Received: 2 June 2023 | Revised: 3 May 2024 | Accepted: 7 June 2024

DOI: 10.1002/malq.202300024

MATHEMATICAL LOGIC QUARTERLY

ORIGINAL PAPER

Implications of Ramsey Choice principles in ZF

Lorenz Halbeisen[1] | **Riccardo Plati**[1] | **Saharon Shelah**[2,3]

[1]Department of Mathematics, ETH Zentrum, Zurich, Switzerland

[2]Einstein Institute of Mathematics, The Hebrew University of Jerusalem, Yerushalayim, Israel

[3]Department of Mathematics, Hill Center – Busch Campus, Rutgers, State University of New Jersey, Highland Park, USA

Correspondence
Lorenz Halbeisen, Department of Mathematics, ETH Zentrum, Rämistrasse 101, 8092 Zurich, Switzerland.
Email: lorenz.halbeisen@math.ethz.ch

Funding information
Israel Science Foundation, Grant/Award Number: 1838/19; NSF, Grant/Award Number: DMS 1833363

The Ramsey Choice principle for families of n-element sets, denoted RC_n, states that every infinite set X has an infinite subset $Y \subseteq X$ with a choice function on $[Y]^n := \{z \subseteq Y : |z| = n\}$. We investigate for which positive integers m and n the implication $\mathrm{RC}_m \implies \mathrm{RC}_n$ is provable in ZF. It will turn out that beside the trivial implications $\mathrm{RC}_m \implies \mathrm{RC}_m$, under the assumption that every odd integer $n > 5$ is the sum of three primes (known as ternary Goldbach conjecture), the only non-trivial implication which is provable in ZF is $\mathrm{RC}_2 \implies \mathrm{RC}_4$.

1 | INTRODUCTION

For positive integers n, the Ramsey Choice principle for families of n-element sets, denoted RC_n, is defined as follows: For every infinite set X there is an infinite subset $Y \subseteq X$ such that the set $[Y]^n := \{z \subseteq Y : |z| = n\}$ has a choice function. The Ramsey Choice principle was introduced by Montenegro [5] who showed that for $n = 2, 3, 4$, $\mathrm{RC}_n \implies \mathrm{C}_n^-$. where C_n^- is the statement that every infinite family of n-element has an infinite subfamily with a choice function. However, the question of whether or not $\mathrm{RC}_n \to \mathrm{C}_n^-$ for $n \geq 5$ is still open (for partial answers to this question see [2, 3]).

In this paper, we investigate the relation between RC_n and RC_m for positive integers n and m. First, for each positive integer m we construct a permutation models $\mathbf{MOD}_m$ in which RC_m holds, and then we show that RC_n fails in $\mathbf{MOD}_m$ for certain integers n. In particular, assuming the ternary Goldbach conjecture, which states that every odd integer $n > 5$ is the sum of three primes, and by the transfer principles of Pincus [6], we we obtain that for $m, n \geq 2$, the implication $\mathrm{RC}_m \implies \mathrm{RC}_n$ is not provable in ZF except in the case when $m = n$, or when $m = 2$ and $n = 4$.

Fact 1.1. The implications $\mathrm{RC}_m \implies \mathrm{RC}_m$ (for $m \geq 1$) and $\mathrm{RC}_2 \implies \mathrm{RC}_4$ are provable in ZF.

Proof. The implication $\mathrm{RC}_m \implies \mathrm{RC}_m$ is trivial. To see that $\mathrm{RC}_2 \implies \mathrm{RC}_4$ is provable in ZF, we assume RC_2. If X is an infinite set, then by RC_2 there is an infinite subset $Y \subseteq X$ such that $[Y]^2$ has a choice function f_2. Now, for any $z \in [Y]^4$, $[z]^2$ is a 6-element subset of $[Y]^2$, and by the choice function f_2 we can select an element from each 2-element subset of z.

