QUITE FREE COMPLICATED ABELIAN GROUPS, PCF AND BLACK BOXES SH1028

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Abstract. We would like to build Abelian groups (or R-modules) which on the one hand are quite free (say, $\aleph_{\omega+1}$-free) and on the other hand are complicated in some suitable sense. We choose as our test problem having no non-trivial homomorphism to $\mathbb{Z}$ (known classically for $\aleph_1$-free, recently for $\aleph_n$-free). We succeed to prove the existence of even $\aleph_{\omega_1 \cdot n}$-free ones. This requires building n-dimensional black boxes, which are quite free. This combinatorics is of interest in its own right, and we believe it will be also useful for other purposes. On the other hand, modulo suitable large cardinals, we prove that it is consistent that every $\aleph_{\omega_1 \cdot \omega}$-free Abelian group has non-trivial homomorphisms to $\mathbb{Z}$.

Annotated Content

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Key words and phrases. cardinal arithmetic, pcf, black box, Abelian groups, λ-free, the TDC, the trivial dual conjecture, trivial endomorphism conjecture, forcing, independence results.

The author thanks the Israel Science Foundation for support of this paper, Grant No. 1053/11. Publication 1028. References like [Shea, Th0.2$_{=\text{Ly5}}$] means the label of Th 0.2 is y5. The author thanks Alice Leonhardt for the beautiful typing. The reader should note that the version in my website is usually more updated that the one in the mathematical archive. First typed January 25, 2012.

§ 0. Introduction

§ 0(A). **Answer to Daniel Kronberg.** I hope you don't go crazy. At least, not as much as I do trying to decipher that file. Is there any chance you can help? I have compiled quite a list in need of clarifying (first number is page in the proof, number in parentheses is page in the pdf file you sent):

p.4(5), line 16–I cannot decypher.
— add after "and each α"
kind of a

p.5(6), comment before ", then"–I cannot decypher.
– add
are regular

p.8(10), lines 9,10—I believe there is some reply concerning my comment about the braces, but I cannot get it.
– the and marked by yellow – no change needed
- the "is I is a ∂-complete ideal, then" should be replaces by
} is

p.11(13)–Should we remove the comma and replace it with "¡boldk,"? What is the last symbol in the second comment?
– replace " $\nu_{\bar\eta=\Lambda_{\mathbf{x}_1}}, n \in \mathbf{k}$"
by
$\nu_{\bar\eta} = \Lambda_{\mathbf{x}_1,<\mathbf{k}}, n < \omega$

p.25(27)–I cannot decypher.
add after " Hence as in earlier cases" add
(see 1.16(1)(C))

p.29(34), line -4–Which of the subscripts/superscripts are we to remove?
replace "$J^{\rm bd}_\partial$
by
J

p.30(35), line -11–It seems as sentence should read '...is stronger covered by "first" whereas...". I have a hard time making sense.
–*– replace "stronger " by
covered by "first""

line -10–I cannot decypher the citation in the replacement text.
– the citation is (in the notation you are using
[She13a,0.6(g),(g')]

p.30A(36)–I cannot decypher.
–**- add
Let $S_\bullet \in \hat I_\sigma[\lambda]$ be stationary such that $\delta \in S_\bullet \to \mu^\omega|\delta$
and for $\delta \in S_\bullet$ we let $\rho_\delta \in {}^\sigma\delta$ be increasing with limit δ .
Now let $f'_\delta \in {}^\sigma\delta$ be such that $i < \partial \wedge j < \sigma \to f'_\delta(\sigma i + 1) = \mu f_\delta(i) + \rho_\delta(j)$.

p.32(38), last comment–Does it say "and anyhow not used"? I do not see the sense.
add "is" after "the proof" make sense to me- even if the reader doubt the proof no harm done because we do not use it

p.43(52), last comment–I cannot decypher.
— add
so may write $(\partial, J, \mathbf{k}, \theta)$

p.44(53), line -5–The word "this" is underlined. Why?

- ignore- do not underline

p.45(55), line -3–I cannot decypher the added text.
– add
but we elaborate

p.48(59), line 1–Should we replace "...and..." by "...be an..."?
– yes replace

p.49(60), line 6–With what should we replace the word "hence"?
–replace by
better

p.54(66), line 11–I cannot decypher.
– add after "R-ring"
closed under isomorphism

p.55(67), comment at top of page–I cannot decypher.
— replace the line starting with "equivalently satisfies" by:
equivalently satisfies the condition inside 3.2(0)

p.57(69), line 15 (item (b))–With what should we replace the G?
by (twice in subscript)
$\mathfrak{A}$

p.59(71), line -8–I cannot decypher the two last words in the added text.
— the added text is
and ∂ is the critical cardinal

p.60(72), line 1–I cannot decypher the last word of the added text.
– it is
below
line 17–I cannot decypher the text added at the end of the line.
—add after "sufficient condition for" add
the SCH, that is
line 18–What are the superscripts in the formula?
-*- the formula should be
$\partial = \mathrm{cf}(\partial) = \mathrm{cf}(\mu) \wedge 2^\partial < \mu \Rightarrow \mu^\partial = \mu^+$
Hopefully that's it.

§ 0(B). **Abelian Groups.**

We would like to determine the supremum of all λ for which we can prove TDC_λ, the *trivial dual conjecture for* λ, where:

(TDC_λ) There is a λ-free Abelian group G such that $\mathrm{Hom}(G,\mathbb{Z}) = 0$.

Contrapositively, this is the minimal λ such that consistently we have NTDC_λ, the *negation* of TDC_λ.

This seems the weakest algebraic statement of this kind; it is consistent that the number is ∞, as if $\mathbf{V} = \mathbf{L}$ then TDC_λ holds for every λ (see e.g. [GT12]). On the one hand, by Magidor-Shelah [MS94], NTDC_λ is consistent for

$$\lambda = \min\{\lambda : \lambda = \aleph_\lambda\}$$

(that is, the first fixed point cardinal), as that paper proves the consistency of "λ-free $\Rightarrow$ free".

On the other hand, [$\mathsf{TDC}_{\aleph_1}$ **has been known for a long time,**] and recently by [She07] we know that for $\lambda = \aleph_n$ (for every n) there are examples using the n-BB (n-dimensional black boxes) introduced there. Subsequently those were used for more complicated algebraic relatives in Göbel-Shelah [GS09], Göbel-Shelah-Strüngman [GSS13] and Göbel-Herden-Shelah [GHS14]. In [She13b] we have several close approximations to proving in ZFC the existence for $\aleph_\omega$ (that is, $\mathsf{TDC}_{\aleph_\omega}$) using 1-black boxes.

Here we finally fully prove that $\mathsf{TDC}_{\aleph_\omega}$ holds, and much more; $\lambda = \aleph_{\omega_1 \cdot \omega}$ is the first cardinal for which TDC_λ cannot be proved in ZFC. The existence proof for $\lambda' < \lambda$ is a major result here, relying on existence proof of quite free n-black boxes (in §1) which use results on pcf (see [She13a]). For complementary consistency results, we start with the universe forced in [MS94] and then we force with a ccc forcing notion making "$\mathsf{MA} + 2^{\aleph_0}$ large," but we have to work to show the desired result.

Of course, we can get better results (μ^+-free) when $\mu \in \mathbf{C}_\theta$ (see Definition 0.2) is so-called '1-solvable,' or $\Upsilon < 2^\mu = 2^{<\Upsilon} < 2^\Upsilon$.

Note a point which complicates our work relative to previous ones: the amount of freeness (i.e. the κ such that we demand κ-free) and the cardinality of the structure are markedly different. In [She13b] this point is manifested when we construct, say, G of cardinality λ which is μ^+-free where $\mu \in \mathbf{C}_{\aleph_0}$ or $\mu \in \mathbf{C}_{\aleph_1}$ and $\lambda = 2^\mu$ or $\min\{\lambda : 2^\lambda > 2^\mu\}$. The "distance" is even larger in [She07].

An interesting point here is that for many non-structure problems we naturally end up with two incomparable proofs. One is when we have a μ^+-free $\mathscr{F} \subseteq {}^\partial\mu$ of cardinality λ, λ as above. In this case the amount of freeness is large. In the other, we use the black box from Theorem 1.26. But we may like to use more sophisticated black boxes: say, start with λ_ℓ and μ_ℓ (for $\ell \le \mathbf{k}$), a black box $\mathbf{x}$ as in Theorem 1.26, and combine it with [She05]. The quotients $G/G_{\delta+1}$ for δ a limit ordinal are close to being $\lambda_{\mathbf{k}}^+$-free, replacing 'free' by direct sums of small subgroups.

Recall from [She22, §3] : if we are given BB approximating models with universe, e.g. κ_2 by "guesses of cardinality κ_1", and usually models $\kappa_2 = \kappa_2^{\kappa_1}$ then we can construct models of cardinality κ_2 quite freely except the "corrections" toward avoiding, e.g. undesirable endomorphisms, i.e. for each approximation of such endomorphisms given by the BB is seen as a "task" how to avoid that in the end there will be an endomorphism extending the one given by the approximation. The "price" is that we make the construction not free, but between the various approximations there is little interaction. This will hopefully help in [S+a], which follows [She08] to use $\partial > \aleph_0$ and here to try to sort out the complicated cases like $\mathrm{End}(G) \cong R$. Maybe we can get a neater proof.

In [She75b], [She75c] we suggested that combinatorial proofs from [She78, Ch.VIII], [She90, Ch.VIII], should be useful for proving the existence of many non-isomorphic structures, as well as rigid and indecomposable ones. The most successful case were black boxes applied to Abelian groups and modules first applied in [She84a], [She84b]. That is:

(A) For separable Abelian p-groups G, proving the existence of ones of cardinality $\lambda = \lambda^{\aleph_0}$ with only so-called small endomorphisms. ([She84a])

(B) Let R be a ring whose additive group R^+ is cotorsion-free; i.e. R^+ is reduced and has no subgroups isomorphic to $\mathbb{Z}/p\mathbb{Z}$ or to the p-adic integers. For $\lambda = \lambda^{\aleph_0} > |R|$ there is an abelian group G of cardinality λ whose endomorphism ring is isomorphic to R, and as an R-module it is $\aleph_1$-free. ([She84b, Th.0.1, pg.40])

We can relax the demands on R^+ and may require that G extends a suitable group G_0 such that R is realized as $\mathrm{End}(G)$ modulo a suitable ideal of "small" endomorphisms.

(C) Let R be a ring whose additive group is the completion of a direct sum of copies of the p-adic integers. If $\lambda^{\aleph_0} \geq |R|$ then there exists a separable Abelian p-group G with a so-called *basic* subgroup of cardinality λ and $R = \mathrm{End}(G)/\mathrm{End}_s(G)$. As usual we get $\mathrm{End}(G) = \mathrm{End}_s(G) \oplus R$ [She84b, Th.0.2, pg.41].

On previous history of those algebraic problems see [EM02] and [GT12]. Quite a few works using black boxes follow, starting with Corner-Göbel [CG85], see again [EM02], [GT12]. On Black Boxes in set theory with weak versions of choice see [She16, §3A], with no choice [She16, §3B], and for **k**-dimensional [S⁺b] will hopefully be [useful / available].

On further applications of those black boxes continuing the present work, mainly representation of a ring R and the endomorphism ring of a quite free Abelian group, see [Sheb].

Discussion 0.1. 1) Note that usually, the known constructions were either for λ-free R-module of cardinality λ using a non-reflecting $S \subseteq S^{\lambda}_{\aleph_0}$ with diamond <u>or</u> $\aleph_1$-free of some cardinality λ (mainly $\lambda = (\mu^{\aleph_0})^+$ but also in some other cases) many times using a black box (see [She22]) or "the elevator" (see [GT12]). In the former we use induction on $\alpha < \lambda$ and each α has some kind of "one task".

That is, using black boxes in the nice versions, we have for each $\delta \in S$ a perfect set of pairwise isomorphic tasks.

To deal with getting an $\aleph_n$-free Abelian group G with $\mathrm{Hom}(G, \mathbb{Z}) = 0$, the n-dimensional black boxes actually constructed and used in [She07] were products of black boxes from [She22]; each black box separately is only $\aleph_1$-free but the product of k gives $\aleph_k$-freeness. Here things are more complicated.

2) Here cardinality and freeness differ.

3) Note that the versions of freeness of BB in [She13b] and here are not the same.

§ 0(C). Notation.

Recall the following: [1]

[1] On pp, see [She94] (but the reader can just use 0.3 below).

Definition 0.2. Let $\mathbf{C} = \{\mu : \mu \text{ strong limit singular and } \mathrm{pp}(\mu) =^+ 2^\mu\}$

$$\mathbf{C}_\kappa = \{\mu \in \mathbf{C} : \mathrm{cf}(\mu) = \kappa\}.$$

Claim 0.3.

(a) *$\mu \in \mathbf{C}$ if μ is strong limit singular of uncountable cofinality.*

(b) *If $\mu = \beth_\delta > \mathrm{cf}(\mu)$ and $\delta = \omega_1$ (or just $\mathrm{cf}(\delta) > \aleph_0$), then $\mu \in \mathbf{C}_{\mathrm{cf}(\mu)}$ and for a club (a* closed unbounded subset*) of $\alpha < \delta$ we have $\beth_\alpha \in \mathbf{C}$.*

Proof. Clause (a) holds by [She94, Ch.II,§2] and clause (b) by [She94, Ch.IX,§5]. $\square_{0.3}$

Explanation 0.4. 1) A reader, particularly one with algebraic background, may wonder how the ideals defined in Definition 0.5 below are used in the algebraic construction. For an ideal J on a set S we may try to find an Abelian group G_1 extending the free Abelian group $G_0 = \bigoplus\{\mathbb{Z}x_s : s \in S\}$ such that the quotient $G_1/\bigoplus\{\mathbb{Z}_s : s \in S_1\}$ is free for every $S_1 \in J$. In particular, we would like to have some $h_0 \in \mathrm{Hom}(G_0, \mathbb{Z})$ which cannot be extended to a homomorphism from G_1 to $\mathbb{Z}$. Copies of such tuples (S, J, G_1, G_0, h_0) are used as "the building block" in the constructions, so finding such examples is crucial; see §2 and more in [Sheb].

2) Concerning Observation 0.6, note that the product $J_1 \times J_2$ is not symmetric (even up to isomorphism). E.g. if $\partial < \kappa$ are regular then

$$J_\partial \times J_\kappa = \big\{A \subseteq \partial \times \kappa : \text{for some } i < \partial,\ j < \kappa \text{ we have } A \subseteq (i \times \kappa) \cup (j \times \partial)\big\},$$

but $J_\kappa \times J_\partial$ has no such representation.

[Don't i and j both have to be $< \partial$? Or should that second term be $(\partial \times j)$?]

Definition 0.5. 1) For a set S of ordinals with no last member, let J^{bd}_S be the ideal consisting of the bounded subsets of S.

2) If J_ℓ is an ideal on S_ℓ for $\ell = 1, 2$, then $J_1 \times J_2$ is the ideal on $S_1 \times S_2$ consisting of the $S \subseteq S_1 \times S_2$ such that

$$\big\{s_1 \in S_1 : \{s_2 \in S_2 : (s_1, s_2) \in S\} \notin J_2\big\}$$

belongs to J_1.

3) If δ_1, δ_2 are limit ordinals, J_ℓ is an ideal on δ_ℓ, and $\delta_1 \cdot \delta_2 = \delta_3$ then

$$J_1 * J_2 := \big\{\{\delta_1 \cdot i + j : i < \delta_2,\ j < \delta_1, \text{ and } (j, i) \in A\} : A \in J_1 \times J_2\big\}.$$

4) If δ_1, δ_2 are limit ordinals, J_ℓ is an ideal on δ_ℓ for $\ell = 1, 2$, and $\delta_1 \cdot \delta_2 = \delta_3$, then

$$J_1 \odot J_2 := \big\{\{\delta_1 \cdot i + j : i < \delta_2,\ j < \delta_1, \text{ and } (i, j) \in A\} : A \in J_2 \times J_1\big\}.$$

[Isn't this LITERALLY identical to $J_1 * J_2$?]

[Whichever one you keep, you'll need to change the last clause to $A \subseteq J_1 \times J_2$. Also, δ_3 is not used anywhere in either definition.]

Observation 0.6. *If $\partial \geq \kappa$ are regular cardinals then $J^{\mathrm{bd}}_\partial \times J^{\mathrm{bd}}_\kappa$ is isomorphic to $J^{\mathrm{bd}}_\partial * J^{\mathrm{bd}}_\kappa$ which include in $J^{\mathrm{bd}}_\partial \odot J^{\mathrm{bd}}_\kappa$ which is isomorphic to $J^{\mathrm{bd}}_\kappa \times J^{\mathrm{bd}}_\partial$.*

Proof. Should be clear, but we elaborate on the first equivalence.

Why is $J' = J^{\mathrm{bd}}_\partial \times J^{\mathrm{bd}}_\kappa$ isomorphic to $J'' = J^{\mathrm{bd}}_\partial * J^{\mathrm{bd}}_\kappa$?

Note that J' is an ideal on $\partial \times \kappa$ and J'' is an ideal on $\partial \cdot \kappa$. We define a function $\pi : \partial \times \kappa \to \partial \cdot \kappa$ as follows:

$(*)$ $(i, j) \mapsto \partial \cdot j + i$.

So π is a one-to-one function from $\partial \times \kappa$ onto $\partial \cdot \kappa$ by the rules of ordinal division.

It suffices to prove that $A \in J' \Leftrightarrow \pi''(A) \in J''$ for any $A \subseteq \partial \times \kappa$.

Fix $A \subseteq \partial \times \kappa$, and consider the four statements below:

- $\bullet_1$ $A \in J'$
- $\bullet_2$ $\{s_1 \in \partial : \{s_2 \in \kappa : (s_1, s_2) \in A\} \notin J^{\rm bd}_\kappa\} \in J^{\rm bd}_\partial$
- $\bullet_3$ $\{i < \partial : \{j < \kappa : \partial j + i \in \pi''(A)\} \notin J^{\rm bd}_\kappa\} \in J^{\rm bd}_\partial$
- $\bullet_4$ $\pi''(A) \in J''$.

$\bullet_1 \Leftrightarrow \bullet_2$ by the definition of J', $\bullet_2 \Leftrightarrow \bullet_3$ by the choice of π, and $\bullet_3 \Leftrightarrow \bullet_4$ by the definition of J''. Hence $\bullet_1 \Leftrightarrow \bullet_4$, and we are done. $\square_{0.6}$

Definition 0.7. 1) We say $\mathscr{F} \subseteq {}^S X$ is (θ, J)-*free* when[2] J is an ideal on S and for every $\mathscr{F}' \subseteq \mathscr{F}$ of cardinality $< \theta$ there is a sequence $\langle w_\eta : \eta \in \mathscr{F}' \rangle$ such that

$$\eta \in \mathscr{F}' \Rightarrow w_\eta \in J$$

and

$$\eta_1 \neq \eta_2 \in \mathscr{F}' \wedge s \in S \setminus (w_{\eta_1} \cup w_{\eta_2}) \Rightarrow \eta_1(s) \neq \eta_2(s).$$

2) We say $\mathscr{F} \subseteq {}^S X$ is $[\theta, J]$-free when J is an ideal on S and for every $\mathscr{F}' \subseteq \mathscr{F}$ of cardinality $< \theta$ there is a list $\langle \eta_\alpha : \alpha < \alpha_* \rangle$ of $\mathscr{F}'$ such that if $\alpha < \alpha_*$ then the set

$$w_\alpha := \{s \in S : \eta_\alpha(s) \in \{\eta_\beta(s) : \beta < \alpha\}\}$$

belongs to J.

3) We omit J and write θ-free (or (θ)-free) when $J = J^{\rm bd}_S$.

4) We say μ is 1-*solvable* when μ is singular strong limit and there is a μ^+-free family $\mathscr{F} \subseteq {}^{{\rm cf}(\mu)}\mu$ of cardinality 2^μ.

5) We say μ is $(\theta, 1)$-solvable when above we weaken "μ^+-free" to "θ-free".

6) We say $\mathscr{F} \subseteq {}^S X$ is *weakly ordinary* when each $\eta \in \mathscr{F}$ is a one-to-one function. We say $\mathscr{F} \subseteq {}^\gamma{\rm Ord}$ is *ordinary* when each $\eta \in \mathscr{F}$ is an increasing function.

Claim 0.8. *Assume $\theta > \partial$ and ∂ is regular, J is an ideal on ∂ extending $[\partial]^{<\partial}$, $\mathscr{F} \subseteq {}^\partial{\rm Ord}$, and*[3]

- $\eta \neq \nu \in \mathscr{F} \Rightarrow |\{i < \partial : \eta(i) \in {\rm rang}(\nu)\}| < \partial$.

1) *The set $\mathscr{F}$ is (θ, J)-free iff $\mathscr{F}$ is $[\theta, J]$-free.*

2) *If every $\eta \in \mathscr{F}$ is one-to-one, then we can add*

$$\text{“}\eta_\alpha(s) \notin \{\eta_\beta(t) : \beta < \alpha,\, t \in S\}\text{”}$$

in Definition 0.7(2).

Remark 0.9. 1) We may consider only the case $i \neq j \Rightarrow \eta(i) \neq \nu(j)$ in 0.7(1), 1.2(6), 1.11(1).

2) Compare with [She94], [She13b].

3) Thanks to 0.8, the difference between (θ, J)-free and $[\theta, J]$-free is not serious. For $\mathbf{k}$-c.p. $\mathbf{x}$ see Definition 1.5; there we use only the latter version so we do not write $[\theta, J]$.

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[2] E.g. in [She94], this version is used. Sometimes we even demand

$$\alpha < \alpha_* \Rightarrow \{s \in S : \eta_\alpha(s) \in \{\eta_\beta(t) : \beta < \alpha,\, t \in I\}\} \in J.$$

But in the main case "J is a θ-complete filter on θ," the versions in 0.7(1),(2) are equivalent (see 1.16).

[3] We can replace "$< \partial$" by "$\in J'$" when $J' \subseteq J$ is a ∂-complete ideal.

Proof. 1) It is enough to prove for every $\mathscr{F} \subseteq {}^{\partial}\mathrm{Ord}$ of cardinality $< \theta$, $\mathscr{F}$ is (θ, J)-free iff $\mathscr{F}$ is $[\theta, J]$-free.

First, if $\mathscr{F}$ is $[\theta, J]$-free, then there is a sequence $\langle \eta_\alpha : \alpha < \alpha_* \rangle$ enumerating $\mathscr{F}$ as in Definition 0.7(2); i.e.

$$\alpha < \alpha_* \Rightarrow w^1_\alpha := \big\{ i < \partial : \eta_\alpha(i) \in \{\eta_\beta(i) : \beta < \alpha\} \big\} \in J.$$

Define w_η by $\eta = \eta_\alpha \Rightarrow w_\eta := w^1_\alpha$; easily $\langle w_\eta : \eta \in \mathscr{F} \rangle$ is as required in Definition 0.7(1).

Second, if $\mathscr{F}$ is (θ, J)-free, then there is $\langle w_\eta : \eta \in \mathscr{F} \rangle$ which is as required in Definition 0.7(1).

Let $\langle \eta^1_\alpha : \alpha < \alpha_* \rangle$ list $\mathscr{F}$, and for each α we define $u_{\alpha,n}$ by induction on n as follows:

$(*)^1_\alpha$ (a) $u_{\alpha,0} := \{\alpha\}$
(b) $u_{\alpha,n+1} :=$

$$u_{\alpha,n} \cup \big\{ \beta < \alpha_* : \text{for some } i \in \partial \setminus w_\beta \text{ we have } \eta_\beta(i) \in \{\eta_\gamma(i) : \gamma \in u_{\alpha,n}\} \big\}.$$

Now

$(*)^2_\alpha$ $|u_{\alpha,n}| \leq \partial$ and $u_{\alpha,n} \subseteq \alpha_*$.

[Why? Trivially, $u_{\alpha,n} \subseteq \alpha_*$. Also $|u_{\alpha,0}| = 1 \leq \partial$, and if $|u_{\alpha,n}| \leq \partial$ then

$$\begin{aligned} |u_{\alpha,n+1}| &\leq |u_{\alpha,n}| + \sum_{i<\partial} \sum_{\gamma \in u_{\alpha,n}} \Big| \big\{ \beta < \alpha_* : i \notin w_\beta \wedge \eta_\beta(i) = \eta_\gamma(i) \big\} \Big| \\ &= |u_{\alpha,n}| + \sum_{i<\partial} \sum_{\gamma \in u_{\alpha,n}} 1 \\ &\leq \partial + \partial \cdot \partial \cdot 1 = \partial \end{aligned}$$

and the inductive step holds.]

We define u_α by induction on $\alpha < \alpha_*$ as follows:

$$u_\alpha := \bigcup_n u_{\alpha,n} \setminus \bigcup_{\beta<\alpha} u_\beta,$$

so $\langle u_\alpha : \alpha < \alpha_* \rangle$ is a partition of α_* to sets each of cardinality $\leq \partial$, so we can let $\langle \beta_{\partial\alpha+i} : i < i_\alpha \leq \partial \rangle$ list u_α. Let

$$\mathscr{U} := \big\{ \partial\alpha + i : \alpha < \alpha_*,\ i < i_\alpha, \text{ and } \beta_{\partial\alpha+i} \notin \bigcup_{\gamma<\alpha} u_\gamma \big\},$$

so $\{\beta_\gamma : \gamma \in \mathscr{U}\}$ lists α_* with no repetitions and easily $\langle \eta_{\beta_\zeta} : \zeta \in \mathscr{U} \rangle$ is a list as required in Definition 0.7(2). That is, let $\beta := \beta_{\partial\alpha+i} = \beta(\gamma_\alpha + i)$ for $i < i_\alpha$. So

$$\big\{ i < \partial : \eta_\beta(j) \in \{\eta_\gamma(j) : \gamma \in \mathscr{U} \cap \beta\} \big\} = w^2_\beta \cup \bigcup_{k<i} w^2_{\beta,k},$$

where

$$w^2_\beta := \big\{ j < \partial : \eta_\beta(j) \in \{\eta_\gamma(j) : \gamma \in \mathscr{U} \cap \beta_{\partial,\alpha}\} \big\}$$

and

$$w^2_{\beta,k} := \big\{ j < \partial : \eta_\beta(j) = \eta_{\partial\alpha+k}(j) \big\}.$$

Now each of those sets belong to J.

[Why? w^2_β by the choice of the $u_{\gamma,n}$-s and the u_γ-s; $w^2_{\beta,k}$ as it is included in $w_{\eta_{\beta(\partial\alpha+i)}}$.]

So if J is a ∂-complete ideal we are done. If not, then by the bullet in the assumption of the claim,

$$(\forall k < i)\big[|w^2_{\beta,k}| < \partial \big].$$

So recalling that ∂ is regular, $\bigcup\limits_{k<i} w^2_{\beta,k}$ has cardinality $< \partial$ and hence belongs to J, so as J is an ideal we are done.

Pedantically, $\langle \eta'_\gamma : \gamma < \mathrm{otp}(\mathscr{U})\rangle$ is such a list when we define η'_γ by $\eta'_{\mathrm{otp}(\zeta\cap\mathscr{U})} := \eta_{\beta_\zeta}$ for $\gamma < \mathrm{otp}(\mathscr{U})$.

2) Similarly to the $\Leftarrow$ implication in the proof of 0.8(1), except that $(*)^1_\alpha$(b) is changed to:

(b)$'$ $u_{\alpha,n+1} :=$

$$u_{\alpha,n} \cup \big\{\beta < \alpha_* : \text{for some } i \in \partial \setminus w_\beta,\ \eta_\beta(i) = \{\eta_\gamma(j) : \gamma \in u_{\alpha,n}, j < \partial\}\big\}.$$

$\square_{0.8}$

Question 0.10. 1) If μ is strong limit and $\aleph_0 = \mathrm{cf}(\mu) < \mu$ (but not necessarily $\mu \in \mathbf{C}$), can we get the freeness results of [She13a]?

2) In the cases we have can we strengthen the χ-BB by having $F : \Lambda_{\mathbf{x}} \to \chi$ and demand $\eta_m(i) \in F(\bar\eta \restriction (m, < i))$?

2A) Is this preserved by products?

§ 1. Black Boxes

We generalize the $\mathbf{k}$-dimensional black box from [She07] (where we dealt with the special case when $\ell < \mathbf{k} \Rightarrow \partial_\ell = \aleph_0$) because this seems natural for Abelian groups. The black boxes before that paper were for $\mathbf{k} = 1$.

But here, for Abelian groups, the most interesting cases are when

$$\{\partial_\ell : \ell < \mathbf{k}\} \subseteq \{\aleph_0, \aleph_1\}.$$

In the cases we prove existence, the $\mathbf{k}$-dimensional black box is the product of black boxes (i.e. the ones for $\mathbf{k} = 1$).

The main result is Theorem 1.26 telling us that there are $\mathbf{k}$-dimensional black boxes which are quite free.

The central notion here is of the combinatorial parameters. Those objects ($\mathbf{x}$) consist of the relevant finitely many cardinals ($\langle \partial_\ell : \ell < \mathbf{k}\rangle$), sets ($\langle S_\ell : \ell < \mathbf{k}\rangle$), and a family ($\Lambda$) of sequences $\langle \eta_\ell : \ell < \mathbf{k}\rangle$ with η_ℓ a sequence of length ∂_ℓ of members of S_ℓ. Such objects are used in the construction of Abelian groups G. The point is that on the one hand, the relevant (algebraic) freeness of the Abelian group G is deduced from (set-theoretic) freeness of $\mathbf{x}$ (i.e. of Λ). And on the other hand, e.g. $\mathrm{Hom}(G, \mathbb{Z}) = 0$ is deduced by using the $\mathbf{x}$ having a black box (which is used in the construction). (See more in 1.4.)

Convention 1.1. 1) $\bar\partial$ will denote a sequence $\langle \partial_\ell : \ell < \mathbf{k}\rangle$ of regular cardinals (or just limit ordinals) of length $\mathbf{k} \geq 1$ and then $\partial(\ell) = \partial_\ell$ (but note that $k = \mathbf{k} - 1$ was used in [She07]). A major case is where $\bar\partial$ is constant; i.e. $\bigwedge_\ell [\partial_\ell = \partial]$ for some ∂.

2) Let $\mathbf{x}, \mathbf{y}, \mathbf{z}$ denote combinatorial parameters; see Definition 1.5 below.

Notation 1.2. 0) Here $\bar S = \langle S_\ell : \ell < \mathbf{k}\rangle$ and $\bar\partial = \langle \partial_\ell = \partial(\ell) : \ell < \mathbf{k}\rangle$.

1) Let $\bar S^{[\bar\partial]} := \prod_{\ell < \mathbf{k}} {}^{\partial_\ell}(S_\ell)$ and $\bar S^{[\bar\partial, u]} := \prod_{\ell \in u} {}^{\partial_\ell} S_\ell$ for $u \subseteq \{0, \dots, \mathbf{k} - 1\}$.

If each S_ℓ is a set of ordinals let $\bar S^{\langle\bar\partial\rangle} := \big\{\bar\eta \in \bar S^{[\bar\partial]} : \text{each } \eta_\ell \text{ is increasing}\big\}$, and similarly for $\bar S^{\langle\bar\partial, u\rangle}$.

2) If $\bar\eta \in \bar S^{[\bar\partial]}$, $m < \mathbf{k}$, and $i < \partial_m$ then[4]

$$\bar\eta \upharpoonleft (m, i) = \bar\eta \upharpoonleft_{\mathbf{x}} (m, i)$$

is the sequence $\langle \eta'_\ell : \ell < \mathbf{k}\rangle$, where

$$\eta'_\ell := \begin{cases} \eta_\ell & \text{if } \ell < \mathbf{k} \wedge \ell \neq m \\ \eta_\ell \restriction \{i\} & \text{if } \ell = m. \end{cases}$$

This is close to, but not the same as in,[5] [She07].

Also, for $w \subseteq \partial_m$, $\bar\eta \upharpoonleft (m, {=}w)$ is defined as $\langle \eta'_\ell : \ell < \mathbf{k}\rangle$, where

$$\eta'_\ell := \begin{cases} \eta_\ell & \text{if } \ell < \mathbf{k} \wedge \ell \neq m \\ \eta_\ell \restriction w & \text{if } \ell = m. \end{cases}$$

Let $\bar\eta \upharpoonleft (m) := \langle \eta_\ell : \ell \neq m, \ell < \mathbf{k}\rangle$.

Note that $\bar\eta \upharpoonleft (m, {=}i) \neq \bar\eta \upharpoonleft (m, i)$ for $i \in (0, \partial_m)$.

---

[4] It is sometimes natural to replace "$i < \partial_\ell$" by "i a subset of ∂_ℓ from some family $\mathscr{P}_\ell$ and $\eta'_\ell = \eta_\ell \restriction i$ when $\ell = m$;" say, using $J^{\mathrm{bd}}_{\aleph_1} * J^{\mathrm{bd}}_{\aleph_1}$ as in [She13b]. In [She07] this version was used.

[5] But if we use a tree like $\Lambda \subseteq \bar S^{[\partial]}$ (see 1.2(6)) the difference is small; what we use there is called here $\bar\eta \upharpoonleft (m, {=}i)$.

3) If $\Lambda \subseteq \bar{S}^{[\bar{\partial}]}$, $m < \mathbf{k}$, and $i < \partial_m$ then

$$\Lambda \upharpoonleft_{\mathbf{x}} (m,i) := \{\bar{\eta} \upharpoonleft (m,i) : \bar{\eta} \in \Lambda\}.$$

$\Lambda \upharpoonleft_{\mathbf{x}} (\eta, {=}w)$ is defined similarly.

4) If $\Lambda \subseteq \bar{S}^{[\bar{\partial}]}$, $m < \mathbf{k}$, and $i \leq \partial_m$ then

$$\Lambda \upharpoonleft_{\mathbf{x}} (m, < i) := \bigcup_{i_1 < i} \Lambda \upharpoonleft (m, i_1).$$

5) For $u \subseteq \{0, \ldots, \mathbf{k}-1\}$,

$$\Lambda_{\mathbf{x}, \in u} := \bigcup_{m \in u} \bigcup_{i < \partial_m} \Lambda_{\mathbf{x}} \upharpoonleft (m,i).$$

We may write "$< m$" instead of "$\in m$" when $u = \{0, \ldots, m-1\}$, and let $\Lambda_{\mathbf{x},m} := \Lambda_{\mathbf{x}, \in \{m\}}$.

