$\aleph_1$-FREE ABELIAN NON-ARCHIMEDEAN POLISH GROUPS

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Abstract. An uncountable $\aleph_1$-free group can not admit a Polish group topology but an uncountable $\aleph_1$-free abelian group can, as witnessed e.g. by the Baer-Specker group $\mathbb{Z}^\omega$. In this paper we study $\aleph_1$-free abelian non-Archimedean Polish groups. We prove two results. The first is that being $\aleph_1$-free abelian is a complete co-analytic problem in the space of closed subgroups of S_∞. The second is that there are continuum many $\aleph_1$-free abelian non-Archimedean Polish groups which are not topologically isomorphic to product groups and are pairwise not continuous homomorphic images of each other.

1. Introduction

Recall that a topological group G is said to be Polish if its group topology is Polish, i.e., separable and completely metrizable, and that it is said to be non-Archimedean Polish if in addition it has a basis at the identity that consists of open subgroups. In a meeting in Durham in 1997, D. Evans asked if a non-Archimedean Polish group can be an uncountable free group. Around the same time H. Becker and A. Kechris asked the same question without the assumption of non-Archimedeanity (cf. [1]). In [12] the second author settled the first question, and in the subsequent paper [13] he settled also the second question. It was later discovered that some of the the impossibility results from [13] followed already from an old important result of Dudley [3]. All these results were later vastly generalized in [10] by both authors of the present paper in the context of graph product of groups. In another direction, Khelif proved in [8] that no uncountable $\aleph_1$-free groups can be Polish, where we recall that a group G is said to be $\aleph_1$-free (resp. $\aleph_1$-free abelian) if every countable subgroups of G is free (resp. free abelian). Furthermore, a Polish group cannot be free abelian either, by [13]. In contrast, as well-known, the Baer-Specker group $\mathbb{Z}^\omega$ when topologized with the product topology is non-Archimedean Polish and $\aleph_1$-free (but not free). This interesting state of affairs motivated us to investigate the structure of the class of $\aleph_1$-free abelian non-Archimedean Polish groups, wondering in particular: how complicated are these topological groups?

This question connects with the recent interest in abelian non-Archimedean Polish groups stemming from the work of L. Ding and S. Gao [2]. In particular, it was shown in [2] that the non-Archimedean Polish groups are exactly the abelian pro-countable groups, i.e., the collection of inverse limits of countable inverse systems of countable abelian groups $(A_i : i \in \mathbb{N})$. In a sense, the $\aleph_1$-free abelian Polish

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non-Archimedean groups are the simplest groups that one can consider, among the reduced torsion-free abelian (TFAB) non-Archimedean Polish groups, where we recall that an abelian group is said to be reduced if it has no non-trivial divisible subgroups (notice that every TFAB splits as a sum $A \oplus B$ with A reduced and B divisible and, as divisible TFAB's are simply sums of $\mathbb{Q}$, it seems reasonable to only look at reduced TFAB's – this assumption is standard in the study of TFAB's).

In this short note we present two results that show that this class of groups is *complicated*, in the appropriate sense. The first result is a purely algebraic result, where by *product group* we mean a group of the form $\prod_{n<\omega} H_n$ (cf. e.g. [11]).

Theorem 1.1. *There are continuum many $\aleph_1$-free abelian non-Archimedean Polish groups which are not topologically isomorphic to product groups and are pairwise not continuous homomorphic images of each other (and so they are not topologically isomorphic). Furthermore, all these groups can be taken to be inverse limits of torsion-free completely decomposable groups (i.e., direct sums of* TFAB*'s of rank* 1*).*

The second negative result that we present is formulated in the framework of invariant descriptive set theory (see [6] for an excellent introduction to this subject). In [7], A. Kechris, A. Nies and K. Tent proposed the following program: as any non-Archimedean Polish group is topologically isomorphic to a closed subgroup of the topological group S_∞ of bijections of $\mathbb{N}$ onto $\mathbb{N}$, we can study classification problems on non-Archimedean Polish groups with respect to the Effros structure on the set of closed subgroups of S_∞, which is a standard Borel space (see [7] for details). At the heart of the program of [7] there is the following two-folded task:

- for natural classes of closed subgroups of S_∞, determine whether they are Borel;
- if a class is Borel, study the Borel complexity of topological isomorphism.