Sh:1243

For any $z \in [Y]^4$ and each $a \in z$, let $\nu_z(a) := |\{x \in [z]^2 : f_2(x) = a\}|$, $m_z := \min\{\nu_z(a) : a \in z\}$, and $M_z := \{a \in z : \nu_z(a) = m_z\}$. Since f_2 is a choice function, we have $\sum_{a \in z} \nu_z(a) = 6$, and since $4 \nmid 6$, the function $f : [Y]^4 \to Y$ defined by stipulating

$$f(z) := \begin{cases} a & \text{if } M_z = \{a\}, \\ b & \text{if } z \setminus M_z = \{b\}, \\ c & \text{if } |M_z| = 2 \text{ and } f_2(M_z) = c, \end{cases}$$

is a choice function on $[Y]^4$, which shows that RC_4 holds. □

2 | A MODEL IN WHICH RC_m HOLDS

In this section we construct a permutation model $\mathbf{MOD}_m$ in which RC_m holds. According to [1, p. 211 ff.], the model $\mathbf{MOD}_m$ is a *Shelah Model of the Second Type*.

Fix an integer $m \geq 2$ and let $\mathcal{L}_m$ be the signature containing the relation symbol Sel_m. Let T_m be the $\mathcal{L}_m$-theory containing the following axiom-schema:

> For all pairwise different $x_1, \dots, x_m$, there exists a unique index $i \in \{1, \dots, m\}$ such that, whenever $\{b_1, \dots, b_m\} = \{1, \dots, m\}$,
>
> $$\mathsf{Sel}_m(x_{b_1}, \dots, x_{b_m}, x_b) \iff b = i.$$

In other words, Sel_m is a selecting function which selects an element from each m-element set $\{x_1, \dots, x_m\}$. In any model of the theory T_m, the relation Sel_m is equivalent to a function Sel which selects a unique element from any m-element set.

For a model $\boldsymbol{M}$ of T_m with domain M, we will simply write $M \vDash \mathsf{T}_m$. Let

$$\widetilde{C} = \{M : M \in \mathrm{fin}(\omega) \wedge M \vDash \mathsf{T}_m\}.$$

Evidently $\widetilde{C} \neq \emptyset$. Partition $\widetilde{C}$ into maximal isomorphism classes and let C be a set of representatives. We proceed with the construction of the set of atoms for our permutation model. With the next result, taken from [1], we give an explicit construction of the Fraïssé limit of the finite models of T_m.

Proposition 2.1. Let $m \in \omega \setminus \{0\}$. There exists a model $\mathbf{F} \vDash \mathsf{T}_m$ with domain ω such that:

1. Given a non empty $M \in C$, $\mathbf{F}$ admits infinitely many submodels isomorphic to M.
2. Any isomorphism between two finite submodels of $\mathbf{F}$ can be extended to an automorphism of $\mathbf{F}$.

Proof. The construction of $\mathbf{F}$ is made by induction. Let $F_0 = \emptyset$. F_0 is trivially a model of T_m and, for every element M of C with $|M| \leq 0$, F_0 contains a submodel isomorphic to M. Let F_n be a model of T_m with a finite initial segment of ω as domain and such that for every $M \in C$ with $|M| \leq n$, F_n contains a submodel isomorphic to M. Let

1. $\{A_i : i \leq p\}$ be an enumeration of $[F_n]^{\leq n}$,
2. $\{R_k : k \leq q\}$ be an enumeration of all the $M \in C$ such that $1 \leq |M| \leq n+1$,
3. $\{j_l : l \leq u\}$ be an enumeration of all the embeddings $j_l : F_n|_{A_i} \hookrightarrow R_k$, where $i \leq p$, $k \leq q$ and $|R_k| = |A_i| + 1$.

For each $l \leq u$, let $a_l \in \omega$ be the least natural number such that $a_l \notin F_n \cup \{a_{l'} : l' < l\}$. The idea is to add a_l to F_n, extending $F_n|_{A_i}$ to a model $F_n|_{A_i} \cup \{a_l\}$ isomorphic to R_k, where $j_l : F_n|_{A_i} \hookrightarrow R_k$. Define $F_{n+1} := F_n \cup \{a_l : l \leq u\}$ and make F_{n+1} into a model of T_m by choosing a way of defining the function Sel on the missing subsets. The desired model is finally given by $\mathbf{F} = \bigcup_{n \in \omega} F_n$.