6) We say $\Lambda \subseteq \bar{S}^{[\bar{\partial}]}$ is *tree-like* when

$$\bar{\eta}, \bar{\nu} \in \Lambda \wedge \bar{\eta} \upharpoonleft (m,i) = \bar{\nu} \upharpoonleft (m,j) \ \Rightarrow \ \eta_m \restriction i = \nu_m \restriction j$$

(so in particular, it implies $i = j$).

7) We say $\Lambda \subseteq \bar{S}^{\langle \bar{\partial} \rangle}$ is *normal* when

$$\bar{\eta}, \bar{\nu} \in \Lambda \wedge m < \mathbf{k} \wedge i,j < \partial_m \wedge \eta_m(i) = \nu_m(j) \ \Rightarrow \ i = j$$

(hence each ν_m is one-to-one; this follows from being tree-like).

We now define the standard $\mathbf{x}$ in Definition 1.3 below, as it is more transparent than the general case (in 1.5). However, we will not use it because the ZFC-existence results are not standard (see the explanation after Definition 1.3). The main difference is that in the general (i.e. not necessarily standard) version, we have the extra parameter J_ℓ, an ideal on ∂_ℓ.

Definition 1.3. 1) We say $\mathbf{x}$ is a *standard* $\bar{\partial}$*-c.p.* (combinatorial $\bar{\partial}$-parameter) when

$$\mathbf{x} = (\mathbf{k}, \bar{\partial}, \bar{S}, \Lambda) = (\mathbf{k}_{\mathbf{x}}, \bar{\partial}_{\mathbf{x}}, \bar{S}_{\mathbf{x}}, \Lambda_{\mathbf{x}})$$

satisfies:

(a) $\mathbf{k} \in \{1, 2, \ldots\}$ and let $k = k_{\mathbf{x}} := \mathbf{k} - 1$. (This is to fit the notation in [She07].)
[Can I just say $\mathbf{k} < \omega$?]

(b) $\bar{\partial} = \langle \partial_\ell : \ell < \mathbf{k} \rangle$ is a sequence of regular cardinals, so $\partial_\ell = \partial_{\mathbf{x},\ell}$.

(c) $\bar{S} = \langle S_\ell : \ell < \mathbf{k} \rangle$, with S_ℓ a set of ordinals (so $S_\ell = S_{\mathbf{x},\ell}$).

(d) $\Lambda \subseteq \bar{S}^{[\bar{\partial}]} = \prod_{\ell < \mathbf{k}} {}^{\partial_\ell}(S_\ell)$ (See 1.2(1).)

2) If $\ell < \mathbf{k} \Rightarrow \partial_\ell = \partial$ we may write ∂ instead of $\bar{\partial}$ in $(\mathbf{k}, \bar{\partial}, \bar{S}, \Lambda)$, and may say 'combinatorial $(\partial, \mathbf{k})$-parameter.' If $\ell < \mathbf{k} \Rightarrow \partial_\ell = \aleph_0$ we may omit $\bar{\partial}$ and write "$\mathbf{x}$ is a combinatorial $\mathbf{k}$-parameter." If $\ell < \mathbf{k} \Rightarrow S_\ell = S$ we may write S instead of $\bar{S}$. Also, we may write $\mathbf{k}(\mathbf{x})$ for $\mathbf{k}_{\mathbf{x}}$.

3) We say $\mathbf{x}$ (or Λ) is *ordinary* when (each S_ℓ is a set of ordinals and)

$$\bar{\eta} \in \Lambda \Rightarrow \text{each } \eta_\ell \text{ is increasing.}$$

We say $\mathbf{x}$ (or Λ) is *weakly ordinary* when

$$\bar{\eta} \in \Lambda \wedge m < \ell g(\bar{\eta}) \Rightarrow \eta_m \text{ is one-to-one.}$$

We say $\mathbf{x}$ is *disjoint* when $\langle S_{\mathbf{x},m} : m < \mathbf{k}\rangle$ is a sequence of pairwise disjoint sets. We say $\mathbf{x}$ is *ordinarily full* when it is ordinary and

$$\Lambda_{\mathbf{x}} = \{\langle \eta_\ell : \ell < \mathbf{k}\rangle : \eta_\ell \in {}^{\partial_\ell}(S_\ell) \text{ is increasing for } \ell < \mathbf{k}\}.$$

Similarly for weakly ordinary.

4) We say $\mathbf{y}$ is a permutation of $\mathbf{x}$ when for some permutation π of $\{0, \dots, \mathbf{k}-1\}$ we have

$$m < k \Rightarrow \partial_{\mathbf{x},m} = \partial_{\mathbf{y},\pi(m)} \wedge S_{\mathbf{x},m} = S_{\mathbf{y},\pi(m)}$$

and

$$\Lambda_{\mathbf{y}} = \big\{\langle \eta_{\pi(m)} : m < \mathbf{k}\rangle : \langle \eta_m : m < \mathbf{k}\rangle \in \Lambda_{\mathbf{x}}\big\}.$$

5) We say $\overline{\pi}$ is an isomorphism from $\mathbf{x}$ onto $\mathbf{y}$ when:

- (a) $\mathbf{k}_{\mathbf{y}} = \mathbf{k}_{\mathbf{x}}$ (Call it $\mathbf{k}$.)
- (b) $\overline{\pi} = \langle \pi_m : m \le \mathbf{k}\rangle$
- (c) $\pi_{\mathbf{k}}$ is a permutation of $\{0, \dots, \mathbf{k}-1\}$.
- (d) $\partial_{\mathbf{x},m} = \partial_{\mathbf{y},\pi_{\mathbf{k}}(m)}$ for $m < \mathbf{k}$.
- (e) π_m is a one-to-one function from $S_{\mathbf{x},m}$ onto $S_{\mathbf{y},\pi_{\mathbf{k}}(m)}$ for $m < \mathbf{k}$.
- (f) $\langle \nu_m : m < \mathbf{k}\rangle \in \Lambda_{\mathbf{y}}$ iff for some $\langle \eta_m : m < \mathbf{k}\rangle \in \Lambda_{\mathbf{x}}$ we have $\nu_{\pi_{\mathbf{k}}(m)} = \langle \pi_m(\eta_m(i)) : i < \partial_{\mathbf{x},m}\rangle$.

Discussion 1.4. It may be helpful to the reader to indicate how such $\mathbf{x}$ helps to construct (e.g.) Abelian groups. For simplicity each ∂_ℓ is $\aleph_0$ (this suffices for constructing an $\aleph_{\omega \cdot n}$-free G, which already is new).

First, let $\langle x_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x}} \restriction (m,i)$ for some m and $i\rangle$ freely generate an Abelian group G_0 and for such $\bar\eta \in \Lambda_{\mathbf{x}}$ we add elements like

$$y_{\bar\eta,n} := \sum_{i \ge n} \sum_{m < \mathbf{k}_{\mathbf{x}}} \frac{i!}{n!}\big(x_{\bar\eta \restriction (m,i)} + a_{\bar\eta,m} x_{\nu_{\bar\eta}}\big)$$

[Do you really want $i \ge n$?]

for some $\nu_{\bar\eta} \in \Lambda_{\mathbf{x},<\mathbf{k}}$, $n < \omega$, and $a_{\bar\eta,m} \in \mathbb{Z}$, getting $G_1 \supseteq G_0$. So we have

$$y_{\bar\eta,n} - {}^{(n+1)} y_{\bar\eta,n+1} = \sum_{m < \mathbf{k}_{\mathbf{x}}} x_{\bar\eta \restriction (m,i)} + a_{\bar\eta,n} x_{\nu_{\bar\eta}}.$$

Now on the one hand we would like G_1 to be θ-free, and on the other hand we would like it to (e.g.) have no non-zero homomorphism into $\mathbb{Z}$. For the second task we need a BB (*Black Box*) property: that is, for each possible $\nu_{\bar\eta}$ to have for each $\bar\eta \in \Lambda$, a homomorphism

$$h_{\bar\eta} : \bigoplus_{\substack{m<\mathbf{k}\\ i<\omega}} \mathbb{Z} x_{\bar\eta \restriction (m,i)} \oplus \mathbb{Z} x_{\nu_{\bar\eta}} \to \mathbb{Z}$$

such that $\{h_{\bar\eta} : \bar\eta \in \Lambda\}$ is dense[6] and choose the $a_{\bar\eta,n}$-s to "defeat" $h_{\bar\eta}$ — i.e. to ensure no $h \in \mathrm{Hom}(G_1, \mathbb{Z})$ extends $h_{\bar\eta}$.

Concerning the first task, we like to ensure $\mathbf{x}$ is θ-free: meaning that for any $\Lambda \subseteq \Lambda_{\mathbf{x}}$ of cardinality $< \theta$, we can list its members as $\langle \bar\eta_\alpha : \alpha < \alpha_*\rangle$ such that for every α, for some m and i, we have

$$j \ge i \Rightarrow \bar\eta_\alpha \restriction (m,j) \notin \{\bar\eta_\beta \restriction (m,j) : \beta < \alpha\}$$

(see Definition 1.7(3)).

In the existence proofs the novel main point is getting enough freeness relying on the pcf theory. I.e. in §1 we prove the existence of a suitable c.p. $\mathbf{x}$.

[6] Or see Definition 1.7(1); it is called $\bar\alpha_{\bar\eta}$ there.

Definition 1.5. 1) We say $\mathbf{x}$ is a $\bar{\partial}$-c.p. (combinatorial $\bar{\partial}$-parameter) <u>when</u>

$$\mathbf{x} = (\mathbf{k}, \bar{\partial}, \bar{S}, \Lambda, \bar{J}) = (\mathbf{k}_{\mathbf{x}}, \bar{\partial}_{\mathbf{x}}, \bar{S}_{\mathbf{x}}, \Lambda_{\mathbf{x}}, \bar{J}_{\mathbf{x}})$$

satisfies:

(a) $\bar{\partial} = \langle \partial_m : m < \mathbf{k} \rangle$, a sequence of limit ordinals.
(b) $\bar{J} = \langle J_m : m < \mathbf{k} \rangle$
(c) J_m is an ideal on ∂_m.
 (In the standard case $J_m \coloneqq \{w \subseteq \partial_\ell : w \text{ is bounded}\}$.)
(d) $\bar{S} = \langle S_m : m < \mathbf{k} \rangle$
 (S_m is a set of ordinals if not said otherwise.)
(e) $\Lambda \subseteq \bar{S}^{[\bar{\partial}]}$.

2) We adopt the conventions and definitions in 1.3(2)-(5).

Convention 1.6. 1) If $\mathbf{x}$ is clear from the context we may write $\mathbf{k}$ for $\mathbf{k}(\mathbf{x})$, k for $k(\mathbf{x})$ and $S, \Lambda, \bar{J}$ instead of $\mathbf{k}_{\mathbf{x}}, k_{\mathbf{x}}, \bar{S}_{\mathbf{x}}, \Lambda_{\mathbf{x}}, \bar{J}_{\mathbf{x}}$, respectively.
2) If not said otherwise, $\mathbf{x}$ is weakly ordinary (see 1.3(3)).

Definition 1.7. Assume $\mathbf{x}$ is a $\bar{\partial}$-c.p.
1) We say $\mathbf{x}$ has [the] $(\bar{\chi}, \mathbf{k}, 1)$-Black Box [or $\bar{\chi}$-pre-Black Box] [property] <u>when</u> some $\bar{\bar{\alpha}}$ is a $(\bar{\chi}, \mathbf{k}, 1)$-Black Box for $\mathbf{x}$ [or $(\mathbf{x}, \bar{\chi})$-pre-Black Box]. *This*, in turn, means:

(a) $\bar{\chi} = \langle \chi_m : m < \mathbf{k}_{\mathbf{x}} \rangle$ is a sequence of cardinals,
(b) $\bar{\bar{\alpha}} = \langle \bar{\alpha}_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\mathbf{x}} \rangle$
(c) $\bar{\alpha}_{\bar{\eta}} = \langle \alpha_{\bar{\eta},m,i} : m < \mathbf{k}_{\mathbf{x}},\ i < \partial_m \rangle$, and $\alpha_{\bar{\eta},m,i} < \chi_m$.
(d) If $h_m : \Lambda_{\mathbf{x},m} \to \chi_m$ for $m < \mathbf{k}_{\mathbf{x}}$ (recalling 1.2(5)) <u>then</u> for some $\bar{\eta} \in \Lambda_{\mathbf{x}}$ we have

$$m < \mathbf{k}_{\mathbf{x}} \wedge i < \partial_m \Rightarrow h_m(\bar{\eta} \upharpoonright \langle m, i \rangle) = \alpha_{\bar{\eta},m,i}.$$

2) For $\Lambda \subseteq \Lambda_{\mathbf{x}}$, we define $\mathbf{x} \restriction \Lambda$ naturally as $(\mathbf{k}_{\mathbf{x}}, \bar{\partial}_{\mathbf{x}}, \bar{S}_{\mathbf{x}}, \Lambda, \bar{J})$.
3) We may write $\bar{\bar{\alpha}}$ as $\mathbf{b}$, a function with domain

$$\big\{(\bar{\eta}, m, i) : \bar{\eta} \in \Lambda_{\mathbf{x}},\ m < \mathbf{k},\ i < \partial_m\big\}$$

such that $\mathbf{b}_{\bar{\eta}}(m, i) = \mathbf{b}(\bar{\eta}, m, i) = \alpha_{\bar{\eta},m,i}$. We may replace $\bar{\chi}$ by χ if the sequence is constant, or by $\bar{C} = \langle C_m : m < \mathbf{k} \rangle$ when $|C_m| = \chi_m$ and $\operatorname{rang}(h_m) \subseteq C_m$. We may replace $\mathbf{x}$ by $\Lambda = \Lambda_{\mathbf{x}}$ (so say $\bar{\bar{\alpha}}$ is a $(\Lambda, \bar{\chi})$-pre-black box).
4) Omitting the "pre" in part (1) means that there is a partition $\bar{\Lambda} = \langle \Lambda_\alpha : \alpha < |\Lambda_{\mathbf{x}}| \rangle$ of $\Lambda_{\mathbf{x}}$ such that each $\mathbf{x} \restriction \Lambda_\alpha$ has $\bar{\chi}$-pre-black box and some $\langle \bar{\nu}_\alpha : \alpha < |\Lambda_{\mathbf{x}}| \rangle$ *witnesses* it. By this we mean:

(a) $\{\bar{\nu}_\alpha : \alpha < |\Lambda_{\mathbf{x}}|\} = \Lambda_{\mathbf{x}}$
(b) Letting μ be maximal such that $(\forall m < \mathbf{k})[2^{<\mu} \leq \chi_m]$, we have

$$\alpha < \beta < \alpha + \mu \Rightarrow \bar{\nu}_\alpha = \bar{\nu}_\beta.$$

(c) If $\alpha \leq \beta < |\Lambda_{\mathbf{x}}|$, $(\alpha, \beta) \neq (0, 0)$, and $\bar{\eta} \in \Lambda_\beta$, then

$$\nu_{\alpha,\mathbf{k}-1} < \eta_{\mathbf{k}-1} \mod J_{\mathbf{x},\mathbf{k}-1}.$$

5) We may write BB instead of Black Box.
6) We say $\mathbf{x}$ *essentially* has a $\bar{\chi}$-black box when some $(\bar{\Lambda}, \mathbf{n})$ witnesses it, which means:[7]

[7] See the proof of 2.10(2).

(a) $\bar{\Lambda} = \langle \Lambda_\alpha : \alpha < |\Lambda_{\mathbf{x}}| \rangle$ is a sequence of pairwise disjoint subsets of $\Lambda_{\mathbf{x}}$.
(b) $\mathbf{x} \restriction \Lambda_\alpha$ has a $\bar{\chi}$-pre-black-box.
(c) $\mathbf{n} = \langle \bar{\nu}_\alpha : \alpha < |\Lambda_{\mathbf{x}}| \rangle$ **[lists the members of $\Lambda_{\mathbf{x}}$.]**
(d) If $\mu := \sup\{\mu : 2^\mu < \min\{|S_{\mathbf{x},\ell}| : \ell < \mathbf{k}_{\mathbf{x}}\}\}$ then

$$\alpha < \beta < \alpha + \mu \Rightarrow \bar{\nu}_\alpha = \bar{\nu}_\beta$$

and

$$\alpha \leq \beta < \lambda \wedge \bar{\eta} \in \Lambda_{\mathbf{x}_\alpha} \Rightarrow \nu_{\alpha,\mathbf{k}-1} <_{J_{\mathbf{x},\ell}} \eta_{\mathbf{k}-1}.$$

(We can use a variant of this, but this suffices presently.)

We shall use freely

Observation 1.8. *If (A) then (B):*

(A) $\mathbf{x}$ *is a* $\bar{\partial}$-c.p. *and* $(\bar{\Lambda}, \mathbf{n})$ *witnesses that* $\mathbf{x}$ *essentially has* $\bar{\chi}$*-black box.*
(B) *There is* $\mathbf{y} = \mathbf{x} \restriction \Lambda$ *for some* $\Lambda \subseteq \Lambda_{\mathbf{x}}$ *which has* $\bar{\chi}$*-black box.*

Proof. We choose $\Omega_n \subseteq \Lambda_{\mathbf{x}}$ by induction on n as follows.

(∗) (a) If $n = 0$ then $\Omega_0 := \Lambda_0 \cup \{\bar{\nu}_0\}$.
(b) If $n = m + 1$ then

$$\Omega_n = \Omega_m \cup \bigcup \{\Lambda_\alpha : \alpha < \lambda = |\Lambda_{\mathbf{x}}| \text{ and } \bar{\nu}_\alpha \in \Omega_m\}.$$

Now $\mathbf{x} \restriction \bigcup_n \Omega_n$ is as required. $\square_{1.8}$

Observation 1.9. 1) *In Definition* 1.7(4) *we may use* $\Lambda_{\mathbf{x}}$ *as the index set of* $\bar{\Lambda}$ *instead of* $|\Lambda_{\mathbf{x}}|$.
2) *If* $\mathbf{x}$ *is a* $\bar{\partial}$-c.p., $\bar{\chi} = \langle \chi_\ell : \ell < \mathbf{k}_{\mathbf{x}} \rangle$, *and* $|\Lambda_{\mathbf{x}}| = \max\{\chi_\ell : \ell < \mathbf{k}_{\mathbf{x}}\}$ *then* $\mathbf{x}$ *has a* $\bar{\chi}$*-black box* <u>*iff*</u> $\mathbf{x}$ *has a* $\bar{\chi}$*-pre-black box.*

Remark 1.10. Concerning the variants below, our aim is to have "$\mathbf{x}$ is (θ)-free" — but to get it we use the other versions.

Definition 1.11. 1) For $\Lambda_* \subseteq \bar{S}^{[\bar{\partial}]}$, we say "$\mathbf{x}$ is (θ, u)-free over Λ_*" <u>when</u>[8] $\mathbf{x}$ is weakly ordinary, $u \subseteq \{0, \ldots, \mathbf{k}_{\mathbf{x}} - 1\}$, and for every $\Lambda \subseteq \Lambda_{\mathbf{x}} \setminus \Lambda_*$ of cardinality $< \theta$ there is a list $\langle \bar{\eta}_\alpha : \alpha < \alpha_* \rangle$ of Λ such that for every α, for some $m \in u$ and $w \in J_{\mathbf{x},m}$, we[9] have

$$\bar{\nu} \in \{\bar{\eta}_\beta : \beta < \alpha\} \cup \Lambda_* \ \wedge \ \bar{\nu} \upharpoonright (m) = \bar{\eta}_\alpha \upharpoonright (m)$$
$$\wedge \ j < \partial_{\mathbf{x},m} \ \wedge \ i \in \partial_{\mathbf{x},m} \setminus w \ \Rightarrow \ \nu_m(j) \neq \eta_{\alpha,m}(i).$$

2) If $\theta > |\Lambda_{\mathbf{x}}|$ we may (in part (1)) write (∞, u)-free or u-free; we may omit "over Λ_*" when $\Lambda_* = \varnothing$.

3) If $u = \{0, \ldots, \mathbf{k} - 1\}$ we may omit it.

4) Suppose we are given cardinals $\theta_1 \leq \theta_2$, combinatorial $\bar{\partial}$-parameter $\mathbf{x}$ and Λ_* (usually $\subseteq \Lambda_{\mathbf{x}}$) and $u \subseteq \{0, \ldots, \mathbf{k}_{\mathbf{x}} - 1\}$.
We say $\mathbf{x}$ is $(\theta_2, \theta_1, u, k)$*-free over* Λ_* <u>when</u>:

(A) $\theta_2 \geq \theta_1 \geq 1$

[8] So if $k_{\mathbf{x}} = 1$ then "$\mathbf{x}$ is $(\theta, \{0\})$-free" has closer meaing to "$\{\eta : \langle \eta \rangle \in \Lambda_{\mathbf{x}}\}$ is $[\theta, J_{\mathbf{x},0}]$-free" than to $(\theta, J_{\mathbf{x},0})$-free (see Definition 0.8).
[9] If $\Lambda_{\mathbf{x}}$ is normal, we can restrict ourselves to $i = j$ and this is the usual case.

(B) $1 \le k \le \mathbf{k}_{\mathbf{x}}$ (If $k = 1$ we may omit it.)
(C) $u \subseteq \{0, \dots, \mathbf{k}_{\mathbf{x}} - 1\}$ has $\ge k$ members.
(D) For every $\Lambda \subseteq \Lambda_{\mathbf{x}} \setminus \Lambda_*$ of cardinality $< \theta_2$ there is a witness $(\bar\Lambda, g, \bar h)$, which means
  (a) $\bar\Lambda = \langle \Lambda_\gamma : \gamma < \gamma_* \rangle$ is a partition of Λ to sets each of cardinality $< \theta_1$ (so γ_* is an ordinal $< \theta_2$).
  (b) $g : \gamma_* \to [u]^k$; when $k = 1$ we usually use $g' : \gamma_* \to u$ where
$$g(\gamma) = \{g'(\gamma)\}$$
  for $\gamma < \gamma_*$, or even use $g'' : \Lambda \to [u]^1$ where
$$\bar\eta \in \Lambda_\gamma \Rightarrow g''(\bar\eta) := g'(\gamma).$$
  Occasionally (when the meaning of $\bar\eta_\beta$ is clear) we may write $g(\bar\eta_\beta)$ or $g'(\bar\eta_\beta)$ instead of $g(\beta)$ and $g'(\beta)$ (so we consider $\Lambda_{\mathbf{x}}$ as the domain of g and g', instead of γ_*).
  (c) $\bar\eta, \bar\nu \in \Lambda_\gamma \wedge m \in \mathbf{k}_{\mathbf{x}} \setminus g(\gamma) \ \Rightarrow\ \eta_m = \nu_m$
  (d) $\bar h = \langle h_m : m \in u \rangle$
  (e) $h_m : \Lambda \to J_m$
  (Really, all that matters is
$$h_m \restriction \big\{\bar\eta \in \Lambda : \gamma < \gamma_* \wedge \bar\eta \in \Lambda_\gamma \Rightarrow m \in g(\gamma)\big\}.$$
  Here again, we may write $h_m(\beta)$ instead of $h_m(\bar\eta_\beta)$.)
  (f) If $\bar\eta \in \Lambda_\beta$, $m \in g(\beta)$,
$$\bar\nu \in \Lambda_* \cup \bigcup_{\alpha<\beta} \Lambda_\alpha,$$
  and $\bar\nu \upharpoonleft (m, {=}\varnothing) = \bar\eta \upharpoonleft (m, {=}\varnothing)$ then
$$i \in \partial_m \setminus h_m(\bar\eta) \Rightarrow \eta_m(i) \neq \nu_m(i).$$

5) In part (4), if $\theta_2 > |\Lambda_{\mathbf{x}}|$ we may write (∞, θ_1, u, k)-free; we may omit Λ_* if $\Lambda_* = \varnothing$ and if $k = 1$ we may omit k.

6) We say $\mathbf{x}$ is (θ, u)-*free over* Λ_*, *respecting* $\bar\Lambda$, when:[10]
$\bar\Lambda = \langle \Lambda_{\bar\nu} : \bar\nu \in \Lambda_{\mathbf{x}} \rangle$, $\Lambda_{\bar\nu} \subseteq \Lambda_{\mathbf{x}}$, and for every $\Lambda \subseteq \Lambda_{\mathbf{x}} \setminus \Lambda_*$ of cardinality $< \theta$ there is a listing $\langle \bar\eta_\alpha : \alpha < \alpha_* \rangle$ of Λ such that:

- $\bullet_1$ If $\bar\eta_\alpha \in \Lambda_{\bar\nu}$ (so $\bar\nu \in \Lambda_{\mathbf{x}}$) then $\bar\nu \in \{\bar\eta_\beta : \beta < \alpha\} \cup \Lambda_*$.
- $\bullet_2$ For every $\alpha < \alpha_*$, for some $m \in u$ and $w \in J_{\mathbf{x},m}$, we have
$$\bar\nu \in \{\bar\eta_\beta : \beta < \alpha\} \cup \Lambda_* \ \wedge\ \bar\nu \upharpoonleft (m) = \bar\eta_\alpha \upharpoonleft m$$
$$\wedge\ j \in \partial_{\mathbf{x},m} \setminus w \wedge i < \partial_{\mathbf{x},m} \ \Rightarrow\ \nu_m(i) \neq \eta_m(j).$$

7) For $\mathbf{x}, \theta_1, \theta_2, \Lambda_*, u$ as in Definition 1.11(4) and a sequence $\bar\Lambda^* = \langle \Lambda^*_{\bar\rho} : \bar\rho \in \Lambda_{\mathbf{x}} \rangle$ of subsets of $\Lambda_{\mathbf{x}}$, we say $\mathbf{x}$ is $(\theta_2, \theta_1, u, k)$-free over Λ_* respecting $\bar\Lambda^*$ <u>when</u> clauses (A)-(D) of Definition 1.11(4) hold, and we add

(D) (g) If $\bar\eta \in \Lambda_\alpha$ and $\bar\eta \in \Lambda^*_{\bar\rho}$, then $\bar\rho \in \Lambda_* \cup \bigcup_{\beta<\alpha} \Lambda_\beta$.

Claim 1.12. *Assume* $\mathbf{x}$ *is a* $\bar\partial$*-c.p. and* $u \subseteq \{0, \dots, \mathbf{k}_{\mathbf{x}} - 1\}$ *is not empty.*

1) $\mathbf{x}$ *is* $(\theta_2, 2, u, 1)$*-free over* Λ_* <u>*iff*</u> $\mathbf{x}$ *is* (θ_2, u)*-free over* Λ_*, [*where* / *and*] $\theta_2 \ge 2$.

2) *If* $\partial > \max\{\partial_\ell : \ell < \mathbf{k}_{\mathbf{x}}\}$, $\mathbf{x}$ *is* (θ, ∂, u)*-free over* Λ_*, *and* $\mathbf{x}$ *is* $(\partial, 2, \{\ell\})$*-free for each* $\ell \in u$, <u>*then*</u> $\mathbf{x}$ *is* $(\theta, 2, u)$*-free over* Λ_* *(equivalently,* (θ, u)*-free over* Λ_**).*

[10] So we may write $\mathbf{k}$ instead of $u = \{\ell : \ell < \mathbf{k}\}$, and θ-free instead of $(\theta, \{\ell : \ell < \mathbf{k}\})$-[free].

Proof. Should be clear, but we elaborate.

1) It is enough to deal with the case $|\Lambda_{\mathbf{x}} \setminus \Lambda_*| < \theta_2$. First, assume $\theta_2 \geq 2$ and $\mathbf{x}$ is (θ_2, u)-free over Λ_*. Let $\langle \bar{\eta}_\alpha : \alpha < \alpha_* \rangle$ listing $\Lambda_{\mathbf{x}} \setminus \Lambda_*$ be as in Definition 1.11(1). Let $\Lambda_\alpha = \{\bar{\eta}_\alpha\}$ for $\alpha < \alpha_*$, and define $g' : \alpha_* \to u$ by setting $g'(\alpha)$ as the minimal $m \in u$ such that for some $w \in J_m$, the condition in Definition 1.11(1) holds.

[I don't see how this depends at all on α.]

By the choice of $\langle \bar{\eta}_\alpha : \alpha < \alpha_* \rangle$ witnessing that $\mathbf{x}$ is (θ_2, u)-free over Λ_*, g' is well defined. Let $g : \alpha_* \to [u]^1$ be defined by $g(\alpha) := \{g'(\alpha)\}$. Also, we define $h_m : \alpha_* \to J_m$ for $m \in u$ such that if $\alpha < \alpha_*$ and $m = g'(\alpha)$ then $h_m(\alpha)$ is any $w \in J_m$ such that the condition in Definition 1.11(1) holds.

[Same.]

Now clearly in Definition 1.11(4), clause (A) holds (letting $\theta_1 = 2$ as $\theta_2 \geq 2 = \theta_1$), clause (B) holds as $k = 1 \in [1, \mathbf{k}_{\mathbf{x}}]$, and clause (C) is obvious. We shall check clauses (D)(a)-(f), hence finishing proving the "if" implication.

Let $\gamma_* = \alpha_*$ and $\bar{\Lambda} = \langle \Lambda_\alpha : \alpha < \alpha_* \rangle$. This definition takes care of (D)(a) and the above definition of g and g' ensures (D)(b). Clause (D)(c) is immediate since each Λ_α is a singleton. Clauses (D)(d),(e) follow from the definition of the h_m-s. Finally, clause (D)(f) follows from Definition 1.11(1).

Second, assume $\mathbf{x}$ is $(\theta_2, 2, u, 1)$-free and let $(\bar{\Lambda}, g, \bar{h})$ witness this (so $\theta_1 = 2$). Note that $\theta_2 \geq 2$, since $\theta_1 = 2$ and $\theta_2 \geq \theta_1$ by Definition 1.11(4)(A). So

$$\bar{\Lambda} = \langle \Lambda_\alpha : \alpha < \alpha_* \rangle, \quad \bar{h} = \langle h_m : m \in u \rangle, \quad \text{and } g : \alpha_* \to [u]^1,$$

so for there is some function $g' : \alpha_* \to u$ such that $\alpha < \alpha_* \Rightarrow g(\alpha) = \{g'(\alpha)\}$. As $|\Lambda_\alpha| < \theta_2 = 2$ we have $|\Lambda_\alpha| \leq 1$. Without loss of generality $\bigwedge\limits_\alpha [\Lambda_\alpha \neq \varnothing]$, hence there is a unique $\bar{\eta}_\alpha \in \Lambda_{\mathbf{x}} \setminus \Lambda_*$ such that $\Lambda_\alpha = \{\bar{\eta}_\alpha\}$. So $\langle \bar{\eta}_\alpha : \alpha < \alpha_* \rangle$ lists $\Lambda_{\mathbf{x}} \setminus \Lambda_*$, and it suffices to check that the condition in Definition 1.11(1) holds for every $\alpha < \alpha_*$. We choose $m := g'(\alpha)$, so $m \in u$, and we choose $w = h_m(\alpha)$, so indeed $w \in J_m$. The condition there holds for m and w by clause (D)(f) of Definition 1.11(4) as $\Lambda_\alpha = \{\bar{\eta}_\alpha\}$ and $\beta < \alpha \Rightarrow \Lambda_\beta = \{\bar{\eta}_\beta\}$.

2) As $\mathbf{x}$ is (θ, ∂, u)-free over Λ_*, there is a triple $(\bar{\Lambda}^*, g^*, \bar{h}^*)$ witnessing it as in Definition 1.11(4), and let $\bar{\Lambda}^* = \langle \Lambda^*_\alpha : \alpha < \alpha_* \rangle$ and $\bar{h}^* = \langle h^*_m : m \in u \rangle$. For each $\ell \in u$ and $\alpha < \alpha_*$ we know that $\mathbf{x}$ is $(\partial, 2, \{\ell\})$-free and Λ^*_α is a subset of $\Lambda_{\mathbf{x}} \setminus \Lambda_*$ of cardinality $< \partial$ hence there is a triple $(\bar{\Lambda}_\alpha, g_\alpha, \bar{h}_\alpha)$ witnessing it.

Let $\bar{\Lambda}_\alpha = \langle \Lambda_{\alpha,\beta} : \beta < \beta_\alpha \rangle$ (and so $|\Lambda_{\alpha,\beta}| < 2$) and without loss of generality $\Lambda_{\alpha,\beta} \neq \varnothing$, so let $\Lambda_{\alpha,\beta} = \{\bar{\eta}_{\alpha,\beta}\}$ and[11] $g_\alpha(\beta) = \{g'_\alpha(\beta)\}$ [for some] $g'_\alpha : \beta_\alpha \to u$, and let $\bar{h}_\alpha = \langle h_{\alpha,m} : m \in u \rangle$.

Let $\gamma_\alpha := \sum\limits_{\iota < \alpha} \beta_\iota$ for $\alpha < \alpha_*$, so clearly $\langle \gamma_\alpha : \alpha \leq \alpha_* \rangle$ is increasing continuous and $\gamma_0 = 0$. Let $\gamma_* := \gamma_{\alpha_*}$; we define $\bar{\eta}_\gamma$ as follows for $\gamma < \gamma_*$. If $\gamma = \gamma_\alpha + \beta$ with $\beta < \beta_\alpha$, then we let $\bar{\eta}_\gamma := \bar{\eta}_{\alpha,\beta}$. Also let $g' : \gamma_* \to [u]^1$ be defined so that $g' \restriction [\gamma_\alpha, \gamma_{\alpha+1})$ is constantly $\{g^*(\alpha)\}$. Let $\bar{\Lambda} = \langle \Lambda_\gamma : \gamma < \gamma_* \rangle$ with $\Lambda_\gamma := \{\bar{\eta}_\gamma\}$, and let $\bar{h} = \langle h_m : m \in u \rangle$ with $h_m : \gamma_* \to J_m$ defined by

$$h_m(\gamma_\alpha + \beta) := h_{\alpha,m}(\beta)$$

if $\alpha < \alpha_*$ and $\beta < \beta_\alpha$. So it is enough to check that $(\bar{\Lambda}, g', \bar{h})$ witnesses that $\Lambda_{\mathbf{x}}$ is $(\theta, 2, u)$-free over Λ_*. (E.g. why clause (f) of Definition 1.11(4)(D) holds.)

Let $\bar{\eta} \in \Lambda_\gamma$, $m \in g'(\gamma)$, and $\bar{\nu} \in \Lambda_* \cup \bigcup\limits_{\alpha < \gamma} \Lambda_\alpha$. So $\bar{\eta} = \bar{\eta}_\gamma$ and one of the following cases occur, letting $\gamma = \gamma_\alpha + \beta$ with $\beta < \beta_\alpha$.

Case 1: $\bar{\nu} \in \Lambda_* \cup \bigcup\limits_{\iota < \alpha} \Lambda^*_\iota$.

[11] As $k = 1$; see the end of 1.11(5).