In this respect, in our second theorem, we prove that the class of $\aleph_1$-free abelian non-Archimedean Polish groups is as complicated as possible (as a co-analytic class).

Theorem 1.2. *Determining if a non-Archimedean Polish group is an $\aleph_1$-free abelian group is a complete co-analytic problem in the space of closed subgroups of S_∞.*

2. The descriptive set theory setting

From here on, as it is customary in set theory, we denote the set of natural numbers by ω. The aim of this section is to fix the coding, i.e., to specify which spaces we refer to in our results, together with introducing the necessary background.

Fact 2.1 ([2, Proposition 2.1])**.** The G be a topological group, then TFAE:

(1) G is a non-Archimedean Polish abelian group;
(2) G is isomorphic (as a topological group) to a closed subgroup of $\prod_{n<\omega} H_n$ (with the product topology), with each H_n abelian, countable and discrete;
(3) G is the inverse limit of an inverse system $(G_n, f_{(n,m)} : m \leqslant n < \omega)$, with each G_n abelian, countable, discrete, where on G we consider the product topology.

Notation 2.2. Let X be a Polish space. The Effros structure on X is the Borel space consisting of the family $\mathcal{F}(X)$ of closed subsets of X together with the σ-algebra generated by the following sets $\mathcal{C}_U$, where, for $U \subseteq X$ open, we let:

$$\mathcal{C}_U = \{D \in \mathcal{F}(X) : D \cap U \neq \emptyset\}.$$

Notation 2.3. We denote by S_∞ the topological group of bijections of ω onto ω, where the topology is the one induced by the following metric:

(1) if $x = y$, then $d(x, y) = 0$;
(2) if $x \neq y$, then $d(x, y) = 2^{-n}$, where n is least such that $x(n) \neq y(n)$.

Definition 2.4. By a tree on ω we mean a non-empty subset of $\omega^{<\omega}$ closed under initial segments (so in particular the empty sequence $\emptyset$ is the root of the tree).

Fact 2.5. Let $\mathcal{T}(S_\infty)$ denote the set of trees on ω with no leafs topologized defining the distance between two trees $T_1 \neq T_2$ as $\frac{1}{2^n}$, where n is least n such that $T_1 \cap \omega^n \neq T_1 \cap \omega^n$. For $C \in \mathcal{F}(S_\infty)$, define:

$$\mathbf{B} : C \mapsto T_C = \{g \restriction n : g \in C \text{ and } n < \omega\} \cup \{g^{-1} \restriction n : g \in C \text{ and } n < \omega\}.$$

Then the map $\mathbf{B}$ is a Borel isomorphism from $\mathcal{F}(S_\infty)$ onto $\mathcal{T}(S_\infty)$.

In virtue of Fact 2.5, we can work with $\mathcal{T}(S_\infty)$ instead of $\mathcal{F}(S_\infty)$.

Fact 2.6. The closed subgroups of S_∞ form a Borel subset of $\mathcal{F}(S_\infty)$ (see [7]), which we denote by $\mathrm{Sgp}(S_\infty)$, together with the Borel structure inherited from $\mathcal{F}(S_\infty)$.

Proviso 2.7. *From now on we only look at inverse systems with binding onto maps.*

Notation 2.8. We can look at an inverse system $A = (A_n, f_{(n,m)} : m \leqslant n < \omega)$ as a first-order structure in a language $L_{\mathrm{inv}} = \{P_n, \cdot_n, f_{(n,m)} : m \leqslant n < \omega\}$, where:
(1) the P_n's are disjoint predicates denoting the groups A_n's;
(2) the $\cdot_n$'s are binary function symbols, interpreted as the operations on the A_n's;
(3) the $f_{(n,m)}$'s are function symbols interpreted as morphisms from A_n to A_m.

Notation 2.9. Of course with respect to the language considered in 2.8 saying that a structure of the form $A = (A_n, f_{(n,m)} : m \leqslant n < \omega)$ is an inverse system is axiomatizable in first-order logic. Thus, the usual setting of invariant descriptive set theory applies and we can consider the Borel space of countable Pro-Groups with domain (a subset of) ω. For concreteness we also assume that our inverse systems $A = (A_n, f_{(n,m)} : m \leqslant n < \omega)$ are such that $\mathrm{dom}(A_n) \subseteq \{p_n^m : 0 < m < \omega\}$, where p_n is the n-th prime number, so that the different sorts A_n's are disjoint.

