phi_{\alpha,\nu}(B\cap I_\nu)$$

and, hence, $\Phi([A]_\kappa)$ is well defined. Since each $\Phi_{\xi,\nu}$ is an automorphism it follows that Φ is an automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$.

To see that Φ is non-trivial, suppose that there is a one-to-one function $F : \kappa \to \kappa$ such that $[F(A)]_\kappa = \Phi([A]_\kappa)$ for all $A \subseteq \kappa$. Consider first the case that there are cofinally many $\nu \in \kappa$ such that $F(I_\nu) \not\subseteq I_\nu$. Since κ is regular, it is then possible to find a cofinal set $W \subseteq \kappa$ and $w_\nu \in I_\nu$ for each $\nu \in W$ such that $F(w_\nu) \notin \bigcup_{\theta\in W} I_\theta$. Let $W^* = \{w_\nu\}_{\nu\in W}$ and note that

$$\Phi([W^*]_\kappa) = \lim_{\xi\to\kappa^+}\left[\bigcup_{\nu\in W}\Phi_{\xi,\nu}(\{w_\nu\})\right]_\kappa \subseteq \left[\bigcup_{\nu\in W} I_\nu\right]_\kappa$$

and $(\bigcup_{\nu\in W} I_\nu) \cap F(W^*) \equiv_\kappa \varnothing$. Since $W^* \not\equiv_\kappa \varnothing$ this contradicts that $\Phi([W^*]_\kappa) =_\kappa [F(W^*)]_\kappa$.

So now assume that $F(I_\nu) \subseteq I_\nu$ for all but an initial segment of ν. Using Hypothesis (4) choose $\xi \in \kappa^+$ for which there is $Z \in [\kappa]^\kappa$ and $A_\nu \in \mathfrak{B}_{\xi,\nu}$ and $z_\nu \in A_\nu$ such that $F(z_\nu) \notin \Phi_{\xi,\nu}(A_\nu)$ for each $\nu \in Z$. Let $A = \bigcup_{\nu\in Z} A_\nu$. It follows from Hypothesis (3) that for any $\eta \geq \xi$

$$\{F(z_\nu) \mid \nu \in Z\} \cap \bigcup_{\nu\in Z}\Phi_{\eta,\nu}(A_\nu) \equiv_\kappa \emptyset$$

and, hence, $[F(A)]_\kappa \neq \Phi[A]_\kappa$. $\square$

3. When are the hypotheses of Lemma 2.1 satisfied?

In answering a question of A. Blass concerning the classification of cardinal invariants of the continuum based on the Borel hierarchy M. Goldstern and S. Shelah introduced a family of cardinal invariants called $\mathbf{c}(f,g)$ defined to be the least number of uniform trees with g-splitting needed to cover a uniform tree with f-splitting [4] and showed that uncountably many of these can be distinct simultaneously. The following definition is very closely related to this as well as to the notion of a *slalom* found in [1].

Definition 3.1. Given functions f and g on κ such that $g(\xi)$ is a cardinal for each $\xi \in \kappa$ define $\mathfrak{d}_{f,g}$ to be the least cardinal of a family $\mathcal{D} \subseteq \prod_{\nu\in\kappa}[f(\nu)]^{g(\nu)}$ such that for every $F \in \prod_{\nu\in\kappa} f(\nu)$ there is $G \in \mathcal{D}$ such that $F(\nu) \in G(\nu)$ for all but an initial segment of $\nu \in \kappa$.

Hypothesis 3.1. Let κ be either inaccessible or ω. The case of $\kappa = \omega$ will require only slightly different arguments from the case that κ is inaccessible. Let f and g be functions from κ to the regular cardinals below κ and let ψ be a function from κ to the cardinals below κ such that for all $\nu \in \kappa$:

(1) $2^{g(\nu)} < f(\nu)$
(2) if $\nu \in \nu^*$ then $|\nu| \leq g(\nu) \leq g(\nu^*)$
(3) $\{G_\xi\}_{\xi\in\kappa^+}$ witnesses that $\mathfrak{d}_{\psi,g} = \kappa^+$

and suppose further that

(4) $\{I_\nu\}_{\nu\in\kappa}$ is a partition of κ such that $|I_\nu| = f(\nu)$
(5) $\{\pi_{\theta,\nu}\}_{\theta\in\psi(\nu)}$ enumerates all one-to-one functions from I_ν to I_ν
(6) $\{E_{\theta,\nu}\}_{\theta\in\psi(\nu)}$ enumerates $\mathcal{P}(I_\nu)$.

In the case that κ is inaccessible it will be assumed that for each $\nu \in \kappa$

(7) $g(\nu)$ is infinite
(8) $\psi(\nu) = 2^{f(\nu)}$

and in the case that $\kappa = \omega$ it will be assumed that for each $k \in \omega$

(9) $f(k) > 3g(k)2^{(g(k)+1)g(k)}$
(10) $\psi(\nu) = f(\nu)!$.

For the purposes of this paper there is no harm in assuming that:

- in the inaccessible case
  - $g(\nu) = \aleph_{\nu+1}$
  - $f(\nu) = \left(2^{\aleph_{\nu+1}}\right)^+$
  - $\psi(\nu) = 2^{\left(2^{\aleph_{\nu+1}}\right)^+}$
- if $\kappa = \omega$
  - $g(k) = k$
  - $f(k) = 3k2^{k^2+k} + 1$
  - $\psi(k) = (3k2^{k^2+k} + 1)!$.

However, if only for notational convenience, the arguments to be presented will deal with the general case.

It will be necessary to recall the definition of Jensen's square sequence about which more can be found in [6] as well as various other sources.

Definition 3.2. For a set of ordinals X let $\mathsf{otp}(X)$ denote the order type of X. Let $\Lambda(\xi)$ denote the limit ordinals in ξ and let $\Lambda_{<\kappa}(\xi)$ denote the ordinals in $\Lambda(\xi)$ of cofinality less than κ. An indexed family $\{C_\mu\}_{\mu\in\Lambda(\kappa^+)}$ is known as a $\square_\kappa$ sequence

- each set C_μ is a closed subset of μ that is cofinal in μ
- $\mathsf{otp}(C_\mu) \leq \kappa$
- if $\xi \in C_\mu$ is a limit of C_μ then $C_\xi = C_\mu \cap \xi$.

The statement that $\square_\kappa$ holds at κ means there is a $\square_\kappa$ sequence.

Lemma 3.1. *Let g be as in Hypothesis 3.1. If κ is inaccessible and $\square_\kappa$ holds then there are $\{Z_{\eta,\zeta}\}_{\eta\in\kappa^+,\zeta\in\kappa}$ satisfying the following:*

(1) $|Z_{\eta,\zeta}| \leq g(\zeta)$
(2) if $\zeta < \zeta^*$ *then* $Z_{\eta,\zeta} \subseteq Z_{\eta,\zeta^*}$
(3) $\bigcup_{\zeta\in\kappa} Z_{\eta,\zeta} = \eta$
(4) if $\eta \in Z_{\eta^*,\zeta}$ *then* $Z_{\eta^*,\zeta} \cap \eta = Z_{\eta,\zeta}$.

Proof. Let $\{C_\mu\}_{\mu\in\kappa^+}$ be a $\square_\kappa$ sequence. Proceed by induction on η to define $Z_{\eta,\zeta}$ for all $\zeta\in\kappa$ as well as a function $B:[\kappa^+]^2\to\kappa$ such that:

$$(\forall\eta^*\in\eta\in\kappa^+)(\forall\zeta\geq B(\{\eta^*,\eta\})\ Z_{\eta,\zeta}\cap\eta^*=Z_{\eta^*,\zeta} \tag{3.1}$$

$$(\forall\eta\in\Lambda_{<\kappa}(\kappa^+))(\forall\nu\in C_\eta)\ B(\{\nu,\eta\})=\sup_{a\in[C_\eta]^2}B(a). \tag{3.2}$$

Let $b:\Lambda_{<\kappa}(\kappa^+)\to\kappa$ be defined by $\sup_{a\in[C_\eta]^2}B(a)=b(\eta)$ so that Equation 3.2 can be rephrased as to say that $B(\{\nu,\eta\})=b(\eta)$ for all $\eta\in\Lambda_{<\kappa}(\kappa^+)$ and $\nu\in C_\eta$. Begin by letting $Z_{0,\zeta}=\varnothing$ for all $\zeta\in\kappa$ and letting $B\restriction 0=\varnothing$. If $B\restriction[\xi]^2$ and $Z_{\eta,\zeta}$ have been defined for $\eta\in\xi$ and $\zeta\in\kappa$ then consider three cases.