Use "$(\bar\Lambda^*, g^*, \bar h^*)$ witnesses that $\Lambda_{\mathbf{x}}$ is (θ, ∂, u)-free over Λ_*."

Case 2: $\bar\nu \in \Lambda^*_\alpha$.

Use "$(\bar\Lambda_\alpha, g_\alpha, \bar h_\alpha)$ witnesses that Λ_α is $(\partial, 2, \{\ell\})$-free" for $\ell = g^*(\alpha)$. $\square_{1.12}$

Definition 1.13. We say $(\mathbf{x}, \bar\Lambda)$ witnesses $\mathrm{BB}^3_{\mathbf{k}}(\lambda, \Theta, \bar\chi, \bar\partial)$ <u>when</u>:

(a) $\mathbf{x}$ is a $\bar\partial$-c.p. with $|\Lambda_{\mathbf{x}}| = \lambda$ and $\mathbf{k} = \mathbf{k}_{\mathbf{x}}$ (i.e. $\mathbf{k} = \ell g(\bar\partial)$).

(b) $\bar\Lambda = \langle \Lambda_{\bar\nu} : \bar\nu \in \Lambda_{\mathbf{x}}\rangle$ is a sequence[12] of pairwise disjoint subsets of $\Lambda_{\mathbf{x}}$.

(c) $\mathbf{x} \restriction \Lambda_{\bar\nu}$ has $\bar\chi$-pre-black box for every $\bar\nu \in \Lambda_{\mathbf{x}}$.

(d) Θ is a collection of cardinals and pairs of cardinals.

(e) If $\theta \in \Theta$ <u>then</u> $\mathbf{x}$ is $(\theta, \mathbf{k})$-free respecting $\bar\Lambda$ (see 1.11(6)), which means that in the list $\langle \bar\eta_\alpha : \alpha < \alpha_*\rangle$ in Definition 1.11(1), we have

$$\alpha > 0 \wedge \bar\eta_\alpha \in \Lambda_{\bar\nu} \Rightarrow \bar\nu \in \{\bar\eta_\beta : \beta < \alpha\}.$$

(f) If $(\theta_2, \theta_1) \in \Theta$ then $\mathbf{x}$ is $(\theta_2, \theta_1, \mathbf{k}, 1)$-free, respecting $\bar\Lambda$(see 1.11(7)).

Remark 1.14. Note that in Definition 1.13, we necessarily have

$$\sum_{\ell < \mathbf{k}} \chi_\ell \le |\Lambda_{\mathbf{x}}|.$$

Clearly,

Claim 1.15. *Assume μ is strong limit $> \mathrm{cf}(\mu) = \partial$, $\mathscr{F} \subseteq {}^\partial\mu$ has cardinality $\lambda = 2^\mu$, and $\mathscr{F}$ is θ-free (i.e. $(\theta, J^{\mathrm{bd}}_\partial)$-free); moreover, it is $[\theta, J^{\mathrm{bd}}_\partial]$-free and weakly ordinary (see* 0.7(1),(2),(6)*).*

<u>*Then*</u> *there is a $\langle\partial\rangle$-c.p. $\mathbf{x}$ with $\Lambda_{\mathbf{x}} = \mathscr{F}$ which is θ-free and has the λ-BB (i.e. $(\langle\lambda\rangle, 1, 1)$-BB).*

Proof. The point is that the set of functions from ${}^{\partial>}\mu$ to λ has cardinality $\lambda = |\mathscr{F}|$; see more in [She13b, 2.2=Ld.6]. $\square_{1.15}$

Claim 1.16. 1) *Assume $\mathbf{x}$ is a $\mathbf{k}$-c.p., $\theta_2 \ge \theta_1 = \mathrm{cf}(\theta_1) > \max\{\partial_{\mathbf{x},\ell} : \ell < \mathbf{k}_{\mathbf{x}}\}$, and $u \subseteq \{0, \dots, \mathbf{k}_{\mathbf{x}} - 1\}$ with $k := |u| \ge 1$.*
<u>*Then*</u> *(A) $\Leftrightarrow$ (B) $\Leftrightarrow$ (C), where:*

(*A*) *$\mathbf{x}$ is $(\theta_2, \theta_1, u, k)$-free over Λ_*.*

(*B*) *As in Definition* 1.11(4), *omitting clause* (D)(c). *In this case we call $(\bar\Lambda, g, \bar h)$ an* almost witness.

(*C*) *For every $\Lambda \subseteq \Lambda_{\mathbf{x}} \setminus \Lambda_*$ of cardinality $< \theta_2$ there is a* weak witness $(g, \bar h)$. *This means we have clauses* (d),(e) *of* 1.11(4)(D) *and*

(b)′ $g : \Lambda \to [u]^k$

(f)′ *If $\bar\eta_1 \in \Lambda$ and $m \in u$, then for all but $< \theta_1$ of the sequences $\bar\eta_2 \in \Lambda$, we have*

- *If $\bar\eta_1 \ne \bar\eta_2$, $m \in g(\bar\eta_1) \cap g(\bar\eta_2)$,*

$$\bar\eta_1 \upharpoonleft (m, =\varnothing) = \bar\eta_2 \upharpoonleft (m, =\varnothing),$$

and $i \in \partial_m \setminus \big(h_m(\bar\eta_1) \cup h_m(\bar\eta_2)\big)$ <u>then</u> $\eta_{1,m}(i) \ne \eta_{2,m}(i)$.

(g)′ *If $\bar\eta_1 \in \Lambda$ and $\bar\eta_2 \in \Lambda_*$ <u>then</u> • of* (f)′ *holds, demanding only $m \in g(\bar\eta_1)$.*

2) *If in addition $\mathbf{x}$ is normal (see* 1.2(7)*), we can add:*

(*D*) *Like (C), but we replace the bullet inside* (f)′ *(and similarly in* (g)′*) by*

[12] In [She07] we use $\Lambda_{\mathbf{x},<k}$ as and index set, which if $k = 1$ may have smaller cardinality. So far this is not a significant difference.

- *If* $\bar\eta_1 \neq \bar\eta_2 \in \Lambda$, $\bar\eta_1 \upharpoonleft (m, =\varnothing) = \bar\eta_2 \upharpoonright (m, \varnothing)$, $m \in g(\bar\eta_1) \cap g(\bar\eta_2)$, *and* $i, j \in \partial_m \setminus \big(h_m(\bar\eta_1) \cup h_m(\bar\eta_2)\big)$ *then* $\eta_{1,m}(i) \neq \eta_{2,m}(j)$.

3) *If in addition* $\Lambda_* \subseteq \Lambda_{\mathbf{x}}$ *and each* $J_{\mathbf{x},\ell}$ *is* σ*-complete, then*

$$\big\{\Lambda \subseteq \Lambda_{\mathbf{x}} \setminus \Lambda_* : \Lambda \text{ is } (\theta_2, \theta_1, u, k)\text{-free over } \Lambda_*\big\}$$

is a σ*-complete ideal on* $\Lambda_{\mathbf{x}} \setminus \Lambda_*$.

Proof. 1) $(A) \Rightarrow (B)$:

Obvious by the formulation of (B).

$(B) \Rightarrow (C)$:

Let $\Lambda \subseteq \Lambda_{\mathbf{x}} \setminus \Lambda_*$ have cardinality $< \theta_2$; by clause (B) we can choose $(\bar\Lambda, g, \bar h)$, an almost witness (for Λ). As $|u| = k$, necessarily g is constantly u, so let $g' : \gamma_* \to [u]^k$ be constantly u. Hence it is enough to prove that $(g', \bar h)$ is a weak witness; clearly clause (b)$'$ of (C) holds. So by the phrasing of (B) and (C) it is enough to prove clauses (f)$'$,(g)$'$ of (C). But clause (g)$'$ follows from the assumptions of (B); i.e. Definition 1.11(4)(D)(f). Now for clause (f)$'$, let $\bar\Lambda = \langle \Lambda_\gamma : \gamma < \gamma_* \rangle$ and assume $\bar\eta_\iota \in \Lambda_{\beta_\iota}$ for $\iota = 1, 2$, $\beta_1 \neq \beta_2 < \gamma_*$, and $m \in u$, and it suffices to prove $\bullet$ of (f)$'$.

[Why? As $|\Lambda_{\beta_\iota}| < \theta_1$.]

Clearly $m \in g'(\beta_1) \cap g'(\beta_2)$. So assuming

$$\bar\eta_1 \upharpoonleft (m, =\varnothing) = \bar\eta_2 \upharpoonleft (m, =\varnothing)$$

and $i \in \partial_m \setminus \big(h_m(\bar\eta_1) \cup h_m(\bar\eta_2)\big)$, we should prove that $\eta_{1,m}(i) \neq \eta_{2,n}(i)$. By symmetry, $\beta_1 < \beta_2$ without loss of generality, and we apply clause (f)$'$ of (B) with $\bar\eta_1, \bar\eta_2, \beta_1, \beta_2, m$ here standing for $\bar\nu, \bar\eta, \beta, m$ there, and get $\eta_{1,m}(i) \neq \eta_{2,m}(i)$ as promised.

$(C) \Rightarrow (A)$:

So assume that $\Lambda \subseteq \Lambda_{\mathbf{x}} \setminus \Lambda_*$ has cardinality $< \theta_2$ and let $(g, \bar h)$ be a weak witness for it;[13] again, necessarily g is constantly u. So for $m \in u$, $i < \partial_m$, and every $\bar\eta \in \Lambda$, let

$$\Omega^1_{i,m,\bar\eta} := \big\{\bar\nu \in \Lambda : \bar\nu \upharpoonleft (m, =\varnothing) = \bar\eta \upharpoonleft (m, =\varnothing),\ i \in \partial_m \setminus h_m(\bar\nu),\ \bar\nu_m(i) = \bar\eta_m(i)\big\}.$$

By the choice of $(g, \bar h)$ and the definition of $\Omega^1_{i,m,\bar\eta}$ we have:

- $\bullet_1$ If $\bar\nu, \bar\rho \in \Omega^1_{i,m,\bar\eta}$ then $\bar\nu \upharpoonright (m, =\varnothing) = \bar\rho \upharpoonleft (m, =\varnothing)$ and $\bar\eta_m(i) = \bar\nu_m(i)$ and $i \in \partial_m \setminus \big(h_m(\bar\nu) \cup h_m(\bar\rho)\big)$.

Hence applying (C)(f)$'$ to any $\bar\eta_1 \in \Omega_{i,m,\bar\eta}$, we have

- $\bullet_2$ $|\Omega^1_{i,m,\bar\eta}| < \theta_1$.

Let[14]

$$\Omega^1_{\bar\eta} := \{\bar\eta\} \cup \bigcup_{m \in u} \bigcup_{i < \partial_m} \Omega^1_{i,m,\bar\eta},$$

so recalling the claim assumption $\theta_1 = \mathrm{cf}(\theta_1) > \sum\limits_m \partial_m$, clearly

- $\bullet_3$ If $\bar\eta \in \Lambda$ then $\Omega^1_{\bar\eta}$ has cardinality $< \theta_1$.

By transitivity of equality,

- $\bullet_4$ If $\bar\nu \in \Omega^1_{\bar\eta}$ then $m < \mathbf{k} \wedge m \notin u \ \Rightarrow\ \bar\nu_m = \bar\eta_m$.

[13] Actually, we have no further use of $|\Lambda| < \theta_2$.

[14] Actually, the '$\{\bar\eta\}$' is redundant.

For $\bar\eta \in \Lambda$ let $\Omega^2_{\bar\eta}$ be the minimal subset $\Omega \subseteq \Lambda$ such that $\bar\eta \in \Omega$ and

$$\bar\nu \in \Omega \Rightarrow \Omega^1_{\bar\nu} \subseteq \Omega.$$

So recalling θ_1 is regular, necessarily $|\Omega^2_{\bar\eta}| < \theta_1$.

Let $\langle \bar\eta^*_\gamma : \gamma < \gamma_* \rangle$ list Λ. We now define

$$\Lambda^1_\gamma := \bigcup_{\beta \le \gamma} \Omega^2_{\bar\eta^*_\beta}$$

for $\gamma < \gamma_*$, so $\{\bar\eta^*_\gamma\} \subseteq \Lambda^1_\gamma \subseteq \Lambda$ and clearly $\bigcup\limits_{\gamma<\gamma_*} \Lambda^1_\gamma = \Lambda$.

Lastly, let $\Lambda^2_\gamma := \Lambda^1_\gamma \setminus \bigcup\limits_{\beta<\gamma} \Lambda^1_\beta$, so obviously $\bar\Lambda^2 = \langle \Lambda^2_\gamma : \gamma < \gamma_* \rangle$ is a partition of Λ. Let $g_* : \gamma_* \to [u]^k$ be constantly u and $\bar h = \langle h_m : m \in u \rangle$, and we shall show that the triple $(\bar\Lambda^2, g_*, \bar h)$ is as required in 1.11(4)(D).

Now clauses (4)(D)(a)-(e) hold by our choices, noting that by $\bullet_4$ we have

$$\bar\eta, \bar\nu \in \Lambda^2_\gamma \wedge m < \mathbf{k} \wedge m \notin u \Rightarrow \bar\eta_m = \bar\nu_m.$$

As for clause (D)(f), let $\bar\eta \in \Lambda_\beta$, $m \in g(\beta)$, $\alpha < \beta$ and $\bar\nu \in \Lambda^2_\alpha$,

$$\bar\nu \upharpoonleft (m, =\varnothing) = \bar\eta \upharpoonleft (m, =\varnothing),$$

and $i \in \partial_m \setminus h_m(\bar\eta)$; we should prove that $\bar\nu_m(i) \neq \bar\eta_m(i)$. But if not, then

$$\bar\eta \in \Omega^1_{\bar\nu} \subseteq \Omega^2_{\bar\eta^*_\alpha} \subseteq \bigcup_{\iota \le \alpha} \Lambda^2_\iota,$$

a contradiction.

2) Similarly.

3) By part (1), as we can use clause (C) as our definition. So assume

$$\Lambda = \bigcup_{i<i_*} \Lambda_i \subseteq \Lambda_{\mathbf{x}} \setminus \Lambda_*,$$

$i_* < \sigma$, $h_{i,m} : \Lambda \to J_m$, and $(g_i, \bar h_i)$ weakly witnesses Λ_i. As $|u| = k$, necessarily $g_0 := \bigcup\limits_i g_i$ is the constant function from Λ into $\{u\}$, and let $h_m : \Lambda \to \mathcal{P}(\partial_m)$ be defined by

$$h_m(\bar\eta) := \bigcup \big\{ h_{i,m}(\bar\eta) : i < i_* \text{ and } \bar\eta \in \Lambda_i \big\}.$$

Now h_m is injective into J_m, as J_m is a σ-complete ideal and $i_* < \sigma$. Lastly, clearly $\big(g_0, \langle h_m : m \in u \rangle\big)$ is a weak witness for Λ, so we are done. $\square_{1.16}$

Remark 1.17. Why the demand $|u| = k$ in the claim?

Our problem is: in (A) we promise that the function g gives (for one fixed γ) the same u for all $\bar\eta \in \Lambda_\gamma$, whereas in clause (C) this is not the case — in fact, it is not even well-defined. It is natural then to divide Λ_γ into $\le 2^{\mathbf{k}}$ cases according to the value of g, but then it is not clear that clause (f) of (A) holds.

[This seems to be referring to 1.11(4)(D)(f), which is an assumption of clause (B) (but not (A)).]

To avoid this we assume $|u| = k$. Maybe 1.16(3) helps but this is not crucial.

Definition 1.18. If $\ell g(\bar\partial_\iota) = \mathbf{k}_\iota$ and $\mathbf{x}_\iota$ is a combinatorial $\bar\partial_\iota$-parameter for $\iota = 1, 2, 3$ then we say $\mathbf{x}_3 := \mathbf{x}_1 \times \mathbf{x}_2$ <u>when</u>:

(a) $\bar\partial_3 = \bar\partial_1 {}^\frown \bar\partial_2$ (Hence $\mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2$.)

(b) $\bar J_{\mathbf{x}_3} = \bar J_{\mathbf{x}_1} {}^\frown \bar J_{\mathbf{x}_2}$

(c) $\bar{S}_{\mathbf{x}_3}$ is $\bar{S}_{\mathbf{x}_1} {}^\frown \bar{S}_{\mathbf{x}_2}$; that is,

$$S_{\mathbf{x}_3,\ell} := \begin{cases} S_{\mathbf{x}_1,\ell} & \text{if } \ell < \mathbf{k}_1 \\ S_{\mathbf{x}_2,\ell-\mathbf{k}_1} & \text{if } \ell \geq \mathbf{k}_1. \end{cases}$$

(d) $\Lambda_{\mathbf{x}_3}$ is the set of $\bar{\eta} \in \prod_{\ell<\mathbf{k}_3} {}^{\partial_3(\ell)}(S_{\mathbf{x}_3,\ell})$ such that for some $\bar{\nu} \in \Lambda_{\mathbf{x}_1}$ and $\bar{\rho} \in \Lambda_{\mathbf{x}_2}$, we have:
- If $\ell < \mathbf{k}_1$ then $\eta_\ell = \nu_\ell$.
- If $\ell \geq \mathbf{k}_1$ then $\eta_\ell = \rho_{\ell-\mathbf{k}_1}$.

Explanation 1.19. What is the role of the next claim? We shall prove, for $(\partial, J) = (\aleph_0, J^{\rm bd}_\omega)$ and $(\aleph_1, J^{\rm bd}_{\aleph_1} \times J^{\rm bd}_{\aleph_0})$, that for many strong limit singular μ there is a 1-c.p. $\mathbf{x}$ such that $(\partial_{\mathbf{x},0}, J_{\mathbf{x},0}) = (\partial, J)$, $\mathbf{x}$ has 2^μ-BB, and $\mathbf{x}$ is quite free. But we do not know how to get one which is even just $\aleph_{\omega+1}$-free, whereas such freeness is needed in §2! However, using long enough finite products we can get enough freeness. More fully, first by 1.20, the product gives a combinatorial parameter of the expected length (the sum) and weak ordinariness, ordinariness and normality are preserved.

Second, by 1.21 the products have the appropriate (pre)-black-box if each product has one.

Third, in 1.21-1.24 we get that [if] each $\mathbf{x}_\ell$ satisfies enough cases of (θ_2, θ_1, u)-freeness conditions then their product satisfies more.

Fourth, in Theorem 1.26 we prove the existence of $\mathbf{x}_\ell$ (for $\ell < \mathbf{k}$) as required, relying on [She13a].

Lastly, in Conclusion 1.28 we get the desired conclusion used in §2.

Claim 1.20. 1) *If $\mathbf{x}_\iota$ is a combinatorial $\bar{\partial}_\iota$-parameter for $\iota = 1, 2$ then there is one and only one combinatorial parameter $\mathbf{x}_3$ such that $\mathbf{x}_1 \times \mathbf{x}_2 = \mathbf{x}_3$.*

2) *The product in Definition 1.18 is associative.*

3) *If $\mathbf{x}_1 \times \mathbf{x}_2 = \mathbf{x}_3$ then $\mathbf{x}_2 \times \mathbf{x}_1$ is a permutation of $\mathbf{x}_3$ (see Definition 1.3(4)).*

4) *If in Definition 1.18 $\mathbf{x}_1, \mathbf{x}_2$ are [weakly] ordinary and/or normal (see 1.3(3), 1.2(7)) then so is $\mathbf{x}_1 \times \mathbf{x}_2$.*

Proof. Straightforward. $\square_{1.20}$

Claim 1.21. 1) *$\mathbf{x}_3$ has $\bar{\chi}_3$-pre-black box when:*

(a) *$\mathbf{x}_\iota$ is a combinatorial $\bar{\partial}_\iota$-parameter for $\iota = 1, 2, 3$.*

(b) *$\mathbf{x}_1 \times \mathbf{x}_2 = \mathbf{x}_3$*

(c) *$\mathbf{x}_\iota$ has $\bar{\chi}_\iota$-pre-black box for $\iota = 1, 2$.*

(d) *$\bar{\chi}_3 = \bar{\chi}_1 {}^\frown \bar{\chi}_2$*

(e) *If $\ell < \ell g(\bar{\partial}_2)$ then $\chi_{2,\ell} = (\chi_{2,\ell})^{|\Lambda_{\mathbf{x}_1}|}$.*

2) *Moreover, $\mathbf{x}_3$ has a $\bar{\chi}_3$-black box when, in addition,*

$(c)^+$ *$\mathbf{x}_2$ has a $\bar{\chi}_2$-black box and $\chi_{2,n} = (\chi_{2,n})^{|\Lambda_{\mathbf{x}_1}|}$.*

Proof. 1) For each $m < \mathbf{k}_{\mathbf{x}_2}$ let $\overline{F}^m = \langle F^m_\alpha : \alpha < \chi_{2,m} \rangle$ list the [injective] functions $\Lambda_{\mathbf{x}_1} \to \chi_{2,m}$. By clause (e) of the assumption, such sequence exists. Let $\bar{\bar{\alpha}}^1$ be a $\bar{\chi}_1$-pre-black box for $\mathbf{x}_1$ and let $\bar{\bar{\alpha}}^2$ be a $\bar{\chi}_2$-pre-black box for $\mathbf{x}_2$; they exist by clause (c) of the assumption.

Lastly, we define

$$\bar{\bar{\alpha}} = \left\langle \bar{\alpha}_{\bar{\eta}} = \langle \alpha_{\bar{\eta},m,i} : m < \mathbf{k}_{\mathbf{x}_3},\ i < \partial_m \rangle : \bar{\eta} \in \Lambda_{\mathbf{x}_3} \right\rangle$$

as follows: for $\bar\eta \in \Lambda_{\mathbf{x}_3}$, $m < \mathbf{k}_{\mathbf{x}_3}$, and $i < \partial_{\mathbf{x}_3,m}$ we let

- If $m < \mathbf{k}_{\mathbf{x}_1}$ then $\alpha_{\bar\eta,m,i} := \alpha^1_{\bar\eta\restriction\mathbf{k}(\mathbf{x}_1),m,i}$.
- If $m = \mathbf{k}_{\mathbf{x}_1} + \ell$ and $\ell < \mathbf{k}_{\mathbf{x}_2}$, then $\alpha_{\bar\eta,m,i} := F^m_{\alpha^2_{\bar\nu,\ell,i}}(\bar\eta \restriction \mathbf{k}_{\mathbf{x}_1})$, where
$$\bar\nu := \bar\eta \restriction [\mathbf{k}_{\mathbf{x}_1}, \mathbf{k}_{\mathbf{x}_3})$$
(i.e. $\bar\nu = \langle \eta_{\mathbf{k}(\mathbf{x}_1)+n} : n < \mathbf{k}_{\mathbf{x}_2}\rangle$).

Clearly $\overline{\overline{\alpha}}$ is of the right form, but is it really a $\bar\chi_3$-pre-black box? So assume $h_m : \Lambda_{\mathbf{x}_3,m} \to \chi_{3,m}$ for $m < \mathbf{k}_{\mathbf{x}_3}$, and we should find $\bar\eta \in \Lambda_{\mathbf{x}_3}$ as in Definition 1.7(1). Now, first we define $h^2_m : \Lambda_{\mathbf{x}_2,m} \to \chi_{2,m}$ for $m < \mathbf{k}_{\mathbf{x}_2}$ as follows: $h^2_m(\bar\nu)$ is the unique $\alpha < \chi_{2,m}$ such that
$$(\forall \bar\rho \in \Lambda_{\mathbf{x}_1})\big[h_{\mathbf{k}(\mathbf{x}_1)+m}(\bar\rho\,\hat{}\,\bar\nu) = F^m_\alpha(\bar\rho)\big];$$
this is possible by the choice of $\overline{F}^m$ above. As $\overline{\overline{\alpha}}^2$ is a $\bar\chi_2$-pre-black box, clearly there is $\bar\nu \in \Lambda_{\mathbf{x}_2}$ such that
$$m < \mathbf{k}_{\mathbf{x}_2} \wedge i < \partial_{\mathbf{x}_2,m} \ \Rightarrow\ h^2_m(\bar\nu \upharpoonright (m,i)) = \alpha^2_{\bar\nu,m,i}.$$

Fix a sequence $\bar\nu \in \Lambda_{\mathbf{x}_2}$ as in the previous paragraph. Now for $m < \mathbf{k}_{\mathbf{x}_1}$ we define $h^1_m : \Lambda_{\mathbf{x}_1,m} \to \chi_{1,m}$ by $h^1_m(\bar\rho) := h_m(\bar\rho\,\hat{}\,\bar\nu)$. It is well defined as by our assumptions on h_m it has domain $\Lambda_{\mathbf{x}_1,m}$, and as $\bar\nu \in \Lambda_{\mathbf{x}_2}$ clearly $\bar\rho\,\hat{}\,\bar\nu \in \Lambda_{\mathbf{x}_3,m}$ by the definition of $\mathbf{x}_3$. As $\overline{\overline{\alpha}}^1$ is a $\bar\chi_1$-pre-black box for $\mathbf{x}_1$ there is $\bar\rho \in \Lambda_{\mathbf{x}_1}$ such that
$$m < \mathbf{k}_{\mathbf{x}_1} \wedge i < \partial_{\mathbf{x}_1,m} \ \Rightarrow\ h^1_m(\bar\rho) = \alpha^1_{\bar\rho,m,i}.$$
We shall show that $\bar\eta := \bar\rho\,\hat{}\,\bar\nu$ is as required.

First, $\bar\eta \in \Lambda_{\mathbf{x}_3}$ because $\mathbf{x}_3 = \mathbf{x}_1 \times \mathbf{x}_2$, $\bar\rho \in \Lambda_{\mathbf{x}_1}$, and $\bar\nu \in \Lambda_{\mathbf{x}_2}$.

Second, if $m < \mathbf{k}_{\mathbf{x}_1}$ and $i < \partial_{\mathbf{x}_3,m} = \partial_{\mathbf{x}_1,m}$ <u>then</u>

$(*)_1$ (a) $h_m(\bar\eta \upharpoonright (m,i)) = h^1_m(\bar\rho \upharpoonright (m,i))$ by the choices of $\bar\eta$ and h^1_m.
(b) $h^1_m(\bar\rho \upharpoonright (m,i)) = \alpha^1_{\bar\rho,m,i}$ by the choice of $\bar\rho$.
(c) $\alpha^1_{\bar\rho,m,i} = \alpha_{\bar\eta,m,i}$ by the choice of $\alpha_{\bar\eta,m,i}$.

So together, $h_m(\bar\eta \upharpoonright (m,i)) = \alpha_{\bar\eta,m,i}$.

Third, if $m \in [\mathbf{k}_{\mathbf{x}_1}, \mathbf{k}_{\mathbf{x}_3}) \wedge i < \partial_{\mathbf{x}_3,n}$ then $m = \mathbf{k}_{\mathbf{x}_1} + \ell$ for some $\ell < \mathbf{k}_{\mathbf{x}_2}$, and use the choices of $\alpha_{\bar\eta,m,i}$ and of $\bar\nu$.

2) We have to deal with the black box case. So recalling Definition 1.7(4) we are assuming:

$(*)_{2.1}$ (a) $\bar\Lambda^2 = \big\langle \Lambda^2_\gamma : \gamma < |\Lambda_{\mathbf{x}_2}|\big\rangle$ is a partition of $\Lambda_{\mathbf{x}_2}$.
(b) If $\gamma < |\Lambda_{\mathbf{x}_2}|$ then $\mathbf{x}_2 \restriction \Lambda^2_\gamma$ has a $\bar\chi_2$-pre-black box.

Now repeating the proof above, note:

(c) $\langle \bar\nu_\alpha : \alpha < |\Lambda_{\mathbf{x}_2}|\rangle$ lists $\Lambda_{\mathbf{x}}$ as required in Definition 1.7(4).

We can choose $\overline{\overline{\alpha}}^2$ such that not only it is a $\bar\chi_2$-pre-black box but also
$$\overline{\overline{\alpha}}^2 \restriction \Lambda^2_\gamma := \langle \bar\alpha^2_{\bar\nu} : \bar\nu \in \Lambda^2_\gamma\rangle$$
is a $\bar\chi_2$-pre-black box for each $\gamma < |\Lambda_{\mathbf{x}_2}|$.

Having defined $\overline{\overline{\alpha}} = \langle \bar\alpha_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x}}\rangle$, note that:

$(*)_{2.2}$ $|\Lambda_{\mathbf{x}_1}| \le \chi_{\mathbf{x}_2,0}$ (by clause (e) of the claim) and $\chi_{\mathbf{x}_2,0} \le |\Lambda_{\mathbf{x}_2}|$ (by 1.14) and $|\Lambda_{\mathbf{x}_2}|$ is infinite (otherwise the $\bar\chi_2$-black box fails), hence
$$|\Lambda_{\mathbf{x}_3}| = |\Lambda_{\mathbf{x}_2}| \times |\Lambda_{\mathbf{x}_2}| = |\Lambda_{\mathbf{x}_2}|.$$

$(*)_{2.3}$ Letting $\Lambda_\gamma := \Lambda_{\mathbf{x}_1} \times \Lambda^2_\gamma$, the sequence $\langle \Lambda_\gamma : \gamma < |\Lambda_{\mathbf{x}_3}|\rangle$ is a partition of $\Lambda_{\mathbf{x}_3}$.

Mainly, we need to prove that if $\gamma < |\Lambda_{\mathbf{x}_3}|$ then $\bar{\bar{\alpha}} \restriction \Lambda_\gamma$ is a $\bar{\chi}$-pre-black box. This proof is exactly as in the proof of the first part.

Lastly, we choose $\langle \bar{\nu}_\alpha : \alpha < |\Lambda_{\mathbf{x}_3}| \rangle$ as required. Toward this, let

$$\mu_\iota := \max\{\mu : (\forall \ell < \mathbf{k}_\iota)[2^{<\mu} \leq \chi_{\iota,\ell}]\}.$$

Note that necessarily $|\Lambda_{\mathbf{x}_2}| = |\Lambda_{\mathbf{x}_3}|$ and $\mu_1 \leq |\Lambda_{\mathbf{x}_1}| < \mu_2$. Now choose

$$\langle \bar{\nu}^1_\alpha : \alpha < |\Lambda_{\mathbf{x}_1}| \rangle$$

to be an enumeration of $\Lambda_{\mathbf{x}_1}$ such that $\beta \in [\alpha, \alpha + \mu_1) \Rightarrow \bar{\nu}^1_\alpha = \bar{\nu}^1_\beta$.

To finish, define $\langle \bar{\nu}_\alpha : \alpha < |\Lambda_{\mathbf{x}_3}| \rangle$ by:

- If $\gamma = |\Lambda_{\mathbf{x}_1}| \cdot \alpha + \beta$ and $\beta < |\Lambda_{\mathbf{x}_1}|$, then $\bar{\nu}_\gamma = \bar{\nu}^1_\beta{}^\frown \bar{\nu}^2_\gamma$.

Recalling $\gamma_2 \in (\gamma_1, \gamma_1 + \mu_2) \Rightarrow \bar{\nu}^2_{\gamma_1} = \bar{\nu}^2_{\gamma_2}$, we are easily done. $\square_{1.21}$

The following definition is somewhat similar to [She07], but with different notation than before.

Definition 1.22. Let $\mathbf{x} = (\mathbf{k}, \bar{\partial}, \bar{S}, \Lambda, \bar{J})$ be disjoint, for notational transparency (see 1.3(3)).

0) For $u \subseteq \{0, \ldots, \mathbf{k} - 1\}$, let $u^\perp := \{\ell < \mathbf{k} : \ell \notin u\}$.

1) For $\mathscr{U} \subseteq \bigcup_{\ell < \mathbf{k}} {}^{\partial_\ell}(S_{\mathbf{x},\ell})$, let

$$\Lambda_{\mathscr{U}} = \Lambda_{\mathbf{x},\mathscr{U}} = \Lambda_{\mathbf{x}}(\mathscr{U}) = \{\bar{\eta} \in \Lambda_{\mathbf{x}} : \eta_\ell \in \mathscr{U} \text{ for every } \ell < \mathbf{k}\}.$$

2) For $\mathscr{U} \subseteq \bigcup_{\ell < \mathbf{k}} {}^{\partial_\ell}(S_{\mathbf{x},\ell})$ and $u \subseteq \{0, \ldots, \mathbf{k} - 1\}$, let:

(a) $\mathrm{add}_{\mathbf{x}}(u) := \{\mathbf{u} \subseteq \bigcup_{\ell \in u} {}^{\partial_\ell}(S_{\mathbf{x},\ell}) : |\mathbf{u} \cap {}^{\partial_\ell}(S_{\mathbf{x},\ell})| = 1 \text{ for } \ell \in u\}$.
(Note that $\mathbf{u} \in \mathrm{add}_{\mathbf{x}}(u) \Rightarrow |\mathbf{u}| = |u|$.)

(b) For $\mathbf{u} \in \mathrm{add}_{\mathbf{x}}(u)$, let

$$\begin{aligned}\Lambda_{\mathscr{U},\mathbf{u}} = \Lambda_{\mathbf{x}}(\mathscr{U}, \mathbf{u}) := \{\bar{\eta} \in \Lambda_{\mathbf{x}} :\ & \text{for some } m \in u, \text{ for all } \ell < \mathbf{k}, \text{ we have} \\ & \ell \neq m \Rightarrow \eta_\ell \in (\mathscr{U} \cup \mathbf{u}) \cap {}^{\partial_\ell}(S_{\mathbf{x},\ell}) \\ & \text{and } \ell = m \Rightarrow \eta_\ell \in \mathscr{U} \cap {}^{\partial_\ell}(S_{\mathbf{x},\ell})\}.\end{aligned}$$

(c) $\Lambda^*_{\mathbf{x}}(\mathscr{U}, \mathbf{u}) := \Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u}) \setminus \Lambda_{\mathbf{x}}(\mathscr{U}, \mathbf{u})$.
This set is interesting (i.e. non-empty) only when $\mathscr{U} \cap \mathbf{u} = \varnothing$, and then it is equal to

$$\{\bar{\eta} \in \Lambda_{\mathbf{x}} : \ell \in u \Rightarrow \eta_\ell \in \mathbf{u} \text{ and } \ell \in \mathbf{k} \setminus u \Rightarrow \eta_\ell \in \mathscr{U}\}.$$

3) For non-empty $u \subseteq \{0, \ldots, \mathbf{k} - 1\}$, we say $\mathbf{x}$ is θ-(u,k)-*free* <u>when</u>: if $\mathscr{U} \subseteq \bigcup_{\ell < \mathbf{k}} {}^{\partial_\ell}(S_{\mathbf{x},\ell})$ has cardinality $< \theta$ and $\mathbf{u} \in \mathrm{add}_{\mathbf{x}}(u^\perp)$ is disjoint to $\mathscr{U}$, then $\Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u})$ is $(\infty, 2, u, k)$-free over $\Lambda_{\mathbf{x}}(\mathscr{U}, \mathbf{u})$, recalling 1.11(4),(5).