Definition 2.10. Let $A = (A_n, f_{(n,m)} : m \leqslant n < \omega)$ be an inverse system as in 2.9, so that in particular $\mathrm{dom}(A_n) \subseteq \{p_n^m : 0 < m < \omega\}$, where p_n is the n-th prime number. For every $\bar{a} = (a_n : n < \omega) \in \varprojlim(A)$ we define a permutation $\pi_{\bar{a}}$ of ω as:

$$\pi_{\bar{a}}(k) = \begin{cases} k +_{A_n} a_n, & \text{if there is } n < \omega \text{ s.t. } k \in A_n, \\ k, & \text{if } k \notin \bigcup_{n<\omega} A_n. \end{cases}$$

Observation 2.11. The map $\bar{a} \mapsto \pi_{\bar{a}}$ from 2.10 is an embedding of $\varprojlim(A)$ into S_∞.

We omit the details of the next proposition as its proof is standard (use e.g. 2.5).

Proposition 2.12. *The map $A \mapsto \varprojlim(A)$ from* Pro-Groups *(recalling 2.9) into* $\mathrm{Sgp}(S_\infty)$ *(recalling the embedding described in 2.10) is Borel.*

3. Preliminary results on TFAB's

The main source of construction of Polish non-Archimedean TFAB's present in this paper is described in the following definition, which we refer to as *the engine*.

Definition 3.1. Let $T \subseteq \omega^{<\omega}$ be a tree with no leafs and $L : T \to \mathcal{P}(\omega \setminus \{0\})$ be s.t.:
(a) if $\eta \trianglelefteq \nu$, then $L(\nu) \subseteq L(\eta)$;

(b) for every $\eta \in T$, $n \in L(\eta)$, and $m \,|\, n \Rightarrow m \in L(\eta)$;
(c) if $\eta \in T$, then $L(\eta) = \bigcup\{L(\nu) : \nu \in T$ and ν is a successor of $\eta\}$.

We define an inverse system $\mathrm{inv}(T, L) = (G_n, f_{(m,n)} : m \leqslant n < \omega)$ as follows:

(i) $G_n \leqslant \sum\{\mathbb{Q}x_\eta : \eta \in T \cap \omega^n\} := H_n$;
(ii) $G_n = \langle \frac{1}{a} x_\eta : \eta \in T \cap \omega^n$ and $a \in L(\eta)\rangle_{H_n}$;
(iii) for $m \leqslant n < \omega$, $f_{(n,m)}$ is the unique homomorphism from G_n into G_m s.t.:

$$\eta \in T \cap \omega^n \;\Rightarrow\; x_\eta \mapsto x_{\eta \restriction m}.$$

Notice that the inverse system $\mathrm{inv}(T, L) = (G_n, f_{(n,m)} : m \leqslant n < \omega)$ is well-defined because of conditions (a) and (b) above. Being it well-defined, we do the following:

$$(T, L) \mapsto \mathrm{inv}(T, L) \mapsto \varprojlim(\mathrm{inv}(T, L)) := G(T, L).$$

So the group $G(T, L)$ is a well-defined Polish non-Archimedean abelian group, when equipped with the topology inherited from the product topology on $\prod_{n<\omega} G_n$.

Remark 3.2. Notice that the inverse systems $\mathrm{inv}(T, L) = (G_n, f_{(n,m)} : m \leqslant n < \omega)$ defined in 3.1 have bounding maps $f_{(n,m)}$ which are onto, this is because we ask that the tree T has no leafs and because we ask that condition (c) from there holds.

Remark 3.3. Notice also that for $\mathrm{inv}(T, L) = (G_n, f_{(n,m)} : m \leqslant n < \omega)$ as in 3.1 we have that the groups G_n's are completely decomposable, i.e., they are direct sums of groups of rank 1, in fact it follows immediately from the definitions that:

$$G_n = \sum_{\eta \in \omega^n \cap T} \langle x_\eta \rangle^*_{G_n},$$

where $\langle X \rangle^*_{G_n}$ denotes pure closure (cf. [5, pg. 151]).