Case One. $\xi=\xi^*+1$

In this case simply define $Z_{\xi,\zeta}=Z_{\xi^*,\zeta}\cup\{\xi^*\}$ and for $\nu\in\xi$ define

$$B(\{\nu,\xi\})=\begin{cases}B(\{\nu,\xi^*\}) & \text{if } \nu\in\xi^*\\ 0 & \text{if } \nu=\xi^*\end{cases}$$

so that the induction hypotheses are all easily verified, as are Equations (3.1) and (3.2).

Case Two. $\xi\in\Lambda_{<\kappa}(\kappa^+)$

In this case let

$$B(\{\nu,\xi\})=\begin{cases}b(\xi) & \text{if } \nu\in C_\xi\\ \max(b(\xi),\sup_{\theta\in C_\xi\setminus\nu}B(\nu,\theta)) & \text{if } \nu\in\xi\setminus C_\xi\end{cases}$$

so that Equation (3.2) holds by construction. Note that from Equation (3.1) it follows that

$$(\forall\zeta\geq b(\xi))(\forall\{\nu^*,\nu\}\in[C_\xi]^2)\text{ if }\nu<\nu^*\text{ then }\ Z_{\nu,\zeta}=Z_{\nu^*,\zeta}\cap\nu. \tag{3.3}$$

Then define

$$Z_{\xi,\zeta}=\begin{cases}\varnothing & \text{if } \zeta\leq\max(b(\xi),|C_\xi|)\\ \bigcup_{\nu\in C_\xi}Z_{\nu,\zeta} & \text{if } \zeta>\max(b(\xi),|C_\xi|)\end{cases}$$

and note that

$$|Z_{\xi,\zeta}|\leq|C_\xi|g(\zeta)\leq|\zeta|g(\zeta)\leq g(\zeta)$$

by (2) of Hypothesis 3.1. Hence Induction Hypothesis (1) holds. To see that Induction Hypothesis (2) holds let $\zeta<\zeta^*$. Then $Z_{\nu,\zeta}\subseteq Z_{\nu,\zeta^*}$ for each $\nu\in C_\xi$ by Induction Hypothesis (2) and it follows that $Z_{\xi,\zeta}\subseteq Z_{\xi,\zeta^*}$. Induction Hypothesis (3) follows from the fact that $\bigcup_{\zeta\in\kappa}Z_{\nu,\zeta}=\nu$ for each $\nu\in C_\xi$ and C_ξ is cofinal in ξ. To see that Induction Hypothesis (4) holds let $\xi^*\in Z_{\xi,\zeta}$. Note that since $Z_{\xi,\zeta}\neq\varnothing$ it follows that $Z_{\xi,\zeta}=\bigcup_{\theta\in C_\xi}Z_{\theta,\zeta}$ and hence $\xi^*\in Z_{\nu,\zeta}$ for some $\nu\in C_\xi$. Then $Z_{\xi^*,\zeta}=Z_{\nu,\zeta}\cap\xi^*$ by Induction Hypothesis (4). By (3.2) and the fact that $\zeta>b(\xi)$ it follows that if $\theta\in C_\xi\setminus\nu$ then $Z_{\nu,\zeta}=Z_{\theta,\zeta}\cap\nu$. On other hand, if $\theta\in C_\xi\cap\nu$ then $Z_{\nu,\zeta}\cap\theta=Z_{\theta,\zeta}$. Keeping in mind that $\xi^*<\nu$ it follows that

$$\begin{aligned}Z_{\xi,\zeta}\cap\xi^*&=\bigcup_{\theta\in C_\xi}Z_{\theta,\zeta}\cap\xi^*=\left(\bigcup_{\theta\in C_\xi\cap\nu}Z_{\theta,\zeta}\cap\xi^*\right)\cup(Z_{\nu,\zeta}\cap\xi^*)\cup\left(\bigcup_{\theta\in C_\xi\setminus\nu}Z_{\theta,\zeta}\cap\xi^*\right)=(Z_{\nu,\zeta}\cap\xi^*)\cup\left(\bigcup_{\theta\in C_\xi\setminus\nu}Z_{\theta,\zeta}\cap\xi^*\right)\\&=(Z_{\nu,\zeta}\cap\xi^*)\cup\left(\bigcup_{\theta\in C_\xi\setminus\nu}(Z_{\theta,\zeta}\cap\nu)\cap\xi^*\right)=(Z_{\nu,\zeta}\cap\xi^*)\cup\left(\bigcup_{\theta\in C_\xi\setminus\nu}Z_{\nu,\zeta}\cap\xi^*\right)=Z_{\nu,\zeta}\cap\xi^*=Z_{\xi^*,\zeta}\end{aligned} \tag{3.4}$$

as required.

To see that Equation (3.1) still holds it will first be shown that

$$(\forall\nu\in C_\xi)(\forall\zeta\geq b(\xi))\ Z_{\xi,\zeta}\cap\nu=Z_{\nu,\zeta}.$$

Fix $\nu \in C_\xi$ and $\zeta \geq \beta(\xi)$ and note that $Z_{\xi,\zeta} = \bigcup_{\theta \in C_\xi} Z_{\theta,\zeta}$ in this case. As in the argument establishing that Induction Hypothesis (4) holds it follows that if $\theta \in C_\xi \setminus \nu$ then $Z_{\nu,\zeta} = Z_{\theta,\zeta} \cap \nu$ and if $\theta \in C_\xi \cap \nu$ then $Z_{\nu,\zeta} \cap \theta = Z_{\theta,\zeta}$. Therefore

$$Z_{\xi,\zeta} \cap \nu = \bigcup_{\theta \in C_\xi} Z_{\theta,\zeta} \cap \nu = \left(\bigcup_{\theta \in C_\xi \cap \nu} Z_{\theta,\zeta} \cap \nu\right) \cup (Z_{\nu,\zeta} \cap \nu) \cup \left(\bigcup_{\theta \in C_\xi \setminus \nu} Z_{\theta,\zeta} \cap \nu\right) \tag{3.5}$$

$$= \left(\bigcup_{\theta \in C_\xi \cap \nu} Z_{\nu,\zeta} \cap \theta \cap \nu\right) \cup Z_{\nu,\zeta} \cup \left(\bigcup_{\theta \in C_\xi \setminus \nu} Z_{\nu,\zeta}\right) = \left(\bigcup_{\theta \in C_\xi \cap \nu} Z_{\nu,\zeta} \cap \theta\right) \cup Z_{\nu,\zeta} = Z_{\nu,\zeta}$$

as required.

Now suppose that $\nu \in \xi \setminus C_\xi$ and $\zeta \geq B(\nu, \xi)$. Let $\eta \in C_\xi \setminus \nu$. It has already been established that $Z_{\xi,\zeta} \cap \eta = Z_{\eta,\zeta}$ since $\zeta \geq B(\nu, \xi) \geq b(\xi)$. Moreover, since $\zeta \geq B(\nu, \eta)$ it follows that $Z_{\eta,\zeta} \cap \nu = Z_{\nu,\zeta}$. Since $\nu < \eta$ it follows that $Z_{\xi,\zeta} \cap \nu = Z_{\nu,\zeta}$.

Case Three. $\xi \in \Lambda(\kappa^+) \setminus \Lambda_{<\kappa}(\kappa^+)$ and $\{b(\theta) \mid \theta \in C_\xi \cap \Lambda_{<\kappa}(\xi)\}$ is unbounded in κ.

Using that $\mathsf{otp}(C_\xi) = \kappa$, let $\{\gamma_\theta\}_{\theta \in \kappa} \subseteq \Lambda(\xi) \cap C_\xi$ be a continuous, increasing enumeration of a set on which b is strictly increasing. Using Equation (3.1) and Equation (3.2) it follows that

$$(\forall \theta \in \kappa)(\forall \nu \in C_{\gamma_\theta})(\forall \zeta \geq b(\gamma_\theta))\ Z_{\gamma_\theta,\zeta} \cap \nu = Z_{\nu,\zeta}. \tag{3.6}$$

For $\nu \in \xi$ define

$$B(\{\nu, \xi\}) = \begin{cases} b(\gamma_\theta) & \text{if } \nu = \gamma_\theta \text{ for some } \theta \in \kappa \\ \max(b(\gamma_\theta), B(\nu, \gamma_\theta)) & \text{where } \theta \in \kappa \text{ is minimal such that } \nu < \gamma_\theta \text{ otherwise.} \end{cases}$$

Let $Z_{\xi,\zeta} = \varnothing$ if $\zeta < b(\gamma_0)$ and let $Z_{\xi,\zeta} = Z_{\gamma_\theta,\zeta}$ for all ζ such that $b(\gamma_\theta) \leq \zeta < b(\gamma_{\theta+1})$ and note that the continuity of b ensures that $Z_{\xi,\zeta}$ is defined for all ζ.