3A) If $\theta > |\Lambda_{\mathbf{x}}|$, we may write ∞ instead of θ in part (3).

4) For non-empty $u \subseteq \{0, \ldots, \mathbf{k} - 1\}$, we say $\mathbf{x}$ is (θ_2, θ_1)-(u,k)-*free* <u>when</u>: if $\mathscr{U} \subseteq \bigcup_{\ell < \mathbf{k}} {}^{\partial_\ell}(S_{\mathbf{x},\ell})$ and $\mathbf{u} \in \mathrm{add}_{\mathbf{x}}(u^\perp)$ is disjoint to $\mathscr{U}$, <u>then</u> $\Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u})$ is $(\theta_2, \theta_1, u, k)$-free over $\Lambda_{\mathbf{x}}(\mathscr{U}, \mathbf{u})$ (recalling 1.11(4)).

Observation 1.23. 1) *In Definition* 1.22(3)*, the conclusion is equivalent to* "$\Lambda^*_{\mathbf{x}}(\mathscr{U},\mathbf{u}) := \Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u}) \setminus \Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ *is* $(\infty, 2, u, k)$*-free".*

2) *Similarly in* 1.22(4)*; that is, assume* $u \subseteq \{0, \ldots, \mathbf{k}-1\}$*,* $\mathscr{U} \subseteq \bigcup\limits_{\ell<\mathbf{k}} {}^{\partial_\ell}(S_{\mathbf{x},\ell})$*, and* $\mathbf{u} \in \mathrm{add}(u^\perp)$ *is disjoint to* $\mathscr{U}$*, then:* $\Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u})$ *is* $(\theta_2, \theta_1, u, k)$*-free over* $\Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ *iff* $\Lambda^*_{\mathbf{x}}(\mathscr{U},\mathbf{u}) := \Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u}) \setminus \Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ *is* $(\theta_2, \theta_1, u, k)$*-free.*

3) If $\mathbf{x}$ *is* θ*-*(u,k)*-free then* $\mathbf{x}$ *is* (θ, u, k)*-free; see Definitions* 1.22(3)*,* 1.11(4)*,*(5)*, respectively.*

4) *If* $\mathbf{x}$ *is* (θ_2, θ_1)*-*(u,k)*-free then* $\mathbf{x}$ *is* $(\theta_2, \theta_1, u, k)$*-free; see Definitions* 1.22(4) *and* 1.11(4)*, respectively.*

Proof. 1) As $\bar\eta \in \Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u}) \setminus \Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ and $\bar\nu \in \Lambda_{\mathbf{x}}(\mathscr{U},\mathbf{u})$, as $\mathbf{u} \in \mathrm{add}_{\mathbf{x}}(u^\perp)$, it follows that $(\exists m \in u^\perp)[\eta_m \neq \nu_m]$.
2) Similarly.
3) Why? The assumption tells us that for every $\rho \in \prod\limits_{\ell \in u^\perp} S_\ell$, the set

$$\Lambda_\rho := \{\nu \in \Lambda : \nu \restriction u^\perp = \rho\}$$

is (θ, u)-free. Clearly this will suffice.
4) Similarly. $\square_{1.23}$

The gain in the following theorem is that [when] taking products of combinatorial parameters, we gain new cases of freeness.

Theorem 1.24 (**The Freeness Theorem**)**.** *If* $\boxplus$ *below holds, then* $\mathbf{x}$ *is* $(\theta_{\mathbf{m}}, \theta_0^+)$*-*$(u,1)$*-free. If in addition every* $\mathbf{x}_\ell$ *is* θ_0^+*-free, then* $\mathbf{x}$ *is* $(\theta_{\mathbf{m}}, u)$*-free.*

$\boxplus$ (*a*) $\mathbf{x}_\ell$ *is a combinatorial* $\langle \partial_\ell \rangle$*-parameter for* $\ell < \mathbf{k}$*.*
(*b*) $\mathbf{x} = \mathbf{x}_0 \times \ldots \times \mathbf{x}_{\mathbf{k}-1}$
(*c*) $u \subseteq \{0, \ldots, \mathbf{k}-1\}$ *and* $\mathbf{m} = |u| > 0$ *(hence* $\mathbf{m} \le \mathbf{k}$*).*
(*d*) $\theta_0 < \theta_1 < \ldots < \theta_{\mathbf{m}}$ *are regular, except possibly* θ_0*.*
(*e*) $\partial_{\mathbf{x}_\ell} \le \theta_0$ *for* $\ell < \mathbf{k}$*.*
(*f*) $\mathbf{x}_k$ *is* $(\theta_{m+1}, \theta_m^+)$*-free when* $k \in u \wedge m < \mathbf{m}$*.*

Remark 1.25. Concerning the sequence of θ-s in 1.24$\boxplus$, we can use $\theta_\ell := \theta_0^{+\ell}$; in this case clause (f) always holds.

Proof. Without loss of generality $\mathbf{x}$ is disjoint (i.e. the sets $S_\ell := S_{\mathbf{x},\ell}$ are pairwise disjoint for $\ell < \mathbf{k}$). We prove the claim by induction on $\mathbf{m}$ (so fix $\mathbf{k}$, but we vary u and the θ_m-s). Let $\mathbf{u} \in \mathrm{add}_{\mathbf{x}}(u^\perp)$ and $\mathscr{U} \subseteq \bigcup\limits_{\ell<\mathbf{k}} {}^{\partial_\ell}(S_\ell)$ has cardinality $< \theta_{\mathbf{m}}$ and we shall prove that $\Lambda^*_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ is $(\infty, \theta_0^+, u, 1)$-free. Clearly this suffices for the first phrase and the second follows recalling 1.12(2), 1.23(2).

Case 1: $\mathbf{m} = 1$.
As $|u| = 1$, let $u := \{\ell\}$; hence $\bar\eta \mapsto \eta_\ell$ is a one-to-one function from $\Lambda^*_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ onto $\mathscr{U}_\ell := \mathscr{U} \cap \Lambda_{\mathbf{x}_\ell}$. We know that $\mathbf{x}_\ell$ is (θ_1, θ_0^+)-free and $|\mathscr{U}_\ell| < \theta_1$, hence there is a partition $\langle \mathscr{U}_{\ell,\alpha} : \alpha < \alpha_* \rangle$ of $\mathscr{U}_\ell$ to sets each of cardinality $\le \theta_0$ (where $\alpha_* \le |\mathscr{U}_\ell| < \theta_1$). $h_\ell : \mathscr{U}_\ell \to J_{\mathbf{x}_\ell}$ is such that

$$\alpha < \beta < \alpha_* \wedge \eta \in \mathscr{U}_{\ell,\alpha} \wedge \nu \in \mathscr{U}_{\ell,\beta} \wedge \partial_\ell > i \notin h_\ell(\nu) \Rightarrow \eta(i) \neq \nu(i).$$

For $\alpha < \alpha_*$, let $\Lambda_\alpha := \{\bar\eta \in \Lambda^*_{\mathbf{x}}(\mathscr{U},\mathbf{u}) : \eta_\ell \in \mathscr{U}_{\ell,\alpha}\}$. Clearly $\langle \Lambda_\alpha : \alpha < \alpha_* \rangle$ is a partition of $\Lambda^*_{\mathbf{x}}(\mathscr{U},\mathbf{u})$ to sets each of cardinality $\le \theta_0$. Let the function g from α_*

to $[u]^1 = \{\{\ell\}\}$ be defined by $g(\alpha) = \{\ell\}$. Clearly the partition $\langle \Lambda_\alpha : \alpha < \alpha_* \rangle$ and the functions g, h_ℓ witness that $\Lambda^*_{\mathbf{x}}(\mathscr{U}, \mathbf{u})$ is $(\theta_{\mathbf{m}}, \theta_0^+)$-free, as required.

Case 2: $\mathbf{m} > 1$.

Let $m := \mathbf{m} - 1$. As $\mathbf{m} > 1$, clearly m is ≥ 1. So for $k \in u$, the c.p. $\mathbf{x}_k$ is $(|\mathscr{U}|^+, \theta_m^+)$-free, and let $\mathscr{U}_k := \mathscr{U} \cap {}^{\partial(k)}(S_k) \subseteq \Lambda_{\mathbf{x}_k}$. By the induction hypothesis, without loss of generality $|\mathscr{U}| \geq \theta_m$. Hence as in earlier cases (see 1.16(1)(C)) we can find a function $h^*_k : \mathscr{U}_k \to J_{\mathbf{x}_k}$ such that in the directed graph $(\mathscr{U}_k, R_k)$, each node has out-degree $\leq \theta_m$. That is, $(\forall \eta \in \mathscr{U}_k)(\exists^{\leq \theta_m} \nu \in \mathscr{U}_k)[\eta \, R_k \, \nu]$, where

$(*)_1$ $R_k = R_{k,h_k} := \big\{(\eta, \nu) \in \mathscr{U}_k \times \mathscr{U}_k : (\exists i < \partial_k)\big[i \notin h^*_k(\nu) \wedge \eta(i) = \nu(i)\big]\big\}$

$(*)_2$ Let Λ_* be $\Lambda^*_{\mathbf{x}}(\mathscr{U}, \mathbf{u}) = \Lambda_{\mathbf{x}}(\mathscr{U} \cup \mathbf{u}) \setminus \Lambda_{\mathbf{x}}(\mathscr{U}, \mathbf{u})$.

$(*)_3$ Let $R_* := \big\{(\bar\eta, \bar\nu) \in \Lambda_* \times \Lambda_* :$ for some $k \in u$ we have $\eta_k \, R_k \, \nu_k$ and $\ell < \mathbf{k} \wedge \ell \neq k \Rightarrow \eta_\ell = \nu_\ell\big\}$.

Clearly,

$(*)_4$ (Λ_*, R_*) is a directed graph with each node having out-degree $\leq \theta_m$.

Let $\bar\Lambda = \langle \Lambda_\gamma : \gamma < \gamma_* \rangle$ be such that:

$(*)_5$ (a) $\bar\Lambda$ is a partition of Λ_*.

(b) Each Λ_γ has cardinality $\leq \theta_m$.

(c) If $\bar\eta \in \Lambda_\beta$, $\bar\nu \in \Lambda_\gamma$, and $\beta < \gamma < \gamma_*$ <u>then</u> $\neg[\bar\eta \, R_* \, \bar\nu]$.
That is,

- If $\ell \in u$ and $\bar\eta \upharpoonright (\ell, < 0) = \bar\nu \upharpoonright (\ell, < 0)$ <u>then</u> $\neg[\eta_\ell \, R_\ell \, \nu_\ell]$.

[Why? Let $\langle \bar\eta_\alpha : \alpha < |\Lambda_*| \rangle$ list Λ_* with no repetition. For $\alpha < |\Lambda_*|$ we define $u_{\alpha,n} \in \big[|\Lambda_*|\big]^{\leq \theta_m}$ by induction on n, increasing with n by $u_{\alpha,0} := \{\alpha\}$ and

$$u_{\alpha,n+1} := u_{\alpha,n} \cup \big\{\beta : (\exists \gamma \in u_{\alpha,n})[\bar\eta_\gamma \, R_* \, \bar\eta_\beta]\big\}.$$

So $u_\alpha := \bigcup_n u_{\alpha,n} \in \big[|\Lambda_*|\big]^{\leq \theta_m}$; [we know] $\alpha \in u_\alpha$ and

$$\bar\eta_\beta \, R_* \, \bar\eta_\gamma \ \wedge \ \beta \in u_\alpha \Rightarrow \gamma \in u_\alpha.$$

Let $\Lambda_\alpha := \big\{\bar\eta_\gamma : \gamma \in u_\alpha,$ but $(\forall \beta < \alpha)[\gamma \notin u_\beta]\big\}$; now check that $\bar\Lambda = \langle \Lambda_\alpha : \alpha < |\Lambda_*| \rangle$ is as required.]

$(*)_6$ It is enough to prove that Λ_γ is $(\infty, \theta_0^+, u, 1)$-free for each $\gamma < \gamma_*$.

[Why? It is enough to prove Λ_* is $(\infty, \theta_0^+, u, 1)$-free.

By the assumption of $(*)_6$, for each $\gamma < \gamma_*$ let $\bar\Lambda_\gamma, g_\gamma, \bar h_\gamma$ witness that Λ_γ is (∞, θ_0^+, u)-free. That is:[15]

- $\bar\Lambda_\gamma = \langle \Lambda_{\gamma,\varepsilon} : \varepsilon < \varepsilon_\gamma \rangle$ is a partition of Λ_γ.
- $\Lambda_{\gamma,\varepsilon}$ has cardinality $\leq \theta_0$.
- $g_\gamma : \varepsilon_\gamma \to u$
- If $\bar\eta, \bar\nu \in \Lambda_{\gamma,\varepsilon}$, $k \in u \subseteq \mathbf{k}$, and $k \neq g_\gamma(\varepsilon)$ then $\eta_k = \nu_k$.
- $\bar h_\gamma = \langle h_{\gamma,k} : k \in u \rangle$
- $h_{\gamma,m}$ is a function from Λ_γ into J_m
- If $\bar\eta \in \Lambda_{\gamma,\varepsilon}$, $\bar\nu \in \bigcup_{\xi < \varepsilon} \Lambda_{\gamma,\xi}$, $m = g_\gamma(\varepsilon)$, $\bar\nu \upharpoonright (m) = \bar\eta \upharpoonright (m)$, and $i \in \partial_k \setminus h_{\gamma,k}(\bar\eta)$ then $\eta_k(i) \neq \nu_k(i)$.

[15] Recall Definition 1.11(4); for $k = 1$, see 1.11(5).

Let

- $\zeta_\gamma := \sum_{\beta<\gamma} \varepsilon_\beta$ for $\gamma \le \gamma_*$.
- $\Lambda'_\varepsilon := \Lambda_{\gamma,\varepsilon-\zeta_\gamma}$ for $\varepsilon \in [\zeta_\gamma, \zeta_{\gamma+1}]$.
- g is the function with domain ζ_{γ_*} such that $g(\varepsilon) := g_\gamma(\varepsilon - \zeta_\gamma)$ when $\varepsilon \in [\zeta_\gamma, \zeta_{\gamma+1})$ and $\gamma < \gamma_*$.
- h_k is the function with domain Λ_* defined as follows: if $\bar\eta \in \Lambda_\zeta$, where $\zeta = \zeta_\gamma + \varepsilon$ and $\varepsilon < \varepsilon_\gamma$, then $h_k(\bar\eta) := h_{\gamma,k}(\bar\eta) \cup h^*_k(\bar\eta)$.

Now check Definition 1.11(4).]

Now let us prove $(*)_6$. Fix $\gamma < \gamma_*$; if $|\Lambda_\gamma| < \theta_m$ the desired statement follows from the induction hypothesis, so assume $|\Lambda_\gamma| = \theta_m$. Let $\langle \eta_{\gamma,\alpha} : \alpha < \theta_m\rangle$ list $\{\nu_k : \bar\nu \in \Lambda_\gamma \text{ and } k \in u\}$.

For $\beta < \theta_m$, let $\mathscr{U}_{\gamma,\beta} := \{\eta_{\gamma,\alpha} : \alpha < \beta\}$ and let $k(\beta)$ be the unique $k \in u$ such that $\eta_{\gamma,\beta} \in {}^{\partial_k}(S_k)$. Clearly $|\mathscr{U}_{\gamma,\beta}| < \theta_m$. Also, $\langle \mathscr{U}_{\gamma,\beta} : \beta < \theta_m\rangle$ is $\subseteq$-increasing continuous with union $\bigcup_{\beta<\theta_m} \Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}) = \Lambda_\gamma$.

We choose $\langle \bar\Lambda_\beta, g_\beta, \bar h^\beta\rangle$ by induction on $\beta < \theta_m$ such that

$(*)_7$
(a) $\bar\Lambda_\beta = \langle \Lambda_{\gamma,\varepsilon} : \varepsilon < \varepsilon_\beta\rangle$ is a partition of $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u})$ (so $\alpha < \beta \Rightarrow \bar\Lambda_\alpha \lhd \bar\Lambda_\beta$).
(b) Each $\Lambda_{\gamma,\varepsilon}$ has cardinality $\le \theta_0$.
(c) $g_\beta : \varepsilon_\beta \to u$ satisfies $\alpha < \beta \Rightarrow g_\alpha \subseteq g_\beta$.
(d) $\bar h^\beta = \langle h^\beta_k : k \in u\rangle$
(e) $h_{\beta,k} : \Lambda_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}) \to J_k$ such that $\alpha < \beta \Rightarrow h^\alpha_k \subseteq h^\beta_k$.
(f) If $\varepsilon < \varepsilon_\beta$, $\bar\eta \in \Lambda_{\gamma,\varepsilon}$, $g_\beta(\bar\eta) = k$ (so $k \in u$), $\bar\nu \in \bigcup_{\zeta<\varepsilon} \Lambda_{\gamma,\zeta}$, and
$$\bar\nu \restriction (k, <0) = \bar\eta \restriction (k, <0)$$
<u>then</u> $i \in \partial_{\mathbf{x},k} \setminus h_{\beta,k}(\bar\eta) \Rightarrow \nu_k(i) \neq \eta_k(i)$.

For $\beta = 0$ we have $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}) = \varnothing$, so this is obvious. For β limit take unions.

Lastly, for $\beta = \beta_* + 1$, it is enough to show that $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u})$ is (∞, θ_0^+, u)-free over $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta_*}, \mathbf{u})$. Now $\mathscr{U}_{\gamma,\beta} \setminus \mathscr{U}_{\gamma,\beta_*} = \{\eta_{\gamma,\beta_*}\}$ with $\eta_{\gamma,\beta_*} \in {}^{\partial_{k(\beta_*)}}(S_{k(\beta_*)})$, hence $\eta_{\gamma,\beta_*} \in \mathscr{U}$.

Let $u_{\gamma,\beta} := u \setminus \{k(\beta_*)\}$ and $\mathbf{u}_{\gamma,\beta} := \mathbf{u} \cup \{\eta_{\gamma,\beta_*}\}$, so $\mathbf{u}_{\gamma,\beta} \in \operatorname{add}_{\mathbf{x}}(u^\perp_{\gamma,\beta})$ and $u_{\gamma,\beta} \subseteq \{0, \ldots, \mathbf{k}_{\mathbf{x}} - 1\}$ has m members, because $|\mathbf{u}| = \mathbf{m} = m + 1$. Recall
$$\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}_{\gamma,\beta}) = \Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}) \setminus \Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta_*}, \mathbf{u}),$$
and by the induction hypothesis on m we know $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}_{\gamma,\beta})$ is $(\infty, \theta_0^+, u_{\gamma,\beta})$-free so there is a witness $(\bar\Lambda^*_{\gamma,\beta}, g^*_{\gamma,\beta}, \bar h^*_{\gamma,\beta})$ (i.e. it is as in 1.11(4)(D) for $k = 1$). In particular:

$(*)_8$ $\bar\Lambda^*_{\gamma,\beta} = \langle \Lambda^*_{\gamma,\beta,\zeta} : \zeta < \zeta_{\gamma,\beta}\rangle$ is a partition of $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}_{\gamma,\beta})$.

We define

$(*)_9$
- $\varepsilon_\beta := \varepsilon_{\beta_*} + \zeta_{\gamma,\beta}$
- $\Lambda_{\varepsilon_{\beta_*}+\zeta} := \Lambda^*_{\gamma,\beta,\zeta}$ for $\zeta < \zeta_{\gamma,\beta}$.
- $g_\beta(\varepsilon_{\beta_*} + \zeta) = g^*_{\gamma,\beta}(\zeta)$ for $\zeta < \zeta_{\gamma,\beta}$; i.e. g_β is the function with domain ε_β extending g_{β_*} and defined on $[\varepsilon_{\beta_*}, \varepsilon_\beta)$ as above.
- $h_{\beta,k}$ is a function with domain $\Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta,\mathbf{u}}) = \bigcup_{\varepsilon<\varepsilon_\beta} \Lambda_\varepsilon$ extending $h_{\gamma,\beta_*,k}$.

- $h_{\beta,k}(\bar\eta) = h^*_{\gamma,\beta,k}(\bar\eta)$ if $\bar\eta \in \Lambda^*_{\mathbf{x}}(\mathscr{U}_{\gamma,\beta}, \mathbf{u}_{\gamma,\beta})$.

Now check: notice that if $\xi < \varepsilon_{\beta_*} \le \varepsilon < \varepsilon_\beta$, $\bar\nu \in \Lambda_{\gamma,\xi}$, $\bar\eta \in \Lambda_{\gamma,\varepsilon} = \Lambda^*_{\gamma,\beta,\varepsilon-\varepsilon_{\beta_*}}$, and $m = g_\beta(\varepsilon) = g^*_{\beta,\gamma}(\varepsilon - \varepsilon_{\beta_*})$ then $m \ne k(\beta_*)$ and $\eta_{k(\beta_*)} \ne \nu_{k(\beta_*)}$, so no problem will arise and the rest should be clear. $\square_{1.24}$

In what follows we assume $\ell < \mathbf{k} \Rightarrow \partial_\ell = \partial$ to simplify things. Anyhow, we have not sorted out what happens to (B)(d) when $\bar\partial$ is not constant, and what we have currently will suffice.

Theorem 1.26. *'If (A) then (B),' where:*

(A) (a) $\bar\partial = \langle \partial_\ell : \ell < \mathbf{k}\rangle$ *such that* $\ell < \mathbf{k} \Rightarrow \partial_\ell = \partial = \mathrm{cf}(\partial)$.

(b) $\mu_\ell \in \mathbf{C}_{\partial_\ell}$ *for* $\ell < \mathbf{k}$ *(see* 0.2, 0.3*).*

(c) $\mu_\ell < \mu_{\ell+1}$ *for* $\ell < \mathbf{k}$.

(d) $\chi_\ell = 2^{\mu_\ell}$

(e) $J^{\mathrm{bd}}_\sigma \odot J^{\mathrm{bd}}_{\partial_\ell}$*, an ideal on ∂_ℓ extending $J^{\mathrm{bd}}_{\partial_\ell}$ isomorphic to for some regular* σ,

$$\bigwedge_{\ell<\mathbf{k}} \left[\sigma < \partial_\ell \wedge J_\ell = J^{\mathrm{bd}}_\sigma \odot J^{\mathrm{bd}}_{\partial_\ell}\right].$$

[I have zero idea how you want me to combine the new and old parts of that sentence.]

(B) *There is* $\mathbf{x}$ *such that:*

(a) $\mathbf{x}$ *is a combinatorial $\bar\partial$-parameter of cardinality* $\le \chi_{\mathbf{k}-1}$*, with* $J_{\mathbf{x},\ell} = J_\ell$.

(b) $\mathbf{x}$ *has a $\bar\chi$-black box.*

(c) $\mathbf{x}$ *is (θ_*, θ^+)-free* <u>*when*</u> $n_* \ge 1$, $\theta = \mathrm{cf}(\theta) \ge \partial$, $\theta_* = \theta^{+\partial\cdot n_*} < \mu_0$*, and* $3n_* + 4 < \mathbf{k}$.

(d) $\mathbf{x}$ *is θ_{**}-free* <u>*when*</u> $\theta_{**} = \partial^{+(\partial\cdot n_* + \partial)} < \mu_0$, $3n_* + 4 < \mathbf{k}$*, and* $n_* \ge 1$.

Remark 1.27. Note that the proof is somewhat easier when we assume $\theta^{+\partial(n_*+1)} < \mu_0$, and the loss is minor.

Proof. For each $\ell < \mathbf{k}$ we can choose $\mathbf{x}_\ell$ such that:

$\oplus$ (a) $\mathbf{x}_\ell$ is a combinatorial $\langle\partial_\ell\rangle$-parameter.

(b) $\mathbf{x}_\ell$ is $(\theta^{+\partial+1}, \theta^{+4}, J_{\mathbf{x},\ell})$-free when $\partial \le \theta < \mu_\ell$.

(c) $\mathbf{x}_\ell$ has a χ_ℓ-pre-black box; moreover,

(c)$^+$ $\mathbf{x}_\ell$ has χ_ℓ-black box.

(d) $\Lambda_{\mathbf{x}_\ell}$ has cardinality χ_ℓ.

(e) $\mathbf{x}_\ell$ is ∂^+-free.

[Why? By [She13a, 0.4,0.5,0.6$_{=\text{Ly19,y22,y40}}$], when we omit clause $\oplus$(c)$^+$; anyhow we elaborate (also, when $\partial = \aleph_0$ we have to say a little more). So let $\ell < \mathbf{k}$ and $\mu := \mu_\ell$, $\lambda := \chi_\ell$.

First, assume that there is a $(\mu^+, J^{\mathrm{bd}}_\partial)$-free subset $\mathscr{F} \subseteq {}^\partial(\mu)$ of cardinality $\lambda = 2^\mu$. We define $\mathbf{x}_\ell$ by $\Lambda_{\mathbf{x}_\ell} := \{\langle\eta\rangle : \eta \in \mathscr{F}\}$ and $J_{\mathbf{x}_\ell} := J^{\mathrm{bd}}_\partial$.

Now $\mathbf{x}_\ell$ has λ-black box.[16] Note also that $\mathbf{x}_\ell$ is tree-like; this is enough for $\oplus$(a)-(e). Without loss of generality there is a list $\langle \eta_\alpha : \alpha < \lambda\rangle$ of the elements of $\mathscr{F}$ such that $\alpha < \beta \Rightarrow \eta_\alpha <_{J^{\mathrm{bd}}_\partial} \eta_\beta$ (see the proof of [She13b, 3.10$_{=\text{L1f.28}}$]. Let $\langle \mathscr{U}_\alpha : \alpha < \lambda\rangle$ be a sequence of pairwise disjoint subsets of λ each of cardinality

[16] By [She13b, §3]; this is easy as the number of functions from ${}^{\partial>}(\mu)$ to λ is $\lambda^\mu = \lambda$.

λ such that $\min(\mathscr{U}_\alpha) > \mu^\omega \cdot \alpha$ and let $\mathscr{F}_\alpha = \{\eta_\beta : \beta \in \mathscr{U}_\alpha\}$ and $\nu_\alpha = \eta_\beta$ when $\alpha \in [\mu \cdot \beta, \mu \cdot \beta + \mu)$ and $\mathscr{F}_* = \bigcup\limits_\alpha \mathscr{F}_\alpha$. Now we choose

$$\Lambda_{\mathbf{x}_\ell} := \{\langle\eta\rangle : \eta \in \mathscr{F}_*\} \text{ and } \Lambda^*_\alpha := \{\langle\eta\rangle : \eta \in \mathscr{F}_\alpha\},$$

so $\langle \nu_\alpha : \alpha < \lambda\rangle$ witnesses that $\mathbf{x}_\ell$ has $\bar{\chi}$-black box.

[What's ν? Everything up to this point has been about ηs.]

Second, assume that there is no $\mathscr{F}$ as above. It follows that $\lambda = 2^\mu$ is regular (see [She13a, 0.4] or [She13b, §3], using the "no hole claim" combining). Note that if there is a $\langle\partial\rangle$-c.p. $\mathbf{x}$ which is (θ_2, θ_1)-free and $\Lambda_{\mathbf{x}} \subseteq {}^\partial\mu$ (pedantically, $\Lambda_{\mathbf{x}} \subseteq \{\langle\eta\rangle : \eta \in {}^\partial\mu\}$, $|\Lambda_{\mathbf{x}}| = 2^\mu$, and $J = J_{\mathbf{x}} := J^{\mathrm{bd}}_\partial$ an ideal on ∂) <u>then</u> there is such $\mathbf{y}$ with $J_{\mathbf{y}} = J^{\mathrm{bd}}_\partial$ and as above both have the λ-BB.

Now as $\lambda = \mathrm{cf}(\lambda) = 2^\mu$ and $\mu \in \mathbf{C}_\partial$, there is a sequence $\langle \lambda_i : i < \partial\rangle$ of regular cardinals $< \mu$ and[17] a ∂-complete ideal $J = J^{\mathrm{bd}}_\partial$ such that $\chi = \mathrm{tcf}(\prod\limits_{i<\partial} \lambda_i, <_J)$: so let $\langle \eta_\alpha : \alpha < \chi\rangle$ be $<_J$-increasing cofinal in $(\prod\limits_{i<\partial} \lambda_i, <_J)$. By [She13a, 0.1=L41] there is $S \in \check{I}_{\theta^+}[\lambda]$ such that if $\delta < \lambda \wedge \mathrm{cf}(\delta) \geq \theta^{+4}$ then

$$\{\delta' < \delta : \mathrm{cf}(\delta') = \theta^{+3} \text{ and } S \cap \delta' \text{ is a stationary subset of } \delta'\}$$

is stationary in δ. (Note that there $\mathrm{cf}(\delta) = \theta^{+4}$, but the general case of $\mathrm{cf}(\delta) \geq \theta^{+4}$ follows.)

Assume $\lambda = \mathrm{cf}(\lambda)$, $S \subseteq \lambda$, and $\sup(S) = \lambda$, and we recall some things from [She13a].

If $\bar{f} = \langle f_\alpha : \alpha < \lambda\rangle$ is $<_J$-increasing, J is an ideal on ∂, $f_\alpha : \partial \to \mathrm{Ord}$ and $u_\alpha \subseteq \alpha$ for $\alpha < \lambda$, we say $\bar{f}$ *obeys* the sequence of sets $\bar{u} = \langle u_\alpha : \alpha < \lambda\rangle$ <u>when</u> for every $\beta \in u_\alpha$ we have

$$\bigwedge_{\gamma<\partial} [f_\beta(\gamma) < f_\alpha(\gamma)]$$

and if $\alpha \in S$ is a limit ordinal then $f_\alpha(\gamma) = \sup\limits_{\beta \in u_\alpha} \big(f_\beta(\gamma) + 1\big)$ for every $\gamma < \partial$.

For $\theta = \mathrm{cf}(\theta) < \lambda$, we say $\bar{u}$ as above is a *witness* for $S \in \check{I}_\theta[\lambda]$ <u>when</u>:

- $\alpha \in S \Rightarrow \mathrm{cf}(\alpha) = \theta$
- $\alpha < \lambda \Rightarrow |u_\alpha| < \theta$
- $\alpha \in u_\beta \Rightarrow u_\alpha = u_\beta \cap \alpha$
- There is a club E of λ such that if $\delta \in S \cap E$ then u_α is an unbounded subset of α of order type θ.

We say $\bar{f}$ is *good in* a limit ordinal $\delta < \lambda$ <u>when</u> there are $u \subseteq \delta = \sup(\delta)$ and $\overline{w} = \langle w_\alpha : \alpha \in u\rangle \in {}^u J$ such that

$$\alpha, \beta \in u \wedge \alpha < \beta \wedge i \in \partial \setminus (w_\alpha \cup w_\beta) \Rightarrow f_\alpha(i) < f_\beta(i).$$

So without loss of generality $\bar{f}$ obeys a witness for $S \in \check{I}_{\partial^+}[\lambda]$, hence it is good in δ when $\delta \in S$ <u>or</u> $S \cap \delta$ is a stationary subset of δ and $\mathrm{cf}(\delta) \in (\theta, \theta^{+\partial})$.

Let $S_\bullet \in \check{I}_\sigma[\lambda]$ be stationary such that

$$\delta \in S_\bullet \Rightarrow \mu^\omega \mid \delta$$

For $\delta \in S_\bullet$, let $\rho_\delta \in {}^\sigma\delta$ be increasing with limit δ. Now let $f'_\delta \in {}^\sigma\delta$ be such that

$$i < \partial \wedge j < \sigma \Rightarrow f'_\delta(\sigma i + j) := \mu \cdot f_\delta(i) + \rho_\delta(j).$$

[17] See [She93a, 6.5].

Hence $\{f_\alpha : \alpha < \lambda\}$ is $(\theta^{+\partial+1}, \theta^{+4})$-free for every $\theta \geq \partial$ (see [She13a, $0.4_{=\mathrm{Ly19}}$][18]). Together we are done except for $\oplus$(c),(c)$^+$, which holds by [She13a, 0.6(g),(g)$'$].]

So we have finished proving $\oplus$.

Let $\mathbf{x} := \mathbf{y}_{\mathbf{k}}$, where for $m \in \{1, \ldots, \mathbf{k}\}$ we let $\mathbf{y}_m = \mathbf{x}_0 \times \mathbf{x}_1 \times \ldots \times \mathbf{x}_{m-1}$ and we shall show it is as required.

Clause (B)(a): "$\mathbf{x}$ is a combinatorial $\bar{\partial}$-parameter of cardinality $\chi_{\mathbf{k}-1}$."

This holds by 1.20(1); i.e. we can prove "$\mathbf{y}_m$ is a $\langle \partial_\ell : \ell < m \rangle$-c.p. of cardinality χ_{m-1}" by induction on $m = 1, \ldots, \mathbf{k}$.

Clause (B)(b): "$\mathbf{x}$ has a $\bar{\chi}$-BB."

This holds by 1.21, that is, again by induction on $m = 1, \ldots, \mathbf{k}$, we can prove that $\mathbf{y}_m$ has the $\langle \chi_\ell : \ell < m \rangle$-BB.

We now shall prove:

Clause (B)(c):

We deduce it from $1.24 + \oplus$(b). We are given θ and n_* as there. Let $\langle \theta_m : m \leq m_* \rangle$ be defined as follows: $m_* := 3n_* + 4$, $\theta_\iota := \theta^{+\iota}$ for $\iota = 0, 1, 2, 3$, and $\theta_{3+3m+\iota} := (\theta_{3+3m})^{+(\partial+\iota)}$ for $\iota = 1, 2, 3$ when $m < n_*$ and $\theta_{m_*} := \theta^{+\partial+1}_{3n_*+4} < \mu_0$;

[You just said $m_* := 3n_* + 4$, so $\theta_{m_*} = \theta_{3n_*+4} \neq (\theta_{3n_*+4})^{+(\partial+1)}$]

the "$\leq \mu_0$" holds by the assumption of clause (B)(c). Note that if $\theta_{m+1} = \theta_m^+$ then "$\mathbf{x}_\ell$ is $(\theta_{m+1}, \theta_m^+)$-free" is trivial.

To apply Theorem 1.24 with $\mathbf{x}_\ell$ as in $\oplus$ above, $\mathbf{x}$ as above, $\mathbf{m} = m_*, u = \{0, \ldots, \mathbf{k}-1\}$ has $\mathbf{m}$ members and θ_ℓ for $\ell \leq \mathbf{m}$ as above; we have to verify clauses (a)-(f) of 1.24$\boxplus$.