We conclude by showing that every isomorphism between finite submodels can be extended to an automorphism of $\mathbf{F}$ with a back-and-forth argument. Let $i_0 : M_1 \to M_2$ be an isomorphism of T_m-models. Let a_1 be the least natural number in $\omega \setminus M_1$. Then $M_1 \cup \{a_1\}$ is contained in some F_n and by construction we can find some $a_1' \in \omega \setminus M_2$ such that $\mathbf{F}|_{M_1 \cup \{a_1\}}$

is isomorphic to $\mathbf{F}|_{M_2 \cup \{a_1'\}}$. Extend i_0 to $l_1 : M_1 \cup \{a_1\} \to M_2 \cup \{a_1'\}$ by imposing $l_1(a_1) = a_1'$. Let b_1' be the least integer in $\omega \setminus (M_2 \cup \{a_1'\})$ and similarly find some $b_1 \in \omega \setminus (M_1 \cup \{a_1\})$ such that we can extend l_1 to an isomorphism $i_1 : M_1 \cup \{a_1, b_1\} \to M_2 \cup \{a_1', b_1'\}$ which maps b_1 to b_1'. Repeating the process countably many times, the desired automorphism of $\mathbf{F}$ is given by $i = \bigcup_{n\in\omega} i_n$. □

Remark 2.2. Let us fix some notations and terminology. The elements of the model $\mathbf{F}$ above constructed will be the atoms of our permutation model. Each element a corresponds to a unique embedding j. We shall call the domain of j the *ground* of a. Moreover, given two atoms a and b, we say that $a < b$ in case $a <_\omega b$ according to the natural ordering. Notice that this well ordering of the atoms will not exist in the permutation model.

Let A be the domain of the model $\mathbf{F}$ of the theory T_m. To build the permutation model $\mathbf{MOD}_m$, consider the normal ideal given by all the finite subsets of A and the group of permutations G defined by

$$\pi \in G \iff \forall X \in [\omega]^m, \pi(\mathsf{Sel}(X)) = \mathsf{Sel}(\pi X).$$

Theorem 2.3. For every positive integer m, $\mathbf{MOD}_m$ is a model for RC_m.

Proof. Let X be an infinite set with support S'. If X is well ordered, the conclusion is trivial, so let $x \in X$ be an element not supported by S' and let S be a support of x, with $S' \subseteq S$. Let $a \in S \setminus S'$. If $\mathrm{fix}_G(S \setminus \{a\}) \subseteq \mathrm{sym}_G(x)$ then $S \setminus \{a\}$ is a support of x, so by iterating the process finitely many times we can assume that there exists a permutation $\tau \in \mathrm{fix}_G(S \setminus \{a\})$ such that $\tau(x) \neq x$. Our conclusion will follow by showing that there is a bijection between an infinite set of atoms and a subset of X, namely between $I = \{\pi(a) : \pi \in \mathrm{fix}_G(S \setminus \{a\})\}$ and $\{\pi(x) : \pi \in \mathrm{fix}_G(S \setminus \{a\})\}$. First, notice that for $\pi \in \mathrm{fix}_G(S \setminus \{a\})$ the function $f : \pi(a) \mapsto \pi(x)$ is well defined on I. Indeed, if for some $\sigma, \pi \in \mathrm{fix}_G(S \setminus \{a\})$ we have $\sigma(x) \neq \pi(x)$, then $\pi^{-1}\sigma(x) \neq x$, which implies $\pi^{-1}\sigma(a) \neq a$ since S is a support of x. To show that f is also injective, suppose towards a contradiction that there are two permutations $\sigma, \sigma' \in \mathrm{fix}_G(S \setminus \{a\})$ such that $\sigma(x) = \sigma'(x)$ and $\sigma(a) \neq \sigma'(a)$. Then, by direct computation, the permutation $\sigma^{-1}\sigma'$ is such that $\sigma^{-1}\sigma'(a) \neq a$ and $\sigma^{-1}\sigma'(x) = x$. Let $b = \sigma^{-1}\sigma'(a)$. Now, by assumption there is a permutation $\tau \in \mathrm{fix}_G(S \setminus \{a\})$ such that $\tau(x) \neq x$. Let $y := \tau(x)$, with $c = \tau(a)$ and $d = \sigma^{-1}\sigma'(c)$. Notice that from $f(a) = f(b)$ we get $f(c) = f(d)$. Let now $e \in A$ be an atom with ground $S \cup \{c\}$ such that e behaves like b with respect to S and like d with respect to $(S \setminus \{a\}) \cup \{c\}$. This is possible by construction of the set of atoms since b and d behave in the same way with respect to $S \setminus \{a\}$. It follows that there are permutations $\pi_b \in \mathrm{fix}_G(S)$ and $\pi_d \in \mathrm{fix}_G((S \setminus \{a\}) \cup \{c\})$ with $\pi_b(b) = e$ and $\pi_d(d) = e$. Let us now consider $f(e)$. On the one hand, since $(S \setminus \{a\}) \cup \{c\}$ is a support of $y = f(d)$, we have $y = \pi_d(f(d)) = f(\pi_d(d)) = f(e)$. On the other hand, since S is a support of $x = f(b)$, we have $x = \pi_b(f(b)) = f(\pi_b(b)) = f(e)$, contradicting the fact that $x \neq y$. □

3 | FOR WHICH n IS $\mathbf{MOD}_m$ A MODEL FOR RC_n?