To see that Induction Hypothesis (1) holds note that for each $Z_{\xi,\zeta}$ there is some θ such that $|Z_{\xi,\zeta}| = |Z_{\gamma_\theta,\zeta}| \leq g(\zeta)$. To see that Induction Hypothesis (2) holds let $\zeta < \zeta^* < \kappa$. Then there are $\theta \leq \theta^*$ such that $Z_{\xi,\zeta} = Z_{\gamma_\theta,\zeta}$ and $Z_{\xi,\zeta^*} = Z_{\gamma_{\theta^*},\zeta^*}$. By Induction Hypothesis (2) it follows that $Z_{\gamma_\theta,\zeta} \subseteq Z_{\gamma_\theta,\zeta^*} \subseteq Z_{\gamma_{\theta^*},\zeta^*}$ with the last inclusion following from (3.6) for $b(\gamma^*_\theta)$, and from the fact that $b(\theta^*) \leq \zeta^*$. Induction Hypothesis (3) follows from the fact that $\bigcup_{\zeta \in \kappa} Z_{\gamma_\theta,\zeta} = \gamma_\theta$ for each $\theta \in \kappa$ and the γ_θ are cofinal in ξ. Since each $Z_{\xi,\zeta}$ is equal to $Z_{\gamma_\theta,\zeta}$ for some θ it is immediate that Induction Hypothesis (4) holds at ξ.

To see that Equation (3.1) holds let $\eta \in \xi$ and $\zeta \geq B(\{\eta, \xi\})$. The first case to consider is that there exists some $\theta \in \kappa$ such that $\eta = \gamma_\theta$. To see that $Z_{\xi,\zeta} \cap \eta = Z_{\eta,\zeta}$ let $\theta^* \in \kappa$ be such that $Z_{\xi,\zeta} = Z_{\gamma_{\theta^*},\zeta}$ and $b(\gamma_{\theta^*}) \leq \zeta < b(\gamma_{\theta^*+1})$. Notice that $\theta \leq \theta^*$ because otherwise $\zeta < b(\gamma_{\theta^*+1}) \leq b(\gamma_\theta) = B(\{\eta, \xi\})$. Then $\eta \leq \gamma_{\theta^*}$ and so $Z_{\xi,\zeta} \cap \eta = Z_{\gamma_{\theta^*},\zeta} \cap \eta = Z_{\eta,\zeta}$ because $\zeta \geq b(\gamma_{\theta^*})$ and (3.2) holds at γ_{θ^*}. On the other hand, if $\nu \in \xi \setminus \{\gamma_\beta\}_{\beta \in \kappa}$ and $\theta \in \kappa$ is minimal such that $\nu < \gamma_\theta$ then by the preceding argument it follows that $Z_{\xi,\zeta} \cap \gamma_\theta = Z_{\gamma_\theta,\zeta}$. Since $\zeta \geq B(\eta, \gamma_\theta)$ it follows that $Z_{\gamma_\theta,\zeta} \cap \eta = Z_{\eta,\zeta}$ and the result follows from the fact that $\eta < \gamma_\theta$.

Finally, note that Equation (3.2) is irrelevant in this case.

Case Four. $\xi \in \Lambda(\kappa^+) \setminus \Lambda_{<\kappa}(\kappa^+)$ and $\{b(\theta) \mid \theta \in C_\xi \cap \Lambda_{<\kappa}(\xi)\}$ is bounded in κ.

In this case, let $\delta \in \kappa$ be such that $B(a) < \delta$ for all $a \in [C_\xi]^2$. For $\nu \in \xi$ define $\Theta(\nu) = \min(C_\xi \setminus \nu)$ and let

$$B(\{\nu, \xi\}) = \begin{cases} \delta & \text{if } \nu \in C_\xi \\ \max(\delta, B(\{\nu, \Theta(\nu)\}), \mathsf{otp}(C_\xi \cap \Theta(\nu))) & \text{otherwise .} \end{cases}$$

Then for $\zeta \in \kappa$ let $\Omega_\zeta \in C_\xi$ be the unique ordinal such that the order type of $C_\xi \cap \Omega(\zeta)$ is ζ. Then define

$$Z_{\xi,\zeta} = \begin{cases} \varnothing & \text{if } \Omega_\zeta \leq \delta \\ \bigcup_{\nu \in C_\xi \cap \Omega_\zeta} Z_{\nu,\zeta} & \text{otherwise} \end{cases}$$

and note that $|Z_{\xi,\zeta}| \leq |\zeta| g(\zeta) \leq g(\zeta)$ by induction and (2) of Hypothesis 3.1. It is then easy to see that Induction Hypotheses (1) to (3) hold. To see that Hypothesis (4) holds note that from Equation (3.1) it follows that

$$(\forall \zeta \geq \delta)(\forall \{\nu, \nu^*\} \in [C_\xi]^2)\ Z_{\nu,\zeta} = Z_{\nu^*,\zeta} \cap \nu. \tag{3.7}$$

If $\nu \in Z_{\xi,\zeta}$ then $Z_{\xi,\zeta} \neq \varnothing$ and so $Z_{\xi,\zeta} = \bigcup_{\rho \in C_\xi \cap \Omega_\zeta} Z_{\rho,\zeta}$ and hence there is some $\rho \in C_\xi \cap \Omega_\zeta$ such that $\nu \in Z_{\rho,\zeta}$. Note that by Equation (3.7) and the fact that $\nu \in \rho$ it follows that $Z_{\xi,\zeta} \cap \nu = Z_{\rho,\zeta} \cap \nu$. Then Induction Hypotheses (4) at ρ yields that $Z_{\xi,\zeta} \cap \nu = Z_{\rho,\zeta} \cap \nu = Z_{\nu,\zeta}$.

Equation (3.2) is again irrelevant and to establish that Equation (3.1) holds let $\eta \in \xi$ where $\zeta \geq B(\{\eta, \xi\})$. Note that $\zeta > \mathsf{otp}(C_\xi \cap \Theta(\eta))$ implies that $\Omega_\zeta > \Theta(\eta)$. Since $Z_{\nu,\zeta} \cap \Theta(\eta) = Z_{\Theta(\eta),\zeta}$ for each $\nu \in C_\xi \cap \Omega_\zeta \setminus \Theta(\eta)$ by Equation (3.7), it follows that

$$Z_{\xi,\zeta} \cap \Theta(\eta) = \bigcup_{\nu \in C_\xi \cap \Omega_\zeta \setminus \Theta(\eta)} Z_{\nu,\zeta} \cap \Theta(\eta) = Z_{\Theta(\eta),\zeta}.$$

Since $\zeta > B(\eta, \Theta(\eta))$ it follows that $Z_{\Theta(\eta),\zeta} \cap \eta = Z_{\eta,\zeta}$ and the result now follows from the fact that $\eta \leq \Theta(\eta)$.

□

The following lemma is the companion to Lemma 3.1 in the case that $\kappa = \omega$

Lemma 3.2. *Let g be as in Hypothesis 3.1. There are $\{Z_{\eta,k}\}_{\eta \in \omega_1, k \in \omega}$ satisfying the following:*

1. $Z_{\eta,k} \subseteq \eta$ *and* $\varnothing \neq Z_{\eta,k}$ *if* $\eta > 0$.
2. $|Z_{\eta,k}| \leq g(k)$
3. *if* $k < k^*$ *then* $Z_{\eta,k} \subseteq Z_{\eta,k^*}$
4. $\bigcup_{k \in \omega} Z_{\eta,k} = \eta$
5. *if* $\eta \in Z_{\eta^*,k}$ *then* $Z_{\eta^*,k} \cap \eta = Z_{\eta,k}$
6. $\lim_{k \to \infty} |Z_{\eta,k}|/g(k) = 0$.

Proof. As in the proof of Lemma 3.1, start by letting $Z_{0,k} = \varnothing$. If $Z_{\eta,k}$ have been defined for $\eta \in \xi$ and $k \in \omega$ consider two cases.