(a) '$\mathbf{x}_\ell$ is a combinatorial $\langle \partial_\ell \rangle$-parameter;' holds by $\oplus$(a).

(b) $\mathbf{x} = \mathbf{x}_0 \times \ldots \mathbf{x}_{\mathbf{k}-1}$; holds by the choice of $\mathbf{x}$ above.

(c) "$u \subseteq \{0, \ldots, \mathbf{k}-1\}$ and $\mathbf{m} = |u| > 0$;" holds by the choice of u and the assumption on m_*.

(d) "$\theta_0 < \ldots < \theta_{\mathbf{m}}$" holds by the choice of the θ_ℓ-s above. Notice that each θ_ℓ (for $\ell > 0$) is a successor and hence regular.

(e) "$\partial_{\mathbf{x}_\ell} \leq \theta_0$ for $\ell < \mathbf{k}$;" this holds because $\theta_0 = \theta \geq \partial = \partial_\ell$ for $\ell < \mathbf{k}$.

(f) "$\mathbf{x}_\ell$ is $(\theta_{m+1}, \theta_m^+)$-free when $\ell \in u$ and $m < \mathbf{m}$." We check this by cases.

Case 1: **[If]** $\theta_{m+1} = \theta_m^+$, (f) holds trivially.

Case 2: $m = 3$, $(\theta_{m+1}, \theta_m^+) = (\theta^{+(\partial+1)}, \theta^{+4})$ holds by clause (b) of $\oplus$.

Case 3: $m = 3n + 3$ where $n < n_*$, so $(\theta_{m+1}, \theta_m^+) = (\theta^{\partial\cdot(n+1)+1}, \theta^{\partial\cdot n+4})$. By clause (b) of $\oplus$ above applied to $\theta = \partial^{+\partial\cdot n}$.

So all clauses of $\boxplus$ of Theorem 1.24 hold, hence its conclusion which says $\mathbf{x}$ in $(\theta_{\mathbf{m}}, \theta_0^+)$-free but $\theta_{\mathbf{m}} = \theta_*$ and $\theta_0 = \partial$, so we are done proving clause (c) of 1.26(B).

Clause (B)(d): "$\mathbf{x}$ is θ_* free, assuming $\theta_* = \partial^{+\partial\cdot(n_*+1)} < \mu_0$, $3m + 4 < \mathbf{k}$, and $\bigwedge_{\ell<\mathbf{k}} [\partial_\ell = \partial]$."

We will deduce it from clause (B)(c); choose $\theta'_* := \theta^{+\partial\cdot n_*+4}, \theta := \partial$, and $m_* := m$. The assumptions in clause (c) hold: $\theta = \partial^+$ so $\theta \geq \partial$, θ'_* is as θ_* there, and $\theta'_* < \mu_0$ by an assumption of clause (d) (which also says $3n_* + 4 < \mathbf{k}$).

[18] In more detail, in [She13a, $0.4_{=\mathrm{Ly19}}$] we conclude (A) or (B): now (A) there is stronger (covered by everything written three paragraphs ago, starting with "First, . . ."), whereas if (B) there holds see [She13a, $0.6(\mathrm{e})_{=\mathrm{Ly40(e)}}$].

So the conclusion of clause (c) holds; i.e. $\mathbf{x}$ is (θ'_*, θ)-free. But $\theta_* \le \theta'_*$ so $\mathbf{x}$ is (θ_*, ∂^+)-free. Also each $\mathbf{x}_\ell$ is ∂^+-free by $\boxplus(e)$ hence by 1.12 the last two statements implies $\mathbf{x}$ is θ_*-free. $\square_{1.26}$

Conclusion 1.28. 1) *If $\sigma < \partial$ are regular, $\chi \ge \partial$, and $n \ge 1$ then for some m there is an $\aleph_{\partial \cdot n}$-free m-c.p. $\mathbf{x}$ which has the χ-BB, with $|\Lambda_{\mathbf{x}}| < \beth_{\partial\cdot\omega}(\chi)$ and $J_{\mathbf{x},m} = J^{\rm bd}_\partial \odot J^{\rm bd}_\sigma$.*
2) *If $\sigma = \partial$ is regular, $\chi \ge \partial$, and $n \ge 1$ then for some m there is an $\aleph_{\partial\cdot n}$-free m-c.p. $\mathbf{x}$ of cardinality $< \beth_{\partial\cdot\omega}(\chi) + \beth_{\omega_1}(\chi)$ which has the χ-BB, is not free,[19] $\Lambda_{\mathbf{x}}$ is not the union of $\le \chi$ free subsets, and $J_{\mathbf{x},m} = J^{\rm bd}_\partial$.*
3) *If $m = 3n+5$, $\sigma = \mathrm{cf}(\sigma) < \partial = \mathrm{cf}(\partial) < \chi < \mu_0 < \ldots < \mu_{m-1}$ with $\mu_\ell \in \mathbf{C}_\partial$ for $\ell < m$, $\lambda_\ell = \mathrm{cf}(2^{\mu_\ell})$, $S_\ell \subseteq \{\delta < \lambda_\ell : \mathrm{cf}(\delta) = \sigma\}$ from $\check I_\sigma[\lambda_\ell]$ [is] stationary, and $J = J^{\rm bd}_\partial \times J^{\rm bd}_\sigma$, then we have (A) or (B), where:*

- (A) (a) *For some ℓ, there is an $\mathscr{F} \subseteq {}^{\partial}(\mu_\ell)$ of cardinality 2^{μ_ℓ} which is μ_ℓ^+-free (i.e. is $(\mu_\ell^+, J^{\rm bd}_\partial)$-free; see Definition 0.7(1)) and even (μ_ℓ^+, J)-free.*
  - (b) *Hence, letting $\mathbf{x}$ be the 1-c.p. such that $\Lambda_{\mathbf{x}} = \{\langle\eta\rangle : \eta \in \mathscr{F}\}$ [and $J_{\mathbf{x}} = J$, $\mathscr{F}$] is a 2^{μ_ℓ}-BB for $\mathbf{x}$ which is μ_ℓ^+-free.*
- (B) *We can choose $\mathbf{x} = \mathbf{x}_0 \times \ldots \times \mathbf{x}_{m-1}$, where $\mathbf{x}_\ell$ is a 1-c.p.,*
$$\Lambda_{\mathbf{x}_\ell} := \{\eta_{\ell,\delta} : \delta \in S_\ell\},$$
and $\lim_{J_{\mathbf{x}_\ell}}(\eta_{\ell,\delta}) = \delta$. Moreover, $\eta_{\ell,\delta}$ is increasing with limit δ, $J_{\mathbf{x}_\ell} = J_\partial \odot J_\sigma$, and $\mathbf{x}_\ell$ has the χ-BB if $\chi < \mu_\ell$.

4) *Given $n, m, \sigma < \partial < \chi$ as in part (3), we can find μ_ℓ (and λ_ℓ, S_ℓ) as there such that:*

- (a) *If $\partial > \aleph_0$ then $\mu_\ell = \beth_{\partial\cdot(1+\ell)}(\chi)$. ([In order to] have "$\mathbf{x}$ is θ_*-free" we need $\chi \ge \theta$.)*
- (b) *If $\partial = \aleph_0$ [then] for some club E of ω_1 [we have] $\mu_\ell \in \big\{\beth_\delta(\chi) : \delta \in E\big\}$.*

Proof. 1) Let $\mathbf{k} := 3n+5$, and for $\ell < \mathbf{k}$ we let $\partial_\ell := \partial$,
$$\mu_\ell := \beth_{\partial\cdot(1+\ell)}\big(\partial^{+(\partial\cdot n+1)} + \chi\big),$$
and $\chi_\ell := 2^{\mu_\ell}$. So each μ_ℓ is strong limit of cofinality $\partial = \mathrm{cf}(\partial) > \sigma \ge \aleph_0$; recalling 0.3, we have $\mu_\ell \in \mathbf{C}_{\partial_\ell}$ (i.e. clause (A)(b) of Theorem 1.26 holds).

Clauses (A)(a),(c)-(e) of 1.26 are obvious, hence there is $\mathbf{x}$ as in clause (B) of 1.26. In particular, it is $\partial^{+(\partial\cdot n+1)}$-free. Also, $\partial^{+(\partial\cdot n+1)} < \beth_{\partial\cdot\omega}(\chi)$ hence also
$$\mu_\ell = \beth_{\partial\cdot(n+2)}(\partial^{+(\partial n+1)} + \chi) < \beth_{\partial\cdot\omega}(\chi)$$
hence $|\Lambda_{\mathbf{x}}| \le 2^{\mu_{\mathbf{k}}-1} < \beth_{\partial\cdot\omega}(\chi)$, so we are done.

2) If $\partial > \aleph_0$, the proof of part (1) holds and $|\Lambda_{\mathbf{x}}| < \beth_{\partial\cdot\omega}(\chi)$. If $\partial = \aleph_0$, we know (see [She94]) that there is a club E of ω_1 consisting of limit ordinals such that $\delta \in E \Rightarrow \beth_\delta(\chi) \in \mathbf{C}_\partial$. We define $\mathbf{k}$, ∂_ℓ as above, and for $\ell < \mathbf{k}$ we let δ_ℓ be the ℓ-th member of E and $\mu_\ell := \beth_{\delta_\ell}(\chi)$, and we continue as in the proof of part (1).
in [S⁺c] and anyhow not used

3) This is straightforward by [She13b], but we elaborate to some extent.
First assume that for some $\ell < \mathbf{k}$ clause (A)(a) of 1.28(3) holds, so $\mathbf{x}$ from (A)(b) is a well defined 1-c.p. and is μ_ℓ^+-free. Letting $\chi := 2^{\mu_\ell}$, there is a χ-BB for $\mathbf{x}$ because the number of functions $h : {}^{\partial>}(\mu_\ell) \to \chi$ is $\le \chi^{\mu_\ell} = \chi$, and by diagonalizing we

[19] Really, this follows.

can choose a χ-pre-BB for $\mathbf{x}$ (see 1.15). To get a χ-BB we work as in the proof of 1.21(2).

So assume there is no such ℓ. Then for each ℓ, we know that $\lambda_\ell = 2^{\mu_\ell}$ is regular (see [She13b, 3.10(3)$_{=\text{L1f.28,pg.39}}$]). By the proof of $\oplus$ in the beginning of the proof of 1.26, there is $\mathbf{x}_{\ell,1}$ as there, so $\Lambda_{\mathbf{x}_{\ell,1}} \subseteq {}^{\partial}(\mu_\ell)$. By [She13b, 3.6$_{=\text{L1f.21}}$], we know that $\alpha < \lambda_\ell \Rightarrow |\alpha|^\sigma < \lambda_\ell$, hence obviously there is a stationary set[20] $S_\ell \subseteq \check{I}_\sigma[\lambda_\ell]$ and without loss of generality $\delta \in S_\ell \Rightarrow (\mu_\ell)^\omega \mid \delta$.

Hence we can find $\bar{\nu} = \langle \nu_\delta : \delta \in S_\ell \rangle$ such that:

- $\nu_\delta \in {}^\sigma\delta$ is increasing with limit δ.
- $\nu_{\delta_1}(i_1) = \nu_{\delta_2}(i_2) \Rightarrow \ell_1 = \ell_2 \wedge \nu_{\delta_1} \restriction i_1 = \nu_{\delta_2} \restriction i_2$.
- $\nu_\delta(i)$ is divisible by μ_ℓ.

Let $\langle \rho_\delta : \delta \in S_\ell \rangle$ list $\Lambda_{\mathbf{x}_\ell,1}$, and for $\delta \in S_\ell$ let $\eta_\delta \in {}^\partial\delta$ be defined by

$$\eta_\delta(\sigma \cdot i + j) = \nu_\delta(j) + \rho_\delta(i)$$

for $i < \partial$ and $j < \sigma$. We define $\mathbf{x}_\ell$ by $\Lambda_{\mathbf{x}_\ell} := \{\eta_\delta : \delta \in S_\ell\}$, $J_{\mathbf{x}_\ell} := J_\partial \odot J^\delta_\sigma$, etc. Now

(∗) $\mathbf{x}_\ell$ is a $\langle \partial \rangle$-pre-BB of cardinality χ_ℓ, with the freeness properties from 1.26.

What about χ-pre-BB? By [She13b, §3] this holds whenever $\chi < \mu_\ell$, which is enough for applying [the definition]. To get χ-BB let $\langle \delta(\zeta) : \zeta < \lambda \rangle$ list S_ℓ in increasing order and let $\langle S_\alpha : \alpha < \lambda_\ell \rangle$ be a sequence of pairwise disjoint stationary subsets of S_ℓ such that $\min(S_\alpha) > \delta(\alpha)$. Let $\nu_\xi := \eta_{\delta(\zeta)}$ when $\xi \in [\zeta \cdot \mu, \zeta \cdot \mu + \mu)$.

We define $\Lambda_\alpha = \Lambda^\ell_\alpha := \{\eta_\delta : \delta \in S_\alpha\}$, so for each α there is a χ_ℓ-pre-BB for Λ_α, and we continue as in the proof of 1.26. We now continue as in part (1) by inside the proof of 1.26.

3) By the proofs above this should be clear. $\square_{1.28}$

Discussion 1.29. 1) The following statement appears in [She13a, 0.4$_{=\text{Ly19}}$]. If $\sigma = \text{cf}(\sigma) < \kappa = \text{cf}(\kappa)$ and $\mu \in \mathbf{C}_\kappa$, then at least one of the following holds:

(A) There exists a μ^+-free $\mathscr{F} \subseteq {}^\kappa\mu$ of cardinality $\lambda = 2^\mu$.

(B) $\lambda = 2^\mu$ is regular, and there is a $(\lambda, \mu, \sigma, \kappa)$-5-solution.

If (A) holds, then we get more than promised (i.e. μ^+_ℓ-freeness). Hence we may assume, without loss of generality, that (B) holds. We shall return to this point (and then recall the definition of '5-solution').

2) We can vary the definition of the BB, using values in χ or using models.

3) We can use products of just two combinatorial parameters, but with any $\mathbf{k}_\mathbf{x}$. At present, this makes no real difference.

Discussion 1.30. Assume $\mathbf{x}$ is a combinatorial $\bar{\partial}$-parameter, $\bar{\partial} = \bar{\partial}_\mathbf{x}$, and $\bar{\partial}' = \langle \partial'_\ell : \ell < \mathbf{k}_\mathbf{x} \rangle$ is a sequence of limit ordinals such that $\ell < \mathbf{k} \Rightarrow \text{cf}(\partial'_\ell) = \partial_\ell$.

It follows that there is a $\mathbf{y}$ such that:

(∗) (a) $\mathbf{y}$ is a combinatorial $\bar{\partial}'$-parameter.

(b) $\mathbf{S}_{\mathbf{y},\ell} = \{\partial'_\ell \cdot \alpha + i : \alpha \in S_{\mathbf{x},\ell}$ and $i < \partial'_\ell\}$

[20] In fact, $\{\delta < \lambda_\ell : \text{cf}(\delta) = \sigma\}$ belongs to $\check{I}_\sigma[\lambda_\ell]$; see [She93a, Claim 2.14].

(c) $\Lambda_{\mathbf{y}} = \{g(\bar{\eta}) : \bar{\eta} \in \Lambda\}$, where $g : \overline{S}^{[\bar{\partial}]}_{\mathbf{x}} \to \overline{S}^{[\bar{\partial}']}_{\mathbf{y}}$ is defined as follows: for each $\ell < \mathbf{k}$, for some increasing continuous sequence $\langle \varepsilon_{\ell,i} : i \leq \partial_\ell \rangle$ of ordinals with $\varepsilon_{\ell,0} = 0$ and $\varepsilon_{\ell,\partial_\ell} = \partial'_\ell$, we have $g(\bar{\eta}) = \bar{\nu}$ iff

$$\varepsilon \in [\varepsilon_{\ell,i}, \varepsilon_{\ell,i+1}) \Rightarrow \nu_\ell(\varepsilon) = \partial'_\ell \cdot \eta_\ell(i) + \varepsilon.$$

(Of course, we could have "economical.")

(d) If $\mathbf{x}$ has $\bar{\chi}$-BB and $\chi_\ell = \chi_\ell^{\partial'_\ell}$ for $\ell < \mathbf{k}$, then $\mathbf{y}$ has $\bar{\chi}$-BB.

Definition 1.31. We say a $\mathbf{k}$-c.p. $\mathbf{x}$ is (θ, σ)-*well-orderable* $(\bar{\chi}, \mathbf{k}, 1)$-BB when there is a witness $\bar{\Lambda}$, which means:

(A) $\bar{\Lambda} = \langle \Lambda_\alpha : \alpha < \delta \rangle$
(B) $\bar{\Lambda}$ is increasing continuous.
(C) $\mathrm{cf}(\delta) \geq \sigma$ and δ is divisible by θ.
(D) If $\alpha < \delta$ then $\mathbf{x} \restriction (\Lambda_{\alpha+1} \setminus \Lambda_\alpha)$ has $\bar{\chi}$-pre-black box.
(E) If $\alpha < \delta$, $\bar{\eta} \in \Lambda_{\alpha+1} \setminus \Lambda_\alpha$, and $m < \mathbf{k}$ then the following set belongs to $J_{\mathbf{x},m}$:

- $\big\{ i < \partial_{\mathbf{x},m} : \text{for some } \bar{\nu} \in \Lambda_\alpha \text{ we have } \bar{\eta} \upharpoonleft (m,i) = \bar{\nu} \upharpoonleft (m,i) \big\}$.

Claim 1.32. 1) *In Theorem* 1.26, *for any* $\theta = \mathrm{cf}(\theta) \leq \chi_{\mathbf{k}-1}$, *clause* (B)(b) *can be strengthened to '*$\mathbf{x}$ *has* θ*-well-orderable* $\bar{\chi}$*-black box.'*
2) *Analogously to Conclusion* 1.28.

§ 2. Building Abelian groups and modules with small dual

For transparency, we restrict ourselves to hereditary rings.

Convention 2.1. 1) All rings R are hereditary; i.e. if M is a free R-module then any pure sub-module N of M is free.
2) An alternative is to interpret "G is a θ-free ring" by demanding $\mathrm{cf}(\theta) > \aleph_0$, and in the game of choosing $A_n \in [G]^{<\theta}$ increasing with n, the even player can guarantee that the sub-module $\langle \bigcup_n A_n \rangle_G$ of G is free.

We shall try to use a $\bar{\partial}$-BB to construct Abelian groups and modules. In 2.2 we present a quite clear case: if $\bigwedge_\ell [\partial_\ell = \aleph_0]$, the ring is $\mathbb{Z}$ (and the equations are simple). Note that the addition of z (in 2.2(1)(b), 2.4(1)(a)) is natural when we are trying to prove $h \in \mathrm{Hom}(G, \mathbb{Z}) \Rightarrow h(z) = 0$ which is central in this section, but is not natural for treating some other questions. When dealing with TDC_λ we may restrict ourselves to G simply derived from $\mathbf{x}$ (see 2.2(3)) so we can ignore 2.2(1A),(2).

Definition 2.2. Let $\mathbf{x}$ be a tree-like[21] combinatorial $\bar{\partial}$-parameter (see Definition 1.2(1)) and let $\mathbf{k} := \mathbf{k}_\mathbf{x}$.
1) If $\ell < \mathbf{k}_\mathbf{x} \Rightarrow \partial_\ell = \aleph_0$, <u>then</u> we say an Abelian group G is *derived from* $\mathbf{x}$ <u>when</u>

- (A) G is generated by $X \cup Y$, where:
  - (α) $X = \{x_{\bar{\eta}\restriction(m,n)} : \bar{\eta} \in \Lambda_\mathbf{x}, m < \mathbf{k}_\mathbf{x} \text{ and } n \in \mathbb{N}\} \cup \{z\}$
  - (β) $Y = \{y_{\bar{\eta},n} : \bar{\eta} \in \Lambda_\mathbf{x} \text{ and } n \in \mathbb{N}\}$
- (B) Moreover, it is freely generated, except the following set of equations

$$\Xi_\mathbf{x} := \Big\{(n+1) \cdot y_{\bar{\eta},n+1} = y_{\bar{\eta},n} - \sum_{m<\mathbf{k}} x_{\bar{\eta}\restriction(m,n)} - a_{\bar{\eta},n} z_{\bar{\eta}} : \bar{\eta} \in \Lambda_\mathbf{x} \text{ and } n \in \mathbb{N}\Big\},$$

where

- $\bullet_1$ $z_{\bar{\eta}} \in \bigoplus\{\mathbb{Z}x_{\bar{\eta}\restriction(m,n)} : \bar{\eta} \in \Lambda_\mathbf{x}, m < \mathbf{k}, n \in \mathbb{N}\} \oplus \mathbb{Z}z$
- $\bullet_2$ $a_{\bar{\eta},n} \in \mathbb{Z}$.

1A) We say the Abelian group G is *canonically derived from* $\mathbf{x}$ <u>when</u> above we omit the $z_{\bar{\eta}}$-s: equivalently, $a_{\bar{\eta},n} = 0$. If we omit z we say *strictly* derived.

2) We say the derivation of G in part (1) is well-orderable (or "G or $\langle z_{\bar{\eta}} : \bar{\eta} \in \Lambda_\mathbf{x} \rangle$ universally respects $\mathbf{x}$") <u>when</u> we replace $\bullet_1$ above by:

- $\bullet_1'$ There is a listing $\langle \bar{\eta}_\alpha : \alpha < \alpha_* \rangle$ of $\Lambda_\mathbf{x}$ such that

$$z_{\bar{\eta}_\alpha} \in \bigoplus \big\{\mathbb{Z}x_{\bar{\eta}_\beta\restriction(m,n)} : \beta < \alpha,\ m < \mathbf{k}\big\} \oplus \mathbb{Z}z$$

  for every $\alpha < \alpha_*$.
  Such a sequence is called a *witness*.

3) We add 'simply' (derived from $\mathbf{x}$) <u>when</u> $z_{\bar{\eta}} = z$ for every $\bar{\eta}$.

Remark 2.3. 1) We can replace $(n+1)y_{\bar{\eta},n+1}$ by $k_{\bar{\eta},n} y_{\bar{\eta},n+1}$ with $k_{\bar{\eta},n} \in \{2, 3, \ldots\}$.
2) By combining Abelian groups, the "simply derived" is enough for cases of the TDC_λ. Instead of "simply derived," we may restrict $\langle z_{\bar{\eta}} : \bar{\eta} \in \Lambda_\mathbf{x} \rangle$ more than in 2.2(2).

A more general case than 2.2 is:

[21] In [She07] this was not necessary, as the definition of $\eta \restriction (m, n)$ there is $\eta \restriction (m, < n+1)$ here.

Definition 2.4. 1) We say an R-module G is derived from a combinatorial $\bar{\partial}$-parameter $\mathbf{x}$ when (R is a ring and):

(a) G_* is an R-module freely generated by
$$X_* := \{x_{\bar{\eta}\upharpoonright(m,i)} : m < \mathbf{k}_{\mathbf{x}}, i < \partial_m \text{ and } \bar{\eta} \in \Lambda_{\mathbf{x}}\} \cup \{z\}.$$
(b) The R-module G is generated by $\bigcup_{\bar{\eta}\in\Lambda_{\mathbf{x}}} G_{\bar{\eta}} \cup X_*$; also, $G_* \subseteq G$.

(c) G/G_* is the direct sum of $\big\langle (G_{\bar{\eta}} + G_*)/G_* : \bar{\eta} \in \Lambda_{\mathbf{x}}\big\rangle$.

(d) $Z_{\bar{\eta}} \subseteq X_* \subseteq G_*$ for $\bar{\eta} \in \Lambda_{\mathbf{x}}$.
(If $Z_{\bar{\eta}} = \{z_{\bar{\eta}}\}$, we may write $z_{\bar{\eta}}$ instead of $Z_{\bar{\eta}}$.)

(e) If $\bar{\eta} \in \Lambda_{\mathbf{x}}$ then the R-submodule $G_{\bar{\eta}} \cap G_*$ of G is generated by (that is, not only included in the submodule generated by)
$$\{x_{\bar{\eta}\upharpoonright(m,i)} : m < \mathbf{k}_{\mathbf{x}} \text{ and } i < \partial_{\mathbf{x},m}\} \cup Z_{\bar{\eta}} \subseteq X_*.$$

1A) We say $\mathfrak{x}$ is an *R-construction* (or $(R,\mathbf{x})$-construction) when it consists of $\mathbf{x}, R, G_*, G$, $\langle x_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\mathbf{x},<\mathbf{k}}\rangle$, $\langle G_{\bar{\eta}}, Z_{\bar{\eta}} : \bar{\eta} \in \Lambda_{\mathbf{x}}\rangle$ as above.[22] We may say $\mathfrak{x}$ is for $\mathbf{x}$ but we may write G rather than $G_{\mathfrak{x}}$, etc. when $\mathfrak{x}$ is clear from the context.

1B) For an R-construction $\mathfrak{x}$ we say "universally respecting $\mathbf{x}$" or "$\mathfrak{x}$ is well-orderable" when we can find a $\bar{\Lambda}$ *obeyed by* $\mathfrak{x}$. By this we mean:

(f) $\bullet_1$ $\bar{\Lambda} = \langle \Lambda_\alpha : \alpha \le \alpha_*\rangle$ is increasing continuous.
$\bullet_2$ $\Lambda_{\alpha_*} = \Lambda_{\mathbf{x}}$ and $\Lambda_0 = \varnothing$.
$\bullet_3$ If $\bar{\eta} \in \Lambda_{\alpha+1} \setminus \Lambda_\alpha$ and $m < \mathbf{k}$, then
$$\big\{i < \partial_m : (\exists \bar{\nu} \in \Lambda_\alpha)\big[\bar{\eta} \upharpoonright (m,i) = \bar{\nu} \upharpoonright (m,i)\big]\big\} \in J_{\mathbf{x},m}.$$
$\bullet_4$ If $\bar{\eta} \in \Lambda_{\alpha+1} \setminus \Lambda_\alpha$ then $Z_{\bar{\eta}} \subseteq \big\langle \{G_{\bar{\nu}} : \bar{\nu} \in \Lambda_\alpha\} \cup \{z\}\big\rangle_G$.

1C) We may say "G is derived from $\mathbf{x}$" and $\mathfrak{x}$ is derived from $\mathbf{x}$.

1D) We add "simple" or "simply derived" when $z_{\bar{\eta}} = z$ (hence $Z_{\bar{\eta}} = \{z\}$ for every $\bar{\eta} \in \Lambda$).

1E) We say $\mathfrak{x}$ is *almost simple* if $\big|Z_{\bar{\eta}} \setminus \{z\}\big| \le 1$.

2) Above, we say $\mathfrak{x}$ is a *locally free derivation* or locally free or G in part (1) is freely derived when in addition:

(g) If $\bar{\eta} \in \Lambda_{\mathbf{x}}$, $m < \mathbf{k}$, and $w \in J_{\mathbf{x},m}$ then $(G_{\bar{\eta}}/G_{\bar{\eta},m,w})$ is a free R-module, where $G_{\bar{\eta},m,w}$ is the R-submodule of G generated by
$$\{x_{\bar{\eta}\upharpoonright(n,i)} : n < k,\, i < \partial_n, \text{ and } n = m \Rightarrow i \in w\} \cup Z_{\bar{\eta}}.$$
So $G_{\bar{\eta}} = G^{\perp}_{\bar{\eta},m,w} \oplus G_{\bar{\eta},m,w}$ for some R-submodule $G^{\perp}_{\bar{\eta},m,w}$, and let $\mathfrak{x}$ determine it.

3) Above, we say $\mathfrak{x}$ is $(<\theta)$-locally free or $\mathfrak{x}$ is a free $(<\theta)$-derivation when[23] in addition to part (1):

(g)$^+$ Like (g), but the quotient $G_{\bar{\eta}}/G_{\bar{\eta},m,w}$ is θ-free.
(h) $\mathbf{x}$ is θ-free.

[22] And we shall write $\Lambda_{\mathfrak{x}} = \Lambda_{\mathbf{x}}$, $G^{\mathfrak{x}}_* = G_*$, $G_{\mathfrak{x}} = G$, $G_{\mathfrak{x},\bar{\eta}} = G_{\bar{\eta}}$, etc., so in 2.2(1) we have a $\mathbb{Z}$-construction with $G_{\bar{\eta}}/(G_{\bar{\eta}} \cap G_*)$ being isomorphic to $(\mathbb{Q},+)$.
[23] So "$\mathfrak{x}$ is locally free" does not imply "$\mathfrak{x}$ is θ-free" because of clause (h).

4) We say $\mathfrak{x}$ is a canonical R-construction (or canonical $(R, \mathbf{x})$-construction) when $\bar\eta \in \Lambda_{\mathbf{x}} \Rightarrow Z_{\bar\eta} = \varnothing$. We say canonically* when we omit z and we write $G^-_{\mathfrak{x}}$.

5) We say $\mathfrak{x}$ (or just $(\mathbf{x}, \bar Z)$, where $\bar Z = \langle Z_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x}}\rangle$) is θ-well-orderable when for every $\Lambda \subseteq \Lambda_{\mathbf{x}}$ of cardinality $< \theta$ there is $\langle \bar\eta_\alpha : \alpha < \alpha_*\rangle$, $\Lambda' \supseteq \Lambda$ witnessing [it], which means:

[Λ' is not used anywhere.]

(a) $\langle \bar\eta_\alpha : \alpha < \alpha_*\rangle \subseteq \Lambda_{\mathbf{x}}$ with no repetitions.

(b) If $\bar\eta \in \Lambda$ then
- $\bar\eta = \bar\eta_\alpha$ for some α.
- $Z_{\bar\eta} \subseteq \{\bar\eta_\beta : \beta < \alpha\}$
- For some $m_* < \mathbf{k}$ and $w \in J_{\mathbf{x},m}$, we have
$$i \in \partial_{m_*} \setminus w \Rightarrow \eta \upharpoonright (m_*, i) \notin \{\bar\eta_\beta \upharpoonright (m_i, j) : m < \mathbf{k},\, j < \partial_m\}.$$

Remark 2.5. In Definition 2.4, we might like $G_{\bar\eta}$ to have more elements from G_*. This can be accomplished by replacing $x_{\bar\nu}$, $\bar\nu \in \Lambda_{\mathbf{x},<\mathbf{k}}$ by $x_{\bar\nu,t}$ for $t \in T_{m,i}$ when $\bar\nu = \bar\eta \upharpoonright (m, i)$ [for some] $\bar\eta \in \Lambda_{\mathbf{x}}$.

However, we can just as well replace ∂_ℓ by $\partial'_\ell := \gamma \cdot \partial_\ell$ for some non-zero ordinal γ (and J_ℓ by $J'_\ell := \big\{w \subseteq \partial'_\ell : (\exists u \in J_\ell)\big[w \subseteq \{\gamma \cdot i + \beta : \beta < \gamma,\, i \in u\}\big]\big\}$).

Claim 2.6. *Assume $\mathfrak{x}$ is a simple R-construction (see* 2.4(1A),(1D)*) which is $(<\theta)$-locally-free (see* 2.4(2)*), $G = G_{\mathfrak{x}}$ (so it is derived from $\mathbf{x}$), and $\mathbf{x}$ is θ-free.*

1) *G is a θ-free R-module.*

2) *If in addition $(R, +)$ (that is, R as an additive – and therefore Abelian – group) is free then $(G, +)$ is θ-free.*

3) *In part* (2) *it suffices that $(R, +)$ is a θ-free Abelian group.*

4) *In* (1)-(3)*, we can replace "derived" by "$(<\theta)$-derived".*

5) *Instead of assuming "$\mathfrak{x}$ is simply derived," we can demand "$\mathfrak{x}$ is well-orderable and almost simple" (see Definition* 2.4(1B),(1E)*).*

Proof. 1) Let $X \subseteq G$ have cardinality $< \theta$. By Definition 2.4(1) there are $\Lambda \subseteq \Lambda_{\mathbf{x}}$ and $\Lambda_* \subseteq \Lambda_{\mathbf{x},<\mathbf{k}}$, each of cardinality $< \theta$, such that
$$X \subseteq \big\langle \{x_{\bar\eta} : \bar\eta \in \Lambda_*\} \cup \{G_{\bar\eta} : \bar\eta \in \Lambda\}\big\rangle_G,$$
recalling $\{z\} = Z_{\bar\eta} \subseteq G_{\bar\eta}$ for every $\bar\eta \in \Lambda_{\mathbf{x}}$. So without loss of generality
$$X = \{x_{\bar\eta} : \bar\eta \in \Lambda_*\} \cup \{Y_{\bar\eta} : \bar\eta \in \Lambda\},$$
where $Y_{\bar\eta} \in [G_{\bar\eta}]^{<\theta}$ for $\bar\eta \in \Lambda$ and
$$m < \mathbf{k} \wedge i < \partial_m \Rightarrow \bar\eta \upharpoonright (m, i) \in \Lambda_*.$$

As $\mathbf{x}$ is θ-free, we can find:

(a) $\langle \bar\eta_\alpha : \alpha < \alpha_*\rangle$ listing Λ,

(b) $m_\alpha < \mathbf{k}_{\mathbf{x}}$ and $w_\alpha \in J_{\mathbf{x},m_\alpha}$ for $\alpha < \alpha_*$,

[such that] if $\alpha < \beta$ and $i \in \partial_{\mathbf{x},m_\beta} \setminus w_\beta$ then $\eta_{\beta,m}(i) \neq \eta_{\alpha,m}(i)$.

For $\alpha \leq \alpha_*$, let $G_\alpha := \big\langle \bigcup_{\beta<\alpha} G_{\bar\eta_\beta}\big\rangle_G$ and let $G_{\alpha_*+1} := \big\langle G_{\alpha_*} \cup \{x_{\bar\eta} : \bar\eta \in \Lambda_*\}\big\rangle_G$.

So $\langle G_\alpha : \alpha \leq \alpha_* + 1\rangle$ is an increasing continuous sequence of sub-modules, $G_0 = 0$, and G_{α_*+1} includes X. Also, $G_{\alpha_*+1}/G_{\alpha_*}$ is free by the choices above.

Lastly, if $\alpha < \alpha_*$ then $G_{\alpha+1}/G_\alpha$ is a θ-free R-module because it is isomorphic to $G_{\bar\eta_\alpha}/G_\alpha = G_{\bar\eta_\alpha}/G_{\bar\eta_\alpha,m_\alpha,w_\alpha}$, which is θ-free by Definition 2.4(3)(g)$^+$.

So clearly we are done.

2,3) Follows.

4) Similarly.

5) Let X, Λ and Λ_* be as in the proof of part (1), and let $\langle \bar\eta_\alpha : \alpha < \alpha_* \rangle$ list Λ.