Notation 3.4. Given a tree $T \subseteq \omega^{<\omega}$ we denote by $[T]$ the set of $\eta \in \omega^\omega$ such that $\eta \restriction n \in T$ for all $n < \omega$, i.e., the set of infinite branches of T. Recall that in 3.1 we only consider trees with no leafs and so, for T's as in 3.1, $[T]$ is always non-empty.

For background on the notions that we now introduce see e.g. [5, Chapter 13].

Definition 3.5. Let $A \in \mathrm{TFAB}$. Let $(p_i : i < \omega)$ be the list of the prime numbers in increasing order. For $a \in A$, we define the characteristic of a, denoted as $\chi(a)$, as follows: $\chi(a) = (h_{p_i}(a) : i < \omega)$, where $h_{p_i}(a)$ is the supremum of the $k < \omega$ such that $p_i^k \,|\, a$, where the supremum is taken in $\omega \cup \{\infty\}$ (so the value ∞ is allowed).

Definition 3.6. (1) Two characteristics $(k_i : i < \omega)$ and $(\ell_i : i < \omega)$ are said to be equivalent if $k_i = \ell_i$ for for almost all $i < \omega$ and both k_i and ℓ_i are finite whenever $k_i \neq \ell_i$. The equivalence classes of characteristics are called *types*.
(2) Given $A \in \mathrm{TFAB}$ we denote by $\mathbf{t}_A(a)$ the type of a in A; we may simply write $\mathbf{t}(a)$ when A is clear from the context.
(3) We denote by $\mathbf{0}$ the type of the characteristic $(0, 0, ...)$.
(4) For $A \in \mathrm{TFAB}$ and $a, b \in A$ we say that $\mathbf{t}(a) \leqslant \mathbf{t}(b)$ if there are characteristics $\chi = (k_i : i < \omega) \in \mathbf{t}(a)$ and $\nu = (\ell_i : i < \omega) \in \mathbf{t}(b)$ such that $(k_i : i < \omega) \leqslant (\ell_i : i < \omega)$, where the order is the pointwise order, i.e., $k_i \leqslant \ell_i$, for all $i < \omega$.

We will need the following easy fact.

Fact 3.7 ([5, (C) on pg. 411])**.** Let G and H be torsion-free abelian and suppose that there is an homomorphism $f : G \to H$, then, for every $g \in G$, $\mathbf{t}(g) \leqslant \mathbf{t}(f(g))$.

Fact 3.8. [4, Theorem 2.3] Let $A \in \mathrm{TFAB}$. Then A is $\aleph_1$-free if and only if, for every finite $S \subseteq A$, the pure closure of S in A (denoted $\langle S\rangle^*_A$) is free abelian.

Lemma 3.9. *Let (T, L) be as in 3.1, then we have that:*

(1) $G(T, L) \in \mathrm{TFAB}$;
(2) if there is $\eta \in [T]$ s.t. $\bigcap\{L(\eta \restriction n) : n < \omega\}$ is infinite, then $G(T, L)$ is not $\aleph_1$-free abelian;
(3) if, for every $\eta \in [T]$, $|\bigcap\{L(\eta \restriction n) : n < \omega\}| < \aleph_0$, then $G(T, L)$ is $\aleph_1$-free abelian.

Proof. Let $G(T, L) = G = \varprojlim(G_n)$ and $\pi_n : G \to G_n$ the canonical projections. Item (1) is clear. Concerning (2), if $\eta \in [T]$ and $L(\eta) = \bigcap\{L(\eta \restriction n) : n < \omega\}$ is infinite, then in G we have that x_η is divisible by a for every $a \in L(\eta)$, contradicting 3.8, as then $\langle x_\eta\rangle^*_G$ is not free. Concerning (3), consider a finite $A \subseteq G$, we will show that $\langle A\rangle^*_G$ is free, as then by 3.8 it follows that G is $\aleph_1$-free. To this extent, we first introduce the following notation, for $S \subseteq [T]$ we let:

$$H_S = \langle x_\eta : \eta \in S\rangle_G \text{ and } G_S = \langle x_\eta : \eta \in S\rangle^*_G.$$

$(*_1)$ Choose finite $S_A \subseteq [T]$ such that $A \subseteq G_{S_A}$.