Case One. $\xi = \xi^* + 1$

In this case let m be so large that $|Z_{\xi^*,k}| + 1 \leq g(k)$ for $k > m$ and define

$$Z_{\xi,k} = \begin{cases} Z_{\xi^*,k} \cup \{\xi^*\} & \text{if } k > m \\ \{\xi^*\} & \text{if } k \leq m. \end{cases}$$

Case Two. ξ is a limit ordinal

In this case let $\{\gamma_n\}_{n \in \omega}$ be an increasing sequence cofinal in ξ and let $\{b(t)\}_{t \in \omega}$ be an increasing sequence of integers such that

$$0 \in Z_{\gamma_0, b(0)} \tag{3.8}$$

$$(\forall t \geq 1)(\forall k \geq b(t))(\forall n \leq t)\ |Z_{\gamma_n,k}|/g(k) \leq 1/t \tag{3.9}$$

$$(\forall t)(\forall n < t)(\forall k \geq b(t))\ Z_{\gamma_t,k} \cap \gamma_n = Z_{\gamma_n,k} \tag{3.10}$$

Let

$$Z_{\xi,k} = \begin{cases} \{0\} & \text{if } k < b(0) \\ Z_{\gamma_t,k} & \text{if } b(t) \leq k < b(t+1). \end{cases}$$

It is clear that Hypothesis (1) holds. To see that Hypothesis (2) holds note that for each $Z_{\xi,k}$ there is some t such that $|Z_{\xi,k}| = |Z_{\gamma_t,k}| \leq g(k)$. The fact that Hypothesis (6) holds follows from Inequality (3.9). To see that Hypothesis (3) holds let $k < k^*$. Then there are $t \leq t^*$ such that $Z_{\xi,k} = Z_{\gamma_t,k}$ and $Z_{\xi,k^*} = Z_{\gamma_{t^*},k^*}$. By Induction Hypothesis (3) it follows that $Z_{\gamma_t,k} \subseteq Z_{\gamma_t,k^*} \subseteq Z_{\gamma_{t^*},k^*}$ with the last inclusion following from the choice of $b(t^*)$, Condition (3.10) and the fact that $b(t^*) \leq k^*$. Hypothesis (4) follows from the fact that $\bigcup_{k \in \omega} Z_{\gamma_t,k} = \gamma_t$ for each $t \in \omega$ and Condition (3.10). Since each $Z_{\xi,k}$ is equal to $Z_{\gamma_t,k}$ for some $t \in \omega$ it is immediate that Hypothesis (5) holds at ξ. □

Lemma 3.3. *Suppose that $\{Z_{\eta,\zeta}\}_{\eta \in \kappa^+, \zeta \in \kappa}$ is the family whose existence is established in Lemma 3.1 or, when $\kappa = \omega$, in Lemma 3.2. Using the notation of Hypothesis 3.1 define*

$$\mathcal{E}_{\eta,\nu} = \{E_{\xi,\nu} \mid \bar{\eta} \in Z_{\eta,\nu} \cup \{\eta\}\ \&\ \xi \in G_{\bar{\eta}}(\nu)\}. \tag{3.11}$$

and suppose further that there are $D_{\eta,\nu}$, $D^*_{\eta,\nu}$, $\varphi_{\eta,\nu}$, $\varphi^*_{\eta,\nu}$ and $b_{\eta,\nu}$ for $\eta \in \kappa^+$ and $\nu \in \kappa$ such that

1. $\varphi_{\eta,\nu} : D_{\eta,\nu} \to D_{\eta,\nu}$ *and* $\varphi^*_{\eta,\nu} : D^*_{\eta,\nu} \to D^*_{\eta,\nu}$ *are bijections*
2. $\bigcup_{\xi \in Z_{\eta,\nu}} \varphi_{\xi,\nu} = \varphi^*_{\eta,\nu} \subseteq \varphi_{\eta,\nu}$
3. $D_{\eta,\nu} = D^*_{\eta,\nu} \cup \{b_{\eta,\nu}, \varphi_{\eta,\nu}(b_{\eta,\nu})\}$ *and* $D^*_{\eta,\nu} \cap \{b_{\eta,\nu}, \varphi_{\eta,\nu}(b_{\eta,\nu})\} = \varnothing$
4. *if* $\theta \in G_\eta(\nu)$ *then* $\varphi_{\eta,\nu}(b_{\eta,\nu}) \neq \pi_{\theta,\nu}(b_{\eta,\nu})$
5. $\varphi_{\eta,\nu}(\{b_{\eta,\nu}, \varphi_{\eta,\nu}(b_{\eta,\nu})\}) = \{b_{\eta,\nu}, \varphi_{\eta,\nu}(b_{\eta,\nu})\}$
6. $\varphi_{\eta,\nu}(b_{\eta,\nu}) \in E$ *if and only if* $b_{\eta,\nu} \in E$ *for every* $E \in \mathcal{E}_{\eta,\nu}$.

Then the hypotheses of Lemma 2.1 hold at κ.

Proof. For each $\eta \in \kappa^+$ and $\nu \in \kappa$ let $\mathfrak{B}_{\eta,\nu}$ be the Boolean subalgebra of $\mathcal{P}(I_\nu)$ generated by $\mathcal{E}_{\eta,\nu} \cup \mathcal{P}(D^*_{\eta,\nu}) \cup \{D_{\eta,\nu}\}$ and let $\Phi_{\eta,\nu} : \mathfrak{B}_{\eta,\nu} \to \mathfrak{B}_{\eta,\nu}$ be the automorphism induced by $\varphi^+_{\eta,\nu}$ where

$$\varphi^+_{\eta,\nu}(\xi) = \begin{cases} \varphi_{\eta,\nu}(\xi) & \text{if } \xi \in D_{\eta,\nu} \\ \xi & \text{otherwise.} \end{cases} \tag{3.12}$$

Note that $\Phi_{\eta,\nu}$ is also induced by $\varphi^{+*}_{\eta,\nu}$ defined by

$$\varphi^{+*}_{\eta,\nu}(\xi) = \begin{cases} \varphi^*_{\eta,\nu}(\xi) & \text{if } \xi \in D^*_{\eta,\nu} \\ \xi & \text{otherwise} \end{cases} \tag{3.13}$$

because of Hypotheses (3), (5) and (6).

It will now be shown that $\mathfrak{B}_{\eta,\nu}$ and $\Phi_{\eta,\nu}$ satisfy the hypotheses of Lemma 2.1. To see that Hypothesis (1) of Lemma 2.1 is satisfied recall that $\{I_\nu\}_{\nu\in\kappa}$ is a partition of κ by construction and $|I_\nu| = f(\nu) < \kappa$ for each $\nu \in \kappa$ by Hypothesis 3.1. That Hypothesis (2) of Lemma 2.1 is satisfied is immediate.

To see that Hypothesis (3) of Lemma 2.1 is satisfied let $\xi \in \eta \in \kappa^+$. Use Conclusions (2), (3) and (3.1) of Lemma 3.1 or Lemma 3.2 to find $\beta \in \kappa$ such that

$$(\forall \nu > \beta)\ \xi \in Z_{\eta,\nu} \tag{3.14}$$

$$(\forall \nu > \beta)\ Z_{\eta,\nu} \cap \xi = Z_{\xi,\nu}. \tag{3.15}$$

Then use Equations (3.14), (3.15) and (3.11) to see that if $\nu > \beta$ then

$$\mathcal{E}_{\xi,\nu} = \{E_{\gamma,\nu} \mid \theta \in Z_{\xi,\nu} \cup \{\xi\}\ \&\ \gamma \in G_\theta(\nu)\} = \{E_{\gamma,\nu} \mid \theta \in Z_{\eta,\nu} \cap \xi + 1\ \&\ \gamma \in G_\theta(\nu)\} \subseteq \mathcal{E}_{\eta,\nu}$$

and so $\mathcal{E}_{\xi,\nu} \subseteq \mathcal{E}_{\eta,\nu}$. Furthermore, it follows from Hypothesis (2) and Equation (3.14) that

$$\varphi^*_{\eta,\nu} = \bigcup_{\theta \in Z_{\eta,\nu}} \varphi_{\theta,\nu} \supseteq \varphi_{\xi,\nu} \tag{3.16}$$

and so $D^*_{\eta,\nu} \supseteq D_{\xi,\nu} \supseteq D^*_{\xi,\nu}$. Since $\mathfrak{B}_{\eta,\nu}$ is generated by

$$\mathcal{E}_{\eta,\nu} \cup \mathcal{P}(D^*_{\eta,\nu}) \cup \{D_{\eta,\nu}\}$$

and $\mathfrak{B}_{\xi,\nu}$ is generated by $\mathcal{E}_{\xi,\nu} \cup \mathcal{P}(D^*_{\xi,\nu}) \cup \{D_{\xi,\nu}\}$, which is contained in $\mathcal{E}_{\eta,\nu} \cup \mathcal{P}(D^*_{\eta,\nu})$, it follows that $\mathfrak{B}_{\eta,\nu} \supseteq \mathfrak{B}_{\xi,\nu}$.