The following result shows that for positive integers m, n which satisfy a certain condition, the implication $\mathrm{RC}_m \implies \mathrm{RC}_n$ is not provable in ZF. Assuming the ternary Goldbach conjecture, it will turn out that all positive integers m, n satisfy this condition, except when $m = n$, or when $m = 2$ and $n = 4$.

Definition 3.1. Given $n \in \omega$, a decomposition of n is a finite sequence $(n_i)_{i\in k}$ with each $n_i \in \omega \setminus \{1\}$ so that $n = \sum_{i\in k} n_i$.

Definition 3.2. Given two natural numbers n and m, a decomposition $(n_i)_{i\in k}$ of n is said to be beautiful for the pair (m, n) if, given any decomposition $(m_i)_{i\in k}$ of m of length k such that for all $i \in k$ we have $m_i \leq n_i$, then there is some $j \in k$ with $\gcd(m_j, n_j) = 1$.

In what follows, when we refer to a decomposition of some n being beautiful, we mean that the decomposition is beautiful for (m, n). It will always be clear from the context to which pair (m, n) we refer.

Proposition 3.3. Let $m, n \in \omega$. If there is a decomposition of n which is beautiful, then the implication $\mathrm{RC}_m \implies \mathrm{RC}_n$ is not provable in ZF.

Remark 3.4. The condition on m and n is somewhat similar to the condition given in Theorem 2.10 of Halbeisen and Schumacher [2]. Let WOC_n^- be the statement that every infinite, well-orderable family $\mathcal{F}$ of sets of size n has an infinite subset $\mathcal{G} \subseteq \mathcal{F}$ with a choice function. Then for every $m, n \in \omega \setminus \{0, 1\}$, the implication $\mathrm{RC}_m \implies \mathrm{WOC}_n^-$ is provable in ZF if an only if the following condition holds: Whenever we can write n in the form

$$n = \sum_{i<k} a_i p_i,$$

where $p_0, \dots, p_{k-1}$ are prime numbers and $a_0, \dots, a_{k-1} \in \omega \setminus \{0\}$, then we find integers $b_0, \dots, b_{k-1} \in \omega$ with

$$m = \sum_{i<k} b_i p_i.$$

Proof of Propostion 3.3. We show that in $\mathbf{MOD}_m$, RC_n fails. Assume towards a contradiction that RC_n holds in $\mathbf{MOD}_m$ and let S be a support of a selection function f on the n-element subsets of an infinite subset X of the set of atoms A.

Given any finite model N of T_m extending S, we can find a submodel of $X \cup S$ isomoprhic to N. Indeed, start by noticing that, since S is a support of f and X is the domain of f, we have that X is symmetric. Then the claim follows directly from the construction in Proposition 2.1, as atoms whose ground includes the support of $X \cup S$ can belong to $X \cup S$ and can behave in arbitrarily chosen ways with respect to each other.

Our conclusion can hence follow from finding a model M of T_m which extends S with $|M \setminus S| = n$ and such that M admits an auotmorphism σ which fixes pointwise S and which does not have any other fixed point, since then $\sigma(f(M \setminus S)) \neq f(M \setminus S)$ but $\sigma(M \setminus S) = M \setminus S$. We start with the following claim:

Claim 3.5. Given a cyclic permutation π on some set P of cardinality $|P| = q$, if a non-trivial power π^r of π fixes a proper subset P' of P, then $\gcd(|P'|, |P|) > 1$.

To prove the claim, notice that π^r is a disjoint union of cycles of the same length $l = \frac{q}{\gcd(q,r)}$. Consider the subgroup of $\langle \pi \rangle$ given by $\langle \pi^r \rangle$. Then P' is a disjoint union of orbits of the form $\mathrm{Orb}_{<\pi^r>}(e)$ with $e \in P'$, all of them with the same cardinality s, with s being a divisor of $l = \frac{q}{\gcd(q,r)}$ and hence of q, from which we deduce the claim.