Let $\bar\Lambda$ witness the well-orderability of $\mathfrak{x}$. Then (recalling Definition 2.4(1B)) there is a function $h : \alpha_* \to \ell g(\bar\Lambda)$ such that:

(c) If $\alpha < \alpha_*$ then $\bar\eta_\alpha \in \Lambda_{h(\alpha)+1} \setminus \Lambda_{h(\alpha)}$.

Let

(d) $Z_{\bar\eta_\alpha} \setminus \{z\} \subseteq \{\nu_\alpha\} \subseteq \{\bar\eta_\beta \restriction (m,i) : \mathbf{n} < \mathbf{k},\ i < \partial_m,\ \text{and } \beta < \alpha\} \subseteq \Lambda_*$.

Also without loss of generality, as in §1

(e) h is non-decreasing.

Now, as Λ is θ-free as in §1, looking carefully at 2.4(1B), without loss of generality $|\Lambda_{\alpha+1} \setminus \Lambda_\alpha| \le 1$, so without loss of generality

(e)′ h is the identity.

The rest is as before. $\square_{2.6}$

Definition 2.7. 1) An Abelian group H is (θ_2, θ_1)-1-free <u>when</u> if $X \in [H]^{<\theta_2}$ then we can find a $\bar G$ such that:

- $\bar G = \langle G_\alpha : \alpha < \alpha_* \rangle$ is a sequence of subgroups of G.
- $G := \sum_{\alpha<\alpha_*} G_\alpha \subseteq H$; both G and H include X.
- G_α is generated by a set of $< \theta_1$ members.
- $G = \bigoplus_{\alpha<\alpha_*} G_\alpha$.

2) Similarly for R-modules.

Claim 2.8. 1) *If $\mathbf{x}$ is a (θ_2, θ_1)-free* $\mathbf{k}$-c.p. *(see* 1.11*), and $\mathfrak{x}$ is a canonical* $(R, \mathbf{x})$-*construction which is locally free and simply derived from* $\mathbf{x}$ <u>*then*</u> G *is* (θ_2, θ_1)-1-*free.*
2) *Similarly for modules, when each $G_{\bar\eta}$ (for $\bar\eta \in \Lambda_{\mathbf{x}}$) has cardinality $< \theta_1$.*

Proof. 1) By (2), using $R = \mathbb{Z}$.
2) Let $G = G_{\mathfrak{x}}$ such that $|\Lambda|, |\Lambda_*| < \theta_2$ and let $X \subseteq G$ be of cardinality $< \theta_2$. Choose Λ, Λ_* as in the proof of 2.6(1). As we are assuming "$\mathbf{x}$ is (θ_2, θ_1)-free" and $\Lambda \subseteq \Lambda_{\mathbf{x}}$ has cardinality $< \theta_2$, there is a sequence $\langle \bar\Lambda, g, \bar h \rangle$ witnessing it, see 1.11(4)(D) such that $\bar\Lambda = \langle \Lambda_\gamma : \gamma < \gamma_* \rangle$ and $\Lambda = \bigcup_\gamma \Lambda_\gamma$. We define the sequence $\langle G_\gamma : \gamma \le \gamma_* + 1 \rangle$ as follows.

For $\gamma < \gamma_*$ let G_γ be the submodule of $G_{\mathbf{x}}$ generated by $\bigcup_{\bar\eta \in \Lambda_\gamma} G^\perp_{\bar\eta, g(\gamma), h_\gamma(\bar\eta)}$. We may assume that $G_0 = \{0\}$. Let G_{γ_*} be the submodule of $G_{\mathbf{x}}$ generated by

$$\big\{ x_{\bar\nu} : \bar\nu \in \Lambda_*,\ \text{but for no } \gamma < \gamma_*,\ \bar\eta \in \Lambda_\gamma,\ \text{or } i \in \partial_{\mathbf{x}, g(\gamma)} \setminus h_\gamma(\bar\eta) \text{ do we have } \bar\nu = \bar\eta \restriction (g(\gamma), i) \big\}.$$

Finally, let $G_{\gamma_*+1} := \sum_{\beta \le \gamma_*} G_\beta$.

For every $\gamma \le \gamma_* + 1$, let $G_{<\gamma}$ be the submodule generated by $\bigcup\limits_{\alpha<\gamma} G_\alpha$. Notice that the sequence $\langle G_{<\gamma} : \gamma \le \gamma_* + 1\rangle$ is increasing and continuous. It suffices to prove that $G_\gamma \cap G_{<\gamma} = \{0\}$. If not, then for some n and pairwise distinct $\bar\eta_0, \dots, \bar\eta_{n-1} \in \Lambda_\gamma$, we have

$$\big(\sum_{\ell<n} G^\perp_{\bar\eta_\ell, g(\gamma), h_\gamma(\bar\eta_\ell)}\big) \cap G_{<\gamma} \ne \{0\}$$

(see 2.4(2)).

If $0 \ne x \in \big(\sum\limits_{\ell\le n} G^\perp_{\bar\eta_\ell,g(\gamma),h_\gamma(\bar\eta_\ell)}\big) \cap G_{<\gamma}$, then $x = \sum\limits_{\ell<n} x_\ell$ for some $x_\ell \in G^\perp_{\bar\eta_0,g(\gamma),h_\gamma(\bar\eta_\ell)}$. Recalling

$$G_{\mathfrak{x}}/G_* = \bigoplus_{\bar\eta\in\Lambda_{\mathbf{x}}} G_{\bar\eta}/(G_{\bar\eta} \cap G_*)$$

necessarily $x \in G_*$. Moreover, recalling 2.4(1)(c), for each $\ell < n$ we have $x_\ell \in G_*$ so

$$x_\ell \in G^\perp_{\bar\eta,g(\gamma),h_\gamma(\bar\eta_\gamma)} \cap G_* \subseteq \bigoplus \big\{x_{\eta_\ell\restriction(g(\gamma),i)} : i \in \partial_{\mathbf{x},g(\gamma)} \setminus h_\gamma(\bar\eta_\ell)\big\} \oplus Rz$$

(see 2.4(2)).

Hence

$$x = \sum_{\ell<n} x_\ell \in H_1 := \bigoplus \big\{x_{\bar\eta_\ell\restriction(g(\gamma),i)} : \ell < n,\ i \in \bigcup_{\ell_1<n} h_\gamma(\eta_{\ell_1})\big\}.$$

By the choice of $(\bar\Lambda, g, \bar h)$,

$$H_2 := G_{<\gamma} \cap G_* \subseteq \bigoplus \big\{Rx_{\bar\nu} : \text{for some } \alpha < \gamma,\ \bar\eta \in \Lambda_\alpha, \text{ and } i \in \partial_{\mathbf{x},g(\alpha)} \setminus h_\alpha(\bar\eta), \\ \text{we have } \bar\nu = \bar\eta \restriction (g(\alpha), i)\big\}.$$

Hence $x \in H_1 \cap H_2 = \{0\}$, a contradiction. $\square_{2.8}$

Claim 2.9. *Assume* $\mathbf{x}$ *is an* $(\aleph_0, \mathbf{k})$-c.p. *with* $(\aleph_0, \mathbf{k})$*-BB.*

1) *There is canonical* $\mathbb{Z}$*-construction* $\mathfrak{x}$ *such that:*

- (a) $G = G_{\mathfrak{x}}$ *(so* G *is an Abelian group of cardinality* $|\Lambda_{\mathbf{x}}|$*).*
- (b) G *is not Whitehead.*
- (c) G *is* θ*-free if* $\mathbf{x}$ *is* θ*-free.*
- (d) G *is* (θ_2, θ_1)*-1-free if* $\mathbf{x}$ *is* (θ_2, θ_1)*-free (see* 2.7(1)*).*
- (e) G *has a* $\mathbb{Z}$*-adic dense subgroup of cardinality* $|\Lambda_{\mathbf{x},<\mathbf{k}}|$.

2) *We can add:*

$(b)^+$ $\mathrm{Hom}(G, \mathbb{Z}) = 0$.

Remark 2.10. Recall that "$\mathbf{b}$ is a $(\chi, \mathbf{k})$-BB" means $\mathbf{b}$ is a function with range $\subseteq \chi$ (see Definition 1.7).

Proof. 1) Let $G_0 = \bigoplus\{\mathbb{Z}x_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x},<\mathbf{k}}\} \oplus \mathbb{Z}z$ and G_1 be the $\mathbb{Z}$-adic closure of G_0, so G_1 is a complete metric space under the $\mathbb{Z}$-adic metric.

For $\bar\eta \in \Lambda_{\mathbf{x}}$, $\bar a \in {}^\omega\mathbb{Z}$, and n_*, in G_1 we let

$$y_{\bar a,\bar\eta,n_*} := \Big(\sum_{n\ge n_*} \frac{n!}{n_*!}\Big)\Big(\sum_{m<\mathbf{k}} x_{\bar\eta\restriction(m,n)} - \sum_{m<\mathbf{k}} \mathbf{b}(\eta, m, n)z + a_n z\Big).$$

Let $\{b_i : i < \omega\}$ list the elements of $\mathbb{Z}$ and let $\bar{\mathbf{c}} = \langle \mathbf{c}_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x}}\rangle$ be an $(\aleph_0, \mathbf{k})$-BB, where $\mathbf{c}_{\bar\eta}$ is a function from $\{\bar\eta \restriction (m, n) : m < \mathbf{k}, n < \omega\}$ to $\mathbb{Z}$. Now for each $\bar\eta \in \Lambda_{\mathbf{x}}$, let

$$G^0_{\bar\eta} := \sum_{\substack{m<\mathbf{k}\\ n<\omega}} \mathbb{Z}x_{\bar\eta\restriction(m,n)} \oplus \mathbb{Z}z$$

and $h_{\bar\eta} \in \mathrm{Hom}(G^0_{\bar\eta}, \mathbb{Z}, z)$ be such that $h_{\bar\eta}(z) = z$ and $h_{\bar\eta}(x_{\bar\eta\restriction(m,n)}) = b_{\mathbf{c}_{\bar\eta}(\bar\eta\restriction(m,n))} z$.

$(*)_1$ We can choose $\bar a = \bar a[\bar\eta] \in {}^\omega\mathbb{Z}$ such that there is no extension of $h_{\bar\eta}$ in $\mathrm{Hom}(G^1_{\bar a,\bar\eta}, \mathbb{Z})$, where $G^1_{\bar a,\bar\eta} := \left\langle G^0_{\bar\eta} \cup \{y_{\bar a,\bar\eta,n} : n < \omega\}\right\rangle_{G_1}$.

[Why? This is well known, but we elaborate. It suffices to prove that

$$\mathcal{A} := \left\{\bar a \in {}^\omega 2 : h_{\bar\eta} \text{ has an extension in } \mathrm{Hom}(G^1_{\bar a,\bar\eta}, \mathbb{Z}) \text{ and } a_0 = a_1 = 0\right\}$$

is a countable subset of ${}^\omega\mathbb{Z}$. (We could have allowed $\bar a \in {}^\omega\mathbb{Z}$, but this seems more transparent to restrict ourselves.) For $\bar a \in \mathcal{A}$, let $h_{\bar a,\bar\eta}$ be an extension witnessing it. Now

- For each $b \in \mathbb{Z}$, the set $\mathcal{A}_b = \{\bar a \in \mathcal{A} \subseteq {}^\omega 2 : h_{\bar a,\bar\eta}(y_{\bar\eta,0}) = b\}$ has at most one member.

  [Why? Toward contradiction, assume $\bar a_1 \neq \bar a_2 \in \mathcal{A}_b$ and let n be minimal such that $a_{1,n} \neq a_{2,n}$. Now $n = 0, 1$ is impossible as $\bar a_1, \bar a_2 \in \mathcal{A}_b \subseteq \mathcal{A}$, so $n \geq 2$.

  Now prove by induction on $\ell < n$ that $h_{\bar a_1,\bar\eta}(y_{\bar\eta,\ell}) = h_{\bar a_2,\eta}(y_{\bar\eta,\ell})$. For $\ell = 0, 1$ use $\bar a_1, \bar a_2 \in \mathcal{A}_b$, and for $\ell = j + 1$ recall

  $$\ell \cdot y_{\eta,\ell} = y_{\eta,j} - \big(\sum_{m<\mathbf{k}} x_{\bar\eta\restriction(m,j)} + a_{\iota,j} z\big)$$

  for $\iota = 1, 2$; apply $h_{\bar a_\iota,\bar\eta}$ and use the induction hypothesis. Now on this equation, for $\ell = n$ and $\iota = 1, 2$ apply $h_{\bar a_\iota,\bar\eta}$. Subtracting, we get that $a_{1,n} - a_{2,n}$ is divisible by ℓ and $\ell \geq 2$. But $a_{1,n} - a_{2,n} \in \{1, -1\}$, a contradiction.]

So clearly there is $\bar a \in {}^\omega 2 \setminus \bigcup\limits_{b\in\mathbb{Z}} \mathcal{A}_b$ such that $a_0 = a_1 = 0$; it is as required. So $(*)_1$ does indeed hold.]

Lastly,

$(*)_2$ Let $G_1 := \left\langle G_0 \cup \{y_{\bar a[\bar\eta],\bar\eta,n} : n < \omega\} : \bar\eta \in \Lambda_{\mathbf{x}}\right\rangle_{G_1}$.

Now G_1 witnesses that $G_2 = G_1/\mathbb{Z}z$ is not a Whitehead group.

[Why? Let $G_2 = G_1/\mathbb{Z}z$, and let h_* be the canonical homomorphism from G_1 onto $G_1/\mathbb{Z}z$ (i.e. $h_*(x) = x + \mathbb{Z}z$ for $x \in G_1$). Toward contradiction, assume G_2 is a Whitehead group: this means that there is a homomorphism g_* from G_2 into G_1 inverting h_*. (That is, $y \in G_2 \Rightarrow h_*(g_*(y)) = y$.)

As $\mathrm{Ker}(g_*) = \mathbb{Z}z$, clearly

$$x \in G_1 \Rightarrow g_*(h_*(x)) - x \in \mathbb{Z}z,$$

so let h_ℓ be the unique function from $\Lambda_{\mathbf{x},<\mathbf{k}}$ into $\mathbb{Z}$ defined by

$$h_0(\bar\nu) = b \;\Leftrightarrow\; \bar\nu \in \Lambda_{\mathbf{x},<\mathbf{k}} \wedge k \in \mathbb{Z} \wedge g_*(h_*(x_{\bar\nu})) - x_{\bar\nu} = bz.$$

By the choice of $\mathbf{b}$, there is $\bar\eta \in \Lambda_{\mathbf{x}}$ such that

$$m < \mathbf{k} \wedge n < \omega \Rightarrow \mathbf{k}(\bar\eta, m, n) = h_\bullet(\bar\eta \restriction (m,n)).$$

So $x \mapsto x - g_*(h_*(x))$ defines a homomorphism from $G_{\bar a(\bar\eta),\bar\eta}$ onto $\mathbb{Z}z$ mapping z to itself and mapping $x_{\bar\eta\restriction(m,n)}$ to $b_{\mathbf{c}_\eta(\bar\eta\restriction(m,n))}z$, contradicting the choice of $\bar a(\eta)$. So $G_2 = G_1/\mathbb{Z}z$ is **[NOT?]** Whitehead indeed.

Now clearly for some canonical* $\mathbb{Z}$-construction $\mathfrak{x}$, $G_{\mathfrak{x}} = G^-_{\mathfrak{x}} \oplus \mathbb{Z}z$, and easily $G_2 \cong G^-_{\mathfrak{x}}$ and G_2 is a direct summand of $G_{\mathfrak{x}}$ so (by well-known facts in group theory) $G_{\mathfrak{x}}$ is also not a Whitehead group. The cardinality and freeness demands are obvious.

[I marked the most obvious typos, but this entire page needs to be rechecked.]

2) For transparency, we ignore the "Whitehead" condition. Recall we assume $\mathbf{x}$ has the $\aleph_0$-black box property, not just the $\aleph_0$-pre-black box (see 1.7(1),(4)).

Let $\langle \Lambda_\alpha : \alpha < |\Lambda_{\mathbf{x}}|\rangle$ and $\langle \bar\nu_\alpha : \alpha < \alpha_*\rangle$ be as in Definition 1.7(4). Let $\langle h_{\bar\eta} : \bar\eta \in \Lambda_\alpha\rangle$ be an $\aleph_0$-BB. We define a $(\mathbb{Z}, \mathbf{x})$-construction $\mathfrak{x}$ by choosing $(z_{\bar\eta}, \bar a_{\bar\eta})$ for $\bar\eta \in \Lambda_\alpha$ by induction on α such that:

- $\bullet_1$ $z_{\bar\eta} = z_0 = z$ if $\alpha = 0$ (alternatively, omit z).
- $\bullet_2$ $z_{\bar\eta} = z_\alpha = x_{\bar\nu_\alpha}$ if $\bar\eta \in \Lambda_{1+\alpha}$
- $\bullet_3$ For $\bar\eta \in \Lambda_\alpha$, $\bar a_{\bar\eta}$ is chosen such that there is no homomorphism h from $G_{\bar\eta}$ into $\mathbb{Z}$ such that $\big(h(x_{\bar\eta\restriction(m,i)}), h(z_\alpha)\big)$ is coded by $h_{\bar\eta}(\bar\eta \restriction (m,i))$.

So if $h \in \mathrm{Hom}(G_{\mathfrak{x}}, \mathbb{Z})$ then $\alpha < \alpha_* \Rightarrow h(z_\alpha) = 0$. But $\bigoplus\limits_\alpha \mathbb{Z} z_\alpha = G_0$ so $h \restriction G_0$ is zero, but $G_{\mathfrak{x}}/G_0$ is divisible hence h is zero.

Alternatively, omitting "$G = G_{\mathfrak{x}}$;" this follows easily by repeated amalgamation of the G constructed in part (1) over pure subgroups isomorphic to $\mathbb{Z}$. (See the proof of 2.12(3) or e.g. [She16, §3].) $\square_{2.9}$

* * *

Now Claim 2.9 (as stated) is enough, when we use §1 to get $\aleph_{\omega\cdot n}$-free $\mathbf{x}$ with χ-BB, (see 1.28(1),(2)). However, it is not enough for $\aleph_{\omega_1\cdot n}$-free, because there we need for $\partial = \aleph_1$, $J = J^{\rm bd}_\kappa \times J^{\rm bd}_\sigma$, and $\sigma < \kappa$ regular: in particular, $(\sigma, \kappa) = (\aleph_0, \aleph_1)$. So we better use the construction from Definition 2.4 rather than 2.2. Also we prefer to have general R-modules and we formalize the relevant property of $R, \bar\partial, \bar J, \theta$. We use ${}_R R$ to denote R as a left R-module.

Definition 2.11. 1) We say that $(\bar\partial, \bar J)$ '*θ-fits R*' (or R is $(\bar\partial, \bar J, \theta)$-fit)[24] <u>when</u>:

(A) (a) R is a ring.
  (b) $\mathbf{k} \geq 1$ is a natural number.
  (c) $\bar\partial = \langle \partial_\ell : \ell < \mathbf{k}\rangle$
  (d) ∂_ℓ is a regular cardinal.
  (e) $\bar J = \langle J_\ell : \ell < \mathbf{k}\rangle$
  (f) J_ℓ is an ideal on ∂_ℓ.

(B) If $G_0 = \bigoplus\{Rx_{m,i} : m < \mathbf{k},\ i < \partial_m\} \oplus Rz$, $h \in \mathrm{Hom}(G_0, {}_R R)$, and $h(z) \neq 0$ <u>then</u> there is G_1 such that
  $(*)$ (α) G_1 is an R-module extending G_0.
    (β) G_1 has cardinality $< \theta$.
    (γ) There is no homomorphism from G_1 to ${}_R R$ (i.e. R as a left R-module) extending h.

1A) We replace "fit" by "weakly fit" when in clause (B) we further demand that $h(x_{m,2i}) = h(x_{m,2i+1})$.

2) We say $(\bar\partial, \bar J)$ *freely θ-fits*[25] R (or R is $(\bar\partial, \bar J, \theta)$-fit) <u>when</u>:

(A) (a)-(f) As above.

(B) As above, adding
  $(*)$ (δ) If $m_* < \mathbf{k} \wedge w \in J_{m_*}$ <u>then</u> G_1 is free over

$$\bigoplus\{Rx_{m,i} : m < \mathbf{k}, i < \partial_m \text{ and } m = m_* \Rightarrow i \in w\} \oplus Rz.$$

[24] But if $\bar\partial = \bar\partial_{\mathbf{x}}$ and $\bar J = \bar J_{\mathbf{x}}$, then we may write $\mathbf{x}$ instead of $(\bar\partial, \bar J)$.

[25] But if $\bar\partial = \bar\partial_{\mathbf{x}}$ and $\bar J = \bar J_{\mathbf{x}}$, then we may write $\mathbf{x}$ instead of $(\bar\partial, \bar J)$.

3) In part (1) above (and also parts (4)-(6) below) we may write $(\partial, J, \mathbf{k})$ instead of $(\bar\partial, \bar J)$ when $\ell < \mathbf{k} \Rightarrow \partial_\ell = \partial \wedge J_\ell = J$. Also, we may omit $\bar J$ (or J) when $\ell < \mathbf{k} \Rightarrow J_\ell = J^{\rm bd}_{\partial_\ell}$.

4) Above, we may replace "J_ℓ is an ideal on ∂_ℓ" by $J_\ell \subseteq \mathcal{P}(\partial_\ell)$.

5) We may omit θ when $\theta = |R|^+ + \max\{\partial^+_m : m < \mathbf{k}\}$.

6) We replace fit by '$\mathbb{I}$-fit' when:

(A) $\mathbb{I}$ is a set of ideals of R, closed under finite intersections, including $I_0 := \{0_R\}$.

(B) We may replace Rz by $(R/I)z$ for $I \in \mathbb{I}$. The default value of $\mathbb{I}$ is

$$\big\{\{a : ab = 0\} : b \in R\big\}.$$

(C) In (1)(B)(∗), if $x \in G_1 \setminus \{0\}$ then $\mathrm{ann}(x, G_1) := \{a \in R : ax = 0\} \in \mathbb{I}$.

Claim 2.12. 1) *Assume* $\mathbf{x}$ *is a* $\mathbf{k}$-c.p., R *is a ring,* $\mathbf{x}$ *does* θ*-fit* R, $\chi^+ \geq \theta + |R|^+$, *and* $\mathbf{x}$ *has* $(\chi, \mathbf{k}, 1)$*-BB.*

There is $\mathfrak{x}$ *such that:*

(a) $\mathfrak{x}$ *is an* $(R, \mathbf{x})$*-construction.*

(b) $G = G_{\mathfrak{x}}$ *is an* R*-module of cardinality* $|\Lambda_{\mathbf{x}}|$.

(c) *There is no* $h \in \mathrm{Hom}(G, {}_RR)$ *such that* $h(z) \neq 0$.

(d) $\mathfrak{x}$ *is* simple*; that is,* $z_{\bar\eta} = z$ *for* $\bar\eta \in \Lambda_{\mathbf{x}}$.

2) *If in addition* $\mathbf{x}$ *freely* θ*-fits* R, *then we can add:*

(e) G *is* σ*-free if* $\mathbf{x}$ *is* σ*-free. (This always holds for* $\sigma = \min(\bar\partial_{\mathbf{x}})$.)

(f) G *is* (θ_2, θ_1)*-1-free if* $\mathbf{x}$ *is* (θ_2, θ_1)*-free.*

3) *In* (2) *we can add:*

(g) $\mathrm{Hom}(G, {}_RR) = 0$.

4) *Above, we can use "weakly fit."*

Proof. Let $G_* := \bigoplus\{Rx_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x},<\mathbf{k}}\} \oplus Rz$. (See more in [S+c].)

1) Let $\{(a^1_\varepsilon, a^2_\varepsilon) : \varepsilon < \chi\}$ list (possibly with repetitions) the members of

$$R \times (R \setminus \{0_R\}),$$

let $\mathbf{b}$ be a $(\chi, \mathbf{k}, 1)$-BB for $\mathbf{x}$, and let $\mathbf{b}', \mathbf{b}''$ be defined such that

$$\varepsilon = \mathbf{b}_{\bar\eta}(m, i) \Rightarrow \mathbf{b}'_{\bar\eta}(m, i) = a^1_\varepsilon \wedge \mathbf{b}''_{\bar\eta}(m, i) := a^2_\varepsilon.$$

For $\bar\eta \in \Lambda_{\mathbf{x}}$, let

$$G^0_{\bar\eta} := \sum_{i<\partial_m} \sum_{m<\mathbf{k}} Rx_{\bar\eta \upharpoonright (m,i)} \oplus Rz \subseteq G_*,$$

let $h_{\bar\eta}$ be the unique homomorphism from $G^0_{\bar\eta}$ into ${}_RR$ satisfying

$$h_{\bar\eta}(x_{\bar\eta \upharpoonright (m,i)}) = \mathbf{b}'_{\bar\eta}(m, i) \quad \text{and} \quad h_{\bar\eta}(z) = \mathbf{b}''_{\bar\eta}(0, 0),$$

and let $G^1_{\bar\eta}$ be an R-module extending $G^0_{\bar\eta}$ such that $(G^1_{\bar\eta}, G^0_{\bar\eta}, h_{\bar\eta})$ here are like (G_1, G_0, h) in Definition 2.11(1)(B)(∗), so in particular there is no homomorphism from $G^1_{\bar\eta}$ into ${}_RR$ extending $h_{\bar\eta}$.

Without loss of generality, $G^1_{\bar\eta} \cap G_0 = G^0_{\bar\eta}$ and $\langle G^1_{\bar\eta} \setminus G^0_{\bar\eta} : \bar\eta \in \Lambda_{\mathbf{x}}\rangle$ is a sequence of pairwise disjoint sets. Let G be the R-module generated by $\bigcup_{\bar\eta \in \Lambda_{\mathbf{x}}} G^1_{\bar\eta} \cup G_0$ extending each $G^1_{\bar\eta}$ and G_*, freely except this.

[Freely generated except what? Except the conditions that it be an extension of the submodules?]

Clearly we have defined an R-construction $\mathfrak{x}$ with $\mathbf{x}_{\mathfrak{x}} = \mathbf{x}$, $G_{\mathfrak{x}} = G$, and $z_{\mathfrak{x},\bar\eta} = \{z\}$, and clauses (a),(b),(d) of the desired conclusion hold. To prove clause (c), assume towards contradiction that $h \in \operatorname{Hom}(G, {}_R R)$ satisfies $h(z) \neq 0$. Let $g : \Lambda_{\mathbf{x},<\mathbf{k}} \to \chi$ be defined by

$$\bar\nu \mapsto \min\{\varepsilon < \chi : \big(h(x_{\bar\nu}), h(z)\big) = (a^1_\varepsilon, a^2_\varepsilon)\}.$$

Clearly the function is well defined, hence as $\mathbf{x}$ has $(\chi, \mathbf{k}, 1)$-BB (that is, by the choice of $\mathbf{b}$) there is $\bar\eta \in \Lambda_{\mathbf{x}}$ such that

$$m < \mathbf{k} \wedge i < \partial_m \Rightarrow g(\bar\eta \upharpoonright (m,i)) = \mathbf{b}_{\bar\eta}(m,i).$$

We get an easy contradiction.

What about the cardinality $|G|$? Note that $|G^1_{\bar\eta}| < \theta$ and $\theta \le \chi^+$.

2) In the proof of part (1), choosing $G^1_{\bar\eta}$ we add the parallel of clause $(*)(\delta)$ of 2.11(2)(B). Now clause 2.12(2)(e) holds by 2.6(1), and clause (f) by 2.8(2).

3) Let G be as constructed in part (1), and let

$$Y := \{y \in G : G/Ry \text{ is } \aleph_1\text{-free, or even } \min(\bar\partial)^+\text{-free}\}$$

(recall 2.6 plus the freeness of $\mathbf{x}$).

So by part (2) the set Y generates G. Let $\langle G_\varrho, h_\rho : \varrho \in {}^{\omega>}Y\rangle$ be such that G_ϱ is an R-module and h_ϱ is an isomorphism from G onto G_ϱ. Without loss of generality $0_{G_\varrho} = 0$ for every ϱ and $G_{\varrho_1} \cap G_{\varrho_2} = \{0\}$ for $\varrho_1 \neq \varrho_2$.

Let $H_1 := \bigoplus_{\varrho \in {}^{\omega>}Y} G_\varrho$, and let H_0 be the R-submodule of H_1 generated by

$$X := \{h_{\varrho\hat{\ }\langle y\rangle}(z) - h_\varrho(y) : \varrho \in {}^{\omega>}Y \text{ and } y \in Y\}.$$

Let $H := H_1/H_0$, and we shall prove that it is as required (on G). The main point is proving $\operatorname{Hom}(H, {}_R R)$ **[is trivial? / empty?]**

That is, toward contradiction $f_0 \in \operatorname{Hom}(H, {}_R R)$ is not zero and $f_1 \in \operatorname{Hom}(H_1, {}_R R)$ be defined by $f_1(x) = h(x + H_0)$,

[Where is f_0 used?]

so also f_1 is not zero but $x \in X \Rightarrow f_1(x) = 0$. By the choice of H_1, there is $\varrho \in {}^{\omega>}Y$ such that $f_1 \upharpoonright G_\varrho$ is not zero. But recall that G is generated by Y, hence G_ϱ is generated by $\{f_1, h_\varrho(y) : y \in Y\}$. Hence for some $n \ge 1$, $y_0, \ldots, y_{n-1} \in Y$, and $b_0, \ldots, b_{n-1} \in R \setminus \{0_R\}$, we have

$$f_1\big(h_\rho\big(\sum_{\ell<n} b_\ell, y_\ell\big)\big) \in R \setminus \{0\},$$

hence

$$f_1(h_\varrho(b_\ell y_\ell)) = f_1(b_\ell h_\varrho(y_\ell)) \neq 0$$

for some $\ell < n$. So letting $y = h_\varrho(y_\ell)$, we have $y \in G_\varrho$ and

$$c?? = f_1(b\, h_\varrho(y_\ell)) = f_2(b\, y))$$

for some $\ell \in R \setminus \{0\}$.

[So these guys aren't b_ℓs? b hasn't been defined yet.]

As said above, we have

$$f_1(y) = f_1(h_\varrho(y_\ell)) = f_1(h_{\varrho\hat{\ }\langle y_\ell\rangle}(z)),$$

so $f_1(h_{\varrho\hat{\ }\langle y_\ell\rangle}(z)) = b \in R\setminus\{0\}$. So $h_{\varrho\hat{\ }\langle y_\ell\rangle} \circ f_1 \in \operatorname{Hom}(G, {}_R R)$ maps z to $b \in R\setminus\{0\}$, a contradiction.

4) Similarly, but replacing $x_{\bar\eta}$ (for $\bar\eta \in \Lambda_{\mathbf{x},<\mathbf{k}}$) by $x_{\bar\eta,\zeta}$ (for $\zeta < |R|^+$). Let us elaborate.

Let $\langle(\alpha_{0,\varepsilon},\alpha_{1,\varepsilon},a_\varepsilon):\varepsilon<\chi\rangle$ list the members of

$$\{(\alpha_0,\alpha_1,a):\alpha_0<\alpha_1<\chi \text{ and } a\in R\setminus\{0_R\}\},$$

possibly with repetitions, let $\mathbf{b}$ be a $(\chi,\mathbf{k},1)$-BB for $\mathbf{x}$, and let $\mathbf{b}^\iota$ for $\iota=0,1,2$ be the functions with the same domain as $\mathbf{b}$ (writing $\mathbf{b}^\iota_{\bar\eta}(m,i)$ or $\mathbf{b}_{\eta,\iota}(m,i)$ for $\mathbf{b}^\iota(\bar\eta,m,i)$) such that $\varepsilon=\mathbf{b}_{\bar\eta}(m,i)$ implies

$$\langle\alpha_{\varepsilon,0},\alpha_{\varepsilon,1},\alpha_{\varepsilon,2}\rangle=\big(\mathbf{b}^0_{\bar\eta}(m,i),\mathbf{b}^1_{\bar\eta}(m,i),\mathbf{b}^2_{\bar\eta}(m,i)\big).$$

[$\alpha_{\varepsilon,2}$ hasn't been defined anywhere.]

Let $G_0:=\bigoplus\{Rx_{\bar\eta,\varepsilon}:\bar\eta\in\Lambda_{\mathbf{x},<\mathbf{k}}$ and $\varepsilon<\chi\}$.

$(*)_1$ For $\bar\eta\in\Lambda_{\mathbf{x}}$, let
- (a) $G^0_{\bar\eta}:=\sum\limits_{\varepsilon<\chi}\sum\limits_{m<\mathbf{k}}\sum\limits_{i<\partial_m}Rx_{\bar\eta\restriction(m,i),\varepsilon}$
- (b) $G^{0,0}_{\bar\eta}=\sum\limits_{m<\mathbf{k}}\sum\limits_{i<\partial_m}\big(Rx_{\bar\eta\restriction(m,i),\mathbf{b}_{\bar\eta,1}(m,i)}-x_{\bar\eta\restriction(m,i),\mathbf{b}_{\bar\eta,0}(m,i)}\big)\oplus Rz$
- (c) $G^{0,1}_{\bar\eta}=G^{0,0}\oplus Rz$
- (d) $h_{\bar\eta}$ is the homomorphism from $G^{0,0}_{\bar\eta}$ into R such that:
  - $h_{\bar\eta}\restriction G^{0,0}$ is constantly zero.
  - $h_{\bar\eta}(z):=\mathbf{b}_{\bar\eta,2}(0,0)\in R\setminus\{0\}$.
- (e) Let $\mathbf{h}_{\bar\eta}$ be the isomorphism from

  $$G_0=\bigoplus\{Rx_{\bar\eta\restriction(m,i)}:m<\mathbf{k},i<\partial_m\}\oplus Rz$$

  [That's not the same as how G_0 was just defined above.]
  onto $G^{0,1}_{\bar\eta}$ such that $\mathbf{h}_{\bar\eta}(z)=z$ and

  $$\mathbf{h}_{\bar\eta}(x_{m,i})=x_{\bar\eta\restriction(m,i),\mathbf{b}_{\bar\eta,1}(m,i)}-x_{\bar\eta\restriction(m,i),\mathbf{b}_{\bar\eta,0}(m,i)}.$$
- (f) Let $G^\bullet_{\bar\eta,1}$ be an R-module extending the R-module $G^\bullet_{\bar\eta}$ such that the triple $G^\bullet_0,G_{\bar\eta,1},h^\bullet_{\bar\eta}\circ\mathbf{h}_{\bar\eta})$ is as in 2.11(1)(B)$(*)$.
  [None of these guys have beend defined anywhere.]
- (g) Let $\mathbf{h}^+_{\bar\eta},G^1_{\bar\eta}$ be such that $G^1_{\bar\eta}$ is an R-module extending $G^0_{\bar\eta}$ and $\mathbf{h}^+_{\bar\eta}$ is an isomorphism from $G^\bullet_{\bar\eta,1}$ onto $G^1_{\bar\eta}$ extending $\mathbf{h}_{\bar\eta}$.