In order to see that if $\xi \in \eta$ then $\Phi_{\eta,\nu} \supseteq \Phi_{\xi,\nu}$ for all but an initial segment of ν, using (3.14) and (3.15), it suffices to show that

$$(\forall \eta \in \kappa^+)(\forall \nu \in \kappa)(\forall \xi \in Z_{\eta,\nu})\ \Phi_{\eta,\nu} \supseteq \Phi_{\xi,\nu}. \tag{3.17}$$

This will be shown by induction on η, so suppose that Condition (3.17) holds for all $\bar\eta \in \eta$ and that $\xi \in Z_{\eta,\nu}$. Let $Y \in \mathfrak{B}_{\xi,\nu}$. Then, since $D_{\xi,\nu} \in \mathfrak{B}_{\xi,\nu}$ it follows that

$$\Phi_{\xi,\nu}(Y) = \Phi_{\xi,\nu}(Y \cap D_{\xi,\nu}) \cup \Phi_{\xi,\nu}(Y \setminus D_{\xi,\nu})$$

and

$$\Phi_{\xi,\nu}(Y \cap D_{\xi,\nu}) = \varphi_{\xi,\nu}(Y \cap D_{\xi,\nu}) = \varphi^*_{\eta,\nu}(Y \cap D_{\xi,\nu}) = \Phi_{\eta,\nu}(Y \cap D_{\xi,\nu})$$

using Equation (3.12) for the first equality, Hypothesis (2) for the second and Equation (3.13) for the third. Hence, it suffices to show that $\Phi_{\eta,\nu}(Y) = \Phi_{\xi,\nu}(Y)$ when $Y \in \mathfrak{B}_{\xi,\nu}$ and $Y \cap D_{\xi,\nu} = \varnothing$. From Equation (3.12) it then suffices to show that $\Phi_{\eta,\nu}(Y) = Y$ when $Y \in \mathfrak{B}_{\xi,\nu}$ and $Y \cap D_{\xi,\nu} = \varnothing$. From Equation (3.13) it then suffices to show that $\Phi_{\eta,\nu}(Y \cap D^*_{\eta,\nu}) = Y \cap D^*_{\eta,\nu}$ provided that $Y \in \mathfrak{B}_{\xi,\nu}$ and $Y \cap D_{\xi,\nu} = \varnothing$.

To this end, let $Y \in \mathfrak{B}_{\xi,\nu}$ be such that $Y \cap D_{\xi,\nu} = \varnothing$. Then

$$\Phi_{\eta,\nu}(Y\cap D^*_{\eta,\nu}) = \varphi^{+*}_{\eta,\nu}(Y\cap D^*_{\eta,\nu}) = \varphi^*_{\eta,\nu}(Y\cap D^*_{\eta,\nu}) = \bigcup_{\bar\eta\in Z_{\eta,\nu}} \varphi_{\bar\eta,\nu}(Y\cap D_{\bar\eta,\nu})) = \bigcup_{\bar\eta\in Z_{\eta,\nu}} \varphi^+_{\bar\eta,\nu}(Y\cap D_{\bar\eta,\nu}) = \bigcup_{\bar\eta\in Z_{\eta,\nu}\setminus\xi} \varphi^+_{\bar\eta,\nu}(Y\cap D_{\bar\eta,\nu}) \tag{3.18}$$

by Hypothesis (2). Note that $\bar\eta \in Z_{\eta,\nu}$ implies that $Z_{\bar\eta,\nu} = Z_{\eta,\nu} \cap \bar\eta$ by Conclusion (4) of Lemma 3.1. Hence if $\xi \in \bar\eta$ then $\xi \in Z_{\eta,\nu} \cap \bar\eta = Z_{\bar\eta,\nu}$ and, hence, $Y \in \mathfrak{B}_{\xi,\nu} \subseteq \mathfrak{B}_{\bar\eta,\nu}$ by Induction Hypothesis (3.17). Therefore, if $\xi \in \bar\eta \in Z_{\eta,\nu}$ then $Y \cap D_{\bar\eta,\nu} \in \mathfrak{B}_{\bar\eta,\nu}$ and so

$$\varphi^+_{\bar\eta,\nu}(Y \cap D_{\bar\eta,\nu}) = \Phi_{\bar\eta,\nu}(Y \cap D_{\bar\eta,\nu})$$

and so Equation (3.18) yields

$$\Phi_{\eta,\nu}(Y \cap D^*_{\eta,\nu}) = \bigcup_{\bar\eta\in Z_{\eta,\nu}\setminus\xi} \Phi_{\bar\eta,\nu}(Y \cap D_{\bar\eta,\nu}) = \bigcup_{\bar\eta\in Z_{\eta,\nu}\setminus\xi} \Phi_{\bar\eta,\nu}(Y) \cap \Phi_{\bar\eta,\nu}(D_{\bar\eta,\nu}) = \bigcup_{\bar\eta\in Z_{\eta,\nu}\setminus\xi} \Phi_{\bar\eta,\nu}(Y) \cap D_{\bar\eta,\nu}. \tag{3.19}$$

Once again using that $\xi \in Z_{\bar\eta,\nu}$ if $\xi \in \bar\eta \in Z_{\eta,\nu}$ and Induction Hypothesis (3.17), it follows that $\Phi_{\xi,\nu}(Y) = \Phi_{\bar\eta,\nu}(Y)$ if $\xi \in \bar\eta \in Z_{\eta,\nu}$. Hence the Equation (3.19) yields that $\Phi_{\eta,\nu}(Y \cap D^*_{\eta,\nu})$ is equal to

$$\bigcup_{\bar\eta \in Z_{\eta,\nu}\setminus\xi} \Phi_{\bar\eta,\nu}(Y) \cap D_{\bar\eta,\nu} = \bigcup_{\bar\eta \in Z_{\eta,\nu}\setminus\xi} \Phi_{\xi,\nu}(Y) \cap D_{\bar\eta,\nu} = \Phi_{\xi,\nu}(Y) \cap \bigcup_{\bar\eta \in Z_{\eta,\nu}\setminus\xi} D_{\bar\eta,\nu} = \Phi_{\xi,\nu}(Y) \cap D^*_{\eta,\nu} = Y \cap D^*_{\eta,\nu}$$

since $Y \cap D_{\xi,\nu} = \varnothing$. This is what is required.

To see that Hypothesis (4) of Lemma 2.1 holds let $F : \kappa \to \kappa$ be one-to-one such that $F(I_\nu) \subseteq I_\nu$ for all but an initial segment of ν. Then $F \restriction I_\nu = \pi_{J(\nu),\nu}$ for some $J(\nu) \in \psi(\nu)$ for a tail of $\nu \in \kappa$. By (3) of Hypothesis 3.1 there is then some $\xi \in \kappa^+$ such that $J(\nu) \in G_\xi(\nu)$ for a final segment of $\nu \in \kappa$. Let $\bar\xi > \xi$. Then it follows from Conclusion (3) to Conclusion (4) of Lemma 3.1 that $Z_{\xi,\nu} \subseteq Z_{\bar\xi,\nu}$ for all but an initial segment of ν. Hence, using Hypotheses (2) and (3) it follows that $b_{\xi,\nu} \in D_{\xi,\nu} \subseteq D^*_{\bar\xi,\nu}$ and $\varphi_{\xi,\nu} \subseteq \varphi^*_{\bar\xi,\nu} \subseteq \varphi_{\bar\xi,\nu}$ for any such ν. Since $\mathcal{P}(D^*_{\bar\xi,\nu}) \subseteq \mathfrak{B}_{\bar\xi,\nu}$ it must be that $\{b_{\xi,\nu}\} \in \mathfrak{B}_{\bar\xi,\nu}$ and so, by Hypotheses (4),

$$F(b_{\xi,\nu}) = \pi_{J(\nu),\nu}(b_{\xi,\nu}) \notin \{\varphi_{\xi,\nu}(b_{\xi,\nu})\} = \{\varphi_{\bar\xi,\nu}(b_{\xi,\nu})\} = \{\varphi^+_{\bar\xi,\nu}(b_{\xi,\nu})\} = \Psi_{\bar\xi,\nu}(\{b_{\xi,\nu}\})$$

for a final segment of $\nu \in \kappa$.