[Why possible? Easy.]

$(*_2)$ H_{S_A} is free.

[Why? As the $(x_\eta : \eta \in S_A)$ are independent (if not, then projecting onto a G_n such that $n < \omega$ is large enough so that $x_{\eta\restriction n} \neq x_{\nu\restriction n}$ for all $\eta \neq \nu \in S_A$, we reach a contradiction.]

$(*_7)$ $G_{S_A} = \langle \frac{1}{a}x_\eta : a \in \bigcap_{n<\omega} L(\eta \restriction n), \eta \in S_A\rangle_G$.

[Why? This is obvious from Definition 3.1.]

$(*_8)$ G_{S_A} is free.

[Why? By $(*_7)$ we have that G_{S_A}/H_{S_A} is finite, as by our assumption (3) we have that, for every $\eta \in S_A$, $\bigcap_{n<\omega} L(\eta \restriction n)$ is finite. Hence, as by $(*_6)$ we have that H_{S_A} is free, then G_{S_A} is free too.]

$(*_9)$ $\langle A\rangle^*_G$ is free.

[Why? Because $\langle A\rangle^*_G \leqslant G_{S_A}$ and G_{S_A} is free by $(*_8)$.] ∎

4. Proof of 1.1

Definition 4.1. Let $\mathbb{P}$ be an infinite set of primes. We define a pair $(\omega^{<\omega}, L_{\mathbb{P}})$ as in 3.1 as follows:

(1) $L(\emptyset) = \mathbb{P}$, where $\emptyset$ denoted the empty sequence (i.e., the root of $\omega^{<\omega}$);
(2) for every $\eta \in \omega^{<\omega}$, $\{p_{(\eta,\ell)} : \ell < \omega\}$ enumerates $L(\eta)$ in increasing order;
(3) for every $\eta \in \omega^{<\omega}$, $L(\eta^\frown(n)) = \{p_{(\eta,\ell)} : p_{(\eta,\ell)} \geqslant \lg(\eta) \text{ or } \ell = n\}$.

Observation 4.2. For $(\omega^{<\omega}, L_{\mathbb{P}})$ as in 4.1 we have that the group $G(\omega^{<\omega}, L_{\mathbb{P}})$ (cf. 3.1) is $\aleph_1$-free, this is by 3.9, as, for every $\eta \in \omega^\omega$, $\bigcap\{L_{\mathbb{P}}(\eta \restriction n) : n < \omega\}$ is finite.

We need the following definition from [9].

Definition 4.3. For a category $\mathcal{C}$ by Pro-$\mathcal{C}$ we denote the category of ω-inverse systems where the morphisms are given as sequences of onto maps $f_n : A_{\varphi(n)} \to B_n$ where $\varphi : \omega \to \omega$ is increasing, and all is commutative, that is, letting $A =$

$(A_n, \pi^A_{(n,m)} : m \leqslant n < \omega)$ and $B = (B_n, \pi^B_{(n,m)} : m \leqslant n < \omega)$ in Pro-$\mathcal{C}$, for every $n \leqslant m < \omega$, we have that $f_m \circ \pi^A_{(\varphi(n),\varphi(m))} = \pi^B_{(n,m)} \circ f_n$.

Lemma 4.4. *Let $A = (A_n, \pi^A_{(n,m)} : m \leqslant n < \omega)$ and $B = (B_n, \pi^B_{(n,m)} : m \leqslant n < \omega)$ be inverse systems of groups and let $G = \varprojlim(A)$ and $H = \varprojlim(B)$. Then if there is a continuous surjection from G onto H, then there is a morphism of Pro-Groups from A to B.*

Proof. Suppose that $\gamma : G \to H$ is a continous surjection (epimorphism). Recall that procountable group means that there is a nbhd basis of e_G of normal open subgroups. So if G is procountable, D is discrete and $f : G \to D$ is a continuous epimorphism, then the kernel of f being an open subgroup contains a subgroup from this nbhd basis. This yields the desired morphism of Pro-Groups from A to B. ■