Now, given a beautiful decomposition $(n_i)_{i \in k}$ of n, we want to show that we can find a model M of T_m, which extends S with $|M \setminus S| = n$ and such that it admits an automorphism σ which fixes pointwise S and acts on $M \setminus S$ as a disjoint union of k cycles, each of length n_i for $i \in k$. This can be done as follows. Pick an m-element subset P of M for which $\mathsf{Sel}(P)$ has not been defined yet. If $P \cap S \neq \emptyset$ then let $\mathsf{Sel}(P)$ be any element in $P \cap S$. Otherwise, by our the assumptions, there is a cycle C_j of length n_j for some $j \in k$ such that $\gcd(|P \cap C_j|, |C_j|) = 1$. Define $\mathsf{Sel}(P)$ as an arbitrarily fixed element of $P \cap C_j$ and, for all permutations π in the group generated by σ, define $\mathsf{Sel}(\pi(P)) = \pi(\mathsf{Sel}(P))$. We need to argue that this is indeed well defined, i.e. that for two permutations $\pi, \pi' \in \langle \sigma \rangle$ we have that $\pi(P) = \pi'(P)$ implies $\pi(\mathsf{Sel}(P)) = \pi'(\mathsf{Sel}(P))$. Problems can arise only when $P \cap S = \emptyset$, in which case we notice that $\pi(P) = \pi'(P)$ implies $\pi(P \cap C_j) = \pi'(P \cap C_j)$, which in turn by the claim implies that $\pi^{-1} \circ \pi'$ fixes $P \cap C_j$ pointwise, from which we deduce $\pi(\mathsf{Sel}(P)) = \pi'(\mathsf{Sel}(P))$. □

Proposition 3.3 allows us to immediately deduce the following results.

Corollary 3.6. If $m > n$, then RC_m does not imply RC_n.

Proof. The decomposition $n = \sum_{i \in 1} n_i$ with $n_0 = n$ is clearly beautiful, so we can directly apply Proposition 3.3. □

Corollary 3.7. If there is a prime p for which $p \mid n$ but $p \nmid m$, then RC_m does not imply RC_n.

Proof. Given the assumption, the decomposition of n given by $n = \sum_{i \in \frac{n}{p}} n_i$, where each $n_i = p$, is beautiful, so we can apply Proposition 3.3. □

Moreover, we can show the following:

Theorem 3.8. For any positive integers m and n, the implication $\mathrm{RC}_m \implies \mathrm{RC}_n$ is provable in ZF only in the case when $m = n$ or when $m = 2$ and $n = 4$.

The proof of Theorem 3.8 is given in the following results, where in the proofs we use two well-known number-theoretical results: The first one is Bertrand's postulate, which asserts that for every positive integer $m \geq 2$ there is a prime p with $m < p < 2m$, and the second one is ternary Goldbach conjecture (assumed to be proven by Helfgott [4]), which asserts that every odd integer $n > 5$ is the sum of three primes.

Proposition 3.9. If m is prime and $n \neq m$ with $(m, n) \neq (2, 4)$, then the implication $\mathrm{RC}_m \implies \mathrm{RC}_n$ is not provable in ZF.

Proof. Given Corollary 3.7, we can assume that $n = m^k$ for some natural number $k > 1$. Let p be a prime such that $m < p < 2m$, whose existence is guaranteed by Bertrand's postulate. Then clearly $m \nmid n - p$, from which, considering that because of parity reasons $n - p \neq 1$, we get that the decomposition $n = p + (n - p)$ is beautiful. □

Proposition 3.10. If n is odd and $m \neq n$, then the implication $\mathrm{RC}_m \implies \mathrm{RC}_n$ is not provable in ZF.

Proof. By the ternary Goldbach conjecture, let us write n as sum of three primes $n = p_0 + p_1 + p_2$. Given Proposition 3.9, we can assume that $m = p_0 + p_1$, since otherwise the decomposition $n = p_0 + p_1 + p_2$ would be beautiful.