Finally, to see that Hypothesis (5) of Lemma 2.1 holds let $A \subseteq \kappa$. Then $A \cap I_\nu = E_{J(\nu),\nu}$ for some $J(\nu) \in \psi(\nu)$ for all $\nu \in \kappa$. By (3) of Hypothesis 3.1 there is $\eta \in \kappa^+$ such that $J(\nu) \in G_\eta(\nu)$ for a final segment of $\nu \in \kappa$. It follows that for a final segment of $\nu \in \kappa$

$$E_{J(\nu),\nu} \in \{E_{\xi,\nu} \mid \bar\eta \in \{\eta\} \ \& \ \xi \in G_{\bar\eta}(\nu)\} \subseteq \{E_{\xi,\nu} \mid \bar\eta \in Z_{\eta,\nu} \cup \{\eta\} \ \& \ \xi \in G_{\bar\eta}(\nu)\} = \mathcal{E}_{\eta,\nu}$$

by Equation (3.11). It follows that $E_{J(\nu),\nu} \in \mathcal{E}_{\eta,\nu}$ for all but an initial segment of ν and so, since $\mathfrak{B}_{\eta,\nu} \supseteq \mathcal{E}_{\eta,\nu}$, it follows that $A \cap I_\nu = E_{J(\nu),\nu} \in \mathfrak{B}_{\eta,\nu}$ for all but an initial segment of $\nu \in \kappa$. $\square$

Lemma 3.4. *The hypotheses of Lemma 2.1 hold at κ when κ is inaccessible and $\square_\kappa$ holds.*

Proof. It will be shown that the hypotheses of Lemma 3.3 hold. Construct $D_{\eta,\nu}$, $D^*_{\eta,\nu}$, $\varphi_{\eta,\nu}$, $\varphi^*_{\eta,\nu}$ and $b_{\eta,\nu}$ for each $\nu \in \kappa$ satisfying Hypotheses (1) to (6) of Lemma 3.3 by induction on $\eta \in \kappa^+$. Begin by letting $\{Z_{\eta,\zeta}\}_{\eta\in\kappa^+,\zeta\in\kappa}$ be the family whose existence is established in Lemma 3.1 and defining $D_{0,\nu} = \varnothing = \varphi_{0,\nu}$ for each ν in κ. Now assume that $D_{\eta,\nu}$, $D^*_{\eta,\nu}$, $\varphi_{\eta,\nu}$, $\varphi^*_{\eta,\nu}$ and $b_{\eta,\nu}$ have been defined for all $\xi \in \eta$ and $\nu \in \kappa$ and that the following induction hypothesis is also satisfied:

$$(\forall \xi \in \eta)(\forall \nu \in \kappa)\ |D_{\xi,\nu}| \leq g(\nu). \tag{3.20}$$

Observe that by Condition (4) of Lemma 3.1 it follows that if $\xi \in \xi^*$ and $\{\xi, \xi^*\} \subseteq Z_{\eta,\nu}$ then $Z_{\eta,\nu} \cap \xi^* = Z_{\xi^*,\nu}$ and so $\xi \in Z_{\xi^*,\nu}$. Now applying the induction hypothesis that Condition (2) of Lemma 3.3 holds it follows that

$$(\forall \xi \in \xi^*) \text{ if } \{\xi, \xi^*\} \subseteq Z_{\eta,\nu} \text{ then } \varphi_{\xi,\nu} \subseteq \varphi_{\xi^*,\nu}. \tag{3.21}$$

Since $|Z_{\eta,\nu}| \leq g(\nu)$ by Condition (1) of Lemma 3.1, it follows from Induction Hypothesis (3.20) and the fact $g(\nu)$ is infinite that

$$\left| \bigcup_{\xi \in Z_{\eta,\nu}} \varphi_{\xi,\nu} \right| \leq g(\nu)g(\nu) = g(\nu). \tag{3.22}$$

By (3.21), letting $\varphi^*_{\eta,\nu} = \bigcup_{\xi\in Z_{\eta,\nu}} \varphi_{\xi,\nu}$ and letting $D^*_{\eta,\nu}$ be the domain of $\varphi^*_{\eta,\nu}$, it follows that $\varphi^*_{\eta,\nu}$ is a bijection of $D^*_{\eta,\nu}$. From Inequality (3.22), it follows that $|D^*_{\eta,\nu}| \leq g(\nu)$.

Now let $\mathcal{A}_{\eta,\nu}$ be the partition of I_ν generated by $\mathcal{E}_{\eta,\nu}$ as defined by (3.11) of Lemma 3.3. Since

$$|\mathcal{A}_{\eta,\nu}| \leq 2^{|\mathcal{E}_{\eta,\nu}|} \leq 2^{(|Z_{\eta,\nu}|+1)g(\nu)} \leq 2^{g(\nu)g(\nu)} = 2^{g(\nu)}$$

by Condition (1) of Lemma 3.1 it follows that $|\mathcal{A}_{\eta,\nu}| \leq 2^{g(\nu)} < f(\nu) = |I_\nu|$ by Hypothesis 3.1. Using that $|D^*_{\eta,\nu}| \leq g(\nu)$ and $f(\nu)$ is regular it is possible to find for each $\nu \in \kappa$ some $A_{\eta,\nu} \in \mathcal{A}_{\eta,\nu}$ such that $|A_{\eta,\nu} \setminus D^*_{\eta,\nu}| = f(\nu)$. Then let $b_{\eta,\nu} \in A_{\eta,\nu} \setminus D^*_{\eta,\nu}$ and let $B_\nu = \{\pi_{\theta,\nu}(b_{\eta,\nu}) \mid \theta \in G_\eta(\nu)\}$. Then, since

$$|B_\nu| \leq |G_\eta(\nu)| \leq g(\nu) < f(\nu) = |A_{\eta,\nu} \setminus D^*_{\eta,\nu}|$$

it is again possible to choose $b'_{\eta,\nu} \in A_{\eta,\nu} \setminus (D^*_{\eta,\nu} \cup B_\nu)$. Now let $\varphi : \{b_{\eta,\nu}, b'_{\eta,\nu}\} \to \{b_{\eta,\nu}, b'_{\eta,\nu}\}$ be the involution sending $b_{\eta,\nu}$ to $b'_{\eta,\nu}$.

In order to satisfy (1) and (3) of Lemma 3.3 define

$$D_{\eta,\nu} = D^*_{\eta,\nu} \cup \{b_{\eta,\nu}, b'_{\eta,\nu}\} \tag{3.23}$$

and

$$\varphi_{\eta,\nu} = \varphi^*_{\eta,\nu} \cup \varphi \tag{3.24}$$

observing that (5) of Lemma 3.3 is now satisfied. Moreover

$$(\forall \nu \in \kappa)(\forall \theta \in G_\eta(\nu))\ \varphi_{\eta,\nu}(b_{\eta,\nu}) \neq \pi_{\theta,\nu}(b_{\eta,\nu}) \tag{3.25}$$

and hence (4) of Lemma 3.3 is also satisfied. To see that (6) of Lemma 3.3 is satisfied note that $\{b_{\eta,\nu}, b'_{\eta,\nu}\} \subseteq A_{\eta,\nu}$ and that $A_{\eta,\nu}$ is an element of the partition generated by $\mathcal{E}_{\eta,\nu}$. The fact that (2) of Lemma 3.3 holds follows immediately from the construction of $\varphi^*_{\eta,\nu}$. $\square$

Corollary 3.1. *If κ is inaccessible and $\square_\kappa$ holds and $2^\kappa = \kappa^+$ there is a non-trivial automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$.*

Proof. Let f and g be functions on κ satisfying Conditions (1) and (2) of Hypothesis 3.1. Let $\{H_\xi\}_{\xi\in\kappa^+}$ enumerate $\prod_{\nu\in\kappa} 2^{f(\nu)}$. Let $e_\xi : \kappa \to \xi$ be a bijection for $\xi \in \kappa^+ \setminus \kappa$. Then let $G_\xi \in \prod_{\nu\in\kappa}[2^{f(\nu)}]^{g(\nu)}$ be defined by

$$G_\xi(\nu) = \left\{ H_{e_\xi(\eta)}(\nu) \ \mid \eta \in g(\nu) \right\}.$$

In order to apply Lemma 3.3 it will be shown that $\{G_\xi\}_{\xi\in\kappa^+}$ witnesses that $\mathfrak{d}_{\psi,g} = \kappa^+$. To see this, let $H \in \prod_{\nu\in\kappa} 2^{f(\nu)}$. Then $H = H_\xi$ for some $\xi \in \kappa^+$ and if $\xi \in \theta \in \kappa^+$ it follows that there is some $\rho \in \kappa$ such that $e_\theta(\rho) = \xi$. Then for all but an initial segment of ν it must be that $\rho \in g(\nu)$ and, hence, that $H = H_\xi \in G_\theta(\nu)$.