Lemma 4.5. *Let $\mathbb{P}_1$ and $\mathbb{P}_2$ be infinite sets of primes such that $\mathbb{P}_1 \setminus \mathbb{P}_2$ is infinite, and let $(\omega^{<\omega}, L_{\mathbb{P}_\ell})$ be as in 4.1 and $G_\ell = G(\omega^{<\omega}, L_{\mathbb{P}_\ell})$ (recall 3.1). Then there is no continuous surjection from G_1 onto G_2.*

Proof. Let $G_1 = \varprojlim(G_{(1,n)})$ and $G_2 = \varprojlim(G_{(2,n)})$, where the $G_{(\ell,n)}$'s as from the inverse system $\mathrm{inv}(\omega^{<\omega}, L_{\mathbb{P}_\ell})$, of course. Suppose that there is a continuous surjection from G_1 onto G_2, then by 4.4 there is $n < \omega$ and a surjection from $G_{(1,n)}$ onto $G_{(2,0)}$, but by 3.7 we reach a contradiction as we can find $a \in G_{(1,n)}$ such that $\mathbf{t}(a) \not\leqslant \mathbf{t}(b)$ for all $b \in G_{(2,0)}$ recalling that by assumption $\mathbb{P}_1 \triangle \mathbb{P}_2$ is finite, and so there are infinitely many primes in $\mathbb{P}_1 \setminus \mathbb{P}_2$ (as by assumption $\mathbb{P}_1$ is infinite). ■

We recall that by *product group* we mean a group of the form $\prod_{n<\omega} H_n$.

Proof of 1.1. Relying on 4.5, it suffices to take a collection $\{\mathbb{P}_\alpha : \alpha < 2^{\aleph_0}\}$ of almost disjoint subsets of the set of prime numbers, where almost A, B are disjoint if $A \cap B$ is finite. We are only left to show that each $G(\omega^{<\omega}, L_{\mathbb{P}_\ell})$ is not topologically isomorphic to a product group. Clearly a $\aleph_1$-free abelian *product* group has to have the form $H = \prod_{n<\omega} H_n$ with each H_n countable and free abelian. Suppose that there is surjective continuous homomorphism from $G = G(\omega^{<\omega}, L_{\mathbb{P}_\alpha}) = \varprojlim(G_n)$ onto H for some $\alpha < 2^{\aleph_0}$, then by 4.4 there is $n < \omega$ and a surjective homomorphism from G_n onto a free abelian group, but, by 3.7, this leads to a contradiction, as there is $g \in G_n$ such that $\mathbf{t}(g) \neq \mathbf{0}$, while every $h \in H$ is such that $\mathbf{t}(h) = \mathbf{0}$. ■

5. Proof of 1.2

Definition 5.1. Let $T \subseteq \omega^{<\omega}$ be a tree (in this case, crucially, possibly with leafs). We define a pair $(\omega^{<\omega}, L_T)$ as in 3.1 as follows:

(1) $L(\emptyset) = \{p : \text{ is a prime }\}$, where $\emptyset$ denoted the empty sequence;
(2) for every $\eta \in \omega^{<\omega}$, $\{p_{(\eta,\ell)} : \ell < \omega\}$ enumerates $L(\eta)$ in increasing order;
(3) for every $\eta \in \omega^{<\omega}$, we define:

$$L(\eta^\frown(n)) = \begin{cases} \{p_{(\eta,\ell)} : p_{(\eta,\ell)} \geqslant \lg(\eta) \text{ or } \ell = n\}, & \text{if } \eta^\frown(n) \notin T \\ L(\eta), & \text{if } \eta^\frown(n) \in T. \end{cases}$$

Lemma 5.2. *For $(\omega^{<\omega}, L_T)$ as in 5.1, $G(\omega^{<\omega}, L_T)$ is $\aleph_1$-free iff T is well-founded.*

Proof. This is by 3.9 as if T is such that there is $\eta \in [T]$, then $\bigcap\{L_T(\eta \restriction n) : n < \omega\}$ is infinite, and otherwise, for every $\eta \in \omega^\omega$, $\bigcap\{L_T(\eta \restriction n) : n < \omega\}$ is finite. ■

Proof of 1.2. This is by 2.12 and 5.2, together with the observation that the map $T \mapsto \mathrm{inv}(\omega^{<\omega}, L_T)$ is Borel, when considering models whose domain is $\subseteq \omega$. ∎

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