Lastly,

$(*)_2$ Without loss of generality $G^1_{\bar\eta}\cap G_0=G^{0,0}_{\bar\eta}$, $\langle G^1_{\bar\eta}\setminus G^{0,0}_{\bar\eta}:\bar\eta\in\Lambda_{\mathbf{x}}\rangle$ are pairwise disjoint, and G^*_1 is an R-module extending G_0 and $G^1_{\bar\eta}$ for $\bar\eta\in\Lambda_{\mathbf{x}}$, and generated by their union freely (except the equations implicit in "extending" above).

Note:

$(*)_3$ If $h\in\operatorname{Hom}(G,{}_RR)$ satisfies $h(z)\neq 0_R$ then we define a function $\mathbf{c}:\Lambda_{\mathbf{x},<\mathbf{k}}\to\chi$ as follows. $\mathbf{c}(\bar\eta?)$ is the minimal $\varepsilon<\chi$ such that:
- $h(x_{\bar\eta,\alpha_{\varepsilon,0}})=h(x_{\bar\eta,\alpha_{\varepsilon,1}})$
- $h(z)=a_\varepsilon$.

So indeed, $H:=H_1/H_0$ is as required. $\square_{2.12}$

Remark 2.13. We can use a 2^χ-BB $\mathbf{b}$, and then let $\mathbf{c}(\bar\eta)$ code

$$\big(h\restriction\{x_{\bar\eta,\varepsilon}:\varepsilon<\chi\},h(z)\big).$$

Remark 2.14. 1) There is an alternative to the proof of 2.12(3): assume that $\mathbf{x}$ has $\aleph_0$-well-orderable $(\chi, \mathbf{k}, 1)$-BB $\bar{\bar{\alpha}}$, as witnessed by $\bar{\Lambda}$ (see Definition 1.31). We then can find a $(R, \mathbf{x})$-construction obeying $\bar{\Lambda}$ (see 2.4(1B)).

2) It may suffice for us to prove in 2.12 that $\mathfrak{x}$ is simple and Rz is not a direct summand of the R-module $G_{\mathfrak{x}}$. For this, we can weaken the demand in Definition 2.11(1)(B) demanding $h(z) = 1_R$.

Claim 2.15. 1) *If $\partial = \aleph_0$, $J = J^{\rm bd}_\partial$, and $\mathbf{k} = 1$, then $(\partial, J, \mathbf{k}, \theta)$ freely fits R when:*

$\boxplus_1$ (a) *R is an infinite ring.*

(b) *If $D \in R \setminus \{0\}$ and $\bar{d} \in {}^\omega R$, then we can find $a^\iota_n \in R$ for $\iota = 1, 2, 3$ and $n < \omega$ such that the following set Γ of equations cannot be solved in R:*

$$\Gamma = \{a_n x_{n+2} = x_n + d_n + b_n D : n < \omega\}.$$

[Why aren't there any a^ι_n-s? I just see a_n and b_n.]

2) *For $\partial, J, \mathbf{k}$ as above, $(\partial, J, \mathbf{k}, \theta)$ weakly freely fits R when:*

$\boxplus_2$ (a) *As above.*

(b) *For every $D \in R \setminus \{0\}$, letting $\bar{d}$ be constantly 0_R, the demand in $\boxplus_1$ above holds. I.e. there are $a_n, b_n \in R$ for $n < \omega$ such that the following set Γ of equations is not solved in R:*

$$\Gamma = \{a_n x_{n+1} = x_n + b_n D : n < \omega\}.$$

[That's not the same set of equations.]

3) *If R is an infinite ring, then $\boxplus_1$ holds when:*

$\boxplus_3$ (a) *As above.*

(b) *$(R, +)$ is $\aleph_1$-free (or at least $\bigcap_{n \geq 2} nR = \{0\}$).*

Proof. 1) We should check all the clauses in Definition 2.11(2). First, Clause (A) is obvious: R is a ring by $\boxplus_1$(a) and $\mathbf{k} = 1 > 0$ by our assumption. Of course (letting $\bar{\partial} = \langle \partial \rangle$ and $\bar{J} = \langle J^{\rm bd}_\partial \rangle$) $\partial = \aleph_0$ is regular and $J^{\rm bd}_\partial = J^{\rm bd}_{\aleph_0}$ is an ideal on ∂.

Second, toward proving Clause (B), assume[26]

$$G_0 := \bigoplus_{i < \partial} R x_{0,i} \oplus \mathbb{Z} z,$$

$h_0 \in \mathrm{Hom}(G_0, {}_R R)$, and $D := h_0(z) \neq 0_R$, and let $D - n = h(x_{0,n})$. We should find G_1 satisfying (B)(∗) there. Let $\langle (a^\bullet_n, b_n) : n < k \rangle$ be as guaranteed by $\boxplus_1$(b) of the claim for D and $\langle d_n : n < \omega \rangle$ from above.

For each $n < \partial$, let $G^*_n := G_0 \oplus R y_n$ be an R-module; clearly there is an embedding $g_n : G^*_n \to G^*_{n+1}$ such that $g_n \restriction G_0 = \mathrm{id}_{G_0}$ and

$$g_n(y_n) = a^\bullet_n y_{n+1} + x_{0,n} + b_n z,$$

where the $a_n, b_n \in R$ are from $\boxplus_1$(b) for our h.

By renaming, without loss of generality $G^*_n \subseteq G^*_{n+1}$ and g_n is the identity on G^*_n. Lastly, let $G_1 := \bigcup_n G^*_n$ and it suffices to prove that (∗) of Definition 2.11(2)(B) is satisfied. Clearly G_1 is an R-module extending G_0; i.e. (∗)(α) holds. Also,

$$|G_1| \leq \aleph_0 + |G_0| = \aleph_0 + \aleph_0 \cdot |R| = |R| < |R|^+ = \theta$$

(recalling R is an infinite ring), so also (∗)(β) holds.

[26] Note that $m = 0$ is fixed below, as $m < \mathbf{k} := 1$.

Lastly, to prove $(*)(\gamma)$, toward contradiction assume $h_2 \in \mathrm{Hom}(G_1, {}_R R)$ extends h. Let $c_n := h_2(y_n) \in R$. Now:

$(*)_1$ (a) $\bar{c} = \langle c_n : n < \omega \rangle \in {}^\omega R$

(b) $a_n c_{n+1} = a_n h_1(y_{n+1}) = h_2(a^1_\eta y_{n+1}) = h_2(y_n + x_{0,n} + b_n z)$

$$= h_2(y_n) + h_1(x_{0,n}) + b_n h_2(z) = c_n + d_n + b_n D.$$

So $\bar{c}$ solves (in R) the set of equations $\Gamma = \{a_n z_{n+1} = z_n + d_n + b_n D : n < \omega\}$, contradicting the choice of $\big\langle (a_n, h_n) : n < \omega \big\rangle$.

We still have to justify the "freely;" i.e. clause (δ) of 2.11(2)(B)$(*)$. So let $m_* < \mathbf{k}$ (i.e. $m_* = 0$) and $w \in J_0 = J^{\rm bd}_\partial$ (so w is finite) and let $G_0 := \bigoplus\limits_{i \in w} R x_{0,i}$. Let n_* be such that $\sup(w) < n_*$, and we easily finish by noting:

$(*)_2$ The sequence $\langle y_n : n > n_* \rangle\hat{}\langle x_{0,m} : m \leq n^* \rangle\hat{}\langle z \rangle$ generates G_1.

[Why? Freely, it generates G_1 because $x_{0,m} = a_n y_{m+2} - b_m y_m$ for $m > n_*$; use $y_n = a_n y_{n+1} - x_{0,n} - b_n z$ by downward induction on $n \leq n_*$. Translating the equations, they become trivial.]

2) Similarly, but we choose g_n such that

$$g_n(y_n) = a_n y_{n+1} + (x_{0,2n} - x_{0,2n+1}) + b_n z_n.$$

3) Choose $b_n = 1_R$, $a_n : n! \cdot 1_R$.
[I don't know what this means.] $\square_{2.15}$

Claim 2.16. 1) *The quadruple $(\partial, J, \mathbf{k}, \theta)$ freely fits $\mathbb{Z}$ <u>when</u>:*

(a) *$\theta = \aleph_2$, $\partial = \aleph_2$ and $\mathbf{k} > 0$.*

(b) *$J = J^{\rm bd}_{\aleph_1} \times J^{\rm bd}_{\aleph_0}$ (But pedantically, we use the isomorphic copy*

$$J_{\aleph_1 * \aleph_0} = \{A : \textit{for some } n_\alpha < \omega,\ \textit{for } \alpha < \omega_1 \textit{ and } i_* < \omega_1,\ \textit{we have } A \subseteq \{\omega \cdot i + n : i < i_* \vee n < n_\alpha\}\};$$

hence it is better to use $J = J^{\rm bd}_{\aleph_1} \odot J^{\rm bd}_{\aleph_0}$.)

[I don't see what i_* and n_α are doing for this. If A just has to be contained in some set of this form for some n_α and i_*, why can't we just say '$\mathcal{P}(\{\omega \cdot i + n : i < \omega_1,\ n < \omega\})$?']

2) *The quadruple $(\aleph_1, J, \mathbf{k}, \theta)$ freely fits R <u>when</u>:*

(a), (b) *As above.*

(c) *$\theta = \aleph_2$* ***[This is in (1)(a), so it's already assumed.]***

(d) *Given $b_{\alpha,n} \in R$ for $\alpha < \omega_1$ and $n < \omega$, and $t \in R \setminus \{0_R\}$, there are pairwise distinct $\rho_\alpha \in {}^\omega 2$ for $\alpha < \omega_1$ and $a_{\alpha,n}, d_{\alpha,n} \in R$ such that the following set of equations is not solvable in R:*

- $d_{\alpha,n+1} \cdot y^1_{\alpha,n+1} = y^1_{\alpha,n} - y^2_{\rho_\alpha \restriction n} - b_{\alpha,n} - a_{\alpha,n} t.$

3) Similarly for "weakly fit."

Remark 2.17. 1) Probably we can use $\bar{\partial} = \langle \partial_\ell : \ell < \mathbf{k} \rangle$ with $\partial_\ell \in \{\aleph_0, \aleph_1\}$, but there is no real need so far.

2) This is essentially [She80, §4] and [She13b, 4.10(C)$_{=\mathrm{L5e.28}}$].

Proof. 1) Proving clause (A) of 2.11(1) and clause $(*)(\delta)$ of 2.11(2)(B) is easy as in 2.15, so we concentrate on 2.11(1)(B).

So let G_0, h be as in 2.11(1)(B). Choose p_n by induction on n as follows: $p_0 = 2$, and p_{n+1} will be the first prime $> p_n + n$ such that

$$\frac{p_{n+1}!}{c_{n+1} - n} > \sqrt{p_{n+1}!},$$

where $c_n := \prod_{m<n} p_m!$.

[Wouldn't it be easier to say $p_{n+1}! > (c_{n+1} - n)^2$?]

Now observe that:

⊞ For $n \geq 100$, there is $C_n \subseteq \{0, 1, \ldots, (p_n!) - 1\}$ such that if $b \in \mathbb{Z}$ and $t \in \mathbb{Z}$ satisfies $0 < |t| < n$, then for some $a_0, a_1 \in \mathbb{Z}$ we have
- $b + c_n a_0 t \in \bigcup\{i + (p_{n+1}! - 1)\mathbb{Z} : i \in C_n\}$
- $b + c_n a_1 t \notin \bigcup\{i + (p_{n+1}!)\mathbb{Z} : i \in C_n\}$.

[Why? It suffices to consider $b \in \{0, \ldots, p_n! - 1\}$ and $t \in \{\ell, -\ell : 0 < \ell \leq n\}$, and let

$$A_{b,t} := \{b + c_n a t : a \in \mathbb{Z}\} \cap \{0 \ldots, p_{n+1}! - 1\}.$$

Clearly

$$|A_{b,t}| = \frac{p_n!}{c_n \cdot |t|} > \sqrt{p_n!}.$$

The family

$$\{A_{b,t} : b \in \{0, \ldots, p_{n+1}! - 1\},\ t \in \{\ell, -\ell : 0 < \ell \leq n\}\}$$

has at most $2n(p_n!)$ members. Easily, the number of $C \subseteq \{0, \ldots, p_n! - 1\}$ such that

$$(\exists A_{b,t})[C \supseteq A_{b,t} \vee C \cap A_{b,t}) = \varnothing]$$

is[27] $< 2^{\sqrt{p_{n+1}!}}$, hence there is C_n as required.]

Let $\Omega \subseteq {}^{\omega}2$ be of cardinality $\aleph_1$ and $\langle \rho_\alpha : \alpha < \omega_1 \rangle$ list Ω without repetitions.

Let G be freely generated by

$$\{x_{m,\alpha} : \alpha < \aleph_1,\ m < \mathbf{k}\} \cup \{y^1_{\rho,n} : \rho \in \Omega,\ n < \omega\} \cup \{y^2_\varrho : \varrho \in {}^{\omega>}2\} \cup \{z\},$$

except for the equations:

$$(*)^1_{\alpha,n} \quad p_n! \cdot y^1_{\alpha,n+1} = y^1_{\alpha,n} - y^2_{\rho_\alpha \restriction n} - \sum_{m<\mathbf{k}} x_{m,\omega\cdot\alpha+n} - a_{\alpha,n} z$$

where $a_{\alpha,n} \in \mathbb{Z}$ are chosen below.

Let $\bar{a} := \langle a_{\alpha,n} : \alpha < \omega_1,\ n < \omega \rangle$ (so really $G = G_{\bar{a}}$) and let $\bar{a}_{\alpha,<n} := \langle a_{\alpha,i} : i < n \rangle$.

Note that in G,

$$(*)^2_{\alpha,n} \quad y^1_{\alpha,0} = c_n y^1_{\alpha,n} + \sum_{i<n} c_i \cdot \big(y^2_{\rho_\alpha \restriction i} + \sum_{m<\mathbf{k}} x_{m,\omega\cdot\alpha+n} + a_{\alpha,i} z\big).$$

Define

$$(*)^3_{\alpha,n} \quad b_{\alpha,n} := \sum_{i \leq n} h\big(\sum_{m<\mathbf{k}} c_i \cdot x_{m,\omega\cdot\alpha+n}\big) \in \mathbb{Z}.$$

[27] In other words, for each b, t above a random $C \subseteq \{0, \ldots, p_{n+1}! - 1\}$ has probability

$$\leq 2^{1-|A_{b,t}|} \leq 2^{1-\sqrt{p_n!}}$$

to include $A_{b,t}$ or to be disjoint to it. So the probability that this occurs for some pair (b, t) in $\leq 2 \cdot |\{A_{b,t} : b, t \text{ is as above}\}| \,/\, 2^{\sqrt{p_n!}} \leq 4n(p_n!) \,/\, 2^{\sqrt{p_n!}}$, which is $\ll 1$.

Recall that G_0, h are as in 2.11(1)(B). Let $n_* := |h(z)|$ (so $n_* > 0$). We choose $a_{\alpha,n} \in \mathbb{Z}$ by induction on n such that if $n > |h(z)|$ then

$(*)^5_{\alpha,n}$ $\rho_\alpha(n) = 1$ iff $b_{\alpha,n} + \sum_{i\le n} c_i a_{\alpha,i} h(z)$ is equal to some $a \in C_n$ modulo N, for some $N < p_n!$.

[Why is this possible? Arriving at n, the sum on the right side is

$$b_{\alpha,n} + \sum_{i\le n} c_i a_{\alpha,i} h(z) \in \mathbb{Z},$$

with the first two summands being already determined; i.e. they are computable from $\bar{a}_{\alpha,<n}$ and $|h(z)| \le n$. Applying $\boxplus$ with $\big(n, h(z), b_{\alpha,n} + \sum_{i<n} c_i a_{\alpha,i} h(z)\big)$ here standing for (n,t,b) there, we get there a_0 and a_1. Let

$$a_{\alpha,n} := \begin{cases} a_0 & \text{if } \rho_\alpha(n) = 0 \\ a_1 & \text{if } \rho_\alpha(n) = 1. \end{cases}$$

So for every n, $a_{\alpha,n}$ is as required and can be chosen.]

Having chosen $\bar{a} = \langle a_{\alpha,m} : \alpha < \omega_1, m < \omega\rangle$, the Abelian group $G = G_{\bar{a}}$ is chosen. Hence we just have to prove that G is as required in clause (B) of 2.11(1),(2). First, for 2.11(1)(B):

- $\odot$ Toward contradiction, assume that $f \in \mathrm{Hom}(G,\mathbb{Z})$ extends h and $n_* = |f(z)|$ is > 0.

hence (for every α and n, applying f_n to the equation in $(*)^2_{\alpha,n}$):

$(*)^6_{\alpha,n}$ $f(y^1_{\alpha,0}) = c_n! f(y^1_{\alpha,n}) + \sum_{i<n} c_i f(y^2_{\rho_\alpha \restriction i}) + \sum_{i<n}\sum_{m<\mathbf{k}} c_i f(x_{m,\omega\cdot\alpha+i}) + \sum_{i<n} c_i a_{\alpha,n} f(z)$.

So recalling $|h(z)| = n_*$, for some $\rho_* \in {}^{n_*+100}2$ and $a_* \in \mathbb{Z}$, we have $|S| = \aleph_1$, where

$$S := \{\alpha < \aleph_1 : f(y^1_{\alpha,0}) \equiv a_* \text{ and } \rho_\alpha \restriction (n_*+1) = \rho_*\}.$$

So choose $\alpha < \beta$ from S and let $n := \min\{\ell : \rho_\alpha(\ell) \ne \rho_\beta(\ell)\}$; clearly we have $n > n_*$ hence $n \ge n_* + 1 \ge 2$. Subtracting the equations $(*)^6_{\alpha,n+1}$, $(*)^6_{\beta,n+1}$, in the left side we get a multiple of c_{n+1} (so a number divisible by $p_n!$) and in the right side we get the sum of the following four differences:

- $\odot_1$ $f(y^1_{\alpha,0}) - f(y^1_{\beta,0})$, which is zero by the choice of S and the demand $\alpha,\beta \in S$.
- $\odot_2$ $\sum_{i\le n} c_i\, f(y^2_{\rho_\alpha\restriction i}) - \sum_{i\le n} c_i\, f(y^2_{\rho_\beta\restriction i})$, which is zero as $i \le n \Rightarrow \rho_\alpha \restriction i = \rho_\beta \restriction i$.
- $\odot_3$ $\sum_{i\le n}\sum_{m<\mathbf{k}} c_i\, f(x_{m,\omega\cdot\alpha+n}) - \sum_{i\le n}\sum_{m<\mathbf{k}} c_i\, f(x_{m,\omega\cdot\beta+n})$, which (recalling $(*)^3_{\alpha,n} + (*)^3_{\beta,n}$) is equal to $b_{\alpha,n} - b_{\beta,n}$ by the choice of $b_{\alpha,n}, b_{\beta,n}$, as f and h agree on G_0.
- $\odot_4$ $\sum_{i\le n} c_i\, a_{\alpha,i} f(z) - \sum_{i\le n} c_i\, a_{\beta,i} f(z)$.

Hence (recalling $f(z) = h(z)$)

$\boxdot$ $b_{\alpha,n} + \sum_{i\le n} c_i\, a_{\alpha,i} f_\alpha(z) - \big(b_{\beta,n} + \sum_{i\le n} c_i\, a_{\beta,n} f(z)\big)$ is divisible by $p_n!$ in $\mathbb{Z}$.

But by the choice of $a_{\alpha,n}$ (i.e. by $(*)^5_{\alpha,n}$) we know that $b_{\alpha,n} + \sum_{i \le n} c_i\, a_{\alpha,n} f(z)$ is equal to some $i \in C_n$ modulo $p_n!$ iff $\rho_\alpha(n) = 1$. Similarly for β, but $\rho_\alpha(n) \neq \rho_\beta(n)$, in contradiction to $\boxdot$. So indeed, $\odot$ leads to contradiction. This means that the demand in 2.11(1)(B) is satisfied. Second, recall that we need to verify the "freely fit." This means that

$\circledast_1$ For $\bar{a}$ as above and $w \in J$, the Abelian group $G_{\bar{a}} / \bigoplus_{\alpha \in w} \mathbb{Z} x_\alpha$ is free.

$\circledast_2$ $G_{\bar{a}}$ is free.

[Why? Easy.]

Hence

$\circledast_3$ Without loss of generality $w = \{\omega\alpha + n : \alpha < \alpha_* \text{ or } \alpha < \omega_1 \restriction n < n^*_\alpha\}$ for some $\alpha_* < \omega_1$ and $n^*_\alpha < \omega$ for $\alpha < \omega_1$.

[Not doing anything drastic to a paper that's existed for a while, but this is the exact reason why lowercase italic w needs to be banned as a variable, without exceptions.]

Now,

$\circledast_4$ Letting

$$G_* := \bigoplus_{\varrho \in {}^{\omega >}2} \mathbb{Z} y^2_\varrho \oplus \bigoplus_{\alpha < \omega\alpha_*} \mathbb{Z} X_\alpha$$

and $B_\omega := \bigoplus \{\mathbb{Z} X_\alpha : \alpha \in \omega, \alpha \geq \omega\alpha_*\}$,

[How can α be both $\in \omega$ AND be bigger than a multiple of ω?]

we have

(a) $G_\omega + G_* = G_\omega \oplus G_*$

[Why? Check.]

(b) It suffices to prove $G_{\bar{a}}/(G_\omega \oplus G_*)$ is free.

[Why? By (a).]

(c) $G_{\bar{a}}/(G_\omega \oplus G_*)$ is the direct sum of

$$\langle H'_\alpha := H_\alpha + (G_\omega \oplus G_*)/G_\omega \oplus G_* : \alpha \in [\omega\alpha_*, \omega_1] \rangle,$$

where H_α is the subgroup of $G_{\bar{a}}$ generated by

$$\{X_{\omega\alpha+n} : n < \omega\} \cup \{y^1_{\alpha,n} : n < \omega\} \cup \{y^2_{\rho_\alpha \restriction n} : n < \omega\}.$$

[Why? Check.]

(d) It suffices to prove each H'_α is a free Abelian group.

[Why? By (c).]

(e) H'_α is isomorphic to

$$H_\alpha / \bigoplus \big(\bigcup_{n < \omega} \mathbb{Z} y^2_{\rho_\alpha \restriction n} \cup \bigcup_{n < n_\alpha} \mathbb{Z} X_{\omega\alpha, n} \big).$$

[Why? Check.]

(f) H'_α is indeed free.

[Why? By the same proof as in 2.15.]

So $(\partial, J, \mathbf{k}, \theta)$ does indeed freely fit $\mathbb{Z}$.

2) We can fix $G_0 := \bigoplus \{R X_{m,i} : m < \mathbf{k}, i < \partial_m\} \oplus Rz$ and $h \in \mathrm{Hom}(G_0, {}_R R)$ such that $h(z) \neq 0$. Let $\Omega, \langle \rho_\alpha : \alpha < \omega_1 \rangle$ be as in the proof of part (1).

We are given $b_{\alpha,n} = h(x_{m,\omega\alpha+n})$ (for $\alpha < \aleph_1$, $n \in \mathbb{N}$) and $t = h(z)$ from R. We shall choose

$$\langle (a_{\alpha,n}, d_{\alpha,m}) : \alpha < \omega_1, n < \omega \rangle$$

and will let G be the R-module generated by

$$\{x_{m,\alpha} : \alpha < \aleph_1,\, m < \mathbf{k}\} \cup \{y^1_{\alpha,n} : \alpha < \aleph_1,\, n < \omega\} \cup \{y^2_\varrho : \varrho \in {}^{\omega>}2\} \cup \{z\};$$

freely, except the equations

$$(*)_{\alpha,n}\quad d_{\alpha,n}\, y^1_{\alpha,n+1} = y^1_{\alpha,n} + y^2_{\rho_\alpha \restriction n} + \sum_{m<\mathbf{k}} x_{m,\omega\cdot\alpha+n} - a_{\alpha,n} z.$$

Hence

$$(*)'_{\alpha,n}\quad y^1_{\alpha,0} = \big(\prod_{\ell<n} d_{\alpha,\ell}\big) y^1_{\alpha,n} + \sum_{i<n} \big(\prod_{\ell=i}^{n-1} d_{\alpha,\ell}\big) y^2_{\rho_\alpha\restriction\ell} + \sum_{i<n}\sum_{m<\mathbf{k}} \big(\prod_{\ell=i}^{n-1} d_{\alpha,\ell}\big) x_{m,\omega\cdot\alpha+n} + \sum_{i<n} \big(\prod_{\ell=i}^{n-1} d_\ell\big) a_{\alpha,i} z.$$

Now continue as in the proof of part (1). $\square_{2.16}$

We now can put things together.

Theorem 2.18. 1) *For every $k \geq 1$ there is an $\aleph_{\omega_1\cdot k}$-free Abelian group G which is not Whitehead, and even* $\mathrm{Hom}(G,\mathbb{Z}) = 0$.

2) *If the ring R satisfies the demands in clause (2)(c) from* 2.16

[Clause** (2)(c) **is '$\theta = \aleph_2$;' it's not a demand on** R **at all.]

then for every k there is an $\aleph_{\omega_1\cdot k}$-free R-module such that $\mathrm{Hom}(G, {}_R R) = 0$ *and* $\mathrm{Ext}(G, {}_R R) \neq 0$.

Proof. 1) Given k, we use 1.28 to find a c.p. $\mathbf{x}$ which is $\aleph_{\omega_1\cdot k}$-free and has χ-BB, where $\chi := |R| + \aleph_1$ and $J := J^{\mathrm{bd}}_{\aleph_1} \odot J^{\mathrm{bd}}_{\aleph_0}$. Now apply 2.16(1) so $(\aleph_1, J, k, \aleph_1)$ fits $\mathbb{Z}$, and by 2.12(1),(2) we get the desired conclusion.

2) Similarly, but now we use 2.16(2) rather than 2.16(1). $\square_{2.18}$

§ 3. Forcing

The main result of the former section is the existence, in ZFC, of $\aleph_{\omega_1 \cdot n}$-free Abelian groups G (for every $n \in \omega$) such that $\mathrm{Hom}(G, \mathbb{Z}) = 0$. The purpose of this section is to show that this result is best possible in the sense of freeness amount. Assuming the existence of $\aleph_0$-many supercompact cardinals in the ground model, we shall force the following statement. 'For every non-trivial $\aleph_{\omega_1 \cdot \omega}$-free Abelian group G, $\mathrm{Hom}(G, \mathbb{Z}) \neq 0$.'

This section is divided into two subsections. §3(A), like §1, is combinatorial; we describe a general framework for dealing with freeness of R-modules (this continues [She85], [She96] and [She19]; but have to work more).

In §3(B) we rely on forcing. We focus on $R = \mathbb{Z}$ (hence R-modules are simply Abelian groups), and we prove the main consistency result in Theorem 3.9 which relies on Magidor-Shelah [MS94]. The proof is based on the context of §3(A), with double meaning.

§ 3(A). Freeness Classes.

Context 3.1. 1) R is a ring with no zero divisors and is hereditary (see 2.1(1A)).

2) $\mathbf{K}$ is the class of R-rings, closed under sums.
[R *is* **a ring. Did you mean 'R-module?'**]

3) $\mathbf{K}_*$ will denote a subclass of $\mathbf{K}$.

Definition 3.2. 0) $\mathbf{K}^{\mathrm{W}} \subseteq \mathbf{K}$ will denote the class of *Whitehead modules.*
That is, M is a Whitehead module if $\mathrm{Ext}(M, {}_R R) = 0$. Equivalently, if $N_1 \subseteq N_2$ are R-modules, $N_2/N_1 \cong M$, and $h_1 \in \mathrm{Hom}(N_1, {}_R R)$ then there is $h_2 \in \mathrm{Hom}(N_2, {}_R R)$ extending h_1.

1) We say $\mathbf{K}_*$ is a λ-*freeness class* inside $\mathbf{K}$ when:

(a) $\mathbf{K}_* \subseteq \mathbf{K}_{<\lambda}$, where for any cardinality θ we define

$$\mathbf{K}_{<\theta} := \{M \in \mathbf{K} : \|M\| < \theta\}.$$

(b) $\mathbf{K}_*$ is closed under isomorphisms.

For simplicity, $\lambda > |R|$.

1A) We say $\mathbf{K}_*$ is *hereditary* when $\mathbf{K}_*$ is closed under pure submodules; i.e.

$$M \subseteq_{\mathrm{pr}} N \in \mathbf{K}_* \Rightarrow M \in \mathbf{K}_*.$$

In part (1), we may omit $\mathbf{K}$ when clear from context.
[K was defined in 3.1(2). It's not going to change or vary.]

2) We say $M \in \mathbf{K}$ is $\mathbf{K}_*$-free when there is $\overline{M} = \langle M_\alpha : \alpha \leq \alpha_* \rangle$ which is [purely / strictly] increasing continuous, M_0 is the zero module,

$$\alpha < \alpha_* \Rightarrow M_{\alpha+1}/M_\alpha \in \mathbf{K}_*,$$

and $M_{\alpha_*} = M$.

2A) $M \in \mathbf{K}$ is $(\lambda, \mathbf{K}_*)$-free when every $M' \subseteq_{\mathrm{pr}} M$ of cardinality $< \lambda$ is $\mathbf{K}_*$-free.

3) $\mathbf{K}^*_{<\theta} := \mathbf{K}_* \cap \mathbf{K}_{<\theta}$, for any cardinal θ.

4) The class $\mathbf{K}_*$ is called a (λ, κ)-freeness class when $\mathbf{K}_*$ is a λ-freeness class, $\mathbf{K}_*$ is hereditary, and if $M \in \mathbf{K}_{<\lambda} \setminus \mathbf{K}_*$ then there is $N \subseteq_{\mathrm{pr}} M$ from $\mathbf{K}_{<\kappa} \setminus \mathbf{K}_*$.

The main example here is:

Claim 3.3. *Assume $R = \mathbb{Z}$, $\lambda \geq \aleph_1$, and $\mathbf{K}$ = the class of R-modules, and let $\mathbf{K}_{whu} = \mathbf{K}_* := \mathbf{K}^{\mathrm{W}} \cap \mathbf{K}_{<\lambda}$ (the class of Whitehead modules of cardinality $< \lambda$) and $\mathbf{K}_{\mathrm{fr}} := \{M \in \mathbf{K}_{<\aleph_1} : M \text{ free}\}$.*

0) *$\mathbf{K}_{\mathrm{fr}}$ is a hereditary $\aleph_1$-freeness class.*

1) *If $\lambda > \aleph_2$ and $\mathsf{MA}_{<\lambda}$ then $\mathbf{K}_*$ is a hereditary $(\lambda, \aleph_2)$-freeness class.*

2) *If $M \in \mathbf{K}$ is $\mathbf{K}_*$-free then M is a Whitehead group.*

3) *If $M_1 \subseteq_{\mathrm{pr}} M_2$, M_2/M_1 is $\mathbf{K}_*$-free, and $h_1 \in \mathrm{Hom}(M_1, {}_RR)$ then there is $h_2 \in \mathrm{Hom}(M_2, {}_RR)$ extending h_1.*

4) $\mathbf{K}_{**} :=$

$$\big\{M \in K_{<\lambda} : \text{for every ccc forcing } \mathbb{P}_1, \text{ for some ccc forcing notion } \mathbb{P}_2 \text{ satisfying } \mathbb{P}_1 \lessdot \mathbb{P}_2, \text{ we have } \Vdash_{\mathbb{P}_2} \text{“}M \text{ is a Whitehead group”}\big\}$$

is a $(\lambda, \aleph_2)$-freeness class.

Proof. 0) Obvious, as $\mathbb{Z}$ is countable.

1) The first property in 3.2(4) holds trivially by the choice of $\mathbf{K}_*$. As for the second property, it is well known that $\mathbf{K}_*$ is a hereditary class; see [Fuc73]. The third property in 3.2(4) follows from the full characterization of being Whitehead for Abelian group G of cardinality $< \lambda$ when $\mathsf{MA}_{<\lambda}$ holds, (not just proving “strongly $\aleph_1$-free is enough”). In particular, G is Whitehead if every subgroup of cardinality $\leq \aleph_1$ is Whitehead (see [EM02]).

2) Follows by (3).

3) Without loss of generality let $M = M_2/M_1$ and $\pi \in \mathrm{Hom}(M_2, M)$ be surjective with kernel M_1. Let $\langle M'_\alpha : \alpha \leq \alpha_* \rangle$ be as in 3.2(4) for M and let $N_\alpha = \pi^{-1}(M'_\alpha)$: so $\langle N_\alpha : \alpha \leq \alpha_* \rangle$ is purely increasing continuous, $N^*_0 = M_1$, $N_{\alpha_*} = M_2$, and $N_{\alpha+1}/N_\alpha \in \mathbf{K}_*$.

Given $h_1 \in \mathrm{Hom}(M_1, {}_RR)$, we choose $f_\alpha \in \mathrm{Hom}(N_\alpha, {}_RR)$ by induction on α, increasing continuously with α. For $\alpha = 0$ let $f_\alpha = h_1$; for α limit let $f_\alpha = \bigcup_{\beta<\alpha} f_\beta$; and for $\alpha = \beta + 1$ use $N_\alpha/N_\beta \cong M'_\alpha/M'_\beta \in \mathbf{K}_*$ and the choice of $\mathbf{K}_*$.

Lastly, $h_2 = f_{\alpha_*}$ is as required.

4) Easy. $\square_{3.3}$

On these freeness contexts, see [She75a] (or even better, [She19] and history there). Note that in §3(B) we shall use 3.7(B)(c), and for this we need witnesses $\mathbf{s}$ from those references.

Recall (see [She19]):

Definition 3.4. 1) We say $\mathbf{c}$ is a *pre-1-freeness context* <u>when</u> $\mathbf{c}$ consists of:

(a) $\mathscr{U}$ is a fixed set (we shall deal with subsets of it) or $\mathfrak{U}$ is an algebra with universe $\mathscr{U}$ (maybe with the set of functions empty). Let $c\ell_{\mathbf{c}}(A)$ be the closure of the set $A \subseteq \mathscr{U}$ in the algebra $\mathfrak{U}$; but we may sometimes say $\mathscr{U}$ instead of $\mathfrak{U}$.

(b) $\mathscr{F}$ a family of pairs of subsets of $\mathscr{U}$; we may write “A/B is free” or “A is free over B” for (A, B) in $\mathscr{F}$.