$\square$

Corollary 3.2. *If κ is inaccessible then it is consistent that $2^\kappa > \kappa^+$ and there is a non-trivial automorphism of $\mathcal{P}(\kappa)/[\kappa]^{<\kappa}$.*

Proof. Fix functions f and g on κ satisfying Conditions (1) and (2) of Hypothesis 3.1. Let $\mathbb{P}$ be the partial order consisting of pairs $(d, \mathcal{H})$ such that:

- there is some $\alpha \in \kappa$ such that $d \in \prod_{\nu\in\alpha}[f(\nu)]^{g(\nu)}$
- $\mathcal{H} \subseteq \prod_{\nu\in\kappa} f(\nu)$
- $|\mathcal{H}| \leq g(\alpha)$

and define $(d, \mathcal{H}) \leq (d^*, \mathcal{H}^*)$ if

- $\mathcal{H} \supseteq \mathcal{H}^*$
- $d \supseteq d^*$
- $h(\eta) \in d(\eta)$ for all $h \in \mathcal{H} \setminus \mathcal{H}^*$ and η in the domain of $d \setminus d^*$.

It is routine to see that if $G \subseteq \mathbb{P}$ is generic over V and $d_G \in \prod_{\nu\in\alpha}[f(\nu)]^{g(\nu)}$ is defined by $d_G(\alpha) = d(\alpha)$ for some $(d, \mathcal{H}) \in G$ then for any $h \in \prod_{\nu\in\kappa} f(\nu) \cap V$ there is some $\beta \in \kappa$ such that $h(\alpha) \in d_G(\alpha)$ for all $\alpha > \beta$. It is then standard to obtain Condition (3) and $2^\kappa > \kappa^+$ by iterating this forcing κ^+ times over a model where $2^\kappa > \kappa^+$ and $\square_\kappa$ holds.

To get a model where $2^\kappa > \kappa^+$ and $\square_\kappa$ holds use §4 of [6] to force over the model where κ is inaccessible to first get $\square_\kappa$ to hold while preserving all cardinals. Then add more than κ^+ subsets of κ with κ closed forcing, noting that this preserves $\square_\kappa$ and the inaccessibility of κ. $\square$

Lemma 3.5. *The hypotheses of Lemma 2.1 hold at ω.*

Proof. Let $\{Z_{\eta,k}\}_{\eta\in\omega_1,k\in\omega}$ be the family whose existence is established in Lemma 3.2. A key difference between the general case and the specific case when $\kappa = \omega$ is that each $Z_{\eta,k}$ is finite and it is possible to let $\zeta_{\eta,k}$ be the maximal element of $Z_{\eta,k}$, provided that $\eta > 0$. In order to show that the hypotheses of Lemma 3.3 hold construct $D_{\eta,k}$, $D^*_{\eta,k}$, $\varphi_{\eta,k}$, $\varphi^*_{\eta,k}$ and $b_{\eta,k}$ for each $k \in \omega$ satisfying Hypotheses (1) to (6) of Lemma 3.3 by induction on $\eta \in \omega_1$.

Now assume that $D_{\xi,k}$, $D^*_{\xi,k}$, $\varphi_{\xi,k}$, $\varphi^*_{\xi,k}$ and $b_{\xi,k}$ have been defined all $\xi \in \eta$ and $k \in \omega$ and that the following induction hypothesis is also satisfied:

$$(\forall \xi \in \eta)(\forall k \in \omega)\ |D_{\xi,k}| \leq 2|Z_{\xi,k}|. \tag{3.26}$$

Let $\varphi^*_{\eta,k} = \varphi_{\zeta_{\eta,k},k}$ and let $D^*_{\eta,k}$ be the domain of $\varphi^*_{\eta,k}$. It follows from Inequality (3.26) that $|D^*_{\eta,k}| \leq 2|Z_{\zeta_\eta,k}| \leq 2g(k)$.

Now let $\mathcal{A}_{\eta,k}$ be the partition of I_k generated by $\mathcal{E}_{\eta,k}$ as defined by (3.11) of Lemma 3.3. Since

$$|\mathcal{A}_{\eta,k}| \leq 2^{|\mathcal{E}_{\eta,k}|} \leq 2^{(|Z_{\eta,k}|+1)g(k)} \leq 2^{(g(k)+1)g(k)}$$

it is possible to find for each $k \in \omega$ some $A_{\eta,k} \in \mathcal{A}_{\eta,k}$ such that

$$|A_{\eta,k}| > \frac{|I_k|}{2^{(g(k)+1)g(k)}} = \frac{f(k)}{2^{(g(k)+1)g(k)}} > 3g(k)$$

with the last inequality following from Hypothesis 3.1 in the case $\kappa = \omega$. Then $|A_{\eta,k} \setminus D^*_{\eta,k}| > g(k)$ and it is possible to find $b_{\eta,\nu} \in A_{\eta,k} \setminus D^*_{\eta,k}$. Moreover,

$$|\{\pi_{\theta,k}(b_{\eta,k}) \ \mid \theta \in G_\eta(k)\}| \leq g(k)$$

and hence it is possible to extend $\varphi^*_{\eta,k}$ to a bijection $\varphi_{\eta,k}$ with domain $D^*_{\eta,k} \cup \{b_{\eta,k}, \varphi_{\eta,k}(b_{\eta,k})\}$ such that

$$\varphi(b_{\eta,k}) \in A_{\eta,k} \setminus D^*_{\eta,k}$$

and such that

$$(\forall k \in \omega)(\forall \theta \in G_\eta(k))\ \varphi_{\eta,k}(b_{\eta,k}) \neq \pi_{\theta,k}(b_{\eta,k}). \tag{3.27}$$

Now note that Condition (3.26) is satisfied because of $|Z_{\eta,k}| > |Z_{\zeta_\eta,k}|$. The verification that the other induction hypotheses hold is the same as in the proof Lemma! 3.4.

□

Corollary 3.3. *If Hypothesis 3.1 holds at $\kappa = \omega$ for some functions f, g and ψ then then there is a nontrivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$.*

Definition 3.3. Given functions f and g from ω to ω define and a filter $\mathcal{F}$ on ω define $\mathfrak{d}_{f,g}(\mathcal{F})$ to be the least cardinal of a family $\mathcal{D} \subseteq \prod_{k\in\omega}[f(k)]^{g(k)}$ such that for every $F \in \prod_{k\in\omega} f(k)$ there is $G \in \mathcal{D}$ such that $\{k \in \omega \mid F(k) \in G(k)\} \in \mathcal{F}^+$.

Remark 1. It can be verified that if $\mathcal{F}$ is generated by a $\subseteq^*$ descending tower of length ω_1 and then in order to obtain the conclusion of Lemma 3.5 it suffices to have the equality $\mathfrak{d}_{\psi,g}(\mathcal{F}) = \aleph_1$. This yields the following corollary.

Corollary 3.4. *If there is an $\aleph_1$-generated filter $\mathcal{F}$ such that $\mathfrak{d}_{\psi,g}(\mathcal{F}) = \aleph_1 \neq \mathfrak{d}$ then there is a nontrivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$.*

Proof. Let $\mathcal{F}$ be generated by $\{X_\xi\}_{\xi\in\omega_1}$. Use Rothberger's argument and $\aleph_1 \neq \mathfrak{d}$ to construct a $\subseteq^*$-descending sequence $\{Y_\xi\}_{\xi\in\omega_1}$ all of whose terms are $\mathcal{F}$ positive and such that $Y_\xi \subseteq X_\xi$. Let $\mathcal{F}'$ be generated by $\{Y_\xi\}_{\xi\in\omega_1}$ and note that $\mathfrak{d}_{\psi,g}(\mathcal{F}') = \aleph_1$. □

4. Remarks and questions

For the illustrative purposes of this last section, fix functions $g(k) = k$ and $\psi(k) = (3k2^{k^2+k} + 1)!$ as in Hypothesis 3.1. The first thing to note is that there are models where $\mathfrak{d}_{\psi,g} = \aleph_1 < 2^{\aleph_0}$ for f and g satisfying the Hypothesis 3.1 — for example, this is true in the model obtained by either iteratively adding ω_2 Sacks reals[1] or adding more than $\aleph_1$ Sacks reals side-by-side. Of course $\mathfrak{d} = \aleph_1$ also in these models. It is therefore of interest to note that the Laver property implies that $\mathfrak{d}_{\psi,g} = \aleph_1$ in the Laver model as well, yet $\mathfrak{d} = \aleph_2$ in this model. It should also be observed that it is possible for $\mathfrak{d}_{\psi,g}$ to be larger that $\mathfrak{d}$. For example, iteratively forcing ω_2 times with perfect trees T that are cofinally f branching will yield such a model.