We first deal with the case in which $p_0 = p_1 = p_2$ holds, for which we rename $p = p_0$. By hand we can exclude the case $p = 2$, and now we want to show that the decomposition $n = n_0 + n_1 = (3p - 2) + 2$ is beautiful. Notice that $\gcd(3p - 2, 2p - 2) \in \{1, p\}$, from which we deduce that necessarily if $m = m_0 + m_1$ is a decomposition of m with $m_0 \leq 3p - 2$ and $m_1 \leq 2$, then $m_1 = 0$. To conclude this first case, it suffices to notice that, since p is a prime grater than 2, $\gcd(3p - 2, 2p)$ necessarily equals 1.

We can now assume that it is not true that $p_0 = p_1 = p_2$. Since n is odd, $p_0 + p_1 \nmid p_2$. If $p_2 \nmid p_0 + p_1$, then the decomposition $n = n$ is actually beautiful. So, given $p_2 \mid p_0 + p_1$, without loss of generality let us assume that $p_2 < p_0$. By $p_2 \mid p_0 + p_1$ we deduce that $p_1 \neq p_2$, and we now consider the decomposition $n = n_0 + n_1 = (p_1 + p_2) + p_0$. We can't have $m_1 = p_0$ since $\gcd(p_1, p_1 + p_2) = 1$. On the other hand, we can't even have $m_1 = 0$ since $p_0 + p_1 > p_1 + p_2$, which proves that the assumptions of Proposition 3.3 are satisfied. □

Proposition 3.11. Let $m > 2$ be an even natural number and $k \in \omega$ such that $2^k + 1$ is prime. If $n = m + 2^k$, then the implication $\mathrm{RC}_m \implies \mathrm{RC}_n$ is not provable in ZF.

Proof. We consider the decomposition $n = n_0 + n_1 = (m - 1) + (2^k + 1)$. It directly follows from the assumptions of the proposition that in order to have a decomposition $m = m_0 + m_1$ which disproves the fact that the above decomposition of n is beautiful, since $n_0 < m$, necessarily $m_1 = 2^k + 1$, from which we deduce $m_0 = m - 2^k - 1$. This immediately gives a contradiction in the case $2^k + 1 > m$, so let us assume $2^k + 1 < m$. We get again a contradiction by the fact that $\gcd(m_0, n_0) = \gcd(m - 2^k - 1, m - 1) = \gcd(2^k, m - 1) = 1$, where we used that m is even. We can hence conclude that the decomposition $n = (m - 1) + (2^k + 1)$ is indeed beautiful. □

Proposition 3.12. Let m and n be even natural numbers such that there is an odd prime p with $m < p < n$ and $n > p + 1$. Then the implication $\mathrm{RC}_m \implies \mathrm{RC}_n$ is not provable in ZF.

Proof. If $n = p + 3$ or $n = p + 5$ the decomposition $n = p + (n - p)$ is already beautiful. Otherwise, by the ternary Goldbach conjecture, write $n - p$ as sum of three primes $n - p = p_0 + p_1 + p_2$. Consider now the decomposition $n = \sum_{i \in 4} n_i = p + p_0 + p_1 + p_2$. In order to write $m = \sum_{i \in 4} m_i$, necessarily $m_0 = 0$. If $n - p < m$ we can already conclude that $n = p + p_0 + p_1 + p_2$ is a beautiful decomposition. Otherwise, we find ourselves in the assumptions of Proposition 3.10, which again allows us to conclude that RC_m does not imply RC_n. □

The following result deals with all the remaining cases and completes the proof of Theorem 3.8.

Sh:1243

Proposition 3.13. Let m and n be even natural numbers with $3 \le \frac{n}{2} \le m < n$ such that if there is a prime p with $m < p < n$, then $p = n - 1$. Then the implication $\mathrm{RC}_m \implies \mathrm{RC}_n$ is not provable in ZF.