(c) χ, μ will be fixed cardinals such that $|\tau(\mathfrak{U})| \leq \chi < \mu \leq \infty$ and

$$(A, B) \in \mathscr{F} \Rightarrow |A| + |B| < \mu.$$

But if $\mu = \infty$ (equivalently, $\mu > |\mathscr{U}|$) we may omit it.

2) For a property P, we say "$P(X)$ for the χ-majority of $X \subseteq A$" <u>when</u> there is an algebra $\mathfrak{B}$ with universe A and χ functions, such that any $X \subseteq A$ closed under those functions satisfies P. We can replace $X \subseteq A$ by $X \in \mathcal{P}(A)$ or $X \in \mathcal{P}_{<\lambda}(A)$; alternatively, we may say $\{X \subseteq A : P(A)\}$ is a χ-majority.

3) We say $\mathbf{c}$ is a freeness context <u>when</u> in addition to (a)-(c) of part (1), it satisfies the following [axioms]:[28]

Ax.II$_\mu$:

(a) A/B is free <u>iff</u> $(A \cup B)/B$ is free.

(b)$_\mu$ A/B is free <u>when</u> $|B| < \mu$ and $A \subseteq B$.

Ax.III: [2-transitivity.]
If A/B and B/C are free and $C \subseteq B \subseteq A$, <u>then</u> A/C is free.

Ax.IV$_{\lambda,\mu}$: [Continuous transitivity.]
If $\langle A_i : i < \lambda \rangle$ is increasing, $A_\gamma / \bigcup\limits_{j<i} (A_j \cup B)$ is free for [some / all] $i \leq \gamma < \lambda$, $\lambda < \mu$, and $|\bigcup\limits_{i<\lambda} A_i| < \mu$ <u>then</u> $\bigcup\limits_{i<\lambda} A_i/B$ is free.

Let $\mathbf{Ax.IV}_{<\lambda,\mu}$ mean '$\theta < \lambda \Rightarrow \mathbf{Ax.IV}_{\theta,\mu}$.' $\mathbf{Ax.IV}_\mu$ will mean $\mathbf{Ax.IV}_{<\mu,\mu}$, and **Ax.IV** means $\mathbf{Ax.IV}_\infty$.

Ax.VI: If A is free over $B \cup C$, <u>then</u> for the $\chi_{\mathbf{c}}$-majority of $X \subseteq A \cup B \cup C$, the pair $A \cap X/\big((B \cap X) \cup C\big)$ is free.

Ax.VII: If A is free over B, <u>then</u> for the $\chi_{\mathbf{c}}$-majority of $X \subseteq A \cup B$, the pair $A/\big((A \cap X) \cup B\big)$ is free.

4) We say $\mathbf{c}$ is a freeness$^+$ context <u>when</u>, in addition,

Ax.I**: If A/B is free and $A^* \subseteq A$, <u>then</u> A^*/B is free.

5) We say $\mathbf{c}$ is a (λ, κ)-freeness context <u>when</u> in addition, $\chi_{\mathbf{c}} \leq \kappa$, **Ax.I****, and if A/B is not $\mathbf{c}$-free and $|A| < \lambda$ then for some $A' \subseteq A$ of cardinality $< \kappa$, A'/B is not $\mathbf{c}$-free.

Definition 3.5. For a λ-freeness class $\mathbf{K}_*$ and R-module G and $\chi \geq |R| + \aleph_0$ (if equal then χ may be omitted) we define what we call a *pre-freeness context*

$$\mathbf{c} = \mathbf{c}_G = \mathbf{c}_{\mathbf{K}_*,G,\chi}$$

(this is proved in 3.6) as the tuple $(\mathscr{U}, \mathfrak{A}, \mathscr{F}, \chi) = (\mathscr{U}_{\mathbf{x}}, \mathfrak{A}_{\mathbf{c}}, \mathscr{F}_{\mathbf{c}}, \chi_{\mathbf{c}})$, where:

(a) $\mathscr{U} = G$ as a set, and $\mathfrak{A}$ is an expansion of G by $\langle F_a^{\mathfrak{A}} : a \in R \rangle$ such that if $G \models ax = y$ and $y' = F_a(y)$, then $G \models ay' = y$. Furthermore, if $g \in aG$ then $F_a(g) = 0$.

(b) $\mathscr{F} := \{A/B : B, A \subseteq \mathscr{U}$ and $\langle A \cup B \rangle_{\mathfrak{A}} / \langle B \rangle_{\mathfrak{A}}$ is $\mathbf{K}_*$-free$\}$.
We may say 'A/B is $\mathbf{c}$-free,' so A/B stands for the formal quotient. So pedantically, this is just the pair (A, B), where $\langle B \rangle_G$ is the minimal pure[29] sub-module of G which includes B.

(c) $\chi_{\mathbf{c}} = \chi$ (so $\geq |R| + \aleph_0$). (And $\mu_{\mathbf{c}} = \infty$.)

[28] Adding a superscript$^+$ to an axiom means that whenever "$A/B \in \mathscr{F}$" or its negation appear in the assumption then we demand B to be free over $\varnothing$. Of course, $\mathscr{F}_{\mathbf{c}} = \mathbf{F}$, $\chi_{\mathbf{c}} = \chi$, etc.

[29] Our modules are torsion free; i.e. $a \in R \wedge x \in G \Rightarrow [ax = 0 \Leftrightarrow (a = 0_R \vee x = 0_G)]$. This holds when $R = \mathbb{Z}$, and we have no problem. Otherwise, recall we have expanded G to an algebra $\mathfrak{A}$ such that $A = \mathit{cl}_{\mathscr{U}}(A) \Rightarrow A \subseteq_{\mathrm{pr}} G$.

Fact 3.6. Assume $\mathbf{K}_*$ is a hereditary λ-freeness class and $\chi := |R| + \aleph_0$.

1) Being $\mathbf{K}_*$-free has compactness in singular cardinals $> \lambda$.

2) For any R-module G_*, $\mathbf{c} = \mathbf{c}_{\mathbf{K}_*, G_*, \chi}$ as defined in 3.5 above is a freeness context and satisfies $\mathbf{Ax.I}^{**}$.

3) Moreover, if $\mathbf{K}_*$ is a (λ, κ)-freeness class (see 3.2(4)) then $\mathbf{c}$ is a (λ, κ)-freeness context (see 3.4(5)).

Proof. 1) By part (2) and [She19], see history there.

2) Check.

3) Easy. $\square_{3.6}$

Claim 3.7. *'If (A) then (B),' where:*

- (A) (a) $\mathbf{K}_*$ *is a* (λ, κ)*-freeness class (see Definition* 3.2(4)*).*
  - (b) $G \in \mathbf{K}$ *is* $(\mathbf{K}_*, \lambda)$*-free [but] not* $\mathbf{K}_*$*-free (see Definition* 3.2(2),(2A)*). Fix such* G *of minimal cardinality, [and call that cardinality]* μ*.*
  - (c) $\mathbf{c} = \mathbf{c}_{\mathbf{K}_*, G, \kappa}$ *(see Definition* 3.5(1)*).*
- (B) *There is a witness* $\mathbf{s}$ *for* G *in the context*[30] $\mathbf{c}$ *such that:*
  - (a) $B^{\mathbf{s}}_{\langle\,\rangle} = \varnothing$ *and* $B^{\mathbf{s}}_{\langle\,\rangle^+} \subseteq G$*, so* $\lambda(\langle\,\rangle, S_{\mathbf{s}}) \leq \|M\|$*.*
  - (b) *If* $\eta \notin \mathrm{fin}(S_{\mathbf{s}})$ *then* $\lambda_{\mathbf{s},\eta} \geq \lambda$*.*
  - (c) *If* $\eta\hat{\ }\langle\delta\rangle \in S_{\mathbf{s}}$ *then* $\mathrm{cf}(\delta) \notin [\kappa, \lambda)$*.*
  - (d) *If* $\eta \in \mathrm{fin}(S_{\mathbf{s}})$ *then* $B_{\mathbf{s},\eta^+} \setminus B_{\mathbf{s},\eta}$ *has cardinality* $< \kappa$*.*

Proof. By 3.6, we can apply 3.8 below. $\square_{3.7}$

Claim 3.8. *'If (A) then (B),' where:*

- (A) (a) $\mathbf{c}$ *is a freeness context satisfying* $\mathbf{Ax.I}^{**}$*.*
  - (b) $\mathbf{c}$ *is a* (λ, κ)*-freeness context.*
  - (c) A/B *is a* λ*-free (but not free) pair, with* $|A|$ *minimal.*
- (B) *There is a witness* $\mathbf{s}$ *such that*
  - (a) $B^{\mathbf{s}}_{\langle\,\rangle} = B$ *and* $B^{\mathbf{s}}_{\langle\,\rangle^+} \subseteq A$ *(so* $\lambda \leq \lambda(\langle\,\rangle, S_{\mathbf{s}}) \leq |A|$*).*
  - (b) *If* $\eta \notin \mathrm{fin}(S_{\mathbf{s}})$ *then* $\lambda_{\mathbf{s},\eta} \geq \lambda$*.*
  - (c) *If* $\eta\hat{\ }\langle\delta\rangle \in S_{\mathbf{s}}$ *then* $\mathrm{cf}(\delta) \notin [\kappa, \lambda)$*.*
  - (d) *if* $\eta \in \mathrm{fin}(S_{\mathbf{s}})$ *then* $B_{\mathbf{s},\eta^+} \setminus B_{\mathbf{s},\eta}$ *has cardinality* $< \kappa$*.*

Proof. Now,[31] there is a disjoint witness $\mathbf{s}$ for A/B being non-$\mathbf{c}$-free. So without loss of generality ($n = n(\mathbf{s})$ is well-defined, and . . .) for some $\bar{\lambda}^* = \langle \lambda^*_\ell : \ell < n \rangle$, $\bar{\kappa}^* = \langle \kappa^*_\ell : \ell < n \rangle$ we have:

- $(*)_1$ (a) For each $\ell < n$, one of the following holds.
  - (α) λ_ℓ is a regular cardinal and $\eta \in S_{\mathbf{s},\ell} \Rightarrow \lambda(\eta, S_{\mathbf{s}}) = \lambda^*_\ell$.
  - (β) $\ell = 0$, $\lambda_\ell = *$, and $\eta \in S_{\mathbf{s},\ell} \Rightarrow \lambda(\eta, S_{\mathbf{s}})$ is inaccessible (possibly weakly inaccessible).
  - (b) For each $\ell < n$, either κ_ℓ is a regular cardinal and
  $$\eta \in S_{\mathbf{s},\ell} \wedge \delta \in W(\eta, S_{\mathbf{s}}) \Rightarrow \mathrm{cf}(\delta) = \kappa_\ell,$$
  or $\kappa_\ell = *$ and $\lambda_{\ell+1}$ is $*$.

[30] See [She85, §2] (and better, [She96, §3]).

[31] See [She96, §3], or better yet, see [S+d, 4.5=Ld15].

Naturally, without loss of generality

$(*)_2$ $\mathbf{s}$ is *minimal*, which means that: (fixing A and B)

(a) $n = n(\mathbf{s})$ is minimal.

(b) Under condition (a), $\bar{\lambda}^*$ is minimal under the lexicographical order.

(c) Under (a)+(b), $\bar{\kappa}^*$ is minimal under the lexicographical order.

Now

$(*)_3$ If $\eta \in \text{ini}(S_{\mathbf{s}})$ then $\lambda(\eta, S_{\mathbf{s}}) \geq \lambda$.

[Why? Otherwise, chose a counterexample η with $\lambda(\eta, S_{\mathbf{s}})$ minimal. So by the definition of a witness, as $\chi_{\mathbf{c}} \leq \kappa$, we have that $B^{\mathbf{s}}_{\eta+}/B^{\mathbf{s}}_{\leq\eta}$ is not free (for (c)) and $B^{\mathbf{s}}_{\eta+} \setminus B^{\mathbf{s}}_{\eta}$ has cardinality $\lambda(\eta, S_{\mathbf{s}})$ (so $< \lambda$).

Recalling "$\mathbf{c}$ is a (λ, κ)-freeness context"[32] there is $C_\eta \subseteq B^{\mathbf{s}}_{\eta+}$ of cardinality $\leq \kappa$ such that $C_\eta/B^{\mathbf{s}}_{\leq\eta}$ is not $\mathbf{c}$-free. So (this follows by the minimality of $\mathbf{s}$) we get a contradiction, so $\lambda(\eta, S_{\mathbf{s}}) \geq \lambda$ as promised in $(*)_3$.]

$(*)_4$ If $\eta\hat{\ }\langle\delta\rangle \in S_{\mathbf{s}}$ then $\text{cf}(\delta) \notin [\kappa, \lambda)$.

[Why? As in the proofs in [She85], [She99], for each $\eta \in S_{\mathbf{s}}$ satisfying $\text{cf}(\delta) \geq \kappa$, by the minimality [we have]

$$\text{cf}(\delta) \in \{\lambda(\nu, S_{\mathbf{s}}) : \nu \in S_{\mathbf{s}},\ \eta \triangleleft \nu\},$$

so $(*)_4$ follows by $(*)_3$.]

So we are done. $\square_{3.8}$

§ 3(B). The Main Independence Result.

Below, it is reasonable to assume that the ring R is $\mathbb{Z}$, and we assume this is the nice version. Note that we prove that a non-Whitehead group has a *non-free* subgroup of small cardinality, not necessarily a non-Whitehead one. This is connected to the black boxes here having cardinality (much) bigger than the amount of freedom. For simplicity, presently we deal with freeness only in hereditary cases.

Recall that μ is *supercompact* iff for every ∂ there exists an elementary embedding $j : \mathbf{V} \to M$ such that M is a transitive class satisfying ${}^{\partial}M \subseteq M$ and ∂ is the critical cardinal.

[The definition says 'for every ∂.' Do you mean '. . . the critical cardinal for j?]

Theorem 3.9. *If in* $\mathbf{V}$ *there are* $\aleph_0$*-many supercompact cardinals, then in some forcing extension, for* $\mu_* := \aleph_{\omega_1\cdot\omega}$*, we have:*

$\oplus_{\mu_*}$ (a) *If G is a nontrivial μ_*-free Abelian group then* $\text{Hom}(G, \mathbb{Z}) \neq 0$.

(b) *If $G \subseteq H$ are Abelian groups, H/G is μ_*-free, and $h \in \text{Hom}(G, \mathbb{Z})$, then h can be extended to a homomorphism from H to $\mathbb{Z}$.*
(This is an equivalent definition of "H/G is Whitehead;" the reader may use it here as a definition.)

This will be proved shortly. As usual in such proofs, we make the large cardinal[s collapse] into quite small ones: so they cannot be *really* large, but some remnant of their early largeness remains and is enough for our purpose. This is the rationale of Definition 3.10 below.

[32] See Definition 3.4(5) and 3.6(3).

Definition 3.10. Let $\mathrm{Pr}_{\lambda_*,\mu_*,\kappa_*}$ mean[33]

(A) (a) $\lambda_* > \mu_* > \kappa_*$
 (b) λ_*, κ_* are regular uncountable cardinals.
 (c) μ_* is a limit cardinal.

(B) 'If (a) then (b),' where
 (a) (α) λ is a regular cardinal $\geq \lambda_*$.
  (β) $\chi > \lambda$, $\mu < \mu_*$, and $x \in \mathcal{H}(\chi)$.
  (γ) $S \subseteq \{\delta < \lambda : \mathrm{cf}(\delta) < \kappa_*\}$ is a stationary subset of λ.
  (δ) $u_\alpha \in [\alpha]^{\leq\mu}$ for $\alpha \in S$.
 (b) There exist a regular $\lambda' \in (\mu + \kappa_*, \mu_*)$ and an increasing continuous sequence $\langle \alpha_\varepsilon : \varepsilon < \lambda' \rangle$ of ordinals $< \lambda$ such that the set

$$\{\varepsilon < \lambda' : \alpha_\varepsilon \in S \text{ and } u_{\alpha_\varepsilon} \subseteq \{\alpha_\zeta : \zeta < \varepsilon\}\}$$

 is a stationary subset of λ'.

On the strong hypothesis above, see [She93b]; it is a sufficient condition for the SCH — i.e.

$$\partial = \mathrm{cf}(\mu) \wedge 2^\partial < \mu \Rightarrow \mu^\sigma \leq \mu^+ \wedge 2^\partial = \partial^+.$$

[I don't know what σ is.]

Definition 3.11. We say the universe $\mathbf{V}$ *satisfies the strong hypothesis above* λ <u>when</u>: if

$$\chi > \mathrm{cf}(\chi) \wedge \lambda > \lambda + \mu_1 \Rightarrow \mathrm{cf}\big([\chi]^{<\mu_1}, \subseteq\big) \leq \chi^+,$$

<u>then</u> $\lambda_1 = \chi^+$ and $\mathrm{cf}(\chi) < \mu_1$.

Theorem 3.12. 1) *Assume that in $\mathbf{V}_0$ there are infinitely many supercompact cardinals $> \theta$, where $\theta = \mathrm{cf}(\theta) \in [\aleph_1, \aleph_{\omega_1})$. Let $\lambda_* = \mathrm{cf}(\lambda_*) := \mu_*^+$, $\mu_* := \aleph_{\theta\cdot\omega}$, and $\kappa_* := \theta^+$.*

<u>Then</u> for some forcing notion $\mathbb{Q}$ which does not add new subsets to θ, $\mathbf{V}_1 := \mathbf{V}_0^{\mathbb{Q}}$ satisfies $\mathrm{Pr}_{\lambda_*,\mu_*,\kappa_*}$.

1A) *We can (by preliminary forcing) assume that the universe $\mathbf{V}_1$ above also satisfies the GCH above θ (we just use "above μ_*") and $\diamondsuit^*_\lambda$ holds for every regular uncountable λ above μ_*.*

2) *If $\mathrm{Pr}_{\lambda_*,\mu_*,\kappa_*}$ holds in $\mathbf{V}$ and the ccc forcing $\mathbb{P}$ has cardinality λ_*, <u>then</u> $\mathrm{Pr}_{\lambda_*,\mu_*,\kappa_*}$ still holds in $\mathbf{V}^{\mathbb{P}}$.*

3) *Part* (1) *holds for any freeness$^+$ context (see Definition* 3.4(3),(4)*).*

Proof. 1,1A) Similarly to [MS94, §4,Th.1,pg.807]. As there, let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of supercompact cardinals. Without loss of generality GCH holds above $\mu := \sum_n \kappa_n$ (called κ there) and $\diamondsuit^*_\chi$ holds for every $\chi = \mathrm{cf}(\chi) > \mu$. Also for each n, the supercompactness of κ_n is preserved by forcing notions which are κ_n-directed closed.

We proceed as there, but now in the interval (κ_{n-1}, κ_n), the set of cardinals we do not collapse has order type $h_*(n) + 1$.

[You just deleted the definition of h_*.]

2,3) Easy. $\square_{3.12}$

[33] We may allow $\lambda_* = \mu_*$ here and in 3.13, but then we have to say somewhat more.

Proof. **Proof of 3.9.**

Let $\mathbf{V}_1 := \mathbf{V}_0^{\mathbb{Q}}$ be as in 3.12(1)(1A) with $\theta = \aleph_1$ (so $\kappa_* = \aleph_2$, $\mu_* = \aleph_{\omega_1 \cdot \omega}$, $\lambda_* = \mu_*^+$) and in $\mathbf{V}_1$, let $\mathbb{P}$ be a ccc forcing notion of cardinality λ_* such that $\Vdash_{\mathbb{P}}$ "$\mathsf{MA} + (2^{\aleph_0} = \lambda_*)$". The result follows from Theorem 3.13 below. Clause (d) there holds because $\mathbf{V} = \mathbf{V}_1^{\mathbb{P}}$; see 3.12(2). $\square_{3.9}$

Recall that the strong hypothesis says that $\mathrm{pp}(\lambda) = \lambda^+$ for every singular cardinal λ: we rely on §3(A).

Theorem 3.13. *The statement* $\oplus_{\mu_*}$ *from* 3.9 *holds* *when* $\mathbf{V}$ *satisfies:*

(a) *The statement* $\mathrm{Pr}_{\lambda_*,\mu_*,\kappa_*}$ *from Definition* 3.10.

(b) $\lambda_* = \lambda_*^{<\lambda_*} > \mu_*$

(c) $\kappa_* = \aleph_2$

(d) $\mathsf{MA} + {}$'$2^{\aleph_0} = \lambda_*$', *and* $\mathbf{V}$ *satisfies the strong hypothesis above* λ_* *(see* 3.11 *or* [She93b]*).*

Proof. We rely on 3.1-3.8. First, clause (b) of $\oplus_{\mu_*}$ implies clause (a).

[Why? Because if H is a μ_*-free Abelian group, let $x \in H \setminus \{0_H\}$ and without loss of generality x is not divisible by any $n \in \{2, 3, \dots\}$ hence $K := \mathbb{Z}x$ is a pure subgroup of H. Let h be an isomorphism from K onto $\mathbb{Z}$. As H is μ_*-free, easily also H/K is μ_*-free, hence by $\oplus_{\mu_*}$(b) there is a homomorphism $h^+ : H \to \mathbb{Z}$ extending h. So $h^+(x) \neq 0_{\mathbb{Z}}$ hence $h^+ \in \mathrm{Hom}(H, \mathbb{Z})$ is non-zero, as required.]

So it suffices to prove clause (b) of $\oplus_{\mu_*}$. Let $R = \mathbb{Z}$ and let $\mathbf{K}, \mathbf{K}_*$ be as in Claim 3.3 for λ_*, so $\mathbf{K}_*$ is a hereditary $(\mu_*, \aleph_2)$-freeness class[34] by 3.3(1).

So toward contradiction, assume $G \in \mathbf{K}$ is a counterexample of minimal cardinality (call that λ) so G is μ_*-free.

[Where is this cardinality referenced in this proof?]

To get a contradiction and finish the proof, it suffices to assume $G_1 \subseteq_{\mathrm{pr}} G_2$, $G_2/G_1 \cong G$, and $h_1 \in \mathrm{Hom}(G_1, \mathbb{Z})$, and prove that there is $h_2 \in \mathrm{Hom}(G_2, \mathbb{Z})$ extending h_1. If G is $\mathbf{K}_*$-free (see Definition 3.2(2)) then by 3.3(3) a homomorphism h_2 exists as required.

Hence G is not $\mathbf{K}_*$-free. Let $\mathbf{c} := \mathbf{c}_{\mathbf{K}_*,G,\theta}$ (see Definition 3.5) so by 3.6(3), $\mathbf{c}$ is a (λ_*, κ_*)-freeness context and by 3.7(2),(3) (with λ_*, κ_* here standing for λ, κ there) there is a witness $\mathbf{s}$ as there. By 3.3(1) we have $\lambda(\langle\,\rangle, S_{\mathbf{s}}) \geq \lambda_*$.

Let $\mathbf{c}_1 := \mathbf{c}_{\mathbf{K}_{\mathrm{fr}},G,\theta}$; it is a $(\lambda, \aleph_1)$-freeness context.

[Why? By 3.6, with K_{fr} (see 3.3) playing the role of K_*.]

Let $S_1 := W(\langle\,\rangle, S_{\mathbf{s}})$, so $B^{\mathbf{s}}_{\langle\delta+1\rangle}/B^{\mathbf{s}}_{\langle\delta\rangle}$ is not free for $\mathbf{c}$ for any $\delta \in S_1$, so [it] cannot be μ_*-free for $\mathbf{c}_1$ (as we have chosen a counter-example of minimal cardinality). Hence there is $A_\delta \subseteq B^{\mathbf{s}}_{\langle\delta+1\rangle}$ of cardinality $< \mu_*$ such that $A_\delta/B^{\mathbf{s}}_{\langle\delta\rangle}$ is not free for $\mathbf{c}_1$.

Let $B'_\delta \subseteq B^{\mathbf{s}}_{\langle\delta\rangle}$ be of cardinality $\leq |A_\delta| + \kappa_*$ such that

$$B'_\delta \subseteq B' \subseteq B^{\mathbf{s}}_{\langle\delta\rangle} \Rightarrow A_\delta/B' \text{ is not free for } \mathbf{c}_1,$$

This exists by properties of Abelian groups, as $B^{\mathbf{s}}_{\langle\delta\rangle} \subseteq B^{\mathbf{s}}_{\langle\delta+1\rangle}$ are free and $A_\delta/B^{\mathbf{s}}_{\langle\delta\rangle}$ is not (for $\mathbf{c}_1$).

So for some $\mu < \mu_*$, the set

$$S_2 := \{\delta \in S_1 : |A_\delta \cup B'_\delta| + \kappa_* = \mu\}$$

is a stationary subset of $\lambda(\langle\,\rangle, S_{\mathbf{s}})$. Let $h : \lambda(\langle\,\rangle, S_{\mathbf{s}}) \to B^{\mathbf{s}}_{\langle\lambda\rangle}$ be one-to-one function and onto, and let

$$C := \{\delta < \lambda(\langle\,\rangle, S_{\mathbf{s}}) : h \text{ maps } \delta \text{ onto } B^{\mathbf{s}}_{\langle\delta\rangle}\}.$$

[34] See Definition 3.2(1),(1A),(4).

It is a club of $\lambda(\langle\ \rangle, S_{\mathbf{s}})$, hence $S_3 := S_2 \cap C$ is a stationary subset of $\lambda(\langle\ \rangle, S_{\mathbf{s}})$. Also, for $\delta \in S_3$ let $u_\delta := \{\alpha < \delta : h(\alpha) \in B'_\delta\}$.

By clause (B)(c) of 3.7 (i.e. the choice of $\mathbf{s}$) without loss of generality one of the following occurs:

(a) $\delta \in S_3 \Rightarrow \mathrm{cf}(\delta) = \kappa_1$ for some regular $\kappa_1 < \kappa_*$.

(b) Every $\delta \in S_3$ has cofinality $\geq \lambda_*$.

Case 1: $\kappa_1 < \kappa_*$ is as in clause (a).

Just use $\mathrm{Pr}_{\lambda_*,\mu_*,\kappa_*}$ for $\lambda, S_3, \langle u_\delta : \delta \in S_3\rangle$ to prove G is not a μ_*-free, a contradiction.

Case 2: Clause (b) above holds.

For $\delta \in S_3$, clearly

$$|u_\delta| = |A_\delta \cup B'_\delta| + \kappa_* = \mu < \mu_* \leq \lambda_* \leq \mathrm{cf}(\delta)$$

hence there is $\gamma_\delta < \delta$ such that $u_\delta \subseteq \gamma_\delta$, hence for some $\gamma_* < \lambda$ the set

$$S_4 := \{\delta \in S_3 : u_\delta \subseteq \gamma_*\}$$

is stationary.

Subcase 2A: $\mathrm{cf}([\gamma_*]^{\leq \mu_*}, \subseteq) < \lambda(\langle\ \rangle, S_{\mathbf{s}})$.

So for some $u_* \in [\gamma_*]^{\leq \mu}$, the set $S_5 = \{\delta < \lambda : u_\delta \subseteq u_*\}$ is a stationary subset of λ. Let $S_6 \subseteq S_5$ be of cardinality μ^+ and let

$$A^* := \bigcup_{\delta \in S_6} A_\delta \cup \{h(\alpha) : \alpha \in u_*\}.$$

Clearly $A^* \subseteq G$ is of cardinality $< \mu$ and $A^*/\varnothing$ is not free for $\mathbf{c}_1$.

So G has a non-free subgroup of cardinality $< \mu_*$, in contradiction to the assumption "$G = G_2/G_1$ is μ_*-free".

Subcase 2B: $\mathrm{cf}([\gamma_*]^{\leq \mu}, \subseteq) \geq \lambda(\langle\ \rangle, S_{\mathbf{s}})$.

Note that because $\mathbf{V}$ satisfies the strong hypothesis (see [She93b]), necessarily for some cardinal ∂ of cofinality $< \kappa_*$ we have $\lambda(\langle\ \rangle, S_{\mathbf{s}}) = \partial^+$.

In any case, clearly for every $\alpha \in [\gamma_*, \lambda)$, letting $\beta_\alpha := \min(S_4 \setminus \alpha)$, the pair $A_{\beta_\alpha}/B_{\langle \alpha \rangle}$ is not $\mathbf{c}_1$-free. So by renaming, without loss of generality

$$\alpha \geq \gamma_* \wedge \mathrm{cf}(\alpha) = \aleph_0 \Rightarrow \langle \alpha \rangle \in S,$$

and we continue as in Case 1 (so this also works in Subcase 2A). $\square_{3.13}$

References

[CG85] A. L. S. Corner and Rüdiger Göbel, *Prescribing endomorphism algebras, a unified treatment*, Proceedings of the London Mathematical Society. Third Series **50** (1985), 447–479.

[EM02] Paul C. Eklof and Alan Mekler, *Almost free modules: Set theoretic methods*, North–Holland Mathematical Library, vol. 65, North–Holland Publishing Co., Amsterdam, 2002, Revised Edition.

[Fuc73] Laszlo Fuchs, *Infinite abelian groups*, vol. I, II, Academic Press, New York, 1970, 1973.

[GHS14] Rüdiger Göbel, Daniel Herden, and Saharon Shelah, *Prescribing endomorphism algebras of $\aleph_n$-free modules*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 9, 1775–1816. MR 3273308

[GS09] Rüdiger Göbel and Saharon Shelah, *$\aleph_n$-free modules with trivial duals*, Results Math. **54** (2009), no. 1-2, 53–64. MR 2529626

[GSS13] Rüdiger Göbel, Saharon Shelah, and Lutz H. Strüngmann, *$\aleph_n$-free modules over complete discrete valuation domains with almost trivial dual*, Glasg. Math. J. **55** (2013), no. 2, 369–380. MR 3040868

[GT12] Rüdiger Göbel and Jan Trlifaj, *Approximations and endomorphism algebras of modules, vol. i, ii*, de Gruyter Expositions in Mathematics, Walter de Gruyter, 2012.

[MS94] Menachem Magidor and Saharon Shelah, *When does almost free imply free? (For groups, transversals, etc.)*, J. Amer. Math. Soc. **7** (1994), no. 4, 769–830. MR 1249391

[S+a] S. Shelah et al., *Tba*, In preparation. Preliminary number: Sh:F918.

[S+b] ______, *Tba*, In preparation. Preliminary number: Sh:F1303.

[S+c] ______, *Tba*, In preparation. Preliminary number: Sh:F1968.

[S+d] ______, *Tba*, In preparation. Preliminary number: Sh:F13.

[Shea] Saharon Shelah, *Dependent dreams: recounting types*, arXiv: 1202.5795.

[Sheb] ______, *Quite free Abelian groups with prescribed endomorphism ring.*

[She75a] ______, *A compactness theorem for singular cardinals, free algebras, Whitehead problem and transversals*, Israel J. Math. **21** (1975), no. 4, 319–349. MR 0389579

[She75b] ______, *The lazy model-theoretician's guide to stability*, Logique et Analyse (N.S.) **18** (1975), no. 71-72, 241–308. MR 0539969

[She75c] ______, *Why there are many nonisomorphic models for unsuperstable theories*, Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), vol. 1, Canad. Math. Congress, Montreal, Que., 1975, pp. 259–263. MR 0422015

[She78] ______, *Classification theory and the number of nonisomorphic models*, Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam-New York, 1978. MR 513226

[She80] ______, *Whitehead groups may not be free, even assuming CH. II*, Israel J. Math. **35** (1980), no. 4, 257–285. MR 594332

[She84a] ______, *A combinatorial principle and endomorphism rings. I. On p-groups*, Israel J. Math. **49** (1984), no. 1-3, 239–257. MR 788269

[She84b] ______, *A combinatorial theorem and endomorphism rings of abelian groups. II*, Abelian groups and modules (Udine, 1984), CISM Courses and Lect., vol. 287, Springer, Vienna, 1984, pp. 37–86. MR 789808

[She85] ______, *Incompactness in regular cardinals*, Notre Dame J. Formal Logic **26** (1985), no. 3, 195–228. MR 796637

[She90] ______, *Classification theory and the number of nonisomorphic models*, 2nd ed., Studies in Logic and the Foundations of Mathematics, vol. 92, North-Holland Publishing Co., Amsterdam, 1990, Revised edition of [Sh:a]. MR 1083551

[She93a] ______, *Advances in cardinal arithmetic*, Finite and infinite combinatorics in sets and logic (Banff, AB, 1991), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 411, Kluwer Acad. Publ., Dordrecht, 1993, arXiv: 0708.1979, pp. 355–383. MR 1261217

[She93b] ______, *More on cardinal arithmetic*, Arch. Math. Logic **32** (1993), no. 6, 399–428, arXiv: math/0406550. MR 1245523

[She94] ______, *Cardinal arithmetic*, Oxford Logic Guides, vol. 29, The Clarendon Press, Oxford University Press, New York, 1994. MR 1318912

[She96] ______, *If there is an exactly λ-free abelian group then there is an exactly λ-separable one in λ*, J. Symbolic Logic **61** (1996), no. 4, 1261–1278, arXiv: math/9503226. MR 1456106

[She99] ______, *Borel sets with large squares*, Fund. Math. **159** (1999), no. 1, 1–50, arXiv: math/9802134. MR 1669643

[She05] ______, *Super black box (ex. Middle diamond)*, Arch. Math. Logic **44** (2005), no. 5, 527–560, arXiv: math/0212249. MR 2210145

[She07] ______, *$\aleph_n$-free abelian group with no non-zero homomorphism to $\mathbb{Z}$*, Cubo **9** (2007), no. 2, 59–79, arXiv: math/0609634. MR 2354353

[She08] ______, *Theories with Ehrenfeucht-Fraïssé equivalent non-isomorphic models*, Tbil. Math. J. **1** (2008), 133–164, arXiv: math/0703477. MR 2563810

[She13a] ______, *Non-reflection of the bad set for $\check{I}_\theta[\lambda]$ and pcf*, Acta Math. Hungar. **141** (2013), no. 1-2, 11–35, arXiv: 1206.2048. MR 3102967

[She13b] ______, *Pcf and abelian groups*, Forum Math. **25** (2013), no. 5, 967–1038, arXiv: 0710.0157. MR 3100959

[She16] ______, *ZF + DC + AX_4*, Arch. Math. Logic **55** (2016), no. 1-2, 239–294, arXiv: 1411.7164. MR 3453586

[She19] ______, *Compactness in singular cardinals revisited*, Sarajevo J. Math. **15(28)** (2019), no. 2, 201–208, arXiv: 1401.3175. MR 4069744

[She22] ______, *Black boxes*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **65** (2022), 69–130, arXiv: 0812.0656 Ch. IV of The Non-Structure Theory" book [Sh:e]. MR 4636538

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