To be a bit more precise, given $f : \omega \to \omega$ define $\mathbb{S}(f)$ to consist of all trees $T \subseteq \bigcup_{n\in\omega}\prod_{j\in n} f(j)$ such that or each $t \in T$ there is $s \supseteq t$ such that $s^\frown j \in T$ for all $j \in f(|s|)$. So Sacks forcing is just $\mathbb{S}(2)$ where 2 is the constant 2 function. The same proof as for Sacks forcing shows that $\mathbb{S}(f)$ is proper and adds no reals unbounded by the ground model. Iterating $\mathbb{S}(f)$ with countable support ω_2 times then yields model in which $\mathfrak{d} = \aleph_1$. However, if $g : \omega \to \omega$ and $\mathcal{H} \subseteq \prod_{n\in\omega}[f(n)]^{g(n)}$ has cardinality $\aleph_1$ then there is some model containing g and $\mathcal{H}$ and there is $\Gamma \in \prod_{n\in\omega} f(n)$ which is generic over this model. This genericity ensures that for all $h \in \mathcal{H}$ there are infinitely many j such that $\Gamma(j) \notin h(j)$.

It has already been shown in Corollary 3.2 that the hypotheses of Lemma 3.3 can be satisfied for uncountable cardinals, but it is worth noting that the generalization of Sacks reals to uncountable cardinals in [5] provides an alternate argument. It also has to be noted that the hypothesis of Corollary 3.4 is not vacuous in the sense that there are models of set theory in which it holds. For example, in the model obtained by iterating Miller reals ω_2 times the following hold:

- $\mathfrak{d} = \aleph_2$ because the Miller reals themselves are unbounded by the ground model
- $\mathfrak{d}_{f,g} = \aleph_1$ for appropriate f and g because the Miller partial order satisfies the Laver property
- $\mathfrak{u} = \aleph_1$ because P-points from the ground model generate ultrafilters in the extension.

However there does not seem to be any model demonstrating that the assumption that $\aleph_1 \neq \mathfrak{d}$ in Corollary 3.4 is essential. It is shown in [3] that it is consistent with set theory that $\mathfrak{d} = \aleph_1$ yet all automorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ are trivial. However, $\mathfrak{u} = \aleph_2$ in that model because random reals are added cofinally often. This motivates the following question.

Question 4.1. Does the existence of a nontrivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ follow from the assumption that there is an $\aleph_1$-generated filter $\mathcal{F}$ such that $\mathfrak{d}_{\psi,g}(\mathcal{F}) = \aleph_1$?

It is worth observing that the isomorphism of Lemma 2.1 is trivial on some infinite sets — indeed, if $\xi \in \kappa^+$ and $X \subseteq \mathbb{N}$ are such that $\{x\}$ belongs to some $\mathfrak{B}_{\xi,\nu}$ for each $x \in X$ then Φ is trivial on $\mathcal{P}(X)$. However, if $\mathcal{T}(\Phi)$ is defined to be the ideal $\{X \subseteq \mathbb{N} \mid \Phi \restriction \mathcal{P}(X) \text{ is trivial}\}$ then $\mathcal{T}(\Phi)$ is a small ideal in the sense that the quotient algebra $\mathcal{P}(\mathbb{N})/\mathcal{T}(\Phi)$ has large antichains, even modulo the ideal of finite sets — in the terminology of [2], the ideal $\mathcal{T}(\Phi)$ is not ccc by fin. To see

[1]See [1] for definitions of terms not defined in this section as well as for details of proofs.

this, simply observe that the proof of Lemma 2.1 actually shows that Hypothesis 4 of Lemma 2.1 can be strengthened to: For any one-to-one $F : \mathbb{N} \to \mathbb{N}$ there is $\xi \in \omega_1$ such that for all but finitely many $k \in \omega$ there is an atom $a \in \mathfrak{B}_{\xi,k}$ and $\iota \in a$ such that $F(\iota) \notin \Phi_{\xi,k}(a)$. It follows that if $Z \subseteq \mathbb{N}$ is infinite then $Z^* = \bigcup_{k\in Z} I_k \notin \mathcal{T}(\Phi)$. Hence, if $\mathcal{A}$ is an almost disjoint family of subsets of $\mathbb{N}$ then $\{A^* \mid A \in \mathcal{A}\}$ is an antichain modulo the ideal of finite sets.

One should not, therefore, expect to get a nowhere trivial isomorphism by these methods. It is nevertheless, conceivable that there are some other cardinal invariants similar to $\mathfrak{d}_{f,g}$ that would, when small, imply the existence of nowhere trivial isomorhisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$. In this context it is interesting to note that it is at least consistent with small $\mathfrak{d}$ that there are nowhere trivial isomorphisms.

Proposition 4.1. *It is consistent that $\aleph_1 = \mathfrak{d} \neq 2^{\aleph_0}$ and there is a nowhere trivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$.*

Sketch of proof. The partial order defined in Definition 2.1 of §2 of [13] will be used[2]. Begin with a model V satisfying $2^{\aleph_0} > \aleph_1$ and construct a tower of permutations $\{(A_\xi, F_\xi, \mathfrak{B}_\xi)\}_{\xi\in\mathrm{Lim}(\omega_1)}$ such that, letting $\mathfrak{S}_\eta = \{(A_\xi, F_\xi, \mathfrak{B}_\xi)\}_{\xi\in\mathrm{Lim}(\eta)}$ and $\mathbb{P}_\eta$ be the finite support iteration of partial orders that are $\mathbb{Q}(\mathfrak{S}_\xi)$ for $\xi \in \mathrm{Lim}(\eta)$ and Hechler forcing if ξ is a successor, the following holds for each η and G that is $\mathbb{P}_{\omega_1}$ generic over V:

- $A_\eta = A_{\mathfrak{S}_\eta}[G \cap \mathbb{Q}(\mathfrak{S}_\eta)]$
- $F_\eta = F_{\mathfrak{S}_\eta}[G \cap \mathbb{Q}(\mathfrak{S}_\eta)]$
- $\mathfrak{B}_\eta = \mathcal{P}(\mathbb{N}) \cap V[G \cap \mathbb{P}_\eta]$

The proof of Theorem 2.1 in [13] shows that there is a nowhere trivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ in this model and, since $\mathbb{P}_{\omega_1}$ is ccc, it is also true that $2^{\aleph_0}$ remains larger than $\aleph_1$ in the generic extension. The Hechler reals guarantee that $\mathfrak{d} = \aleph_1$. □

It should also be noted that Lemma 2.1 actually yields $2^{(\kappa^+)}$ isomorphisms. It is shown in [15] that it is possible to have non-trivial isomorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ without having $2^{\mathfrak{c}}$ such isomorphisms. This motivates the following, somewhat vague, question.

Question 4.2. Can there be some variant of $\mathfrak{d}_{f,g}$ which, when small, yields a non-trivial isomorphism of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ without yielding the maximal possible number of such?

Given the remarks following Corollary 3.4 it is natural to ask the following.

Question 4.3. Is it consistent that $\mathfrak{d}_{\psi,g} = \mathfrak{d}$ for f and g satisfying the Hypothesis 3.1 and to have $\mathfrak{u} = \aleph_1$ and to have that all isomorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ are trivial?

A positive answer to his question would require a model in which there is an ultrafilter of character $\aleph_1$ yet there are no P-points of character $\aleph_1$ since, as has already been mentioned in the introduction, it was shown by W. Rudin in [8, 9] that if there are P-points of character $\aleph_1$ then there are non-trivial isomorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$. It is an interesting, and well known, problem on its own whether or not the existence of an ultrafilter of character $\aleph_1$ implies the existence of a P-point of character $\aleph_1$.

As a final remark it will be noted that Corollary 3.3 shows that Theorem 3.1 of [13] cannot be improved to show that in models obtained by iterating Sacks or Silver reals all isomorphisms of $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\aleph_0}$ are trivial because the equality $\mathfrak{d}_{\psi,g} = \aleph_1$ holds in these models for the necessary f and g.

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[2] The reader is warned that the word "finite" should be removed from (3) of Definition 2.1 in [13].

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