Proof. By Bertrand's postulate, let p be a prime with $\frac{n}{2} < p < n$. This implies by the assumption $\frac{n}{2} < p < m$ or $p = n - 1$. If we are in the latter case, apply again Bertrand's postulate to find a further prime $\frac{n}{2} - 1 < p' < n - 2$ (notice that by our assumption we have $2 \le \frac{n}{2} - 1$). Since m is not prime we necessarily have $p' \ne m$, which together with the present assumptions makes us able to assume without loss of generality that $\frac{n}{2} < p < m$. Given that $n - m$ is even, by Proposition 3.11 we can assume $n - m > 4$, which in turn implies $n - p > 5$. Since by the ternary Goldbach conjecture we can write $n = p + p_0 + p_1 + p_2$ with $m > p_0 + p_1 + p_2$, notice that by the fact that n and m are even, we can assume that $m - p$ equals some odd prime p', since otherwise the decomposition $n = p + p_0 + p_1 + p_2$ would already be beautiful. Now, either $n = p + (n - p)$ is beautiful, or $n - p$ is a multiple of p'. We distinguish two cases, namely when $n - p$ is a power of p' and when it is not. In the second case, let p'' be a prime distinct from p' such that $p'' \mid n - p$. The decomposition of n given by $n = n_0 + \sum_{i \in \frac{n-p}{p''}} n_i = p + \sum_{i \in \frac{n-m}{p''}} p''$ is beautiful, as $n - p < m$ and hence if $m = m_0 + \sum_{i \in \frac{n-m}{p'}} m_i$ then $m_0 = p$. For the last case, without loss of generality assume that $p_0 + p_1 + p_2 = p_0^k$ for some natural number $k > 1$. If $p_0 = p_1 = p_2 = 3$, we decompose $9 = n - p$ as $5 + 2 + 2$, so we can assume $p_0^{k-1} - 2 \ne 1$. Now we get $p_2 \ne p_0$, since otherwise we would have $p_1 = p_0^k - 2p_0 = p_0(p_0^{k-1} - 2)$, which is a contradiction, and similarly we obtain $p_1 \ne p_0$. We finally assume wlog that $p_1 > p_0$, which allows us to conclude that the decomposition $n = p + p_1 + (p_0 + p_2)$ is in this case beautiful, concluding the proof. □

For the sake of completeness, we summarise the proof of our main theorem:

Proof of Theorem 3.8. Let m and n be two distinct positive integers.

$$\mathrm{ZF} \vdash \mathrm{RC}_m \implies \mathrm{RC}_n \overset{\text{Cor. 3.4}}{\implies} m \le n \overset{\text{Prp. 3.8}}{\implies} n \text{ is even} \overset{\text{Cor. 3.5}}{\implies} m \text{ is even}$$

Now, if m and n are both even, we have the following two cases:

$$m < \frac{n}{2} \overset{\text{Prp. 3.10}}{\implies} \mathrm{ZF} \nvdash \mathrm{RC}_m \implies \mathrm{RC}_n$$

$$m \ge \frac{n}{2} \ge 3 \underset{\text{Prp. 3.10}}{\overset{\text{Prp. 3.11}}{\implies}} \mathrm{ZF} \nvdash \mathrm{RC}_m \implies \mathrm{RC}_n$$

Thus, by Fact 1.1, the implication $\mathrm{RC}_m \implies \mathrm{RC}_n$ is provable in ZF if and only if $m = n$ or $m = 2$ and $n = 4$. □

Remark 3.14. The proof of the implication $\mathrm{RC}_2 \implies \mathrm{RC}_4$ (Fact 1.1) is very similar to the proof of the implication $\mathrm{C}_2 \implies \mathrm{C}_4$, where C_n states that every family n-element sets has a choice function. Moreover, similar to the proof of $\mathrm{C}_2 \wedge \mathrm{C}_3 \implies \mathrm{C}_6$ one can proof the implication $\mathrm{RC}_2 \wedge \mathrm{RC}_3 \implies \mathrm{RC}_6$. So, it might be interesting to investigate which implications of the form

$$\mathrm{RC}_{m_1} \wedge \cdots \wedge \mathrm{RC}_{m_k} \implies \mathrm{RC}_n$$

are provable in ZF and compare them with the corresponding implications for C_n's. Since $\mathrm{C}_4 \implies \mathrm{C}_2$ but $\mathrm{RC}_4 \not\Rightarrow \mathrm{RC}_2$, the conditions for the RC_n's are clearly different from the conditions for the C_n's (cf. Halbeisen and Tachtsis [3] for some results in this direction).

ACKNOWLEDGMENTS

We would like to thank the referee for her or his careful reading and the numerous comments that helped to improve the quality of this article. Saharon Shelah was partially supported by the Israel Science Foundation, grant no. 1838/19 and by NSF grant DMS 1833363.

Sh:1243

CONFLICT OF INTEREST STATEMENT

The authors declare no conflicts of interest.

ORCID

Lorenz Halbeisen https://orcid.org/0000-0001-6078-7237

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